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Algebraic Tools in Combinatorial Geometry and Topology

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Abstract: In this thesis we study combinatorial problems through the lenses of the exterior algebra. This algebra is a natural object to model set systems as well as simplicial complexes. Moreover, it is possible to translate the classical operations from the combinatorial setting to the setting of exterior algebra. For example, set intersection and the classical boundary map from topology. Often this point of view makes it possible to translate the combinatorial problem to a problem regarding the dimension of certain vector space. The latter one might turn up to be easier since we can study the dimension of a vector space with linear maps. We follow such approach in this thesis.

In the first part we study the weak saturation problem introduced by Bollobás in the 60's. This problem consists in, given a host graph F and a pattern graph H, to determine the minimum number of infected edges of F one has to start with in order for the infection to spread, according to the pattern H, to the whole host graph F. We study this problem when the host and the pattern are complete uniform multipartite hypergraphs.

Next, we work on a generalization of a theorem by Helly regarding intersecting patterns of convex sets. Concretely, given a finite family of convex sets in $\mathbb{R}^d$ partitioned into $d+1$ colors classes such that a fraction α of the colorful $(d+1)$-tuples intersect, what is the size of the largest monochromatic intersecting subfamily that we can guaranty? We answer this question.

In the third part we study the notion of volume-rigidity for simplicial complexes. This is a generalization of (generic) rigidity for graphs to higher dimensional objects. We relate volume-rigidity with exterior algebraic shifting and show that compact surfaces of small genus without boundary are volume-rigid.

In the last part we study a generalization of a classical theorem by Erdős, Ko and Rado on the maximal size of a pairwise-intersecting family. Concretely we are interested in charaterizing the family achieving the maximal size when it is restricted to a simplicial complex.

Keywords: Helly theorem Erdős-Ko-Rado rigidity simplicial complex algebraic shifting exterior algebra weak saturation

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1. Introduction

In combinatorics one is usually interested in counting, or enumerating, objects that satisfy certain properties. For example, what is the number of trees on n vertices? What is the largest size of a pairwise-intersecting family of k-subsets from a ground set with n elements? One of the possible ways to answer these and similar questions is with the aid of linear algebra. In this setting counting the desired quantity translates into computing the dimension of a certain vector space. It often turns out that the later one is easier to compute since one can study the dimension of a vector space through linear maps which capture the properties satisfied by the objects of interest.

In this thesis we will mainly work with the exterior algebra of a vector space since, among other properties, it is a natural framework to model the set intersection property as well as the classical boundary map from topology. Moreover, one can associate elements of a set family with basis elements in this algebra in a very simple way. This process associates a combinatorial object with an algebraic object. We will be also interested in the opposite direction, that is from a vector space we obtain a combinatorial object which is in some sense, to be made precise later, the simplest object satisfying the desired properties. This last part refers to a compression operation, so called algebraic shifting, introduced by Kalai [50] and it turns out that one can read structural and geometric information from the compressed object.

Results of the thesis. Bollobás [15] introduced the problem of weak saturation on graphs. Given a pattern graph H and a host graph F, the objective is to determine the minimum number of infected edges that one has to start with in order to propagate the infection according to H to all the edges of F. In Chapter 3 we solve this problem for the case when the pattern and the host graphs are complete uniform multipartite hypergraphs.

The celebrated theorem of Helly [38] states that every finite family of convex sets in $\mathbb{R}^d$ with every $(d+1)$-tuple of its elements having non-empty intersection satisfies the condition that all of its members have non-empty intersection. Kalai [46] and Eckhoff [26] showed independently the optimal size of an intersecting subfamily if one has only that a certain fraction of the $(d+1)$-tuples is intersecting. In Chapter 4 we extend this result for the case of colorful families of convex sets.

Asimow and Roth [5, 6] introduced the notion of graph rigidity. Lee [57, 67] showed that the membership of a certain edge in the symmetric algebraic shift of the corresponding graph is equivalent with the graph being (generically) rigid. In Chapter 5 we establish an analogous link between volume-rigidity and the membership of a face in the exterior algebraic shifting of the corresponding simplicial complex. Moreover, we show that any triangulation of a small genus surface is volume-rigidity.

What is the largest cardinality of a family of pairwise-intersecting sets? The celebrated theorem of Erdős, Ko and Rado [28] states that the family achieving the maximal size is given by subsets all containing a fixed element. In Chapter 6 we show that the statement is still true when the pairwise-intersecting family is restricted to a sequentially Cohen-Macaulay near-cone. This family of simplicial complexes is quite large, for example it contains the simplicial complex which faces are given by the independence sets of a chordal graph that contains an isolated vertex.

Bibliographic remarks. The results of this thesis have been published or are being considered for publication. The relation of the chapters with the published work is as follows. In Chapter 2 we establish the necessary preliminaries about simplicial complexes and exterior algebra. This chapter is based on the respective section from the articles [19, 22]. Chapter 3 is based on a joint work with Mykhaylo Tyomkyn and Martin Tancer [22]. Chapter 4 is based on a joint work with Afshin Goodarzi and Martin Tancer [19]. Chapter 5 is based on a joint work with Eran Nevo and Yuval Peled [20]. Chapter 6 is based on a joint work with Russ Woodroofe and some of the results appeared in [21]. The presentation of this last chapter is different from the already published work and it contains unpublished results which will be part of the full version.

2. Preliminaries

2.1 Simplicial complex

A *(finite abstract) simplicial complex* is a set system K on a finite *set of vertices* N such that whenever $A \in K$ and $B \subseteq A$, then $B \in K$. The elements of N represent vertices and we will typically denote them by letters such as v or w. The elements of K are *faces* (a.k.a., *simplices*) of K. The *dimension* of a face $A \in K$ is defined as $\dim A = |A| - 1$; this corresponds to representing A as an $(|A| - 1)$-dimensional simplex. The dimension of K, denoted $\dim K$, is the maximum of the dimensions of faces in K. A face of dimension k is a *k-face* in short. *Vertices* of K are usually identified with 0-faces, that is, $v \in N$ is identified with $\{v\} \in K$. (Though the definition of simplicial complex allows that $\{v\} \notin K$ for $v \in N$, in our applications we will always have $\{v\} \in K$ for $v \in N$.) The *k-skeleton of K*, denoted by $K^{(k)}$, is the simplicial complex formed by the set of faces from K whose dimension is at most k. The *f-vector of K* is defined by $f(K) = (f_{-1}(K), f_0(K), \ldots, f_d(K))$ where $f_k(K)$ is the number of k-faces in K with the convention that $f_{-1}(K) = 1$ if K is not the empty complex.

Example 1. *Let K be the simplicial complex depicted in Figure 2.1, that is*

$$
\begin{aligned}
K = \{ & \\
& \emptyset, \\
& \{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\} \text{ 0-}\textit{faces, a.k.a., vertices,} \\
& \{a,b\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{c,e\} \text{ 1-}\textit{faces, a.k.a., edges,} \\
& \{a,b,d\} \text{ 2-}\textit{faces} \\
& \}.
\end{aligned}
$$

Since the simplicial complex is determined by its maximal faces, called facets, we will say that the simplicial complex is generated by its set of facets. In this example, K is generated by $\{\{f\}, \{b,c\}, \{c,d\}, \{c,e\}, \{a,b,d\}\}$. Its f-vector is $f(K) = (1, 6, 6, 1)$.

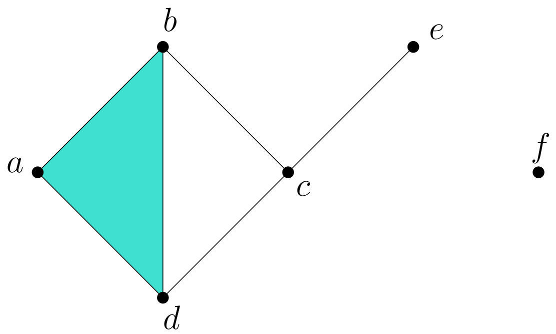

Figure 2.1: Example of a simplicial complex.

2.2 Exterior algebra

Let N be a set of size n, ordered with a total order $<$ and let $V = \mathbb{R}^N$ be an n-dimensional real vector space with a basis $(e_v)_{v\in N}$. The *exterior algebra* of V, denoted by $\bigwedge V$, is a 2^n-dimensional vector space with basis $(e_S)_{S\subseteq N}$ and an associative bilinear product operation, denoted by $\wedge$, that satisfies

(i) $e_\emptyset$ is the neutral element, i.e., $e_\emptyset \wedge e_S = e_S = e_S \wedge e_\emptyset$;

(ii) $e_S = e_{s_1} \wedge \cdots \wedge e_{s_k}$ for $S = \{s_1 < \cdots < s_k\} \subseteq N$;

(iii) $e_v \wedge e_w = -e_w \wedge e_v$ for all $v, w \in N$.

For $0 \le k \le n$ we denote by $\bigwedge^k V$ the subspace of $\bigwedge V$ with basis $(e_S)_{S\in\binom{N}{k}}$. Denote by $\langle\cdot,\cdot\rangle$ the standard inner product (dot product) on V as well as on $\bigwedge V$ with respect to the basis $(e_v)_{v\in N}$ and $(e_S)_{S\subseteq N}$ respectively; that is, for every pair of sets $S, T \subseteq N$, the inner product $\langle e_S, e_T\rangle$ is 1 if $S = T$ and 0 otherwise.

If $(f_v)_{v\in N}$ is another basis of V, then $(f_S)_{S\subseteq N}$ is a new basis of $\bigwedge V$, where f_S stands for $f_{s_1} \wedge \cdots \wedge f_{s_k}$ for $S = \{s_1 < \cdots < s_k\} \subseteq N$. Similarly, $(f_S)_{S\in\binom{N}{k}}$ is a basis of $\bigwedge^k V$ for $k \in \{0, \ldots, n\}$. The formulas (i), (ii) and (iii) remain valid for the basis $(f_v)_{v\in N}$ due to the definition of f_S and bilinearity of $\wedge$.

In particular, $\bigwedge V$ and $\bigwedge^k V$ do not depend on the initial choice of the basis. Using (ii) and (iii) iteratively, for $S, T \subseteq N$ we get

$$f_S \wedge f_T = \begin{cases} \operatorname{sgn}(S,T) f_{S\cup T} & \text{if } S \cap T = \emptyset \\ 0 & \text{if } S \cap T \neq \emptyset, \end{cases} \tag{2.1}$$

where $\operatorname{sgn}(S,T)$ is the sign of the permutation of $S \cup T$ obtained by first placing the elements of S (in our total order $<$) and then the elements of T. Equivalently, $\operatorname{sgn}(S,T) = (-1)^{\alpha(S,T)}$ where $\alpha(S,T) = |\{(s,t) \in S \times T : t < s\}|$ is the number of transpositions.

Let $A = (a_{vw})_{v,w\in N}$ be the transition matrix from $(e_v)_{v\in N}$ to $(f_v)_{v\in N}$, meaning that $f_v = \sum_{w\in N} a_{vw} e_w$. Then, for $S \subseteq N$ of size k, f_S can be expressed as

$$f_S = \sum_{T\in\binom{N}{k}} \det(A_{S|T}) e_T, \tag{2.2}$$

where $A_{S|T}$ is the submatrix of A formed by rows in S and columns in T, i.e., $A_{S|T} = (a_{vw})_{v\in S, w\in T}$.

As noted in [46], it follows from the Cauchy-Binet formula that if the basis $(f_v)_{v\in N}$ is orthonormal then $(f_S)_{S\subseteq N}$ is orthonormal as well. For completeness, we provide a short explanation. Let $S, L \subseteq N$ be a pair of subsets. If $|S| \neq |L|$, then f_S and f_L belong to two orthogonal subspaces of $\bigwedge V$, namely $\bigwedge^{|S|} V$ and $\bigwedge^{|L|} V$, and so $\langle f_S, f_L\rangle = 0$. On the other hand, if $|S| = |L| =: k$, then by writing f_S and f_L in the standard basis $(e_T)_{T\subseteq N}$ we have that

$$\langle f_S, f_L\rangle = \sum_{T\in\binom{N}{k}} \det(A_{S|T}) \det(A^t_{L|T}) = \det(A_{S|N} A^t_{L|N}),$$

where B^t stands for the transpose matrix of B (and expressions like $A^t_{L|T}$ stand for $(A_{L|T})^t$), and the last equality holds by the Cauchy-Binet formula (see e.g. Section

1.2.4 of [37]). Notice that for any $u \in S$ and $w \in L$ we have $(A_{S|N}A^t_{L|N})_{u,w} = \langle f_u, f_w \rangle$, and since $(f_v)_{v\in N}$ is orthonormal this is 1 if $u = w$ and 0 otherwise. Therefore, if $S = L$, the product $A_{S|N}A^t_{L|N}$ is the identity matrix and consequently the determinant will be 1. On the other hand, if $S \neq L$, the product $A_{S|N}A^t_{L|N}$ will have a zero column, and so the determinant will be 0. The above claim follows.

We say that the change of basis from $(e_v)_{v\in N}$ to $(f_v)_{v\in N}$ is *generic* if $\det(A_{S|T}) \neq 0$ for every $S, T \subseteq N$ of the same size; that is, every square submatrix of A has full rank. It is known (see e.g. [46, Section 2]) that $(f_v)_{v\in N}$ can be chosen to be both generic and orthonormal. For a basis $(f_v)_{v\in N}$ generic with respect to $(e_v)_{v\in N}$ and a pair of sets $S, T \in \binom{N}{k}$ we have

$$\langle f_S, e_T \rangle \overset{(2.2)}{=} \langle \sum_{T' \in \binom{N}{k}} \det(A_{S|T'}) e_{T'}, e_T \rangle = \sum_{T' \in \binom{N}{k}} \det(A_{S|T'}) \langle e_{T'}, e_T \rangle = \det A_{S|T} \neq 0. \tag{2.3}$$

The following lemma is implicitly contained in [46].

Lemma 2. *If the columns of an $m \times n$ matrix A are linearly independent, then the columns of $C_k(A)$ are linearly independent as well.*

Proof. If columns of A are linearly independent, then $n \leq m$. Consider an arbitrary square submatrix B of rank n. Considering B as a transition matrix from $(e_i)_{i\in N}$ to $(f_i)_{i\in N}$, we get that $C_k(B)$ is a transition matrix from $(e_S)_{S\in\binom{N}{k}}$ to $(f_S)_{S\in\binom{N}{k}}$, thus $C_k(B)$ has full rank. However, $C_k(B)$ is also a submatrix of $C_k(A)$ with all $\binom{n}{k}$ columns. ☐

2.2.1 Left interior product

In this subsection we recall the *left interior product* $g \llcorner f$ of g and f. We refer to Section 2.2.6 of [72] for a more extensive coverage of the topic.

Lemma 3. *For any $f, g \in \bigwedge V$ there exists a unique element $g \llcorner f \in \bigwedge V$ that satisfies*

$$\langle h, g \llcorner f \rangle = \langle h \wedge g, f \rangle \text{ for all } h \in \bigwedge V. \tag{2.4}$$

Furthermore, assuming $f \in \bigwedge^s V$ and $g \in \bigwedge^t V$, if $t > s$ then $g \llcorner f = 0$, while if $t \leq s$ then $g \llcorner f \in \bigwedge^{s-t} V$.

Proof. For $f, g \in \bigwedge V$ we set

$$g \llcorner f := \sum_{S \subseteq N} \langle e_S \wedge g, f \rangle e_S.$$

To verify that this satisfies (2.4) let $h \in \bigwedge V$ be arbitrary. By bilinearity of $\langle \cdot, \cdot \rangle$ and $\wedge$, and orthonormality of $(e_S)_{S\subseteq N}$ we have

$$\begin{aligned} \langle h, g \llcorner f \rangle &= \langle h, \sum_{S \subseteq N} \langle e_S \wedge g, f \rangle e_S \rangle = \sum_{S \subseteq N} \langle e_S \wedge g, f \rangle \langle h, e_S \rangle \\ &= \Big\langle \sum_{S \subseteq N} \langle h, e_S \rangle (e_S \wedge g), f \Big\rangle = \Big\langle \Big(\sum_{S \subseteq N} \langle h, e_S \rangle e_S \Big) \wedge g, f \Big\rangle \\ &= \langle h \wedge g, f \rangle. \end{aligned}$$

To show uniqueness, suppose that z is an element in $\bigwedge V$ that satisfies (2.4). Then for each $T \subseteq N$ we have

$$\langle e_T, z\rangle \overset{(2.4)}{=} \langle e_T \wedge g, f\rangle \overset{(2.4)}{=} \langle e_T, g \llcorner f\rangle.$$

Therefore z and $g \llcorner f$ are identical, as their inner products with all basis elements coincide.

Now assume that $f \in \bigwedge^s V$ and $g \in \bigwedge^t V$, and let $S \subseteq N$ be arbitrary. By (2.4) we have

$$\langle e_S, g \llcorner f\rangle = \langle e_S \wedge g, f\rangle.$$

Observe that $e_S \wedge g \in \bigwedge^{|S|+t}$ while $f \in \bigwedge^s V$ and these spaces are orthogonal unless $|S| + t = s$. Hence, $g \llcorner f = 0$ for $t > s$ and $g \llcorner f \in \bigwedge^{s-t} V$ otherwise. $\square$

It is straightforward to check from the definition that the left interior product is bilinear:

- $(f + g) \llcorner h = (f \llcorner h) + (g \llcorner h)$,
- $f \llcorner (g + h) = (f \llcorner g) + (f \llcorner h)$,

and satisfies

$$h \llcorner (g \llcorner f) = (h \wedge g) \llcorner f. \tag{2.5}$$

Lemma 4. *Let $(f_v)_{v \in N}$ be an orthonormal basis of V. Then, for any $S, T \subseteq N$ we have*

$$f_T \llcorner f_S = \begin{cases} \operatorname{sgn}(S \setminus T, T) f_{S \setminus T} & \text{if } T \subseteq S, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Put $s := |S|$ and $t := |T|$. If $t > s$ then by Lemma 3 we have $f_T \llcorner f_S = 0$ and the conclusion follows. So we may assume that $s \geq t$, and by the same lemma it follows that $f_T \llcorner f_S \in \bigwedge^{s-t} V$. Since the basis $(f_v)_{v \in N}$ is orthonormal, so is the basis $(f_L)_{L \in \binom{N}{s-t}}$ of $\bigwedge^{s-t} V$. Expressing $f_T \llcorner f_S$ in this basis and using (2.4), we obtain

$$f_T \llcorner f_S = \sum_{L \in \binom{N}{s-t}} \langle f_L, f_T \llcorner f_S\rangle f_L = \sum_{L \in \binom{N}{s-t}} \langle f_L \wedge f_T, f_S\rangle f_L.$$

Due to (2.1) and orthonormality of $(f_v)_{v \in N}$ we have $\langle f_L \wedge f_T, f_S\rangle = 0$ unless $T \subseteq S$ and $L = S \setminus T$. Therefore, using (2.1) again we get

$$f_T \llcorner f_S = \begin{cases} \langle f_{S \setminus T} \wedge f_T, f_S\rangle f_{S \setminus T} = \operatorname{sgn}(S \setminus T, T) f_{S \setminus T} & \text{if } T \subseteq S, \\ 0 & \text{if } T \not\subseteq S. \end{cases}$$

$\square$

Lemma 5. *Let $(f_v)_{v \in N}$ be a generic orthonormal basis of V with respect to $(e_v)_{v \in N}$. For a pair of sets $T, R \subseteq N$ of sizes t and r, respectively, such that $r \geq t$ we have*

$$f_T \llcorner e_R = \sum_{S \in \binom{N \setminus T}{r-t}} \lambda_S f_S,$$

where all the coefficients λ_S are non-zero.

Proof. By Lemma 3 we have that $f_T \llcorner e_R \in \bigwedge^{r-t} V$. Since $(f_S)_{S \in \binom{N}{r-t}}$ is an orthonormal basis of $\bigwedge^{r-t} V$, we can write

$$f_T \llcorner e_R = \sum_{S \in \binom{N}{r-t}} \langle f_S, f_T \llcorner e_R \rangle f_S.$$

Applying (2.4) and (2.1) gives

$$\langle f_S, f_T \llcorner e_R \rangle = \langle f_S \wedge f_T, e_R \rangle = \begin{cases} \pm \langle f_{S \cup T}, e_R \rangle & \text{if } S \cap T = \emptyset, \text{, i.e., if } S \in \binom{N \setminus T}{r-t}, \\ 0 & \text{otherwise.} \end{cases}$$

Setting $\lambda_S = \langle f_S \wedge f_T, e_R \rangle$ for $S \in \binom{N \setminus T}{r-t}$, we thus obtain

$$f_T \llcorner e_R = \sum_{S \in \binom{N \setminus T}{r-t}} \lambda_S f_S,$$

as claimed. In addition, since we assumed that $(f_v)_{v \in N}$ is generic with respect to $(e_v)_{v \in N}$, we have $\lambda_S = \pm \langle f_{S \cup T}, e_R \rangle \neq 0$ by (2.3) for all $S \in \binom{N \setminus T}{r-t}$. □

3. Weak saturation

Let F and H be q-uniform hypergraphs (q-graphs for short); we identify hypergraphs with their edge sets. We say that a subgraph $G \subseteq F$ is *weakly H-saturated* in F if the edges of $F \setminus G$ can be ordered as $e_1, \ldots, e_k$ such that for all $i \in [k]$ the hypergraph $G \cup \{e_1, \ldots, e_i\}$ contains an isomorphic copy of H which in turn contains the edge e_i. We call such $e_1, \ldots, e_k$ an *H-saturating sequence* of G in F. The *weak saturation number* of H in F, $\mathrm{wsat}(F, H)$ is the minimum number of edges in a weakly H-saturated subgraph of F. When F is complete of order n, we simply write $\mathrm{wsat}(n, H)$.

Weak saturation was introduced by Bollobás [15] in 1968 and is related to (strong) graph saturation: G is H-saturated in F if adding any edge of $F \setminus G$ would create a new copy of H. However, a number of properties of weak saturation make it a more natural object of study. Firstly, it follows from the definition that any graph G achieving $\mathrm{wsat}(F, H)$ has to be H-free (we could otherwise remove an edge from a copy of H in G resulting in a smaller example), while for strong saturation H-freeness may or may not be imposed, resulting in two competing notions (see [64] for a discussion). Secondly, a short subadditivity argument originally due to Alon [2] shows that for every 2-uniform H, $\lim_{n\to\infty} \mathrm{wsat}(n, H)/n$ exists. Whether the same holds for strong saturation is a longstanding conjecture of Tuza [83]. And thirdly, weak saturation lends itself to be studied via algebraic methods, thus offering insight into algebraic and matroid structures underlying graphs and hypergraphs.

The most natural case when F and H are cliques was the first to be studied. Let K_r^q denote the complete q-graph of order r. Confirming a conjecture of Bollobás, Frankl [34], and Kalai [47, 48] independently proved that $\mathrm{wsat}(n, K_r^q) = \binom{n}{q} - \binom{n-r+q}{q}$. Another proof has been given by Alon [2] and in hindsight this conjecture could be also derived from an earlier paper of Lovász [60]. While the upper bound is a construction that is easy to guess (a common feature in weak saturation problems), all of the above lower bound proofs rely on algebraic or geometric methods, and no purely combinatorial proof is known to this date.

In the subsequent years weak saturation has been studied extensively [2, 85, 27, 69, 84, 64, 70, 73, 18, 74, 30, 8, 9, 63]. Despite this, our understanding of weak saturation numbers is still rather limited. For instance we do not know whether for $q \geq 3$ we have a similar limiting behavior as in the graph case, in that $\lim_{n\to\infty} \mathrm{wsat}(n, H)/n^{q-1}$ always exists; this has been conjectured by Tuza [85].

In this chapter we address the case when $H = K^q_{r_1,\ldots,r_d}$ is a complete d-partite q-graph for arbitrary $d \geq q > 1$. That is, $V(H)$ is a disjoint union of sets $R_1, \ldots, R_d$ with $|R_i| = r_i$ and

$$E(H) = \left\{ e \in \binom{V(H)}{q} : |e \cap R_i| \leq 1 \text{ for all } i \in [d] \right\},$$

in particular, for $q = 2$ we recover the usual complete multipartite graphs. This is perhaps the next most natural class of hypergraphs to consider after the cliques.

For the host graph F, besides the clique it is natural to consider a larger complete d-partite q-graph $K^q_{n_1,\ldots,n_d}$. In the latter case we have a choice between the *undirected* and *directed* versions of the problem. The former follows the definition of weak saturation given at the beginning, while in the latter we additionally impose that the new copies of H in F created in every step “point the same way”, i.e. have r_i vertices in the i-th partition class for all $i \in [d]$ (see below for a formal definition).

All three above versions have been studied in the past. For $q = 2$, Kalai [48] determined $\mathrm{wsat}(n, K_{r,r})$ for large enough n. Kronenberg, Martins and Morrison [54] recently extended it to $\mathrm{wsat}(n, K_{r,r-1})$ and asymptotically to all $\mathrm{wsat}(n, K_{s,t})$. No other values $\mathrm{wsat}(n, K^q_{r_1,\ldots,r_d})$ are known except for $r_1 = \cdots = r_d = 1$ when H is a clique and a handful of closely related cases, e.g., when all r_i but one are 1 [70]. When both H and F are complete d-partite, for $d = q$ Alon [2] solved the problem in the directed setting. Moshkovitz and Shapira [64], building on Alon's work, settled the undirected case, determining $\mathrm{wsat}(K^d_{n_1,\ldots,n_d}, K^d_{r_1,\ldots,r_d})$. There has been no progress for $d > q$.

In our main contribution in this chapter we settle completely the directed case for all q and d. To state the problem formally, let $\mathbf{r} = (r_1, \ldots, r_d)$ and $\mathbf{n} = (n_1, \ldots, n_d)$ be integer vectors such that $1 \le r_i \le n_i$. Suppose $N = N_1 \sqcup \cdots \sqcup N_d$ where $|N_i| = n_i$ and $\sqcup$ denotes a disjoint union. Let $K^q_{\mathbf{n}}$ be the complete d-partite q-graph on N whose partition classes are the N_i, and let $K^q_{\mathbf{r}}$ be an unspecified complete d-partite q-graph on the same partition classes, with r_i vertices in each N_i. Given a subgraph G of $K^q_{\mathbf{n}}$, a sequence of edges $e_1, \ldots, e_k$ in $K^q_{\mathbf{n}}$ is a *(directed) $K^q_{\mathbf{r}}$-saturating sequence of G in $K^q_{\mathbf{n}}$* if: (i) $K^q_{\mathbf{n}} \setminus G = \{e_1, \ldots, e_k\}$; (ii) for every $j \in [k]$ there exists $H_j \subseteq G \cup \{e_1, \ldots, e_j\}$ isomorphic to $K^q_{\mathbf{r}}$ such that $e_j \in H_j$ and $|V(H_j) \cap N_i| = r_i$ for all $i \in [d]$. The q-graph G is said to be *(directed) weakly $K^q_{\mathbf{r}}$-saturated in $K^q_{\mathbf{n}}$* if it admits a $K^q_{\mathbf{r}}$-saturating sequence in the latter. The *(directed) weak saturation number of $K^q_{\mathbf{r}}$ in $K^q_{\mathbf{n}}$*, in notation $\mathrm{w}(K^q_{\mathbf{n}}, K^q_{\mathbf{r}})$, is the minimal number of edges in a weakly $K^q_{\mathbf{r}}$-saturated subgraph of $K^q_{\mathbf{n}}$.

Theorem 6. *For all $d \ge q \ge 2$, $\mathbf{n}$ and $\mathbf{r}$ we have*

$$\mathrm{w}(K^q_{\mathbf{n}}, K^q_{\mathbf{r}}) = \sum_{I \in \binom{[d]}{q}} \prod_{i \in I} n_i - \sum_{I \in \binom{[d]}{\le q}} \prod_{i \in I} (n_i - r_i).$$

In the above formula $\binom{[d]}{\le q}$ stands for the set of all subsets of $[d]$ of size at most q, and we use the convention that $\prod_{i \in \emptyset}(n_i - r_i) = 1$.

As mentioned, the $d = q$ case of Theorem 6 was proved by Alon [2]. Hence our result generalizes Alon's theorem to arbitrary $d \ge q$. When H is balanced, that is when $r_1 = \cdots = r_d$, there is no difference between the directed and undirected partite settings. Writing $K^q(r; d)$ for $K^q_{r,\ldots,r}$ (d times), Theorem 6 thus determines the weak saturation number of $K^q(r; d)$ in complete d-partite q-graphs.

Corollary 7. *For all $d \ge q \ge 2$ and $n_1, \ldots, n_d \ge r \ge 1$ we have*

$$\mathrm{wsat}(K^q_{n_1,\ldots,n_d}, K^q(r; d)) = \sum_{I \in \binom{[d]}{q}} \prod_{i \in I} n_i - \sum_{I \in \binom{[d]}{\le q}} \prod_{i \in I} (n_i - r).$$

Our proof of Theorem 6 combines exterior algebra techniques in the spirit of [48] with a new ingredient: the use of the colorful exterior algebra inspired by the recent work of Bulavka, Goodarzi and Tancer on the colorful fractional Helly theorem [19].

Kronenberg, Martins and Morrison ([54], Section 5) remarked that while the values $\mathrm{wsat}(n, K_{t,t})$ and $\mathrm{wsat}(K_{\ell,m}, K_{t,t})$ for $\ell + m = n$, which were determined in separate works, are of the same order of magnitude, it is not obvious if there is any direct connection. In our second contribution in this chapter we establish such a connection using a tensoring trick. As we have mentioned earlier, 2-graphs H satisfy $\mathrm{wsat}(n, H) = c_H n + o(n)$, and Alon's proof of this fact [2] can be straightforwardly adjusted to show that $\mathrm{wsat}(K_{n,n}, H) = c'_H \cdot 2n + o(n)$ when H is bipartite. We show that in fact $c_H = c'_H$. A minor adjustment to our proof gives that, for any rational $0 < \alpha < 1$, the quantities $\mathrm{wsat}(n, H)$ and $\mathrm{wsat}(K_{\alpha n,(1-\alpha)n}, H)$, when $\alpha n \in \mathbb{Z}$, are of the same order of magnitude. Setting $H = K_{t,t}$ answers the above question of [54].

For $q \geq 3$ while we do not have (yet) the same knowledge of limiting constants, a similar method determines asymptotically the weak saturation number of complete d-partite d-graphs in the clique, generalizing Theorem 4 of [54].

Theorem 8. *For every bipartite 2-uniform graph H we have*

$$\lim_{n\to\infty} \frac{\mathrm{wsat}(n, H)}{n} = \lim_{n\to\infty} \frac{\mathrm{wsat}(K_{n,n}, H)}{2n}. \tag{3.1}$$

Furthermore, for any $d \geq 2$ and $1 \leq r_1 \leq \cdots \leq r_d$ we have

$$\mathrm{wsat}(n, K^d_{r_1,\dots,r_d}) = \frac{r_1 - 1}{(d-1)!} n^{d-1} + O(n^{d-2}). \tag{3.2}$$

The rest of the chapter is organized as follows. In Section 3.1 we give a construction for the upper bound in Theorem 6. In Section 3.2 we provide the necessary background for this chapter. In Section 3.3 we provide the proof for the lower bound in Theorem 6. In Section 3.4 we discuss weak saturation in the clique and prove Theorem 8.

3.1 The upper bound

In this section we prove the upper bound in Theorem 6 by exhibiting a weakly $K^q_{\mathbf{r}}$-saturated q-graph G. Fix a subset $R \subseteq N$ such that $|R \cap N_i| = r_i$ for every $i \in [d]$ and set

$$\Sigma := \left\{ S \in \binom{N \setminus R}{\leq q} : |S \cap N_i| \leq 1 \text{ for each } i \in [d] \right\}.$$

We define G via its complement in $K^q_{\mathbf{n}}$ as follows. For every $S \in \Sigma$ choose an edge $\lambda(S) \in K^q_{\mathbf{n}}[R \cup S]$ satisfying $S \subseteq \lambda(S)$. Note that the assignment λ is injective, as $\lambda(S) \cap (N \setminus R) = S$. Recall that we associate hypergraphs with their edge sets. Define

$$G := K^q_{\mathbf{n}} \setminus \bigcup_{S \in \Sigma} \lambda(S),$$

so that

$$|E(G)| = \sum_{I \in \binom{[d]}{q}} \prod_{i \in I} n_i - \sum_{I \in \binom{[d]}{\leq q}} \prod_{i \in I} (n_i - r_i).$$

Notice that the choices of $\lambda(S)$ are not unique, but as the next lemma shows, each of them yields a weakly $K^q_{\mathbf{r}}$-saturated q-graph. Such non-uniqueness is a common occurrence in weak saturation: for instance, every n-vertex tree is an extremal example for weak triangle saturation in K_n.

