Report on the doctoral thesis

Properties of function spaces and operators acting on them

by H. Turčinová

The submitted thesis consists of an Introduction and the four published papers

I A. Nekvinda and H. Turčinová. Characterization of functions with zero traces via the distance function and Lorentz spaces. J. Math. Anal. Appl. 529, no. 1, Paper no. 127567, 2024. Doi: 10.1016/j.jmaa.2023.127567

II H. Turčinová. Basic functional properties of certain scale of rearrangement-invariant spaces. Math. Nachr. 296, no. 8, 3652–3675, 2023. Doi: 10.1002/mana.202000463

III M. Křepela, Z. Mihula, and H. Turčinová. Discretization and antidiscretiza- tion of Lorentz norms with no restrictions on weights. Rev. Mat. Complut. 35, no. 2, 615–648, 2022. Doi: 10.1007/s13163-021-00399-7

IV A. Gogatishvili, Z. Mihula, L. Pick, H. Turčinová, and T. Ünver. Weighted inequalities for a superposition of the Copson operator and the Hardy op- erator. J. Fourier Anal. Appl. 28, no. 2, Paper no. 24, 24 pp., 2022. Doi: 10.1007/s00041-022-09918-6

Paper I. This first paper is around the equivalence

$$u \in W_0^{1,p}(\Omega) \qquad \Longleftrightarrow \qquad \frac{u}{d} \in L_a^{1,\infty}(\Omega) \quad \text{and} \quad |\nabla u| \in L_p(\Omega)\,.$$

It is clear that $|\nabla u| \in L_p(\Omega)$ is the main ingredient but that does not mean that one can really relax the condition $u \in L_p(\Omega)$. Here progress has been made in two directions:

(a) The authors have replaced $L^{1,q}(\Omega)$, $q < \infty$, by the strictly larger space $L_a^{1,\infty}(\Omega)$.

(b) The restrictions for the underlying domain has been weakened.

Both is of interest. Also of some importance are the investigations regarding optimality of the found sufficient conditions. In summary, for me this is an impressive paper on an interesting subject, which will be not only of interest for the community *function spaces*, but also for those which deal with elliptic partial differential equations on non-smooth domains in Kondratyev spaces.

Paper II. In interpolation theory, in the last century, various authors were working with the so-called p-convexification of a Banach lattice X. The constructions showing up in this paper are of the same spirit. For X being a rearrangement-invariant space of measurable functions which are finite a.e. and $\alpha > 0$ the set $X^{\{\alpha\}}$ is given as the collection of all such f satisfying

$$\Big\| |f|^{\alpha} \Big\|_X^{1/\alpha} < \infty\,.$$

Similarly, the set $X^{\langle\alpha\rangle}$ is defined as collection of all f satisfying

$$\Big\| \Big((|f|^{\alpha})^{**} \Big)^{1/\alpha} \Big\|_{\overline{X}} < \infty\,,$$

where $\overline{X}$ denotes the representation space of X. The author investigates some basic properties of $X^{\{\alpha\}}$ (Banach function norm, fundamental function, embeddings). The considered problems are definitely not simple, but the object is rather special. However, it has found applications (it is used in Paper III).

Paper III and IV. These papers deal with weighted inequalities. In Paper III the three weight inequality

$$\Big(\int_0^L (f^*(t))^q w(t)dt\Big)^{1/q} \leq C\Big(\int_0^L \Big(\int_0^t u(s)ds\Big)^{-p}\Big(\int_0^t f^*(s)u(s)ds\Big)^p v(t)dt\Big)^{1/p}$$

is investigated. Here $L \in (0,\infty]$, $p, q \in (0,\infty)$ and u, v, w are locally integrable weights, u being stricly positive. Here the authors extend earlier studies and remove some degeneracy assumptions on the weights needed before. In Paper IV the three weight inequality

$$\Big(\int_a^b \Big(\int_t^b \Big(\int_a^s f(\tau)^p v(\tau)d\tau\Big)^{q/p} u(s)ds\Big)^{r/q} w(t)dt\Big)^{1/r} \leq C\int_a^b f(t)dt$$

is investigated. Here $a, b \in [-\infty,\infty]$, $a < b$, $q, r \in (0,\infty)$ and $p \in (0,1]$. The authors characterize all triples u, v, w of weights on (a,b) such that this inequality holds for every nonnegative measurable function f. In both cases the obtained results have a final character. This is impressive. The used methods are nontrivial and refined over decades. All variants, relatives of the Hardy inequality have been proved to be extremely useful. Here I have no doubts what concerns applications in future.

All four papers contain new results. The papers have appeared in good journals, partly in very good journals, which can be understood as a quite positive evaluation as well.

With this thesis H. Turčinová has proved here ability for creative scientific work. At the same time she has shown an extensive knowledge in function spaces and related fields. The four publications deal with three different topics. I have no doubts that the thesis will meet all standards at the Faculty of Mathematics and Physics of the Charles University. Therefore I strongly recommend acceptance of this doctoral thesis.

Sincerely yours