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Ladislav Drážný

Optimal function spaces in weighted Sobolev embeddings with monomial weight

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Supervisor of the master thesis: RNDr. Zdeněk Mihula, Ph.D.
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Author: Ladislav Drážný

Department: Department of Mathematical Analysis

Supervisor: RNDr. Zdeněk Mihula, Ph.D., Department of Mathematics, Faculty of Electrical Engineering, Czech Technical University in Prague

Abstract: In this thesis we study a weighted Sobolev-type inequality for functions from a certain Sobolev-type space that is built upon a rearrangement-invariant space. Considered rearrangement-invariant spaces are defined on the space $\mathbb{R}^n$ endowed with the measure that is given by a monomial weight. We prove a so-called reduction principle for the Sobolev-type inequality. The reduction principle represents a method of how to characterize the rearrangement-invariant spaces that satisfy the Sobolev-type inequality by means of one-dimensional inequalities. Next, for a fixed domain rearrangement-invariant space, we describe the optimal, i.e. the smallest target rearrangement-invariant space such that the Sobolev-type inequality holds. Finally, we describe some concrete examples. We describe the optimal spaces for Lorentz–Karamata spaces.

Keywords: rearrangement-invariant function spaces, Sobolev embeddings, monomial weights, optimal function spaces, Lorentz–Karamata spaces

Contents

Introduction

Various Sobolev spaces and inequalities have played important roles in mathematics for decades. Their applications include (but are not limited to) analysis of partial differential equations, calculus of variations, or harmonic analysis. In this thesis, we study a certain Sobolev inequality with monomial weights, which has recently become quite fashionable. We will establish, in a sense, an optimal version of the inequality.

One of the most standard Sobolev inequalities can be written in the following way (e.g., [22, Theorem 11.2]). Let $m, n \in \mathbb{N}$, $1 \leq m < n$. Let $p \in [1, n/m)$. There is a finite positive constant C such that

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C \|\nabla^m u\|_{L^p(\mathbb{R}^n)} \quad \text{for every } u \in V_0^m L^p(\mathbb{R}^n). \tag{1}$$

Here $p^* = np/(n - mp)$, $\nabla^m u$ is the vector of all mth order weak derivatives of u, and $V_0^m L^p(\mathbb{R}^n)$ is a Sobolev-type space consisting of m times weakly differentiable functions in $\mathbb{R}^n$ whose mth order (weak) gradients belong to the Lebesgue space $L^p(\mathbb{R}^n)$ and that together with their weak derivatives up to order $m-1$ "vanish at infinity" (see Definition 1.14 for the precise definition). Although Lebesgue spaces and Sobolev spaces built upon them play a prominent role, there are situations when finer scales of function spaces are needed. An example of such a finer scale is the scale of Lorentz spaces $L^{p,q}(\mathbb{R}^n)$. This scale is indeed finer because, on the one hand, for $p \in [1, \infty]$ we have $L^p(\mathbb{R}^n) = L^{p,p}(\mathbb{R}^n)$, and, on the other hand, for $p \in (1, \infty)$ and $1 \leq q_1 < q_2 \leq \infty$, we have $L^{p,q_1}(\mathbb{R}^n) \subsetneq L^{p,q_1}(\mathbb{R}^n)$. The interested reader can find more information on Lorentz spaces in [33, Chapter 8]. Having Lorentz space at our disposal, the classical inequality (1) can be improved. The Lebesgue L^{p^*} norm on the left-hand side of (1) can be replaced by the Lorentz $L^{p^*,p}$ norm ([32]). Since $1 \leq p < p^* < \infty$, the Lorentz norm is indeed essentially stronger. An imminent question is, can the inequality be improved any further? Or is it optimal? It turns out that the Lorentz space $L^{p^*,p}(\mathbb{R}^n)$ is in a sense optimal, but we first need to agree on what we mean by optimal, and on how general function spaces we allow.

What not only the Lebesgue and Lorentz norms but also other common function norms (such as the Orlicz or Lorentz–Zygmund ones) have in common is that they depend only on the measure of level sets. By that we mean that if u and v are two measurable functions such that the measures of the sets $\{x : |u(x)| > \lambda\}$ and $\{x : |v(x)| > \lambda\}$ are the same for every $\lambda > 0$, then their norms are equal. For example, for the Lebesgue norm, this follows from the well-known layer cake representation formula ([23, Theorem 1.13]). Such function spaces belong to the class of so-called rearrangement-invariant function spaces. In this thesis, we will consider function spaces from the class of rearrangement-invariant function space because it is not only quite general (and so it contains function spaces appearing in various delicate situations) but also reasonably pleasant to work with. Roughly speaking, rearrangement-invariant function spaces are usually suitable for measuring integrability, but they are not useful for measuring regularity or oscillation of functions. We can now precisely formulate in what sense the Lorentz space $L^{p^*,p}(\mathbb{R}^n)$ is optimal ([20, 25]). On the one hand, (1) is valid with $L^p(\mathbb{R}^n)$ replaced by $L^{p^*,p}(\mathbb{R}^n)$, and, on the other hand, $L^{p^*,p}(\mathbb{R}^n)$ is the smallest rearrangement-

-invariant function space that $L^p(\mathbb{R}^n)$ can be replaced by (i.e., if $Y(\mathbb{R}^n)$ is a rearrangement-invariant function space such that (1) is valid with $L^p(\mathbb{R}^n)$ replaced by $Y(\mathbb{R}^n)$, then $L^{p^*,p}(\mathbb{R}^n) \subseteq Y(\mathbb{R}^n)$).

In this thesis, we will study the question of optimality in a considerably more general Sobolev inequality in the setting of rearrangement-invariant function spaces. Let $n \in \mathbb{N}$, $n \geq 2$, and $A_1, \dots, A_n \in [0, \infty)$. Set $D = n + A_1 + \cdots + A_n$. Let $m \in \mathbb{N}$, $1 \leq m < D$, and let μ_D be the measure on $\mathbb{R}^n$ whose density with respect to the Lebesgue measure is the monomial weight $x^A = |x_1|^{A_1} \cdots |x_n|^{A_n}$, that is,

$$d\mu_D(x) = x^A dx.$$

Namely, we will study the Sobolev inequality of the form

$$\|u\|_{Y(\mathbb{R}^n,\mu_D)} \leq C \|\nabla^m u\|_{X(\mathbb{R}^n,\mu_D)} \quad \text{for every } u \in V_0^m X(\mathbb{R}^n, \mu_D), \tag{2}$$

where $X(\mathbb{R}^n, \mu_D)$ and $Y(\mathbb{R}^n, \mu_D)$ are rearrangement-invariant function spaces on $\mathbb{R}^n$ endowed with the measure μ_D and $V_0^m X(\mathbb{R}^n, \mu_D)$ is a suitable Sobolev-type space built upon $X(\mathbb{R}^n, \mu_D)$. We will characterize when, for a given $X(\mathbb{R}^n, \mu_D)$, there is a rearrangement-invariant function space $Y(\mathbb{R}^n, \mu_D)$ with which (2) is valid, and we will describe the optimal (i.e., the smallest) such a rearrangement-invariant function space $Y(\mathbb{R}^n, \mu_D)$. We will also provide concrete examples of the optimal rearrangement-invariant function spaces in (2) when $X(\mathbb{R}^n, \mu_D)$ is a Lorentz–Karamata space. Lorentz–Karamata spaces form a wide subclass of rearrangement-invariant function spaces that contains not only Lebegue and Lorentz spaces but also Lorentz–Zygmund spaces and some Orlicz spaces. Sobolev-type inequalities with monomial weights have drawn a lot of attention lately (e.g., [3, 6, 7, 19, 21, 37]). The study of weighted Sobolev inequalities with monomial weights was initially motivated by [5], where the regularity of stable solutions to certain planar reaction-diffusion problems was studied. Noteworthily, arguments based on symmetrization can often be successfully used even though the monomial weight is not radially symmetric (unless $A_1 = \cdots = A_n = 0$). We, too, will exploit this quite surprising feature.

The question of optimal rearrangement-invariant function spaces for a large number of Sobolev inequalities in various settings has been intensively studied for more than two decades (e.g., [1, 8, 10, 11, 12, 13, 14, 16, 20, 25, 26]). Despite that, our main results appear to be new (with the exception of the trivial case $A_1 = \cdots = A_n = 0$, which was studied in [25]), and they answer the question of optimality in an actively developing setting of Sobolev-type inequalities with monomial weights. To achieve that, we will combine and make use of a lot of different techniques developed and improved over time together with results from both classical and contemporary theory of (rearrangement-invariant) function spaces.

This thesis is structured as follows. In Chapter 1 we will recall some aspects of the theory of rearrangement-invariant function spaces and Sobolev spaces built upon rearrangement-invariant function spaces, which will be used in the thesis. We will also introduce properties of the Lorentz–Karamata spaces that we will exploit in the thesis. In Chapter 2 we will prove a reduction principle for the Sobolev inequality (2). The reduction principle represents a method of how to characterize the rearrangement-invariant function spaces $X(\mathbb{R}^n, \mu_D)$ and $Y(\mathbb{R}^n, \mu_D)$ that satisfy (2) by means of inequalities involving just one-dimensional functions. Thus, it reduces the question of what spaces satisfy (2) from n dimensions to the real line. In Chapter 3 we will obtain the characterization of the optimal target space in (2) for a given rearrangement-invariant function space $X(\mathbb{R}^n, \mu_D)$, as it was already mentioned above. We will also describe the optimal space for a Lorentz–Karamata space.

1. Preliminaries

In the whole thesis we use the convention $\frac{1}{\infty} = 0$ and $0 \cdot \infty = 0$. When $E \subseteq (0, \infty)$ is (Lebesgue) measurable, we denote by $\lambda(E)$ its Lebesgue measure.

Let (R, μ) be a σ-finite nonatomic measure space. By $\mathcal{M}(R, \mu)$ we will denote the class of all μ-measurable functions on R whose values lie in $\mathbb{R} \cup \{-\infty, \infty\}$. We will denote the class of all μ-measurable functions on R whose values lie in $[0, \infty]$ by $\mathcal{M}^+(R, \mu)$. And the class of all functions in $\mathcal{M}(R, \mu)$ that are finite μ-almost everywhere in R will be denoted by $\mathcal{M}_0(R, \mu)$.

1.1 Rearrangement-invariant function spaces

Now we introduce rearrangement-invariant Banach function spaces and their basic poperties. The theory that is presented here follows the first three chapters of [2].

Definition 1.1 (Banach function norm)**.** *Let ρ be a real valued nonnegative mapping on $\mathcal{M}^+(R, \mu)$. We say that ρ is a* Banach function norm *if all the following properties are satisfied for all $f, g \in \mathcal{M}^+(R, \mu)$, $\{f_k; k \in \mathbb{N}\} \subseteq \mathcal{M}^+(R, \mu)$, $c \in [0, \infty)$ and $A \subseteq R$ such that A is μ-measurable.*

1. *the* norm axiom*: $\rho(f) = 0$ if and only if $f = 0$ μ-almost everywhere in R, $\rho(cf) = c\rho(f)$, $\rho(f + g) \leq \rho(f) + \rho(g)$;*
2. *the* lattice axiom*: if $g \leq f$ μ-almost everywhere in R, then $\rho(g) \leq \rho(f)$;*
3. *the* Fatou axiom*: if $f_k \uparrow f$ μ-almost everywhere in R, then $\rho(f_k) \uparrow \rho(f)$;*
4. *the* nontriviality axiom*: if $\mu(A) < \infty$, then $\rho(\chi_A) < \infty$;*
5. *the* local embedding in L^1*: if $\mu(A) < \infty$, then*

$$\int_A f \, d\mu \leq K_A \rho(f), \tag{1.1}$$

   where $K_A \geq 0$ is a real constant which may depend on A but which does not depend on f.

Definition 1.2 (Banach function space)**.** *Let ρ be a Banach function norm. The collection of all functions $f \in \mathcal{M}(R, \mu)$ such that $\rho(|f|) < \infty$ is called a* Banach function space*. We will denote it by $X(\rho)$, $X(R, \mu)$ or just by X, depending on what we want to stress.*

As their name suggests, Banach function spaces are Banach spaces. Textbook examples of Banach function spaces are the Lebesgue spaces $L^p(R, \mu)$ for $p \in [1, \infty]$. Every Banach function space contains simple functions (i.e., linear combinations of characteristic functions of μ-measurable sets of finite measure) and is contained in $\mathcal{M}_0(R, \mu)$.

To every Banach function space, there is associated another Banach function space, which is related to its dual space, but which is usually more useful in the theory of Banach function spaces.

Definition 1.3 (Associate norm, associate space)**.** *Let $X(\rho)$ be a Banach function space. We say that the mapping ρ' defined on $\mathcal{M}^+(R,\mu)$ by*

$$\rho'(g) = \sup_{f\in\mathcal{M}^+(R,\mu),\rho(f)\le 1} \int_R fg\,d\mu, \quad g\in\mathcal{M}^+(R,\mu), \tag{1.2}$$

is the associate norm *of the function norm ρ. We say that the space $X(\rho')$ is the associate space to the space $X(\rho)$ and we detone this space by X'.*

For example, when $X = L^p(R,\mu)$ for $p\in[1,\infty]$, then $X' = L^{p'}(R,\mu)$. Here $p'\in[1,\infty]$ is the dual index defined by $\frac{1}{p}+\frac{1}{p'}=1$.

The associate norm of a Banach function norm is a Banach function norm. An important property of Banach function spaces is that, if X is a Banach function space, then

$$(X')' = X. \tag{1.3}$$

For every Banach function norm ρ, *the Hölder inequality*

$$\int_R |fg|\,d\mu \le \rho(f)\rho'(g) \tag{1.4}$$

holds for every $f,g\in\mathcal{M}(R,\mu)$.

Definition 1.4 (Continuous embedding)**.** *Let X,Y be Banach function spaces over (R,μ). We say that $X(R,\mu)$ is* continuously embedded *into $Y(R,\mu)$ if for every function $u\in X(R,\mu)$ it holds that $u\in Y(R,\mu)$ and that $\|u\|_{Y(R,\mu)} \le C\|u\|_{X(R,\mu)}$, where C is a constant that does not depend on u. We denote the fact that $X(R,\mu)$ is continuously embedded into $Y(R,\mu)$ by $X(R,\mu)\hookrightarrow Y(R,\mu)$.*

In fact, inclusion between Banach function spaces is always continuous in the sense that

$$X(R,\mu)\hookrightarrow Y(R,\mu) \quad \text{if and only if} \quad X(R,\mu)\subseteq Y(R,\mu).$$

If $X(R,\mu), Y(R,\mu)$ are Banach function spaces, it holds that

$$X(R,\mu)\hookrightarrow Y(R,\mu) \text{ if and only if } Y'(R,\mu)\hookrightarrow X'(R,\mu). \tag{1.5}$$

Both embeddings hold with the same constant.

Definition 1.5 (Equimeasurable functions)**.** *Let (R,μ) and (S,ν) be σ-finite nonatomic measure spaces. Let $f\in\mathcal{M}(R,\mu)$, $g\in\mathcal{M}(S,\nu)$. We say that the functions f,g are* equimeasurable *if*

$$\mu(\{x\in R; |f(x)|>\lambda\}) = \nu(\{x\in S; |g(x)|>\lambda\})$$

for every $\lambda\ge 0$.

In the rest of this thesis, we will be interested in an important subclass of Banach function spaces whose norms are invariant with respect to certain rearrangements.

Definition 1.6 (Nonincreasing rearrangement)**.** *Let $f \in \mathcal{M}(R,\mu)$. The* nonincreasing rearrangement *of the function f is the function $f^*_\mu\colon (0,\infty) \to [0,\infty]$ defined by*

$$f^*_\mu(t) = \inf_{\lambda>0} \left(\mu(\{x \in R; |f(x)| > \lambda\}) \le t\right) \quad t \in (0,\infty). \tag{1.6}$$

The function f^*_μ is nonincreasing and right-continuous. The functions f and f^*_μ are equimeasurable. If $f \in \mathcal{M}(R,\mu)$, $g \in \mathcal{M}(S,\nu)$ are equimeasurable, then

$$f^*_\mu(t) = g^*_\nu(t), \quad t \in (0,\infty). \tag{1.7}$$

When it is obvious what the measure that the rearrangement is taken with respect to is, we often omit the subscript.