Lemma 9. *The q-graph G defined above is weakly $K^q_{\mathbf{r}}$-saturated. Therefore,*

$$\mathrm{w}(K^q_{\mathbf{n}}, K^q_{\mathbf{r}}) \le |E(G)| = \sum_{I\in\binom{[d]}{q}} \prod_{i\in I} n_i - \sum_{I\in\binom{[d]}{\le q}} \prod_{i\in I}(n_i - r_i).$$

Proof. For each $0 \le k \le q$ let

$$G_k := G \cup \{T \in K^q_{\mathbf{n}} \colon |T \setminus R| \le k\},$$

and put $G_{-1} := G$. We claim that adding any new edge $L \in K^q_{\mathbf{n}}$ with $|L \setminus R| = k$ to G_{k-1} creates a new copy of $K^q_{\mathbf{r}}$ containing L. This gives rise to a $K^q_{\mathbf{r}}$-saturating sequence between G_{k-1} and G_k and, by extension, between $G = G_{-1}$ and $G_q = K^q_{\mathbf{n}}$.

First, notice that G_0 is obtained from G_{-1} by adding the sole missing edge $\lambda(\emptyset)$. Doing so creates a new copy of $K^q_{\mathbf{r}}$, namely $K^q_{\mathbf{n}}[R]$. For an arbitrary k, suppose that L is a missing edge in G_{k-1} such that $S := L \setminus R$ is of size k. Observe that every $T \in K^q[R \cup S]$ is an edge in G_{k-1} unless $T = L$. Indeed, if $|T \setminus R| < k$ then this holds by definition of G_{k-1}. While otherwise we have $T \setminus R = S$. Hence, by the definition of G, we have $L = \lambda(S)$, so that either $T = L$ or $T \in G \subseteq G_{k-1}$. Therefore, adding L to G_{k-1} creates a new copy of $K^q_{\mathbf{n}}[R \cup S]$ containing L and a fortiori also a new copy of $K^q_{\mathbf{r}}$ containing L, as desired. □

3.2 Colorful exterior algebra.

In this section we introduce the linear algebra tools needed for the proof of the lower bound in Theorem 6.

Before we start explaining the algebraic background, we will try to sketch why algebraic tools can be useful in this context. This sketch should be understood loosely—we do not provide any guarantees for the claims in this sketch. In particular, many important technical details are skipped in the sketch. Understanding this sketch is not required in the following text, thus it can be skipped.

Consider first the somewhat trivial case of providing the lower bound on $\mathrm{wsat}(n, K_3)$, the weak saturation number of the complete graph K_3 in K_n. Consider a subgraph G of K_n and a saturating sequence $e_1, \ldots e_k$ of edges in $E(K_n) \setminus E(G)$. Let $G_i := G \cup \{e_1, \ldots, e_i\}$. Because the sequence is saturating, we know that G_i contains a copy of K_3 containing e_i. This means that the dimension of the cycle space of G_i is strictly larger than the dimension of the cycle space of G_{i-1}. Because the final dimension of the cycle space of K_n equals $\binom{n-1}{2}$, we may perform at most $\binom{n-1}{2}$ such steps. In other words $k \le \binom{n-1}{2}$ and thus $|E(G)| \ge \binom{n}{2} - \binom{n-1}{2}$ as required.

In the language of algebraic topology (which we however do not use in the proofs, no topological background is required), the property that the dimension of the cycle space increases can be phrased so that a new copy of K_3 in each step belongs to the kernel of the standard boundary operator. For more complicated (hyper)graphs than K_3 it is actually useful to use several independent boundary operators in order to generalize the aforementioned approach. Using such independent operators can be actually efficiently phrased in terms of exterior algebra (without mentioning algebraic topology). They correspond to the *left interior product*, which we discussed in subsection 2.2.1, subject to some suitable independence (genericity) condition.[1]

Let $V = \mathbb{R}^N$, in Section 2.2 we have introduced the exterior algebra $\bigwedge V$. As we are interested in multipartite hypergraphs it is natural to assume in addition that the set N is partitioned as a disjoint union $N = N_1 \sqcup N_2 \sqcup \cdots \sqcup N_d$; consistently with the beginning of this chapter $n_i := |N_i|$. Here each N_i is ordered by a total order $<_i$. We extend these orders to the whole N as follows, for $x \in N_i$ and $y \in N_j$, we say that

$$x < y \text{ if } i < j \text{ or if } i = j \text{ and } x <_i y.$$

Given the standard basis $(e_v)_{v \in N}$ of V we say that a basis $(f_v)_{v \in N}$ is *colorful* with respect to this partition if $(f_v)_{v \in N_i}$ generates the same subspace of $V = \mathbb{R}^N$ as $(e_v)_{v \in N_i}$ for every $i \in [d]$; we denote this subspace V_i. Put differently, the transition matrix A from $(e_v)_{v \in N}$ to $(f_v)_{v \in N}$ is a block-diagonal matrix with blocks $N_i \times N_i$ for $i \in [d]$. We also say that $(f_v)_{v \in N}$ is *colorful generic* (with respect to this partition) if the basis change from $(e_v)_{v \in N_i}$ to $(f_v)_{v \in N_i}$ is generic for every $i \in [d]$. Remember that $(e_v)_{v \in N_i}$ to $(f_v)_{v \in N_i}$ is generic if $\langle f_S, e_T \rangle \neq 0$ for every $S, T \subseteq N_i$ with $|S| = |T|$. It is possible to choose a basis which is simultaneously colorful generic with respect to a given partition and orthonormal by choosing each change of basis from $(e_v)_{v \in N_i}$ to $(f_v)_{v \in N_i}$ generic and orthonormal.

By $\bigwedge V_i$ we denote the subalgebra of $\bigwedge V$ generated by e_S for $S \subseteq N_i$ and by $\bigwedge^k V_i$ the subspace of $\bigwedge V_i$ with basis $(e_S)_{S \in \binom{N_i}{k}}$; that is, $\bigwedge^k V_i = \bigwedge^k V \cap \bigwedge V_i$.

The following formula expresses a colorful product as a linear combination of colorful elements. Let $s_1, \dots, s_d$ be integers with $0 \leq s_i \leq |N_i|$. Suppose that for each $i \in [d]$ we are given

$$h_i = \sum_{S_i \in \binom{N_i}{s_i}} \lambda_{S_i} f_{S_i}$$

for $\lambda_{S_i} \in \mathbb{R}$ (so that $h_i \in \bigwedge^{s_i} V$). Then by bilinearity of $\wedge$ and (2.1) we get

$$h_1 \wedge \cdots \wedge h_d = \sum_{\substack{(S_1, \dots, S_d) \in \\ \binom{N_1}{s_1} \times \cdots \times \binom{N_d}{s_d}}} \left(\prod_{i \in [d]} \lambda_{S_i} \right) f_{S_1} \wedge \cdots \wedge f_{S_d} \tag{3.3}$$

$$= \sum_{\substack{(S_1, \dots, S_d) \in \\ \binom{N_1}{s_1} \times \cdots \times \binom{N_d}{s_d}}} \pm \left(\prod_{i \in [d]} \lambda_{S_i} \right) f_{S_1 \cup \cdots \cup S_d}. \tag{3.4}$$

[1]Perhaps the closest relation between the boundary operators and the left interior product can be seen in Lemma 5 interpreting e_R as a simplex with set of vertices R, and $f_{T^\llcorner}$ as an operator removing t times the top-dimensional simplices, yielding a linear combination of simplices f_S with $r - t$ vertices. (However, for this relation, it would be even better to express the right hand side using e_S so that all possible e_S would appear.) Adding a colorful aspect (in our case) then makes it easier to work with multipartite (hyper)graphs rather than complete ones.

We claim that the left interior product behaves nicely with respect to a colorful partition. To see this, we first need an auxiliary lemma about signs.

Lemma 10. *Let U and T be disjoint subsets of N and for all $i \in [d]$ let $U_i := U \cap N_i$, $T_i := T \cap N_i$, $u_i := |U_i|$ and $t_i := |T_i|$. Then*

$$\operatorname{sgn}(U,T) = (-1)^c \operatorname{sgn}(U_1,T_1) \cdots \operatorname{sgn}(U_d,T_d),$$

where c depends only on $u_1,\dots,u_d$ and $t_1,\dots,t_d$.

Proof. The value $\operatorname{sgn}(U,T)$ is -1 to the number of transpositions in the permutation π of $U \cup T$ where we first place the elements of U (in our given order on N) and then the elements of T (in the same order). Considering that for $i < j$, U_i precedes U_j and T_i precedes T_j, the order of the blocks $U_1,\dots,U_d,T_1,\dots,T_d$ in π is

$$(U_1,\dots,U_d,T_1,\dots,T_d).$$

After c transpositions where c depends only on $u_1,\dots,u_d,t_1,\dots,t_d$, we get a permutation π' with the following order of blocks

$$(U_1,T_1,U_2,T_2,\dots,U_d,T_d).$$

By the above, the sign of π' equals $(-1)^c \operatorname{sgn}(U,T)$. On the other hand, as T_i precedes U_j for $i < j$ in our order on N, the sign of π' is also equal the product $\operatorname{sgn}(U_1,T_1)\cdots\operatorname{sgn}(U_d,T_d)$. Equating these two expressions gives the desired identity. □

In the following proposition, the f_i are not necessarily coming from a colorful generic basis. However, we intend to apply it in this setting. With a slight abuse of notation, we use $\bigwedge$ both for the exterior algebra as well as for the wedge product of multiple elements. (This can be easily distinguished from the context.)

Proposition 11. *Suppose that $s_1,\dots,s_d$ and $t_1,\dots,t_d$ are nonnegative integers with $t_i \le s_i \le n_i$ for every $i \in [d]$. Suppose further that $f_i \in \bigwedge^{t_i} V_i$ and $h_i \in \bigwedge^{s_i} V_i$ for all $i \in [d]$. Then*

$$\left(\bigwedge_{i=1}^{d} f_i\right) \llcorner \left(\bigwedge_{i=1}^{d} h_i\right) = \pm \bigwedge_{i=1}^{d} (f_i \llcorner h_i).$$

Proof. We will show that

$$\left(\bigwedge_{i=1}^{d} f_i\right) \llcorner \left(\bigwedge_{i=1}^{d} h_i\right) = (-1)^c \bigwedge_{i=1}^{d} (f_i \llcorner h_i) \tag{3.5}$$

where c comes from Lemma 10; in particular, it depends only on $t_1,\dots,t_d$ and $s_1,\dots,s_d$.

By bilinearity of $\llcorner$ and $\wedge$ it is sufficient to prove (3.5) in the case when the f_i and the h_i are basis elements of $\bigwedge^{t_i} V_i$ and $\bigwedge^{s_i} V_i$ respectively. So, assume for each $i \in [d]$ that $f_i = e_{T_i}$ and $h_i = e_{S_i}$ where $T_i \in \binom{N_i}{t_i}$ and $S_i \in \binom{N_i}{s_i}$, and let $T := T_1 \cup \dots \cup T_d$ and $S := S_1 \cup \dots \cup S_d$. Then $\bigwedge_{i=1}^{d} f_i = e_T$ and $\bigwedge_{i=1}^{d} h_i = e_S$ by the definition of the exterior product $\wedge$. If $T_i \not\subseteq S_i$ for some $i \in [d]$, then $T \not\subseteq S$ and both sides of (3.5) vanish by Lemma 4. Therefore, it remains to check the case that $T_i \subseteq S_i$ for every $i \in [d]$. Here by Lemma 10 (with $U = S \setminus T$) and Lemma 4 we get

$$\begin{aligned} e_T \llcorner e_S &= \operatorname{sgn}(S \setminus T, T) e_{S \setminus T} \\ &= (-1)^c \operatorname{sgn}(S_1 \setminus T_1, T_1) \cdots \operatorname{sgn}(S_d \setminus T_d, T_d) e_{S_1 \setminus T_1} \wedge \cdots \wedge e_{S_d \setminus T_d} \\ &= (-1)^c (e_{T_1} \llcorner e_{S_1}) \wedge \cdots \wedge (e_{T_d} \llcorner e_{S_d}), \end{aligned}$$

as required. □

3.3 The lower bound

In this section we prove the lower bound in Theorem 6. Our proof follows a strategy similar to [8] and [48]. Viewing the edges of $K^q_{\mathbf{n}}$ as elements of the exterior algebra of $\mathbb{R}^N$, we will define a linear mapping closely related to the weak saturation process and lower-bound $\mathrm{w}(K^q_{\mathbf{n}}, K^q_{\mathbf{r}})$ by the rank of the corresponding matrix.

As outlined in Section 2.2, let V be an n-dimensional real vector space with a basis $(e_v)_{v \in N}$, equipped with a standard inner product $\langle \cdot, \cdot \rangle$ with respect to this basis, that is, $(e_v)_{v \in N}$ is orthonormal. Using the exterior product notation of Section 2.2, define

$$\operatorname{span} K^q_{\mathbf{n}} := \operatorname{span}\{e_T : T \in E(K^q_{\mathbf{n}})\} \subseteq \bigwedge\nolimits^q V.$$

For an element $m \in \bigwedge^k V$ the *support of* m is the set

$$\operatorname{supp}(m) = \left\{ S \in \binom{N}{k} : \langle e_S, m \rangle \neq 0 \right\}.$$

The following lemma, which converts the problem at hand into a constructive question in linear algebra, is analogous to Lemma 3 in [8].[2]

Lemma 12. *Let Y be a real vector space and $\Gamma : \operatorname{span} K^q_{\mathbf{n}} \to Y$ a linear map such that for every subset $R \subseteq N$ with $|R \cap N_i| = r_i$ for all $i \in [d]$ there exists an element $m \in \ker \Gamma$ with* $\operatorname{supp}(m) = E(K^q_{\mathbf{n}}[R])$. *Then*

$$\mathrm{w}(K^q_{\mathbf{n}}, K^q_{\mathbf{r}}) \geq \operatorname{rank} \Gamma.$$

Proof. Suppose the q-graph G_0 is weakly $K^q_{\mathbf{r}}$-saturated in $K^q_{\mathbf{n}}$ and $|E(G_0)| = \mathrm{w}(K^q_{\mathbf{n}}, K^q_{\mathbf{r}})$. Denote by $\{L_1, \ldots, L_k\}$ a corresponding saturating sequence and by H_i a new copy of $K^q_{\mathbf{r}}$ that appears in $G_i = G_0 \cup \{L_1, \ldots, L_i\}$ with $L_i \in E(H_i)$. Let $Y_i = \operatorname{span}\{\Gamma(e_T) : T \in E(G_i)\}$, and note that $Y_k = \Gamma(\operatorname{span} K^q_{\mathbf{n}})$. By assumption, for each $i = 1, \ldots, k$ there exist non-zero coefficients $\{c_T : T \in E(H_i)\}$ such that $\sum_{T \in E(H_i)} c_T \Gamma(e_T) = 0$. Therefore,

$$\Gamma(e_{L_i}) = -\frac{1}{c_{L_i}} \sum_{T \in E(H_i) \setminus L_i} c_T \Gamma(e_T) \in Y_{i-1}.$$

[2] Put equivalently in the language of [8], we map each edge of K^q_n to vector in a certain vector space $\tilde{W}$, so that for each copy of $K^q_{\mathbf{r}}$ in K^q_n the underlying vectors are linearly dependent with all coefficients involved being non-zero. This implies $\mathrm{w}(K^q_n, K^q_{\mathbf{r}}) \geq \dim \tilde{W}$.

We conclude that $Y_i = Y_{i-1}$. By repeating this procedure we obtain

$$\mathrm{w}(K_{\mathbf{n}}^q, K_{\mathbf{r}}^q) = |E(G_0)| \geq \dim Y_0 = \dim Y_k = \operatorname{rank}\Gamma.$$

□

Our goal now is to define a linear map Γ as in Lemma 12. For this purpose let us fix an orthonormal colorful generic basis $(f_v)_{v\in N}$ of V with respect to the partition of N, as described in Section 3.2. Next, for each $i \in [d]$ choose a set $J_i \subseteq N_i$ with $|J_i| = r_i - 1$ and a vertex $w_i \in N_i \setminus J_i$. Put $J := \bigcup_{i\in[d]} J_i$ and $W := \{w_i : i \in [d]\}$. Finally, set $s := d - q$ and

$$g := \sum_{T\in\binom{W}{s}} f_T. \tag{3.6}$$

We can now state the following auxiliary lemma.

Lemma 13. *Let z be an integer with $d \geq z \geq s$ and let $Z \in \binom{N}{z}$. Then*

(i) $g \llcorner f_Z = 0$ if $|Z \cap W| < s$.

(ii) If $z = s$, then $\langle g, f_Z\rangle = \begin{cases} \pm 1 & \text{if } Z \subseteq W, \\ 0 & \text{if } Z \not\subseteq W. \end{cases}$

Proof. By (3.6), bilinearity of $\llcorner$, and Lemma 4 we get

$$g \llcorner f_Z = \sum_{W'\in\binom{W}{s}} f_{W'} \llcorner f_Z = \sum_{W'\in\binom{W\cap Z}{s}} \pm f_{Z\setminus W'}. \tag{3.7}$$

The last expression is 0 if $|Z \cap W| < s$; this shows (i).

Now, assume that $z = s$. Then

$$\langle g, f_Z\rangle = \langle f_\emptyset \wedge g, f_Z\rangle = \langle f_\emptyset, g \llcorner f_Z\rangle \overset{(3.7)}{=} \sum_{W'\in\binom{W\cap Z}{s}} \pm\langle f_\emptyset, f_{Z\setminus W'}\rangle. \tag{3.8}$$

If $Z \not\subseteq W$, then $|Z \cap W| < z = s$, so $g \llcorner f_Z = 0$ from (i), and thus (3.8) evaluates to 0. On the other hand, if $Z \subseteq W$, then $\binom{W\cap Z}{s} = \{Z\}$. It follows that

$$\langle g, f_Z\rangle \overset{(3.8)}{=} \pm\langle f_\emptyset, f_\emptyset\rangle = \pm 1,$$

yielding (ii). □

We define the subspace

$$U := \operatorname{span}\{g \llcorner f_T : T \in E(K_{\mathbf{n}}^d[N \setminus J]), |T \cap W| \geq s\}, \tag{3.9}$$

and observe first that $U \subseteq \operatorname{span} K_{\mathbf{n}}^q$. Indeed, for each T in (3.9) and $W' \in \binom{W}{s}$, we have by Lemma 4 that $f_{W'} \llcorner f_T = 0$ if $W' \not\subseteq T$ and $f_{W'} \llcorner f_T = \pm f_{T\setminus W'}$ if $W' \subseteq T$. In the latter case note that $T \setminus W' \in E(K_{\mathbf{n}}^q)$, and the claim follows by bilinearity of $\llcorner$.

Let Y be the orthogonal complement of U in $\operatorname{span} K_{\mathbf{n}}^q$ and let $\Gamma\colon \operatorname{span} K_{\mathbf{n}}^q \to \operatorname{span} K_{\mathbf{n}}^q$ be the orthogonal projection on Y. Our main technical lemma states that Γ satisfies the assumptions of Lemma 12.

Lemma 14. *Suppose that $R \subseteq N$ satisfies $|R \cap N_i| = r_i$ for every $i \in [d]$. Then, there exists $m \in \ker \Gamma$ such that $\operatorname{supp}(m) = E(K^q_{\mathbf{n}}[R])$.*

Deferring the proof of Lemma 14, let us first compute $\operatorname{rank} \Gamma$ and conclude the proof of Theorem 6 assuming Lemma 14.

Notice that the sets $T \in K^d_{\mathbf{n}}[N \setminus J]$ with $|T \cap W| \geq s$ are in bijective correspondence with the sets $T \setminus W \in K^p_{\mathbf{n}}[N \setminus (J \cup W)]$ with $p \leq q$. Using this bijection,

$$\dim U \overset{(3.9)}{\leq} |\{T \in K^d_{\mathbf{n}}[N \setminus J] \colon |T \cap W| \geq s\}| = \sum_{\substack{I \subseteq [d] \\ |I| \leq q}} \prod_{i \in I} (n_i - r_i).$$

Consequently,

$$\operatorname{rank} \Gamma = \dim(\operatorname{span} K^q_{\mathbf{n}}) - \dim U \geq \sum_{I \in \binom{[d]}{q}} \prod_{i \in I} n_i - \sum_{\substack{I \subseteq [d] \\ |I| \leq q}} \prod_{i \in I} (n_i - r_i). \tag{3.10}$$

Proof of Theorem 6. On the one hand, by Lemma 14 the map Γ satisfies the assumptions of Lemma 12. Therefore,

$$\mathrm{w}(K^q_{\mathbf{n}}, K^q_{\mathbf{r}}) \geq \operatorname{rank} \Gamma \overset{(3.10)}{\geq} \sum_{I \in \binom{[d]}{q}} \prod_{i \in I} n_i - \sum_{\substack{I \subseteq [d] \\ |I| \leq q}} \prod_{i \in I} (n_i - r_i).$$

On the other hand, Lemma 9 gives the same upper bound. □

Proof of Lemma 14. We claim that

$$m = (g \wedge f_J) \llcorner e_R$$

is the desired element.[3] Let $R_i := R \cap N_i$ for each $i \in [d]$.

First, we verify that $m \in \ker \Gamma = U$. By Proposition 11 we have

$$f_J \llcorner e_R = \pm (f_{J_1} \llcorner e_{R_1}) \wedge \cdots \wedge (f_{J_d} \llcorner e_{R_d}).$$

By Lemma 5 we can write each of these terms as

$$f_{J_i} \llcorner e_{R_i} = \sum_{v \in N_i \setminus J_i} \lambda_v f_v \quad \text{with all} \quad \lambda_v \neq 0. \tag{3.11}$$

Combining this with (3.3) gives

$$f_J \llcorner e_R = \sum_{Z \in E(K^d_{\mathbf{n}}[N \setminus J])} \pm \left(\prod_{v \in Z} \lambda_v\right) f_Z. \tag{3.12}$$

[3]Let us briefly sketch the topological idea hidden behind this choice: As it can be easily deduced from the computations below, m can be also expressed as $\pm g \llcorner \big((f_{J_1} \llcorner e_{R_1}) \wedge \cdots \wedge (f_{J_d} \llcorner e_{R_d})\big)$. In the terminology of simplicial complexes interpreting loosely (i) e_{R_i} as a full simplex on the vertex set R_i, (ii) $\wedge$ as a join of simplicial complexes and (iii) $\llcorner$ as an operator taking the skeleton of appropriate dimension, we gradually get the following: $f_{J_i} \llcorner e_{R_i}$ corresponds to the 0-skeleton of the simplex on R_i, that is, the vertices of R_i. Then $(f_{J_1} \llcorner e_{R_1}) \wedge \cdots \wedge (f_{J_d} \llcorner e_{R_d})$ corresponds to the join of the sets R_i, that is, the complete d-partite complex on $R_1, \ldots, R_d$. Finally, applying $g \llcorner$ to this element takes the skeleton again reducing the dimension so that the corresponding hypergraph is the required $K^q_{\mathbf{n}}[R]$.

Therefore, we get

$$m = (g \wedge f_J) \llcorner e_R \overset{(2.5)}{=} g \llcorner (f_J \llcorner e_R) \overset{(3.12)}{=} \sum_{Z \in E(K^d_{\mathbf{n}}[N \setminus J])} (\prod_{v \in Z} \lambda_v) g \llcorner f_Z$$
$$= \sum_{\substack{Z \in E(K^d_{\mathbf{n}}[N \setminus J]) \\ |Z \cap W| \geq s}} (\prod_{v \in Z} \lambda_v) g \llcorner f_Z,$$

where the last equality follows by Lemma 13(i) with $z = d$. Thus $m \in U$ as wanted.

Next, we show that $\operatorname{supp}(m) = E(K^q_{\mathbf{n}}[R])$. As we just have shown, $m \in U \subseteq \operatorname{span} K^q_{\mathbf{n}}$, i.e. $\operatorname{supp}(m) \subseteq E(K^q_{\mathbf{n}})$. Now, for $T \in E(K^q_{\mathbf{n}})$ we have

$$\langle e_T, m \rangle \overset{(2.4)}{=} \langle e_T \wedge (g \wedge f_J), e_R \rangle = \pm \langle (g \wedge f_J) \wedge e_T, e_R \rangle \overset{(2.4)}{=} \pm \langle g \wedge f_J, e_T \llcorner e_R \rangle. \quad (3.13)$$

If $T \notin E(K^q_{\mathbf{n}}[R])$, then $T \not\subseteq R$ and by Lemma 4 we have $e_T \llcorner e_R = 0$, and consequently $\langle e_T, m \rangle = 0$. Hence, $T \notin \operatorname{supp}(m)$.

Now assume that $T \in E(K^q_{\mathbf{n}}[R])$, i.e., $T \subseteq R$. By (3.13) and Lemma 4 we have

$$\langle e_T, m \rangle = \pm \langle g \wedge f_J, e_{R \setminus T} \rangle \overset{(2.4)}{=} \pm \langle g, f_J \llcorner e_{R \setminus T} \rangle. \quad (3.14)$$

Let $P := \{i \in [d] \colon T \cap N_i \neq \emptyset\}$ and $P' := [d] \setminus P$. Using this notation we can write

$$e_{R \setminus T} = \pm \left(\bigwedge_{i \in P} e_{R_i \setminus \tau_i} \right) \wedge \left(\bigwedge_{i \in P'} e_{R_i} \right),$$

where for each $i \in P$ the set $\tau_i = T \cap N_i$ contains a single vertex. Applying Proposition 11, we deduce

$$f_J \llcorner e_{R \setminus T} = \pm \left(\bigwedge_{i \in P} f_{J_i} \llcorner e_{R_i \setminus \tau_i} \right) \wedge \left(\bigwedge_{i \in P'} f_{J_i} \llcorner e_{R_i} \right). \quad (3.15)$$

Since $|J_i| = r_i - 1 = |R_i \setminus \tau_i|$, by Lemma 3 for every $i \in P$ we have $f_{J_i} \llcorner e_{R_i \setminus \tau_i} \in \bigwedge^0 V$. Thus

$$f_{J_i} \llcorner e_{R_i \setminus \tau_i} = \langle e_\emptyset, f_{J_i} \llcorner e_{R_i \setminus \tau_i} \rangle e_\emptyset = \langle e_\emptyset \wedge f_{J_i}, e_{R_i \setminus \tau_i} \rangle e_\emptyset = \langle f_{J_i}, e_{R_i \setminus \tau_i} \rangle e_\emptyset,$$

and notice that $\langle f_{J_i}, e_{R_i \setminus \tau_i} \rangle \neq 0$ because $(f_v)_{v \in N_i}$ is generic with respect to $(e_v)_{v \in N_i}$. Plugging it into (3.15) yields

$$f_J \llcorner e_{R \setminus T} = \pm \left(\bigwedge_{i \in P} \langle f_{J_i}, e_{R_i \setminus \tau_i} \rangle e_\emptyset \right) \wedge \left(\bigwedge_{i \in P'} f_{J_i} \llcorner e_{R_i} \right) \quad (3.16)$$
$$= \pm \left(\prod_{i \in P} \langle f_{J_i}, e_{R_i \setminus \tau_i} \rangle \right) \bigwedge_{i \in P'} f_{J_i} \llcorner e_{R_i}. \quad (3.17)$$

Turning to P', denote $N' := \bigcup_{i \in P'} N_i \setminus J_i$. We have

$$\bigwedge_{i \in P'} f_{J_i} \llcorner e_{R_i} \overset{(3.11)}{=} \bigwedge_{i \in P'} \left(\sum_{v \in N_i \setminus J_i} \lambda_v f_v \right) \overset{(3.3)}{=} \sum_{Z \in E(K^s_{\mathbf{n}}[N'])} \pm \left(\prod_{v \in Z} \lambda_v \right) f_Z. \quad (3.18)$$

Therefore,

$$\langle g, \bigwedge_{i\in P'} f_{J_i} \llcorner e_{R_i}\rangle = \sum_{Z\in E(K^s_{\mathbf{n}}[N'])} \pm\Big(\prod_{v\in Z}\lambda_v\Big)\langle g, f_Z\rangle = \pm \prod_{v\in W\cap N'} \lambda_v, \tag{3.19}$$

where the second equality is due to Lemma 13(ii), using that there is exactly one $Z \in E(K^s_{\mathbf{n}}[N'])$ with $Z \subseteq W$, namely $Z = W \cap N'$. Putting it all together,

$$\begin{aligned}\langle e_T, m\rangle &\overset{(3.14)}{=} \pm\langle g, f_J \llcorner e_{R\setminus T}\rangle \overset{(3.16)}{=} \pm\Big(\prod_{i\in P}\langle f_{J_i}, e_{R_i\setminus\tau_i}\rangle\Big)\langle g, \bigwedge_{i\in P'} f_{J_i}\llcorner e_{R_i}\rangle \\ &\overset{(3.19)}{=} \pm\Big(\prod_{i\in P}\langle f_{J_i}, e_{R_i\setminus\tau_i}\rangle\Big)\prod_{v\in W\cap N'}\lambda_v \neq 0,\end{aligned}$$

and consequently $T \in \operatorname{supp}(m)$. □

3.4 Weak saturation in the clique

In this section we prove Theorem 8. Let H be a q-graph where $q \geq 2$ without isolated vertices. We recall the notion of a *link hypergraph* of a vertex $v \in V(H)$: it is the $(q-1)$-graph (possibly with isolated vertices) defined via

$$L_H(v) := \{e \setminus \{v\} \colon e \in E(H), v \in e\}.$$

The *co-degree* of a set W of $q-1$ vertices in H is

$$d_H(W) := |\{e \in E(H) : W \subset e\}|.$$

Define the *minimum positive co-degree* of H, in notation $\delta^*(H)$, as

$$\delta^*(H) := \min\Big\{d_H(W) \colon W \in \binom{V(H)}{q-1}, d_H(W) > 0\Big\}.$$

Notice that $\delta^*(H) \leq \delta^*(L_H(v))$ for all $v \in V(H)$, and equality holds for some v.

Lemma 15. $\operatorname{wsat}(n, H) \leq (\delta^*(H) - 1)\binom{n}{q-1} + O_H(n^{q-2})$.

Proof. We apply induction on q. For $q = 2$ this is a well-known fact ([31], Theorem 4). Suppose now that $q \geq 3$ and the statement holds for all smaller values. Let H be a q-graph and let $W = \{v_1, \ldots, v_{q-1}\}$ be a set satisfying $d_H(W) = \delta^*(H)$. Let $H_1 = L_H(v_1)$ be the link hypergraph of v_1, and observe that $\delta^*(H_1) = \delta^*(H)$. A weakly H-saturated q-graph on $[n]$ is obtained as follows. Take a minimum weakly H_1-saturated $(q-1)$-graph on $[n-1]$ and insert n into each edge; take a union of the resulting q-graph with a minimum weakly H-saturated q-graph on $[n-1]$. We therefore obtain

$$\operatorname{wsat}(n, H) \leq \operatorname{wsat}(n-1, H) + \operatorname{wsat}(n-1, H_1).$$

Iterating and applying the induction hypothesis,

$$\begin{aligned}\operatorname{wsat}(n, H) &\leq \operatorname{wsat}(|V(H)|, H) + \sum_{m=|V(H)|}^{n-1} \operatorname{wsat}(m, H_1) \\ &\leq (\delta^*(H_1) - 1)\sum_{m=q-2}^{n-1}\binom{m}{q-2} + O_H(n^{q-2}) \\ &= (\delta^*(H) - 1)\binom{n}{q-1} + O_H(n^{q-2}).\end{aligned}$$

□

The *tensor product* of two q-graphs G and J, $G \times J$ is defined having the vertex set $V(G) \times V(J)$ and the edge set

$$E(G \times J) = \Big\{\{(v_1, w_1), \ldots (v_q, w_q)\} : \{v_1, \ldots, v_q\} \in E(G), \{w_1, \ldots, w_q\} \in E(J)\Big\}.$$

(Note that every pair of edges in the original graphs produces $q!$ edges in the product.)

Lemma 16. *Let $H = K^d_{r_1,\ldots,r_d}$, and let F^d_n be the copy of $K^d(n; d)$ between the vertex sets $[n] \times \{1\}, \ldots, [n] \times \{d\}$. Then there exists a d-graph $E^d(n, H) \subseteq F^d_n \setminus (K^d_{[n]} \times K^d_{[d]})$ of size $O_H(n^{d-2})$ such that*

$$G(n, H) := (K^d_{[n]} \times K^d_{[d]}) \sqcup E^d(n, H)$$

is weakly H-saturated in F^d_n.

Proof. It suffices to prove the above statement when $r_1 = \cdots = r_d =: r$, i.e. when $H = K^d(r; d)$, as every edge creating a new copy of $K^d(\max\{r_1, \ldots, r_d\}; d)$ creates in particular a new copy of $K^d_{r_1,\ldots,r_d}$.

We apply induction on d and n. For $d = 2$ and any $n \geq |V(H)|$ the graph $K_{[n]} \times K_{[2]}$ misses only a matching from F^2_n, making it already H-saturated in F^2_n, as can be easily checked. Moreover, for every fixed H we can assume the statement to hold for all n less than some large $C(H)$.

For the induction step, fix (n, d) and suppose that the statement holds for all (n', d') with $d' < d$ and all (n'', d) with $n'' < n$. It suffices to show that $O_H(n^{d-3})$ edges can be added to $G(n-1, H)$ to satisfy the assertion; these edges will be as follows.

