FACULTY OF MATHEMATICS AND PHYSICS
Charles University

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Dalimil Peša

Fine properties of certain specific function spaces

Department of Mathematical Analysis

Supervisor of the doctoral thesis: prof. RNDr. Luboš Pick, CSc., DSc.
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Title: Fine properties of certain specific function spaces

Author: Dalimil Peša

Department: Department of Mathematical Analysis

Supervisor: prof. RNDr. Luboš Pick, CSc., DSc., Department of Mathematical Analysis

Abstract: This thesis is focused on studying the properties of function spaces from three distinct angles: abstract classes of function spaces, one particular class of function spaces, and some specific applications of the function space theory. It contains six papers; two for each of the above mentioned topics.

The first paper studies the properties of quasi-Banach function spaces. We prove several results that provide useful tools for working with concrete examples of spaces belonging to this class.

The second paper studies the so-called Wiener–Luxemburg amalgam spaces, an abstract framework that allows for constructing a space where the conditions on local and global behaviour of its functions is prescribed separately. We describe the fundamental properties of such constructed spaces and develop tools for working with them.

The third paper provides a thorough and comprehensive treatment of Lorentz–Karamata spaces. We consider a wide variety of topics (e.g. normability, absolute continuity of the (quasi)norm, associate spaces) and investigate each of them extensively.

The fourth paper is focused on proving the result, that for every slowly varying function b there exists an equivalent (hence also slowly varying) function c which has continuous classical derivatives of all orders.

The fifth paper is dedicated to proving a so-called reduction principle, which states that operators of a certain form are bounded if and only if their restriction to the cone of non-increasing functions is bounded.

In the sixth paper we develop a new strongly non-linear version of the Gagliardo–Nirenberg inequality which we then use to obtain a priori estimates for several examples of non-linear PDEs. The paper also contains several variants of the main result and other applications.

Keywords: Banach function spaces, rearrangement-invariant spaces, amalgam spaces, quasi-Banach function spaces, Gagliardo–Nirenberg inequality, reduction principle, Lorentz–Karamata spaces, slowly varying functions

Contents

Introduction

1. Prologue

Function spaces have long been studied, because they appear most naturally in many different settings, e.g. harmonic analysis or theory of partial differential equations, and proper understanding of their properties often leads to significant advances in the related fields. It is therefore only natural that the study of function spaces constitutes a significant part of the field of functional analysis. This thesis attempts to contribute to this process by studying the properties of function spaces from three distinct angles: studying abstract classes of function spaces, studying a concrete class of function spaces, and developing some specific applications of the function space theory. It consists of six papers, a note containing some improvements to two of the results contained in one of the papers, and this Introduction, where we provide some basic context for the papers and a brief overview of their content.

This Introduction is divided into three sections, each of them corresponding to one of the main topics and covering the two relevant papers. In Section 2 we present Papers (I) and (II), which concern themselves, respectively, with the properties of (r.i.) quasi-Banach function spaces and Wiener–Luxemburg amalgams constructed upon them. This section is distinguished by the fact that the studied objects are abstract in nature, defined via axioms or an abstract construction. Paper (II) is further complemented by Note (II') which contains improvements to two of its results that have been developed after publication. In Section 3 we redirect our focus to the class of Lorentz–Karamata spaces, a specific class of function spaces that are studied in great detail in Paper (III). The second paper of the section, Paper (IV), provides a positive answer to the question whether slowly varying functions can be a priori assumed to be smooth; this problem is closely related to the study of Lorentz–Karamata spaces, as it appeared naturally during our work on characterising their corresponding endpoint spaces and associate spaces in some limiting cases. Finally, the focus of Section 4 is on applications of the theory of function spaces. The Papers (V) and (VI) cover very distinct topics; both are however related to problems from the theory of Sobolev spaces.

To conclude this section, let us briefly discuss the relationship of some of the papers to the previous theses by the author.

- Some of the results of Paper (I) have been included in the author's Master thesis, as they were necessary prerequisites for the work done there. However, the submitted version of the paper contains several more results, including the most involved ones concerning separability and especially the boundedness of dilation operator, which is arguably the most interesting result of the paper.

- Paper (II) is based on the principal part of the author's Master thesis, but compared to it it contains significant additions and improvements. The most important one is the description of associate spaces, in which the proof in the thesis required an additional assumption that greatly weakened the result. Proving this description without the extra assumption is the most involved part of the paper and it made the theory significantly stronger and more useful, as one of the consequences is that the Wiener–Luxemburg amalgams of r.i. Banach function spaces are always normable. Yet another essential difference from the thesis is that the paper also covers the amalgams of quasi-Banach function spaces and includes many new results in this context.