For every $f, g \in \mathcal{M}(R,\mu)$, $f_k \in \mathcal{M}(R,\mu)$, $k \in \mathbb{N}$, and $\alpha \in \mathbb{R}$, we have the following facts. Firstly, if $|f| \le |g|$ μ-almost everywhere in R, then

$$f^*_\mu(t) \le g^*_\mu(t), \quad t \in (0,\infty). \tag{1.8}$$

Secondly,

$$(\alpha f)^*_\mu(t) = |\alpha|\, f^*_\mu(t), \quad t \in (0,\infty). \tag{1.9}$$

Finally, if $|f_k| \uparrow |f|$ μ-almost everywhere in R, then

$$(f_k)^*_\mu(t) \uparrow f^*_\mu(t), \quad t \in (0,\infty). \tag{1.10}$$

Other important properties of the nonincreasing rearrangement are the following. For every $f \in \mathcal{M}(R,\mu)$ and $t \in (0,\infty)$, we have

$$\mu(\{x \in R; |f(x)| > f^*_\mu(t)\}) \le t. \tag{1.11}$$

Furthermore, if $f^*_\mu(t) < \infty$ and $\mu(\{x \in R; |f(x)| > f^*_\mu(t) - \varepsilon\}) < \infty$ for some $\varepsilon > 0$, then

$$\mu(\{x \in R; |f(x)| \ge f^*_\mu(t)\}) \ge t. \tag{1.12}$$

Now, we finally introduce rearrangement-invariant norms and spaces.

Definition 1.7 (Rearrangement-invariant norm and space)**.** *Let $X(\rho)$ be a Banach function space. If it holds that $\rho(f) = \rho(g)$ whenever $f^*_\mu = g^*_\mu$, $f, g \in \mathcal{M}^+(R,\mu)$, then we say that the norm ρ is a* rearrangement-invariant norm *and that the space X is a* rearrangement-invariant space*.*

It follows from the layer cake representation formula that $L^p(R,\mu)$ spaces, $p \in [1,\infty]$, are rearrangement-invariant spaces. When X is a rearrangement-invariant space, so is its associate space. Furthermore, if $g \in \mathcal{M}(R,\mu)$ and if ρ is a rearrangement-invariant norm, we have

$$\rho'(g) = \sup_{f \in \mathcal{M}^+(R,\mu), \rho(f) \le 1} \int_R f^* g^* \, d\mu. \tag{1.13}$$

The Hardy–Littlewood inequality is very important in the theory of rearrangement-invariant spaces. It states that

$$\int_R |fg| \, d\mu \leq \int\limits_0^\infty f^*(t) g^*(t) \, dt$$

for every $f, g \in \mathcal{M}(R, \mu)$. In particular, by taking $g = \chi_E$, we have

$$\int_E |f| \, d\mu \leq \int\limits_0^{\mu(E)} f^*(t) \, dt \tag{1.14}$$

for each μ-measurable set $E \subseteq R$.

Another important result in the theory of rearrangement-invariant spaces is the so-called *Hardy–Littlewood–Pólya principle.* For every rearrangement-invariant norm ρ, if $f, g \in \mathcal{M}^+(R, \mu)$ are such that

$$\int_0^t f^*(\tau) \, d\tau \leq \int_0^t g^*(\tau) \, d\tau$$

for every $t \in (0, \infty)$, then

$$\rho(f) \leq \rho(g). \tag{1.15}$$

We will also need the following fact. For every $t \in (0, \mu(R))$ and for every $f \in \mathcal{M}(R, \mu)$, we have

$$\int_0^t f^*(\tau) \ d\tau = \sup\left(\left\{\int_E |f| \ d\mu; E \subseteq R, E \ \mu\text{-measurable}, \mu(E) = t\right\}\right). \tag{1.16}$$

Each rearrangement-invariant space on (R, μ) can be represented as a rearrangement-invariant space on $(0, \mu(R))$. More precisely, if $X(R, \mu)$ is a rearrangement-invariant space, then there exists a rearrangement-invariant space $X(0, \mu(R))$ such that for every function $f \in X(R, \mu)$ it holds that

$$\|f\|_{X(R,\mu)} = \left\|f^*_\mu\right\|_{X(0,\mu(R))}.$$

The rearrangement-invariant space $X(0, \mu(R))$ is called *the representation space* of $X(R, \mu)$. For example, if $X(R, \mu) = L^p(R, \mu)$, then $X(0, \mu(R)) = L^p(0, \mu(R))$.

On the other hand, for every $f \in \mathcal{M}(0, \mu(R))$, there exists a function $u \in \mathcal{M}(R, \mu)$ such that

$$f^*_\lambda(t) = u^*_\mu(t), \quad t \in (0, \infty).$$

Therefore, for every function $f \in X(0, \mu(R))$ there exists a function $u \in X(R, \mu)$ such that

$$\|f\|_{X(0,\mu(R))} = \|u\|_{X(R,\mu)}. \tag{1.17}$$

Closely related to the nonincreasing rearrangement is *the maximal nonincreasing rearrangement.*

Definition 1.8 (Maximal nonincreasing operator)**.** *The* maximal nonincreasing operator

$$P_\mu \colon \mathcal{M}(R,\mu) \to \mathcal{M}^+(0,\infty)$$

is defined by

$$P_\mu(f)(t) = \frac{1}{t}\int_0^t f_\mu^*(\tau)\,d\tau, \quad f \in \mathcal{M}(R,\mu), t \in (0,\infty). \tag{1.18}$$

The image of a function $f \in \mathcal{M}(R,\mu)$ under the maximal nonincreasing operator P_μ is also commonly denoted by f_μ^{**}, and it is called the maximal nonincreasing function.

Note that if it is clear what the measure that the maximal nonincreasing operator is taken with respect to is, we often write just f^{**} or P for short, analogously as in the case of the nonincreasing rearrangement.

The maximal nonincreasing function is nonincreasing and we have $f^* \le f^{**}$. The maximal noncreasing function also have the following four properties that are similar to the the properties (1.8)–(1.10) of the nonincreasing rearrangement. For every $f, g \in \mathcal{M}(R,\mu)$, $f_k \in \mathcal{M}(R,\mu)$, $k \in \mathbb{N}$, and $\alpha \in \mathbb{R}$, the following facts hold. Firstly, if $|f| \le |g|$ μ-almost everywhere in R, then

$$f_\mu^{**}(t) \le g_\mu^{**}(t), \quad t \in (0,\infty). \tag{1.19}$$

Secondly,

$$(\alpha f)_\mu^{**}(t) = |\alpha|\, f_\mu^{**}(t), \quad t \in (0,\infty), \tag{1.20}$$

Finally, if $|f_k| \uparrow |f|$ μ-almost everywhere in R, then

$$(f_k)_\mu^{**}(t) \uparrow f_\mu^{**}(t), \quad t \in (0,\infty). \tag{1.21}$$

Another important property of the maximal nonincreasing function is the subadditivity. It means that for every $f, g \in \mathcal{M}(R,\mu)$ we have

$$(f+g)_\mu^{**}(t) \le f_\mu^{**}(t) + g_\mu^{**}(t), \quad t \in (0,\infty). \tag{1.22}$$

Note that the nonincreasing rearrangement function is not subadditive.

If $X(0,\mu(R))$ is a rearrangement-invariant space and $h \in \mathcal{M}^+(0,\mu(R))$ is a nonincreasing function, we know thanks to (1.13) that

$$\|h\|_{X'(0,\mu(R))} = \sup_{g \in \mathcal{M}^+(0,\mu(R)), \|g\|_{X(0,\mu(R))} \le 1} \int_0^{\mu(R)} h(t) g^*(t)\,dt.$$

In general, when $h \in \mathcal{M}^+(0,\mu(R))$ is not necessarily nonincreasing, we only have

$$\|h\|_{X'(0,\mu(R))} \ge \sup_{g \in \mathcal{M}^+(0,\mu(R)), \|g\|_{X(0,\mu(R))} \le 1} \int_0^{\mu(R)} h(t) g^*(t)\,dt \tag{1.23}$$

owing to (1.2). However, it follows from [14, Theorem 9.5] and [30, Theorem 3.10] that

$$\|t^\alpha f^{**}(t)\|_{X'(0,\mu(R))} \le 4 \sup_{g \in \mathcal{M}^+(0,\mu(R)), \|g\|_{X(0,\mu(R))} \le 1} \int_0^{\mu(R)} t^\alpha f^{**}(t) g^*(t)\,dt \tag{1.24}$$

for every $\alpha \in [0,1]$ and $f \in \mathcal{M}(R,\mu)$. Inequalities (1.23) and (1.24) mean that the norm of $t^\alpha f^{**}(t)$ can be approached, up to a multiplicative constant, by nonincreasing functions in this case even though the function $t^\alpha f^{**}(t)$ does not have to be nonincreasing.

We conclude this subsection by introducing the dilation operator.

Definition 1.9 (Dilation operator)**.** *Let $\alpha \in (0,\infty)$. The* dilation operator

$$D_\alpha \colon \mathcal{M}^+(0,\infty) \to \mathcal{M}^+(0,\infty)$$

is the operator defined by

$$(D_\alpha f)(t) = f(\alpha t), \quad f \in \mathcal{M}^+(0,\infty),\ t \in (0,\infty).$$

The dilation operator is bounded on every rearrangement-invariant space over $(0,\infty)$. More precisely, there exists a constant $0 < C \le \max\{1, \frac{1}{\alpha}\}$ such that

$$\|D_\alpha f\|_{X(0,\infty)} \le C \|f\|_{X(0,\infty)} \tag{1.25}$$

for every $f \in \mathcal{M}^+(0,\infty)$, every $\alpha \in (0,\infty)$ and every rearrangement-invariant space $X(0,\infty)$.

1.2 Lorentz–Karamata spaces

Now we introduce the theory of Lorentz–Karamata spaces. For proofs and more information see [31]. Lorentz–Karamata spaces contain a lot of classical function spaces, such as the Lebesgue spaces, Lorentz spaces or Lorentz-Zygmund spaces. In this subsection, we assume that $\mu(R) = \infty$.

Definition 1.10 (Equivalent functions)**.** *Let $f, g\colon (0,\infty) \to (0,\infty)$ be functions. We say that the function f is* equivalent *to the function g if there exist constants $C_1 > 0$ and $C_2 > 0$ such that*

$$C_2 g(t) \le f(t) \le C_1 g(t)$$

for every $t \in (0,\infty)$. We denote equivalent functions by

$$f \approx g.$$

Definition 1.11 (Slowly varying function)**.** *Let $b\colon (0,\infty) \to (0,\infty)$ be a continuous function. We say that b is* slowly varying *if for every $\varepsilon > 0$ there exists a nondecreasing function φ_ε and a nonincreasing function $\varphi_{-\varepsilon}$ such that $t^\varepsilon b(t)$ is equivalent to $\varphi_\varepsilon(t)$ on $(0,\infty)$ and that $t^{-\varepsilon} b(t)$ is equivalent to $\varphi_{-\varepsilon}$ on $(0,\infty)$.*

A positive continuous function on $(0,\infty)$ that is equivalent to a positive constant function is a trivial example of a slowly varying function. Functions of logarithmic type constitute less trivial and very important examples of slowly varying functions. For $k \in \mathbb{N}$, the function $\ell_k\colon (0,\infty) \to (0,\infty)$ defined as

$$\ell_k(t) = \begin{cases} 1 + |\log t| & \text{if } k = 1, \\ 1 + \log \ell_{k-1}(t) & \text{if } k > 1, \end{cases}$$

$t \in (0,\infty)$, is slowly varying. More generally, the function $\ell_k^{\mathbb{A}} \colon (0,\infty) \to (0,\infty)$ defined as

$$\ell_k^{\mathbb{A}}(t) = \begin{cases} \ell_k^{\alpha_0}(t) & \text{if } t \in (0,1), \\ \ell_k^{\alpha_\infty}(t) & \text{if } t \in [1,\infty), \end{cases}$$

where $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$, is slowly varying.

We now list some important properties of slowly varying functions. If b_1 and b_2 are slowly varying functions, then so are $b_1 + b_2$ and $b_1 \cdot b_2$. If b is a slowly varying function, then for every $r \in \mathbb{R}$ the function b^r is also slowly varying. If b is a slowly varying function and $\alpha \in (0,\infty)$, then

$$b(t) \approx b(\alpha t) \tag{1.26}$$

on $(0,\infty)$. Furthermore, for every slowly varying function b and for every $\alpha \in (0,\infty)$ we have

$$\int_0^t \tau^{\alpha-1} b(\tau)\,d\tau \approx t^{\alpha} b(t) \tag{1.27}$$

on $(0,\infty)$,

$$\int_t^\infty \tau^{-\alpha-1} b(\tau)\,d\tau \approx t^{-\alpha} b(t) \tag{1.28}$$

on $(0,\infty)$,

$$\sup_{\tau \in (0,t)} \tau^{\alpha} b(\tau) \approx t^{\alpha} b(t) \tag{1.29}$$

on $(0,\infty)$ and

$$\sup_{\tau \in (t,\infty)} \tau^{-\alpha} b(\tau) \approx t^{-\alpha} b(t) \tag{1.30}$$

on $(0,\infty)$.

We now define Lorentz–Karamata spaces.

Definition 1.12 (Lorentz–Karamata space). *Let b be a slowly varying function. Let $p, q \in [1,\infty]$. We define the Lorentz–Karamata functionals by*

$$\|f\|_{p,q,b} = \left\| t^{\frac{1}{p}-\frac{1}{q}} b(t) f_\mu^*(t) \right\|_{L^q(0,\infty)}$$

and by

$$\|f\|_{(p,q,b)} = \left\| t^{\frac{1}{p}-\frac{1}{q}} b(t) f_\mu^{**}(t) \right\|_{L^q(0,\infty)}$$

for every $f \in \mathcal{M}(R,\mu)$. The Lorentz–Karamata spaces *are defined as*

$$L^{p,q,b}(R,\mu) = \left\{ f \in \mathcal{M}(R,\mu); \|f\|_{p,q,b} < \infty \right\}$$

and as

$$L^{(p,q,b)}(R,\mu) = \left\{ f \in \mathcal{M}(R,\mu); \|f\|_{(p,q,b)} < \infty \right\}.$$

If we take $p = q$ and $b \equiv 1$, we obtain the Lebesgue spaces. More generally, we obtain the Lorentz spaces $L^{p,q}(R,\mu)$ and $L^{(p,q)}(R,\mu)$ by taking $b \equiv 1$. Furthermore, Lorentz–Karamata spaces also include the Lorentz–Zygmund spaces, which were thoroughly studied in [29]. We obtain them by taking slowly varying functions of logarithmic type as above.

Even though we refer to Lorentz–Karamata spaces as spaces, they are not always rearrangement-invariant spaces.

The space $L^{(p,q,b)}(R,\mu)$ is a rearrangement-invariant Banach function space if and only if $q \in [1,\infty]$ and one of the following conditions holds:

1. $p \in (1,\infty)$,
2. $p = 1$ and $\left\| t^{-\frac{1}{q}} b(t) \chi_{(1,\infty)}(t) \right\|_{L^q(0,\infty)} < \infty$,
3. $p = \infty$ and $\left\| t^{-\frac{1}{q}} b(t) \chi_{(0,1)}(t) \right\|_{L^q(0,\infty)} < \infty$.

The Lorentz–Karamata functional $\|\cdot\|_{p,q,b}$ is equivalent to a rearrangement-invariant Banach function norm if and only if $q \in [1,\infty]$ and one of the following conditions is satisfied:

1. $p \in (1,\infty)$,
2. $p = q = 1$ and b is equivalent to a nonincreasing function on $(0,\infty)$,
3. $p = \infty$ and $\left\| t^{-\frac{1}{q}} b(t) \chi_{(0,1)}(t) \right\|_{L^q(0,\infty)} < \infty$.

By the equivalence, we mean that there are a rearrangement-invariant function norm ρ and constants $C_1 > 0$ and $C_2 > 0$ such that

$$C_1 \rho(f) \leq \|f\|_{p,q,b} \leq C_2 \rho(f) \quad \text{for every } f \in \mathcal{M}^+(R,\mu). \tag{1.31}$$

Since we will not be interested in exact values of constants, we treat $L^{p,q,b}(R,\mu)$ as a rearrangement-invariant space whenever $q \in [1,\infty]$ and one of these conditions is satisfied.

The spaces $L^{p,q,b}(R,\mu)$ and $L^{(p,q,b)}(R,\mu)$ are closely related to each other. We always have

$$\|f\|_{p,q,b} \leq \|f\|_{(p,q,b)} \quad \text{for every } f \in \mathcal{M}^+(R,\mu).$$

If $p > 1$, then there is a constant $C > 0$ such that

$$\|f\|_{(p,q,b)} \leq C \|f\|_{p,q,b} \quad \text{for every } f \in \mathcal{M}^+(R,\mu).$$

In other words, if $p > 1$, then

$$\|\cdot\|_{p,q,b} \approx \|\cdot\|_{(p,q,b)}. \tag{1.32}$$

Moreover, when either $p \in (1,\infty)$ or $p = \infty$ and $\left\| t^{-\frac{1}{q}} b(t) \chi_{(0,1)}(t) \right\|_{L^q(0,\infty)} < \infty$, we can take $\rho(\cdot) = \|\cdot\|_{(p,q,b)}$ in (1.31).

