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Weighted inequalities, Limiting real interpolation and Function spaces

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Title: Weighted inequalities, Limiting real interpolation and Function spaces

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Abstract: This thesis is focused on studying limiting interpolation spaces with weight functions of slowly varying type and properties of operators defined on them.

In Paper 1 we establish conditions under which K-spaces in the limiting real interpolation involving slowly varying functions can be described by means of J-spaces and we also solve the reverse problem. Further, we apply our results to obtain density theorems for the corresponding limiting interpolation spaces.

In paper 2 we study the properties of compactness of operators defined on limiting interpolation spaces and derive the quantitative estimates of measure of non-compactness.

In paper 3 we estimate dual spaces of limiting interpolation spaces that involve weight functions of slowly varying type.

Keywords: Banach function spaces, Theory of real interpolation, Weighted inequalities, Slowly-varying functions, K- and J-spaces, Compactness, Measure of non-compactness, Duality

Contents

Introduction

1 Prologue

The theory of the real interpolation, now regarded as an independent branch of mathematics, originated from the classical work of Riesz (1927) [R] and Marcinkiewicz (1939) [M39]. It was only until 1960s, through the work of many famous mathematicians such as Aronszajn [AG], Calderón, and Gagaliardo ([GE60, GE63]), that its importance was realized. The method of real interpolation has many applications in the fields of harmonic analysis, approximation theory, theoretical numerical analysis, geometry of Banach spaces, and functional analysis.
In Section 3, we have provided a brief introduction to slowly varying functions along with a few examples, followed by the introduction of the Lorentz- Karamata spaces. Sections 4, 5, and 6 discuss Papers 1, 2, and 3 respectively.
In section 4, we have defined a few important functionals and spaces that play a pivotal role in this thesis. We have also mentioned two of the main results of Paper 1.
The next section is related to the behaviour of operators on limiting interpolation spaces. In this section, we have defined the measure of non-compactness followed by one of the main results of Paper 2. In Paper 2, we work with a set of quasi-Banach spaces.
In section 6, we study the dual spaces of limiting interpolation spaces. We have mentioned an important assertion that plays a key role in Paper 3, followed by one of the main results of the paper.

2 Notations and Definitions

Let $\mathcal{M}(0,\infty)$ be the set of all Lebesgue-measurable functions on $(0,\infty)$. The set $\mathcal{M}^+(0,\infty)$ is a subset of $\mathcal{M}(0,\infty)$ consisting of all non-negative functions. For $f \in \mathcal{M}(0,\infty)$, we define the *distribution function* μ_f, for $\lambda \geq 0$, as

$$\mu_f(\lambda) = \mu\{x \in (0,\infty); |f(x)| > \lambda\}.$$

The *non-increasing rearrangement* of f is the function $f^* : [0,\infty) \to [0,\infty]$, defined as

$$f^*(t) = \inf\{\lambda; \mu_f(\lambda) \leq t\}.$$

The *maximal function* $f^{**} : (0,\infty) \to [0,\infty)$ of f^* is defined by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\, ds.$$

To know more about these functions, one can refer the book [BS].
Next we define a compatible couple. Given two Banach (or quasi-Banach) spaces, let's say X_0 and X_1, the pair (X_0, X_1) is called a *compatible couple* if there is some Hausdorff topological vector space in which each of these spaces is continuously embedded. Further, the space X is said to be an intermediate space between X_0 and X_1, if X is continuously embedded between $X_0 \cap X_1$ and $X_0 + X_1$, i.e.,

$$X_0 \cap X_1 \hookrightarrow X \hookrightarrow X_0 + X_1.$$

An intermediate space X of a compatible couple (X_0, X_1) is said to be an *interpolation space* for (X_0, X_1) if every admissible operator T (i.e., T maps X_0 into X_0 and X_1 into X_1) maps X onto itself. For example, every rearrangement-invariant Banach function space over a resonant measure space is an interpolation space for (L^1, L^∞) (cf. [BS], p. 106, Theorem 2.2). We also define the set $\mathcal{W}(0,\infty)$ of weights on the interval $(0,\infty)$ as

$$\mathcal{W}(0,\infty) := \{w \in \mathcal{M}^+(0,\infty), w < \infty \text{ a.e. on } (0,\infty)\}.$$

For two non-negative expressions (i.e. functions or functionals) A, B the symbol $A \lesssim B$ (or $A \gtrsim B$) means that $A \leq c\,B$ (or $c\,A \geq B$), where c is a positive constant independent of significant quantities involved in A and B. If $A \lesssim B$ and $A \gtrsim B$, we write $A \approx B$ and say that A and B are equivalent.