- Paper (III) has roots in the author's Bachelor thesis, but it has since been revised in most significant ways: the very definition of slowly varying functions has been changed to the modern and more general version presented below as Definition 3.1 (see Section 3 for discussion of the various definitions of slowly varying functions that appear in literature; the definition presented there as (D1) has been used in the thesis), many of the proofs have been changed to accommodate the modified definition, the gaps that were present in many of the results (e.g. characterisation of embeddings and normability and description of associate spaces) have been filled, and many entirely new topics have been added.

Notation

Let us now establish some basic notation that is common through the entirety of this Introduction (other notation and definitions will appear later, as necessary).

By (R, μ) we denote some arbitrary (totally) sigma-finite measure space. Given a μ-measurable set $E \subseteq R$ we will denote its characteristic function by χ_E. By $\mathcal{M}(R, \mu)$ we will denote the set of all extended complex-valued μ-measurable functions defined on R. As is customary, we will identify functions that coincide μ-almost everywhere. We will further denote by $\mathcal{M}_+(R, \mu)$ the subset of $\mathcal{M}(R, \mu)$ containing the non-negative functions.

When there is no risk of confusing the reader, we will abbreviate μ-almost everywhere, $\mathcal{M}(R, \mu)$, and $\mathcal{M}_+(R, \mu)$ to μ-a.e., $\mathcal{M}$, and $\mathcal{M}_+$, respectively.

When X is a set and $f, g : X \to \mathbb{C}$ are two maps satisfying that there is some positive and finite constant C, depending only on f and g, such that $|f(x)| \leq C|g(x)|$ for all $x \in X$, we will denote this by $f \lesssim g$. We will also write $f \approx g$, or sometimes say that f and g are equivalent, whenever both $f \lesssim g$ and $g \lesssim f$ are true at the same time.

When X, Y are two topological linear spaces, we will denote by $Y \hookrightarrow X$ that $Y \subseteq X$ and that the identity mapping $I : Y \to X$ is continuous.

As for some special cases, we will denote by λ^n the classical n-dimensional Lebesgue measure, with the exception of the 1-dimensional case in which we will simply write λ. We will further denote by m the counting measure over $\mathbb{N}$. When $p \in (0, \infty]$ we will denote by L^p the classical Lebesgue space (of functions in $\mathcal{M}(R, \mu)$) defined by

$$L^p = \left\{ f \in \mathcal{M}(R, \mu);\ \int_R |f|^p \, d\mu < \infty \right\}$$

equipped with the customary (quasi-)norm

$$\|f\|_p = \left(\int_R |f|^p \, d\mu \right)^{\frac{1}{p}}$$

with the usual modifications when $p = \infty$. In the special case when $(R, \mu) = (\mathbb{N}, m)$ we will denote this space by l^p.

Note that in this Introduction we consider 0 to be an element of $\mathbb{N}$.

2. Function spaces in the abstract

The first two papers deal with function spaces from an abstract point of view, in the sense that they study classes of spaces defined either via axioms (in the case of Paper (I)) of via an abstract construction (in the case of Paper (II)), in contrast to concrete spaces equipped with specific norms.

Axiomatic approach to function spaces

The starting point of our work are the standard Banach function spaces, which were developed as a useful abstract framework by taking several significant properties of Lebesgue spaces and using them as the defining axioms for the norm that will then induce the space itself:

Definition 2.1. Let $\|\cdot\| : \mathcal{M} \to [0, \infty]$ be some non-negative functional such that it holds for all $f \in \mathcal{M}$ that $\|f\| = \| \, |f| \, \|$. We then say that $\|\cdot\|$ is a Banach function norm if its restriction to $\mathcal{M}_+$ satisfies the following conditions:

(P1) $\|\cdot\|$ is a norm, i.e.

- (a) it is positively homogeneous, i.e. $\forall a \in \mathbb{C} \; \forall f \in \mathcal{M}_+ : \|a \cdot f\| = |a| \|f\|$,
- (b) it satisfies $\|f\| = 0 \Leftrightarrow f = 0$ μ-a.e.,
- (c) it is subadditive, i.e. $\forall f, g \in \mathcal{M}_+ : \|f + g\| \leq \|f\| + \|g\|$.

(P2) $\|\cdot\|$ has the lattice property, i.e. if some $f, g \in \mathcal{M}_+$ satisfy $f \leq g$ μ-a.e., then also $\|f\| \leq \|g\|$.

(P3) $\|\cdot\|$ has the Fatou property, i.e. if some $f_n, f \in \mathcal{M}_+$ satisfy $f_n \uparrow f$ μ-a.e., then also $\|f_n\| \uparrow \|f\|$.

(P4) $\|\chi_E\| < \infty$ for all $E \subseteq R$ satisfying $\mu(E) < \infty$.

(P5) For every $E \subseteq R$ satisfying $\mu(E) < \infty$ there exists some finite constant C_E, dependent only on E, such that for all $f \in \mathcal{M}_+$ the inequality $\int_E f \, d\mu \leq C_E \|f\|$ holds.