We end this subsection by describing the associate space of $L^{p,q,b}(R,\mu)$. Recall that for every $p \in [1,\infty]$, the dual index $p' \in [1,\infty]$ is defined by $\frac{1}{p} + \frac{1}{p'} = 1$. It holds that

$$\left(L^{p,q,b}(R,\mu) \right)' = L^{p',q',b^{-1}}(R,\mu) \tag{1.33}$$

if one of the following conditions is satisfied:

1. $p \in (1, \infty)$ and $q \in [1, \infty]$,

2. $p = q = 1$ and b is equivalent to a nonincreasing function on $(0, \infty)$.

By $\left(L^{p,q,b}(R,\mu)\right)' = L^{p',q',b^{-1}}(R,\mu)$, we mean that both embeddings

$$\left(L^{p,q,b}(R,\mu)\right)' \hookrightarrow L^{p',q',b^{-1}}(R,\mu) \quad \text{and} \quad L^{p',q',b^{-1}}(R,\mu) \hookrightarrow \left(L^{p,q,b}(R,\mu)\right)'$$

are true. To avoid confusion, we stress that b^{-1} is the function $1/b$, not the inverse function of b. The case $p = \infty$ is more complicated, but we will not need it.

1.3 Sobolev spaces built on rearrangement-invariant spaces

In this subsection, we define suitable weighted Sobolev spaces built on rearrangement-invariant spaces. We start with some notation and conventions used in the rest of this thesis.

Conventions. *Throughout the rest of this thesis, we assume that $n \in \mathbb{N}$, $n \geq 2$, is the dimension of $\mathbb{R}^n$. We also assume that $A_1, \dots, A_n \in [0, \infty)$ are fixed nonnegative numbers. We set*

$$D = n + A_1 + \cdots + A_n.$$

Finally, we assume that $m \in \mathbb{N}$ is such that

$$1 \leq m < n.$$

Definition 1.13 (Monomial weight and weighted measure μ_D)**.** *For every $x \in \mathbb{R}^n$, we set*

$$x^A = |x_1|^{A_1} \cdots |x_n|^{A_n}.$$

We define the weighted measure μ_D on $\mathbb{R}^n$ as

$$\mu_D(E) = \int_E x^A \, dx$$

for every Lebesgue measurable set $E \subseteq \mathbb{R}^n$.

Note that the measure μ_D is absolutely continuous with respect to the n-dimensional Lebesgue measure λ_n, i.e.,

$$\lambda_n(E) = 0 \Rightarrow \mu_D(E) = 0 \tag{1.34}$$

for every Lebesgue measurable set $E \subseteq \mathbb{R}^n$.

We now defined the Sobolev spaces that we will work with.

Definition 1.14 (Sobolev spaces $V^k X(\mathbb{R}^n, \mu_D)$ and $V_0^k X(\mathbb{R}^n, \mu_D)$)**.** *Let $k \in \mathbb{N}$ and let u be a k-times weakly differentiable function in $\mathbb{R}^n$ (i.e., it has all weak derivatives up to the k-th order). We denote by $\nabla^l u$, $l \in \{1, \dots, k\}$, the vector of all l-th order weak derivatives of u. We also set $\nabla^0 u = u$.*

Let $X(\mathbb{R}^n, \mu_D)$ be a rearrangement-invariant space. We say that u belongs to the space $V^k X(\mathbb{R}^n, \mu_D)$ if

$$\left|\nabla^k u\right| \in X(\mathbb{R}^n, \mu_D).$$

We say that u belongs to the space $V_0^k X(\mathbb{R}^n, \mu_D)$ if $u \in V^k X(\mathbb{R}^n, \mu_D)$ and for every $l \in \{0, 1, \dots, k-1\}$ and for every $\lambda > 0$ it holds that

$$\mu_D\left(\left\{x \in \mathbb{R}^n; \left|\nabla^l u(x)\right| > \lambda\right\}\right) < \infty.$$

For short, we will write $\|\nabla^k u\|_{X(\mathbb{R}^n,\mu_D)}$ instead of $\|\,|\nabla^k u|\,\|_{X(\mathbb{R}^n,\mu_D)}$.

If $u \in V_0^1 X(\mathbb{R}^n, \mu_D)$, then we have

$$u_\mu^*(t) < \infty \tag{1.35}$$

for every $t \in (0, \infty)$ and, furthermore

$$\lim_{t\to\infty} u_\mu^*(t) = 0. \tag{1.36}$$

We will also encounter Sobolev space $W^{1,1}(\mathbb{R}^n, \mu_D)$, which is a weighted counterpart of the classical Sobolev space $W^{1,1}(\mathbb{R}^n)$.

Definition 1.15 (Sobolev space $W^{1,1}(\mathbb{R}^n, \mu_D)$)**.** *We say that a function u belongs to the space $W^{1,1}(\mathbb{R}^n, \mu_D)$ if it is weakly differentiable in $\mathbb{R}^n$, $u \in L^1(\mathbb{R}^n, \mu_D)$ and $|\nabla u| \in L^1(\mathbb{R}^n, \mu_D)$.*

We will need to use an isoperimetric inequality for the weighted measure μ_D.

Convention. *Let $x \in \mathbb{R}^n$ and $r \in (0, \infty)$. We denote*

$$B_r(x) = \{y \in \mathbb{R}^n; |x - y| \leq r\}.$$

Definition 1.16 (Points of density and essential boundary)**.** *Let $t \in [0, 1]$ and let $E \subseteq \mathbb{R}^n$ be a Lebesgue measurable set. We say that E has* density t *in a point $x \in \mathbb{R}^n$ if*

$$\lim_{r\downarrow 0} \frac{\lambda_n(E \cap B_r(x))}{\lambda_n(B_r(x))} = t.$$

We denote the set containing all points where E has density t by E^t. The essential boundary *of E is defined as*

$$\mathbb{R}^n \setminus \left(E^0 \cup E^1\right).$$

We denote the essential boundary of E by $\partial^* E$.

We have $\partial^* E \subseteq \partial E$. Furthermore, the sets E^t are Borel sets for every $t \in [0, 1]$. So $\partial^* E$ is a Borel set.

The isoperimetric inequality that we will need follows from [4, Theorem 1.3] (see also [3, Theorem 1.4], cf. [15, Theorem 1.1]).

Theorem 1.17 (Isoperimetric inequality)**.** *For each Borel set $E \subseteq \mathbb{R}^n$ that satisfies $\mu_D(E) < \infty$ it holds that*

$$P_{\mu_D}(E) = \int_{\partial^* E} x^A \, d\mathcal{H}^{n-1}(x) \geq C_{iso} \mu_D(E)^{\frac{D-1}{D}},$$

where $C_{iso} > 0$ is a constant that depends only on D.

2. Reduction principle

The goal of this chapter is to prove a suitable reduction principle. First of all we derive a variant of the Pólya–Szegő inequality. Our proof is based on the proofs of theorems [9, Lemma 4.1], [10, Lemma 3.3] and [36, Lemma 1.E].

Theorem 2.1 (Pólya–Szegő inequality)**.** *Let X be a rearrangement-invariant space over $(\mathbb{R}^n, \mu_D)$ and $u \in V_0^1 X(\mathbb{R}^n, \mu_D)$. Then $u^*_{\mu_D}$ is a locally absolutely continuous function on the interval $(0, \infty)$, and it holds that*

$$C_{iso} \left\| t^{\frac{D-1}{D}} \frac{du^*_{\mu_D}}{dt}(t) \right\|_{X(0,\infty)} \leq \|\nabla u\|_{X(\mathbb{R}^n,\mu_D)}, \tag{2.1}$$

where C_{iso} is the constant from the isoperimetric inequality (Theorem 1.17).

Proof. Firstly, we prove the theorem for nonnegative u. We start with the proof of the local absolute continuity of the function $u^*_{\mu_D}$. Let $\{(a_m, b_m)\}_{m \in M}$ be a countable system of pairwise disjoint nonempty bounded intervals. For each $m \in M$ define the function $f_m \colon \mathbb{R} \to \mathbb{R}$ in the following way:

$$f_m(y) = \begin{cases} 0 & \text{if } y \leq u^*_{\mu_D}(b_m), \\ y - u^*_{\mu_D}(b_m) & \text{if } u^*_{\mu_D}(b_m) < y < u^*_{\mu_D}(a_m), \\ u^*_{\mu_D}(a_m) - u^*_{\mu_D}(b_m) & \text{if } u^*_{\mu_D}(a_m) \leq y. \end{cases}$$

Note that for every $m \in M$, the function f_m is Lipschitz continuous and nonnegative. For each $m \in M$ we now set $v_m = f_m \circ u$. Choose an arbitrary $m \in M$. The function v_m is well-defined since the function $u^*_{\mu_D}$ is finite everywhere in $\mathbb{R}$ owing to (1.35). The function v_m is also μ_D-measurable since u is μ_D-measurable and f_m is continuous. Now we prove that $v_m \in W^{1,1}(\mathbb{R}^n, \mu_D)$. We have

$$v_m \leq u^*_{\mu_D}(a_m) - u^*_{\mu_D}(b_m) < \infty$$

μ_D-almost everywhere in $\mathbb{R}^n$. For μ_D-almost every x in the set

$$\{x \in \mathbb{R}^n; u(x) \leq u^*_{\mu_D}(b_m)\},$$

it holds that $v_m(x) = 0$. It also holds that

$$\mu_D(\{x \in \mathbb{R}^n; u(x) > u^*_{\mu_D}(b_m)\}) < \infty \tag{2.2}$$

since $u \in V_0^1 X(\mathbb{R}^n, \mu_D)$. So, the function v_m is bounded, and it can be nonzero in a set of finite μ_D-measure only. We obtain $v_m \in L^1(\mathbb{R}^n, \mu_D)$. We know that the function f_m is Lipschitz continuous and that u is weakly differentiable in $\mathbb{R}^n$. So, we can use the chain rule for Sobolev functions (see [38, Theorem 2.1.11]). We obtain v_m is weakly differentiable in $\mathbb{R}^n$ and

$$\nabla v_m = \nabla u \chi_{\{u^*_{\mu_D}(b_m) < u < u^*_{\mu_D}(a_m)\}}$$

μ_D-almost everywhere (see (1.34)) in $\mathbb{R}^n$. From this equality we get $|\nabla v_m| = 0$ μ_D-almost everywhere in the set

$$\{x \in \mathbb{R}^n; u(x) \leq u^*_{\mu_D}(b_m)\}.$$

We know that the function $\nabla u \in X(\mathbb{R}^n, \mu_D)$. So, by virtue of (1.1) it follows that $\nabla u \in L^1(E, \mu_D)$ for every μ_D-measurable set $E \subseteq \mathbb{R}^n, \mu_D(E) < \infty$. We can now again use (2.2) to obtain $\nabla v_m \in L^1(\mathbb{R}^n, \mu_D)$. To conclude, we have just proved that for all $m \in M$ it holds that $v_m \in W^{1,1}(\mathbb{R}^n, \mu_D)$.

Now we can use the coarea formula (see [35], [24]) for the functions v_m, $m \in M$. We obtain

$$\begin{aligned}
&\int_{\bigcup_{m\in M}\{u^*_{\mu_D}(b_m)<u<u^*_{\mu_D}(a_m)\}} |\nabla u(x)|\, d\mu_D(x) = \sum_{m\in M}\int_{\mathbb{R}^n} |\nabla v_m(x)|\, d\mu_D(x) \\
&= \sum_{m\in M}\int_{-\infty}^{\infty} P_{\mu_D}(\{x \in \mathbb{R}^n; v_m(x) > t\})\, dt \\
&= \sum_{m\in M}\int_0^{u^*_{\mu_D}(a_m)-u^*_{\mu_D}(b_m)} P_{\mu_D}(\{x \in \mathbb{R}^n; v_m(x) > t\})\, dt \\
&\geq C_{iso}\sum_{m\in M}\int_0^{u^*_{\mu_D}(a_m)-u^*_{\mu_D}(b_m)} \mu_D(\{x \in \mathbb{R}^n; v_m(x) > t\})^{\frac{D-1}{D}}\, dt \\
&= C_{iso}\sum_{m\in M}\int_{u^*_{\mu_D}(b_m)}^{u^*_{\mu_D}(a_m)} \mu_D(\{x \in \mathbb{R}^n; u(x) > t\})^{\frac{D-1}{D}}\, dt.
\end{aligned} \tag{2.3}$$

The first equality holds because the sets $\{u^*_{\mu_D}(b_m) < u < u^*_{\mu_D}(a_m)\}, m \in M$, are pairwise disjoint. In the second equality we used the coarea formula. Now we verify that the third equality holds. If $t < 0$, then

$$\{x \in \mathbb{R}^n; v_m(x) > t\} = \mathbb{R}^n,$$

and it is true that $P_{\mu_D}(\mathbb{R}^n) = 0$ since $\partial\mathbb{R}^n = \emptyset$. If $t > u^*_{\mu_D}(a_m) - u^*_{\mu_D}(b_m)$, then

$$\{x \in \mathbb{R}^n; v_m(x) > t\} = \emptyset.$$

So, the third equality is correct. The first inequality holds by virtue of the isoperimetric inequality (Theorem 1.17) since

$$\mu_D(\{x \in \mathbb{R}^n; v_m(x) > t\}) < \infty$$

for every $t \in (0, u^*_{\mu_D}(a_m) - u^*_{\mu_D}(b_m))$. The fourth equality is true owing to the definition of the function v_m and a change of variables $t \mapsto t + u^*_{\mu_D}(b_m)$.

Now we derive an upper estimate of

$$\int_{\bigcup_{m\in M}\{u^*_{\mu_D}(b_m)<u<u^*_{\mu_D}(a_m)\}} |\nabla u(x)|\, d\mu_D(x).$$

We obtain

$$\begin{aligned}
&\int_{\bigcup_{m\in M}\{u^*_{\mu_D}(b_m)<u<u^*_{\mu_D}(a_m)\}} |\nabla u(x)|\, d\mu_D(x) \\
&\leq \int_0^{\mu_D\left(\bigcup_{m\in M}\{u^*_{\mu_D}(b_m)<u<u^*_{\mu_D}(a_m)\}\right)} |\nabla u|^*_{\mu_D}(t)\, dt \\
&= \int_0^{\sum_{m\in M}\mu_D(\{u^*_{\mu_D}(b_m)<u<u^*_{\mu_D}(a_m)\})} |\nabla u|^*_{\mu_D}(t)\, dt \leq \int_0^{\sum_{m\in M}(b_m-a_m)} |\nabla u|^*_{\mu_D}(t)\, dt.
\end{aligned} \tag{2.4}$$

The first inequality holds by virtue of the Hardy–Littlewood inequality (1.14). The last inequality we can verify in the following way. We have

$$\begin{aligned}&\mu_D(\{u^*_{\mu_D}(b_m) < u < \mu^*_{\mu_D}(a_m)\}) = \mu_D(\{u^*_{\mu_D}(b_m) < u\}) - \mu_D(\{u^*_{\mu_D}(a_m) \le u\})\\ &\le b_m - a_m,\end{aligned}$$

where we used (1.11) and (1.12) in the inequality.

