Report on the Ph.D. thesis
"Properties of function spaces and operators acting on them"
by Hana Turčinová

This thesis is largely concerned with various important properties of function spaces and operators acting on them. It begins with a very helpful account of the principal topics to be discussed, including such matters as Banach function spaces, Sobolev spaces, the reduction principle and inequalities of Hardy type; the state of current knowledge is briefly outlined. Following this is a brief description of the four published papers that constitute the heart of the thesis, and finally these papers are given in full.

Paper 1 is concerned with the Sobolev space $W^{1,p}(\Omega)$ (where Ω is an open subset of $\mathbb{R}^n$ and $p \in [1, \infty)$) and its subspace $W_0^{1,p}(\Omega)$ (the closure of $C_0^\infty(\Omega)$).The importance of this subspace stems from its connection with Dirichlet problems and accordingly much attention has been devoted to the discovery of criteria adequate to detect elements of it; the distance function d ($d(x) = dist.(x, \partial\Omega)$) plays a prominent role in this activity. It is shown that, under mild conditions on Ω, $f \in W_0^{1,p}(\Omega)$ if and only if

$$u/d \in L_a^{1,\infty}(\Omega) \text{ and } |\nabla u| \in L^p(\Omega),$$

where $L_a^{1,\infty}(\Omega)$ is the space of functions in the Lorentz space $L^{1,\infty}(\Omega)$ with absolutely continuous $L^{1,\infty}(\Omega)-$ quasisnorm. This is an excellent result, improving much earlier work. It is shown that replacement of $L_a^{1,\infty}(\Omega)$ by $L^{1,\infty}(\Omega)$ is not allowed.

The second paper is concened with the construction and properties of certain r.i. spaces. Let X be an r.i. Banach function space over a non-atomic $\sigma-$finite measure space $(\mathcal{R}, \mu)$ and let $\alpha \in (0, \infty)$.The space $X^{\langle\alpha\rangle}$ is defined to be the set of all $\mu-$measurable scalar functions f on $(\mathcal{R}, \mu)$ for which

$$\|f\|_{X^{\langle\alpha\rangle}} := \left\| \left((|f|^\alpha)^{**}\right)^{1/\alpha} \right\|_{\overline{X}(0,\mu(\mathcal{R}))} < \infty;$$

here $\overline{X}$ denotes the representation space. Various properties of such spaces are exhibited: it is shown that $X^{\langle\alpha\rangle}$ is continuously embedded in X; that the converse holds iff the Hardy averaging operator is bounded on an appropriate space; and, most importantly, that the $X^{\langle\alpha\rangle}$ give rise to a 1-parameter path of function spaces proceeding from a Lebesgue space to a Zygmund class.

In the third paper a new technique is provided for discretisation and antidiscretisation for weighted r.i. norms without the restriction on the weights that appeared in much earlier work of this nature. Application of these results is made in paper four to derive an inequality for the superposition of the Copson and Hardy operators under weakened conditions on the three weights involved..

My opinion is that this thesis presents results of exceptional interest; and that in producing it Hana Turčinová has displayed both creative ability of a high order and a remarkable command of technique. I believe that her work is well above the standard normally required of a Ph.D. thesis and have absolutely no hesitation in recommending that she be awarded the Ph.D. degree.

D. E. Edmunds

D. E. Edmunds

Department of Mathematics, University of Sussex
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