Furthermore, when $\|\cdot\|$ is a Banach function norm, then the set

$$X = \{f \in \mathcal{M}; \; \|f\| < \infty\}$$

equipped with the norm $\|\cdot\|$ will be called a Banach function space.

The definition above is adapted from [3, Definition 1.1], but the concept itself is older, originating from [28]. We would like to note that while this definition is standard, there are also competing approaches: the Fatou property (P3) is often not required and the properties (P4) and (P5) are sometimes used in a weaker form (e.g. by considering a smaller class of admissible sets E) or even omitted entirely. We shall stick to this definition, as the Fatou condition is not very restrictive and implies many interesting and useful results, while the properties (P4) and (P5) become natural when combined with the property of rearrangement invariance, which we define below, and we are interested mainly in spaces satisfying this extra condition.

Definition 2.2. We say that a Banach function norm $\|\cdot\|$ is rearrangement invariant, abbreviated r.i., if it satisfies the following additional condition:

(P6) If two functions $f, g \in \mathcal{M}_+$ satisfy $f^* = g^*$ then $\|f\| = \|g\|$.

Here, f^* is the non-increasing rearrangement of a function $f \in \mathcal{M}$, which is defined as the generalised inverse of the distribution function, that is, for every $t \in [0, \infty)$ we put

$$f^*(t) = \inf\{s \in [0, \infty);\ \mu_f(s) \leq t\},$$

where the distribution function μ_f of a function $f \in \mathcal{M}$ is defined for $s \in [0, \infty)$ by

$$\mu_f(s) = \mu(\{t \in R;\ |f(t)| > s\}).$$

The class of r.i. Banach function spaces is a wide one. In most of the cases, Lebesgue spaces and their various generalisations (e.g. Lorentz spaces, Orlicz spaces, etc.) belong to this class. However, one does not have to go far to find spaces that do not fit to this class, the notable examples being Lebesgue spaces L^p for $p \in (0, 1)$ and the space $L^{1,\infty}$ which appears naturally in weak-type estimates of many operators. We are thus motivated to consider the following more general class of function spaces:

Definition 2.3. Let $\|\cdot\| : \mathcal{M} \to [0, \infty]$ be some non-negative functional such that it holds for all $f \in \mathcal{M}$ that $\|f\| = \|\,|f|\,\|$. We then say that $\|\cdot\|$ is a quasi-Banach function norm, if it satisfies the axioms (P2), (P3), and (P4) from the definition of r.i. Banach function norm and also a weaker version of (P1), namely

(Q1) $\|\cdot\|$ is a quasinorm, i.e.

- (a) it is positively homogeneous, i.e. $\forall a \in \mathbb{R} \forall f \in \mathcal{M}_+ : \|a \cdot f\| = |a|\|f\|$,
- (b) it satisfies $\|f\| = 0 \Leftrightarrow f = 0$ μ-a.e.,
- (c) it is subadditive up to a constant, i.e. there is some finite constant C such that $\forall f, g \in \mathcal{M}_+ : \|f + g\| \leq C(\|f\| + \|g\|)$.

If $\|\cdot\|$ further satisfies (P6) we say that it is an r.i. quasi-Banach function norm.

Moreover, when $\|\cdot\|$ is an (r.i.) quasi-Banach function norm, then the set

$$X = \{f \in \mathcal{M};\ \|f\| < \infty\}$$

equipped with the norm $\|\cdot\|$ will be called an (r.i.) quasi-Banach function space.

This class now includes all the above mentioned examples (as it includes all non-trivial Lorentz spaces) and more. However, while the theory of (r.i.) Banach function spaces is very well developed, the theory of the more general (r.i.) quasi-Banach function spaces was until recently almost untouched. Even the most basic property of completeness has only been proved in 2016 in the paper [6]. This is the motivation for Paper (I), where we aim to take the first steps in developing a comprehensive theory.

Amalgam approach to function spaces

When examining spaces of functions where the underlying domain is in some sense non-atomic and infinite, it quickly becomes apparent that in many cases the space in fact imposes two separate conditions, one concerning the local behaviour of its functions and the other concerning their global behaviour (global is meant in the sense "behaviour near infinity"). This is evident for Lebesgue spaces (over non-atomic measure space of infinite measure), where the local condition prescribes how fast a blow-up can be while the global condition prescribes how fast the decay must be, and where these conditions are clearly separate; this is well illustrated by the fact that the local condition gets stronger as p increases, while the global condition gets weaker at the same time. This phenomenon is present in all r.i. (quasi-)Banach function spaces, but it is not limited to them. For example, consider some topological space G which is locally compact, non-compact, and without isolated points. Consider further the classical-one point compactification of G (with the added point denoted by ∞). Then the space $C_0(G)$ of continuous functions f satisfying the extra condition that

$$\lim_{s\to\infty} f(s) = 0,$$

prescribes continuity as the local condition, which is totally separate from the global condition of decay.