In this part of the proof, we will assume that all the intervals $(a_m, b_m), m \in M$, are contained in an interval $[a, b] \subseteq (0, \infty)$. We prove that the function $u^*_{\mu_D}$ is absolutely continuous on the interval $[a, b]$. We can assume that $u^*_{\mu_D}(a) > 0$. Otherwise $u^*_{\mu_D}$ is equal to 0 on the whole interval $[a, b]$ since $u^*_{\mu_D}$ is nonincreasing. We set

$$K = \mu_D(\{x \in \mathbb{R}^n; u(x) \ge u^*_{\mu_D}(a)\}).$$

Then we have $K < \infty$ since $u \in V^1_0 X(\mathbb{R}^n, \mu_D)$. Since $u^*_{\mu_D}(a) > 0$, we can use (1.12) to obtain $K > 0$. Owing to (2.3) we obtain

$$\begin{aligned}&\int_{\bigcup_{m\in M}\{u^*_{\mu_D}(b_m)<u<u^*_{\mu_D}(a_m)\}} |\nabla u(x)| \, d\mu_D(x)\\ &\ge C_{iso} \sum_{m\in M} \int_{u^*_{\mu_D}(b_m)}^{u^*_{\mu_D}(a_m)} \mu_D(\{x \in \mathbb{R}^n; u(x) > t\})^{\frac{D-1}{D}} \, dt\\ &\ge C_{iso} \sum_{m\in M} \int_{u^*_{\mu_D}(b_m)}^{u^*_{\mu_D}(a_m)} \mu_D(\{x \in \mathbb{R}^n; u(x) \ge u^*_{\mu_D}(a_m)\})^{\frac{D-1}{D}} \, dt\\ &\ge C_{iso} \sum_{m\in M} \int_{u^*_{\mu_D}(b_m)}^{u^*_{\mu_D}(a_m)} \mu_D(\{x \in \mathbb{R}^n; u(x) \ge u^*_{\mu_D}(a)\})^{\frac{D-1}{D}} \, dt\\ &= C_{iso} K^{\frac{D-1}{D}} \sum_{m\in M} (u^*_{\mu_D}(a_m) - u^*_{\mu_D}(b_m)).\end{aligned} \tag{2.5}$$

It follows that

$$\begin{aligned}&\sum_{m\in M} (u^*_{\mu_D}(a_m) - u^*_{\mu_D}(b_m))\\ &\le C^{-1}_{iso} K^{\frac{1-D}{D}} \int_{\bigcup_{m\in M}\{u^*_{\mu_D}(b_m)<u<u^*_{\mu_D}(a_m)\}} |\nabla u(x)| \, d\mu_D(x)\\ &\le C^{-1}_{iso} K^{\frac{1-D}{D}} \int_0^{\sum_{m\in M}(b_m - a_m)} |\nabla u|^*_{\mu_D}(t) \, dt.\end{aligned} \tag{2.6}$$

The first inequality holds due to (2.5). The second inequality holds by virtue of (2.4).

Next we want to prove that

$$\int_0^t |\nabla u|^*_{\mu_D}(\tau) \, d\tau < \infty \tag{2.7}$$

for every $t \in (0, \infty)$. Firstly, we show that the function $|\nabla u|^*_{\mu_D}$ is integrable over the interval $(0, 1)$. This is equivalent to the fact that $|\nabla u| \in (L^1 + L^\infty)(\mathbb{R}^n, \mu_D)$ (see [2, Chapter 2, Theorem 6.4]). But the latter is satisfied since we know that $|\nabla u| \in X(\mathbb{R}^n, \mu_D)$ and that $X(\mathbb{R}^n, \mu_D) \subseteq (L^1 + L^\infty)(\mathbb{R}^n, \mu_D)$ (see [2, Chapter 2, Theorem 6.6]). So, we have the fact that the function $|\nabla u|^*_{\mu_D}$ is integrable over the interval $(0,1)$. If $t \in (0,1]$, we have also proved (2.7) since the function $|\nabla u|^*_{\mu_D}$ is nonnegative. Now assume that $t \in (1,\infty)$. We exploit the fact that the function $|\nabla u|^*_{\mu_D}$ is nonincreasing to obtain

$$\begin{aligned}\int_0^t |\nabla u|^*_{\mu_D}(\tau)\,d\tau &= \int_0^1 |\nabla u|^*_{\mu_D}(\tau)\,d\tau + \int_1^t |\nabla u|^*_{\mu_D}(\tau)\,d\tau\\ &\leq \int_0^1 |\nabla u|^*_{\mu_D}(\tau)\,d\tau + (t-1)\,|\nabla u|^*_{\mu_D}(1) < \infty.\end{aligned}$$

So, we have proved (2.7) for all $t \in (0,\infty)$.

Since we know that $|\nabla u|^*_{\mu_D}$ is integrable over an arbitrary bounded interval $(0,t)$, we can use (2.6) to obtain the fact that the function $u^*_{\mu_D}$ is absolutely continuous on the interval $[a,b]$. It follows that it is locally absolutely continuous on the interval $(0,\infty)$, which is the desired result.

It remains to prove the inequality (2.1). From now we do not anymore assume that the intervals (a_m,b_m) are contained in $[a,b]$. Note that since the function $u^*_{\mu_D}$ is nonincreasing, it is also differentiable almost everywhere in $(0,\infty)$ and $\frac{du^*_{\mu_D}}{dt}(t) \leq 0$, wherever the derivative exists. In particular, it means that the left-hand side of the inequality makes sense. Define the function $\phi\colon (0,\infty) \to [0,\infty)$ by $\phi(t) = -C_{iso}t^{\frac{D-1}{D}}\frac{du^*_{\mu_D}}{dt}(t),\ t \in (0,\infty)$. We show that

$$\int_0^t \phi^*(\tau)\,d\tau \leq \int_0^t |\nabla u|^*_{\mu_D}(\tau)\,d\tau, \quad t \in (0,\infty). \tag{2.8}$$

Choose $t \in (0,\infty)$ arbitrarily. By virtue of (1.16), we know that it is enough to prove that for every measurable set $E \subseteq (0,\infty)$ such that $\lambda(E) = t$, it holds that

$$\int_E \phi(\tau)\,d\tau \leq \int_0^t |\nabla u|^*_{\mu_D}(\tau)\,d\tau. \tag{2.9}$$

Now choose an arbitrary $m \in M$. We have

$$\int_{a_m}^{b_m} \phi(\tau)\,d\tau = -\int_{a_m}^{b_m} C_{iso}\tau^{\frac{D-1}{D}}\frac{du^*_{\mu_D}}{d\tau}(\tau)\,d\tau. \tag{2.10}$$

We use the change of variables theorem (see [34, page 156]). Since the function $u^*_{\mu_D}$ is absolutely continuous and nonincreasing on the interval $[a_m,b_m]$, and the function $\tau \mapsto \mu_D(\{x \in \mathbb{R}^n; u(x) > \tau\})^{\frac{D-1}{D}}, \tau \in (0,\infty)$, is nonnegative on the interval (a_m,b_m), we obtain

$$\begin{aligned}&\int_{u^*_{\mu_D}(b_m)}^{u^*_{\mu_D}(a_m)} C_{iso}\mu_D(\{x \in \mathbb{R}^n; u(x) > s\})^{\frac{D-1}{D}}\,ds\\ &= -\int_{a_m}^{b_m} C_{iso}\mu_D(\{x \in \mathbb{R}^n; u(x) > u^*_{\mu_D}(\tau)\})^{\frac{D-1}{D}}\frac{du^*_{\mu_D}}{d\tau}(\tau)\,d\tau.\end{aligned} \tag{2.11}$$

Now we prove that

$$\begin{aligned}&\int_{a_m}^{b_m} C_{iso}\mu_D(\{x \in \mathbb{R}^n; u(x) > u^*_{\mu_D}(\tau)\})^{\frac{D-1}{D}}\frac{du^*_{\mu_D}}{d\tau}(\tau)\,d\tau\\ &= \int_{a_m}^{b_m} C_{iso}\tau^{\frac{D-1}{D}}\frac{du^*_{\mu_D}}{d\tau}(\tau)\,d\tau.\end{aligned} \tag{2.12}$$

Owing to (1.11) we know that for every $\tau \in (a_m, b_m)$ it holds that

$$\mu_D(\{x \in \mathbb{R}^n; u(x) > u^*_{\mu_D}(\tau)\}) \leq \tau.$$

The function $u^*_{\mu_D}$ is differentiable almost everywhere in the interval (a_m, b_m). Choose an arbitrary $\tau \in (a_m, b_m)$ such that the function $u^*_{\mu_D}$ is differentiable at τ. Assume that

$$\mu_D(\{x \in \mathbb{R}^n; u(x) > u^*_{\mu_D}(\tau)\}) < \tau.$$

Then there exists $\delta \in \mathbb{R}, \delta > 0$, such that for every $s \in (\tau - \delta, \tau)$ it holds that

$$\mu_D(\{x \in \mathbb{R}^n; u(x) > u^*_{\mu_D}(\tau)\}) < s. \tag{2.13}$$

From the inequality (2.13) we obtain for each $s \in (\tau - \delta, \tau)$

$$\begin{aligned} &u^*_{\mu_D}(s) = \inf \{\alpha \in (0, \infty); \mu_D(\{x \in \mathbb{R}^n; u(x) > \alpha\}) \leq s\} \\ &\leq u^*_{\mu_D}(\tau). \end{aligned}$$

Because the function $u^*_{\mu_D}$ is nonincreasing it holds that $u^*_{\mu_D}(s) = u^*_{\mu_D}(\tau)$ for all $s \in (\tau - \delta, \tau)$. Since the function $u^*_{\mu_D}$ is differentiable at τ, we have $\frac{du^*_{\mu_D}}{d\tau}(\tau) = 0$. So, we have just proved (2.12). By (2.10), (2.11) and (2.12) we have

$$\int_{a_m}^{b_m} \phi(\tau)\, d\tau = \int_{u^*_{\mu_D}(b_m)}^{u^*_{\mu_D}(a_m)} C_{iso}\mu_D(\{x \in \mathbb{R}^n; u(x) > \tau\})^{\frac{D-1}{D}}\, d\tau. \tag{2.14}$$

Now we obtain

$$\begin{aligned} &\int_{\bigcup_{m\in M}(a_m,b_m)} \phi(\tau)\, d\tau = \sum_{m\in M} \int_{u^*_{\mu_D}(b_m)}^{u^*_{\mu_D}(a_m)} C_{iso}\mu_D(\{x \in \mathbb{R}^n; u(x) > \tau\})^{\frac{D-1}{D}}\, d\tau \\ &\leq \int_{\bigcup_{m\in M}\{u^*_{\mu_D}(b_m)<u<u^*_{\mu_D}(a_m)\}} |\nabla u(x)|\, d\mu_D(x) \\ &\leq \int_0^{\sum\limits_{m\in M}(b_m-a_m)} |\nabla u|^*_{\mu_D}(\tau)\, d\tau. \end{aligned} \tag{2.15}$$

The equality holds thanks to (2.14) and to the fact that the intervals $(a_m, b_m), m \in M$, are pairwise disjoint. The first inequality holds thanks to (2.3). The second inequality is true by virtue of (2.4). Now choose an arbitrary measurable set $E \subseteq (0, \infty), \lambda(E) = t$. Then for every $\varepsilon \in \mathbb{R}, \varepsilon > 0$, there exists a countable system $\{(a_m, b_m)\}_{m\in M}$ of pairwise disjoint nonempty bounded intervals such that

$$E \subseteq \bigcup_{m\in M}(a_m, b_m) \quad \text{and} \quad \lambda\left(\bigcup_{m\in M}(a_m, b_m) \setminus E\right) < \varepsilon.$$

So, choose an arbitrary $\varepsilon \in \mathbb{R}, \varepsilon > 0$, and let $\{(a_m, b_m)\}_{m\in M}$ be such a system of intervals. From (2.15) we obtain

$$\int_E \phi(\tau)\, d\tau \leq \int_{\bigcup_{m\in M}(a_m,b_m)} \phi(\tau)\, d\tau \leq \int_0^{t+\varepsilon} |\nabla u|^*_{\mu_D}(\tau)\, d\tau.$$

We already know from (2.7) that for all $s \in (0, \infty)$ the function $|\nabla u|^*_{\mu_D}$ is integrable over $(0, s)$. So, we obtain

$$\int_E \phi(\tau)\, d\tau \leq \int_0^t |\nabla u|^*_{\mu_D}(\tau)\, d\tau$$

thanks to the Lebesgue dominated convergence theorem. This means that (2.9) is true. So, we have just proved (2.8). The inequality (2.1) now follows from the Hardy–Littlewood–Pólya principle (1.15).

We have proved the theorem for nonnegative u. Now let $u \in V_0^1 X(\mathbb{R}^n, \mu_D)$ be general. Since u is weakly differentiable in $\mathbb{R}^n$, it is true (see [38, Corollary 2.1.8]) that $|u|$ is weakly differentiable in $\mathbb{R}^n$ and that $|\nabla u| = |\nabla |u||$ almost everywhere in $\mathbb{R}^n$. It means that $|u| \in V_0^1 X(\mathbb{R}^n, \mu_D)$. So, the theorem holds for $|u|$. Since $u^*_{\mu_D} = |u|^*_{\mu_D}$, the theorem holds also for the function u.

□

In the remaining part of this chapter we prove the reduction principle. Recall that the parameteres m and D were introduced in Section 1.3 and that we have $m \in \mathbb{N}, 1 \leq m < D$.

Theorem 2.2 (Reduction principle)**.** *Let X and Y be rearrangement-invariant spaces over $(\mathbb{R}^n, \mu_D)$. Then the following three statements are equivalent.*

1. *For all functions $v \in V_0^m X(\mathbb{R}^n, \mu_D)$ it holds that*

$$\|v\|_{Y(\mathbb{R}^n,\mu_D)} \leq C_1 \|\nabla^m v\|_{X(\mathbb{R}^n,\mu_D)}. \tag{2.16}$$

2. *For all functions $f \in \mathcal{M}^+(0, \infty)$ it holds that*

$$\left\| \int_t^\infty f(\tau)\tau^{\frac{m}{D}-1}\, d\tau \right\|_{Y(0,\infty)} \leq C_2 \|f\|_{X(0,\infty)}. \tag{2.17}$$

3. *For all functions $g \in \mathcal{M}^+(0, \infty)$ it holds that*

$$\left\| t^{\frac{m}{D}} g^{**}(t) \right\|_{X'(0,\infty)} \leq C_2 \|g\|_{Y'(0,\infty)}. \tag{2.18}$$

Here, C_1 and C_2 are positive constants such that C_1 depends only on C_2, m and D, and C_2 depends only on C_1, m and D.

The proof of this theorem will be divided into three steps. The first step will be Proposition 2.3, the second step will be Proposition 2.4 and the third one will be Proposition 2.11. In Proposition 2.3 we prove the equivalence of the second and the third statement of Theorem 2.2.

Proposition 2.3. *Let X, Y be rearrangement-invariant spaces over $(\mathbb{R}^n, \mu_D)$. Let $\varphi \in \mathcal{M}^+(0, \infty)$. Then the following two statements are equivalent.*

1. *For all functions $f \in \mathcal{M}^+(0, \infty)$ it holds that*

$$\left\| \int_t^\infty f(\tau)\varphi(\tau)\tau^{-1}\, d\tau \right\|_{Y(0,\infty)} \leq C \|f\|_{X(0,\infty)}. \tag{2.19}$$

2. For all functions $g \in \mathcal{M}^+(0,\infty)$ it holds that

$$\|\varphi(t)g^{**}(t)\|_{X'(0,\infty)} \leq C\,\|g\|_{Y'(0,\infty)}\,. \tag{2.20}$$

Here, C is a positive constant.

Proof. The inequality (2.19) is equivalent to the inequality

$$\sup_{f\in\mathcal{M}^+(0,\infty),\|f\|_{X(0,\infty)}\leq 1} \left\|\int_t^\infty f(\tau)\varphi(\tau)\tau^{-1}\,d\tau\right\|_{Y(0,\infty)} \leq C. \tag{2.21}$$

The inequality (2.20) is equivalent to the inequality

$$\sup_{g\in\mathcal{M}^+(0,\infty),\|g\|_{Y'(0,\infty)}\leq 1} \|\varphi(t)g^{**}(t)\|_{X'(0,\infty)} \leq C. \tag{2.22}$$

For the proof of the equivalence of (2.19) and (2.20) it is enough to show that (2.21) is equivalent to (2.22). Firstly, assume that (2.21) holds. For all functions $f, g \in \mathcal{M}^+(0,\infty)$, $\|f\|_{X(0,\infty)} \leq 1$, $\|g\|_{Y'(0,\infty)} \leq 1$, it holds that

$$\begin{aligned}
&\int_0^\infty f(\tau)\varphi(\tau)g^{**}(\tau)\,d\tau = \int_0^\infty g^*(t)\int_t^\infty f(\tau)\varphi(\tau)\tau^{-1}\,d\tau\,dt\\
&\leq \|g\|_{Y'(0,\infty)}\left\|\int_t^\infty f(\tau)\varphi(\tau)\tau^{-1}\,d\tau\right\|_{Y(0,\infty)}\\
&\leq C\,\|g\|_{Y'(0,\infty)} \leq C.
\end{aligned}$$

We used the Fubini theorem in the equality. The first inequality holds by virtue of the Hölder inequality (1.4). The second inequality is true thanks to (2.21). This proves (2.22) thanks to (1.2).

It remains to prove that (2.22) implies (2.21). Assume that (2.22) holds. For all functions $f, g \in \mathcal{M}^+(0,\infty)$, $\|f\|_{X(0,\infty)} \leq 1$, $\|g\|_{Y'(0,\infty)} \leq 1$, it holds that

$$\begin{aligned}
&\int_0^\infty g^*(t)\int_t^\infty f(\tau)\varphi(\tau)\tau^{-1}\,d\tau\,dt = \int_0^\infty f(\tau)\varphi(\tau)g^{**}(\tau)\,d\tau\\
&\leq \|f\|_{X(0,\infty)}\|\varphi(\tau)g^{**}(\tau)\|_{X'(0,\infty)} \leq C\,\|f\|_{X(0,\infty)} \leq C.
\end{aligned}$$

The first inequality is true by virtue of the Hölder inequality. The second inequality holds by (2.22). It follows that (2.21) holds owing to (1.2).