3 Slowly Varying Functions

The notion of the slowly varying function, first introduced by J. Karamata, were inspired from the work of Issai Schur and R. Schmidt. J. Karamata (cf. [K30]) defined this function, let's say L, given on the interval $[a, \infty)$, where $a > 0$, as slowly varying if L is a positive continuous function satisfying $\lim\limits_{x \to \infty} \frac{L(tx)}{L(x)} = 1$ for all $t > 0$. Another, but the equivalent definition, was used by Zygmund [ZYG57]. A positive continuous function L, defined on the interval $[a, \infty)$, where $a > 0$, is called a slowly varying if, for any $\delta > 0$, $L(x)x^{\delta}$ is a non-decreasing, and $L(x)x^{-\delta}$ is a non-increasing, function of x for x large enough. We refer to [EE] or [NEV02] for another definitions.

The definition used in this thesis was introduced by Gogattishvili, Opic, and Trebels in the paper [GOT]. They say:

A positive, finite and Lebesgue-measurable function b is *slowly varying* on $(0, \infty)$, and write $b \in SV(0, \infty)$, if, for each $\varepsilon > 0$, $t^{\varepsilon} b(t)$ is equivalent to a non-decreasing function on $(0, \infty)$ and $t^{-\varepsilon} b(t)$ is equivalent to a non-increasing function on $(0, \infty)$. An example of such a function is a broken logarithmic function, denoted by $\ell^{\mathbb{A}}(t)$ for $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$, and defined by

$$\ell^{\mathbb{A}}(t) := \begin{cases} (1 + |\log t|)^{\alpha_0} & \text{if } \ 0 < t \le 1, \\ (1 + |\log t|)^{\alpha_\infty} & \text{if } \ 1 < t < \infty. \end{cases} \tag{1}$$

One can find much theory related to this particular example of slowly varying function in the papers [EO, EOP, CCKU, CS]. Another example of slowly varying functions are powers of iterated logarithms (see [EGO]) and $\exp(|\log t|^{\alpha})$, $\alpha \in (0, 1)$. To know about the various properties of SV functions, one can refer to the papers [GOT, GNO] [ZYG57, Chap 5, p. 186], [BGT, MAR, NEV02].

With the help of slowly varying functions, we define the *Lorentz-Karamata spaces*. Let (R, μ) be a non-atomic σ-finite measure space. For $0 < p,\ q \le \infty$ and $b \in SV(0, \infty)$, the *Lorentz-Karamata space* $L_{p,q;b}(R)$ is formed by all (equivalent classes of) measurable functions f on R such that

$$\|f\|_{L_{p,q;b}} := \|t^{1/p - 1/q} b(t) f^*(t)\|_{q,(0,\infty)} < \infty.$$

These spaces generalise many other important spaces. If $b(t) = 1$ for all $t > 0$, then the Lorentz-Karamata space coincides with the *Lorentz space* $L_{p,q}(R)$. If b is the broken logarithm from (1), then the Lorentz-Karamata space is the *generalised Lorentz-Zygmund space* $L_{p,q}(\log L)_{\mathbb{A}}(R)$ from [OP]. If $b(t) = (1 + |\log t|)^{a}$, $t > 0$, we get *Lorentz-Zygmund space* $L_{p,q}(\log L)_{a}(R)$ (see [BR, BS]). Moreover, if $a = 0$ and $p = q$, we obtain the *Lebesgue space* $L_p(R)$.

4 Paper 1

The introduction of K- and J-functionals by J. Peetre in [CP], [P68] played a key role in the advancement of the real interpolation theory. Note also that similar ideas appeared in the work of E. Gagliardo [GE59, GE60].