It turns out that it is often desirable to make the distinctness of the local and global conditions explicit, i.e. to prescribe them separately when specifying a function space. To this end, a framework called Wiener amalgam spaces has been developed during the last century, starting with the work of Wiener in [33] and culminating with the introduction of their most general form by Feichtinger in [13]. As the full definition is rather involved and requires significant preparation, we shall present here only a much simpler definition for the case of Lebesgue spaces:

Definition 2.4. Let $p, q \in [1, \infty]$. Consider the classical Lebesgue spaces L^p of functions belonging to $\mathcal{M}([0,\infty), \lambda)$ and l^q of sequences belonging to $\mathcal{M}(\mathbb{N}, m)$. We then define, for all $f \in \mathcal{M}([0,\infty), \lambda)$, the Wiener norm $\|\cdot\|_{W(L^p,l^q)}$ by

$$\|f\|_{W(L^p,l^q)} = \left(\sum_{n=0}^{\infty} \|f\chi_{[n,n+1)}\|_{L^p}^q \right)^{1/q} \qquad \text{for } q \in [1,\infty),$$

$$\|f\|_{W(L^p,l^q)} = \sup_{n\in\mathbb{N}} \|f\chi_{[n,n+1)}\|_{L^p} \qquad \text{for } q = \infty,$$

and the corresponding Wiener amalgam space (or just Wiener amalgam) by

$$W(L^p, l^q) = \{f \in M([0,\infty), \lambda);\ \|f\|_{W(L^p,l^q))} < \infty\}.$$

Wiener amalgams are very useful and they accumulated a wide range of applications, great reviews of which have been conducted by Fournier and Stewart in [18] and by Feichtinger in [14] and [15]. However, they are not very well compatible with the class of r.i. Banach function spaces; even the amalgams of Lebesgue spaces as defined above (except for the trivial case $p = q$) fail to satisfy the property (P6) and either (P4) (for $p > q$) or (P5) (for $p < q$). This limitation motivates Paper (II), in which we develop an alternative approach that is more compatible with the class of r.i. (quasi-)Banach function spaces.

2.1 Paper (I): On the properties of quasi-Banach function spaces

In this paper we take the first steps in developing a comprehensive theory of quasi-Banach function spaces. We were motivated by the fact, that even though quasi-Banach function spaces appear naturally in many settings, the literature on this topic lacks even some of the most basic results. This is in direct contrast to the case of Banach function spaces, where the theory is very well developed and extremely useful, as it provides many great tools for the study of particular cases of function spaces. This paper thus aims at providing such tools even for the more general quasi-Banach function spaces, with emphasis on those which we needed ourselves in our research.

The main part of the paper (i.e. Section 3.) can be roughly divided into three parts. The first two subsections cover some basic, but very useful, results that translate more or less directly from corresponding results for Banach function spaces. The third subsection then treats separability, where we provide a precise treatment using a direct construction. Finally, in the last subsection, we show that the dilation operator is bounded on any r.i. quasi-Banach function space over $\mathcal{M}(\mathbb{R}^n, \lambda^n)$. This part of the paper is the most involved one, as all the known techniques for Banach function spaces failed and an entirely new approach had to be developed.

2.2 Paper (II): Wiener–Luxemburg amalgam spaces

This paper is dedicated to introducing the so-called Wiener–Luxemburg amalgam spaces and developing their theory. Said spaces are constitute abstract construction which provides a rigorous framework for the following simple idea: having two r.i. quasi-Banach function spaces A and B, construct a space that measures blow-ups in the same way as A and decay in the same way as B, and do so in such a way that the result is, once again, an r.i. quasi-Banach function space.

We were greatly inspired by the more classical Wiener amalgam spaces (which are discussed above), which provide a solution to this task, except that the resulting space is generally not an r.i. quasi-Banach function space. As this behaviour often leads to rather counter-intuitive situations, we decided to develop an alternative construction that would be more compatible with this setting.

The principal tool of our construction is the non-increasing rearrangement (as defined above), which naturally separates the local behaviour of a function (i.e. blow-ups) from its global behaviour (i.e. decay). This allows the construction to be very simple, but it also leads to some significant technical difficulties when working with the resulting space. To be more specific, it is on one hand easy to show that the construction preserves all the axioms of quasi-Banach function spaces and that the local integrability condition (P5) is inherited from the local component, while on the other hand it is not immediately apparent whether a Wiener–Luxemburg amalgam of two r.i. Banach function spaces is normable; this result is in fact proved indirectly through the description of associate spaces of Wiener–Luxemburg amalgams, which is the most involved result of the paper.