For each $i \in [d]$ let the $(d-1)$-graph E'_i be an isomorphic copy of $E^{d-1}(n-1, K^{d-1}(r; d-1))$ between the sets $[n-1] \times \{j\}$ for $j \in [d] \setminus \{i\}$, such that $(K^{d-1}_{[n-1]} \times K^{d-1}_{[d]\setminus\{i\}}) \sqcup E'_i$ is weakly $K^{d-1}(r; d-1)$-saturated in the complete $(d-1)$-partite $(d-1)$-graph between the sets $[n-1] \times \{j\}$ for $j \in [d] \setminus \{i\}$. Let

$$E_i := \{e \sqcup \{(n, i)\} : e \in E'_i\}.$$

By the induction hypothesis $|E_i| = |E'_i| = O_H(n^{d-3})$.

Similarly, for each $\{i_1, i_2\} \in \binom{[d]}{2}$ apply Corollary 7 to obtain a $(d-2)$-graph E'_{i_1,i_2} of size $O_H(n^{d-3})$ which is weakly $K^{d-2}(r; d-2)$-saturated in the copy of $K^{d-2}(n-1; d-2)$ between the sets $[n-1] \times \{j\}$ for $j \in [d] \setminus \{i_1, i_2\}$ (for $d = 3$ take any $r-1$ vertices in $[n-1] \times [d] \setminus \{i_1, i_2\}$). As above, insert (n, i_1) and (n, i_2) into each edge of E'_{i_1,i_2}; let the resulting edge set be called E_{i_1,i_2}.

Finally, take all edges of F^d_n containing at least three vertices with n as their first coordinate, and let E_0 be this edge set; clearly $|E_0| = O_H(n^{d-3})$ as well. Put

$$G(n, H) := G(n-1, H) \cup \bigcup_{i \in [d]} E_i \cup \bigcup_{\{i_1,i_2\} \in \binom{[d]}{2}} E_{i_1,i_2} \cup E_0,$$

and

$$E^d(n, H) := G(n, H) \setminus (K^d_{[n]} \times K^d_{[d]}).$$

By the induction hypothesis and the bounds on the $|E_i|$, the $|E_{i_1,i_2}|$ and $|E_0|$, we have $|E^d(n, H)| = O_H(n^{d-2})$. To see that $G(n, H)$ is weakly H-saturated, first note that by induction hypothesis $G(n-1,H)$ is weakly H-saturated in F^d_{n-1}, hence the d-graph $G(n-1,H)\cup(K^d_{[n]}\times K^d_{[d]})\subseteq G(n,H)$ is weakly H-saturated in $J_0:=F^d_{n-1}\cup(K^d_{[n]}\times K^d_{[d]})$. Furthermore, let

$$K_1:=\{e\in F^d_n\colon |e\cap(\{n\}\times[d])|=1\},$$

and

$$K_2:=\{e\in F^d_n\colon |e\cap(\{n\}\times[d])|=2\}.$$

Let $J_1:=J_0\cup K_1$ and $J_2:=J_1\cup K_2$. By construction, $J_0\cup\bigcup_{i\in[d]}E_i$ is weakly H-saturated in J_1, $J_1\cup\bigcup_{\{i_1,i_2\}\in\binom{[d]}{2}}E_{i_1,i_2}$ is weakly H-saturated in J_2 and $J_2\cup E_0=F^d_n$. Thus, $G(n,H)$ is weakly H-saturated in F^d_n as desired. This proves the induction step, and the statement of the lemma follows. □

Proof of Theorem 8. For the first statement, suppose that $G\subseteq K_{n,n}$ is weakly H-saturated in $K_{n,n}$. Placing two $|V(H)|$-cliques on the parts of G is easily seen to produce a weakly H-saturated graph in K_{2n}. Therefore,

$$\mathrm{wsat}(2n,H)\le\mathrm{wsat}(K_{n,n},H)+|V(H)|^2. \tag{3.20}$$

Conversely, suppose that $G=G_0$ is weakly H-saturated in $K_{[n]}$ via a saturating sequence $e_1=\{i_1,j_1\},\dots,e_k=\{i_k,j_k\}$. For $1\le\ell\le k$ let $G_\ell=G_0\cup\{e_1,\dots e_\ell\}$, and let H_ℓ be a copy of H in G_ℓ containing e_ℓ.

Let $G^{bip}=G\times K_{[2]}$, i.e., $V(G^{bip})=[n]\times\{1,2\}$ and

$$E(G^{bip})=\{\{(i,1),(j,2)\}:\{i,j\}\in E(G)\}.$$

We claim that G^{bip} is weakly H-saturated in $K^{bip}_{[n]}=K_{[n]}\times K_{[2]}$ via the H-saturating sequence $f_1,f_1',\dots,f_k,f_k'$, where, for each $\ell\in[k]$, $f_\ell=\{(i_\ell,1),(j_\ell,2)\}$ and $f_\ell'=\{(i_\ell,2),(j_\ell,1)\}$, and that $G^{bip}_{\ell-1}\cup\{f_\ell,f_\ell'\}=G^{bip}_\ell$ for all $\ell\in[k]$ (where G^{bip}_ℓ is defined analogously, i.e., $G^{bip}_\ell=G_\ell\times K_{[2]}$). Indeed, let (A,B) be a bipartition of $V(H_\ell)$ with $i_\ell\in A$ and $j_\ell\in B$, and consider the analogous graph H^b_ℓ between $A\times\{1\}$ and $B\times\{2\}$, i.e., for every $(i,j)\in A\times B$ we have $\{(i,1),(j,2)\}\in E(H^b_\ell)$ if and only if $\{i,j\}\in E(H_\ell)$. Note that $f_\ell\in E(H^b_\ell)$ is the only edge of H^b_ℓ not already present in $G^{bip}_{\ell-1}$, therefore we can add it to the latter creating a new copy of H, namely H^b_ℓ. Symmetrically, taking a graph H'^b_ℓ between $A\times\{2\}$ and $B\times\{1\}$ allows to add f_ℓ'. Since $G_\ell=G_{\ell-1}\cup e_\ell$, we have $G^{bip}_{\ell-1}\cup\{f_\ell,f_\ell'\}=G^{bip}_\ell$. Finally, note that $G^{bip}\cup\{f_1,\dots,f_k'\}=G^{bip}_k=K^{bip}_{[n]}$.

Note that $K^{bip}_{[n]}$ is isomorphic to $K_{n,n}$ minus a perfect matching, and it is a straightforward check that this graph is H-saturated in $K_{n,n}$ (we can assume that $|V(H)|\le n$). We have thus shown

$$\mathrm{wsat}(K_{n,n},H)\le 2\,\mathrm{wsat}(n,H). \tag{3.21}$$

Combining (3.20) and (3.21) gives

$$\frac{\mathrm{wsat}(2n,H)}{2n}-o(1)\le\frac{\mathrm{wsat}(K_{n,n},H)}{2n}\le\frac{\mathrm{wsat}(n,H)}{n},$$

and taking the limit, (3.1) follows readily.

For the second statement, denote $H = K^d_{r_1,\ldots,r_d}$ where $1 \leq r_1 \leq \cdots \leq r_d$. Observe that the upper bound in (3.2) holds by Lemma 15, as $\delta^*(H) = r_1$. To prove the lower bound, suppose G is weakly H-saturated in $K^d_{[n]}$, and that $|E(G)| = \text{wsat}(n, H)$. Let $G^{mult} = G \times K^d_{[d]}$, that is, $V(G^{mult}) = [n] \times [d]$ and

$$E(G^{mult}) = \{\{(i_1, 1), \ldots, (i_d, d)\} : \{i_1, \ldots, i_d\} \in E(G)\}.$$

Essentially the same argument as for G^{bip} before shows that G^{mult} is weakly H-saturated in $K^d_{[n]} \times K^d_{[d]}$. By Lemma 16 adding further $O_H(n^{d-2})$ edges creates a weakly H-saturated d-graph in $K^d(n; d)$. Hence,

$$\text{wsat}(K^d(n; d), H) \leq |E(G^{mult})| + O(n^{d-2}) = d!\,\text{wsat}(n, H) + O(n^{d-2}). \quad (3.22)$$

On the other hand, Moshkovitz and Shapira [64] proved that $\text{wsat}(K^d(n; d), H) = d(r_1 - 1)n^{d-1} + O(n^{d-2})$. Combining this with (3.22) yields the lower bound in (3.2). □

4. Colorful Fractional Helly Theorem

The target of this chapter is to provide optimal bounds for the colorful fractional Helly theorem first stated by Bárány, Fodor, Montejano, Oliveros, and Pór [11], and then improved by Kim [53]. In order to explain the colorful fractional Helly theorem, let us briefly survey the preceding results.

The starting point, as usual in this context, is the Helly theorem:

Theorem 17 (Helly's theorem [38]). *Let $\mathcal{F}$ be a finite family of at least $d+1$ convex sets in $\mathbb{R}^d$. Assume that every subfamily of $\mathcal{F}$ with exactly $d+1$ members has a nonempty intersection. Then all sets in $\mathcal{F}$ have a nonempty intersection.*

Helly's theorem admits numerous extensions and two of them, important in our context, are the fractional Helly theorem and the colorful Helly theorem. The fractional Helly theorem of Katchalski and Liu covers the case when only some fraction of the $d+1$ tuples in $\mathcal{F}$ has a nonempty intersection.

Theorem 18 (The fractional Helly theorem [52]). *For every $\alpha \in (0,1]$ and every non-negative integer d, there is $\beta = \beta(\alpha, d) \in (0,1]$ with the following property. Let $\mathcal{F}$ be a finite family of $n \geq d+1$ convex sets in $\mathbb{R}^d$ such that at least $\alpha\binom{n}{d+1}$ of the subfamilies of $\mathcal{F}$ with exactly $d+1$ members have a nonempty intersection. Then there is a subfamily of $\mathcal{F}$ with at least βn members with a nonempty intersection.*

An interesting aspect of the fractional Helly theorem is not only to show the existence of $\beta(\alpha, d)$ but also to provide the largest value of $\beta(\alpha, d)$ with which the theorem is valid. This has been resolved independently by Eckhoff [26] and by Kalai [46] showing that the fractional Helly theorem holds with $\beta(\alpha, d) = 1-(1-\alpha)^{1/(d+1)}$; yet another simplified proof of this fact has been subsequently given by Alon and Kalai [3]. It is well known that this bound is sharp by considering a family $\mathcal{F}$ consisting of $\approx (1-(1-\alpha)^{1/(d+1)})n$ copies of $\mathbb{R}^d$ and $\approx (1-\alpha)^{1/(d+1)}n$ hyperplanes in general position; see, e.g., the introduction of [46].

The colorful Helly theorem of Lovász covers the case where the sets are colored by $d+1$ colors and only the 'colorful' $(d+1)$-tuples of sets in $\mathcal{F}$ are considered. Given families $\mathcal{F}_1, \ldots, \mathcal{F}_{d+1}$ of sets in $\mathbb{R}^d$ a family of sets $\{F_1, \ldots, F_{d+1}\}$ is a *colorful* $(d+1)$*-tuple* if $F_i \in \mathcal{F}_i$ for $i \in [d+1]$, where $[n] := \{1, \ldots, n\}$ for a non-negative integer $n \geq 1$. (The reader may think of $\mathcal{F}$ from preceding theorems decomposed into color classes $\mathcal{F}_1, \ldots, \mathcal{F}_{d+1}$.)

Theorem 19 (The colorful Helly theorem [59, 10]). *Let $\mathcal{F}_1, \ldots, \mathcal{F}_{d+1}$ be finite nonempty families of convex sets in $\mathbb{R}^d$. Let us assume that every colorful $(d+1)$-tuple has a nonempty intersection. Then one of the families $\mathcal{F}_1, \ldots, \mathcal{F}_{d+1}$ has a nonempty intersection.*

Both the colorful Helly theorem and the fractional Helly theorem with optimal bounds imply the Helly theorem. The colorful one by setting $\mathcal{F}_1 = \cdots = \mathcal{F}_{d+1} = \mathcal{F}$ and the fractional one by setting $\alpha = 1$ giving $\beta(1, d) = 1$.

The preceding two theorems can be merged into the following colorful fractional Helly theorem:

Theorem 20 (The colorful fractional Helly theorem [11]). *For every $\alpha \in (0,1]$ and every non-negative integer d, there is $\beta_{\text{col}} = \beta_{\text{col}}(\alpha, d) \in (0,1]$ with the following property. Let $\mathcal{F}_1, \ldots, \mathcal{F}_{d+1}$ be finite nonempty families of convex sets in $\mathbb{R}^d$ of sizes $n_1, \ldots, n_{d+1}$ respectively. If at least $\alpha n_1 \cdots n_{d+1}$ of the colorful $(d+1)$-tuples have a nonempty intersection, then there is $i \in [d+1]$ such that $\mathcal{F}_i$ contains a subfamily of size at least $\beta_{\text{col}} n_i$ with a nonempty intersection.*

Bárány et al. proved the colorful fractional Helly theorem with the value $\beta_{\text{col}}(\alpha, d) = \frac{\alpha}{d+1}$ and they used it as a lemma [11, Lemma 3] in a proof of a colorful variant of a (p,q)-theorem. Despite this, the optimal bound for β_{col} seems to be of independent interest. In particular, the bound on β_{col} has been subsequently improved by Kim [53] who showed that the colorful fractional Helly theorem is true with $\beta_{\text{col}}(\alpha, d) = \max\{\frac{\alpha}{d+1}, 1-(d+1)(1-\alpha)^{1/(d+1)}\}$. On the other hand, the value of $\beta_{\text{col}}(\alpha, d)$ cannot go beyond $1-(1-\alpha)^{1/(d+1)}$ because essentially the same example as for the standard fractional Helly theorem applies in this setting as well—it is sufficient to set $n_1 = n_2 = \cdots = n_{d+1}$ and take $\approx (1-(1-\alpha)^{1/(d+1)})n_i$ copies of $\mathbb{R}^d$ and $\approx (1-\alpha)^{1/(d+1)} n_i$ hyperplanes in general position in each color class.[1] (Kim [53] provides a slightly different upper bound example showing the same bound.)

Coming back to the lower bound on $\beta_{\text{col}}(\alpha, d)$, Kim explicitly conjectured that $1-(1-\alpha)^{1/(d+1)}$ is also a lower bound, thereby an optimal bound for the colorful fractional Helly theorem. He also provides a more refined conjecture, that we discuss slightly later on (see Conjecture 24), which implies this lower bound. We prove the refined conjecture, and therefore the optimal bounds for the colorful fractional Helly theorem.

Theorem 21 (The optimal colorful fractional Helly theorem). *Let $\mathcal{F}_1, \ldots, \mathcal{F}_{d+1}$ be finite nonempty families of convex sets in $\mathbb{R}^d$ of sizes $n_1, \ldots, n_{d+1}$ respectively. If at least $\alpha n_1 \cdots n_{d+1}$ of the colorful $(d+1)$-tuples have a nonempty intersection, for $\alpha \in (0,1]$, then there is $i \in [d+1]$ such that $\mathcal{F}_i$ contains a subfamily of size at least $(1-(1-\alpha)^{1/(d+1)})n_i$ with a nonempty intersection.*

In the proof we follow the exterior algebra approach which has been used by Kalai [46] in order to provide optimal bounds for the standard fractional Helly theorem. We have to upgrade Kalai's proof to the colorful setting. This requires guessing the right generalization of several steps in Kalai's proof (in particular guessing the statement of Theorem 26 below). However, we honestly admit that after making these 'guesses' we follow Kalai's proof quite straightforwardly.

Let us also compare one aspect of our proof with the previous proof of the weaker bound by Kim [53]: Kim's proof uses the colorful Helly theorem as a blackbox while our proof includes the proof of the colorful Helly theorem.

Last but not least, the exterior algebra approach actually allows to generalize Theorem 21 in several different directions. The extension to so called d-collapsible complexes is essentially mandatory for the well working proof while the other generalizations that we will present just follow from the method. We will discuss this in detail in forthcoming sections.

[1] At the end of Section 4.3 we discuss this example in full detail in more general context. However, in this special case, it is perhaps much easier to check directly that β_{col} cannot be improved due to this example.

4.1 d-representable and d-collapsible complexes

The nerve and d-representable complexes. The important information in Theorems 17, 18, 19, 20, and 21 is which subfamilies have a nonempty intersection. This information can be efficiently stored in a simplicial complex called the nerve.

Given a family of sets $\mathcal{F}$, the *nerve* of $\mathcal{F}$, is the simplicial complex whose vertex set is $\mathcal{F}$ and whose faces are subfamilies with a nonempty intersection. A simplicial complex is *d-representable* if it is the nerve of a finite family of convex sets in $\mathbb{R}^d$.

As a preparation for the d-collapsible setting, we now restate Theorem 21 in terms of d-representable complexes. For this we need two more notions. Given a simplicial complex K and a subset U of the vertex set N, the *induced subcomplex* $K[U]$ is defined as $K[U] := \{A \in K : A \subseteq U\}$. Now, let us assume that the vertex set N is split into $d+1$ pairwise disjoint subsets $N = N_1 \sqcup \cdots \sqcup N_{d+1}$ (we can think of this partition as coloring each vertex of N with one of the $d+1$ possible colors). Then a *colorful d-face* is a d-face A, such that $|A \cap N_i| = 1$ for every $i \in [d+1]$.

Theorem 22 (Theorem 21 reformulated)**.** *Let K be a d-representable simplicial complex with the set of vertices $N = N_1 \sqcup \cdots \sqcup N_{d+1}$ divided into $d+1$ disjoint subsets. Let $n_i := |N_i|$ for $i \in [d+1]$ and assume that K contains at least $\alpha n_1 \cdots n_{d+1}$ colorful d-faces for some $\alpha \in (0,1]$. Then there is $i \in [d+1]$ such that* $\dim K[N_i] \geq (1-(1-\alpha)^{1/(d+1)})n_i - 1$.

Theorem 22 is indeed just a reformulation of Theorem 21: Considering $\mathcal{F}$ as disjoint union[2] $\mathcal{F} = \mathcal{F}_1 \sqcup \cdots \sqcup \mathcal{F}_{d+1}$, then K corresponds to the nerve of $\mathcal{F}$, colorful d-faces correspond to colorful $(d+1)$-tuples with nonempty intersection and the dimension of $K[N_i]$ corresponds to the size of largest subfamily of $\mathcal{F}_i$ with nonempty intersection minus 1. (The shift by minus 1 between size of a face and dimension of a face is a bit unpleasant; however, we want to follow the standard terminology.)

d-collapsible complexes. In [87] Wegner introduced an important class of simplicial complexes, called d-collapsible complexes. They include all d-representable complexes, which is the main result of [87], while they admit quite simple combinatorial description which is useful for induction.

Given a simplicial complex K, we say that a simplicial complex K' arises from K by an *elementary d-collapse*, if there are faces $L, M \in K$ with the following properties: (i) $\dim L \leq d-1$; (ii) M is the unique inclusion-wise maximal face which contains L; and (iii) $K' = K \setminus \{A \in K : L \subseteq A\}$. A simplicial complex K is *d-collapsible* if there is a sequence of simplicial complexes $K_0, \ldots, K_\ell$ such that $K = K_0$; K_i arises from K_{i-1} by an elementary d-collapse for $i \in [\ell]$; and K_ℓ is the empty complex.

We will prove the following generalization of Theorem 22 (equivalently of Theorem 21).

[2] If there are any repetitions of sets in $\mathcal{F}$, which we generally allow for families of sets, then each repetition creates a new vertex in the nerve.

Theorem 23 (The optimal colorful fractional Helly theorem for d-collapsible complexes)**.** *Let K be a d-collapsible simplicial complex with the set of vertices $N = N_1 \sqcup \cdots \sqcup N_{d+1}$ divided into $d+1$ disjoint subsets. Let $n_i := |N_i|$ for $i \in [d+1]$ and assume that K contains at least $\alpha n_1 \cdots n_{d+1}$ colorful d-faces for some $\alpha \in (0, 1]$. Then there is $i \in [d+1]$ such that* $\dim K[N_i] \geq (1 - (1 - \alpha)^{1/(d+1)})n_i - 1$.

4.1.1 Kim's refined conjecture and further generalization

As a tool for a possible proof of Theorem 21, Kim [53, Conjecture 4.2] suggested the following conjecture. (The notation k_i in Kim's statement of the conjecture is our $r_i + 1$.)

Conjecture 24 ([53])**.** *Let n_i be positive and r_i non-negative integers for $i \in [d+1]$ with $n_i \geq r_i + 1$. Let $\mathcal{F}_1, \ldots, \mathcal{F}_{d+1}$ be families of convex sets in $\mathbb{R}^d$ such that $|\mathcal{F}_i| = n_i$ and there is no subfamily of $\mathcal{F}_i$ of size $r_i + 1$ with non-empty intersection for every $i \in [d+1]$. Then the number of colorful $(d+1)$-tuples with nonempty intersection is at most*

$$n_1 \cdots n_{d+1} - (n_1 - r_1) \cdots (n_{d+1} - r_{d+1}).$$

We explicitly prove this conjecture in a slightly more general setting for d-collapsible complexes. (Note that the condition 'no subfamily of size $r_i + 1$' translates as 'no r_i-face', that is, 'the dimension is at most $r_i - 1$'.)

Proposition 25. *Let n_i be positive and r_i non-negative integers for $i \in [d+1]$ with $n_i \geq r_i + 1$. Let K be a d-collapsible simplicial complex with the set of vertices $N = N_1 \sqcup \cdots \sqcup N_{d+1}$ divided into $d+1$ disjoint subsets. Assume that $|N_i| = n_i$ and that* $\dim K[N_i] \leq r_i - 1$ *for every $i \in [d+1]$. Then K contains at most*

$$n_1 \cdots n_{d+1} - (n_1 - r_1) \cdots (n_{d+1} - r_{d+1}).$$

colorful d-faces.

Our main technical result. Now, let us present our main technical tool for a proof of Proposition 25 and consequently for a proof of Theorem 23 as well.

We denote by $\mathbb{N}$ the set of positive integers whereas $\mathbb{N}_0$ is the set of non-negative integers. Let us consider $c \in \mathbb{N}$ and vectors $\mathbf{k} = (k_1, \ldots, k_c), \mathbf{r} = (r_1, \ldots, r_c) \in \mathbb{N}_0^c$ and $\mathbf{n} = (n_1, \ldots, n_c) \in \mathbb{N}^c$ such that $\mathbf{k}, \mathbf{r} \leq \mathbf{n}$. (Here the notation $\mathbf{a} \leq \mathbf{b}$ means that $\mathbf{a}$ is less or equal to $\mathbf{b}$ in every coordinate.) We will also use the notation $k := k_1 + \cdots + k_c$, $n := n_1 + \cdots + n_c$, and $r := r_1 + \cdots + r_c$. Let N be a set with n elements partitioned as $N = N_1 \sqcup \cdots \sqcup N_c$ where $|N_i| = n_i$ for $i \in [c]$. By $\binom{N}{\mathbf{k}}$ we denote the set of all subsets A of N such that $|A \cap N_i| = k_i$ for every $i \in [c]$. Note that $\binom{N}{\mathbf{k}} \subseteq \binom{N}{k}$ where $\binom{N}{k}$ denotes the set of all subsets of N of size k.

Let K be a simplicial complex with the vertex set N as above. We say that a face A of K is $\mathbf{k}$-colorful if $A \in \binom{N}{\mathbf{k}}$, that is, $|A \cap N_i| = k_i$ for every $i \in [c]$. The earlier notion of colorful face corresponds to setting $c = d + 1$ and $\mathbf{k} = \mathbf{1} := (1, \ldots, 1) \in \mathbb{N}^c$. By $f_{\mathbf{k}} = f_{\mathbf{k}}(K)$ we denote the $\mathbf{k}$-*colorful f-vector* of K, that is, the number of $\mathbf{k}$-colorful faces in K.

Let us further assume that we are given sets $R_i \subseteq N_i$ with $|R_i| = r_i$ for every $i \in [c]$. Let $R = R_1 \sqcup \cdots \sqcup R_c$ and $\bar{R} := N \setminus R$. Then, we define the set system

$$P_{\mathbf{k}}(\mathbf{n}, d, \mathbf{r}) = \left\{ S \in \binom{N}{\mathbf{k}} : |S \cap \bar{R}| \leq d \right\}.$$

We remark that $P_{\mathbf{k}}(\mathbf{n}, d, \mathbf{r})$ is not a simplicial complex, as it contains only sets in $\binom{N}{\mathbf{k}}$. However, this set system is useful for estimating the number of $\mathbf{k}$-colorful faces in a d-collapsible complex. By $p_{\mathbf{k}}(\mathbf{n}, d, \mathbf{r})$ we denote the size of $P_{\mathbf{k}}(\mathbf{n}, d, \mathbf{r})$, that is, $p_{\mathbf{k}}(\mathbf{n}, d, \mathbf{r}) := |P_{\mathbf{k}}(\mathbf{n}, d, \mathbf{r})|$.

Theorem 26. *For integers $c, d \geq 1$, let K be a d-collapsible simplicial complex with vertex partition $N = N_1 \sqcup \cdots \sqcup N_c$ and let $\mathbf{n} = (n_1, \ldots, n_c) \in \mathbb{N}^c$ be the vector with $n_i = |N_i|$. For $\mathbf{r} = (r_1, \ldots, r_c) \in \mathbb{N}^c$ such that $\dim K[N_i] \leq r_i - 1$ for $i \in [c]$ and $\mathbf{k} \in \mathbb{N}_0^c$ such that $\mathbf{k} \leq \mathbf{n}$ it follows that*

$$f_{\mathbf{k}}(K) \leq p_{\mathbf{k}}(\mathbf{n}, d, \mathbf{r}).$$

Theorem 26 is proved in Section 4.2. Here we show the implications Theorem 26 ⇒ Proposition 25 and Proposition 25 ⇒ Theorem 23. In addition, we advertise that Theorem 26 yields further generalizations of Theorem 23. We explain this last part in Section 4.3.

Proof of Proposition 25 modulo Theorem 26. We use Theorem 26 with $c = d + 1$ and $\mathbf{k} = \mathbf{1}$. Then it is sufficient to compute $p_{\mathbf{1}}(\mathbf{n}, d, \mathbf{r})$. On the one hand, the size of $\binom{N}{\mathbf{1}}$ is $n_1 \ldots n_{d+1}$. On the other hand, A belongs to $\binom{N}{\mathbf{1}} \setminus P_{\mathbf{1}}(\mathbf{n}, d, \mathbf{r})$ if and only if $|A \cap (N_i \setminus R_i)| = 1$ for every $i \in [d+1]$. Then, the number of such A is $(n_1 - r_1) \cdots (n_{d+1} - r_{d+1})$. Combining these observations we obtain the required formula

$$p_{\mathbf{1}}(\mathbf{n}, d, \mathbf{r}) = n_1 \ldots n_{d+1} - (n_1 - r_1) \cdots (n_{d+1} - r_{d+1}).$$

□

Proof of Theorem 23 modulo Proposition 25. By contradiction, let us assume that for every $i \in [d+1]$ we get $\dim K[N_i] < (1 - (1 - \alpha)^{1/(d+1)})n_i - 1$. Let us set $r_i := \dim K[N_i] + 1 < (1 - (1 - \alpha)^{1/(d+1)})n_i$. Then Proposition 25 gives that the number of colorful d-faces is at most

$$\prod_{i=1}^{d+1} n_i - \prod_{i=1}^{d+1} (n_i - r_i) < \prod_{i=1}^{d+1} n_i - (1 - (1 - (1 - \alpha)^{1/(d+1)}))^{d+1} \prod_{i=1}^{d+1} n_i = \alpha \prod_{i=1}^{d+1} n_i$$

which is a contradiction due to the strict inequality on the first line. □

4.2 Colorful exterior algebra

Now we extend the tools introduced in Section 2.2 to the colorful setting. Set $V = \mathbb{R}^N$. From now on, let us assume that N is an n-element set decomposed into c-color classes, $N = N_1 \sqcup \cdots \sqcup N_c$. (The total order on N in this case starts with elements of N_1, then continues with elements of N_2, etc.) Let A

be an $N \times N$ transition matrix from $(e_j)_{j \in N}$ to $(g_j)_{j \in N}$[3]. We pick A so that it is a block-diagonal matrix with blocks corresponding to individual N_i. That is, $A_{N_i|N_j}$ is a zero matrix whenever $i \neq j$. On the other hand, as shown by Kalai [46, Section 2], it is possible to pick each $A_{N_i|N_i}$ so that $(g_j)_{j \in N_i}$ is an orthonormal basis of the subspace of V generated by $(e_j)_{j \in N_i}$ and each square submatrix of $A_{N_i|N_i}$ has full rank. Therefore, from now on, we assume that we picked A and the vectors g_j this way. (Such a block matrix, for $c = 2$, is previously mentioned in [66].)

Similarly as in the beginning of the chapter, let us set $\mathbf{n} = (n_1, \ldots, n_c)$ so that $n_i = |N_i|$ for $i \in [c]$; for simplicity, let us assume that each N_i is nonempty—that is, $\mathbf{n}$ is a c-tuple of positive integers. Let us also consider another c-tuple $\mathbf{k} = (k_1, \ldots, k_c)$ of non-negative integers such that $\mathbf{k} \leq \mathbf{n}$ and we set $k := k_1 + \cdots + k_c$. Then by $\bigwedge^{\mathbf{k}} V$ we mean the subspace of $\bigwedge V$ generated by $(e_S)_{S \in \binom{N}{\mathbf{k}}}$; recall that $\binom{N}{\mathbf{k}}$ is the set of all subsets A of N such that $|A \cap N_i| = k_i$ and that $\binom{N}{\mathbf{k}} \subseteq \binom{N}{k}$. Thus we also get that $\bigwedge^{\mathbf{k}} V$ is a subspace of $\bigwedge^{k} V$. In addition, due to our choice of $(g_j)_{j \in N}$ we get that $g_S \in \bigwedge^{\mathbf{k}} V$ if $S \in \binom{N}{\mathbf{k}}$. In addition $\det A_{S|T} = 0$ if $T \in \binom{N}{k} \setminus \binom{N}{\mathbf{k}}$ because $A_{S|T}$ is in this case a block matrix such that some of the blocks is not a square. Thus the formula (2.2) simplifies to

$$g_S = \sum_{T \in \binom{N}{\mathbf{k}}} \det A_{S|T} e_T. \tag{4.1}$$

Proof of Theorem 26. For $\mathbf{k} \in \mathbb{N}^c$ such that $k \leq d$ we have that $P_{\mathbf{k}}(\mathbf{n}, d, \mathbf{r}) = \binom{N}{\mathbf{k}}$, thus the theorem follows trivially. On the other hand, if $k > r$, then $k_i > r_i$ for some i and consequently $f_{\mathbf{k}}(K) = 0$ due to our assumption $\dim K[N_i] \leq r_i - 1$; therefore the theorem again follows trivially. From now on we assume $d + 1 \leq k \leq r$. (We also use the notation for the sets R, $\bar{R}$ and R_i with $|R_i| = r_i$ as in the definition of $P_{\mathbf{k}}(\mathbf{n}, d, \mathbf{r})$.)

Let us define the following subspaces of $\bigwedge^{\mathbf{k}} V$

$$A_{\mathbf{k}} := \left\{ m \in \bigwedge^{\mathbf{k}} V : \left(\forall T \in \binom{R}{k-d} \right) \; g_T \llcorner m = 0 \right\},$$

and

$$W_{\mathbf{k}} := \operatorname{span} \left\{ e_S \in \bigwedge^{\mathbf{k}} V : S \in \binom{N}{\mathbf{k}} \text{ and } S \in K \right\},$$

from the definition it follows that the colorful f-vector and the dimension of $W_{\mathbf{k}}$ coincide, i.e. $f_{\mathbf{k}} = \dim(W_{\mathbf{k}})$.

We claim that

$$\dim(A_{\mathbf{k}}) \geq \left| \binom{N}{\mathbf{k}} \right| - p_{\mathbf{k}}(\mathbf{n}, d, \mathbf{r}).$$

Indeed, if $S \in \binom{N}{\mathbf{k}}$ such that $S \notin P_{\mathbf{k}}(\mathbf{n}, d, \mathbf{r})$, then $|S \cap \bar{R}| > d$. As $S \subseteq R \sqcup \bar{R} = N$ and $|S| = k$ we have that $|S \cap R| < k - d$. If $T \in \binom{R}{k-d}$ we have that $S \not\supseteq T$; therefore $g_T \llcorner g_S = 0$. From this it follows that $g_S \in A_{\mathbf{k}}$ and finally the claim because $g_S \in \bigwedge^{\mathbf{k}} V$.

[3]Although in Section 2.2 we have chosen to use f's to denote a new basis, in this chapter we will use g's to avoid possible confusion with the f-vector.

The core of the proof is to show $A_{\mathbf{k}} \cap W_{\mathbf{k}} = \{0\}$. Once we have this, we get $f_k(K) = \dim(W_k) \leq \dim \bigwedge^{\mathbf{k}} V - \dim A_{\mathbf{k}} \leq p_{\mathbf{k}}(\mathbf{n}, d, \mathbf{r})$ which proves the theorem.