Let (X_0, X_1) be a compatible couple. The K-functional is defined, for each $f \in X_0 + X_1$ and for all $t > 0$, as

$$K(f,t) := K(f,t;X_0,X_1) = \inf\{\|f_0\|_{X_0} + t\|f_1\|_{X_1} : f = f_0 + f_1\},$$

where the infimum extends over all representations of $f = f_0 + f_1$ with $f_0 \in X_0$ and $f_1 \in X_1$. The J-functional is defined, for all $f \in X_0 \cap X_1$ and for all $t > 0$, by

$$J(f,t) := J(f,t;X_0,X_1) = \max\{\|f\|_{X_0}, t\|f\|_{X_1}\}.$$

One immediately gets interested in knowing how these functionals would look like for some particular compatible couple. If we consider the couple (L_1, L_∞), then the $K(f,t;L_1,L_\infty) = tf^{**}(t)$ for all $f \in L_1 + L_\infty$ and $t > 0$. This result was then generalized by P. Kree [K67], who found K-functionals for the compatible couples (L_s, L_∞), $(L_{s,\infty}, L_\infty)$ with $s > 0$.
Using the K-functionals, we define *K-spaces*. Let (X_0, X_1) be a compatible couple. Let $0 < \theta < 1$, $1 \le q \le \infty$, and $v \in \mathcal{W}(0,\infty)$. The K- space $(X_0, X_1)_{\theta,q,v;K}$ is defined by

$$(X_0, X_1)_{\theta,q,v;K} := \{f \in X_0 + X_1 : \|f\|_{\theta,q,v;K} < \infty\},$$

where

$$\|f\|_{\theta,q,v;K} := \|t^{-\theta-1/q}v(t)K(f,t)\|_{q,(0,\infty)}.$$

The *J-space* $(X_0, X_1)_{\theta,q,v;J}$ is the set of all those $f \in X_0 + X_1$, for which there exists a Bochner integral representation such that

$$f = \int_0^\infty u(s)\,\frac{ds}{s} \quad \text{(convergence in } X_0 + X_1) \tag{2}$$

where $u : (0,\infty) \to X_0 \cap X_1$ is strongly measurable function and for which the functional

$$\|f\|_{\theta,q,v;J} := \inf \|t^{-\theta-1/q}v(t)J(u(t),t)\|_{q,(0,\infty)}$$

is finite (the infimum is taken over all the representations (2) of f).
If $1 \le q \le \infty$ and X_0, X_1 are Banach spaces, then the K- and J-spaces are also Banach-spaces. In the classical case when $v(t) = 1$ for all $t > 0$, $0 < \theta < 1$, and $1 \le q \le \infty$ the equivalence theorem (see, e.g., [BS, Chapter 5, Theorem 2.8]) shows that the constructions $(X_0, X_1)_{\theta,q,1;K}$ and $(X_0, X_1)_{\theta,q,1;J}$ give the same spaces. If $\theta \in (0,1)$ and $v \in SV(0,\infty)$, then the corresponding space $(X_0, X_1)_{\theta,q,v}$ is a particular case of an interpolation space with a function parameter and, by [Gu78, Theorem 2.2], the equivalence theorem continues to hold. However, some problems in mathematical analysis have motivated the investigation of the real interpolation with the limiting values $\theta = 0$ or $\theta = 1$. If θ takes the limiting value i.e., $\theta \in \{0,1\}$, then, in order for the K- and J-spaces to be meaningful, we need some extra conditions on the parameters and the weight functions (cf., Paper 1, Theorem 2.3 and Theorem 2.4).
Now a natural question arises: Given $\theta \in \{0,1\}, q \in [1,\infty]$ and $v \in SV(0,\infty)$,