Beside the above mentioned results, we also characterise the embeddings of Wiener–Luxemburg amalgams, describe their relation with the spaces $A+B$ and $A \cap B$, introduce a new concept of the so-called integrable associate spaces which we then use to describe the associate space of Wiener–Luxemburg amalgams of quasi-Banach function spaces in the case when the local component has the property (P5) while the global component does not, and provide a much more precise treatment of the classical result that if X is an r.i. Banach function space, then

$$L^1 \cap L^\infty \hookrightarrow X \hookrightarrow L^1 + L^\infty. \tag{2.1}$$

Let us consider the last topic in more detail as it is rather interesting. It turns out that the embeddings (2.1) contain in fact four separate pieces of information: L^1 is both locally largest of r.i. Banach function spaces and globally smallest of them, while L^∞ is in the same setting the locally smallest and the globally largest space. When expanding the setting to the class of r.i. quasi-Banach function spaces, we find out that the two statements concerning L^∞ remain valid, while the local embedding of L^1 into any given r.i. quasi-Banach function spaces X holds if and only if X has the property (P5). The most interesting is the remaining statement, i.e. that L^1 is in the global sense smaller than any given r.i. quasi-Banach function space X. For this we provide only sufficient conditions, the weakest of which is that the statement holds provided that the space X satisfies the Hardy–Littlewood–Pólya principle, i.e. that it holds for any pair of functions $f, g \in \mathcal{M}$ that

$$\left(\forall t \in (0, \infty) : \int_0^t g^* \, d\lambda \leq \int_0^t f^* \, d\lambda \right) \implies \|g\|_X \leq \|f\|_X.$$

Given that there are clearly spaces that are, in the global sense, smaller than L^1 while still having the property (P5) (e.g. the Lorentz space $L^{1,\frac{1}{2}}$), this provides a positive answer to the previously open question whether there are r.i. quasi-Banach spaces with the property (P5) which violate the Hardy–Littlewood–Pólya principle.

We complement this paper with Note (II') where we provide essential improvements to two results of the paper which were developed after its publication, as we believe that these improvements provide better insights and understanding of the matter.

3. Lorentz–Karamata spaces

The second two papers concern themselves with a specific class of spaces, namely the Lorentz–Karamata spaces, and the closely related concept of slowly varying functions. Said spaces were introduced by Edmunds, Kerman, and Pick in [12], and represent a further generalisation of Lorentz–Zygmund spaces that has found a wide range of applications, for example in relation to Sobolev embeddings and interpolation (e.g. [1], [2], [9], [22], and [31]).

Slowly varying functions

The defining distinction of Lorentz–Karamata spaces, in contrast to (generalised) Lorentz–Zygmund spaces, is that instead of being defined using functions of logarithmic or broken-logarithmic type, they employ the so-called slowly varying functions. In our work, we always consider the following modern and general definition:

Definition 3.1. Let $b : (0, \infty) \to (0, \infty)$ be a measurable function. Then b is said to be slowly varying, abbreviated s.v., if for every $\varepsilon > 0$ there exists a non-decreasing function b_ε and a non-increasing function $b_{-\varepsilon}$ such that $t^\varepsilon b(t) \approx b_\varepsilon(t)$ on $(0, \infty)$ and $t^{-\varepsilon} b(t) \approx b_{-\varepsilon}(t)$ on $(0, \infty)$.

We make this distinction because there are three different definitions of slowly varying functions that can be found in the literature. The three defining conditions (for $b \in M((0, \infty), \lambda)$) are:

(D1) For every $\varepsilon > 0$ the function $t^\varepsilon b(t)$ is non-decreasing on some neighbourhoods of zero and infinity while the function $t^{-\varepsilon} b(t)$ is non-increasing on some neighbourhoods of zero and infinity.

(D2) For every $\varepsilon > 0$ there exists a non-decreasing function b_ε and non-increasing function $b_{-\varepsilon}$ such that

$$\lim_{t \to 0} \frac{t^\varepsilon b(t)}{b_\varepsilon(t)} = \lim_{t \to \infty} \frac{t^\varepsilon b(t)}{b_\varepsilon(t)} = \lim_{t \to 0} \frac{t^{-\varepsilon} b(t)}{b_{-\varepsilon}(t)} = \lim_{t \to \infty} \frac{t^{-\varepsilon} b(t)}{b_{-\varepsilon}(t)} = 1.$$

(D3) b satisfies Definition 3.1.

The condition (D1) was used in the original definition of Lorentz–Karamata spaces in [12] and the functions satisfying it are said to belong to the Zygmund class $\mathcal{Z}$ (see [34] for further details). The condition (D2) is the closest to the original definition of slowly varying functions as given by Karamata in [25] and [26]—while it is not the original definition, it is equivalent to it. Functions satisfying this condition are treated thoroughly in [4]. The condition (D3) is the one currently used in most papers concerning Lorentz–Karamata spaces and it first appeared in either [20] or [21] (the idea to use only the equivalence with monotone functions appeared originally in the paper [31], but in this case the function was also required to behave the same way near zero as it does near infinity, so the definition was significantly less general).