For contradiction, let $m \in A_k \cap W_k$ be a non-zero element. Because $m \in W_{\mathbf{k}}$, it can be written as $m = \sum \alpha_S e_S$ where the sum is over all $S \in \binom{N}{\mathbf{k}}$ such that $S \in K$. Let $K_0, \ldots, K_\ell$ be a sequence of simplicial complexes showing d-collapsibility of K as in the definition of d-collapsible complex. In addition, due to [46, Lemma 3.2], it is possible to assume that K_i arises from K_{i-1} by so called *special* elementary d-collapse which is either a removal of a maximal face of dimension at most $d-1$ or the minimal face (the face L in the definition) has dimension exactly $d-1$. Now let us consider the first step from K_{i-1} to K_i such that a face $U \in \binom{N}{\mathbf{k}}$ with non-zero α_U is eliminated. Denote by L and M the faces determining the collapse as in the definition. We have $L \subseteq U \subseteq M$, $|M| \geq |U| = k > d$ and therefore $|L| = d$ (equivalently, $\dim L = d-1$), because the collapse is special. For $T \in \binom{R}{k-d}$ let $\mathbf{t} = (t_1, \ldots, t_c) \in \mathbb{N}^c$ be such that $t_i = |T \cap N_i|$. Then $g_T = \sum_{P \in \binom{N}{\mathbf{t}}} \det(A_{T|P}) e_P$ via (4.1). We also need to simplify the expression $\langle e_L, g_T \llcorner e_S \rangle$ for $S \in \binom{N}{\mathbf{k}}$. We obtain

$$\langle e_L, g_T \llcorner e_S \rangle = \langle e_L \wedge g_T, e_S \rangle = \sum_{P \in \binom{N}{\mathbf{t}}} \det(A_{T|P}) \langle e_L \wedge e_P, e_S \rangle \tag{4.2}$$

If $S \not\supseteq L$ then $\langle e_L \wedge e_P, e_S \rangle = 0$ for all P, and therefore $\langle e_L, g_T \llcorner e_S \rangle = 0$. If $S \supseteq L$ then $\langle e_L \wedge e_P, e_S \rangle = 0$ unless $P = S \setminus L$ and therefore $\langle e_L, g_T \llcorner e_S \rangle = \langle e_L \wedge e_{S \setminus L}, e_S \rangle \det(A_{T|S \setminus L})$.

Since $m \in A_k$, for arbitrary $T \in \binom{R}{k-d}$ we get

$$\begin{aligned} 0 = \langle e_L, g_T \llcorner m \rangle &= \sum_{S \in \binom{N}{\mathbf{k}} : S \in K} \alpha_S \langle e_L, g_T \llcorner e_S \rangle = \sum_{S \in \binom{N}{\mathbf{k}} : S \in K_{i-1}} \alpha_S \langle e_L, g_T \llcorner e_S \rangle \\ &= \sum_{S \in \binom{N}{\mathbf{k}} : S \supseteq L} \alpha_S \langle e_L, g_T \llcorner e_S \rangle = \sum_{S \in \binom{N}{\mathbf{k}} : M \supseteq S \supseteq L} \alpha_S \langle e_L \wedge e_{S \setminus L}, e_S \rangle \det(A_{T|S \setminus L}) \end{aligned}$$

where the third equality follows from the fact that $\alpha_S = 0$ for $S \in K \setminus K_{i-1}$ due to our choice of K_{i-1} and the last two equalities follow from our earlier simplification of $\langle e_L, g_T \llcorner e_S \rangle$. (We also use that the expressions $S \supseteq L$ and $M \supseteq S \supseteq L$ are equivalent as M is the unique maximal face containing L.)

We also have $U \in \binom{N}{\mathbf{k}}$ with $M \supseteq U \supseteq L$ for which $\alpha_U \neq 0$ as well as $\langle e_L \wedge e_{U \setminus L}, e_U \rangle$ is nonzero (the latter one equals ± 1). Therefore the expression above is a linear dependence of the columns of $C_{k-d}(A_{R|M \setminus L})$. However, we will also show that the columns of $C_{k-d}(A_{R|M \setminus L})$ are linearly independent, thereby getting a contradiction. Via Lemma 2, it is sufficient to check that the columns of $A_{R|M \setminus L}$ are linearly independent. Because A is a block-matrix with blocks $A_{N_i|N_i}$, we get that $A_{R|M \setminus L}$ is a block matrix with blocks $A_{R_i|(M \setminus L) \cap N_i}$. Thus it is sufficient to check that the columns are independent in each block. But this follows from our assumptions of how we picked A in each block, using that $|R_i| = r_i \geq |(M \setminus L) \cap N_i|$ as $|M \cap N_i| \leq r_i$ due to our assumption $\dim K[V_i] \leq r_i - 1$. $\square$

4.3 k-colorful fractional Helly theorem

Theorem 26 allows to generalize Theorem 23 in two more directions.

The first generalization of Theorem 23 is already touched in the beginning of the chapter. We can deduce an analogy of Theorem 23 for $\mathbf{k}$-colorful faces (instead of just colorful d-faces) where $\mathbf{k} = (k_1, \dots, k_c) \in \mathbb{N}_0^c$ is some vector with $c \geq 1$. For example, if $d = 2$, $\mathbf{k} = (2, 1, 1)$ and we understand the partition of $N = N_1 \sqcup N_2 \sqcup N_3$ as coloring the vertices of K red, green, or blue. Then we seek for number of faces that contain two red vertices, one green vertex and one blue vertex.

For the second generalization, let us first observe that in the conclusion of Theorem 23 there is the same coefficient $1 - (1 - \alpha)^{1/(d+1)}$ independently of i. However, in the notation of Theorem 23, we may also seek for i such that $\dim K[N_i] \geq \beta_i n_i + 1$ where $\boldsymbol{\beta} = (\beta_1, \dots, \beta_c) \in (0, 1]^c$ is some fixed vector. Then for given $\boldsymbol{\beta}$, we want to find the lowest $\alpha \in (0, 1]$ with which we reach the conclusion analogous as in Theorem 23. This is a natural analogy of various Ramsey type statements: for example, if the edges of a complete graph G with at least 9 vertices are colored blue or red, then the graph contains either a blue copy of the complete graph on 3 vertices or a red copy of the complete graph on 4 vertices.

For the purpose of stating the generalization, let us set

$$L_{\mathbf{k}}(d) := \{\boldsymbol{\ell} = (\ell_1, \cdots \ell_c) \in \mathbb{N}_0^c \colon \ell_1 + \cdots + \ell_c \leq d \text{ and } \ell_i \leq k_i \text{ for } i \in [c]\} \tag{4.3}$$

and

$$\alpha_{\mathbf{k}}(d, \boldsymbol{\beta}) := \sum_{\boldsymbol{\ell} = (\ell_1, \dots, \ell_c) \in L_{\mathbf{k}}(d)} \prod_{i=1}^{c} \binom{k_i}{\ell_i} (1 - \beta_i)^{\ell_i} (\beta_i)^{k_i - \ell_i}. \tag{4.4}$$

Theorem 27. *Let $c, d \geq 1$ and $\mathbf{k} = (k_1, \dots, k_c) \in \mathbb{N}_0^c$ be such that $k := k_1 + \cdots + k_c \geq d + 1$. Let K be a d-collapsible simplicial complex with the set of vertices $N = N_1 \sqcup \cdots \sqcup N_c$ divided into c disjoint subsets. Let $n_i := |N_i|$ for $i \in [c]$ and assume that K contains at least $\alpha_{\mathbf{k}}(d, \boldsymbol{\beta}) \left| \binom{N}{\mathbf{k}} \right|$ $\mathbf{k}$-colorful faces for some $\boldsymbol{\beta} = (\beta_1, \dots, \beta_c) \in (0, 1]^c$. Then there is $i \in [c]$ such that $\dim K[N_i] \geq \beta_i n_i - 1$.*

The formula (4.4) for $\alpha_{\mathbf{k}}(d, \boldsymbol{\beta})$ in Theorem 27 is, unfortunately, a bit complicated. However, this is the optimal value for α in the theorem. We first prove Theorem 27 and then we will provide an example showing that for every d, $\mathbf{k}$ and $\boldsymbol{\beta}$ as in the theorem, the value for α cannot be improved. The remark below is a probabilistic interpretation of (4.4). (This, for example, easily reveals that $\alpha_{\mathbf{k}}(d, \boldsymbol{\beta}) \in (0, 1]$ for given parameters and will help us with checking monotonicity in $\boldsymbol{\beta}$.)

Remark 28. *Consider a random experiment where we gradually for each i pick k_i numbers $x_1^i, \dots, x_{k_i}^i$ in the interval $[0, 1]$ independently at random (with uniform distribution). Let ℓ_i be the number of x_j^i which are greater than β_i and let us consider the event $A_{\mathbf{k}}(d, \boldsymbol{\beta})$ expressing that $\ell_1 + \cdots + \ell_c \leq d$. Then $\alpha_{\mathbf{k}}(d, \boldsymbol{\beta})$ is the probability $\mathbf{P}[A_{\mathbf{k}}(d, \boldsymbol{\beta})]$.*

Indeed, the probability that the number of x_j^i which are greater than β_i is exactly ℓ_i is given by the expression beyond the sum in (4.4). Therefore, we need to sum this over all options giving $\ell_1 + \cdots + \ell_c \leq d$ and $\ell_i \leq k_i$.

In the proof of Theorem 27 we will need the following slightly modified proposition. We relax 'at least' to 'more than' while we aim at strict inequality in the conclusion—this innocent change will be a significant advantage in the proof. On the other hand, after this change we can drop the assumption $k \geq d+1$. But this is only a cosmetic change, because the proposition below is vacuous if $\alpha_{\mathbf{k}}(d, \boldsymbol{\beta}) = 1$ which in particular happens if $k < d+1$.

Proposition 29. *Let $c, d \geq 1$ and $\mathbf{k} = (k_1, \dots, k_c) \in \mathbb{N}_0^c$. Let K be a d-collapsible simplicial complex with the set of vertices $N = N_1 \sqcup \cdots \sqcup N_c$ divided into c disjoint subsets. Let $n_i := |N_i|$ for $i \in [c]$ and assume that K contains more than $\alpha_{\mathbf{k}}(d, \boldsymbol{\beta}) \left|\binom{N}{\mathbf{k}}\right|$ $\mathbf{k}$-colorful faces for some $\boldsymbol{\beta} = (\beta_1, \dots, \beta_c) \in (0,1]^c$. Then there is $i \in [c]$ such that $\dim K[N_i] > \beta_i n_i - 1$.*

First we show how Theorem 27 follows from Proposition 29 by a limit transition. Then we prove Proposition 29.

Proof of Theorem 27 modulo Proposition 29. Let us consider $\varepsilon > 0$ such that $\boldsymbol{\beta} - \boldsymbol{\varepsilon} \in (0,1]^c$ for $\boldsymbol{\varepsilon} = (\varepsilon, \dots, \varepsilon) \in (0,1]^c$.

First, we need to check $\alpha_{\mathbf{k}}(d, \boldsymbol{\beta}) > \alpha_{\mathbf{k}}(d, \boldsymbol{\beta} - \boldsymbol{\varepsilon})$. For this we will use Remark 28 and we also use $k \geq d+1$. It is easy to check $A_{\mathbf{k}}(d, \boldsymbol{\beta}) \supseteq A_{\mathbf{k}}(d, \boldsymbol{\beta} - \boldsymbol{\varepsilon})$ which gives $\alpha_{\mathbf{k}}(d, \boldsymbol{\beta}) \geq \alpha_{\mathbf{k}}(d, \boldsymbol{\beta} - \boldsymbol{\varepsilon})$. In order to show the strict inequality, it remains to show that $A_{\mathbf{k}}(d, \boldsymbol{\beta}) \setminus A_{\mathbf{k}}(d, \boldsymbol{\beta} - \boldsymbol{\varepsilon})$ has positive probability. Consider the output of the experiment when each $x_i^j \in (\beta_i - \varepsilon, \beta_i)$. This output has positive probability ε^k. In addition, this output belongs to $A_{\mathbf{k}}(d, \boldsymbol{\beta})$ whereas it does not belong to $A_{\mathbf{k}}(d, \boldsymbol{\beta} - \boldsymbol{\varepsilon})$ (because $k \geq d+1$) as required.

This means, that we can apply Proposition 29 with $\alpha_{\mathbf{k}}(d, \boldsymbol{\beta} - \boldsymbol{\varepsilon})$ as we know that K has at least $\alpha_{\mathbf{k}}(d, \boldsymbol{\beta}) \left|\binom{N}{\mathbf{k}}\right|$ $\mathbf{k}$-colorful faces by assumptions of Theorem 27 which is more than $\alpha_{\mathbf{k}}(d, \boldsymbol{\beta} - \varepsilon) \left|\binom{N}{\mathbf{k}}\right|$. We obtain $\dim K[N_i] > (\beta_i - \varepsilon) n_i - 1$. By letting ε to tend to 0, we obtain the required $\dim K[N_i] \geq \beta_i n_i - 1$. $\square$

Boosting the complex. In the proof of Proposition 29, we will need the following procedure for boosting the complex. For a given complex K with vertex set $N = N_1 \sqcup \cdots \sqcup N_c$ partitioned as usual, and a non-negative integer m we define the complex $K_{\langle m \rangle}$ as a complex with the vertex set $N \times [m] = N_1 \times [m] \sqcup \cdots \sqcup N_c \times [m]$ whose maximal faces are of the form $S \times [m]$, where S is a maximal face of K. We will also use the notation $\delta_{\mathbf{k}}(K) := f_{\mathbf{k}}(K)/|\binom{N}{\mathbf{k}}|$ for the density of $\mathbf{k}$-colorful faces of K.

Lemma 30. *Let K be a simplicial complex with vertex partition $N = N_1 \sqcup \cdots \sqcup N_c$ and $\mathbf{k} = (k_1, \dots, k_c) \in \mathbb{N}_0^c$, then*

(i) $\delta_{\mathbf{k}}(K_{\langle m \rangle}) \geq \delta_{\mathbf{k}}(K)$; and

(ii) if K is d-collapsible, then $K_{\langle m \rangle}$ is d-collapsible as well.

Proof. Let us start with the proof of (i). If $\delta_{\mathbf{k}}(K) = 0$ there is nothing to prove. Thus we may assume that $\delta_{\mathbf{k}}(K) > 0$ (equivalently $f_{\mathbf{k}}(K) > 0$) and consequently we have that $|N_i| \geq k_i$. Let us interpret $\delta_{\mathbf{k}}(K)$ as the probability that a random $\mathbf{k}$-tuple of vertices in N is a simplex of K, and we interpret $\delta_{\mathbf{k}}(K_{\langle m \rangle})$ analogously. Let $\pi \colon N \times [m] \to N$ be the projection to the first coordinate. Now, let U be a $\mathbf{k}$-tuple of vertices in $N \times [m]$ taken uniformly at random. Considering the set $\pi(U) \subseteq N$, it need not be a $\mathbf{k}$-tuple (this happens exactly when two points in

U have the same image under π) but it can be extended to a $\mathbf{k}$-tuple W using that $|N_i| \geq k_i$ for every i. Let W be an extension of $\pi(U)$ to a $\mathbf{k}$-tuple, taken uniformly at random among all possible choices. Because of the choices we made, W is in fact a $\mathbf{k}$-tuple of vertices in N taken uniformly at random. (Note that the choices done in each N_i or $N_i \times [m]$ are independent of each other.) Altogether, using $\mathbf{P}$ for probability, we get

$$\delta_{\mathbf{k}}(K_{\langle m \rangle}) = \mathbf{P}[U \in K_{\langle m \rangle}] = \mathbf{P}[\pi(U) \in K] \geq \mathbf{P}[W \in K] = \delta_{\mathbf{k}}(K).$$

This shows (i).

For (ii), we follow the idea of splitting a vertex from [4, Proposition 14(i)] which proves a similar statement for d-Leray complexes. For a complex K and a vertex $v \in K$ let $K^{v \to v_1, v_2}$ be a complex obtained from K by splitting the vertex v into two newly introduced vertices v_1 and v_2. That is, if V is the set of vertices of K, then the set of vertices of $K^{v \to v_1, v_2}$ is $(V \cup \{v_1, v_2\}) \setminus \{v\}$ assuming $v_1, v_2 \notin V$. The maximal simplices of $K^{v \to v_1, v_2}$ are obtained from maximal simplices S of K by replacing v with v_1 and v_2, if S contains v (otherwise S is kept as it is). Our aim is to show that if K is d-collapsible, then $K^{v \to v_1, v_2}$ is d-collapsible as well. This will prove (ii) because $K_{\langle m \rangle}$ can be obtained from K by repeatedly splitting some vertex. For the proof, we extend the notation $K^{v \to v_1, v_2}$ by setting $K^{v \to v_1, v_2} = K$ if v does not belong to K.

Let $K_0 = K, K_1, \ldots, K_\ell = \emptyset$ be a sequence such that K_i arises from K_{i-1} by an elementary d-collapse. Our task is to show that $K_{i-1}^{v \to v_1, v_2}$ d-collapses to $K_i^{v \to v_1, v_2}$ for $i \in [\ell]$. This will show the claim as $K_\ell^{v \to v_1, v_2} = \emptyset$. For simplicity of the notation, we will treat only the elementary d-collapse from K to K_1 as other steps are analogous. We will assume $v \in K$, as there is nothing to do if $v \notin K$.

Let L and M be the faces from the elementary d-collapse. That is, $\dim L \leq d - 1$; M is the unique maximal face in K which contains L and K_1 is obtained from K by removing all faces that contain L, including L. We will distinguish three cases according to whether $v \in L$ or $v \in M$.

If $v \notin M$ (which implies $v \notin L$), then M is the unique maximal face containing L in $K^{v \to v_1, v_2}$ and the elementary d-collapse removing L and all its superfaces yields $K_1^{v \to v_1, v_2}$.

If $v \in M$ while $v \notin L$, then $(M \cup \{v_1, v_2\}) \setminus \{v\}$ is the unique maximal face containing L in $K^{v \to v_1, v_2}$ and the elementary d-collapse removing L and all its superfaces yields $K_1^{v \to v_1, v_2}$.

Finally, if $v \in M$ and $v \in L$, then we need to perform the d-collapse from $K^{v \to v_1, v_2}$ to $K_1^{v \to v_1, v_2}$ by two elementary steps; see Figure 4.1. First we realize that $(M \cup \{v_1, v_2\}) \setminus \{v\}$ is the unique maximal face containing $(L \cup \{v_1\}) \setminus \{v\}$ in $K^{v \to v_1, v_2}$. Because $\dim (L \cup \{v_1\}) \setminus \{v\} = \dim L$, we can perform an elementary d-collapse removing $(L \cup \{v_1\}) \setminus \{v\}$ and all its superfaces obtaining a complex K'. In K' we have that $(M \cup \{v_2\}) \setminus \{v\}$ is the unique maximal face containing $(L \cup \{v_2\}) \setminus \{v\}$. After removing $(L \cup \{v_2\}) \setminus \{v\}$ and all its superfaces, we get desired $K_1^{v \to v_1, v_2}$ (note that in this case $K_1^{v \to v_1, v_2}$ is indeed obtained from $K^{v \to v_1, v_2}$ by removing $(L \cup \{v_1\}) \setminus \{v\}$, $(L \cup \{v_2\}) \setminus \{v\}$ and all their superfaces). □

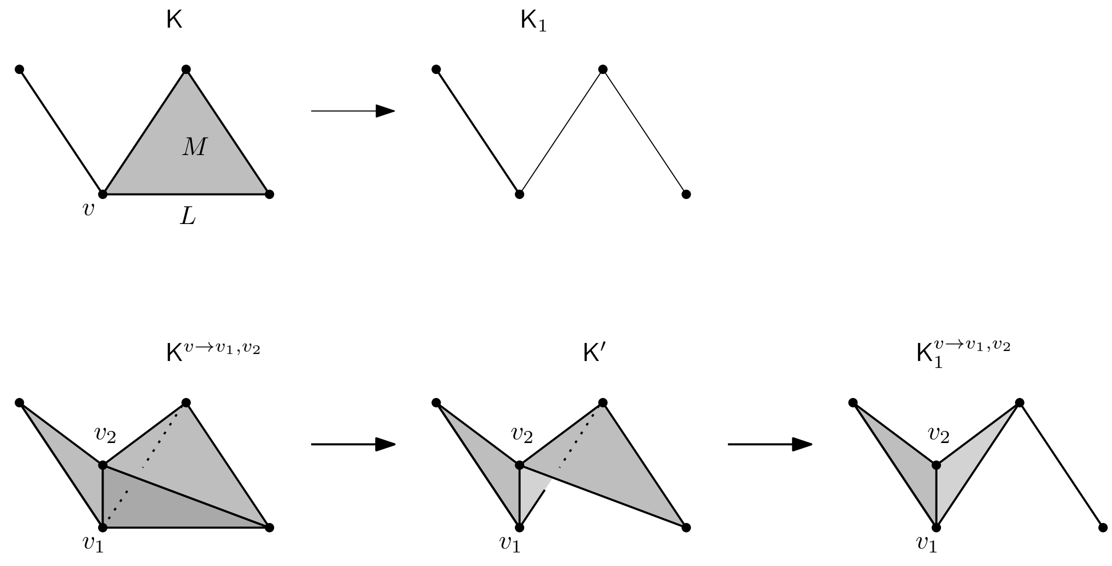

Figure 4.1: Collapses from $K^{v\to v_1,v_2}$ to $K_1^{v\to v_1,v_2}$ if $v \in L$.

Density of $P_{\mathbf{k}}(\mathbf{n}, d, \mathbf{r})$. Now, we will provide a formula for the density of $P_{\mathbf{k}}(\mathbf{n}, d, \mathbf{r})$. In the following computations we also set

$$\delta_{\mathbf{k}}(\mathbf{n}, d, \mathbf{r}) = p_{\mathbf{k}}(\mathbf{n}, d, \mathbf{r})/\left|\binom{N}{\mathbf{k}}\right|$$

using the notation from the definition of $P_{\mathbf{k}}(\mathbf{n}, d, \mathbf{r})$. We get

$$\begin{aligned} p_{\mathbf{k}}(\mathbf{n}, d, \mathbf{r}) &= \left|\left\{S \in \binom{N}{\mathbf{k}} : |S \cap \bar{R}| \le d\right\}\right| \\ &= \sum_{\boldsymbol{\ell}=(\ell_1,\dots,\ell_c)\in L_{\mathbf{k}}(d)} \left|\left\{S \in \binom{N}{\mathbf{k}} : |S_i \cap \bar{R}_i| = l_i\right\}\right| \\ &= \sum_{\boldsymbol{\ell}=(\ell_1,\dots,\ell_c)\in L_{\mathbf{k}}(d)} \prod_{i=1}^{c} \binom{n_i - r_i}{l_i}\binom{r_i}{k_i - l_i}. \end{aligned}$$

Then, using $(x)_m := x \cdot (x-1)\cdots(x-(m-1))$, the density is given by

$$\delta_{\mathbf{k}}(\mathbf{n}, d, \mathbf{r}) = \frac{p_{\mathbf{k}}(\mathbf{n}, d, \mathbf{r})}{\prod_{i=1}^{c}\binom{n_i}{k_i}} = \frac{\sum\limits_{\boldsymbol{\ell}=(\ell_1,\dots,\ell_c)\in L_{\mathbf{k}}(d)} \prod\limits_{i=1}^{c} \binom{k_i}{\ell_i}(n_i - r_i)_{\ell_i}(r_i)_{k_i-\ell_i}}{\prod_{i=1}^{c}(n_i)_{k_i}}. \tag{4.5}$$

Proof of Proposition 29. For contradiction, let us assume that for every $i \in [c]$ we have that $\dim(K[V_i]) \le \beta_i n_i - 1$. Let us set $r_i := \dim(K[V_i]) + 1 \le \beta_i n_i$. Note that the conclusion of Theorem 26 can be restated as $\delta_{\mathbf{k}}(K) \le \delta_{\mathbf{k}}(\mathbf{n}, d, \mathbf{r})$.

Now we get

$$\begin{aligned}
\delta_{\mathbf{k}}(K) &\leq \liminf_{m\to\infty} \delta_{\mathbf{k}}(K_{\langle m\rangle}) \text{ by Lemma 30(i)} \\
&\leq \liminf_{m\to\infty} \delta_{\mathbf{k}}(m\mathbf{n}, d, m\mathbf{r}) \text{ by Theorem 26 using Lemma 30(ii)} \\
&\leq \liminf_{m\to\infty} \delta_{\mathbf{k}}(m\mathbf{n}, d, \lfloor mn_i\beta_i \rfloor),\ r_i \leq \beta_i n_i \text{ and } p_{\mathbf{k}}(\mathbf{n}, d, \mathbf{r}) \text{ is monotone in } \mathbf{r} \\
&= \liminf_{m\to\infty} \frac{\sum_{\boldsymbol{\ell}=(\ell_1,\dots,\ell_c)\in L_{\mathbf{k}}(d)} \prod_{i=1}^{c} \binom{k_i}{\ell_i}(mn_i - \lfloor mn_i\beta_i\rfloor)_{\ell_i}(\lfloor mn_i\beta_i\rfloor)_{k_i-\ell_i}}{\prod_{i=1}^{c}(mn_i)_{k_i}} \text{ by (4.5)} \\
&= \sum_{\boldsymbol{\ell}=(\ell_1,\dots,\ell_c)\in L_{\mathbf{k}}(d)} \prod_{i=1}^{c} \binom{k_i}{\ell_i}(1-\beta_i)^{\ell_i}(\beta_i)^{k_i-\ell_i} \\
&= \alpha_{\mathbf{k}}(d, \boldsymbol{\beta})
\end{aligned}$$

which is a contradiction with the assumptions. □

Remark 31. *It would be much more natural to try to avoid boosting the complex and show directly $\delta_k(K) \leq \delta_k(\mathbf{n}, d, \mathbf{r}) \leq \alpha_{\mathbf{k}}(d, \boldsymbol{\beta})$ in the proof of Proposition 29. The former inequality follows from Theorem 26. However, the latter inequality turned out to be somewhat problematic for us when we attempted to show it directly from the definition of $\alpha_{\mathbf{k}}(d, \boldsymbol{\beta})$ and from* (4.5)*. Thus, in our computations, we take an advantage of the fact that the computations in the limit are easier.*

Tightness of Theorem 27. We conclude this section by showing that the bound given in Theorem 27 is tight.

Let us fix $c, d \in \mathbb{N}$, $\mathbf{k} = (k_1, \dots, k_c) \in \mathbb{N}_0^c$ with $k := k_1 + \cdots + k_c \geq d+1$ and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_c) \in (0,1]^c$ as in the statement of Theorem 27. Let $0 \leq \alpha' < \alpha_{\mathbf{k}}(d, \boldsymbol{\beta})$. We will find a complex K which contains at least $\alpha'|\binom{N}{\mathbf{k}}|$ $\mathbf{k}$-colorful faces while $\dim K[N_i] < \beta_i n_i - 1$ for every $i \in [c]$ (using the notation from the statement of Theorem 27).

Similarly as in the proof of Theorem 27 let us consider $\varepsilon > 0$ such that $\boldsymbol{\beta} - \boldsymbol{\varepsilon} \in (0,1]^c$ for $\boldsymbol{\varepsilon} = (\varepsilon, \dots, \varepsilon) \in (0,1]^c$. In addition, because $\alpha_{\mathbf{k}}(d, \boldsymbol{\beta})$ is continuous in $\boldsymbol{\beta}$ due to its definition (4.4), we may pick ε such that $\alpha' < \alpha_{\mathbf{k}}(d, \boldsymbol{\beta} - \boldsymbol{\varepsilon})$. For simplicity of notation, let $\boldsymbol{\beta}' = (\beta'_1, \dots, \beta'_c) := \boldsymbol{\beta} - \boldsymbol{\varepsilon}$.

Now we pick a positive integer m and set $\mathbf{n} = (m, \dots, m) \in \mathbb{N}^c$, that is, $n_1 = \cdots = n_c = m$ and $n = cm$ in our standard notation. We also set $\mathbf{r} = (r_1, \dots, r_c)$ so that $r_i := \lfloor \beta'_i m \rfloor$.[4] We assume that m is large enough so that $r_i \geq k_i$ for each $i \in [c]$. We define families N_i of convex sets in $\mathbb{R}^d$ so that each N_i contains r_i copies of $\mathbb{R}^d$ and $m - r_i$ hyperplanes in general position. We also assume that the collection of all hyperplanes in $N_1, \dots, N_c$ is in general position. We set K to be the nerve of the family $N = N_1 \sqcup \cdots \sqcup N_c$. In particular K is d-representable (therefore d-collapsible as well).

First, we check that $\dim K[N_i] < \beta_i m - 1$ provided that m is large enough. A subfamily of N_i with nonempty intersection contains at most d hyperplanes from N_i. Therefore $\dim K[N_i] < r_i + d = \lfloor \beta'_i m \rfloor + d < \beta_i m - 1$ for m large enough.

[4]This choice of $\mathbf{n}$ will yield a counterexample where each color class has equal size. It would be also possible to vary the sizes.

Next we check that K contains at least $\alpha'|\binom{N}{\mathbf{k}}|$ $\mathbf{k}$-colorful faces provided that m is large enough. Partitioning N_i so that R_i is the subfamily of the copies of $\mathbb{R}^d$ and $\bar{R}_i$ is the subfamily of hyperplanes, we get

$$f_{\mathbf{k}}(K) = p_{\mathbf{k}}(\mathbf{n}, d, \mathbf{r})$$

from the definition of $p_{\mathbf{k}}(\mathbf{n}, d, \mathbf{r})$. Therefore (4.5) gives

$$\delta_{\mathbf{k}}(K) = \frac{\sum\limits_{\boldsymbol{\ell}=(\ell_1,\dots,\ell_c)\in L_{\mathbf{k}}(d)} \prod\limits_{i=1}^{c} \binom{k_i}{\ell_i}(m - \lfloor \beta_i' m \rfloor)_{\ell_i}(\lfloor \beta_i' m \rfloor)_{k_i-\ell_i}}{\prod_{i=1}^{c}(m)_{k_i}}.$$

Passing to the limit (considering the dependency of K on m), we get

$$\lim_{m\to\infty} \delta_{\mathbf{k}}(K) = \sum_{\boldsymbol{\ell}=(\ell_1,\dots,\ell_c)\in L_{\mathbf{k}}(d)} \prod_{i=1}^{c} \binom{k_i}{\ell_i}(1-\beta_i')^{\ell_i}(\beta_i')^{k_i-\ell_i} = \alpha_{\mathbf{k}}(d, \boldsymbol{\beta}').$$

Therefore, for m large enough K contains at least $\alpha'|\binom{N}{\mathbf{k}}|$ $\mathbf{k}$-colorful as $\alpha' < \alpha_{\mathbf{k}}(d, \boldsymbol{\beta}')$.

4.4 A topological version?

A simplicial complex K is d-*Leray* if the ith reduced homology group $\tilde{H}_i(L)$ (over $\mathbb{Q}$) vanishes for every induced subcomplex $L \leq K$ and every $i \geq d$. As we already know, every d-representable complex is d-collapsible, and in addition every d-collapsible complex is d-Leray [87]. Helly-type theorems usually extend to d-Leray complexes and such extensions are interesting because they allow topological versions of Helly-type when collections of convex sets are replaced with good covers. We refer to several concrete examples [39, 51, 4] or to the survey [82].

We believe that it is possible to extend Theorem 23 to d-Leray complexes:

As we mentioned in the beginning of the chapter we conjecture that it should be possible to extend Theorem 23 to d-Leray complexes and probably Theorem 26 as well. Here we state the conjectured generalization of Theorem 23.

Conjecture 32 (The optimal colorful fractional Helly theorem for d-Leray complexes)**.** *Let K be a d-Leray simplicial complex with the set of vertices $N = N_1 \sqcup \cdots \sqcup N_{d+1}$ divided into $d+1$ disjoint subsets. Let $n_i := |N_i|$ for $i \in [d+1]$ and assume that K contains at least $\alpha n_1 \cdots n_{d+1}$ colorful d-faces for some $\alpha \in (0, 1]$. Then there is $i \in [d+1]$ such that* $\dim K[N_i] \geq (1-(1-\alpha)^{1/(d+1)})n_i - 1$.

In fact, our original approach how to prove Theorem 23 was to prove directly Conjecture 32. Indications that this could be possible are that both the optimal fractional Helly theorem [4, 50] and the colorful Helly theorem [51] hold for d-Leray complexes. In addition, there is a powerful tool, algebraic shifting, developed by Kalai [50], which turned out to be very useful in attacking similar problems.

In the remainder of this section we briefly survey a possible approach towards Conjecture 32 but also the difficulty that we encountered. Because we do not really prove any new result in this section, our description is only sketchy.

Our starting point is the proof of the optimal fractional Helly theorem for d-Leray complexes. The key ingredient is the following theorem of Kalai [4, Theorem 13].

Theorem 33. *Let K be a d-Leray complex and $f_0(K) = n$. Then $f_d(K) > \binom{n}{d+1} - \binom{n-r}{d+1}$ implies $f_{d+r}(K) > 0$ (where $f(K)$ denotes the f-vector of K).*

As far as we can judge, the only proof of Theorem 33 in the literature follows from the first and the third sentence in the following remark in [50]:

"It is not hard to see (although it has been overlooked for a long time) that the class of d-Leray complexes (for some d) with complete $(d-1)$-dimensional skeletons is precisely the Alexander dual of the class of Cohen-Macaulay complexes. This observation implies that the fact that shifting preserves the Leray property easily follows from the fact that shifting preserves the Cohen-Macaulay property. Moreover, it shows that the characterization of face numbers of d-Leray complexes follows from the corresponding characterization for Cohen-Macaulay complexes."