can we describe the space $(X_0, X_1)_{\theta,q,v;K}$ as a $(X_0, X_1)_{\theta,q,w;J}$ space with a convenient $w \in SV(0,\infty)$?
If v is of logarithmic form, then the answer is given in [CK] for the case that a pair (X_0, X_1) of Banach spaces X_0 and X_1 is ordered, and in [CS] and [BCFC20] for a general pair (X_0, X_1) of Banach spaces X_0 and X_1.
The aim of this paper is to answer the given question for a general $v \in SV(0,\infty)$. We also study the reverse problem, i.e., we establish conditions on $w \in SV(0,\infty)$ that ensure that the space $(X_0, X_1)_{\theta,q,w;J}$ coincides with some $(X_0, X_1)_{\theta,q,v;K}$ space. In paper 1, we have proved results for the limiting case $\theta = 0$ only. The results for the limiting case $\theta = 1$ would follow, since $(X_0, X_1)_{0,q,v;K} = (X_1, X_0)_{1,q,u;K}$, or $(X_0, X_1)_{0,q,v;J} = (X_1, X_0)_{1,q,u;J}$, where $u(t) = v(1/t)$ for all $t > 0$ (which is a consequence of the fact that $K(f, t; X_0, X_1) = tK(f, t^{-1}; X_1, X_0)$ if $f \in X_0 + X_1$ and $t > 0$, or $J(f, t; X_0, X_1) = tJ(f, t^{-1}; X_1, X_0)$ if $f \in X_0 \cap X_1$ and $t > 0$, and a change of variables).
One of the main results of our paper reads as follows:

Theorem 4.1. *Let (X_0, X_1) be a compatible couple and $1 \le q < \infty$. If $b \in SV(0,\infty)$ satisfies*

$$\int_x^\infty t^{-1} b^q(t)\, dt < \infty \quad \text{for all } x > 0, \qquad \int_0^\infty t^{-1} b^q(t)\, dt = \infty, \tag{3}$$

and $a \in SV(0,\infty)$ is defined by

$$a(x) := b^{-q/q'}(x) \int_x^\infty t^{-1} b^q(t)\, dt \quad \text{for all } x > 0, \tag{4}$$

then there are constants $c_1, c_2 > 0$ such that

$$c_1 \|f\|_{0,q,b;K} \le \|f\|_{0,q,a;J} \le c_2 \|f\|_{0,q,b;K} \quad \text{for all } f \in X_0 + X_1. \tag{5}$$

In particular,

$$(X_0, X_1)_{0,q,b;K} = (X_0, X_1)_{0,q,a;J}. \tag{6}$$

Some important applications of the equivalence theorem includes the density and the duality theorems (cf. Paper 3). For example, Theorem 4.1 can be applied to get the following density result:

Theorem 4.2 (Density theorem)**.** *Let (X_0, X_1) be a compatible couple and $1 \le q < \infty$. If $b \in SV(0,\infty)$ satisfies (3), then the space $X_0 \cap X_1$ is dense in $(X_0, X_1)_{0,q,b;K}$.*

5 Paper 2

In Paper 2 we work with compatible couples of quasi-Banach spaces (unlike Paper 1 and Paper 3). Studying the behaviour of operators on function spaces can be regarded as one of the important applications of the interpolation theory. First of all boundedness of operators but then also other useful properties of operators. For example, techniques used by Davis, Figiel, Johnson and Pelczyński [DFJP] in the proof of their famous factorization theorem for weakly compact operators motivated the investigation of the behaviour of weak compactness under interpolation.

If $T : X \to Y$ is a bounded linear operator between quasi-Banach spaces X and Y, then T is *compact* if it maps every bounded set in X into a set with compact closure in Y. The behaviour under interpolation of compactness have been also deeply studied (see [Cw, CKS, C]). Quantitative estimates in terms of the measure of non-compactness have been established, too.

In this paper we determine estimates for the measure of non-compactness of operators interpolated by the limiting perturbations of the real method involving slowly varying functions. The (ball) *measure of non-compactness* can be defined as follows:

Let A, B be a quasi-Banach spaces and $T \in \mathcal{L}(A, B)$ (i.e., T is a bounded linear operator from A to B). The (ball) *measure of non-compactness* $\beta(T) = \beta(T : A \to B)$ is defined to be the infimum of the set of numbers $\sigma > 0$ for which there is a finite subset $\{z_1, \dots, z_n\} \subseteq B$ such that

$$T(U_A) \subseteq \bigcup_{j=1}^{n} \{z_j + \sigma U_B\}.$$

Here U_A, U_B are the closed unit balls of A and B, respectively. Note that $\beta(T) \leq \|T\|_{A,B}$ and that $\beta(T) = 0$ if and only if T is compact.