Lorentz–Karamata spaces

Definition 3.2. Let $p \in (0,\infty], q \in (0,\infty]$ and let b be an s.v. function. We then define the Lorentz–Karamata functionals $\|\cdot\|_{p,q,b}$ and $\|\cdot\|_{(p,q,b)}$, for $f \in M$, as follows:

$$\|f\|_{p,q,b} = \|t^{\frac{1}{p}-\frac{1}{q}} b(t) f^*(t)\|_q,$$
$$\|f\|_{(p,q,b)} = \|t^{\frac{1}{p}-\frac{1}{q}} b(t) f^{**}(t)\|_q,$$

where $\|\cdot\|_q$ is the classical Lebesgue functional on $(0,\infty)$.

We then define the corresponding Lorentz–Karamata spaces $L^{p,q,b}$ and $L^{(p,q,b)}$ as

$$L^{p,q,b} = \{f \in M;\ \|f\|_{p,q,b} < \infty\},$$
$$L^{(p,q,b)} = \{f \in M;\ \|f\|_{(p,q,b)} < \infty\}.$$

The definition of Lorentz–Karamata spaces is conceptually similar to that of Lorentz and (generalised) Lorentz–Zygmund spaces which preceded them; needless to say, these predecessors are included in the class (among many others). On the other hand, Lorentz–Karamata spaces are themselves included in the wider class of classical Lorentz spaces. The main usefulness of the Lorentz–Karamata spaces lies in the fact, that they are on one hand quite general, while on the other hand their definition is specific enough to make the spaces manageable; in this regard they are similar to their above-mentioned predecessors.

3.1 Paper (III): Lorentz–Karamata spaces

This paper is a comprehensive study of Lorentz–Karamata spaces which provides complete characterisation for many of their fundamental properties. This is in contrast with the existing literature, where the papers are usually focused on applications (e.g. [2] or [20]) and thus contain only partial results necessary for their particular topic of interest. There are some exceptions which go deeper (e.g. [11] or [31]), however those consider a much more restricted definition of s.v. functions (and thus the derived Lorentz–Karamata spaces) than is appropriate, either by requiring that the behaviour near zero be the same as that near infinity or by using the condition (D1).

We cover a wide range of topics, including a complete characterisation of normability, description of the corresponding fundamental functions and endpoint spaces, Boyd indices, embeddings, absolute continuity of the quasinorm, and description of the associate spaces. We would like to stress especially the description of associate spaces in limiting and sublimiting cases (i.e. when either $p \in \{1,\infty\}$ or when $q \in (0,1] \cup \{\infty\}$) which is rather involved and, to our knowledge, only few of the cases have been treated before even when considering the more restrictive definitions of s.v. functions as outlined above.

Throughout the paper, we employ a variety of techniques, opting for direct and elementary proofs when possible, resorting to the powerful abstract tools only when necessary to provide optimal results.

3.2 Paper (IV): On the smoothness of slowly varying functions

This paper is directly motivated by our work in Paper (III), where it turned out that in some endpoint cases the description of the Lorentz endpoint spaces could be made significantly more elegant if the s.v. functions defining those spaces were assumed to be differentiable in the classical sense. The question then naturally arose, whether such an assumption could be made without loss of generality. Paper (IV) provides a positive answer to this question, as its main result shows that for every s.v. function b there is an equivalent (and thus also slowly varying) function c (i.e. we have $c \approx b$) which has continuous classical derivatives of all orders.

We prove our result by using a mollification technique, which we have suitably modified to be compatible with this setting, and new techniques for decomposing and combining s.v. functions. To be more precise, the direct mollification of an s.v. function yields an equivalent (and thus s.v.) function only when the original function has finite positive limit at zero. Hence, an arbitrary s.v. function b has to be first decomposed into two parts, each containing either the information about the behaviour of b near zero or near infinity and with both satisfying this extra assumption. These can than be mollified and combined together to obtain the desired smooth function c. Needless to say, a careful application of the properties of s.v. function is necessary in every step (decomposition, mollification, recombination) to ensure that the function c indeed has the desired properties.