For completeness we add that the characterization of face numbers of Cohen-Macaulay complexes has been done by Stanley [76]. Given a simplicial complex K on vertex set N its *Alexander dual* is a simplicial complex defined as $K^* := \{T \subseteq N : N \setminus T \notin K\}$. We skip the definition of Cohen-Macaulay complex because we will only use it implicitly but we refer, for example, to [50, §4] for more details.

A simplicial complex K on vertex set $[n]$ is called *shifted* if for all integers i and j with $1 \leq i < j \leq n$ and all faces A of K such that $j \in A$ and $i \notin A$, the set $(A \setminus \{j\}) \cup \{i\}$ is a face of K. *Exterior algebraic shifting* is a function that associates to a simplicial complex K a shifted complex $\Delta(K)$, while preserving many interesting invariants of K. Below we list some properties of exterior algebraic shifting that we will use. A simplicial complex is *pure* if all its inclusion-maximal faces have the same dimension.

Theorem 34. *(i) [50, Theorem 2.1] Exterior algebraic shifting preserves the f-vector.*

(ii) [50, Theorem 4.1] If K is Cohen-Macaulay, then $\Delta(K)$ is Cohen-Macaulay, in particular, pure.

(iii) [50, 3.5.6] The following equality holds $\Delta(K^*) = \Delta(K)^*$.

The next lemma is a possible replacement of the third sentence in Kalai's remark how to prove Theorem 33. We prove it as motivation for the tools we would need in the colorful scenario.

Lemma 35. *Let K be a d-Leray complex on $[n]$ with complete $(d-1)$-skeleton and let $D = \dim(K) + 1$. Then $K^e \subseteq \Delta_{D-d-1} * \Delta^{(d-1)}_{n-D+d-1}$.*

Proof. By the first sentence of Kalai's remark, the Alexander dual K^* of K is a Cohen-Macaulay complex. By the definition of Alexander dual, it has dimension $n-d-2$ and contains complete $(n-D-2)$-skeleton. Hence, properties (i) and (ii) of Theorem 34 imply that the exterior algebraic shifting $(K^*)^e$ of K^* is a pure shifted complex of dimension $n-d-2$ with complete $(n-D-2)$-skeleton. If we take any subset A of size $n-D-1$ in $(K^*)^e$, then A is a face and by purity there must be a face of size $n-d-1$ that contains A. Now, since $(K^*)^e$ is shifted we have that $\{1, 2, \ldots, D-d\} \cup A \in (K^*)^e$. This implies that $\Delta_{D-d-1} * \Delta^{(n-D-2)}_{n-D+d-1} \subseteq (K^*)^e = (K^e)^*$, by Theorem 34(iii). Taking Alexander dual from both sides proves the first part of the statement. (Using that $(L^*)^* = L$; $L_1 \subseteq L_2 \Rightarrow L_2^* \subseteq L_1^*$; and $(\Delta_{D-d-1} * \Delta^{(n-D-2)}_{n-D+d-1})^* = \Delta_{D-d-1} * \Delta^{(d-1)}_{n-D+d-1}$.) $\square$

For completeness, Theorem 33 quickly follows from Lemma 35. Indeed, if K is d-Leray such that $f_{d+r}(K) = 0$, then $D := \dim K + 1 \leq d + r$. In addition, we can assume without loss of generality that K contains complete $(d-1)$-skeleton. Consequently, Lemma 35 gives $f_d(K) = f_d(K^e) \leq f_d(\Delta_{D-d-1} * \Delta^{(d-1)}_{n-D+d-1}) = \binom{n}{d+1} - \binom{n-D+d}{d+1} \leq \binom{n}{d+1} - \binom{n-r}{d+1}$.

Now, in order to attack Conjecture 32, we would like to do something similar in colorful setting. In particular, we need to preserve the colorful f-vector. Babson and Novik [7] give a definition of colorful algebraic shifting which preserves the colorful f-vector. Nevertheless, the conjecture does not follow immediately from their result as the Alexander dual of a d-Leray complex is not in general balanced.

We show next why a shifting operator preserving d-Leray and colorful f-vector is enough. For it let us introduce the following definition. Let $N = N_1 \sqcup \cdots \sqcup N_{d+1}$ with each N_i having a total order $<_i$. A simplicial complex K with vertex set N is *color-shifted* if for every $F \in K$, $i \in [d+1]$ and $v \in F \cap N_i$, if $w <_i v$ then $(F \setminus \{v\}) \cup \{w\} \in K$.

Proposition 36. *Let K be a color-shifted d-Leray simplicial complex with vertex partition $N = N_1 \sqcup \cdots \sqcup N_{d+1}$ divided into $d+1$ disjoint sets. Let $n_i = |N_i|$ for $i \in [d+1]$ and assume that K contains at least $\alpha n_1 \cdots n_{d+1}$ for some $\alpha \in (0,1]$. Then there is $i \in [d+1]$ such that* $\dim K[N_i] \geq (1-(1-\alpha)^{1/d+1})n_i - 1$.

Proof. As observed previously it is enough to show if $r_1, \ldots, r_{d+1}$ are positive integers such that $\dim K[N_i] \leq r_i - 1$ then $f_{\mathbf{1}}(K) \leq n_1 \cdots n_{d+1} - (n_1 - r_1) \cdots (n_{d+1} - r_{d+1})$. Let R_i denote the first r_i vertices in N_i for each $i \in [d+1]$. Then it is enough to verify that every colorful d-face intersects at least one of the R_i's. Let us assume that this is not the case and let $T = (t_1, \ldots, t_{d+1})$ be such a face, with $t_i \in N_i \setminus R_i$ for every $i \in [d+1]$. Let T_i denote the first t_i element in N_i, in particular $t_i = |T_i| > |R_i| = r_i$. Since K is color-shifted and $T \in K$ we have that $T_1 \times \cdots \times T_{d+1} \subseteq K$. Then $K[\bigcup_{i \in [d+1]} T_i]$ is d-Leray and contains every colorful d-face. By the colorful Helly theorem of Kalai and Meshulam [51] we can conclude that there exists $i \in [d+1]$ such that $T_i \in K$. This is a contradiction since $r_i - 1 \geq \dim K[N_i] \geq \dim K[T_i] = t_i - 1 > r_i - 1$. $\square$

We close this chapter with the following question.

Problem 37. *Is there a shifting operation preserving the d-Leray property as well as the colorful f-vector such that the resulting complex is color-shifted?*

5. Volume rigidity

Let K be an n-vertex $(d-1)$-dimensional simplicial complex and $\mathbf{p} : N(K) \to \mathbb{R}^{d-1}$ be a generic mapping of its vertices, in the sense that its $(d-1)n$ coordinates are algebraically independent over $\mathbb{Q}$. This chapter deals with the infinitesimal version of the following problem: is there a *non-trivial* continuous motion of the vertices starting at $\mathbf{p}$ that preserves the volumes of all the $(d-1)$-simplices in K? By "non-trivial" we mean that, for some $(d-1)$-simplex on $N(K)$ that is not in K, its volume would change along the motion. It is easy to show that the continuous and infinitesimal versions coincide for generic embeddings, as is the case for graph rigidity [5].

Volume Rigidity. The signed volume of a $(d-1)$-face $S = \{v_1, \ldots, v_d\} \in K$ with respect to $\mathbf{p}$ is given by the determinant of the $d \times d$ matrix

$$M_{\mathbf{p},S} = \begin{pmatrix} \mathbf{p}(v_1) & \ldots & \mathbf{p}(v_d) \\ 1 & \ldots & 1 \end{pmatrix}.$$

Observe that, for every $1 \le i \le d-1,\ 1 \le j \le d$, the derivative of the signed volume $\det M_{\mathbf{p},S}$ with respect to the i-th coordinate of $\mathbf{p}(v_j)$ is given by the cofactor $C_{i,j}(M_{\mathbf{p},S})$ — that is, the determinant of the submatrix obtained by removing the i-th row and j-th column multiplied by $(-1)^{i+j}$.

The *volume-rigidity* matrix $\mathfrak{V}(K, \mathbf{p})$ of the pair $(K, \mathbf{p})$ is a $(d-1)n \times f_{d-1}(K)$ matrix, where the columns are indexed by the $(d-1)$-faces of K, and every vertex is associated with a block of $(d-1)$ rows. The column vector $\mathbf{v}_S$ corresponding to a $(d-1)$-face $S = \{v_1, \ldots, v_d\} \in K$ is defined by

$$(\mathbf{v}_S)_{v_i,j} = C_{i,j}(M_{\mathbf{p},S}) \ ,\ i \in [d],\ j \in [d-1],$$

and 0 elsewhere. Here $(\mathbf{v}_S)_{v_i,j}$ denotes the j-th coordinate of $\mathbf{v}_S$ in the block of v_i. In words, $\mathfrak{V}$ is the Jacobian of the function $\mathbf{p} \mapsto (\det M_{\mathbf{p},S})_{S \in K}$, viewing $\mathbf{p}$ as a $(d-1)n$-dimensional vector.

This matrix was introduced in [61, Appendix A] along with the description of a *trivial* (d^2-d-1)-subspace of the left kernel of $\mathfrak{V}(K, \mathbf{p})$, arising from the volume-preserving transformations of $\mathbb{R}^{d-1}$. Concretely, the trivial subspace consists of all $(d-1)n$-dimensional vectors z obtained by choosing a $(d-1) \times (d-1)$ matrix A whose trace is zero and a vector $u \in \mathbb{R}^{d-1}$, and letting $z_v = A \cdot \mathbf{p}(v) + u$ for every vertex v. The following definition suggests itself.

Definition 38. *An n-vertex $(d-1)$-dimensional simplicial complex K is called* volume-rigid *if*

$$\operatorname{rank}(\mathfrak{V}(K, \mathbf{p})) = (d-1)n - (d^2 - d - 1),$$

for a generic $\mathbf{p} : N(K) \to \mathbb{R}^{d-1}$.

Exterior shifting. Algebraic shifting was introduced by Kalai (see e.g. [49] and the survey [50]) and has been studied extensively in algebraic combinatorics. Here we present a variant of exterior shifting. The standard basis $(e_i)_{i \in [n]}$ of $\mathbb{R}^n$ induces the basis $(e_S)_{S \subseteq [n]}$ of its exterior algebra $\bigwedge \mathbb{R}^n$. Consider a generic basis $(f_1, \ldots, f_n)$ of $\mathbb{R}^n$, where without loss of generality we assume that $f_1 = \mathbf{1} \in \mathbb{R}^n$, namely, the other $n^2 - n$ coordinates in this basis are algebraically independent over $\mathbb{Q}$. Consider the *exterior face ring* $\bigwedge K = \bigwedge \mathbb{R}^n \ / \ (e_S \colon S \notin K)$, and let q denote the natural quotient map. Given a partial order $<$ on the power set of $[n]$, define

$$\Delta^{<}(K) = \{S \subseteq [n] \colon q(f_S) \notin \operatorname{span}_{\mathbb{R}}\{q(f_T) \colon T < S,\ |T| = |S|\}\}. \qquad (5.1)$$

Of special importance in our case is the partial order $<_p$ defined by

$$S = \{s_1 < ... < s_m\} \leq_p T = \{t_1 < ... < t_{m'}\}$$

if $m = m'$ and $s_i \leq t_i$, $\forall i \in [m]$. Corollary 44 asserts that $\Delta^p(K) := \Delta^{<_p}(K)$ is a shifted simplicial complex independent of the generic choice of f. (Note that $\Delta^p(K)$ may have more faces than K.)

5.0.1 Main results.

Our main result is a characterization of volume rigidity in the setting of Kalai's exterior shifting.

Theorem 39. *Fix $d \geq 3$. An n-vertex $(d-1)$-dimensional simplicial complex K is volume-rigid if and only if $\{1, 3, 4, ..., d, n\} \in \Delta^p(K)$.*

In the 2-dimensional case we are able to derive the volume rigidity of triangulations of the following surfaces.

Corollary 40. *Every triangulation of the 2-sphere, the torus, the projective plane or the Klein bottle is volume rigid. In addition, every triangulation of the 2-sphere and the torus minus a single triangle is also volume-rigid. In particular, every simplicial disc with a 3-vertex boundary is* minimally *volume-rigid.*

In the case of the 2-sphere we give a complete mathematical proof. For the other surfaces , we reduce — via edge contractions á la Whiteley [88] — to irreducible triangulations, whose volume-rigidity we verify numerically.

Hypergraph sparsity was introduced by Streinu and Theran [78], generalizing results on graph sparsity, prominently by White and Whiteley [89] who studied it from a matroid perspective. We say that a $(d-1)$-complex is $(d-1, d^2-d-1)$-sparse (resp. tight) if every subset A of its vertices of cardinality at least d spans at most $(d-1)|A| - (d^2-d-1)$ simplices of dimensions $d-1$ (resp. and equality holds when A equals the entire vertex set).

Clearly, a vertex subset A spanning more $(d-1)$-simplices induces a non trivial linear dependence between the columns of $\mathfrak{V}(K, \mathbf{p})$, and it is natural to ask whether this characterizes all the linear dependencies in the volume rigidity matrix. Using Theorem 39, we show that the answer is negative, hence a Laman-type condition for volume-rigidity does not hold true [1].

Corollary 41. *For every $d \geq 3$, there exists a $(d-1, d^2-d-1)$-tight $(d-1)$-complex that is not volume-rigid.*

[1]Corollary 41 shows that Prop.1 in the preprint [77] from 2007 is a misstatement.

5.0.2 Relation to previous works.

The maximal independent sets of columns of $\mathfrak{V}(\binom{[n]}{d}, \mathbf{p})$, for all generic embeddings $\mathbf{p}$, form the bases of the same matroid. For $d = 2$ they correspond to spanning trees, namely the bases in the graphic matroid on $\binom{[n]}{2}$. Kalai [48] introduced for every integer $k \geq 1$ the k-hyperconnectivity matroid on $\binom{[n]}{2}$ [2], where $k = 1$ corresponds to the graphic matroid, and identified its bases in terms of exterior shifting (w.r.t. the lexicographic order): G is a basis if and only if the edges of $\Delta^{\text{lex}}(G)$ form the initial segment that ends with $\{k, n\}$, w.r.t. the lex-order.

Here, in Theorem 39, rather then increasing the dimension of the embedding space and staying with graphs, we increase also the dimension of the pure complex, by the same number, and characterize the bases of the resulted d-volume-rigidity matroid in terms of exterior shifting w.r.t. the partial order $<_p$.

The fact that $(d-1, d^2-d-1)$-sparse complexes form the independent sets of a matroid on $\binom{[n]}{d}$ was asserted in [58, 89]. Additional matroidal and algorithmic properties of sparsity matorids were studied by Streinu and Theran in [78, 79]. By Corollary 41, the $(d-1, d^2-d-1)$-sparsity matroid strictly contains the $(d-1)$-volume-rigidity matroid for all $d \geq 3$. It would be interesting to find further combinatorial conditions that once imposed on the bases of the sparsity-matroid would give the bases of the volume-rigidity matroid.

The remainder of the chapter is organized as follows. In Section 5.1 we establish the connection between volume rigidity and exterior shifting, and prove Theorem 39. Afterwards, in Section 5.2 we investigate the effect of local moves on volume rigidity and prove Corollary 40. In the following Section 5.3 we prove Corollary 41, and we conclude in Section 5.4 with some related open problems.

5.1 Volume rigidity and $\Delta^p(\cdot)$

This section is devoted to studying the basic properties of the shifted complex $\Delta^p(K)$, and to establishing the connection between $\Delta^p(K)$ and K's volume rigidity.

5.1.1 Basic properties of $\Delta^p(\cdot)$

We start by briefly exploring some useful properties of the complex $\Delta^p(K)$ that appears in Theorem 39. Given a partially ordered set (poset) $(P, <)$ and an element $x \in P$ we denote by $P_{<,x}$ the prefix $\{y \in P \ : \ y \leq x\}$.

Claim 42. *Let $(P, <)$ be a poset and $x \in P$, then there exists a linear extension $<_l$ of $<$ such that $P_{<_l,x} = P_{<,x}$.*

Proof. View the sets $A = P_{<,x}$ and $B = P \setminus P_{<,x}$ as posets with the partial order induced by $<$. Extend each of these posets linearly, and concatenate the extensions such that the elements in A are smaller than those in B. □

[2]The k-hyperconnectivity matroid is derived from an embedding of the vertex set into $\mathbb{R}^k$. Studying higher hyperconnectivity translates to increasing the dimension of the embedding space.

We will mainly work with the partial order $<_p$ on the power set of $[n]$ and denote the set of its linear extensions by $\mathcal{L}$. We usually denote an element in $\mathcal{L}$ by $<_l$ and the corresponding shifted complex by $\Delta^l(K) := \Delta^{<_l}(K)$.

Claim 43. $\Delta^p(K) = \bigcup_{<_l \in \mathcal{L}} \Delta^l(K)$.

Proof. On the one hand, if $S \in \Delta^l(K)$ for some $<_l \in \mathcal{L}$ then $q(f_S)$ is not spanned by $B_{l,S} := \{q(f_T) \ : \ T <_l S\}$, which contains the vector set $B_{p,S}$. Therefore, by the definition of $\Delta^<$ in (5.1), we find that $\Delta^p(K) \supseteq \Delta^l(K)$. On the other hand, for every $S \in \Delta^p(K)$, there exists by the previous claim a linear extension $<_l \in \mathcal{L}$ satisfying $B_{p,S} = B_{l,S}$ hence $S \in \Delta^l(K)$. □

Corollary 44. *For every simplicial complex K there holds that $\Delta^p(K)$ is a shifted simplicial complex independent of the choice of the generic basis f. In addition, $\Delta^p(K) = K$ if K is shifted.* □

That $\Delta^p(K)$ is downwards closed follows exactly as in the proof for $\Delta^{\text{lex}}(K)$. The rest of Corollary 44 follows immediately from the above decomposition of $\Delta^p(K)$ and the fact that the basic properties of algebraic shifting in [49] – being shifted, and independence the the generic f chosen– hold in every linear extensions of $<_p$, as remarked in [49, p.58].

5.1.2 Volume rigidity and $\Delta^p(\cdot)$

We are now ready to prove Theorem 39. We denote $S_0 = \{1, 3, ..., d, n\}$ and observe that the prefix $B := \{T \leq_p S_0 \ : \ |T| = d\}$ consists of the subsets $[d]$ and $[d] \setminus \{i\} \cup \{v\}$ for $2 \leq i \leq d$ and $d+1 \leq v \leq n$. We define a linear transformation $\psi : \bigoplus_{i=2}^{d} \bigwedge^1 \mathbb{R}^n \to \bigwedge^d \mathbb{R}^n$ given by

$$\psi(m_2, \ldots, m_d) = \sum_{i=2}^{d} f_{[d]\setminus\{i\}} \wedge m_i.$$

Lemma 45. *The image of ψ is spanned by $\{f_T \ : \ T \in B\}$, and its kernel is $(d^2 - d - 1)$-dimensional.*

Proof. The fact that $f_T \in \text{im}(\psi)$ for every $T \in B$ can be shown directly. Indeed, $\psi(0, ..., 0, f_d) = f_{[d]}$ and by taking $m_i = f_v, \ m_{i'} = 0 \ \forall i' \neq i$ we have that $\psi(0, ..., f_v, ..., 0) = f_{[d]\setminus\{i\}\cup\{v\}}$ for $2 \leq i \leq d$ and $d+1 \leq v \leq n$. To show that these $1+(n-d)(d-1)$ linearly independent vectors span the image of ψ, we will construct d^2-d-1 linearly independent vectors in $\ker \psi$ which actually completes the proof by the rank-nullity theorem since $(1+(n-d)(d-1)) + (d^2-d-1) = n(d-1)$.

First, for every $2 \leq i \leq d$ and $j \in [d] \setminus \{i\}$ consider the vector defined by setting $m_i = f_j$ and $m_{i'} = 0$ for every $i' \neq i$. Then,

$$\psi(m_2, ..., m_d) = f_{[d]\setminus\{i\}} \wedge f_j = 0$$

since $j \in [d] \setminus \{i\}$. This amounts to $(d-1)^2$ vectors in $\ker \psi$, and the remaining $d-2$ are given by vectors of the form $m_i = a_i f_i, \ 2 \leq i \leq d$, where the scalars $a_2, ..., a_d$ satisfy $\sum_{i=2}^{d} (-1)^i a_i = 0$. Indeed,

$$\psi(m_2, ..., m_d) = \sum_{i=2}^{d} a_i f_{[d]\setminus\{i\}} \wedge f_i = \sum_{i=2}^{d} a_i (-1)^{d-i} f_{[d]} = 0\,.$$

The linear independence of these d^2-d-1 vectors follows directly from the linear independence of $f_1, ..., f_d$. $\square$

Proof of Theorem 39. Identify the vertices of K with the set $[n]$. W.l.o.g. assume that $f_{d-1}(K) \geq (d-1)n - (d^2-d-1)$, as otherwise K is not volume-rigid and $\{1,3,4,...,d,n\} \notin \Delta^p(K)$. In addition, suppose that the generic embedding $\mathbf{p} : N(K) \to \mathbb{R}^{d-1}$ is obtained from the vectors $f_2, ..., f_d$ in the generic basis of $\mathbb{R}^n$ by taking $(f_i)_v = \mathbf{p}(v)_{i-1}$ for every $2 \leq i \leq d$ and $v \in [n]$.

Consider the (i,v)-unit vector $e_{i,v} = (m_2, ..., m_d)$ in the domain of ψ, for $2 \leq i \leq d$ and $v \in [n]$, defined by $m_i = e_v$ and $m_{i'} = 0$, $\forall i' \neq i$. Then,

$$\begin{aligned}\psi(e_{i,v}) &= f_{[d]\setminus\{i\}} \wedge e_v \\ &= (-1)^{d-i} f_1 \wedge \cdots \wedge f_{i-1} \wedge e_v \wedge f_{i+1} \wedge \cdots \wedge f_d \\ &= (-1)^{d-i+d-1} f_2 \wedge \cdots \wedge f_{i-1} \wedge e_v \wedge f_{i+1} \wedge \cdots \wedge f_d \wedge f_1.\end{aligned}$$

Let $S = \{v_1, ..., v_d\} \subset [n]$. Clearly, for the inner product on $\bigwedge \mathbb{R}^n$ with orthonormal basis $\{e_S : S \subset [n]\}$, we have $\langle e_S, \psi(e_{i,v})\rangle = 0$ if $v \notin S$. Otherwise, by the identification of $\mathbf{p}$ with $f_2, ..., f_d$ above and $f_1 = \mathbf{1}$, if $v = v_j$ then $\langle e_S, \psi(e_{i,v})\rangle$ is equal to $(-1)^{i-1}$ times the determinant of the matrix that is obtained from $M_{\mathbf{p},S}$ by replacing its $(i-1)$-th row with the j-th d-dimensional all-ones row vector. Consequently,

$$\langle e_S, \psi(e_{i,v})\rangle = (-1)^{i-1} C_{i-1,j}(M_{\mathbf{p},S}).$$

Thus, by letting $q : \bigwedge \mathbb{R}^n \longrightarrow \bigwedge K$ be the natural quotient map, and by choosing the basis $\{e_S : S \in K\}$ for $\bigwedge K$, we find that the $f_{d-1}(K) \times (d-1)n$ matrix representation of $q \circ \psi$ is equal — up to multiplying some of its columns by -1 and reordering them — to the transpose of the volume-rigidity matrix $\mathfrak{V}(K, \mathbf{p})$. Therefore, K is volume-rigid if and only if $\dim \ker(q \circ \psi) = d^2 - d - 1$. In other words, K is *not* volume-rigid if and only if there exists a non-zero $f \in \operatorname{im}(\psi)$ such that $q(f) = 0$. By the characterization of ψ's image in Lemma 45, f can be written as a non-trivial linear combination $f = \sum_{T \in B} \lambda_T f_T$.

To conclude the proof we claim that $q(f) = 0$ for some $f \in \operatorname{span}\{f_T : T \in B\}$ if and only if $S_0 \notin \Delta^p(K)$. Indeed, on one direction, $q(\sum_{T\in B} \lambda_T f_T) = 0$ implies that for some $T \in B$, $q(f_T)$ is a linear combination of its predecessors in $<_p$. By (5.1), $T \notin \Delta^p(K)$ and since $\Delta^p(K)$ is shifted then $S_0 \notin \Delta^p(K)$. On the other hand, by manipulating the linear combination which asserts that $S_0 \notin \Delta^p(K)$, we obtain a non-zero vector $f = \sum_{T \in B} \lambda_T f_T$ satisfying $q(f) = 0$. $\square$

5.2 Volume rigidity, local moves and homology

We turn to study the effect of local combinatorial moves on volume rigidity. We start by proving a volume-rigidity analog of Whiteley's vertex splitting [88], by which he showed that every triangulation of the 2-sphere has a 3-rigid 1-skeleton.

Lemma 46 (Edge contraction)**.** *Let K be a pure $(d-1)$-dimensional simplicial complex, $e = \{u, w\} \in K$ such that at least $(d-1)$ facets in K contain e. Let K' to be the simplicial complex obtained from K by contracting the edge e, i.e. by identifying the vertex u with w, and removing duplicates. If K' is volume rigid then so is K.*

Proof. Without loss of generality assume that $u < w$ are the first among the n vertices of K, as the vertex labels do not effect volume-rigidity. We will construct an auxiliary $(d-1)n \times f_{d-1}(K)$ matrix A such that

$$\operatorname{rank} \mathfrak{V}(K, \mathbf{p}) \geq \operatorname{rank}(A) = (d-1)n - (d^2 - d - 1).$$

First, we replace the position of the vertex w, i.e. $\mathbf{p}(w)$, by the position of the vertex u, i.e. $\mathbf{p}(u)$. Formally we define a new (non-generic) placement of vertices $\mathbf{p}'$ that coincides with $\mathbf{p}$ on all vertices except w on which we set it to equal to $\mathbf{p}(u)$. Clearly, since $\mathbf{p}$ is generic, there holds $\operatorname{rank} \mathfrak{V}(K, \mathbf{p}) \geq \operatorname{rank} \mathfrak{V}(K, \mathbf{p}')$. To obtain A, we add the rows in $\mathfrak{V}(K, \mathbf{p}')$ corresponding to the vertex u to the rows corresponding to the vertex w, an operation that does not change the rank.

We first claim that the submatrix of A which corresponds to the columns of the facets L containing $e = \{u, w\}$ is supported on the rows corresponding to u. Indeed, if $e \subseteq S = \{v_1, \ldots, v_d\} \in L$ then for $v_j \in S$ such that $v_j \neq \{u, w\}$ we have that each entry $A_{v_j,i;S} = C_{i,j}(M_{\mathbf{p}',S}) = 0$ because $M_{\mathbf{p}',S}$ has two identical columns as $\mathbf{p}'(u) = \mathbf{p}'(w)$. On the other hand, because we added the rows corresponding to vertex u to the rows corresponding to vertex w, we have that

$$A_{w,i;S} = C_{i,1}(M_{\mathbf{p}',S}) + C_{i,2}(M_{\mathbf{p}',S}) = 0.$$

This follows from our assumption that u and w are the first two vertices hence their cofactors in $M_{\mathbf{p}',S}$ have opposite signs, and they in fact cancel-out since and $\mathbf{p}'(u) = \mathbf{p}'(w)$.

Second, we claim that the submatrix $A_{u,L}$ of A corresponding to the $d-1$ rows of u and the columns of L has a full rank of $d-1$. We derive this claim by the assumption that $|L| \geq d-1$ and the fact that $(\mathbf{p}'(v) \ : v \neq u)$ is generic. Indeed, consider a vector $x \in \mathbb{R}^{d-1}$ in the left kernel of $A_{u,L}$. A brief calculation yields that the orthogonality of x and the column in $A_{u,L}$ corresponding to the facet $S = \{u, w, v_2, ..., v_d\}$ is equivalent to x being in the span of $\mathbf{p}'(v_i) - \mathbf{p}'(w),\ i = 2, ..., d$. By the assumption that $\mathbf{p}$ is generic, such $|L| \geq d-1$ constraints are only satisfied by $x = 0$ hence

$$\operatorname{rank}(A_{u,L}) = d - 1. \tag{5.2}$$

Third, consider the complement submatrix $A' := A_{\{u\}^c, L^c}$ whose rows correspond to all the vertices except u, and columns to all the facets that are not in L. We observe that A' contains as a submatrix the generic volume rigidity matrix $\mathfrak{V}(K', \mathbf{p})$ — where $\mathbf{p}$ is viewed here as a generic embedding of $N(K') = N(K) \backslash \{u\}$ into $\mathbb{R}^{d-1}$. Indeed, every facet S of K' arises from a facet $\hat{S}$ of K.

- If $u \notin \hat{S}$ then $S = \hat{S}$ and the columns in $\mathfrak{V}(K', \mathbf{p})$ and A' corresponding to S are clearly equal.
- Otherwise, if $u \in \hat{S}$ then $S = \hat{S} \cup \{w\} \setminus \{u\}$, and by the construction of A — in which $\mathbf{p}'(u) = \mathbf{p}'(w)$ and the rows of u are added to the rows of w — we have that the column in A' created from $\hat{S}$ is equal to the column of S in $\mathfrak{V}(K', \mathbf{p})$.

Note that A' may contain some duplicate columns — in case there are two facets that differ only in the vertices of e — but, regardless, our observation that A' contains $\mathfrak{V}(K', \mathbf{p})$ as a submatrix implies that

$$\operatorname{rank}(A') = (n-1)(d-1) - (d^2 - d - 1). \tag{5.3}$$

In conclusion, A takes the form

$$A = \begin{array}{c} \\ u \\ \{u\}^c \end{array} \begin{array}{c} \begin{array}{cc} L & L^c \end{array} \\ \begin{pmatrix} A_{u,L} & * \\ 0 & A' \end{pmatrix} \end{array},$$

and by combining (5.2) and (5.3) we find that

$$\operatorname{rank}(\mathfrak{V}(K,\mathbf{p}) \geq \operatorname{rank}(A) = \operatorname{rank}(A_{u,L}) + \operatorname{rank}(A') = n(d-1) - (d^2 - d - 1)\,,$$

as claimed.

□

The next two lemmas are direct analogs of basic results in graph rigidity [6] asserting that gluing preserves volume-rigidity. We include their proofs for completeness.

Lemma 47. *Let K be $(d-1)$-volume-rigid, $v \notin N(K)$ and $S \subseteq N(K)$ such that $|S| \geq d$, then $K \cup (v * \binom{S}{d-1})$ is $(d-1)$-volume-rigid.*

Proof. The vertex v is in at least $d-1$ facets of $L = K \cup (v * \binom{S}{d-1})$. The volume rigidity matrix of L is of the form

$$\begin{array}{c} \\ V \\ v \end{array} \begin{array}{c} \begin{array}{cc} K & S * v \end{array} \\ \begin{pmatrix} \mathfrak{V}(K,\mathbf{p}) & * \\ 0 & N \end{pmatrix} \end{array},$$

where the matrix N has $d-1$ rows and at least $d-1$ columns. Because of general position N has full rank, i.e. $rank(N) = d-1$. Because K is $(d-1)$-volume-rigid we have that $rank(\mathfrak{V}(K,\mathbf{p})) = n(d-1) - d(d-1) + 1$. Then, $rank(\mathfrak{V}(L,\mathbf{p})) = rank(\mathfrak{V}(K,\mathbf{p})) + rank(N) = (n+1)(d-1) - d(d-1) + 1$ and consequently L is $(d-1)$-volume-rigid. □

Lemma 48 (Union of volume-rigid complexes)**.** *Let K and L be $(d-1)$-volume-rigid complexes such that $|N(K) \cap V(L)| \geq d$. Then $K \cup L$ is $(d-1)$-volume-rigid.*

Proof. Because K and L are $(d-1)$-volume-rigid we can assume that each of them has a complete $(d-1)$-skeleton on its respective vertex set. Then $K \cup L$ contains the vertex spanning subcomplex Q obtained from K by adding one vertex v at a time, adding the facets $\binom{S}{d-1} * v$ at step v, where $S = N(L) \cap N(K)$. This subcomplex is $(d-1)$-volume-rigid at each step by application of the previous lemma. In particular, Q is $(d-1)$-volume-rigid, hence so is $K \cup L$. □

5.2.1 Volume rigidity of surfaces

Barnette and Edelson [13, 14] proved that every compact surface without boundary admits only finitely many *irreducible* triangulations, namely, triangulations where every edge contraction would result in a simplicial complex not homeomorphic to the given surface. Thus, by Lemma 46, in order to conclude that for a given surface S every simplicial complex that triangulates it is volume-rigid, it is enough to verify if for the irreducible triangulations of S. Those are known for the surfaces indicated in Corollary 40: one for the 2-sphere (namely the boundary of a tetrahedron), two for the projective plane [12], 21 for the torus [55] and 29 for the Klein bottle [56, 80]. Clearly the boundary of the tetrahedron is volume-rigid, and we verified by computer that the irreducible triangulations K of the other surfaces mentioned above are volume-rigid – for this it was enough to find some embedding $\mathbf{p}_K : N(K) \longrightarrow \mathbb{R}^2$ such that $\text{rank}(\mathfrak{V}(K, \mathbf{p}_K) = 2|N(K)| - 5$.