□

Note that if we set $\varphi(t) = t^{\frac{m}{D}}$, $t \in (0,\infty)$, then we obtain the equivalence of (2.17) and (2.18) in the reduction principle (Theorem 2.2).

Now we aim to prove the following proposition.

Proposition 2.4. *Let X, Y be rearrangement-invariant spaces over $(\mathbb{R}^n, \mu_D)$. Assume that there exists a positive constant C_1 such that for all functions $v \in V_0^m X(\mathbb{R}^n, \mu_D)$ it holds that*

$$\|v\|_{Y(\mathbb{R}^n,\mu_D)} \leq C_1\,\|\nabla^m v\|_{X(\mathbb{R}^n,\mu_D)}\,.$$

Then there exists a positive constant C_2 such that for all functions $f \in \mathcal{M}^+(0,\infty)$ it holds that

$$\left\|\int_t^\infty f(\tau)\tau^{\frac{m}{D}-1}\,d\tau\right\|_{Y(0,\infty)} \leq C_2\,\|f\|_{X(0,\infty)}\,. \tag{2.23}$$

The constant C_2 depends only on the constant C_1, on m and on D.

Owing to this proposition the n-dimensional part of the reduction principle implies the one-dimensional part. Before the proof of the proposition, we need prove the following two lemmata.

Convention. *Let ν be a measure on $\mathbb{R}^n$ that is absolutely continuous with respect to the Lebesgue measure. Then we denote by B_ν the weighted measure of the unit ball in $\mathbb{R}^n$, i.e.,*

$$B_\nu = \nu(\{x \in \mathbb{R}^n; |x| \leq 1\}).$$

Lemma 2.5. *Let ν be a measure on $\mathbb{R}^n$ that is absolutely continuous with respect to the Lebesgue measure with a positive locally integrable density ω. Let $\alpha \geq 0$ be a constant and assume that the density ω is an α-homogeneous function, i.e., $\omega(rx) = r^\alpha \omega(x)$ for every $x \in \mathbb{R}^n$ and for every $r > 0$. Then the mapping $\sigma \colon \mathbb{R}^n \to [0, \infty)$ defined by*

$$\sigma(x) = B_\nu \left|x\right|^{\alpha+n}, \quad x \in \mathbb{R}^n,$$

has the following property. For every Lebesgue measurable set $E \subseteq [0, \infty)$, it is true that $\sigma^{-1}(E)$ is a ν-measurable set and that $\nu(\sigma^{-1}(E)) = \lambda(E)$.

The mapping σ with the property mentioned in the statement is called a *measure-preserving transformation* of the measure spaces $(\mathbb{R}^n, \nu)$ and $([0, \infty), \lambda)$.

Proof. Firstly, observe that the mapping σ is continuous. It means that it is also Lebesgue measurable. So, for every measurable set $F \subseteq [0, \infty)$, we have the fact that the set

$$\sigma^{-1}(F) \subseteq \mathbb{R}^n$$

is ν-measurable. It remains to prove that

$$\nu\left(\sigma^{-1}(F)\right) = \lambda(F). \tag{2.24}$$

Choose an arbitrary $r > 0$. Set

$$M = \sigma^{-1}([0, r)) = \left\{x \in \mathbb{R}^n; |x| < \left(\frac{r}{B_\nu}\right)^{\frac{1}{\alpha+n}}\right\}.$$

We obtain

$$\nu\left(\sigma^{-1}([0, r))\right) = \int_M \omega(x)\, dx = \int_{B_1(0)} \left(\frac{r}{B_\nu}\right)^{\frac{n}{\alpha+n}} \left(\frac{r}{B_\nu}\right)^{\frac{\alpha}{\alpha+n}} \omega(x)\, dx = r,$$

where the second equality holds owing to the change of variables and to the α-homogeneity of ω. Recall that by $B_1(0)$ we denote the closed unit ball in $\mathbb{R}^n$ with the center at the origin. It follows that (2.24) holds for $F = [0, r)$. Now choose an arbitrary $0 < q < r$. We obtain

$$\sigma^{-1}([0, r)) = \sigma^{-1}([0, q)) \cup \sigma^{-1}([q, r)). \tag{2.25}$$

So,

$$\nu\left(\sigma^{-1}([q, r))\right) = r - q$$

since the sets on the right-hand side of (2.25) are disjoint. It means that (2.24) holds with $F = [q, r)$. Finally, choose an arbitrary $s \in [0, \infty)$. We obtain

$$\nu\left(\sigma^{-1}(\{s\})\right) = \nu\left(\left\{x \in \mathbb{R}^n; |x| = \left(\frac{s}{B_\nu}\right)^{\frac{1}{\alpha+n}}\right\}\right) = 0$$

since the Lebesgue measure of this sphere is equal to zero. It follows that (2.24) holds for $F = \{s\}$. Consequently, we have proved that (2.24) holds for every open subset of $[0, \infty)$. Now choose an arbitrary compact set $K \subseteq [0, \infty)$. Find $r > 0$ such that $K \subseteq [0, r)$. Since $[0, r)$ and $[0, r) \setminus K$ are open in $[0, \infty)$ and since K and $[0, r) \setminus K$ are disjoint, we obtain (2.24) for $F = K$. Now choose an arbitrary measurable set $F \subseteq [0, \infty)$. Let $\{K_k\}_{k=1}^\infty$ be a sequence of compact subsets of $[0, \infty)$ such that $K_k \subseteq F$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} \lambda(K_k) = \lambda(F)$. Let $\{G_k\}_{k=1}^\infty$ be a sequence of open subsets of $[0, \infty)$ such that $F \subseteq G_k$ for every $k \in \mathbb{N}$ and $\lim_{k\to\infty} \lambda(G_k) = \lambda(F)$. For every $k \in \mathbb{N}$ we obtain

$$\lambda(K_k) = \nu\left(\sigma^{-1}(K_k)\right) \le \nu\left(\sigma^{-1}(F)\right) \le \nu\left(\sigma^{-1}(G_k)\right) = \lambda(G_k),$$

so (2.24) holds for an arbitrary measurable F.

□

Remark. It follows from the preceding lemma that every function $h \in \mathcal{M}^+(0, \infty)$ is equimeasurable with the μ_D-measurable function $x \mapsto h(B_\mu |x|^D)$, $x \in \mathbb{R}^n$.

The following lemma is just a technical tool that we will use in the proof of Proposition 2.4.

Lemma 2.6. *Let $f \in \mathcal{M}^+(0, \infty) \cap L^\infty(0, \infty)$ be a function with a bounded support. Let $g\colon [0, \infty) \to [0, \infty)$ be the function defined by*

$$g(t) = \int_t^\infty f(\tau)\tau^{\frac{m}{D}-m}(\tau - t)^{m-1}\, d\tau, \quad t \in [0, \infty). \tag{2.26}$$

Then $g \in \mathcal{C}^{m-1}(0, \infty)$ and

$$g^{(j)}(t) = (-1)^j \frac{(m-1)!}{(m-j-1)!} \int_t^\infty f(\tau)\tau^{\frac{m}{D}-m}(\tau - t)^{m-j-1}\, d\tau, \quad t \in (0, \infty) \tag{2.27}$$

for every $j \in \{1, \dots, m-1\}$. Moreover, $g^{(m-1)}$ is locally Lipschitz on $(0, \infty)$ and

$$g^{(m)}(t) = (-1)^m (m-1)! f(t) t^{\frac{m}{D}-m} \tag{2.28}$$

for almost every $t \in (0, \infty)$.

Proof. Recall that $m < D$. Assume that $m > 1$ and that $j = 1$. Choose an arbitrary $t \in (0, \infty)$. We have

$$\begin{aligned} g'(t) &= \lim_{h\to 0} \frac{1}{h}\left(\int_{t+h}^\infty f(\tau)\tau^{\frac{m}{D}-m}(\tau - t - h)^{m-1}\, d\tau - \int_t^\infty f(\tau)\tau^{\frac{m}{D}-m}(\tau - t)^{m-1}\, d\tau\right) \\ &= \lim_{h\to 0} \frac{1}{h} \int_{t+h}^\infty f(\tau)\tau^{\frac{m}{D}-m}\left((\tau - t - h)^{m-1} - (\tau - t)^{m-1}\right) d\tau \\ &\quad - \lim_{h\to 0} \frac{1}{h} \int_t^{t+h} f(\tau)\tau^{\frac{m}{D}-m}(\tau - t)^{m-1}\, d\tau, \end{aligned} \tag{2.29}$$

provided that the limits exist. For the first term on the right-hand side of (2.29), it holds that

$$\begin{aligned}
&\lim_{h\to 0}\frac{1}{h}\int_{t+h}^{\infty} f(\tau)\tau^{\frac{m}{D}-m}\left((\tau-t-h)^{m-1}-(\tau-t)^{m-1}\right)\,d\tau\\
&=\lim_{h\to 0}\frac{1}{h}\int_{t+h}^{\infty} f(\tau)\tau^{\frac{m}{D}-m}(m-1)(-h)(\tau-t)^{m-2}\,d\tau\\
&\quad+\lim_{h\to 0}\frac{1}{h}\int_{t+h}^{\infty} f(\tau)\tau^{\frac{m}{D}-m}\sum_{i=2}^{m-1}\binom{m-1}{i}(\tau-t)^{m-1-i}(-h)^i\,d\tau \qquad (2.30)\\
&=-(m-1)\lim_{h\to 0}\int_{t+h}^{\infty} f(\tau)\tau^{\frac{m}{D}-m}(\tau-t)^{m-2}\,d\tau\\
&=-(m-1)\int_{t}^{\infty} f(\tau)\tau^{\frac{m}{D}-m}(\tau-t)^{m-2}\,d\tau.
\end{aligned}$$

The second equality holds since

$$\begin{aligned}
&\left|\lim_{h\to 0}\frac{1}{h}\int_{t+h}^{\infty} f(\tau)\tau^{\frac{m}{D}-m}\sum_{i=2}^{m-1}\binom{m-1}{i}(\tau-t)^{m-1-i}(-h)^i\,d\tau\right|\\
&\leq\lim_{h\to 0}\sum_{i=2}^{m-1}\binom{m-1}{i}h^{i-1}\int_{\frac{t}{2}}^{\infty} f(\tau)\tau^{\frac{m}{D}-m}\left|\tau-t\right|^{m-1-i}\,d\tau=0.
\end{aligned}$$

The last equality holds owing to the Lebesgue dominated convergence theorem. Observe that we want to show that $g'(t)$ is equal to the last term of (2.30). It means that it remains to prove that the second term on the right-hand side of (2.29) is equal to zero. For every $h\in(-t,t)$ we obtain

$$\begin{aligned}
0&\leq\frac{1}{|h|}\left|\int_{t}^{t+h} f(\tau)\tau^{\frac{m}{D}-m}\left|\tau-t\right|^{m-1}\,d\tau\right|\leq\frac{1}{|h|}\left|\int_{t}^{t+h} f(\tau)\tau^{\frac{m}{D}-m}\left|h\right|^{m-1}\,d\tau\right|\\
&=|h|^{m-2}\left|\int_{t}^{t+h} f(\tau)\tau^{\frac{m}{D}-m}\,d\tau\right|.
\end{aligned}\qquad (2.31)$$

The last term of (2.31) converges to zero as h approaches zero by virtue of the fact that $m-2\geq 0$ and thanks to the Lebesgue dominated convergence theorem. It means that

$$\lim_{h\to 0}\frac{1}{h}\int_{t}^{t+h} f(\tau)\tau^{\frac{m}{D}-m}(\tau-t)^{m-1}\,d\tau=0,$$

so (2.27) holds with $j=1$. The fact that (2.27) holds for $1<j\leq m-1$ can be proved in a similar way as for $j=1$. Thanks to (2.27) we have the fact that $g^{(j)}$ is differentiable and hence also continuous on $(0,\infty)$ for every $j\in\{0,\dots,m-2\}$. Note that we use the notation $g=g^{(0)}$.

Now let $1\leq m<D$ be arbitrary. Owing to (2.26) and to (2.27), we have

$$g^{(m-1)}(t)=(-1)^{m-1}(m-1)!\int_{t}^{\infty} f(\tau)\tau^{\frac{m}{D}-m}\,d\tau,\quad t\in(0,\infty).$$

This function is locally Lipschitz on $(0,\infty)$ since for every closed interval $[t_1,t_2]\subseteq(0,\infty)$ and for every $s_1,s_2\in[t_1,t_2]$, $s_1\leq s_2$, we have

$$\int_{s_1}^{s_2} f(\tau)\tau^{\frac{m}{D}-m}\,d\tau\leq\|f\|_{L^\infty(0,\infty)}\,t_1^{\frac{m}{D}-m}(s_2-s_1).$$

Hence the function $g^{(m-1)}$ is differentiable almost everywhere in $(0,\infty)$ and we obtain (2.28) for almost every $t \in (0,\infty)$.

$\square$

Now we finally prove Proposition 2.4.

Proof of Proposition 2.4. Choose an arbitrary function $f \in \mathcal{M}^+(0,\infty)$. Observe that it is enough to prove the theorem for $\|f\|_{X(0,\infty)} < \infty$. Firstly, assume that the function f belongs to $L^\infty(0,\infty)$ and that it has a bounded support. Define the function $v\colon \mathbb{R}^n \to [0,\infty)$ by

$$v(x) = \int_{B_{\mu_D}|x|^D}^{\infty} f(\tau)\tau^{\frac{m}{D}-m}\left(\tau - B_{\mu_D}|x|^D\right)^{m-1} d\tau, \quad x \in \mathbb{R}^n. \tag{2.32}$$

Now define the function $\sigma\colon \mathbb{R}^n \to [0,\infty)$ by

$$\sigma(x) = B_{\mu_D}|x|^D, \quad x \in \mathbb{R}^n.$$

It holds that

$$\sigma \in \mathcal{C}^\infty(\mathbb{R}^n \setminus \{0\}). \tag{2.33}$$

Observe that

$$v(x) = (g \circ \sigma)(x), \quad x \in \mathbb{R}^n, \tag{2.34}$$

where g is the function from (2.26).

