

# Bachelor Thesis Review

Faculty of Mathematics and Physics, Charles University

**Thesis author** Lukáš Kriško  
**Thesis title** Tennenbaum phenomena in models of arithmetic  
**Year submitted** 2024  
**Study program** Computer Science  
**Specialization** Logic

**Review author** Neil Thapen Advisor  
**Department** Institute of Mathematics CAS

## Overall

	good	OK	poor	insufficient
Assignment difficulty	X			
Assignment fulfilled	X			
Total size <i>... text and code, overall workload</i>	X			

## Thesis Text

	good	OK	poor	insufficient
Form <i>... language, typography, references</i>	X			
Structure <i>... context, goals, analysis, design, evaluation, level of detail</i>	X			

Tennenbaum's theorem is a classical result that no nonstandard model of Peano Arithmetic can be recursive. In particular, in such an object neither the addition  $+$  or the multiplication function  $\cdot$  can be recursive. The task set was to present this result and to explore some extensions of it, such as, to what extent it remains true for theories much weaker than PA, or what can be said about arithmetical functions other than  $+$  and  $\cdot$ .

The extension to weaker theories is well-studied in the literature on Tennenbaum's theorem, and it is not reasonable to expect original research on it at the undergraduate level. Lukáš presents a version of the theorem for the subtheory  $I\Delta_0$  of PA, due to McAloon. This is in Sections 4 and 5, and is preceded by three sections developing the necessary background in basic Peano arithmetic and computability theory. These are in general clearly presented and show a thorough understanding of the mathematics. They are drawn from several sources and combined in an intelligent way. The same goes for Section 6, which again is not an original result, showing that there is a model of PA in which the order relation and successor function are recursive.

The rest of the thesis, Part II, is original work addressing whether models exist in which the quotient and remainder functions are recursive, either individually or in various combinations, and either as full binary functions or restricted to a fixed standard divisor. Addressing first Sections 7 to 9, the questions studied are not difficult but it is novel work, the mathematics is carried out correctly, and there is at least one somewhat complex construction, of a model in which order and remainder are recursive, adapting a construction from Section 6.

What I have described so far is already 100 pages, and I believe by itself forms an excellent, long, undergraduate thesis, showing a good understanding of a difficult topic and the capability

of doing independent work. It is followed by two more sections, another 35 pages, which are last-minute additions with one of them, Section 10, extremely technical. Lukáš has not discussed this material with me and I have not been able to check everything thoroughly in the time since submission. It reads like a draft of material probably suitable for a masters thesis, and in particular contains a fairly sophisticated construction studying countable nonstandard models of PA restricted to a single unary function, namely division by a standard constant, showing that these are isomorphic to a canonical such structure, in the style of Cantor's theorem, and in this way constructing a model in which this function is recursive.

Some weaknesses: the thesis is very long – partly this is that Lukáš has chosen to go into careful detail about some of the mathematical basics, and partly that in later sections he has a lot to say; the discussion of “computably codable sets” at the end of Section 3 is too informal; Sections 10 and 11 should maybe not have been included, as they are; there is sometimes confusion about what it means for elements of a structure to be findable by an algorithm; some of the technical parts could use more text motivating what they are doing.

<b>Overall grade</b>	Excellent
<b>Award level thesis</b>	No

Date

26/1/24

Signature