Concerning the real method, the first result in this direction is due to Edmunds and Teixeira [TE]. The case of general Banach couples has been studied by Cobos, Fernández-Martínez and Martínez [CMM]. Results for the real method with a function parameter and $0 < \theta < 1$ are due to Cordeiro [C99], Szwedek [S06] and Cobos, Fernández-Cabrera and Martínez [CCM07]. Besides, the case of limiting methods involving logarithms have been considered by Cobos, Fernández-Cabrera and Martínez [CCM12, CCM16] and Besoy and Cobos [BC].

Next, we state one of the important result that plays a key role in this paper: Let $\overline{A} = (A_0, A_1)$ and $\overline{B} = (B_0, B_1)$ be compatible couples of quasi-Banach spaces. Let $T \in \mathcal{L}(\overline{A}, \overline{B})$, i.e., T is a bounded linear operator from $A_0 + A_1$ into $B_0 + B_1$ such that the restrictions $T : A_j \to B_j$ are bounded for $j = 0, 1$ and let $b \in SV(0, \infty)$. Then the restriction

$$T : (A_0, A_1)_{0,q;b} \to (B_0, B_1)_{0,q;b}$$

is also bounded. If M_j is bigger than or equal to the norm of $T : A_j \to B_j$, $j = 0, 1$, then

$$\|T\|_{\bar{A}_{0,q;b}, \bar{B}_{0,q;b}} \leq \begin{cases} M_0 & \text{if } M_1 \leq M_0, \\ cM_0 \bar{b}\left(\frac{M_0}{M_1}\right) & \text{if } M_0 < M_1, \end{cases} \tag{7}$$

where $c > 0$ is a constant depending only on b. One of the main result of this paper states:

Theorem 5.1. *Let* $\overline{A} = (A_0, A_1)$, $\overline{B} = (B_0, B_1)$ *be quasi-Banach couples and let* $T \in \mathcal{L}(\overline{A}, \overline{B})$. *Let* $0 < q \leq \infty$ *and* $b \in SV(0, \infty)$ *satisfying*

$$\|t^{-1/q} b(t)\|_{q,(1,\infty)} < \infty \tag{8}$$

and

$$\begin{cases} (\int_0^1 b(t)^q \, dt/t)^{1/q} = \infty & \text{if } 0 < q < \infty, \\ \lim_{t \to 0} b(t) = \infty & \text{if } \ q = \infty. \end{cases} \tag{9}$$

Then we have

(i) $\beta(T : \overline{A}_{0,q;b} \to \overline{B}_{0,q;b}) = 0$ *if* $\beta(T : A_0 \to B_0) = 0$,

(ii) $\beta(T : \overline{A}_{0,q;b} \to \overline{B}_{0,q;b}) \leq C\beta(T : A_0 \to B_0)$ *if* $0 \leq \beta(T : A_1 \to B_1) < \beta(T : A_0 \to B_0)$,

(iii) $\beta(T : \overline{A}_{0,q;b} \to \overline{B}_{0,q;b}) \leq C \max \Big\{ \beta(T : A_0 \to B_0),$

$$\beta(T : A_0 \to B_0) \overline{b} \left(\frac{\beta(T : A_0 \to B_0)}{\beta(T : A_1 \to B_1)} \right) \Big\}$$

if $0 < \beta(T : A_0 \to B_0) \leq \beta(T : A_1 \to B_1)$.

Here C *is a constant independent of* T.

In section 4 of this paper, as applications we have derived estimates for the measure of non-compactness of operators acting between certain Lorentz-Karamata spaces. In particular, one of our results can be considered as a quantitative extension of a compactness result of Edmunds and Opic [EO] for operators acting between Lorentz-Zygmund spaces.