Remark 49. *The fact that every triangulation K of the 2-sphere is volume-rigid follows also from combining the 3-hyperconnectivity of its graph with the Cohen-Macaulay property. Indeed, the first property says that $\{3, |N(K)|\} \in \Delta^{\text{lex}}(K)$, and as K is Cohen-Macaluay then $\Delta^{\text{lex}}(K)$ is pure and hence $\{1, 3, |N(K)|\} \in \Delta^{\text{lex}}(K)$ as $\Delta^{\text{lex}}(K)$ is shifted, which implies, by Claim 43, that $\{1, 3, |N(K)|\} \in \Delta^{p}(K)$, and we are done by Theorem 39.*

To prove the second part of Corollary 40 we are left to show that removing one triangle from a triangulated 2-sphere or torus preserves volume-rigidity, done next. A pure simplicial complex is a *minimal cycle* (over some coefficients commutative ring $\mathbb{F}$) if there exists an $\mathbb{F}$-linear combination of its facets whose boundary vanishes, and no proper nonempty subset of its facets has this property. For example, every triangulation of a compact connected surface (resp. and orientable) is a minimal cycle over $\mathbb{Z}_2$ (resp. $\mathbb{Z}$).

Lemma 50. *If K is a $(d-1)$-dimensional volume rigid minimal cycle over $\mathbb{Z}$, then $K \setminus S$ is volume rigid for every $S \in K$.*

We first give short proof for the special case $d = 3$ and conclude the proof of Corollary 40. Afterwards, we give a more technical proof for the general case.

Proof of Lemma 50 ($d = 3$). As K is a minimal cycle over $\mathbb{Z}$, its 2-dimensional homology with $\mathbb{R}$-coefficients is one dimensional, and for each facet S of K, for $K \setminus S$ this homology vanishes. By the translation of homology in terms of algebraic shifting, $\Delta^{\text{lex}}(K) \ni \{2, 3, 4\} \notin \Delta^{\text{lex}}(K \setminus S)$, and as shifting preserves containment we conclude

$$\Delta^{\text{lex}}(K \setminus S) = \Delta^{\text{lex}}(K) \setminus \{\{2, 3, 4\}\}.$$

Note that in this dimension[3] $T <_p \{1, 3, n\}$ iff $T <_{\text{lex}} \{1, 3, n\}$, and thus: $\{1, 3, n\} \in \Delta^p(K)$ (by Theorem 39), hence $\{1, 3, n\} \in \Delta^{\text{lex}}(K)$, and by the displayed equality above also $\{1, 3, n\} \in \Delta^{\text{lex}}(K \setminus S)$, so finally $\{1, 3, n\} \in \Delta^p(K \setminus S)$, equivalently, $K \setminus S$ is volume-rigid. □

Proof of Corollary 40. This is immediate from Lemma 46, Lemma 50 for the case $d = 3$, and the discussion in the beginning of §5.2.1. □

We conclude this section with a more direct proof of Lemma 50 for all $d \geq 3$.

Proof of Lemma 50. For two subsets S and $T = S \setminus \{v\}$ of $[n]$ that differ by one element, denote $\text{sign}(T, S) := (-1)^j$ if v is the j-th element in S. We prove the following stronger statement. Let $z \in \mathbb{Z}^{f_{d-1}(K)}$ be a $(d-1)$-dimensional chain in

[3]For $d > 3$, $\{T : T <_p \{1, 3, 4, \ldots, d, n\}\}$ is smaller than $\{T : T <_{\text{lex}} \{1, 3, 4, \ldots, d, n\}\}$.

K, and $\partial z \in \mathbb{Z}^{f_{d-2}(K)}$ be its boundary, i.e. $(\partial z)_T = \sum_{T\subset S} \operatorname{sign}(T,S)\cdot z_S$. Then, for every vertex v and every $i \in [d-1]$

$$(\mathfrak{V}(K,\mathbf{p})\cdot z)_{v,i} = (-1)^{d+i}\sum_F \operatorname{sign}(F, F\cup\{v\})\cdot \det N_{\mathbf{p},F,i}\cdot(\partial z)_{F\cup\{v\}}\,, \tag{5.4}$$

where (i) the summation is over the $(d-3)$-faces F of K that belong to the link of v and, (ii) the $(d-2)\times(d-2)$ matrix $N_{\mathbf{p},F,i}$ is obtained from the $(d-1)\times(d-2)$ matrix $(\mathbf{p}(v)\ :\ v\in F)$ by removing its i-th row. In particular, if z is a generator of the $(d-1)$-homology of a minimal cycle K, then z is also a non-trivial linear dependence between the columns of $\mathfrak{V}(K,\mathbf{p})$. Therefore, removing a $(d-1)$-face from K does not change the rank of the volume rigidity matrix, hence if K is volume rigid then so is $K\setminus\{S\}$. To derive (5.4), note that

$$\begin{aligned}(\mathfrak{V}(K,\mathbf{p})\cdot z)_{v,i} &= \sum_{v\in S\in K} z_S\cdot C_{i,j}M_{\mathbf{p},S}\\ &= \sum_S z_S\cdot(-1)^{i+j}\sum_{j'\in[d]\setminus\{j\}}(-1)^{d-1+j'-\mathbf{1}_{j'>j}}\det N_{\mathbf{p},S\setminus\{v_j,v'_j\},i}\end{aligned} \tag{5.5}$$

Indeed, suppose that $S=\{v_1,\dots,v_d\}$ and $v=v_j$ and expand the (i,j)-th minor of $M_{\mathbf{p},S}$ by the last row (of ones). Denote $T := S\setminus\{v_{j'}\}$ and $F := T\setminus\{v_j\}$, and we easily observe that $\operatorname{sign}(F,T) = (-1)^{j-\mathbf{1}_{j'<j}}$. Therefore, by changing the order of summation in (5.5) we find that

$$(\mathfrak{V}(K,\mathbf{p})\cdot z)_{v,i} = (-1)^{d+i}\sum_F \operatorname{sign}(F,T)\det N_{\mathbf{p},F,i}\sum_{T\subset S}\operatorname{sign}(T,S)\cdot z_S,$$

as claimed. □

5.3 Volume rigidity and sparsity

Proof of Corollary 41. Let $d\geq 3$ and K be the $(d-1)$-dimensional simplicial complex obtained from the graph $K_{3,3}$ by iterating the cone operation $d-2$ times. Then K is $(d-1,d^2-d-1)$-sparse. (Indeed, $K_{3,3}$ is $(2,3)$-sparse, and if a pure $(k-1)$-dimensional simplicial complex is $(k-1,k^2-k-1)$-sparse then its cone is $(k,(k+1)^2-(k+1)-1)$-sparse.) Thus, by completing it to a basis in the $(d-1,d^2-d-1)$-sparsity matroid, we find a basis K' containing K, so K' is $(d-1,d^2-d-1)$-tight.

In order to show that K' is not volume-rigid, by Theorem 39 it is enough to show that

$$\{1,2,\dots,d-2,d+1,d+2\}\in\Delta^p(K'),$$

as $\{1,2,\dots,d-2,d+1,d+2\}\not\leq_p\{1,3,4,\dots,d,n\}$ and using tightness.

The displayed equation above follows from basic properties of this shifting operator, proved in the same way as for exterior shifting w.r.t. the lex-order:

- If K is a subcomplex of K' then $\Delta^p(K)\subseteq\Delta^p(K')$.
- Cone and Δ^p commute, namely, if $K=v*L$ for a simplicial complex L then $\Delta^p(K)=1*(\Delta^p(L)+1)$.

Here for a family F of subsets of $[m]$, $F+1 := \{B+1:\ B\in F\}$, and $B+1 := \{i+1:\ i\in B\}$ (so $\emptyset+1=\emptyset$). To conclude the proof it is left to note that $\{3,4\}\in\Delta^{\mathrm{lex}}(K_{3,3})$ and hence, by Claim 43, also $\{3,4\}\in\Delta^p(K_{3,3})$. □

5.4 Concluding remarks

We end up with some related open problems. An obvious one is to extend Corollary 40 to include all surface triangulations.

Conjecture 51. *Every triangulation of a compact connected surface without boundary, minus a single triangle, is volume-rigid.*

The problem we face in applying Fogelsanger's decomposition [32] (see also [23, Sec.3.3]) to volume rigidity of surfaces is that the pieces in the decomposition include triangle faces not existing in the original triangulation, and thus the gluing lemmas we could prove, e.g. Lemma 48, are not strong enough to settle Conjecture 51.

It is known that for every triangulation K of the 2-sphere on n vertices, minus a single triangle, its exterior shifting $\Delta^{\text{lex}}(K) = \Delta^p(K)$ consists exactly of the triangle $13n$ and all the triangles that are smaller than it in the lex-order, and their subsets. This is a sufficient condition for volume rigidity by Theorem 39. The following conjecture deals with a higher-dimensional counterpart of this fact.

Conjecture 52. *For every $d \geq 3$, every triangulation K of the $(d-1)$-sphere minus a single $(d-1)$-simplex is volume rigid.*

It is also natural to ask whether the stronger property of $\{1, 3, 4, \dots, d, n\} \in \Delta^{\text{lex}}(K)$ holds true. This is known, and tight, for stacked spheres [65] (also [67, Example 2.1.8]). Let us remark that the conclusion $\{1, 3, 4, \dots, d, n\} \in \Delta^s(K)$ for Kalai's symmetric shifting operator $\Delta^s(\cdot)$ is equivalent to the hard-Lefschetz isomorphism from degree 1 to degree $d-1$ in a generic Artinian reduction of the Stanley-Reisner ring of K over the field of reals; the later isomorphism was proved recently by Adiprasito [1].

Back to general complexes,

Problem 53. *For every dimension, find a combinatorial characterization of the corresponding volume-rigidity matroid.*

The combinatorial characterization problem is important for the d-rigidity matroid (and is open for $d \geq 3$). The d-rigidity of a graph G on n vertices is equivalent to $\{d, n\} \in \Delta^s(G)$. In view of this fact, we ask:

Problem 54. *Define a version of symmetric shifting $\Delta^{sp}(\cdot)$ and find a matroid on $\binom{[n]}{d}$ such that its bases K are exactly those satisfying $\Delta^{sp}(K) = \{T : T \leq_p \{1, 3, 4, \dots, d, n\}\}$.*

An additional direction to explore is the volume rigidity of a $(d-1)$ dimensional simplicial complex K in $\mathbb{R}^{d'}$ for $d' \geq d-1$. That is, let $\mathbf{p} : N(K) \to \mathbb{R}^{d'}$ be generic, and ask whether there is a non-trivial motion of the vertices that preserves all the volumes of K's $(d-1)$-dimensional simplices in $\mathbb{R}^{d'}$. The case $d = 2$ corresponds to the standard framework rigidity in $\mathbb{R}^{d'}$, and the case $d' = d-1$ is the volume rigidity notion we study in this chapter. Several natural questions on the remaining cases $2 < d \leq d'$ arise: what are the trivial motions in this setting? Is there a characterization of $(d-1)$-volume rigidity in $\mathbb{R}^{d'}$ in terms of algebraic shifting?

6. Erdős-Ko-Rado for Simplicial Complexes

A set system $\mathcal{F}$ is said to be *pairwise-intersecting* if for every pair of its members $F, F' \in \mathcal{F}$ we have that $F \cap F' \neq \emptyset$. What is the largest cardinality of a family of pairwise-intersecting sets? A now-classic result of Erdős, Ko, and Rado answers this question if the sets all have the same number of elements, and are otherwise unrestricted.

Theorem 55 (Erdős, Ko, and Rado [28])**.** *Let $k \leq n/2$. If $\mathcal{F}$ is a family of pairwise-intersecting subsets of $[n]$, each with k elements, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$.*

If $|\mathcal{F}|$ achieves the upper bound and $k < n/2$, then $\mathcal{F}$ consists of all the k-element subsets containing some fixed element.

That is, under the above hypotheses a family of maximal size of pairwise-intersecting objects is given by a family with a common intersection, i.e., $\bigcap \mathcal{F} = \bigcap_{F \in \mathcal{F}} F \neq \emptyset$. Moreover, under slightly stronger hypotheses, these are the only such families. Hilton and Milner [40] later gave an improved upper bound on the size of pairwise-intersecting families that do not all contain a common element.

Theorem 56 (Hilton and Milner [40])**.** *Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be a pairwise-intersecting family such that $\bigcap \mathcal{F} = \emptyset$. Then,*

$$|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-(k+1)}{k-1} + 1.$$

Hilton and Milner introduced the following generalization of pairwise-intersecting families. Two set systems $\mathcal{F}$ and $\mathcal{G}$ are said to be *cross-intersecting* if for every $F \in \mathcal{F}$ and $G \in \mathcal{G}$ we have that $F \cap G \neq \emptyset$. In the same article Hilton and Milner provided the following upper bound for cross-intersecting families of sets all of which members are of the same size.

Theorem 57. *Let $k \leq n/2$ and $\mathcal{F}, \mathcal{G} \subseteq \binom{[n]}{k}$ be a pair of non-empty cross-intersecting families. Then,*

$$|\mathcal{F}| + |\mathcal{G}| \leq \binom{n}{k} - \binom{n-k}{k} + 1.$$

There is a large number of generalizations of the above theorems. We focus on one in particular. Holroyd and Johnson asked at the 1997 British Combinatorial Conference [45] about whether an analogue of the Erdős-Ko-Rado property holds for independent sets in cyclic and similar graphs. Talbot showed the answer to be "yes" in a strong sense.

Theorem 58 (Talbot [81])**.** *Let n, k, r be positive integers such that $k \leq n/(r+1)$. Let G be the undirected graph with vertex set $\mathbb{Z}_n$ and edges consisting of those x, y such that either $x - y \mod (n)$ or $y - x \mod (n)$ is in $\{1, \ldots, r\}$.*

If $\mathcal{F}$ is a family of pairwise-intersecting independent sets of G, each with k elements, then $|\mathcal{F}|$ is at most the size of the family $\mathcal{B}$ of all independent sets with k elements containing 0. If $|\mathcal{F}|$ achieves the upper bound and $n \neq 2k+2$, then $\mathcal{F}$ is $\mathcal{B}$ up to relabeling the vertices.

Holroyd and Talbot asked whether similar results hold for independent sets in other graphs G. Since the independent sets of a graph form a simplicial complex, Borg extended this question to arbitrary simplicial complexes. Before stating the conjecture, let us restate Theorems 55, 56 and 57 in the language of simplicial complexes. For this purpose, we will first recall some basic definitions for simplicial complexes. Let K be a simplicial complex and v a vertex in K, the link of v in K is the simplicial complex $\operatorname{lk}(v, K) = \{F \in K : v \notin F, F \cup \{v\} \in K\}$. For example, let $\Delta_{n-1} = \{F \subseteq [n]\}$ denote the $(n-1)$-simplex, then

$$\operatorname{lk}(1, \Delta_{n-1}) = \{F \subseteq [n] : 1 \notin F, F \cup 1 \subseteq [n]\}.$$

This last set system is in bijection with Δ_{n-2}. Consequently,

$$f_{k-2}(\operatorname{lk}(1, \Delta_{n-1})) = f_{k-2}(\Delta_{n-2}) = \binom{n-1}{k-1}.$$

Where for a simplicial complex K, $f_k(K)$ denotes the number of k-faces of K. Now we can restate Theorem 55.

Theorem 59 (Restatement of Theorem 55)**.** *Let $k \leq n/2$. If $\mathcal{F} \subseteq K = \Delta_{n-1}$ is a family of pairwise-intersecting $(k-1)$-faces, then $|\mathcal{F}| \leq f_{k-2}(\operatorname{lk}(1, K)) = \binom{n-1}{k-1}$.*

If $|\mathcal{F}|$ achieves the upper bound and $k < n/2$, then $\mathcal{F}$ consists of all the $(k-1)$-faces containing some fixed vertex. That is, $\mathcal{F} = \{F \cup \{v\} : F \in \operatorname{lk}(v, K),\ |F| = k-1\}$ for some $v \in N$.

Now we need to recall the notion of an induced simplicial complex. Let K be a simplicial complex with vertex set N and let $S \subseteq N$. The induced simplicial complex of K on S is $K[S] = \{F \in K : F \subseteq S\}$. Finally, we can restate Theorems 56 and 57.

Theorem 60 (Restatement of Theorem 56)**.** *Let $\mathcal{F} \subseteq K = \Delta_{n-1}$ be a pairwise-intersecting family of $(k-1)$-faces such that $\bigcap \mathcal{F} = \emptyset$. Then,*

$$|\mathcal{F}| \leq f_{k-2}(\operatorname{lk}(1, K)) - f_{k-2}(\operatorname{lk}(1, K[[n] \setminus [k+1]])) + 1.$$

Theorem 61 (Restatement of Theorem 57)**.** *Let $\mathcal{F}, \mathcal{G} \subseteq K = \Delta_{n-1}$ be a pair of non-empty cross-intersecting families of $(k-1)$-faces with $k \leq n/2$. Then,*

$$|\mathcal{F}| + |\mathcal{G}| \leq f_{k-1}(K) - f_{k-1}(K[[n] \setminus [k]]) + 1.$$

It is natural to ask if one can replace in the above reformulations the underlying $(n-1)$-simplex K by any arbitrary simplicial complex. There are counterexamples for k around the size of the minimal facet cardinality, but not for somewhat smaller k. We recall that a facet is an inclusion-wise maximal face. The following conjecture states this requirement precisely.

Conjecture 62 (Holroyd and Talbot [42], extended by Borg to arbitrary simplicial complexes [16])**.** *Let K be a simplicial complex whose smallest facet has d vertices, and let $k \leq d/2$. If $\mathcal{F}$ is a family of pairwise-intersecting faces of K, each with k elements, then there is some vertex v of K so that $|\mathcal{F}| \leq f_{k-2}(\operatorname{lk}(v, K))$. If $k < d/2$ and $|\mathcal{F}|$ achieves the upper bound, then $\mathcal{F}$ consists of the faces containing some vertex v.*

If a simplicial complex K satisfies the upper bound of Conjecture 62 at a specified value of k, then we say that K is *k-EKR*. If every pairwise-intersecting family of maximum size has a common intersection, then we say that K is *strictly k-EKR*. Since originally this conjecture was posed for the independence complex of a graph, i.e., the simplicial complex whose faces are the independence sets of a graph, we extend these definitions to graphs as follows. We abuse terminology to say that a graph is (strictly) k-EKR if its independence complex has the same property. For a graph G, we will denote its independence complex by $\mathrm{Ind}(G)$.

There has been considerable work on Conjecture 62. Hurlbert and Kamat showed [43] that any chordal graph with an isolated vertex satisfies the upper bound of Conjecture 62. Holroyd, Spencer and Talbot showed [41] that the following families of graphs satisfy the above upper bound: the disjoint union of $n \geq k$ complete graphs, each of order at least 2; the r-th power of a path on n vertices; the disjoint union of $n \geq 2k$ complete graphs, cycles and paths. Holroyd and Talbot showed [42] that the independence complex of a disjoint union of cliques and disjoint union of a pair of complete multipartite graphs satisfy the above upper bound as well. Borg showed [16] that the conjecture is true for shifted simplicial complexes. Moreover, he showed that if the minimal facet cardinality of a simplicial complex K is at least $(k-1)\binom{3k-3}{2}+k$, then K satisfies k-EKR and strict k-EKR. See also [68] for EKR type property on facets of flag complexes. Woodroofe showed more generally [91] that any sequentially Cohen-Macaulay near-cone satisfies the upper bound of Conjecture 62. Regarding this last result we want to remark that the class of sequentially Cohen-Macaulay simplicial complexes is a broad class that includes the independence complexes of chordal graphs and many others [24, 62, 90]. In particular the independence complex of a chordal graph with an isolated vertex is a sequentially Cohen-Macaulay near-cone. These notions will be defined in subsections 6.2.2 and 6.2.3. Neither Hurlbert and Kamat nor Woodroofe addressed the strict k-EKR property for this type of simplicial complexes.

The main purpose of the current chapter is to fill in this gap. We show:

Theorem 63. *Let $2 \leq k < d/2$. If the simplicial complex K is a sequentially Cohen-Macaulay near-cone with minimal facet cardinality d, then K is strictly k-EKR. That is, the pairwise-intersecting families of maximum size consist of all $(k-1)$-faces containing an apex vertex.*

Under stronger hypothesis on the underlying simplicial complex, so called t-fold near-cone, we are able to generalize Theorem 56 and Theorem 57. For example, the independence complex of a chordal graph with t isolated vertices $\{w_1, \ldots, w_t\}$ is a sequentially Cohen-Macaulay t-fold near-cone w.r.t. $(w_1, \ldots, w_t)$.

Theorem 64. *Let $2 \leq k < d/2$. Let K be a sequentially Cohen-Macaulay $(k+1)$-fold near-cone w.r.t. $W = (w_1, \ldots, w_{k+1})$ and with minimal facet cardinality d. Let $\mathcal{F} \subseteq K$ be a pairwise-intersecting family of $(k-1)$-faces with $\bigcap \mathcal{F} = \emptyset$, then*

$$|\mathcal{F}| \leq f_{k-2}(\mathrm{lk}(w_1, K)) - f_{k-2}(\mathrm{lk}(w_1, K)[N \setminus W]) + 1.$$

Theorem 65. *Let $k \leq d/2$. Let K be a sequentially Cohen-Macaulay k-fold near-cone w.r.t. W and with minimal facet cardinality d. Let $\mathcal{F}, \mathcal{G} \subseteq K$ be a pair* *of non-empty cross-intersecting families of* $(k-1)$*-faces, then*

$$|\mathcal{F}| + |\mathcal{G}| \leq f_{k-1}(K) - f_{k-1}(K[N \setminus W]) + 1.$$

In fact, we are able to show a stronger version of these theorems using an algebraic notion to lower bound the minimal facet cardinality, so called depth of a simplicial complex. This notion will be defined in subsection 6.2.3.

Theorem 66. *Let* $k \geq 2$*. If* K *is a near-cone with apex vertex* w *and* $k < \frac{\operatorname{depth}(K)+1}{2}$*, then* K *is strictly* k*-EKR. That is, the pairwise-intersecting families of maximum size consist of all* $(k-1)$*-faces containing an apex vertex.*

Theorem 67. *Let* K *be a* $(k+1)$*-fold near-cone w.r.t.* $W = (w_1, \dots, w_{k+1})$ *and* $2 \leq k \leq \frac{\operatorname{depth}(K)+1}{2}$*. Let* $\mathcal{F} \subseteq K$ *be a pairwise-intersecting family of* $(k-1)$*-faces such that* $\bigcap \mathcal{F} = \emptyset$*. Then,*

$$|\mathcal{F}| \leq f_{k-2}(\operatorname{lk}(w_1, K)) - f_{k-2}(\operatorname{lk}(w_1, K)[N \setminus W]) + 1.$$

Theorem 68. *Let* K *be a* k*-fold near-cone w.r.t.* W *and* $k \leq \frac{\operatorname{depth}(K)+1}{2}$*. Let* $\mathcal{F}, \mathcal{G} \subseteq K$ *be a pair of non-empty cross-intersecting families of* $(k-1)$*-faces, then*

$$|\mathcal{F}| + |\mathcal{G}| \leq f_{k-1}(K) - f_{k-1}(K[N \setminus W]) + 1.$$

The novelty of our techniques is to combine algebraic and combinatorial shifting operations. We also make use of some of the ideas behind recent proofs of the Hilton-Milner theorem [36, 44].

This chapter is organized as follows. In Sections 6.1 we review graph classes which independence complex falls into the class of sequentially Cohen-Macaulay complexes, in particular proving Conjecture 62 for such graph classes. In Section 6.2 we recall the necessary background for the current chapter. In Section 6.3 we give the proofs of Theorems 64 and 67. In Section 6.4 we present the proofs of Theorems 63 and 66. In Section 6.5 we present the proofs of Theorems 65 and 68.

Notation. Throughout this chapter we will be using N to denote the vertex set of a simplicial complex when an order on the vertices is not required. On the other hand, when a total order on the vertices is needed we will use $[n] = \{1, \dots, n\}$. We will usually use letters such as v, w to refer to vertices coming from N and i, j for those coming from $[n]$.

6.1 Applications

In this section we recall some families of simplicial complexes that belong to the class of sequentially Cohen-Macaulay simplicial complexes. By applying our main results of this chapter we are able to derive the strict k-EKR property in the cases for which only the k-EKR property was previously known. In particular, proving completely the conjecture for such families. Under additional hypothesis we are able to derive a Hilton-Milner type result as well as an upper bound on the size of cross-intersecting families for such families of simplicial complexes.

Let us first recall the definition of a join for simplicial complexes. Given a pair of simplicial complexes L and K, their *join*, denoted by $L * K$, is the simplicial complex given by $\{F \sqcup T \colon F \in L,\ T \in K\}$ where $\sqcup$ denotes the disjoint union.

We would like to emphasize that even though the vertices of K and L might not be disjoint, we make them disjoint when considering their join. For example, by taking $N(K) \times \{1\}$ and $N(L) \times \{2\}$ to be the new vertex sets for the simplicial complexes K and L. Here we have denoted the vertex set of a simplicial complex K by $N(K)$. If L is only a single vertex, then the join $\{v\} * K$ is called a *cone with apex vertex* v and we will usually denote it by $v * K$. Through this subsection we will use the fact that a cone is a near-cone, in spite of the fact that we have not introduced near-cones yet. Similarly, we will use the fact that $\Delta_{t-1} * K$ is a t-fold near-cone and postpone the definition until subsection 6.2.2. Let us recall that Δ_{t-1} stands for the $(t-1)$-simplex.

Let us now recall several families of simplicial complexes that belong to the class of sequentially Cohen-Macaulay simplicial complexes. One might argue that the graph class of trees is the simplest class of graph. One definition of a tree is that of a graph with every subgraph having a leaf. Faridi [29] generalized this to simplicial complexes, so called simplicial trees, and showed that these are sequentially Cohen-Macaulay. Further generalization were obtained to clutters with the free vertex property [86]. Yet another example of a sequentially Cohen-Macaulay simplicial complex is the simplicial complex given by the independent sets of a matroid. In fact, this last one satisfies an even stronger property, so called vertex decomposability [71].

Several classes of graphs whose independence complex is sequentially Cohen-Macaulay have been found. For example: chordal graphs [33]; graphs with no induced cycles of length other than 3 or 5 [90]; bipartite graphs satisfying the following recursive condition: there exist a degree 1 vertex v with its unique neighbor w such that $G \setminus N[v]$ and $G \setminus N[w]$ satisfy the same condition [86]. Here we have denotes by $N[v]$ the closed neighborhood of v in G, i.e., $N[v] = \{v\} \cup \{w \in N \colon vw \in G\}$.

We notice that for two graphs G_1 and G_2 it follows from the definition that $\mathrm{Ind}(G_1 \sqcup G_2) = \mathrm{Ind}(G_1) * \mathrm{Ind}(G_2)$. In particular, $\mathrm{Ind}(\{w\} \sqcup G) = w * \mathrm{Ind}(G)$ and this is a near-cone. In other words, if G has an isolated vertex we have that $\mathrm{Ind}(G)$ is a near-cone. We notice that adding an isolated vertex to any of the above graph classes will create a graph whose independence complex is a sequentially Cohen-Macaulay near-cone.

Hurlbert and Kamat [43] showed that chordal graphs with an isolated vertex satisfy the k-EKR property. We can now conclude, using Theorem 63, that chordal graphs with an isolated vertex completely satisfy Conjecture 62.

Corollary 69. *Let G be a chordal graph with an isolated vertex. Let d denote the minimal facet cardinality of* $\mathrm{Ind}(G)$*. Then, for $k < d/2$ we have that G satisfies strict k-EKR.*

Proof. Since G is a chordal graph with an isolated vertex, by the previous discussion, we have that $\mathrm{Ind}(G)$ is a sequentially Cohen-Macaulay near-cone. The conclusion follows now by applying Theorem 63 to $\mathrm{Ind}(G)$. □

Next we derive a Hilton-Milner type result and an upper bound on the size of cross-intersecting families for chordal graphs with several isolated vertices. Before presenting the corollary let us compute the independence complex of a graph with several isolated vertices. Let $W = \{w_1, \ldots, w_t\}$ denote t isolated vertices. Then, by the previous discussion, $\mathrm{Ind}(W \sqcup G) = \mathrm{Ind}(W) * \mathrm{Ind}(G) =$

$\Delta_{t-1} * \mathrm{Ind}(G)$ is a t-fold near-cone w.r.t. $(w_1, \ldots, w_t)$. In other words, if G has t isolated vertices $W = \{w_1, \ldots, w_t\}$ we have that $\mathrm{Ind}(G)$ is a t-fold near-cone w.r.t. $(w_1, \ldots, w_t)$. Moreover, if G is a chordal graph with t isolated vertices $\{w_1, \ldots, w_t\}$ then $\mathrm{Ind}(G)$ is a sequentially Cohen-Macaulay t-fold near-cone w.r.t. $(w_1, \ldots, w_t)$. The following results follow from combining this observation with Theorem 64 and Theorem 65, respectively.

Corollary 70. *Let G be a chordal graph with vertex set N having $k+1$ isolated vertices denoted by $W = \{w_1, \ldots, w_{k+1}\}$. Set $K = \mathrm{Ind}(G)$ and denote by d its minimal facet cardinality. Then, for $k \leq 2d$ and any pairwise-intersecting family $\mathcal{F} \subseteq K$ of $(k-1)$-faces such that $\bigcap \mathcal{F} = \emptyset$, we have that*

$$|\mathcal{F}| \leq f_{k-2}(\mathrm{lk}(w_1, K)) - f_{k-2}(\mathrm{lk}(w_1, K)[N \setminus W]) + 1.$$

Proof. By the above discussion we have that $K = \mathrm{Ind}(G)$ is a sequentially Cohen-Macaulay $(k+1)$-fold near-cone w.r.t. $(w_1, \ldots, w_{k+1})$. The conclusion now follows by applying Theorem 64 to K. □

Corollary 71. *Let G be a chordal graph with vertex set N having k isolated vertices denoted by $W = \{w_1, \ldots, w_k\}$. Set $K = \mathrm{Ind}(G)$ and denote by d its minimal facet cardinality. Then, for $k \leq 2d$ and any pair of non-empty cross-intersecting families $\mathcal{F}, \mathcal{G} \subseteq K$ of $(k-1)$-faces, we have that*

$$|\mathcal{F}| + |\mathcal{G}| \leq f_{k-1}(K) - f_{k-1}(K[N \setminus W]) + 1.$$

Proof. By the above discussion we have that $K = \mathrm{Ind}(G)$ is a sequentially Cohen-Macaulay k-fold near-cone w.r.t. $(w_1, \ldots, w_k)$. The conclusion now follows by applying Theorem 65 to K. □

Holroyd, Spencer and Talbot showed [41, Theorem 8] that a disjoint union of $n \geq 2k$ graphs given by paths, cliques and cycles including an isolated vertex satisfies k-EKR.

Theorem 72. *[41, Theorem 8] Let $2k \leq n$. If $G = \bigsqcup_{i=1}^{n} G_i$ is a disjoint union of non-empty graphs with each G_i being a path, a clique or a cycle and at least one of the G_i being an isolated vertex, then G satisfies k-EKR.*

Woodroofe generalized this result by showing that a disjoint union of $n \geq 2k$ graphs containing an isolated vertex satisfies k-EKR [91, Proposition 4.3].

Theorem 73. *[91, Proposition 4.3] Let $2k \leq n$. If $G = \bigsqcup_{i=1}^{n} G_i$ is a disjoint union of non-empty graphs such that G has an isolated vertex, then G satisfies k-EKR.*

We follow Woodroofe's strategy to show that the strict k-EKR also holds for this type of graphs. For it, first we need to recall how depth behaves w.r.t. disjoint union, see subsection 6.2.3 for the definition of depth of a simplicial complex.

Lemma 74. *[91, Lemma 2.12] Let K_1 and K_2 be simplicial complexes. Then* $\mathrm{depth}(K_1 * K_2) = \mathrm{depth}(K_1) + \mathrm{depth}(K_2) + 1$.

Combining this lemma with Theorem 66 we can deduce the following corollary.

Corollary 75. *Let $2k < n$. If $G = \bigsqcup_{i=1}^{n} G_i$ is a disjoint union of non-empty graphs and G has an isolated vertex, then G satisfies strict k-EKR.*

Proof. On the one hand, since G has an isolated vertex, then by the previous discussion $\mathrm{Ind}(G)$ is a near-cone. On the other hand, by Lemma 74 we have that

$$\begin{aligned}\mathrm{depth}(\mathrm{Ind}(G)) &= \mathrm{depth}(\mathrm{Ind}(G_1 \sqcup \cdots \sqcup G_n)) \\ &= \mathrm{depth}(\mathrm{Ind}(G_1) * \cdots * \mathrm{Ind}(G_n)) \geq n-1.\end{aligned}$$

That is, $\mathrm{Ind}(G)$ is a near-cone with $\mathrm{depth}(\mathrm{Ind}(G)) + 1 \geq n > 2k$. The conclusion now follows by applying Theorem 66 to $\mathrm{Ind}(G)$. □

We can deduce a Hilton-Milner type upper bound as well as an upper bound on the size of cross-intersecting families for independence complexes of union of graphs. The following corollaries follow from combining Lemma 74 with Theorems 67 and 68, respectively.