4. Applications of function spaces

The last two papers contain some applications of function spaces to other problems. Both of the papers are focused on problems related to the important class of Sobolev spaces:

Definition 4.1. Let X be a Banach function space over (R, μ), where $\Omega \subseteq \mathbb{R}^n$ is open and connected and μ is absolutely continuous with respect to the Lebesgue measure. Let further $m \in N$. Then the homogeneous and inhomogeneous Sobolev spaces $V^m X(R, \mu)$ and $W^m X(R, \mu)$, respectively, are defined as

$$\begin{aligned} V^m X(\Omega, \mu) = \{ f \in \mathcal{M};\, & f \text{ is } m\text{-times weakly differentiable}, \\ & |\nabla^m f| \in X \}, \\ W^m X(\Omega, \mu) = \{ f \in \mathcal{M};\, & f \text{ is } m\text{-times weakly differentiable}, \\ & \forall k \in \mathbb{N}, k \leq m : |\nabla^k f| \in X \}, \end{aligned}$$

with the natural norms

$$\begin{aligned} \|f\|_{V^m X(\Omega,\mu)} &= \Big\| \, |\nabla^m f| \, \Big\|_X, \\ \|f\|_{W^m X(\Omega,\mu)} &= \sum_{k=0}^{m} \Big\| \, |\nabla^k f| \, \Big\|_X. \end{aligned}$$

When μ is just the restriction of the usual Lebesgue measure λ^n, we will omit it from the notation. If $X = L^p$ then we will simplify the notation of $V^m L^p$ and $W^m L^p$ to $V^{m,p}$ and $W^{m,p}$, respectively.

Sobolev embeddings via isoperimetric inequalities

Since the pioneering work of Maz'ya in [29] and [30], it has been known that the first-order Sobolev embeddings follow from either isoperimetric or isocapacitary inequalities and that they are equivalent under some additional assumptions on the domain. Recently, it has been shown in [10] that (under some mild assumptions on the domain) the same is true for higher-order embeddings. The core step of their result is the so-called reduction principle, which tells us that, given r.i. Banach function spaces X, Y over (Ω, μ) (normalised such that $\mu(\Omega) = 1$), the embedding

$$V^m X(\Omega, \mu) \hookrightarrow Y(\Omega, \mu)$$

follows, if the operator H_I^m, defined for a given $f \in \mathcal{M}((0,1), \lambda)$ by

$$H_I^m f(t) = \frac{1}{(m-1)!} \int_t^1 \frac{f(s)}{I(s)} \left(\int_t^s \frac{1}{I(r)} \, dr \right)^{m-1} ds, \tag{4.1}$$

is bounded from the representation space of X to the representation space of Y, where both representation spaces being considered with respect to the measure μ. The converse implication again requires some additional assumptions on the domain. Here, I is an increasing function that is smaller near zero than the isoperimetric profile of Ω. The operator H_I^m is the result of composing with itself m-times the simpler operator H_I, defined for a given $f \in \mathcal{M}((0,1),\lambda)$ by

$$H_I f(t) = \int_t^1 \frac{f(s)}{I(s)}\, ds,$$

which appears in the previously known reduction principle for the first-order Sobolev embedding (see [9], [12] or [27]). This corresponds naturally to the fact that the higher-order Sobolev embeddings in [10] were obtained through iteration of first-order Sobolev embeddings that were known to be optimal in the class of r.i. Banach function spaces.

It is worth noting, that the formula for the operator H_I^m simplifies for many cases of domains with specific isoperimetric profiles. For example, if Ω is a John domain ($\mathbb{R}^n, n \geq 2$), then its isoperimetric function I_Ω is known to satisfy

$$I_\Omega(s) \approx s^{\frac{1}{n'}}$$

near zero, where $n' = \frac{n}{n-1}$. If we put

$$I_1 = s^{\frac{1}{n'}}, \qquad\qquad I_m = s^{\frac{n-m}{n}},$$

then it holds for every $f \in \mathcal{M}((0,1),\lambda)$ that

$$H_{I_1}^m f(t) \approx H_{I_m} f(t).$$

However, this is not the case for other choices of I (and thus for domains with more complicated isoperimetric profiles); hence the more general operators of the form (4.1) have to be considered. An important example of a setting where this behaviour is manifested are the Gauss–Sobolev embeddings, see e.g. [9].

In Paper (V) we study operators similar to those defined in (4.1), except that we consider them on $\mathcal{M}((0,\infty),\lambda)$ instead of $\mathcal{M}((0,1),\lambda)$.

Gagliardo–Nirenberg inequality

A pivotal result in the theory of Sobolev spaces and their applications is the much celebrated (to put it mildly) Gagliardo–Nirenberg inequality which was first independently obtained by both Gagliardo in [19] and Nirenberg in [32]. The result, in its most general form for the classical Sobolev spaces over $\mathbb{R}^n$, states the following:

Theorem 4.2. *Let $1 \leq q \leq \infty$, $j, k \in \mathbb{N}$ with $k < m$, and either*

$$\begin{cases} r = 1 \\ \frac{k}{m} \leq \theta \leq 1 \end{cases} \qquad \text{or} \qquad \begin{cases} 1 < r < \infty \\ m - k - \frac{n}{r} \in \mathbb{N} \\ \frac{k}{m} \leq \theta < 1 \end{cases}.$$

If we set

$$\frac{1}{p} = \frac{k}{n} + \theta\left(\frac{1}{r} - \frac{m}{n}\right) + \frac{1-\theta}{q},$$

then there exists a constant C, depending only on the parameters listed above, such that it holds for every $u \in L^q(\mathbb{R}^n) \cap W^{m,r}(\mathbb{R}^n)$ that

$$\|\nabla^k u\|_p \leq C \|\nabla^m u\|_r^\theta \|u\|_q^{1-\theta}.$$

We adopt this formulation from the paper [17] where one can also find a complete elementary proof and interesting historical context.