6 Paper 3

One of the applications of the results from Paper 1 is the description of duals of K- and J-spaces. In paper 3 we found duals of the limiting real interpolation K- and J-spaces $(X_0, X_1)_{0,q,v;K}$ and $(X_0, X_1)_{0,q,v;J}$, where (X_0, X_1) is a compatible couple of Banach spaces, $1 \le q < \infty$, v is a slowly varying function on the interval $(0, \infty)$, and the symbols K and J stand for the Peetre K- and J-functionals.
In the classical case, when $v(t) = 1$ for all $t > 0$, $\theta \in (0, 1)$, and $1 \le q < \infty$, the dual spaces have been described by Lions and Peetre in their fundamental paper [LP] (see also [Li61]).
If $\theta \in (0, 1)$, $q \in [1, \infty)$, and $v \in SV(0, \infty)$, then the description of duals of the given spaces follow from [P86, Theorem 2.4].
If $\theta \in \{0, 1\}, q \in [1, \infty)$, and the weight v is of the logarithmic form, then the duals of the spaces $(X_0, X_1)_{\theta,q,v;K}$ have been determined in [CS].
In Paper 3 we describe the duals of limiting interpolation spaces with $\theta = 0$. Note that the description of duals if $\theta = 1$ follows again from the results with $\theta = 0$.

We also used an important assertion mentioned in [BL, p. 53]: If $X_0 \cap X_1$ is dense in X_0 and X_1, then

$$K(f', t; X_0', X_1') = \sup_{f \in X_0 \cap X_1} \frac{|\langle f', f \rangle|}{J(f, t^{-1}; X_0, X_1)} \text{ for all } f' \in X_0' + X_1' \text{ and } t > 0, \quad (10)$$

and

$$J(f', t; X_0', X_1') = \sup_{f \in X_0 + X_1} \frac{|\langle f', f \rangle|}{K(f, t^{-1}; X_0, X_1)} \text{ for all } f' \in X_0' \cap X_1' \text{ and } t > 0, \quad (11)$$

where $\langle \cdot, \cdot \rangle$ denotes the duality between $X_0 \cap X_1$ and $X_0' + X_1'$ in (10), and between $X_0 + X_1$ and $X_0' \cap X_1'$ in (11).

One of the main result of this paper is the following assertion:

Theorem 6.1 (1. duality theorem for K-spaces and $\theta = 0$). *Let (X_0, X_1) be a compatible couple, $1 \le q < \infty$, and let $X_0 \cap X_1$ be dense in X_0 and X_1. If $b \in SV(0, \infty)$ satisfies*

$$\int_x^\infty t^{-1} b^q(t)\, dt < \infty \quad \textit{for all } x > 0, \qquad \int_0^\infty t^{-1} b^q(t)\, dt = \infty, \quad (12)$$

and $a \in SV(0, \infty)$ is defined by

$$a(x) := b^{-q/q'}(x) \int_x^\infty t^{-1} b^q(t)\, dt \quad \textit{for all } x > 0, \quad (13)$$

then

$$(X_0, X_1)'_{0,q,b;K} = (X_0', X_1')_{0,q',\tilde{b};J} = (X_0', X_1')_{0,q',\tilde{a};K}, \quad (14)$$

where $\tilde{b}(x) := \frac{1}{b(1/x)}$ and $\tilde{a}(x) := \frac{1}{a(1/x)}$ for all $x > 0$.

Bibliography

[AG] N. Aronszajn and E. Gagliardo, Interpolation spaces and Interpolation methods. *Ann. Math. Pura Appl.*, 68 (1965), 51-117.

[BR] C. Benett and K. Rudnick, On Lorentz-Zygmund spaces, *Dissertations Math.*, 175 (1980), 1-72.

[BS] C. Bennett and R. Sharpley, Interpolation of Operators, volume 129 of *Pure and Applied Mathematics*, Academic Press, New York, 1988.

[BL] J. Bergh and J. Löfström, Interpolation spaces. An introduction, volume 223 of *Grundlehren der Mathematischen Wissenschaften*, Springer-Verlag, Berlin-New York, 1976.