Corollary 76. *Let $2k \leq n$ and $G = \bigsqcup_{i=1}^{n} G_i$ be a disjoint union of non-empty graphs with vertex set N such that G contains $k+1$ isolated vertices given by $W = \{w_1, \ldots, w_{k+1}\}$. Let $\mathcal{F} \subseteq \mathrm{Ind}(G)$ be a pairwise-intersecting family of $(k-1)$-faces such that $\bigcap \mathcal{F} = \emptyset$, then*

$$|\mathcal{F}| \leq f_{k-2}(\mathrm{lk}(w, \mathrm{Ind}(G))) - f_{k-2}(\mathrm{lk}(w, \mathrm{Ind}(G))[N \setminus W]) + 1,$$

where w is any of the elements in W.

Proof. On the one hand, since G has $k+1$ isolated vertices given by $W = \{w_1, \ldots, w_{k+1}\}$, by the above discussion, we can conclude that $\mathrm{Ind}(G)$ is a $(k+1)$-fold near-cone w.r.t. $(w_1, \ldots, w_{k+1})$. On the other hand, by Lemma 74 we have that

$$\begin{aligned}\mathrm{depth}(\mathrm{Ind}(G)) &= \mathrm{depth}(\mathrm{Ind}(G_1 \sqcup \cdots \sqcup G_n)) \\ &= \mathrm{depth}(\mathrm{Ind}(G_1) * \cdots * \mathrm{Ind}(G_n)) \geq n-1.\end{aligned}$$

That is, $\mathrm{Ind}(G)$ is a $(k+1)$-fold near-cone with $\mathrm{depth}(\mathrm{Ind}(G)) + 1 \geq n \geq 2k$. The conclusion now follows by applying Theorem 67 to $\mathrm{Ind}(G)$. □

Corollary 77. *Let $2k \leq n$ and $G = \bigsqcup_{i=1}^{n} G_i$ be a disjoint union of non-empty graphs with vertex set N such that G contains k isolated vertices given by $W = \{w_1, \ldots, w_k\}$. Let $\mathcal{F}, \mathcal{G} \subseteq \mathrm{Ind}(G)$ a pair of non-empty cross-intersecting families of $(k-1)$-faces, then*

$$|\mathcal{F}| + |\mathcal{G}| \leq f_{k-1}(\mathrm{Ind}(G)) - f_{k-1}(\mathrm{Ind}(G)[N \setminus W]) + 1.$$

Proof. On the one hand, since G has k isolated vertices given by $W = \{w_1, \ldots, w_k\}$, by the above discussion, we can conclude that $\mathrm{Ind}(G)$ is a k-fold near-cone w.r.t. $(w_1, \ldots, w_k)$. On the other hand, by Lemma 74 we have that

$$\begin{aligned}\mathrm{depth}(\mathrm{Ind}(G)) &= \mathrm{depth}(\mathrm{Ind}(G_1 \sqcup \cdots \sqcup G_n)) \\ &= \mathrm{depth}(\mathrm{Ind}(G_1) * \cdots * \mathrm{Ind}(G_n)) \geq n-1.\end{aligned}$$

That is, $\mathrm{Ind}(G)$ is a k-fold near-cone with $\mathrm{depth}(\mathrm{Ind}(G)) + 1 \geq n \geq 2k$. The conclusion now follows by applying Theorem 68 to $\mathrm{Ind}(G)$. □

6.2 Preliminaries

6.2.1 Combinatorial shifting

Let 2^N denote the set of all subsets of N. Given a set system $\mathcal{F} \subseteq 2^N$, and $F \in \mathcal{F}$. Let $v, w \in N$, the *combinatorial shift* $\mathrm{shift}_{w,v}$ is defined by

$$\mathrm{shift}_{w,v}(F, \mathcal{F}) = \begin{cases} (F \setminus \{v\}) \cup \{w\} & \text{if } v \in F, w \notin F \text{ and} \\ & \quad (F \setminus \{v\}) \cup \{w\} \notin \mathcal{F}, \\ F & \text{otherwise.} \end{cases}$$

$$\mathrm{shift}_{w,v}(\mathcal{F}) = \{\mathrm{shift}_{w,v}(F, \mathcal{F}) \colon F \in \mathcal{F}\}.$$

Combinatorial shifting has been a successful technique to prove upper bounds on the size of cross-intersecting and pairwise-intersecting set systems. We will be using the following properties [35].

Theorem 78. *Let $\mathcal{F}$, $\mathcal{G} \subseteq 2^N$ and $v, w \in N$.*

1. $|\mathrm{shift}_{w,v}(\mathcal{F})| = |\mathcal{F}|$.
2. *If $\mathcal{G} \subseteq \mathcal{F}$, then* $\mathrm{shift}_{w,v}(\mathcal{G}) \subseteq \mathrm{shift}_{w,v}(\mathcal{F})$.
3. *If $\mathcal{F}$ and $\mathcal{G}$ are cross-intersecting, then* $\mathrm{shift}_{w,v}(\mathcal{F})$ *and* $\mathrm{shift}_{w,v}(\mathcal{G})$ *are cross-intersecting. In particular, if $\mathcal{F}$ is pairwise-intersecting, then* $\mathrm{shift}_{w,v}(\mathcal{F})$ *is pairwise-intersecting.*

We will be interested in applying repeatedly combinatorial shifting to a set system until we obtain a set system that is invariant under further aplication. First, let us recall that such a sequence of applications exists. For it, let us assume that the vertex set N is equipped with a total order, i.e., $N = [n]$. A set system $\mathcal{F} \subseteq 2^{[n]}$ is said to be *shifted* if for every $F \in \mathcal{F}$ and $i, j \in [n]$ such that $i < j$, $j \in F$ and $i \notin F$ we have that $(F \setminus \{j\}) \cup \{i\} \in \mathcal{F}$. The property of being shifted can be stated in terms of a partial order $<_p$ defined as follows. Let $S = \{s_1 < \cdots < s_k\}$, $T = \{t_1 < \cdots t_k\} \in \binom{[n]}{k}$, then $S \leq_p T$ if $s_i \leq t_i$ for every $i = 1, \ldots, k$. Then, $\mathcal{F}$ is shifted if and only if for every $F \in \mathcal{F}$ and $G \leq_p F$ we have that $G \in \mathcal{F}$. Moreover, we have the following characterization in terms of combinatorial shifting [35].

Theorem 79. *Let $\mathcal{F} \subseteq 2^{[n]}$. Then, $\mathcal{F}$ is shifted if and only if* $\mathrm{shift}_{i,j}(\mathcal{F}) = \mathcal{F}$ *for every $i, j \in [n]$ with $i < j$.*

By iterating the combinatorial shifting operation, $\mathrm{shift}_{i,j}$ with $1 \leq i < j \leq n$, we will eventually obtain a set system that is shifted [35, Proposition 2.2].

Theorem 80. *[35, Proposition 2.2] Let $\mathcal{F} \subseteq 2^N$ and apply* $\mathrm{shift}_{i,j}$ *to $\mathcal{F}$ for $1 \leq i < j \leq n$ exactly once in an order such that* $\mathrm{shift}_{i,j}$ *is applied before* $\mathrm{shift}_{i',j'}$ *if $j' < j$. The resulting set system is shifted.*

In general we will not have available the entire range of parameters, i.e., $1 \leq i < j \leq n$, to perform combinatorial shifts in order to arrive to a shifted family. In the following we show how we shift a set system in a limited range of parameters in order to obtain another set system invariant under further applications of combinatorial shift in the same range of parameters. Despite this restriction, the same argument as in [35, Proposition 2.2] works in this setting as well. For completeness we present the proof.

Lemma 81. *Let $\mathcal{F} \subseteq 2^N$ and let $W = \{w_1, \dots, w_t\} \subseteq N$. Apply* $\mathrm{shift}_{w_i,v}$ *to $\mathcal{F}$ for $i \in [t]$ and $v \in N \setminus \{w_1, \dots, w_i\}$ exactly once in an order such that* $\mathrm{shift}_{w_i,v}$ *is applied before* $\mathrm{shift}_{w_{i'},w_{j'}}$ *for every $i \in [t]$ and $v \in N \setminus W$ and* shift_{w_i,w_j} *is applied before* $\mathrm{shift}_{w_{i'},w_{j'}}$ *if $j' < j$.*

Proof. We proceed by induction on $|N \setminus W|$. If $N = W$ then the order is the same as the one in the statement of Theorem 80 and the conclusion follows by this theorem. Let $v \in N \setminus W$. First, we apply $\mathrm{shift}_{w_i,v}$ to $\mathcal{F}$ once for every $i \in [t]$ and denote the resulting set system by $\mathcal{H}$. Then

$$\mathcal{H}(v) = \{F \setminus \{v\} \colon F \in \mathcal{H},\ v \in F\} \subseteq \mathcal{H}(w_i)$$

for every $i \in [t]$ and this inclusion is preserved during later shifts. Set $\mathcal{H}(\bar{v}) = \{F \in \mathcal{H} \colon v \notin F\}$. We can apply the inductive hypothesis to $\mathcal{H}(v)$ and $\mathcal{H}(\bar{v})$ since $|(N \setminus \{v\}) \setminus W| < |N \setminus W|$ from which the conclusion follows. □

The same argument yields the following variation of the previous lemma. It will be instrumental to prove Theorems 64 and 67. We state it now for future reference.

Lemma 82. *Let $\mathcal{F} \subseteq 2^N$, let $W = \{w_1, \dots, w_t\} \subseteq N$ and fix some $w_s \in W$. Apply* $\mathrm{shift}_{w_i,v}$ *to $\mathcal{F}$ for $i \in [t] \setminus \{1, s\}$ and $v \in N \setminus \{w_1, \dots, w_i, w_s\}$ exactly once in an order such that* $\mathrm{shift}_{w_i,v}$ *is applied before* $\mathrm{shift}_{w_{i'},w_{j'}}$ *for every $i \in [t] \setminus \{1, s\}$ and $v \in N \setminus W$ and* shift_{w_i,w_j} *is applied before* $\mathrm{shift}_{w_{i'},w_{j'}}$ *if $j' < j$.*

Let K be a simplicial complex on vertex set N and let u be a vertex of K. We will denote by $\mathrm{del}(w, K)$ the simplicial complex $K = [N \setminus \{w\}] = \{T \in K \colon w \notin T\}$. Let us now describe the behavior of combinatorial shifting when applied to $\mathrm{del}(u, K)$ and $\mathrm{lk}(u, K)$.

Lemma 83. *Let K be a simplicial complex and u, v and w be different vertices of K. Then,*

$$\mathrm{shift}_{w,v}(\mathrm{del}(u, K)) = \mathrm{del}(u, \mathrm{shift}_{w,v}(K)) \text{ and}$$

$$\mathrm{shift}_{w,v}(\mathrm{lk}(u, K)) = \mathrm{lk}(u, \mathrm{shift}_{w,v}(K)).$$

Proof. Let us start by showing that the first equality holds. Let $F \in \mathrm{del}(u, K)$, we want to verify that $\mathrm{shift}_{w,v}(F, \mathrm{del}(u, K)) \in \mathrm{del}(u, \mathrm{shift}_{w,v}(K))$.

If $\mathrm{shift}_{w,v}(F, \mathrm{del}(u, K)) = F$, then by definition $v \notin F$, $w \in F$ or $(F \setminus \{v\}) \cup \{w\} \in \mathrm{del}(u, K)$. Since $\mathrm{del}(u, K) \subseteq K$, in each of the three cases we can conclude that $\mathrm{shift}_{w,v}(F, K) = F$. Because $u \notin F$ we have that $F \in \mathrm{del}(u, \mathrm{shift}_{w,v}(K))$.

If $\mathrm{shift}_{w,v}(F, \mathrm{del}(u, K)) = (F \setminus \{v\}) \cup \{w\}$, then $(F \setminus \{v\}) \cup \{w\} \notin \mathrm{del}(u, K)$ and, since it does not contain u, it is also not in K. Then, $\mathrm{shift}_{w,v}(F, K) = (F \setminus \{v\}) \cup \{w\} \in \mathrm{del}(u, \mathrm{shift}_{w,v}(K))$ as wanted.

Now, let $F \in K$ with $\mathrm{shift}_{w,v}(F, K) \in \mathrm{del}(u, \mathrm{shift}_{w,v}(K))$. We will verify that $\mathrm{shift}_{w,v}(F, K) \in \mathrm{shift}_{w,v}(\mathrm{del}(u, K))$ by showing that $\mathrm{shift}_{w,v}(F, K) = \mathrm{shift}_{w,v}(F, \mathrm{del}(u, K))$. If $\mathrm{shift}_{w,v}(F, K) = F$ then $v \notin F$, $w \in F$ or $(F \setminus \{v\}) \cup$

$\{w\} \in K$. If $(F \setminus \{v\}) \cup \{w\} \in K$ then, since it does not contain the vertex u, it is also in $\mathrm{del}(u, K)$. Consequently in all three cases it follows that $\mathrm{shift}_{w,v}(F, \mathrm{del}(u, K)) = F$.

If $\mathrm{shift}_{w,v}(F, K) = (F \setminus \{v\}) \cup \{w\}$ then $(F \setminus \{v\}) \cup \{w\} \notin K$ and consequently not in $\mathrm{del}(u, K)$. Then, $\mathrm{shift}_{w,v}(F, \mathrm{del}(u, K)) = (F \setminus \{v\}) \cup \{w\} \in \mathrm{shift}_{w,v}(\mathrm{del}(u, K))$ as wanted.

Let us show next that the second equality holds. Let $F \in \mathrm{lk}(u, K)$, we want to verify that $\mathrm{shift}_{w,v}(F, \mathrm{lk}(u, K)) \in \mathrm{lk}(u, \mathrm{shift}_{w,v}(K))$. If $\mathrm{shift}_{w,v}(F, \mathrm{lk}(u, K)) = F$ then $v \notin F$, $w \in F$ or $(F \setminus \{v\}) \cup \{w\} \in \mathrm{lk}(u, K)$. If $(F \setminus \{v\}) \cup \{w\} \in \mathrm{lk}(u, K)$ then $(F \setminus \{v\}) \cup \{w, u\} \in K$. Consequently in all three cases we have that $\mathrm{shift}_{w,v}(F \cup \{u\}, K) = F \cup \{u\}$. That is, $F \in \mathrm{lk}(u, \mathrm{shift}_{w,v}(K))$.

If $\mathrm{shift}_{w,v}(F, \mathrm{lk}(u, K)) = (F \setminus \{v\}) \cup \{w\}$ then $(F \setminus \{v\}) \cup \{w\} \notin \mathrm{lk}(u, K)$ and consequently $(F \setminus \{v\}) \cup \{w, u\} \notin K$. Then, $\mathrm{shift}_{w,v}(F \cup \{u\}, K) = (F \setminus \{v\}) \cup \{w, u\} \in \mathrm{shift}_{w,v}(K)$ and consequently $(F \setminus \{v\}) \cup \{w\} \in \mathrm{lk}(u, \mathrm{shift}_{w,v}(K))$ as wanted.

Now, let $F \in \mathrm{lk}(u, \mathrm{shift}_{w,v}(K))$ then $F \cup \{u\} \in \mathrm{shift}_{w,v}(K)$. Let $F' \in \mathrm{lk}(u, K)$ such that $\mathrm{shift}_{w,v}(F' \cup \{u\}, K) = F \cup \{u\}$. If $F = F'$ then $v \notin F'$, $w \in F'$ or $(F' \setminus \{v\}) \cup \{w, u\} \in K$. If $(F' \setminus \{v\}) \cup \{w, u\} \in K$ then $(F' \setminus \{v\}) \cup \{w\} \in \mathrm{lk}(u, K)$. Then, in all three cases we have that $\mathrm{shift}_{w,v}(F', \mathrm{lk}(u, K)) = F' = F \in \mathrm{shift}_{w,v}(\mathrm{lk}(u, K))$ as wanted.

If $F = (F' \setminus \{v\}) \cup \{w\}$ then $(F' \setminus \{v\}) \cup \{w, u\} \notin K$ and consequently $(F' \setminus \{v\}) \cup \{w\} \notin \mathrm{lk}(u, K)$. Therefore, $\mathrm{shift}_{w,v}(F', \mathrm{lk}(u, K)) = (F' \setminus \{v\}) \cup \{w\} = F \in \mathrm{shift}_{w,v}(\mathrm{lk}(u, K))$ as wanted. □

6.2.2 t-fold near-cone

A simplicial complex K is a *near-cone* with apex vertex w if for every face $F \in K$ we have that $(F \setminus \{v\}) \cup \{w\} \in K$ for every $v \in F$. We notice that K is a near-cone with apex vertex w if and only if $\mathrm{shift}_{w,v}(K) = K$ for every $v \in N$. That is, when we apply combinatorial shift $\mathrm{shift}_{w,v}$ to the set system K, this one does not change.

Let K be a simplicial complex with vertex set N and $W = (w_1, \ldots, w_t)$ a sequence of different vertices of K, we say that K *is a t-fold near-cone w.r.t W* if K is a near-cone with apex vertex w_1 and, $\mathrm{del}(w_1, K)$ and $\mathrm{lk}(w_1, K)$ are $(t-1)$-fold near-cones w.r.t. $(w_2, \ldots, w_t)$. The following lemma provides an alternative characterization of t-fold near-cones.

Lemma 84. *Let K be a simplicial complex and $W = (w_1, \ldots, w_t) \subseteq N$ a sequence of different vertices. Then, K is a t-fold near-cone w.r.t. W if and only if $\mathrm{shift}_{w_i,v}(K) = K$ for every $w_i \in W$ and $v \in N \setminus \{w_1, \ldots, w_i\}$.*

Proof. On the one hand, let $w_i \in W$ and $v \in N \setminus \{w_1, \ldots, w_i\}$. We want to show that $\mathrm{shift}_{w_i,v}(K) = K$. For it, let $F \in K$ and assume that $v \in F$, otherwise the combinatorial shift does not affect F. We want to verify that $(F \setminus \{v\}) \cup \{w_i\} \in K$. We will show that $F' = F \setminus \{w_j \in W \cap F : j < i\}$ is in a subcomplex K' of $\mathrm{lk}(\{w_j \in W \cap F : j < i\}, K)$ that is a near-cone with apex vertex w_i. We build K' as follows. First, we initialize $K_0 = K$, then for $j = 1, \ldots, i-1$ we set $K_j = \mathrm{del}(w_j, K_{j-1})$ if $w_j \notin F \cap W$ and $K_j = \mathrm{lk}(w_j, K_{j-1})$ otherwise. By hypothesis it follows that for every $j = 0, \ldots, i-1$ the simplicial complex K_j is a

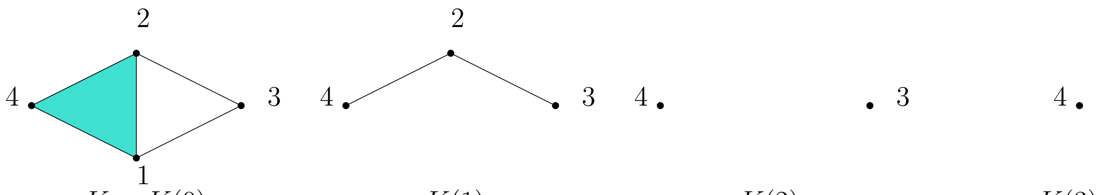

Figure 6.1: Example of a 3-near-cone that it is not a 3-fold near-cone.

near-cone with apex vertex w_{j+1} and $F \setminus \{w_t \in W \cap F : t < j\} \in K_j$. In the end we have that $F' \in K_{i-1} = K'$. Then, since K' is a near-cone with apex vertex w_i we have that

$$(F' \setminus \{v\}) \cup \{w_i\} \in K' \subseteq \mathrm{lk}(\{w_t \in W \cap F : t < i\}, K).$$

Then, $(F \setminus \{v\}) \cup \{w_i\} \in K$.

On the other hand, since $\mathrm{shift}_{w_1,v}(K) = K$ for every $v \in N \setminus \{w_1\}$ we have that K is a near-cone with apex vertex w_1. Now, we want to verify that $\mathrm{del}(w_1, K)$ and $\mathrm{lk}(w_1, K)$ are near-cones with apex vertex w_2. To see this we will show that $\mathrm{shift}_{w_2,v}(\mathrm{del}(w_1, K)) = \mathrm{del}(w_1, K)$ and $\mathrm{shift}_{w_2,v}(\mathrm{lk}(w_1, K)) = \mathrm{lk}(w_1, K)$ for every $v \in N \setminus \{w_1, w_2\}$. By Lemma 83 we have that

$$\mathrm{shift}_{w_2,v}(\mathrm{del}(w_1, K)) = \mathrm{del}(w_1, \mathrm{shift}_{w_2,v}(K)) = \mathrm{del}(w_1, K),$$

and

$$\mathrm{shift}_{w_2,v}(\mathrm{lk}(w_1, K)) = \mathrm{lk}(w_1, \mathrm{shift}_{w_2,v}(K)) = \mathrm{lk}(w_1, K),$$

where in both cases the last equality follows from the assumption $\mathrm{shift}_{w_2,v}(K) = K$ for every $v \in N \setminus \{w_1, w_2\}$. Then, $\mathrm{del}(w_1, K)$ and $\mathrm{lk}(w_1, K)$ are near-cones with apex vertex w_2. By repeating the argument with $\mathrm{del}(w_1, K)$ and $\mathrm{lk}(w_1, K)$ in place of K we can conclude that K is a t-fold near-cone w.r.t. W. ☐

Example 85. *Let G be a graph with t isolated vertices $\{w_1, \dots, w_t\}$, then* $\mathrm{Ind}(G)$ *is a t-fold near-cone w.r.t. $W = (w_1, \dots, w_t)$. In fact, in this case the order of W is not important.*

The following notion is closely related to t-fold near-cones. A simplicial complex K is a *t-near-cone* w.r.t. $W = (w_1, \dots, w_t)$ if there exists a sequence of simplicial complexes

$$K = K(0) \supseteq K(1) \supseteq \cdots \supseteq K(t)$$

such that for every $1 \le j \le t$, $K(j) = \{T \in K(j-1) : w_j \notin T\}$ and $K(j-1)$ is a near-cone with apex vertex w_j.

Lemma 86. *If K is a t-fold near-cone w.r.t. $W = (w_1, \dots, w_t)$, then it is a t-near-cone w.r.t. W.*

Proof. Let $K(0) = K$ and for $j = 1, \dots, t$ set $K(j) = \mathrm{del}(w_j, K(j-1))$. By induction $K(j-1)$ is a near-cone with apex vertex w_j. ☐

Remark 87. *In general a t-near-cone is not a t-fold near-cone. Consider the simplicial complex K with facets $\{1,2,4\},\{1,3\},\{2,3\}$ depicted in Figure 6.1. It is a 3-near-cone w.r.t. $(1,2,3)$ since K is a near-cone with apex 1. The simplicial complex $K(1)$ has facets $\{2,3\}$ and $\{2,4\}$ and it is a near-cone with apex vertex 2, see Figure 6.1. The complex $K(2)$ has facets $\{3\}$ and $\{4\}$ and it is a near-cone with apex vertex 3. Finally, $K(3)$ has a unique facet $\{4\}$. But $\{1,2,3\} \in \mathrm{shift}_{3,4}(K)$ while $\{1,2,3\} \notin K$. By Lemma 84 we can conclude that K is not a 3-fold near-cone w.r.t. $(1,2,3)$ since $\mathrm{shift}_{3,4}(K) \neq K$.*

6.2.3 Depth and the sequentially Cohen-Macaulay property

Let $\mathbb{F}$ denote a field. The i-th Betti number $\beta_i(K,\mathbb{F}) = \dim_{\mathbb{F}} H_i(K,\mathbb{F})$ is the dimension of the i-th homology group. Since we are considering coefficients over a field, this last one is in fact a vector space. One can read-off the Betti number from the near-cone combinatorially as follows.

Theorem 88. *[50, Lemma 3.1] Let K be a near-cone with apex vertex w, then*

$$\beta_{k-1}(K) = |\{S \in K \colon |S| = k,\ S \cup \{w\} \notin K\}|.$$

A simplicial complex K is called *Cohen-Macaulay over* $\mathbb{F}$ if for every face $F \in K$ we have that $\tilde{H}_i(\mathrm{lk}(F,K),\mathbb{F}) = 0$ for $i < \dim(\mathrm{lk}(F,K))$. That is, the reduced homology of every link vanishes on every dimension except possibly the top one. The *pure k-skeleton of K* is the simplicial complex generated by the k-faces of K. A simplicial complex is said to be *sequentially Cohen-Macaulay over* $\mathbb{F}$ if for every k, the pure k-skeleton of K is Cohen-Macaulay over $\mathbb{F}$. From now on we will assume that the field has characteristic 0 and drop it from the notation.

We need to be able to control the behavior of the minimal facet cardinality of a simplicial complex because it plays a key role in Conjecture 62. For this purpose we will use the following definition of *depth of a simplicial complex K*

$$\mathrm{depth}(K) = \max\{d \colon K^{(d)} \text{ is Cohen-Macaulay}\}.$$

We can restate this definition in a more convenient way as follows. The simplicial complex $K^{(d)}$ is Cohen-Macaulay if and only if $\tilde{H}_i(\mathrm{lk}(F,K^{(d)}),\mathbb{F}) = 0$ for every $F \in K^{(d)}$ and $i < \dim \mathrm{lk}(F,K^{(d)})$. Then, for $i < \dim \mathrm{lk}(F,K^{(d)})$ we have that

$$\begin{aligned}\tilde{H}_i(\mathrm{lk}(F,K^{(d)}),\mathbb{F}) &= \tilde{H}_i(\mathrm{lk}(F,K^{(d)})^{(i+1)},\mathbb{F}) = \tilde{H}_i(\mathrm{lk}(F,K)^{(i+1)},\mathbb{F}) \\ &= \tilde{H}_i(\mathrm{lk}(F,K),\mathbb{F}),\end{aligned}$$

where we have used that the k-th homology group only depends on the (k+1)-th skeleton, i.e., $\tilde{H}_k(K^{(k+1)},\mathbb{F}) = \tilde{H}_k(K,\mathbb{F})$, and that $\mathrm{lk}(F,K^{(d)})^{(i)} = \mathrm{lk}(F,K)^{(i)}$ for $i \leq \dim \mathrm{lk}(F,K^{(d)})$. Since $\dim \mathrm{lk}(F,K^{(d)}) \leq d - |F|$ we can conclude the following characterization

$$\mathrm{depth}(K) = \max\{d \colon \tilde{H}_i(\mathrm{lk}(F,K),\mathbb{F}) = 0 \text{ for every } F \in K \text{ and } i < d - |F|\}.$$

Next, we observe that $\operatorname{depth}(K) + 1$ is at most the minimal facet cardinality. To see this let F be a facet of K, then $\operatorname{lk}(F, K) = \{\emptyset\}$ and consequently $\tilde{H}_{-1}(\operatorname{lk}(F, K)) = \tilde{H}_{-1}(\{\emptyset\}) = \mathbb{F}$. That is, $\operatorname{depth}(K) + 1 \leq |F|$ for any facet of K, in particular $\operatorname{depth}(K)+1$ is a lower bound on the minimal facet cardinality. The depth of a simplicial complex is one less than the depth of its Stanley-Reisner ring [75].

6.2.4 Exterior algebraic shifting

Algebraic shifting was introduced by Kalai (see e.g. [49] and the survey [50]) and has been studied extensively in algebraic combinatorics. Here we review the definition of exterior algebraic shifting as well as some of its properties. Let us denote by $(e_i)_{i\in[n]}$ the standard basis of $V = \mathbb{R}^{[n]}$ and consider $(f_i)_{i\in[n]}$ a generic change of basis given by $f_i = \sum_{j\in[n]} a_{ij}e_j$ with the coefficients $(a_{ij})_{i,j\in[n]}$ being algebraically independent. Consider the *exterior face ring*

$$\bigwedge K = \bigwedge V \ / \ (e_T \colon T \notin K),$$

and let q denote the natural quotient map $q\colon \bigwedge V \to \bigwedge K$. The *exterior algebraic shift of* K, denoted by $\Delta(K)$, is defined as

$$\Delta(K) = \{T \subseteq [n] \colon q(f_T) \notin \operatorname{span}_{\mathbb{R}}\{q(f_S) \colon S <_{lex} T, |S| = |T|\}\},$$

where $<_{lex}$ denotes the lexicographical order defined as $S <_{lex} T$ if $|S| < |T|$ or $|S| = |T|$ and $\min((S \cup T) \setminus (S \cap T)) \in S$. Here we merely state the properties we will be using.

Theorem 89. *Let K be a simplicial complex.*

1. *[50, Theorem 2.1.1] Exterior algebraic shifting preserves the f-vector, i.e., $f_k(K) = f_k(\Delta(K))$.*
2. *[50, Theorem 2.1.2] The simplicial complex $\Delta(K)$ is shifted.*
3. *[50, Theorem 2.1.4] If K is shifted, then $\Delta(K) = K$.*
4. *[50, Theorem 2.2.7] If $K \subseteq L$, then $\Delta(K) \subseteq \Delta(L)$.*
5. *[50, Theorem 3.2] Exterior algebraic shifting preserves the Betti numbers, i.e., $\beta_k(K) = \beta_k(\Delta(K))$ for every $k \geq 0$.*
6. *[50, Theorem 4.1] If K is Cohen-Macaulay then $\Delta(K)$ is Cohen-Macaulay.*
7. *[50, Theorem 6.2] If $\mathcal{F} \subseteq \binom{[n]}{a}$ and $\mathcal{G} \subseteq \binom{[n]}{b}$ are cross-intersecting, then $\Delta(\mathcal{F})$ and $\Delta(\mathcal{G})$ are cross-intersecting. In particular, if $\mathcal{F}$ is pairwise-intersecting, then $\Delta(\mathcal{F})$ is pairwise-intersecting.*

We would like to point out that in contrast to combinatorial shifting, exterior algebraic shifting produces a shifted complex in one step rather than after a sequence of applications.

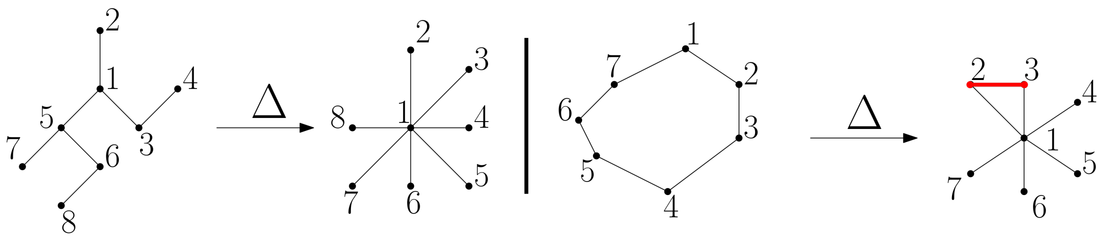

Figure 6.2: Exterior algebraic shifting of a tree and a cycle.

Example 90. *Let G be a tree on n vertices, then $\Delta(G)$ is a connected, acyclic and shifted graph. We claim that the only edges are $\{1, j\} \in \Delta(G)$ for $j = 2, \ldots, n$. On the one hand, these edges must be there: since $\Delta(G)$ is connected for every j there is an edge $\{i, j\} \in \Delta(G)$, then $\{1, j\} \in \Delta(G)$ since the complex is shifted. Since algebraic shifting preserves the number of edges, these are the only ones because $|E(G)| = n - 1$. See the left side of Figure 6.2.*

Now, let G be a cycle on n vertices, then $\Delta(G)$ contains a star on the vertex 1 and the edge $\{2, 3\}$. Since it is connected, it contains the star. On the other hand, since algebraic shifting preserves the first Betti number then $\{2, 3\}$ is in the exterior algebraic shift. By counting the number of edges these must be all of them. See the right side of Figure 6.2.

The following corollary generalizes the above observation regarding the algebraic shift of a cycle. It follows from Theorems 88 and 89.

