THE SUPERVISOR'S REPORT OF THE PH.D. THESIS BY RNDR. HANA TURČINOVÁ

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The Ph.D. Thesis by Hana Turčinová is mainly focused on studying various important properties of function spaces containing measurable functions, and operators acting on them. The thesis consists of four papers. We shall first give a brief description of the contents of each of the papers separately.

1. CHARACTERIZATION OF FUNCTIONS WITH ZERO TRACES VIA THE DISTANCE FUNCTION AND LORENTZ SPACES

This paper is devoted to the improvement of the condition that characterizes when a function belongs to the space $W_0^{1,p}(\Omega)$ via the distance function from the boundary. It is proved here that $u \in W_0^{1,p}(\Omega)$ if and only if $\nabla u \in L^p(\Omega)$ and $u/d \in L_a^{1,\infty}(\Omega)$ which improves all earlier results. Note that $L_a^{1,\infty}(\Omega)$ denotes the set of all functions from $L^{1,\infty}(\Omega)$ with absolutely continuous norms. Of course, a certain regularity of Ω is required.

2. BASIC FUNCTIONAL PROPERTIES OF CERTAIN SCALE OF REARRANGEMENT-INVARIANT SPACES

In this paper the author introduced a new scale of function spaces. Let X be a given rearrangement-invariant Banach function space and $\alpha \in (0, \infty)$. Assume that $X^{\langle\alpha\rangle}$ is a set of all functions with finite functional

$$\|f\|_{X^{\langle\alpha\rangle}} = \|((|f|^\alpha)^{**})^{\frac{1}{\alpha}}\|_{\overline{X}},$$

where $\overline{X}$ is the representation space of X. Many properties of these spaces are investigated here.

3. DISCRETIZATION AND ANTIDISCRETIZATION OF LORENTZ NORMS WITH NO RESTRICTIONS ON WEIGHTS

A new discretization and antidiscretization technique is developed in this paper for weighted rearrangement-invariant norms. The main aim is to eliminate "non-degeneracy" restrictions on the involved weights. A new method is introduced here which enables to provide two-sided estimates of the optimal constant C that renders the inequality

$$\left(\int_0^L (f^*(t))^q w(t)\, dt\right)^{\frac{1}{q}} \leq C \left(\int_0^L \left(\int_0^t u(s)\, ds\right)^{-p} \left(\int_0^t f^*(s)u(s)\, ds\right)^p v(t)\, dt\right)^{\frac{1}{p}}$$

true. This corresponds to the problem of characterizing the embedding of classical Lorentz spaces $\Gamma_u^p(v) \hookrightarrow \Lambda^q(w)$, where $\Gamma_u^p(v)$ is a generalization of $\Gamma^p(v)$ achieved by adding an extra weight u to appropriate places. In the case of weights that would be otherwise excluded by the restrictions, it is shown that the additional

limit terms naturally appear in the characterizations of the optimal C. A weak analogue for $p = \infty$ is also presented.

4. Weighted inequalities for a superposition of the Copson operator and the Hardy operator

In this paper the authors study a three-weight inequality for the superposition of the Hardy operator and the Copson operator, namely

$$\left(\int_a^b \left(\int_t^b \left(\int_a^s f(\tau)^p v(\tau) \, d\tau \right)^{\frac{q}{p}} u(s) \, ds \right)^{\frac{r}{q}} w(t) \, dt \right)^{\frac{1}{r}} \leq C \int_a^b f(t) \, dt,$$

$q, r > 0$ and $p \in (0, 1]$. They focus on characterizing those triples of weight functions (u, v, w) for which this inequality holds for all nonnegative measurable functions f. A new method of discretization and antidiscretization presented in the third paper enables to avoid duality techniques and therefore to remove various restrictions that appear in earlier work.

We shall now present detailed quotations of the papers included in the theses.

References

[1] A. Nekvinda and H. Turčinová, Characterization of functions with zero traces via the distance function and Lorentz spaces. *J. Math. Anal. Appl.* **529, no 1.** (2024), 28 pp.

[2] H. Turčinová, Basic functional properties of certain scale of rearrangement-invariant spaces. *Math. Nachr.* **296** (2022, no 8.), 3652–3675.

[3] M. Křepela, Z. Mihula and H. Turčinová, Discretization and antidiscretization of Lorentz norms with no restrictions on weights. *Rev. Mat. Complut.* **35, no 2.** (2022), 615–648.

[4] A. Gogatishvili, Z. Mihula, L. Pick, H. Turčinová and T. Ünver, Weighted inequalities for a superposition of the Copson operator and the Hardy operator. *J. Fourier Anal. Appl.* **28, no 2.** (2022), 24 pp.

The conclusion

From these four articles, which have already appeared in renowned journals, a considerable erudition of RNDr. Hana Turčinová is evident. She is obviously well established in the field of function spaces, which serves as the basis of the modern theory of partial differential equations. At the same time, I understand the quoted papers as a promise for future scientific work in this area of mathematics.

In conclusion, I recommend this thesis to be accepted, and I strictly recommend RNDr. Hana Turčinová to be awarded the degree Ph.D.

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