[BC] B.F. Besoy and F. Cobos, Logarithmic interpolation methods and measure of non-compactness, *Quart. J. Math*, 71 (2020), 73-95.

[BCFC20] B.F. Besoy, F. Cobos, and L. M. Fernández-Cabrera, On the Structure of a Limit Class of Logarithmic Interpolation Spaces. *Mediterr. J. Math.*, 17 (2020).

[BGT] N. H. Bingham, C. M. Goldie, and J. L. Teugels, Regular variation, volume 27 of *Encyclopedia of Mathematics and Its Applications*, Cambridge University Press, Cambridge, 1989.

[BK] Yu. A. Brudnyĭ, N. Ya. Krugljak, Interpolation functors and interpolation spaces, volume I of *North-Holland Math. Library*, 47. North-Holland Publishing Co., Amsterdam, 1991.

[C] F. Cobos, Interpolation theory and compactness, *Function Spaces, Inequalities and Interpolation*, Edited by J. Lukeš and L. Pick, Paseky, Prague 2009, 31-75.

[CCKU] F. Cobos, L.M. Fernández-Cabrera, T. Kühn, and T. Ullrich, On an extreme class of real interpolation spaces, *J. Funct. Anal.,* 256 (2009), 2321–2366.

[CCM07] F. Cobos, L.M. Fernández-Cabrera, and A. Martínez, Abstract K and J spaces and measure of non-compactness, *Math. Nachr.,* 280 (2007), 1698-1708.

[CCM12] F. Cobos, L.M. Fernández-Cabrera, and A. Martínez, Measure of non-compactness of operators interpolated by limiting real methods, *Operator Theory Advances and Applications*, 219 (2012), 37–54.

[CCM16] F. Cobos, L.M. Fernández-Cabrera, and A. Martínez, Estimates for the spectrum on logarithmic interpolation spaces, *J. Math. Anal. Appl.*, 437 (2016), 292-309.

[CK] F. Cobos and T. Kühn, Equivalence of K- and J-methods for limiting real interpolation spaces, *J. Funct. Anal.*, 261 (2011), 3696–3722.

[CKS] F. Cobos, T. Kühn, and T. Schonbek, One-sided compactness results for Aronszajn-Gagliardo functors, *J. Funct. Anal.*, 106 (1992), 274-313.

[CMM] F. Cobos, P. Fernández-Martínez, and A. Martínez, Interpolation of the measure of non-compactnes by the real method, *Studia Math.*, 135 (1999), 25-38.

[CS] F. Cobos and A. Segurado, Description of logarithmic interpolation spaces by means of the J-functional and applications, *J. Funct. Anal.*, 268 (2015), 2906–2945.

[C99] J.M. Cordeiro, Interpolacíon de Ciertas Clases de Operadores por Métodos Multidimensionales, Ph. D. thesis, *Publicaciones del Depto. de Matemática Aplicada*, Universidad de Vigo, 1999.

[Cw] M. Cwikel, Real and complex interpolation and extrapolation of compact operators, *Duke Math. J.*, 65 (1992), 333-343.

[CP] M. Cwikel and J. Peetre, Abstract K and J spaces. *J. Math. Pures Appl.*, 60 (1981), 1-49.

[DFJP] W.J. Davis, T. Figiel, W.B. Johnson and A. Pelczyński, Factoring weakly compact operators, *J. Funct. Anal.* 17 (1974), 311-327.

[EE] D. E. Edmunds and W. D. Evans, Hardy operators, function spaces and embeddings, *Springer Monographs in Mathematics*, Springer, Berlin, 2004.

[EGO] D. E. Edmunds, P. Gurka, and B. Opic, On embeddings of logarithmic Bessel potential spaces, *J. Funct. Anal.*, 146 (1997), 116–150.

[EO] W.D. Evans and B. Opic, Real interpolation with logarithmic functors and reiteration, *Canad. J. Math.*, 52 (2000), 920-960.

[EOP] W. D. Evans, B. Opic, and L. Pick, Real interpolation with logarithmic functors, *J. Inequal. Appl.*, 7 (2002), 187–269.

