THESIS REPORT

This report is for the Doctoral Thesis of Denys Bulavka for the Department of Applied Mathematics, Faculty of Mathematics and Physics, Charles University.

Thesis title: Algebraic Tools in Combinatorial Geometry and Topology

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Overview. The thesis tackles a variety of combinatorial problems motivated by graph theory and geometry through the use of linear algebra techniques, in particular, the *exterior algebra*. The use of linear algebra techniques in combinatorics is a well-established method, typically where the problem of counting the size of a combinatorial object is transformed into the problem of determining the dimension of an associated vector space. In this thesis, these methods are extended to more general combinatorial settings, thereby developing novel tools which can be applied to current research problems in combinatorics and geometry.

Summary of results. The thesis contains novel results in the areas of

(1) weak-saturation problems for uniform hypergraphs,
(2) face numbers of nerve complexes of convex sets,
(3) volume-rigidity of simplicial complexes, and
(4) Erdös–Ko–Rado properties of simplicial complexes.

Not only are the results interesting in their own right, but the methods of proofs introduce novel techniques which certainly will have an impact on future research in these and related areas.

The first half of the thesis develops the *colorful exterior algebra* which is applied to problems in the areas (1) and (2).

The main result of Chapter 3 determines the minimal number of edges in a weakly $K^q_{(r_1,\dots,r_d)}$-saturated subgraph of $K^q_{(n_1,\dots,n_d)}$. (Here $K^q_{(t_1,\dots,t_d)}$ denotes the complete d-partite q-uniform hypergraph on vertex classes of sizes $t_1, \dots, t_d$.) This is a significant generalization of earlier result of N. Alon (1985), and exhibits the applicability of the colorful exterior algebra.

Chapter 4 is motivated by a recent conjecture of M. Kim (2017) regarding the optimal bounds on colorful fractional Helly theorem from combinatorial convexity. The main result(s) of this chapter completely resolves Kim's conjecture in a very broad sense, and again illustrates the power of the colorful exterior algebra. While the big picture approach of the proof closely follows the earlier work of G. Kalai (1984), there are still many technical difficulties that are dealt with. These results are my personal favorites of the thesis, and I also very much enjoyed the enlightening discussion and conjectures on possible generalizations to d-Leray complexes given in section 4.4.

The second half of the thesis deals with the technique of *exterior algebraic shifting* which is applied to problems in the areas (3) and (4).

The main result of Chapter 5 is a complete characterization of the volume rigidity of a simplicial complex K in terms of the shifted complex $\Delta^p(K)$. Here $\Delta^p(K)$ denotes a particular variant of (exterior) algbraic shifting of a simplicial complex K which is introduced in this thesis. A number of properties of $\Delta^p(K)$ are established, in particular with respect to the volume rigidity matrices of K. I found the results of this chapter to be highly interesting. The main result of the chapter also has a number of appealing corollaries regarding the volume-rigidity of triangulations of the sphere, torus, projective plane, and the Klein bottle.

Chapter 6 is devoted to a conjecture of P. Borg (2009) concerning the Erdös–Ko–Rado property for intersecting families of k-faces of a simplicial complex. The main result of this chapter extends previous work of Woodroofe (2011) and established the strict k-EKR property for simplicial complexes that are sequentially Cohen-Macaulay near-cones. As corollaries one obtains proof of Borg's conjecture for the independence complexes of a number of interesting graph classes. While I found the results of this chapter to be a bit esoteric (for my taste), I was nevertheless fascinated by the proof techniques which uses combinations of combinatorial and algebraic shifting.

Conclusion. The thesis contains new and interesting research in a variety of areas motivated by graph theory and geometry. The results include extensions and generalizations of classical results as well as proofs of recent conjectures. Moreover, the thesis introduces novel techniques and methods which have a strong potential for further application.

In general the thesis is very well-written and organized. My only nitpick would be that I would have preferred that there was a subsection 2.3 which introduces a complete toolbox for the colorful exterior algebra, and a similar subsection 2.4 which deals deals with exterior shifting.

In conclusion, I consider this to be a strong PhD thesis. It is without a shred of doubt that I recommend this for a PhD degree.

Questions to the doctoral candidate. Here are two broad questions/topics related to the research of the thesis. It would be nice if the candidate could express his view on these topics.

- *(On the colorful exterior algebra)* Kalai and Meshulam (2005) replaced the "colors" in the colorful Helly theorem to the setting of a matroid. The case of color classes is the special case of a partition matroid. (See also their recent 2021 article.) Some of your generalizations in Seection 4.3 seem to be dealing with slightly more general partition matroids.
    - Could the colorful exterior algebra be further extended to a "matroidal" exterior algebra ?
    - Less generally, could the colorful exterior algebra at least be extended to some specific type of matroids, other than partition matroid, that could potentially yield interesting applications to weak-saturation problems ?

- *(On the Erdős–Ko-Rado property for simplicial complexes)* Many of the applications of the main results of Chapter 6 (Theorems 63–68, Corollaries 69–77) deal with the independence complexes of *chordal graphs* (with some specified isolated vertices).
  - Since chordal graphs are the 1-skeleta of 1-Leray complexes, this begs the question whether the results and methods of Chapter 6 could be extended to uniform hypergraphs derived from d-Leray complexes ? For instance, the independence complex of the $(d + 1)$-uniform hypergraph consisting of the d-faces of a d-Leray complex (with appropriate isolated vertices etc...)
  - Less generally, could the techniques of the thesis be applied to study the EKR-properties (and in particular Conjecture 62) for nerve complexes of convex sets ?
  - Corollaries 75–77 deal with EKR-properties for independence complexes of disjoint unions of graphs (with the appropriate isolated vertices). Could you discuss the possibility of generalizing these to uniform hypergraphs?