This result, because of its enormous usefulness in the field of partial differential equations, sparked an impressive development which led to a plethora of distinct variants and generalisations of the original inequality. We shall not attempt to provide a summary of these results or an overview of the vast literature, but a good starting point for the reader interested in the topic would probably be the papers [5] and [16], as they cover the state of the art in two of the major directions of research. In Paper (VI), we join this endeavour by proving a certain strongly non-linear version of the inequality on bounded Lipschitz domains.

4.1 Paper (V): Reduction principle for a certain class of kernel type operators

In this paper we prove a reduction principle for Copson-type operators of the form

$$H_I^m f(t) = \frac{1}{(m-1)!} \int_t^\infty \frac{f(s)}{I(s)} \left(\int_t^s \frac{1}{I(r)}\, dr \right)^{m-1} ds,$$

where $m \in \mathbb{N}$ and $I : (0, \infty) \to (0, \infty)$ is a non-decreasing function. We were inspired by [10, Theorem 5.1], where it had been proven that we have for any given pair of Banach function spaces X, Y over the interval $(0, 1)$ that $H_I^m : X \to Y$ if and only if the restriction of H_I^m to the cone of non-increasing function is bounded in the same sense. We show that under some additional assumptions on the function I the result holds also for Banach function spaces over the interval $(0, \infty)$.

The operators of this form are closely related to Sobolev embeddings (with I being some lower bound for the isoperimetric function of the underlying domain), which was the motivation for their study in [10]. The assumptions in question are not too strong and they are satisfied for many of the choices of I that may arise naturally in this context. For example:

(i) if $m = 1$ then the result holds for any non-decreasing I,

(ii) if $I(t) = t^\alpha$ with $\alpha \geq 1$ then the result holds for any $m \in \mathbb{N}$.

Our proof follows roughly the same lines as the original one in [10], with some necessary modifications that address the problems introduced by the unboundedness of the underlying interval. The key ingredients of the proof come from the theory of Banach function spaces; especially the concept of the so-called down-dual space plays a crucial role.

4.2 Paper (VI): Nonlinear Gagliardo–Nirenberg inequality and a priori estimates for nonlinear elliptic eigenvalue problems

This paper is dedicated to obtaining inequalities of the form

$$\int_\Omega |\nabla u(x)|^2 h(u(x))\, dx \le C \int_\Omega \left(\sqrt{|Pu(x)||\mathcal{T}_H(u(x))|}\right)^2 h(u(x))\, dx + \Theta;$$

where $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain, $u \in W^{2,1}_{\mathrm{loc}}(\Omega)$ is non-negative, P is a uniformly elliptic operator in non-divergent form (in general with non-constant coefficients), $\mathcal{T}_H(\cdot)$ is certain transformation of the non-negative continuous function $h(\cdot)$, and Θ is a boundary term which depends on the boundary values of u and ∇u; which holds under some additional assumptions. Several different versions of the inequality are proved; first we prove the comparatively weak version which holds under minimal assumption and then show how different sets of additional assumptions allow for obtaining the stronger versions. In the second half of the paper we then present possible applications of our result, mainly to obtaining a priory estimates for non-linear PDEs but also to some other problems.

The paper follows up on the earlier work done by Kałamajska and Peszek in [23] and [24] and by Choczewski and Kałamajska in [7] and [8], where less general versions were obtained. The notable difference is that our paper considers the operator P with non-constant coefficient which makes the analysis significantly more involved.

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List of publications

(I) A. Nekvinda and D. Peša. On the properties of quasi-Banach function spaces. 2020, arXiv:2004.09435.

(II) D. Peša. Wiener–Luxemburg amalgam spaces. *J. Funct. Anal.*, 282(1):Paper No. 109270, 47, 2022.

(III) D. Peša. Lorentz–Karamata spaces. 2023, arXiv:2006.14455.

(IV) D. Peša. On the smoothness of slowly varying functions. 2023, arXiv:2304.14148.

(V) D. Peša. Reduction principle for a certain class of kernel-type operators. *Math. Nachr.*, 293(4):761–773, 2020.

(VI) A. Kałamajska, D. Peša, and T. Roskovec. Nonlinear Gagliardo–Nirenberg inequality and a priori estimates for nonlinear elliptic eigenvalue problems. 2023, arXiv:2308.00545.

Paper (II) is complemented by the note

(II') D. Peša. Addendum to the paper "Wiener–Luxemburg amalgam spaces".