[GE59] E Gagliardo, Interpolation d'espaces de Banach et applications. III., *C. R. Acad. Sci. Paris*, 248 (1959), 3517–3518.

[GE60] E. Gagliardo, A unified structure in various families of function spaces. Compactness and closure theorems., *Proc. Internat. Sympos. Linear Spaces*, Jerusalem Academic Press, Jerusalem, (1961), 237–241.

[GE63] E. Gagliardo, A common structure in various families of functional spaces. II. Quasilinear interpolation spaces, *Ricerche Mat.*, 12 (1963), 87–107.

[Gu78] J. Gustavsson, A function parameter in connection with interpolation of Banach spaces, *Math. Scand.*, 42 (1978), 289-305.

[GNO] A. Gogatishvili, J.S. Neves, and B. Opic, Optimal embeddings of Bessel-potential-type spaces into generalized Hölder spaces involving k-modulus of smoothness, *Potential Anal.*, 32 (2010), 201–228.

[GOT] A. Gogatishvili, B. Opic, W. Trebels, Limiting reiteration for real interpolation with slowly varying functions, *Math. Nachr.*, 278 (2005), 86–107.

[K30] J. Karamata, Sur un mode de croissance reaguilieare des fontions, *Mathematica (Cluj)*, 4 (1930), 38–53.

[K67] P. Krée, Interpolation d'espaces vectoriels qui ne sont ni normés, ni complets. Applications., *Ann. Inst. Fourier (Grenoble)*, 17 (1967), 137–174.

[Li61] J.L. Lions, Sur les espaces d'interpolation: dualité, *Math. Scand.*, 9 (1961), 147–177.

[LP] J.L. Lions and J. Peetre, Sur une classe déspaces dínterpolation, *Inst. Hautes Études Sci. Publ. Math.*, 19 (1964), 5–68.

[KPS] S.G. Kreĭn, Yu. Ī. Petunīn, E.M. Semënov, Interpolation of linear operators, *American Mathematical Society*, Providence, RI, 1982.

[NEV02] J.S. Neves, Lorentz-Karamata spaces, Bessel and Riesz potentials and embeddings, *Dissertationes Math. (Rozprawy Mat.)*, 405, 2002.

[MAR] V. Marić, Regular Variation and Differential Equations, volume 1726 of *Lecture Notes in Mathematics,* Springer-Verlag, Berlin, 2000.

[M39] J. Marcinkiewicz, Sur Iínterpolation dópkrateurs, *C.R. Acad. Sci. Paris*, 208 (1939), 1272-1273.

[OP] B. Opic and L. Pick, On generalised Lorentz-Zygmund spaces, *Math. Inequal. Appl.*, 2 (1999), 391-467.

[P68] J. Peetre, A theory of interpolation of normed spaces, Notas Mat., *Instituto de Matemática Pura e Aplicada*, Conselho Nacional de Pesquisas, Rio de Janeiro, 1968.

[P86] L.E. Persson., Interpolation with a parameter function, *Math. Scand.*, 59 (1986), 199-222.

[R] M. Riesz, Sur les maxima des formes bilinéaires et sur les fonctionnelles linéaires, *Acta Math*, 49 (1927), 465–497.

[S06] R. Szwedek, Measure of non-compactness of operators interpolated by the real method, *Studia Math*, 175 (2006), 157-174.

[TE] M.F. Teixeira and D.E. Edmunds, Interpolation theory and measures of non-compactness, *Math. Nachr.*, 104 (1981), 129-135.

[ZYG57] A. Zygmund, Trigonometric Series, volume I. *Cambridge University Press,* Cambridge, 1957.

Publications used in the thesis

[1] B. Opic and M. Grover, Description of K-spaces by means of J-spaces and the reverse problem in the limiting real interpolation, *Math. Nachr.*, 296 (2023), 4002-4031.
Doi: 10.1002/mana.202100545

[2] F. Cobos, L.M. Fernández-Cabrera and M. Grover, Measure of non-compactness and limiting interpolation with slowly varying functions (*accepted* by *Banach Journal of Mathematical Analysis*).

[3] M. Grover, B. Opic, Duality of limiting interpolation spaces (*to appear*).