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I am writing to evaluate Denys Bulavka's doctoral thesis "*Algebraic tools in Combinatorial Geometry and Topology*", submitted in 2023 to the doctoral programme at Charles University, Prague.

Beyond the well-written introductory part, the book consists of four, independent yet interconnected chapters, numbered from 3 to 6:

(a) Chapter 3 is on WEAK SATURATION OF HYPERGRAPHS. The main result of this chapter is the exact determination of the weak saturation number of complete d-partite q-uniform hypergraphs. This generalizes a theorem of Alon from 1985, which corresponds to the $d = q$ subcase. This is already a very impressive achievement, and so is the algebraic technique ("left interior products" and exterior algebras) used to prove it. This chapter is joint work with Mykhaylo Tyomkyn and Martin Tancer, the thesis advisor; the results already appeared in the prestigious journal *Combinatorica* last year.

(b) Chapter 4 establishes OPTIMAL BOUNDS FOR THE FRACTIONAL COLORFUL HELLY THEOREM in convex geometry. A foreword: In 1913 Helly proved the famous fact that if you have $n \geq d + 1$ convex subsets in $\mathbb{R}^d$ with the property that any $d + 1$ of these n sets have a point in common, then there is a point common to all n sets. After a century of improvements, strengthenings, and generalizations, we have arrived to the "fractional colorful Helly-theorem", proven in 2014 by Barany, Fodor, Montejano, Oliveros, and Por, and later sharpened by Minki Kim. It reads: "for every $\alpha \in (0, 1]$ and every non-negative integer d, there is a $\beta \in (0, 1]$ with the following property. Let $F_1, \ldots, F_{d+1}$ nonempty families of convex sets in $\mathbb{R}^d$ of sizes $n_1, \ldots, n_{d+1}$, respectively. If at least $\alpha n_1 \cdots n_{d+1}$ of the colorful $(d + 1)$-tuples have a nonempty intersection, then there is an $i \in \{1, \ldots, d + 1\}$ such that F_i contains a subfamily of size at least βn_i with a nonempty intersection". What is novel in this Chapter 4 is the determination of the best possible bound for β, as a function of α and d. This proves a conjecture by Kim. This chapter is joint work with Afshin Goodarzi and again Martin Tancer, the thesis advisor. The results were accepted in the well-known discrete-geometry conference 'SoCG'.

(c) Chapter 5 is on ALGEBRAIC SHIFTING AND VOLUME RIGIDITY. Asimow and Roth developed in the Seventies the notion of *rigidity* of linear embedding of graphs; later this notion has been extended to volume rigidity of linear embeddings of d-dimensional simplicial complexes, in some Euclidean space of dimension *at least* d. The main result is a very elegant characterization of the generic volume rigidity of d-dimensional simplicial complexes in Euclidean space of dimension *exactly* d, via the algebraic notion of exterior algebraic shifting. In addition, it is shown that triangulations of several 2-dimensional surfaces are volume-rigid. It is conjectured that all triangulated surfaces and all triangulated d-spheres become volume-rigid after the removal of (the interior of) a single facet. I find both conjectures fascinating. I should also emphasize that the authors en passant correct a Proposition by Streinu and Theran, which turns out to be a misstatement.
These results, technical and impressive, are joint work with Eran Nevo and Yuval Peled, and were disseminated as a poster presentation at the renowned international conference FPSAC, in UC Davis, 2023.

(d) Chapter 6 is on a version of Erdös–Ko–Rado theorem for certain simplicial complexes. The largest possible family of pairwise-intersecting sets is given by all subsets containing a fixed element. This is a famous theorem by Erdös, Ko, and Rado. The novelty of this chapter is to show that the theorem above remains true if the pairwise-intersecting family is restricted to be a sequentially Cohen-Macaulay near-cone. This part is joint work with Russ Woodroofe. As I understand, some of the results in this chapter have been already submitted and accepted to a conference, while some others have never appeared. However, to the best of my knowledge, the results of this chapter appear to me correct, novel, non-trivial, and interesting. The main technique used is combinatorial shifting and ingenious case analysis.

Summing up, I am positively impressed from the novelty, depth, and variety of Denys Bulavka' results. While reading the findings, which are in my opinion relevant and interesting for our scientific community, I learned several novel methods and approaches as well. Finally, some of the conjectures left open are intriguing — and being able to ask good questions is often as important in research as being able to find the answers. In conclusion, I **strongly recommend** awarding Denys Bulavka the doctoral degree.

Sincerely yours,

Bruno Benedetti