Now, for every $k \in \{1,\dots m\}$ and for the parametres l_1, l_2 satisfying $l_1 \in \mathbb{N}, l_1 \le \min(\{k, m-1\}), l_2 \in \{0,\dots,k\}, 2(l_1+l_2) \ge k$, we define the functions

$$x \mapsto \int_{B_{\mu_D}|x|^D}^{\infty} f(\tau)\tau^{\frac{m}{D}-m}\left(\tau - B_{\mu_D}|x|^D\right)^{m-l_1-1} d\tau\, |x|^{l_1(D-2)-2l_2} \prod_{j=1}^{2(l_1+l_2)-k} x_{i_j} \tag{2.35}$$

for every $x \in \mathbb{R}^n \setminus \{0\}$, and

$$x \mapsto f\left(B_{\mu_D}|x|^D\right)|x|^{-m}\prod_{j=1}^{m} x_{i_j} \tag{2.36}$$

for almost every $x \in \mathbb{R}^n$. Owing to Lemma 2.6, to (2.33) and to the chain rule, we obtain the fact that

$$v \in \mathcal{C}^{m-1}(\mathbb{R}^n \setminus \{0\}).$$

Thanks to (2.26), to (2.27), to the chain rule and to the product rule, we obtain the fact that an arbitrary partial derivative of the function v of the k-th order in every point $x \in \mathbb{R}^n \setminus \{0\}$ and for every $k \in \{1,\dots m-1\}$ is a linear combination of the functions (2.35). By virtue of (2.28) and of (2.33), the partial derivatives of the function v of the m-th order in almost every point $x \in \mathbb{R}^n$ exist and are linear combinations of the functions (2.35) with $k = m$ and of the functions

(2.36). We know that f has a bounded support. Assume that it is contained in $(0, M)$ for some $0 < M < \infty$. We have

$$
\begin{aligned}
&\left| \int_{B_{\mu_D}|x|^D}^{\infty} f(\tau)\tau^{\frac{m}{D}-m} \left(\tau - B_{\mu_D}|x|^D\right)^{m-l_1-1} d\tau\, |x|^{l_1(D-2)-2l_2} \prod_{j=1}^{2(l_1+l_2)-k} x_{i_j} \right| \\
&\leq \left| \int_{B_{\mu_D}|x|^D}^{M} f(\tau)\tau^{\frac{m}{D}-l_1-1}\, d\tau \right| |x|^{l_1 D-k} \\
&\leq \chi_{(0,M)}\left(B_{\mu_D}|x|^D\right) \|f\|_{L^\infty(0,\infty)} \frac{D}{l_1 D - m} |x|^{l_1 D-k} \left(M^{\frac{m}{D}-l_1} + B_{\mu_D}^{\frac{m}{D}-l_1} |x|^{m-l_1 D}\right) \\
&= \chi_{(0,M)}\left(B_{\mu_D}|x|^D\right) \|f\|_{L^\infty(0,\infty)} \frac{D}{l_1 D - m} \left(M^{\frac{m}{D}-l_1} |x|^{l_1 D-k} + B_{\mu_D}^{\frac{m}{D}-l_1} |x|^{m-k}\right)
\end{aligned}
\tag{2.37}
$$

for every $x \in \mathbb{R}^n \setminus \{0\}$ and for every k, l_1, l_2 that the functions (2.35) are defined for. It is also true that

$$
f\left(B_{\mu_D}|x|^D\right)|x|^{-m} \prod_{j=1}^{m} x_{i_j} \leq \|f\|_{L^\infty(0,\infty)} \chi_{(0,M)}\left(B_{\mu_D}|x|^D\right) \tag{2.38}
$$

for almost every $x \in \mathbb{R}^n$. Owing to (2.37), (2.38) and to the facts that $l_1 D - k \geq 0$ and that $m - k \geq 0$, we obtain the fact that all of the partial derivatives of v up to the m-th order belong to the space $L^1(\mathbb{R}^n)$. Each function (2.35) is the product of the functions

$$
\begin{aligned}
x \mapsto &\int_{B_{\mu_D}|x|^D}^{\infty} f(\tau)\tau^{\frac{m}{D}-m} \left(\tau - B_{\mu_D}|x|^D\right)^{m-l_1-1} d\tau \\
&= (-1)^{l_1} \frac{(m-l_1-1)!}{(m-1)!} (g^{(l_1)} \circ \sigma)(x)
\end{aligned}
\tag{2.39}
$$

and

$$
x \mapsto |x|^{l_1(D-2)-2l_2} \prod_{j=1}^{2(l_1+l_2)-k} x_{i_j}. \tag{2.40}
$$

Every function (2.40) is of the class $\mathcal{C}^\infty(\mathbb{R}^n \setminus \{0\})$. If $l_1 < m-1$, then we obtain the fact that the function (2.39) is of the class $\mathcal{C}^1(\mathbb{R}^n \setminus \{0\})$ thanks to Lemma 2.6 and to (2.33). We obtain the fact that these functions are locally absolutely continuous on every line in $\mathbb{R}^n$ that is parallel to the coordinate axes and that does not pass through the origin. If $l_1 = m-1$, then we obtain the same result by virtue of (2.33) and of the fact that the function $g^{(m-1)}$ is locally Lipschitz on $(0, \infty)$ thanks to Lemma 2.6. It means that by [22, Theorem 10.35] we have the fact that all of the weak partial derivatives of v up to the m-th order exist and are equal to the classical partial derivatives of v almost everywhere on $\mathbb{R}^n$.

Now, thanks to (2.35) with $k = m$ and to (2.36), we obtain

$$
|\nabla^m v(x)| \leq C \left(f\left(B_{\mu_D}|x|^D\right) + \sum_{l=1}^{m-1} \int_{B_{\mu_D}|x|^D}^{\infty} f(\tau)\tau^{\frac{m}{D}-l-1}\, d\tau\, |x|^{lD-m} \right) \tag{2.41}
$$

for μ_D-almost every $x \in \mathbb{R}^n$, where C is a positive constant depending only on m and on D.

For every $l \in \{1, \dots m-1\}$, we now define the operator

$$F_l\colon (L^1 + L^\infty)(0,\infty) \to (L^1 + L^\infty)(0,\infty)$$

by

$$F_l(\varphi)(t) = t^{l-\frac{m}{D}} \int_t^\infty \varphi(\tau)\tau^{\frac{m}{D}-l-1}\, d\tau,\ t \in (0,\infty),\ \varphi \in (L^1 + L^\infty)(0,\infty).$$

Choose an arbitrary $l \in \{1, \dots m-1\}$. We have

$$\|F_l\|_{L^\infty \to L^\infty} \le \sup_{t \in (0,\infty)} t^{l-\frac{m}{D}} \int_t^\infty \tau^{\frac{m}{D}-l-1}\, d\tau = \frac{D}{Dl-m}. \tag{2.42}$$

Now choose an arbitrary function $\varphi \in L^1(0,\infty)$. We have

$$\begin{aligned}
&\|F_l(\varphi)\|_{L^1(0,\infty)} \le \int_0^\infty t^{l-\frac{m}{D}} \int_t^\infty |\varphi(\tau)|\, \tau^{\frac{m}{D}-l-1}\, d\tau\, dt \\
&= \int_0^\infty |\varphi(\tau)|\, \tau^{\frac{m}{D}-l-1} \int_0^\tau t^{l-\frac{m}{D}}\, dt\, d\tau = \frac{D}{Dl-m+D} \int_0^\infty |\varphi(\tau)|\, d\tau \\
&= \frac{D}{Dl-m+D} \|\varphi\|_{L^1(0,\infty)}.
\end{aligned}$$

The first equality is valid owing to the Fubini theorem. So,

$$\|F_l\|_{L^1 \to L^1} \le \frac{D}{Dl-m+D}. \tag{2.43}$$

The norm estimates (2.42) and (2.43) have two consequences. The first one is the fact that the range of the operator F_l is indeed a subset of the space $(L^1 + L^\infty)(0,\infty)$. It means that the operator F_l is well defined. The second consequence is the fact that the operator F_l is, owing to [2, Chapter 3, Theorem 2.2], bounded on the space $X(0,\infty)$ and

$$\|F_l\|_{X(0,\infty)} \le \frac{D}{Dl-m+D} \le \frac{D}{2D-m} \tag{2.44}$$

for each $l \in \{1, \dots, m-1\}$.

Now define the functions $h\colon (0,\infty) \to [0,\infty)$ and $w\colon \mathbb{R}^n \to [0,\infty)$. The function h is defined by

$$h(t) = f(t) + \sum_{l=1}^{m-1} F_l(f)(t), \quad t \in (0,\infty). \tag{2.45}$$

The function w is defined by

$$w(x) = (h \circ \sigma)(x), \quad x \in \mathbb{R}^n \setminus \{0\}.$$

Owing to (2.41) we obtain

$$|\nabla^m v|^*_{\mu_D}(t) \le \widetilde{C} w^*_{\mu_D}(t), \quad t \in (0,\infty), \tag{2.46}$$

where $\widetilde{C}$ is a positive constant which depends only on m and on D. By virtue of Lemma 2.5, the functions h and w are equimeasurable. So, we have

$$|\nabla^m v|^*_{\mu_D}(t) \le \widetilde{C} h^*(t), \quad t \in (0,\infty), \tag{2.47}$$

thanks to (1.7). Now we obtain

$$\begin{aligned}\|\nabla^m v\|_{X(\mathbb{R}^n,\mu_D)} &= \left\||\nabla^m v|^*_{\mu_D}\right\|_{X(0,\infty)} \le \widetilde{C}\,\|h^*\|_{X(0,\infty)} = \widetilde{C}\,\|h\|_{X(0,\infty)} \\ &\le \widetilde{C}\left(\|f\|_{X(0,\infty)} + \sum_{l=1}^{m-1}\|F_l(f)\|_{X(0,\infty)}\right) \le K\,\|f\|_{X(0,\infty)}\end{aligned} \tag{2.48}$$

where

$$K = \widetilde{C}\left(1 + \frac{D(m-1)}{2D-m}\right).$$

The first inequality holds by virtue of (2.46) and (2.47). The second inequality is true by (2.45). The third inequality holds owing to (2.44). From (2.48) it follows that $v \in V^m X(\mathbb{R}^n,\mu_D)$. Since the function f has a bounded support, owing to (2.32) and (2.35), we obtain there exists a constant $L > 0$ such that $\left|\nabla^k v(x)\right| = 0$ for every $x \in \mathbb{R}^n, |x| \ge L$, and for every $k \in \{0,\dots,m-1\}$. It means that $v \in V_0^m X(\mathbb{R}^n,\mu_D)$.

Now, thanks to (2.34) and to Lemma 2.5, we have the fact that the functions v and g are equimeasurable. We obtain

$$\begin{aligned}&\|v\|_{Y(\mathbb{R}^n,\mu_D)} = \left\|v^*_{\mu_D}\right\|_{Y(0,\infty)} = \|g^*\|_{Y(0,\infty)} = \|g\|_{Y(0,\infty)} \\ &\ge \left\|\int_{2t}^{\infty} f(\tau)\tau^{\frac{m}{D}-m}(\tau-t)^{m-1}\,d\tau\right\|_{Y(0,\infty)} \ge \left\|\int_{2t}^{\infty} f(\tau)\tau^{\frac{m}{D}-m}\left(\frac{\tau}{2}\right)^{m-1}\,d\tau\right\|_{Y(0,\infty)} \\ &= 2^{1-m}\left\|\int_{2t}^{\infty} f(\tau)\tau^{\frac{m}{D}-m}\tau^{m-1}\,d\tau\right\|_{Y(0,\infty)} \\ &= 2^{1-m}\left\|\int_{2t}^{\infty} f(\tau)\tau^{\frac{m}{D}-1}\,d\tau\right\|_{Y(0,\infty)}.\end{aligned} \tag{2.49}$$

Finally, we have

$$\begin{aligned}&\left\|\int_{t}^{\infty} f(\tau)\tau^{\frac{m}{D}-1}\,d\tau\right\|_{Y(0,\infty)} \le 2\left\|\int_{2t}^{\infty} f(\tau)\tau^{\frac{m}{D}-1}\,d\tau\right\|_{Y(0,\infty)} \le 2^m\,\|v\|_{Y(\mathbb{R}^n,\mu_D)} \\ &\le 2^m C_1\,\|\nabla^m v\|_{X(\mathbb{R}^n,\mu_D)} \le 2^m C_1 K\,\|f\|_{X(0,\infty)}.\end{aligned}$$

The first inequality holds by virtue of (1.25). The second inequality holds thanks to (2.49). The third inequality is true by the assumption of the theorem. The last inequality holds owing to (2.48). So, (2.23) holds with $C_2 = 2^m C_1 K$. We have proved the inequality (2.23) for bounded functions with bounded support. Now let $f \in \mathcal{M}^+(0,\infty)$, $\|f\|_{X(0,\infty)} < \infty$ be general. Define a sequence $\{f_k\}_{k=1}^{\infty}$ of functions from $\mathcal{M}^+(0,\infty)$ by $f_k(t) = \min\{f(t),k\}\chi_{(0,k)}(t), t \in (0,\infty), k \in \mathbb{N}$. Then $\{f_k\}_{k=1}^{\infty}$ is a sequence of bounded functions with bounded support such that $f_k(t) \uparrow f(t)$ for almost every $t \in (0,\infty)$. Since (2.23) holds for every $f_k, k \in \mathbb{N}$, we obtain the fact that (2.23) also holds for f thanks to the Fatou axiom of Banach function norms (3. property in Definition 1.1).

□

It remains to prove that the one-dimensional part of the reduction principle implies the n-dimensional part. We will prove it by induction on m. The first step will be Lemma 2.8. The induction step will be Proposition 2.9. In Proposition 2.11 we will combine the previous results to complete the proof.

In the following technical lemma we prove that a certain n-dimensional mapping is a rearrangement-invariant Banach function norm.

Lemma 2.7. *Let X be a rearrangement-invariant space over $(\mathbb{R}^n, \mu_D)$, let $\varphi \in \mathcal{M}^+(0,\infty)$. Assume that there exists $a > 0$ such that these two conditions hold:*

1. *there exists $A > 0$ such that $0 < \varphi(t) \leq A$ for almost every $t \in (0,a)$;*
2. *$\frac{\varphi(t)}{t}\chi_{(a,\infty)}(t) \in X(0,\infty)$.*

Then the mapping $\sigma \colon \mathcal{M}^+(\mathbb{R}^n, \mu_D) \to [0,\infty]$ defined by

$$\sigma(v) = \left\| \varphi v^{**}_{\mu_D} \right\|_{X(0,\infty)}, \quad v \in \mathcal{M}^+(\mathbb{R}^n, \mu_D),$$

is a rearrangement-invariant Banach function norm.

Proof. The fact that $\sigma(v) = 0$ if and only if $v = 0$ μ_D-almost everywhere in $\mathbb{R}^n$ follows directly from the definition of the nonincreasing rearrangement (1.6) and from the definition of the maximal nonincreasing function (1.18). The positive homogeneity and the subadditivity of σ is valid owing to (1.20) and to (1.22). So, we have proved that σ satisfies the norm axiom of Banach function norms. The fact that σ satisfies the lattice axiom and the Fatou axiom follows from (1.19) and (1.21), respectively.

Now, we prove that σ satisfies the nontriviality axiom. Choose an arbitrary set $E \subseteq \mathbb{R}^n$ such that $0 < \mu_D(E) < \infty$. For every $t \in (0,\infty)$ we have

$$(\chi_E)^{**}_{\mu_D}(t) = \chi_{(0,\mu_D(E))}(t) + \frac{\mu_D(E)}{t}\chi_{[\mu_D(E),\infty)}(t).$$

Assume that $a \leq \mu_D(E)$. We obtain

$$\begin{aligned}
\left\| \varphi \left(\chi_E\right)^{**}_{\mu_D} \right\|_{X(0,\infty)} &\leq \left\| \varphi \chi_{(0,a)} \right\|_{X(0,\infty)} + \left\| \varphi \chi_{(a,\mu_D(E))} \right\|_{X(0,\infty)} \\
&\quad + \left\| \varphi(t) \frac{\mu_D(E)}{t} \chi_{(\mu_D(E),\infty)}(t) \right\|_{X(0,\infty)} \\
&\leq A \left\| \chi_{(0,a)} \right\|_{X(0,\infty)} + 2\mu_D(E) \left\| \frac{\varphi(t)}{t} \chi_{(a,\infty)}(t) \right\|_{X(0,\infty)} < \infty.
\end{aligned}$$

Now, assume that $a > \mu_D(E)$. We obtain

$$\begin{aligned}
\left\| \varphi \left(\chi_E\right)^{**}_{\mu_D} \right\|_{X(0,\infty)} &\leq \left\| \varphi \chi_{(0,\mu_D(E))} \right\|_{X(0,\infty)} + \left\| \varphi(t) \frac{\mu_D(E)}{t} \chi_{(\mu_D(E),a)}(t) \right\|_{X(0,\infty)} \\
&\quad + \left\| \varphi(t) \frac{\mu_D(E)}{t} \chi_{(a,\infty)}(t) \right\|_{X(0,\infty)} \\
&\leq A \left\| \chi_{(0,\mu_D(E))} \right\|_{X(0,\infty)} + A \left\| \chi_{(\mu_D(E),a)} \right\|_{X(0,\infty)} + \mu_D(E) \left\| \frac{\varphi(t)}{t} \chi_{(a,\infty)}(t) \right\|_{X(0,\infty)} \\
&< \infty.
\end{aligned}$$

Now, we prove that σ satisfies the local embedding into L^1. Choose an arbitrary function $v \in \mathcal{M}^+(\mathbb{R}^n, \mu_D)$. Assume that $a \le \mu_D(E)$. We obtain

$$\begin{aligned}
&\int_E v \, d\mu_D \le \int_0^{\mu_D(E)} v^*_{\mu_D}(\tau) \, d\tau = \mu_D(E) v^{**}_{\mu_D}(\mu_D(E)) \\
&\le \left(\left\| \varphi \chi_{(0,a)} \right\|_{X(0,\infty)} \right)^{-1} \mu_D(E) \left\| \varphi \chi_{(0,a)} v^{**}_{\mu_D} \right\|_{X(0,\infty)} \\
&\le \left(\left\| \varphi \chi_{(0,a)} \right\|_{X(0,\infty)} \right)^{-1} \mu_D(E) \left\| \varphi v^{**}_{\mu_D} \right\|_{X(0,\infty)} .
\end{aligned}$$

The first inequality is true owing to the Hardy–Littlewood inequality (1.14). The second inequality is valid since the function $v^{**}_{\mu_D}$ is nonincreasing on $(0, \infty)$. The local embedding into L^1 in the case $a > \mu_D(E)$ can be proved in a similar way as in the previous case.

So, we have proved that σ is a Banach function norm. The fact that σ is rearrangement invariant follows from the definition of the maximal nonincreasing function (1.18).

□

The following lemma, in which we exploit the Pólya–Szegő inequality (Theorem 2.1), is the first step in the induction.

