Report on
Doctoral Thesis:
Combinatorics, group theory, computational complexity & topology
by Michael Skotnica

This thesis explores three topics related to topological combinatorics, from the computational perspective.

The first topic is the relation between shellability and collapsibility of simplicial complexes. Shellability is a decomposition property of simplicial complexes extensively studied since 1980's in combinatorics related to topological methods. A recent breakthrough in the computational complexity of shellability is the NP-completeness proof of the problem of determining shellability of a given simplicial complex by Goaoc, Paták, Patáková, Tancer, and Wagner (2019), which has been a long-standing open problem. Their approach involves reducing the problem of shellability to a criterion related to collapsibility, as presented in Theorem 2.1 (by Hachimori(2008)). The first part of this thesis is about this theorem.

Theorem 2.1 is a theorem for 2-dimensional simplicial complexes, and it contains the equivalence between (ii) the second barycentric subdivision of a 2-dimensional simplicial complex $\mathbf{K}$ is shellable, and (iii) the link of each vertex of $\mathbf{K}$ is connected and $\mathbf{K}$ becomes collapsible after removing $\widetilde{\chi}(\mathbf{K})$ triangles, where $\widetilde{\chi}$ is the reduced Euler characteristic. Though this equivalence can not be generalized to higher dimensions, the result of Chapter 3 of this thesis extends the implication (iii) $\Rightarrow$ (ii) to higher dimensions. To this aim, a higher-dimensional criterion named hereditary removal-collapsibility (HRC) is introduced to replace the link condition in (iii). Another crucial aspect of the result is the introduction of star-decomposability, which clarifies the proof.

The second topic is the PL gemometric category, a PL version of the geometric category. For a topological space, the geometric category of the space is the minimum number of open contractible subspaces needed to cover the space, which is a kind of relaxation of Lusternik-Schnirelmann category. Determining the geometric category, however, is a very hard problem since to determine whether the geometric category is 1 or not is an undecidable problem if the space is a simplicial complex of dimension at least 4. To make it tractable, Borghini (2020) introduced PL geometric category of a compact polyhedron P, plgcat(P), which is a PL version of geometric category. The PL geometric category plgcat(P) is the minimum number of PL collapsible subpolyhedra needed to cover P.

The result of Chapter 3 of this thesis shows that the decision of whether plgcat$(P) \leq 2$ or not is NP-hard. The method for the proof of this result is to use the reduction in the proof of NP-hardness of shellability used by Goaoc, Paták, Patáková, Tancer, and Wagner (2019). By slightly modifying the simplicial complex $\mathbf{K}_\phi$ corresponding to a given 3-CNF ϕ into $\mathbf{K}_\phi^+$, it is shown that plgcat$(\mathbf{K}_\phi^+) \leq 2$ if and only if ϕ is satisfiable.

The third result focuses on the problem called VEST (Vector evaluated after a sequence of transformations). VEST is a problem of computing $M_k = \#\{(i_1, \ldots, i_k) \in \{1, \ldots, m\}^k : ST_{i_k} \ldots T_{i_1}\mathbf{v} = \mathbf{0}\}$, given a list of m $d \times d$ matrices $(T_1, \ldots, T_m)$, a d-dim vector $\mathbf{v}$, and a $h \times d$ matrix S. This problem is related to the computation of the rank of the homotopy group π_k $(k \geq 2)$ and the hardness of VEST implies the hardness of the computation of the homotopy group π_k. The discussion of computational hardness here is from the parametrized complexity theory. Previously, this problem of computing M_k is shown to be #W[1]-hard by Matoušek (2013) when parametrized by k. The result of Chapter 4 of this thesis contains showing that the problem is #W[2]-hard, strengthening the previously known result. Further, this cahpter discusses several variations of VEST and their parametrized complexity.

These three topics are all discussed from the perspective of the computational complexity study of topological combinatorics. Topological combinatorics is a rather new area developed after 1980's. Although computational aspects of the study have been an important topic in topological combinatorics, they were not so widely developed until recently. After NP-completeness result of shellability by Goaoc, Paták, Patáková, Tancer, and Wagner (2019), computational study of this area is now one of hot topics. The results of this thesis can contribute to the next development of new studies.

The first result, which relates shellability and collapsibility in higher dimensional simplicial complexes is not a direct contribution in the computational complexity theory. However, as is explained in Section 2.1, it has the potential to be used for showing hardness of shellability from collapsibility. The result of Theorem 2.2 itself is noteworthy. The statement looks rather plausible, but introducting the condition of (HRC) as the higher-dimensional generalization of the prerequisite condition of Theorem 2.1(iii) is a nontrivial contribution. Furthermore, the proof is more complex than one might expect. The proof given in this thesis organizes it well by introducing the concept of star-decomposability.

The second result is the NP-hardness of PL geometric category. This is an interesting result from several aspects. The result itself is a new computational hardness result for a property of interest in topological combinatorics. The method is showing a new way of the use of the gadget $\mathbf{K}_\phi$, linking 3SAT to simplicial complexes via collapsibility. This may contribute to the further development of computational complexity theory in topological combinatorics. The third result is from the theory of parametric complexity. This is a novel and interesting contribution and highlighting the importance of the perspective of the parmetric complexity theory to the researchers in this area.

Overall, the thesis makes several new contributions to the area of topological combinatorics from the perspective of computational complexity theory. It also has a broader impact that contributes to the further wide development of computational complexity theory in topological combinatorics. This thesis shows the outstanding ability of the author to contribute to the field, and it is strongly recommended to grant the Ph.D degree.

Masahiro Hachimori, Ph.D
University of Tsukuba
Faculty of Engineering,
Information and Systems

The following is a question I'd like to ask the author.

- Other than these three results discussed in this thesis, what kind of topics will be the next topic to be discussed in topological combinatorics from the computational complexity perspective?