Corollary 91. *Let K be a simplicial complex with $\beta_{k-1}(K) > 0$. Then,*

$$\binom{[k+1]}{k} \subseteq \Delta(K).$$

Proof. Let $S \in \Delta(K)$ be a k-face with $1 \notin S$. Such a face exists since

$$|\{T \in \Delta(K) \colon |T| = k, \{1\} \cup S \notin \Delta(K)\}| = \beta_{r-1}(\Delta(K)) = \beta_{r-1}(K) > 0.$$

Because $\{2, \ldots, k+1\} \leq_p S$ and $\Delta(K)$ is shifted we have that $\{2, \ldots, k+1\} \in \Delta(K)$. Using that $[k+1] \setminus \{i\} <_p \{2, \ldots, k+1\}$ for all $i = 2, \ldots k+1$ and that $\Delta(K)$ is shifted we can conclude that $\binom{[k+1]}{k} \subseteq \Delta(K)$. □

Before proceeding let us mention the importance of the preceding corollary. We will use it to guarantee that the algebraic shifting of a pairwise-intersecting family $\mathcal{F}$ satisfies the empty intersection condition, i.e., $\bigcap \Delta(\mathcal{F}) = \emptyset$. Concretely, if $\beta_{k-1}(\langle \mathcal{F} \rangle) > 0$, then $\binom{[k+1]}{k} \subseteq \Delta(\mathcal{F})$. Here we have used the notation $\langle \mathcal{F} \rangle$ to denote the simplicial complex generated by $\mathcal{F}$. In particular, $\bigcap \Delta(\mathcal{F}) \subseteq \bigcap \binom{[k+1]}{k} = \emptyset$. We will use this in several steps when proving Theorem 63 and Theorem 64.

Next we recall how the structure of a t-near-cone behaves w.r.t. algebraic shifting. For it, first let us set a bit of notation. Let $j \in [n]$. For a subset $T \subseteq [n]$ we denote by $T + j$ the set $\{t + j \colon t \in T\}$. For a simplicial complex K with vertex set $[n]$ we denote by $K + j$ the simplicial complex $\{T + j \colon T \in K\}$.

Theorem 92. *[66, Corollary 5.6] Let K be a t-near-cone w.r.t. $W = (w_1, \ldots, w_t)$. Then*

$$\Delta(K) = B \cup \bigsqcup_{j=1}^{t} \{\{j\} \sqcup T \colon T \in (\Delta(\mathrm{lk}(w_j, K(j-1))) + j)\},$$

where $\bigsqcup$ denotes disjoint union and $B = \{F \in \Delta(K) \colon F \cap [t] = \emptyset\}$.

In Example 90 we saw that the degree of a vertex can increase after performing algebraic shifting. The next corollary tells us that if we have a near-cone, the number of faces containing the apex vertex is preserved under algebraic shifting. This is of crucial importance when proving Theorem 63 since we want to transfer the upper bound from the shifted simplicial complex back to the original one.

Corollary 93. *Let K be a near-cone with apex vertex w, then*

$$f_k(\mathrm{lk}(w, K)) = f_k(\Delta(\mathrm{lk}(w, K))) = f_k(\mathrm{lk}(1, \Delta(K))).$$

Proof. Since a near-cone with apex vertex w is a 1-near-cone w.r.t. $W = (w)$, by applying Theorem 92 we obtain

$$\Delta(K) = B \sqcup \{\{1\} \sqcup T \colon T \in (\Delta(\mathrm{lk}(w, K)) + 1)\}$$

with $B = \{F \in \Delta(K) \colon 1 \notin F\}$. Then $\mathrm{lk}(1, \Delta(K)) = \Delta(\mathrm{lk}(w, K)) + 1$ and the desired equality of f-vectors follows. □

For Theorems 64 and 65 we need to relate the number of faces of K disjoint from W with those of $\Delta(K)$ disjoint from $[t]$. This is the purpose of the following proposition.

Proposition 94. *Let K be a t-near-cone w.r.t. $W = (w_1, \ldots, w_t)$. Then*

$$f_k(K[N \setminus W]) = f_k(\Delta(K)[[n] \setminus [t]]).$$

Proof. It is enough to verify the equality for the complement; that is, to verify that

$$f_k(K) - f_k(K[N \setminus W]) = |\{T \in K \colon |T| = k+1,\ T \cap W \neq \emptyset\}|$$

coincides with

$$f_k(\Delta(K)) - f_k(\Delta(K)[[n] \setminus [t]]) = |\{T \in \Delta(K) \colon |T| = k+1,\ T \cap [t] \neq \emptyset\}|$$

because $f_k(K) = f_k(\Delta(K))$. Let $T \in K$ be such that $T \cap W = \{w_{i_1}, \ldots w_{i_k}\}$ with $i_1 < \cdots < i_k$, then $T \in K(i_1 - 1) \setminus (\bigcup_{j > i_1} K(j-1))$. To count such faces only once, we order them by the lowest member of W they contain, that is

$$\begin{aligned} |\{T \in K \colon |T| = k+1,\ T \cap W \neq \emptyset\}| &= \sum_{j=1}^{t} |\{T \in K \colon |T| = k+1, \\ &\qquad \min\{j \colon w_j \in T \cap W\} = j\}| \\ &= \sum_{j=1}^{t} f_{k-1}(\mathrm{lk}(w_j, K(j-1))). \end{aligned}$$

Similarly, and using Theorem 92, we have that

$$\begin{aligned}|\{T\in\Delta(K)\colon |T|=k+1,\ T\cap[t]\neq\emptyset\}| &= \sum_{j=1}^{t}|\{T\in\Delta(K)\colon |T|=k+1,\\ &\qquad \min T\cap[t]=j\}|\\ &= \sum_{j=1}^{t} f_{k-1}(\Delta(\mathrm{lk}(w_j,K(j-1))))\\ &= \sum_{j=1}^{t} f_{k-1}(\mathrm{lk}(w_j,K(j-1))),\end{aligned}$$

where the last equality follows from Theorem 89.1. □

Now we will exhibit the interplay between depth and the minimal facet cardinality after performing algebraic shifting.

Theorem 95 ([25])**.** *The minimum facet dimension of $\Delta(K)$ is at least d if and only if $K^{(d)}$ is Cohen-Macaulay.*

On the one hand, this theorem implies that $\mathrm{depth}(K)+1$ coincides with the minimal facet cardinality of $\Delta(K)$. On the other hand, we have seen at the end of subsection 6.2.3 that $\mathrm{depth}(K)+1$ is at most the minimal facet cardinality of K. Summarizing, we have that

$$\text{minimal facet size of } \Delta(K)=\mathrm{depth}(K)+1\leq \text{ minimal facet size of } K.$$

If K is sequentially Cohen-Macaulay with minimal facet cardinality d, then its pure $(d-1)$-skeleton K' is Cohen-Macaulay. By applying algebraic shifting to such skeleton we obtain a Cohen-Macaulay $(d-1)$-dimensional subcomplex $\Delta(K')\subseteq\Delta(K)$. We notice that for every $k\leq(d-1)$ we have that

$$f_k(\Delta(K'))=f_k(K')=f_k(K)=f_k(\Delta(K))$$

and consequently the shifted subcomplex $\Delta(K')$ contains every face of $\Delta(K)$ with dimension at most $(d-1)$. We can conclude that the minimal facet cardinality of $\Delta(K)$ is $d=\mathrm{depth}(K)+1$ since $\Delta(K')$ is pure of dimension $(d-1)$.

6.3 Hilton-Milner type upper bound

In this section we prove Theorems 64 and 67. For it we adapt the injective proof from [36, 44] to the simplicial complex setting. The first step for this is to show that we can always assume that the pairwise-intersecting family $\mathcal{F}$ with the empty intersectiong property, i.e., $\bigcap\mathcal{F}=\emptyset$, is shifted. This is the content of the following lemma.

Lemma 96. *Let K be a $(k+1)$-fold near-cone w.r.t. $W=(w_1,\dots,w_{k+1})$ and $2\leq k\leq\frac{\mathrm{depth}(K)+1}{2}$. Let $\mathcal{F}\subseteq K$ be a pairwise-intersecting family of $(k-1)$-faces such that $\bigcap\mathcal{F}=\emptyset$ and $|\mathcal{F}|$ is maximal. Then, there exists a pairwise-intersecting shifted family of $(k-1)$-faces $\mathcal{F}'\subseteq\Delta(K)$ with $\bigcap\mathcal{F}'=\emptyset$ and $|\mathcal{F}'|=|\mathcal{F}|$.*

Proof. First, because K is a $(k+1)$-fold near-cone w.r.t. W, by Lemma 84, we have that $\text{shift}_{w_i,v}(K) = K$ for every $w_i \in W$ and $v \in N \setminus \{w_1, \ldots, w_i\}$. In this proof we apply combinatorial shift to $\mathcal{F}$ as well as to K in order to preserve the inclusion. Since we only apply combinatorial shift that does not modify K we avoid mentioning at each step that the result of shifting $\mathcal{F}$ is contained in K

Now, we iteratively apply $\text{shift}_{w_1,v}$ to $\mathcal{F}$ for $v \in N \setminus \{w_1\}$ according to Lemma 81[1] while the resulting family satisfies the empty intersection condition in the statement, otherwise we stop. Let us denote the resulting family by $\mathcal{H}$. We proceed by case analysis.

Case 1: $\text{shift}_{w_1,v}(\mathcal{H}) = \mathcal{H}$ **for all** $v \in N$**.** Since $\mathcal{H}$ satisfies the empty intersection condition, there exists $F \in \mathcal{H}$ such that $w_1 \notin F$ and for every $v \in F$ we have that $(F \setminus \{v\}) \cup \{w_1\} \in \mathcal{H}$. That is, $\binom{F \cup \{w_1\}}{k} \subseteq \mathcal{H}$ and consequently $\beta_{k-1}(\langle \mathcal{H} \rangle) > 0$, where by $\langle \mathcal{H} \rangle$ we denote the simplicial complex with facets given by $\mathcal{H}$. Then by Corollary 91 we have that $\binom{[k+1]}{k} \subseteq \Delta(\mathcal{H})$ and we can conclude that $\mathcal{F}' = \Delta(\mathcal{H}) \subseteq \Delta(K)$ is the desired family.

Case 2: there exists $w_s \in W$ **such that** $\bigcap \text{shift}_{w_1,w_s}(\mathcal{H}) \neq \emptyset$**.** Then $F \cap \{w_1, w_s\} \neq \emptyset$ for all $F \in \mathcal{H}$. By maximality of $|\mathcal{H}|$ we have that

$$\mathcal{T} = \{T \cup \{w_1, w_s\} \colon T \in \text{lk}(\{w_1, w_s\}, K), |T| = k-2\} \subseteq \mathcal{H}.$$

Since $\bigcap \mathcal{H} = \emptyset$ there exist $F_1, F_s \in \mathcal{H}$ such that $w_1 \in F_1, w_s \notin F_1$ and $w_s \in F_s, w_s \notin F_s$

Next, we iteratively apply $\text{shift}_{w_j,v}$, with $w_j \in W \setminus \{w_1, w_s\}$ and $v \in N \setminus \{w_1, \ldots, w_j, w_s\}$ to $\mathcal{H}$ until obtaining a stable family as shown in Lemma 82. Denote the resulting family by $\mathcal{G}$. Because $F_1, F_s \in \mathcal{H}$, there are $G_1, G_s \in \mathcal{G}$ such that $w_1 \in G_1, w_s \notin G_1$ and $w_s \in G_s, w_1 \notin G_s$. Since $\text{shift}_{w,v}(\mathcal{G}) = \mathcal{G}$ for w and v in the above range, we have that $W \setminus w_1,\ W \setminus w_s \in \mathcal{G}$. Moreover, we also have that $\text{shift}_{w,v}(\mathcal{T}) = \mathcal{T}$, for the same range of parameters as above since $\text{shift}_{w,v}(K) = K$. Consequently $\binom{W}{k} \subseteq \mathcal{G}$. The conclusion follows as in the previous case.

Case 3: there exist $v \in N \setminus W$ **and** $w_s \in W$ **such that** $\bigcap \text{shift}_{w_1,v}(\mathcal{H}) \neq \emptyset$ **and** $\bigcap \text{shift}_{w_s,v}(\mathcal{H}) = \emptyset$**.** Let $\mathcal{G} = \text{shift}_{w_s,v}(\mathcal{H})$. We claim that every $F \in \mathcal{G}$ intersects $\{w_1, w_s\}$. If $F \in \mathcal{H}$, then either $w_s \in F$, $v \notin F$, or $(F \setminus \{v\}) \cup \{w_s\} \in \mathcal{H}$. In the last two cases it follows that $w_1 \in F$ since $\bigcap \text{shift}_{w_1,v}(\mathcal{H}) \neq \emptyset$. If $F \notin \mathcal{H}$ then $w_s \in F$. We proceed as in the previous case with $\mathcal{G}$ in place of $\mathcal{H}$.

Case 4: there exists $v \in N \setminus W$ **such that** $\bigcap \text{shift}_{w,v}(\mathcal{H}) \neq \emptyset$ **for all** $w \in W$**.** On the one hand, since $\bigcap \mathcal{H} = \emptyset$ there exists $F \in \mathcal{H}$ such that $v \notin F$. Because $F \cap \{w, v\} \neq \emptyset$ for all $w \in W$ we can conclude that $w \in F$ for all $w \in W$. This is a contradiction with the size of F. □

The next step is to build an injection from the shifted pairwise-intersecting family to a proper subcomplex of the link of the apex vertex. For it we first recall the injection used in the unrestricted case, that is if the underlying simplicial complex is the simplex. We will need the following technical lemma.

Lemma 97. *[36, 44] Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be a pairwise-intersecting shifted family. For every $F \in \mathcal{F}$ there exists $l \geq 1$ such that $|F \cap [2l-1]| \geq l$. Moreover, the maximum such $l = l(F)$ satisfies $|F \cap [2l(F)]| = l(F)$.*

[1]We apply the lemma with $W = \{w_1\}$. Here W denotes the vertex set in the statement of Lemma 81.

Let $2k \leq n$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ be a shifted pairwise-intersecting family with $\bigcap \mathcal{F} = \emptyset$. The following function was previously defined by Frankl [36], see also [44].

$$\alpha\colon \mathcal{F} \to \left\{F \in \binom{[n]}{k} : 1 \in F,\ F \cap [2, k+1] \neq \emptyset\right\} \cup \{[2, k+1]\}$$
$$F \mapsto \begin{cases} F & \text{if } 1 \in F \text{ or } [2, k+1] \subseteq F, \\ (F \cup [2l(F)]) \setminus (F \cap [2l(F)]) & \text{otherwise.} \end{cases}$$

In the same article it was shown that α is well defined and the following properties hold.

Lemma 98. *For $F \in \mathcal{F}$ such that $\alpha(F) \neq F$ we have that:*

- $1 \in \alpha(F)$.
- $\alpha(F) \notin \mathcal{F}$.
- $\alpha(F) \cap [2, k+1] \neq \emptyset$.
- *α is injective.*
- $|\alpha(F)| = |F|$.

Next, we show that α still works in the simplicial complex setting. Before stating the lemma, let us recall the notion of the star of a vertex. Let K be a simplicial complex and v be a vertex in K, then the *star of v in K* is the set $\mathrm{st}(v, K) = \{F \in K : v \in F\}$.

Lemma 99. *Let $k \leq d/2$ and K be a shifted simplicial complex with vertex set $[n]$ and minimal facet cardinality d. Let $\mathcal{F} \subseteq K$ be shifted pairwise-intersecting family of $(k-1)$-faces with $\bigcap \mathcal{F} = \emptyset$. Then, for every $F \in \mathcal{F}$ we have that*

$$\alpha(F) \in (\mathrm{st}(1, K) \setminus \mathrm{st}(1, K[[n] \setminus [2, k+1]])) \cup \{[2, k+1]\}.$$

Proof. It is enough to verify that $\alpha(F) \in K$. First, we notice that $d/2 \geq k \geq |F \cap [2l(F)]| = l(F)$. In particular,

$$|F \cup [2l(F)]| = |F| + |[2l(F)]| - |F \cap [2l(F)]| = k + l(F) \leq 2k \leq d.$$

Since K has minimal facet cardinality d there exist a face T with $k + l(F) \leq d$ vertices containing F. Let $T \setminus F = \{t_1 < \cdots < t_{l(F)}\}$ and $[2l(F)] \setminus F = \{s_1 < \cdots < s_{l(F)}\}$. Then, $s_i \leq t_i$ for every $i \in [l(F)]$. Since K is shifted, it is not affected when applying shift_{s_i,t_i} to it. On the other hand, if we apply iteratively shift_{s_i,t_i} to T exactly once for each $i \in [l(F)]$, we will obtain $F \cup [2l(F)]$. From this we can conclude that the later one is in K. Since $F \subseteq F \cup [2l(F)]$, $(F \cup [2l(F)]) \setminus (F \cap [2l(F)]) \subseteq F \cup [2l(F)]$ and $[2, k+1] \in K$ we conclude that $\alpha(F) \in K$. □

Combining Lemma 98 and Lemma 99 we obtain the following corollary.

Corollary 100. *Let $2 \leq k \leq d/2$ and K be a shifted simplicial complex with vertex set $[n]$ and minimal facet cardinality d. Let $\mathcal{F} \subseteq K$ be a shifted pairwise-intersecting family of $(k-1)$-faces with $\bigcap \mathcal{F} = \emptyset$. Then,*

$$\alpha \colon \mathcal{F} \to (\mathrm{st}(1, K) \setminus \mathrm{st}(1, K[[n] \setminus [2, k+1]])) \cup \{[2, k+1]\}$$

is a well defined injection. In particular,

$$|\mathcal{F}| \leq f_{k-2}(\mathrm{lk}(1, K)) - f_{k-2}(\mathrm{lk}(1, K[[n] \setminus [k+1]])) + 1.$$

Proof of Theorem 67. On the one hand, since $\mathrm{lk}(w_1, K)$ is an k-fold near-cone w.r.t. $(w_2, \dots, w_{k+1})$ we have by Corollary 93 and Proposition 94 that

$$\begin{aligned} f_{k-2}(\mathrm{lk}(1, \Delta(K))[[n] \setminus [k+1]]) =& f_{k-2}(\Delta(\mathrm{lk}(w_1, K))[[n] \setminus [k+1]]) \\ &= f_{k-2}(\mathrm{lk}(w_1, K)[N \setminus W]). \end{aligned}$$

On the other hand, since the minimal facet cardinality of $\Delta(K)$ is $\mathrm{depth}(K) + 1$, by Lemma 96 there exists $\mathcal{F}' \subseteq \Delta(K)$ shifted pairwise-intersecting family such that $|\mathcal{F}'| = |\mathcal{F}|$ and $\bigcap \mathcal{F}' = \emptyset$. Then

$$\begin{aligned} |\mathcal{F}| = |\mathcal{F}'| &\leq f_{k-2}(\mathrm{lk}(1, \Delta(K))) - f_{k-2}(\mathrm{lk}(1, \Delta(K))[[n] \setminus [k+1]]) + 1 \\ &= f_{k-2}(\mathrm{lk}(w_1, K)) - f_{k-2}(\mathrm{lk}(w_1, K)[N \setminus W]) + 1, \end{aligned}$$

where we combined Corollary 93 and Corollary 100. □

As a consequence we obtain a proof for Theorem 64.

Proof of Theorem 64. Since K is a sequentially Cohen-Macaulay simplicial complex with minimal facet cardinality d, it follows from the discussion at the end of subsection 6.2.3 that $\mathrm{depth}(K) + 1 = d$. Since $k \leq d/2 = \frac{\mathrm{depth}(K)+1}{2}$, we can conclude the proof by applying Theorem 67. □

6.4 Strict Erdős-Ko-Rado

In this section we prove Theorems 63 and 66. We do this by reducing the general case to the case where the underlying simplicial complex is shifted. The following corollary for shifted simplicial complexes follows from Corollary 100.

Corollary 101. *Let $2 \leq k < d/2$ and K be a shifted simplicial complex with vertex set $[n]$ and minimal facet cardinality d. Let $\mathcal{F} \subseteq K$ be a shifted pairwise-intersecting family of $(k-1)$-faces with $\bigcap \mathcal{F} = \emptyset$. Then, $|\mathcal{F}| < f_{k-2}(\mathrm{lk}(1, K))$.*

The following proof carries on the reduction mentioned above. When this is not possible we restrict the analysis to a facet. Once we have restricted the situation to a facet, we will exhibit a set that is in the star of an apex vertex but not in the pairwise-intersecting family. This will show that the star of the apex vertex is stricly larger.

Proof of Theorem 66. Let $\mathcal{F} \subseteq K$ be the pairwise-intersecting family of maximal size satisfying the empty intersection condition, i.e., $\bigcap \mathcal{F} = \emptyset$. We iteratively apply $\text{shift}_{w,v}$ to $\mathcal{F}$ for $v \in N \setminus \{w\}$ as shown in Lemma 81[2] while the resulting family satisfies the empty intersection condition, otherwise we interrupt the process and output the previous family. Let $\mathcal{H}$ be the resulting family from this process. Since K is a near-cone with apex vertex w we have that $\text{shift}_{w,v}(K) = K$ for every $v \in N$, see subsection 6.2.2, and consequently $\mathcal{H} \subseteq K$. We proceed by case analysis.

Case 1: $\text{shift}_{w,v}(\mathcal{H}) = \mathcal{H}$ for all $v \in N \setminus \{w\}$. Since $\bigcap \mathcal{H} = \emptyset$, there exists $F \in \mathcal{H}$ such that $w \notin F$. Because $\text{shift}_{w,v}(\mathcal{H}) = \mathcal{H}$ for every $v \in N \setminus \{w\}$ we have that $(F \setminus \{v\}) \cup \{w\} \in \mathcal{H}$ for every $v \in F$. That is $\binom{F \cup \{w\}}{k} \subseteq \mathcal{H}$, and consequently $\beta_{k-1}(\langle \mathcal{H} \rangle) > 0$. By Corollary 91 we can conclude that $\binom{[k+1]}{k} \subseteq \Delta(\mathcal{H}) \subseteq \Delta(K)$. This implies that $\Delta(\mathcal{H})$ is a shifted pairwise-intersecting family that satisfies the empty intersection condition and it is contained in a shifted simplicial complex $\Delta(K)$ with minimal facet cardinality $\text{depth}(K) + 1$, see subsection 6.2.3. Since $k < \frac{\text{depth}(K)+1}{2}$, by Corollary 101 and Corollary 93 we have that

$$|\mathcal{F}| = |\mathcal{H}| = |\Delta(\mathcal{H})| < f_{k-2}(\text{lk}(1, \Delta(K))) = f_{k-2}(\text{lk}(w, K)).$$

Case 2: there exist $v \in N \setminus \{w\}$ and $F \in \binom{N \setminus \{w,v\}}{k-1}$ such that $\bigcap \text{shift}_{w,v}(\mathcal{H}) \neq \emptyset$ and $F \cup \{w\},\ F \cup \{v\} \in \mathcal{H}$. On the one hand, since $\bigcap \text{shift}_{w,v}(\mathcal{H}) \neq \emptyset$ we have that $T \cap \{w, v\} \neq \emptyset$ for every $T \in \mathcal{H}$. On the other hand, because $|\mathcal{H}|$ is maximal we have that

$$\{T \cup \{w, v\} \colon T \in \text{lk}(\{w, v\}, K),\ |T| = k - 2\} \subseteq \mathcal{H}.$$

Then $\binom{F \cup \{w,v\}}{k} \subseteq \mathcal{H}$ because $F \cup \{v\}$ is contained in a facet of size strictly lager than k and consequently we can exchange any of the remaining vertices in this facet for w. Since K is a near-cone with apex vertex w, after this exchange we still have a face. The conclusion follows as in the previous case.

Case 3: there exist $v \in N \setminus \{w\}$ such that $\bigcap \text{shift}_{w,v}(\mathcal{H}) \neq \emptyset$ and for every $F \in \binom{N \setminus \{v,w\}}{k-1}$ if $F \cup \{v\} \in \mathcal{H}$ then $F \cup \{w\} \notin \mathcal{H}$. In this case, we define the function

$$\phi \colon \mathcal{H} \to \{T \in \text{lk}(w, K) \colon |T| = k - 1\}$$
$$F \mapsto \begin{cases} F \setminus \{w\} & \text{if } w \in F, \\ F \setminus \{v\} & \text{if } w \notin F. \end{cases}$$

The map ϕ is injective since if $F = F_1 \setminus \{v\} = F_2 \setminus \{w\}$ for some $F_1, F_2 \in \mathcal{H}$, then F contradicts the assumption of this case. We claim that ϕ is not surjective. The theorem will follow from this since $|\mathcal{H}| = |\operatorname{im} \phi| < f_{k-2}(\text{lk}(w, K))$. Let us assume that ϕ is surjective. We split the analysis into two subcases.

Subcase a: there exist $T \in K$ and $F_v, F_w \in \mathcal{H}$ such that $w \in F_w, v \notin F_w, v \in F_v, w \notin F_v$ and $F_v, F_w \subseteq T$. Without loss of generality we can assume that T is a facet and let $d = |T|$. Then

$$H_T(w\bar{v}) = \{F \setminus \{w\} \subseteq T \colon F \in \mathcal{H}, w \in F, v \notin F\}$$

[2] We apply the lemma with $W = \{w\}$. Here W denotes the vertex set in the statement of Lemma 81.

and $H_T(\bar{w}v)$ are a pair of non-empty cross-intersecting families of $(k-2)$-faces in $T \setminus \{w, v\}$ and $|T \setminus \{w, v\}| = d - 2 \geq \operatorname{depth}(K) - 1$. Since $k < \frac{\operatorname{depth}(K)+1}{2}$, then $(k-1) < \frac{\operatorname{depth}(K)-1}{2} \leq \frac{d-2}{2}$ and consequently by Theorem 57 we have that

$$\begin{aligned}|\mathcal{H}_T(w\bar{v})| + |\mathcal{H}_T(\bar{w}v)| &\leq \binom{d-2}{k-1} - \binom{(d-2)-(k-1)}{k-1} + 1 \\ &= \binom{d-2}{k-1} - \binom{d-k-1}{k-1} + 1 < \binom{d-2}{k-1},\end{aligned}$$

where the last inequality follows since $(d-2) > 2(k-1)$ and consequently $\binom{(d-2)-(k-1)}{k-1} > 1$. On the other hand, the total number of $(k-2)$-faces in $T \setminus \{w, v\}$ is $\binom{d-2}{k-1}$. Then, there exists some $F \in \binom{T\setminus\{w,v\}}{k-1} \setminus (\mathcal{H}_T(v\bar{w}) \cup \mathcal{H}_T(w\bar{v}))$ and consequently $F \notin \operatorname{im}\phi$. This is a contradiction with ϕ being surjective.

Subcase b: for every $T \in K$ containing v and w either $\mathcal{H}_T(w\bar{v}) \neq \emptyset$ or $\mathcal{H}_T(\bar{w}v) \neq \emptyset$, but not both simultaneously. Since $\bigcap \mathcal{H} = \emptyset$ there exist $F_w, F_v \in \mathcal{H}$ such that $w \in F_w, v \notin F_w$ and $v \in F_v, w \notin F_v$. Since $k < \frac{\operatorname{depth}(K)+1}{2}$, there exist a facet T containing F_v and w and set $d = |T|$. Since $F_v \subseteq T$ then $\mathcal{H}_T(w\bar{v}) = \emptyset$ and consequently $F_w \nsubseteq T$. Since

$$|(T \setminus \{w, v\}) \setminus (F_w \setminus \{w\})| \geq (d-2) - (k-1) = d - k - 1 \geq \operatorname{depth}(K) - k > k - 1$$

there exists $G \subseteq (T \setminus \{w, v\}) \setminus F_w$ with $|G| = k - 1$. On the one hand, since ϕ is surjective, then $G \in \operatorname{im}\phi$. On the other hand, since $\mathcal{H}_T(w\bar{v}) = \emptyset$, then $G \cup \{v\} \in \mathcal{H}$. But $(G \cup \{v\}) \cap F_w = \emptyset$ which contradicts $\mathcal{H}$ being pairwise-intersecting. □

As a corollary we obtain Theorem 63.

Proof of Theorem 63. Since K is a sequentially Cohen-Macaulay simplicial complex with minimal facet cardinality d, it follows from the discussion at the end of subsection 6.2.3 that $\operatorname{depth}(K) + 1 = d$. Since $k < d/2 = \frac{\operatorname{depth}(K)+1}{2}$, we can conclude the proof by applying Theorem 66. □

6.5 Cross-intersecting families

In this section we extend Theorem 57 to simplicial complexes. First we show that Theorem 65 holds for shifted simplicial complexes. In particular we give a different proof from the one by Borg [17].

Proposition 102. *Let $k \leq d/2$. Let K be a shifted simplicial complex with vertex set $[n]$ and minimal facet cardinality d. Let $\mathcal{F}, \mathcal{G} \subseteq K$ be a pair of non-empty cross-intersecting families of $(k-1)$-faces, then*

$$|\mathcal{F}| + |\mathcal{G}| \leq f_{k-1}(K) - f_{k-1}(K[[n] \setminus [k]]) + 1.$$

Proof. By Theorem 89 we have that $\Delta(\mathcal{F})$, $\Delta(\mathcal{G}) \subseteq \Delta(K) = K$ are a pair of non-empty shifted cross-intersecting families. We proceed by induction on n and k using that the claim holds if the underlying simplicial complex is the simplex. Set $\mathcal{F}(n) = \{F \setminus \{n\} \colon F \in \Delta(\mathcal{F}), n \in F\}$ and $\mathcal{F}(\bar{n}) = \{F \in \Delta(\mathcal{F}) \colon n \notin F\}$.

We define analogously $\mathcal{G}(n)$ and $\mathcal{G}(\bar{n})$. First we claim that $\mathcal{F}(n)$ and $\mathcal{G}(n)$ are cross-intersecting families of $(k-2)$-faces contained in $\mathrm{lk}(n,K)$. If not, then there exist $F \in \Delta(\mathcal{F})$ and $G \in \Delta(\mathcal{G})$ such that $F \cap G = \{n\}$. Since $2k \leq d$ there exist a facet $T \in K$ strictly containing F. Since $|F \cup G| < 2k \leq d$, then there exist $i \in T \setminus (F \cup G)$. Because $\Delta(\mathcal{F})$ is shifted we have that $(F \setminus \{n\}) \cup \{i\} \in \Delta(\mathcal{F})$. But, the face $(F \setminus \{n\}) \cup \{i\}$ is disjoint from G, which contradicts the cross-intersecting assumption. Then $\mathcal{F}(n)$ and $\mathcal{G}(n)$ are cross-intersecting families of $(k-2)$-faces in $\mathrm{lk}(n,K)$. By induction we can conclude that

$$|\mathcal{F}(n)| + |\mathcal{G}(n)| \leq f_{k-2}(\mathrm{lk}(n,K)) - f_{k-2}(\mathrm{lk}(n,K)[[n-1] \setminus [k-1]]) + 1.$$

Since $\mathcal{F}(\bar{n})$ and $\mathcal{G}(\bar{n})$ are cross-intersecting families of $(k-1)$-faces in $\mathrm{del}(n,K)$ we can conclude by induction that

$$|\mathcal{F}(\bar{n})| + |\mathcal{G}(\bar{n})| \leq f_{k-1}(\mathrm{del}(n,K)) - f_{k-1}(\mathrm{del}(n,K)[[n-1] \setminus [k]]) + 1.$$

The conclusion will follow from the fact that $f_{k-2}(\mathrm{lk}(n,K)) + f_{k-1}(\mathrm{del}(n,K)) = f_{k-1}(K)$ and the following inequality

$$f_{k-1}(K[[n]\setminus[k]]) \leq f_{k-2}(\mathrm{lk}(n,K)[[n-1]\setminus[k-1]]) + f_{k-1}(\mathrm{del}(n,K)[[n-1]\setminus[k]]) - 1.$$

To verify this last inequality consider a $(k-1)$-face $F \in K[[n] \setminus [k]]$, if $n \in F$, then

$$F \setminus \{n\} \in \mathrm{lk}(n,K)[[n-1] \setminus [k]] \subseteq \mathrm{lk}(n,K)[[n-1] \setminus [k-1]]. \tag{6.1}$$

If $n \notin F$, then $F \in \mathrm{del}(n,K)[[n-1] \setminus [k]]$. Moreover, the right hand side of (6.1) also contains the $(k-2)$-face $[k, 2k-2] \in \mathrm{lk}(n,K)[[n-1]\setminus[k-1]]$. This is because $[k, 2k-2] \subseteq [d-1] \cup \{n\} \in K$ since K is shifted with minimal facet cardinality d. Substracting this face gives us the desired inequality. □

Proof of Theorem 68. By Theorem 89 we know that $\Delta(\mathcal{F})$ and $\Delta(\mathcal{G})$ are a pair of non-empty cross-intersecting families of $(k-1)$-faces in $\Delta(K)$. This last one is a shifted simplicial complex with minimal facet cardinality $\mathrm{depth}(K) + 1$. By Proposition 102 we have that

$$\begin{aligned} |\mathcal{F}| + |\mathcal{G}| &= |\Delta(\mathcal{F})| + |\Delta(\mathcal{G})| \\ &\leq f_{k-1}(\Delta(K)) - f_{k-1}(\Delta(K)[[n] \setminus [r]]) + 1 \\ &= f_{k-1}(K) - f_{k-1}(K[N \setminus W]) + 1, \end{aligned}$$

where the last equality follows from Proposition 94. □

As an application we obtain a proof of Theorem 65.

Proof of Theorem 65. Since K is a sequentially Cohen-Macaulay simplicial complex with minimal facet cardinality d, it follows from the discussion at the end of subsection 6.2.3 that $\mathrm{depth}(K) + 1 = d$. Since $k \leq d/2 = \frac{\mathrm{depth}(K)+1}{2}$, we can conclude the proof by applying Theorem 68. □

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