Lemma 2.8. *Let X be a rearrangement-invariant space over $(\mathbb{R}^n, \mu_D)$. Assume that $t^{\frac{1}{D}-1}\chi_{(1,\infty)}(t) \in X'(0,\infty)$. Define the mapping $\sigma \colon \mathcal{M}^+(\mathbb{R}^n, \mu_D) \to [0, \infty]$ by*

$$\sigma(v) = \left\| t^{\frac{1}{D}} v^{**}_{\mu_D}(t) \right\|_{X'(0,\infty)}, \quad v \in \mathcal{M}^+(\mathbb{R}^n, \mu_D).$$

Then σ is a rearrangement-invariant Banach function norm. Denote the respective Banach function space by $Z_1(\mathbb{R}^n, \mu_D)$. Then

$$\|u\|_{Z_1'(\mathbb{R}^n,\mu_D)} \le C_{iso}^{-1} \|\nabla u\|_{X(\mathbb{R}^n,\mu_D)}, \quad u \in V_0^1 X(\mathbb{R}^n, \mu_D),$$

where C_{iso} is the constant from Theorem 2.1.

Proof. The fact that σ is a rearrangement-invariant Banach function norm follows from Lemma 2.7. Observe that for every $g \in \mathcal{M}^+(0,\infty)$, we have

$$\left\| t^{\frac{1}{D}} g^{**}(t) \right\|_{X'(0,\infty)} = \|g\|_{Z_1(0,\infty)}$$

by virtue of (1.17). So, by Proposition 2.3 we obtain

$$\left\| \int_t^\infty f(\tau) \tau^{\frac{1}{D}-1} \, d\tau \right\|_{Z_1'(0,\infty)} \le \|f\|_{X(0,\infty)} \tag{2.50}$$

for every $f \in \mathcal{M}^+(0,\infty)$. In the previous inequality we also used (1.3). Choose an arbitrary function $u \in V_0^1 X(\mathbb{R}^n, \mu_D)$. We have

$$\begin{aligned}
&\|u\|_{Z_1'(\mathbb{R}^n,\mu_D)} = \left\| u^*_{\mu_D} \right\|_{Z_1'(0,\infty)} = \left\| - \int_t^\infty \frac{du^*_{\mu_D}}{d\tau}(\tau) \, d\tau \right\|_{Z_1'(0,\infty)} \\
&= \left\| \int_t^\infty \left(\tau^{\frac{D-1}{D}} \frac{du^*_{\mu_D}}{d\tau}(\tau) \right) \tau^{\frac{1-D}{D}} \, d\tau \right\|_{Z_1'(0,\infty)} \le \left\| t^{\frac{D-1}{D}} \frac{du^*_{\mu_D}}{dt}(t) \right\|_{X(0,\infty)} \\
&\le C_{iso}^{-1} \|\nabla u\|_{X(\mathbb{R}^n,\mu_D)} .
\end{aligned}$$

The second equality holds owing to (1.36) and to the fact that $u^*_{\mu_D}$ is locally absolutely continuous on $(0,\infty)$ (see Theorem 2.1). The first inequality is true thanks to (2.50). The last inequality holds owing to the Pólya–Szegő inequality (Theorem 2.1).

$\square$

The following proposition is the induction step in the induction.

Proposition 2.9. *Let X be a rearrangement-invariant space over $(\mathbb{R}^n, \mu_D)$. Assume that*

$$t^{\frac{m}{D}-1}\chi_{(1,\infty)}(t) \in X'(0,\infty).$$

Define the mapping $\sigma_m \colon \mathcal{M}^+(\mathbb{R}^n, \mu_D) \to [0,\infty]$ by

$$\sigma_m(v) = \left\| t^{\frac{m}{D}} v^{**}_{\mu_D}(t) \right\|_{X'(0,\infty)}, \quad v \in \mathcal{M}^+(\mathbb{R}^n, \mu_D). \tag{2.51}$$

Then σ_m is a rearrangement-invariant Banach function norm. Denote the respective Banach function space by Z_m, i.e.,

$$Z_m(\mathbb{R}^n, \mu_D) = Z_m(\sigma_m). \tag{2.52}$$

Then for every function $u \in V_0^m X(\mathbb{R}^n, \mu_D)$ it holds that

$$\|u\|_{Z'_m(\mathbb{R}^n,\mu_D)} \le K_m \|\nabla^m u\|_{X(\mathbb{R}^n,\mu_D)}, \tag{2.53}$$

where K_m is a positive constant, which depends only on m and on D.

Proof. The fact that σ_m is a rearrangement-invariant Banach function norm is true owing to Lemma 2.7. We prove the inequality (2.53) by induction on m. For $m = 1$ the inequality holds with $K_1 = C_{iso}^{-1}$ thanks to Lemma 2.8. Now, let $m \in \{2, \dots, \lceil D-1 \rceil\}$ and assume that the inequality (2.53) with m replaced by k holds for every $k \in \mathbb{N}, k < m$. Choose an arbitrary function $u \in V_0^m X(\mathbb{R}^n, \mu_D)$ and $i \in \{1, \dots, n\}$. Then the weak partial derivative $\frac{\partial u}{\partial x_i}$ belongs to $V_0^{m-1} X(\mathbb{R}^n, \mu_D)$. So, we can use the induction hypothesis to obtain

$$\left\| \frac{\partial u}{\partial x_i} \right\|_{Z'_{m-1}(\mathbb{R}^n,\mu_D)} \le K_{m-1} \left\| \nabla^{m-1} \frac{\partial u}{\partial x_i} \right\|_{X(\mathbb{R}^n,\mu_D)} \le K_{m-1} \|\nabla^m u\|_{X(\mathbb{R}^n,\mu_D)}.$$

It means that

$$\|\nabla u\|_{Z'_{m-1}(\mathbb{R}^n,\mu_D)} \le n K_{m-1} \|\nabla^m u\|_{X(\mathbb{R}^n,\mu_D)} < \infty. \tag{2.54}$$

It follows that u belongs to $V_0^1 Z'_{m-1}(\mathbb{R}^n, \mu_D)$. Now we show that

$$t^{\frac{1}{D}-1}\chi_{(1,\infty)}(t) \in Z_{m-1}(0,\infty). \tag{2.55}$$

We have

$$\left(\tau^{\frac{1}{D}-1}\chi_{(1,\infty)}(\tau)\right)^*(t) = (t+1)^{\frac{1}{D}-1} \tag{2.56}$$

for every $t \in (0, \infty)$. We obtain

$$
\begin{aligned}
&\left\| t^{\frac{1}{D}-1} \chi_{(1,\infty)}(t) \right\|_{Z_{m-1}(0,\infty)} = \left\| (t+1)^{\frac{1}{D}-1} \right\|_{Z_{m-1}(0,\infty)} \\
&= \left\| t^{\frac{m-1}{D}} \frac{1}{t} \int_0^t (\tau+1)^{\frac{1}{D}-1} \, d\tau \right\|_{X'(0,\infty)} = D \left\| t^{\frac{m-1}{D}-1} \left((t+1)^{\frac{1}{D}} - 1 \right) \right\|_{X'(0,\infty)} \\
&\le D \left\| t^{\frac{m-1}{D}-1} \left((t+1)^{\frac{1}{D}} - 1 \right) \chi_{(0,1)}(t) \right\|_{X'(0,\infty)} \\
&\quad + D \left\| t^{\frac{m-1}{D}-1} \left((t+1)^{\frac{1}{D}} - 1 \right) \chi_{(1,\infty)}(t) \right\|_{X'(0,\infty)} \\
&\le D \left\| t^{\frac{m-1}{D}-1} \left((t+1)^{\frac{1}{D}} - 1 \right) \chi_{(0,1)}(t) \right\|_{X'(0,\infty)} + D \left\| t^{\frac{m}{D}-1} \chi_{(1,\infty)}(t) \right\|_{X'(0,\infty)} < \infty.
\end{aligned}
$$

The first equality holds thanks to (2.56). The second inequality holds since $(t+1)^{\frac{1}{D}} \le t^{\frac{1}{D}} + 1$ for every $t \in (1, \infty)$. The last inequality is true thanks to the fact that the function $t^{\frac{m-1}{D}-1} \left((t+1)^{\frac{1}{D}} - 1 \right)$ is bounded on $(0,1)$ since $\lim_{t \to 0_+} t^{\frac{m-1}{D}-1} \left((t+1)^{\frac{1}{D}} - 1 \right) = 0$. It means that (2.55) holds. Now, we can use Lemma 2.8 with the space $Z'_{m-1}(\mathbb{R}^n, \mu_D)$ to obtain

$$
\|u\|_{W'(\mathbb{R}^n, \mu_D)} \le C_{iso}^{-1} \|\nabla u\|_{Z'_{m-1}(\mathbb{R}^n, \mu_D)}, \tag{2.57}
$$

where $\|u\|_{W(\mathbb{R}^n, \mu_D)} = \left\| t^{\frac{1}{D}} u^{**}_{\mu_D}(t) \right\|_{Z_{m-1}(0,\infty)}$. Owing to [12, Theorem 3.4] and [27, Proposition 5.1] (cf. [14, Theorem 9.5]), we obtain

$$
\|v\|_{W(\mathbb{R}^n, \mu_D)} \le C \left\| t^{\frac{m}{D}} v^{**}_{\mu_D}(t) \right\|_{X'(0,\infty)} = C \|v\|_{Z_m(\mathbb{R}^n, \mu_D)}, \quad v \in \mathcal{M}^+(\mathbb{R}^n, \mu_D), \tag{2.58}
$$

where the positive constant C depends only on m and on D. Thanks to (2.58) and (1.5), we obtain

$$
\|v\|_{Z'_m(\mathbb{R}^n, \mu_D)} \le C \|v\|_{W'(\mathbb{R}^n, \mu_D)}, \quad v \in \mathcal{M}^+(\mathbb{R}^n, \mu_D). \tag{2.59}
$$

By virtue of (2.54), (2.57) and (2.59), we obtain the fact that (2.53) holds with $K_m = nCC_{iso}^{-1} K_{m-1}$.

□

The following lemma guarantees that we can use Proposition 2.9 to prove Proposition 2.11.

Lemma 2.10. *Let X, Y be rearrangement-invariant spaces over $(\mathbb{R}^n, \mu_D)$. Assume that there exists a positive constant C such that for all functions $f \in \mathcal{M}^+(0, \infty)$ it holds that*

$$
\left\| \int_t^\infty f(\tau) \tau^{\frac{m}{D}-1} \, d\tau \right\|_{Y(0,\infty)} \le C \|f\|_{X(0,\infty)}. \tag{2.60}
$$

Then

$$
t^{\frac{m}{D}-1} \chi_{(1,\infty)}(t) \in X'(0, \infty). \tag{2.61}
$$

Proof. We have

$$\begin{aligned}
&\left\|t^{\frac{m}{D}-1}\chi_{(1,\infty)}(t)\right\|_{X'(0,\infty)} = \sup_{f\in\mathcal{M}^+(0,\infty),\|f\|_{X(0,\infty)}\le 1}\int_1^\infty f(\tau)\tau^{\frac{m}{D}-1}\,d\tau\\
&= \left(\left\|\chi_{(0,1)}\right\|_{Y(0,\infty)}\right)^{-1}\sup_{f\in\mathcal{M}^+(0,\infty),\|f\|_{X(0,\infty)}\le 1}\left\|\int_1^\infty f(\tau)\tau^{\frac{m}{D}-1}\,d\tau\chi_{(0,1)}(t)\right\|_{Y(0,\infty)}\\
&\le \left(\left\|\chi_{(0,1)}\right\|_{Y(0,\infty)}\right)^{-1}\sup_{f\in\mathcal{M}^+(0,\infty),\|f\|_{X(0,\infty)}\le 1}\left\|\int_t^\infty f(\tau)\tau^{\frac{m}{D}-1}\,d\tau\chi_{(0,1)}(t)\right\|_{Y(0,\infty)}\\
&\le \left(\left\|\chi_{(0,1)}\right\|_{Y(0,\infty)}\right)^{-1}\sup_{f\in\mathcal{M}^+(0,\infty),\|f\|_{X(0,\infty)}\le 1}\left\|\int_t^\infty f(\tau)\tau^{\frac{m}{D}-1}\,d\tau\right\|_{Y(0,\infty)}\\
&\le C\left(\left\|\chi_{(0,1)}\right\|_{Y(0,\infty)}\right)^{-1},
\end{aligned}$$

where the last inequality holds by virtue of (2.60). It means that (2.61) is true. □

Thanks to Lemma 2.10 we can use Proposition 2.9 to prove the remaining part of the reduction principle.

Proposition 2.11. *Let X, Y be rearrangement-invariant spaces over $(\mathbb{R}^n, \mu_D)$. Assume that there exists a positive constant C_2 such that for all functions $f \in \mathcal{M}^+(0,\infty)$ it holds that*

$$\left\|\int_t^\infty f(\tau)\tau^{\frac{m}{D}-1}\,d\tau\right\|_{Y(0,\infty)} \le C_2\,\|f\|_{X(0,\infty)}\,. \tag{2.62}$$

Then there exists a positive constant C_1 such that for every $u \in V_0^m X(\mathbb{R}^n, \mu_D)$ it holds that

$$\|u\|_{Y(\mathbb{R}^n,\mu_D)} \le C_1\,\|\nabla^m u\|_{X(\mathbb{R}^n,\mu_D)}\,.$$

The constant C_1 depends only on the constant C_2, on m and on D.

Proof. Firstly, we use Lemma 2.10 to obtain the fact that

$$t^{\frac{m}{D}-1}\chi_{(1,\infty)}(t) \in X'(0,\infty).$$

It means that the mapping σ_m defined in (2.51) is a rearrangement-invariant Banach function norm owing to Proposition 2.9. Recall that the respective Banach function space is denoted by $Z_m(\mathbb{R}^n, \mu_D)$ (see (2.52)).

We prove that for every function $u \in V_0^m X(\mathbb{R}^n, \mu_D)$ it holds that

$$\|u\|_{Y(\mathbb{R}^n,\mu_D)} \le C_2\,\|u\|_{Z_m'(\mathbb{R}^n,\mu_D)} \le C_2 K_m\,\|\nabla^m u\|_{X(\mathbb{R}^n,\mu_D)}\,, \tag{2.63}$$

where K_m is the positive constant from the inequality (2.53). Since (2.62) holds, we can use Proposition 2.3 to obtain

$$\|v\|_{Z_m(\mathbb{R}^n,\mu_D)} \le C_2\,\|v\|_{Y'(\mathbb{R}^n,\mu_D)}\,, \quad v \in \mathcal{M}^+(\mathbb{R}^n, \mu_D).$$

So, the first inequality in (2.63) is true owing to (1.5), and the second one holds thanks to Proposition 2.9.

□

Finally, we show how to combine the previous results to obtain the proof of the reduction principle.

Proof of Theorem 2.2. Equivalence of the second statement and the third statement follows from Proposition 2.3. The fact that the first statement implies the second statement follows from Proposition 2.4. The fact that the second statement implies the first one follows from Proposition 2.11.

□

3. Optimality

In this chapter we will firstly prove that for a given rearrangement-invariant Banach function space $X(\mathbb{R}^n, \mu_D)$, the space $Z'_m(\mathbb{R}^n, \mu_D)$ defined in (2.52) is the optimal (i.e., the smallest) target space in the inequality (2.16) among all rearrangement-invariant spaces $Y(\mathbb{R}^n, \mu_D)$. Then we will show some examples. We will describe the optimal space $Z'_m(\mathbb{R}^n, \mu_D)$ when $X(\mathbb{R}^n, \mu_D)$ is a Lorentz–Karamata space $L^{p,q,b}(\mathbb{R}^n, \mu_D)$ with $p \in \left[1, \frac{D}{m}\right]$.

Theorem 3.1. *Let X be a rearrangement-invariant space over $(\mathbb{R}^n, \mu_D)$. Assume that*

$$t^{\frac{m}{D}-1}\chi_{(1,\infty)}(t) \in X'(0,\infty). \tag{3.1}$$

Let $Z_m(\mathbb{R}^n, \mu_D)$ be the rearrangement-invariant Banach function space defined in (2.52). Then there exists a constant $C > 0$, which depends only on m and on D, such that

$$\|u\|_{Z'_m(\mathbb{R}^n,\mu_D)} \leq C \, \|\nabla^m u\|_{X(\mathbb{R}^n,\mu_D)}, \quad u \in V_0^m X(\mathbb{R}^n, \mu_D). \tag{3.2}$$

Moreover, the space $Z'_m(\mathbb{R}^n, \mu_D)$ is the optimal space in the previous inequality among all rearrangement-invariant spaces in the following way. If $Y(\mathbb{R}^n, \mu_D)$ is a rearrangement-invariant space satisfying

$$\|u\|_{Y(\mathbb{R}^n,\mu_D)} \leq \widetilde{C} \, \|\nabla^m u\|_{X(\mathbb{R}^n,\mu_D)}, \quad u \in V_0^m X(\mathbb{R}^n, \mu_D), \tag{3.3}$$

with a positive constant $\widetilde{C}$ that does not depend on u, then

$$Z'_m(\mathbb{R}^n, \mu_D) \hookrightarrow Y(\mathbb{R}^n, \mu_D). \tag{3.4}$$

On the other hand, if (3.1) is not true, then the inequality (3.3) does not hold for any rearrangement-invariant space $Y(\mathbb{R}^n, \mu_D)$.

Proof. The inequality (3.2) is true thanks to Proposition 2.9. Assume that (3.1) holds and that $Y(\mathbb{R}^n, \mu_D)$ is a rearrangement-invariant space satisfying (3.3). Owing to Proposition 2.4 we obtain the fact that $Y(\mathbb{R}^n, \mu_D)$ satisfies the inequality (2.17). It means that we can use the first inequality in (2.63) to obtain the embedding (3.4).

Now, assume that (3.1) is not true. Then owing to Lemma 2.10 there does not exist any rearrangement-invariant space $Y(\mathbb{R}^n, \mu_D)$ such that (2.17) holds. Then by Proposition 2.4 there is not any rearrangement-invariant space $Y(\mathbb{R}^n, \mu_D)$ such that (3.3) is true.

□

In the following proposition we show that the norm of the space $Z_m(\mathbb{R}^n, \mu_D)$ is equivalent to a different rearrangement-invariant function norm provided the boundedness of the maximal nonincreasing operator P. We will use this equivalence in Theorem 3.3.

Proposition 3.2. *Let X be a rearrangement-invariant space over $(\mathbb{R}^n, \mu_D)$. Assume that the maximal nonincreasing operator P is bounded on the space $X(0,\infty)$. Then the functional σ_m defined in (2.51) is equivalent to the functional*

$$u \mapsto \left\| \int_t^\infty u^{**}_{\mu_D}(\tau) \tau^{\frac{m}{D}-1} \, d\tau \right\|_{X'(0,\infty)}, \quad u \in \mathcal{M}^+(\mathbb{R}^n, \mu_D). \tag{3.5}$$

Proof. For every $u \in \mathcal{M}^+(\mathbb{R}^n, \mu_D)$ we have

$$\left\| \int_t^\infty u^{**}_{\mu_D}(\tau) \tau^{\frac{m}{D}-1} \, d\tau \right\|_{X'(0,\infty)} = \sup_{\|g\|_{X(0,\infty)} \le 1} \int_0^\infty g^*(t) \int_t^\infty u^{**}_{\mu_D}(\tau) \tau^{\frac{m}{D}-1} \, d\tau dt$$
$$= \sup_{\|g\|_{X(0,\infty)} \le 1} \int_0^\infty u^{**}_{\mu_D}(\tau) \tau^{\frac{m}{D}-1} \int_0^\tau g^*(t) \, dt d\tau = \sup_{\|g\|_{X(0,\infty)} \le 1} \int_0^\infty u^{**}_{\mu_D}(\tau) \tau^{\frac{m}{D}} g^{**}(\tau) \, d\tau.$$

The first equality holds thanks to (1.13). Since the operator P is bounded on the space $X(0,\infty)$, we obtain

$$\sup_{\|g\|_{X(0,\infty)} \le 1} \int_0^\infty u^{**}_{\mu_D}(\tau) \tau^{\frac{m}{D}} g^{**}(\tau) \, d\tau \le \|P\|_{X(0,\infty) \to X(0,\infty)} \left\| t^{\frac{m}{D}} u^{**}_{\mu_D}(t) \right\|_{X'(0,\infty)}$$

owing to (1.4).

On the other hand, for every $u \in \mathcal{M}^+(\mathbb{R}^n, \mu_D)$, we have

$$\|u\|_{Z_m(\mathbb{R}^n,\mu_D)} = \left\| t^{\frac{m}{D}} u^{**}_{\mu_D}(t) \right\|_{X'(0,\infty)} \le 4 \sup_{\|g\|_{X(0,\infty)} \le 1} \int_0^\infty g^*(\tau) \tau^{\frac{m}{D}} u^{**}_{\mu_D}(\tau) \, d\tau$$
$$\le 4 \sup_{\|g\|_{X(0,\infty)} \le 1} \int_0^\infty g^{**}(\tau) \tau^{\frac{m}{D}} u^{**}_{\mu_D}(\tau) \, d\tau,$$

where the first inequality holds owing to (1.24).

□

Remark. If the condition (3.1) is satisfied, then, by virtue of [17, Theorem 6.3], we obtain the fact that the functional (3.5) is a rearrangement-invariant Banach function norm.

Remark. Assume that $D \in \mathbb{N}$ and that we work on the space $\mathbb{R}^D$ with the Lebesgue measure. The operator I_m defined by

$$I_m(u)(x) = \pi^{\frac{D}{2}} 2^m \frac{\Gamma(\frac{m}{2})}{\Gamma(\frac{D-m}{2})} \int_{\mathbb{R}^D} \frac{u(y)}{|x-y|^{D-m}} \, dy$$

for those functions $u \in \mathcal{M}(\mathbb{R}^D)$ for which the integral exists for almost every $x \in \mathbb{R}^D$ is called the *Riesz potential* of order m. Assume that X is a rearrangement-invariant Banach function space over $\mathbb{R}^D$ such that $t^{\frac{m}{D}-1}\chi_{(1,\infty)}(t) \in X'(0,\infty)$ and that the maximal nonincreasing operator P is bounded on the space $X(0,\infty)$. We say that the rearrangement-invariant Banach function space $Y(\mathbb{R}^D)$ is a target space for the Riesz potential defined on $X(\mathbb{R}^D)$ if the Riesz potential is bounded from $X(\mathbb{R}^D)$ to $Y(\mathbb{R}^D)$. By virtue of Theorem 3.1, Proposition 3.2 and [17, Theorem 6.3], we obtain the following facts. There exists the optimal target space $Y_m(\mathbb{R}^D)$ for the Riesz potential defined on $X(\mathbb{R}^D)$ and this space is equivalent to the space $Z'_m(\mathbb{R}^D)$.

Now we start describing the optimal space for the Lorentz–Karamata space $L^{p,q,b}(\mathbb{R}^n, \mu_D)$. In the following theorem we deal with the case $p \in \left[1, \frac{D}{m}\right)$.

Theorem 3.3. *Let $p \in \left[1, \frac{D}{m}\right)$, let $q \in [1, \infty]$ and let b be a slowly varying function. Furthermore, assume that one of the following conditions is satisfied:*

1. $p \in \left(1, \frac{D}{m}\right)$,
2. $p = q = 1$ *and b is equivalent to a nonincreasing function on $(0, \infty)$.*

Then the optimal space in (3.2) for $X(\mathbb{R}^n, \mu_D)$ equal to the Lorentz–Karamata space $L^{p,q,b}(\mathbb{R}^n, \mu_D)$ is the space $Z'_m(\mathbb{R}^n, \mu_D)$ defined by (2.52). Moreover, the optimal space $Z'_m(\mathbb{R}^n, \mu_D)$ is equivalent to $L^{\frac{Dp}{D-mp},q,b}(\mathbb{R}^n, \mu_D)$.

Proof. Owing to (1.33) we obtain the fact that the space $\left(L^{p,q,b}(\mathbb{R}^n, \mu_D)\right)'$ is equivalent to the space $L^{p',q',b^{-1}}(\mathbb{R}^n, \mu_D)$. Recall that b^{-1} is the function $\frac{1}{b}$. We have

$$\left\| t^{\frac{m}{D}-1} \chi_{(1,\infty)}(t) \right\|_{L^{p',q',b^{-1}}(0,\infty)} = \left\| t^{\frac{1}{p'}-\frac{1}{q'}} (t+1)^{\frac{m}{D}-1} b^{-1}(t) \right\|_{L^{q'}(0,\infty)} < \infty. \tag{3.6}$$

The equality holds since

$$\left(\tau^{\frac{m}{D}-1} \chi_{(1,\infty)}(\tau)\right)^{*}(t) = (t+1)^{\frac{m}{D}-1} \tag{3.7}$$

for every $t \in (0, \infty)$. We show that the inequality is also valid. Recall the fact that, for every $r \in \mathbb{R}$, b^r is a slowly varying function. If $q \in (1, \infty]$, then $p \in \left(1, \frac{D}{m}\right)$. We have

$$\int_0^1 \tau^{\frac{q'}{p'}-1} (\tau+1)^{q'(\frac{m}{D}-1)} b^{-q'}(\tau)\, d\tau < \infty \tag{3.8}$$

owing to (1.27). We also have

$$\int_1^\infty \tau^{\frac{q'}{p'}-1} (\tau+1)^{q'(\frac{m}{D}-1)} b^{-q'}(\tau)\, d\tau < \int_1^\infty \tau^{q'(\frac{1}{p'}+\frac{m}{D}-1)-1} b^{-q'}(\tau)\, d\tau < \infty. \tag{3.9}$$

The second inequality holds thanks to (1.28) since $\frac{1}{p'} + \frac{m}{D} - 1 < 0$. The inequality in (3.6) is now true by virtue of (3.8) and (3.9). If $q = 1$, then $p \in \left[1, \frac{D}{m}\right)$. We have

$$\sup_{t \in (1,\infty)} t^{\frac{1}{p'}} (t+1)^{\frac{m}{D}-1} b^{-1}(t) < \sup_{t \in (1,\infty)} t^{\frac{1}{p'}+\frac{m}{D}-1} b^{-1}(t) < \infty. \tag{3.10}$$

The second inequality holds owing to (1.30) since $\frac{1}{p'} + \frac{m}{D} - 1 < 0$. We also have

$$\sup_{t \in (0,1)} t^{\frac{1}{p'}} (t+1)^{\frac{m}{D}-1} b^{-1}(t) < \infty. \tag{3.11}$$

If $p \in \left(1, \frac{D}{m}\right)$, then the inequality is valid by virtue of (1.29) since $\frac{1}{p'} > 0$. If $p = 1$, then we assume that b is equivalent to a nonincreasing function on $(0, \infty)$. It means that the function b^{-1} is equivalent to a nondecreasing function on $(0, \infty)$.

So, the inequality (3.11) is valid since b^{-1} is bounded on $(0,1)$. The inequality in (3.6) is now valid owing to (3.10) and (3.11). It means that owing to Theorem 3.1, the space $Z_m'(\mathbb{R}^n,\mu_D)$ is the optimal space for $L^{p,q,b}(\mathbb{R}^n,\mu_D)$.

It remains to prove that the space $Z_m'(\mathbb{R}^n,\mu_D)$ is equivalent to the space $L^{\frac{Dp}{D-mp},q,b}(\mathbb{R}^n,\mu_D)$. Firstly, assume that $p\in\left(1,\frac{D}{m}\right)$. Owing to (1.32) the maximal nonincreasing operator P is bounded on the space $L^{p,q,b}(0,\infty)$. It means that by virtue of Proposition 3.2, the norm of the space $Z_m(\mathbb{R}^n,\mu_D)$ is equivalent to the functional

$$u\mapsto\left\|t^{\frac{1}{q}-\frac{1}{p}}b^{-1}(t)\int_t^\infty u^{**}_{\mu_D}(\tau)\tau^{\frac{m}{D}-1}\,d\tau\right\|_{L^{q'}(0,\infty)},\quad u\in\mathcal{M}^+(\mathbb{R}^n,\mu_D).\tag{3.12}$$

The associate space $\left(L^{\frac{Dp}{D-mp},q,b}(\mathbb{R}^n,\mu_D)\right)'$ is equivalent to $L^{\left(\frac{Dp}{D-mp}\right)',q',b^{-1}}(\mathbb{R}^n,\mu_D)$ thanks to (1.33). We have

$$\left(\frac{Dp}{D-mp}\right)'=\frac{\frac{Dp}{D-mp}}{\frac{Dp}{D-mp}-1}=\frac{Dp}{Dp-D+mp}.$$

It means that the norm of $L^{\left(\frac{Dp}{D-mp}\right)',q',b^{-1}}(\mathbb{R}^n,\mu_D)$ is equivalent to the functional

$$u\mapsto\left\|t^{\frac{1}{q}-\frac{1}{p}+\frac{m}{D}}b^{-1}(t)u^{**}_{\mu_D}(t)\right\|_{L^{q'}(0,\infty)},\quad u\in\mathcal{M}^+(\mathbb{R}^n,\mu_D),\tag{3.13}$$

owing to (1.32). By virtue of (1.5) it remains to prove that the functionals (3.12) and (3.13) are equivalent. Since

$$t^{\frac{m}{D}}=\frac{m}{D\left(1-2^{-\frac{m}{D}}\right)}\int_{\frac{t}{2}}^t\tau^{\frac{m}{D}-1}\,d\tau,$$

we obtain

$$t^{\frac{m}{D}}u^{**}_{\mu_D}(t)\le\frac{m}{D\left(1-2^{-\frac{m}{D}}\right)}\int_{\frac{t}{2}}^t u^{**}_{\mu_D}(\tau)\tau^{\frac{m}{D}-1}\,d\tau\le\frac{m}{D\left(1-2^{-\frac{m}{D}}\right)}\int_{\frac{t}{2}}^\infty u^{**}_{\mu_D}(\tau)\tau^{\frac{m}{D}-1}\,d\tau\tag{3.14}$$

for every $u\in\mathcal{M}^+(\mathbb{R}^n,\mu_D)$ and for every $t\in(0,\infty)$. In the first inequality we also used the fact that the function $u^{**}_{\mu_D}$ is nonincreasing. By virtue of (1.26) and (3.14), we obtain existence of a constant $C_1>0$ such that

$$t^{\frac{1}{q}-\frac{1}{p}+\frac{m}{D}}b^{-1}(t)u^{**}_{\mu_D}(t)\le\frac{2^{\frac{1}{q}-\frac{1}{p}}mC_1}{D\left(1-2^{-\frac{m}{D}}\right)}\left(\frac{t}{2}\right)^{\frac{1}{q}-\frac{1}{p}}b^{-1}\left(\frac{t}{2}\right)\int_{\frac{t}{2}}^\infty u^{**}_{\mu_D}(\tau)\tau^{\frac{m}{D}-1}\,d\tau\tag{3.15}$$

for every $u\in\mathcal{M}^+(\mathbb{R}^n,\mu_D)$ and for every $t\in(0,\infty)$. Owing to (1.25) and (3.15) we obtain existence of a constant $C_2>0$ such that

$$\left\|t^{\frac{1}{q}-\frac{1}{p}+\frac{m}{D}}b^{-1}(t)u^{**}_{\mu_D}\right\|_{L^{q'}(0,\infty)}\le C_2\left\|t^{\frac{1}{q}-\frac{1}{p}}b^{-1}(t)\int_t^\infty u^{**}_{\mu_D}(\tau)\tau^{\frac{m}{D}-1}\,d\tau\right\|_{L^{q'}(0,\infty)}$$

for every $u \in \mathcal{M}^+(\mathbb{R}^n, \mu_D)$. Now we prove the opposite inequality, i.e., we prove that there exists a constant $K_1 > 0$ such that

$$\left\| t^{\frac{1}{q}-\frac{1}{p}} b^{-1}(t) \int_t^\infty u^{**}_{\mu_D}(\tau) \tau^{\frac{m}{D}-1}\, d\tau \right\|_{L^{q'}(0,\infty)} \le K_1 \left\| t^{\frac{1}{q}-\frac{1}{p}+\frac{m}{D}} b^{-1}(t) u^{**}_{\mu_D}(t) \right\|_{L^{q'}(0,\infty)}, \tag{3.16}$$

for every $u \in \mathcal{M}^+(\mathbb{R}^n, \mu_D)$. By virtue of the weighted Hardy inequality ([28, Theorem 2]), the inequality (3.16) is valid if

$$\sup_{t\in(0,\infty)} \left\| \tau^{\frac{1}{q}-\frac{1}{p}} b^{-1}(\tau) \right\|_{L^{q'}(0,t)} \left\| \tau^{\frac{1}{p}-\frac{1}{q}-1} b(\tau) \right\|_{L^{q}(t,\infty)} < \infty. \tag{3.17}$$

Again, recall the fact that, for every $r \in \mathbb{R}$, b^r is a slowly varying function. If $q \in (1, \infty]$, then we have

$$\left\| \tau^{\frac{1}{q}-\frac{1}{p}} b^{-1}(\tau) \right\|_{L^{q'}(0,t)} \approx t^{1-\frac{1}{p}} b^{-1}(t) \tag{3.18}$$

on $(0, \infty)$ thanks to (1.27) since $q'(\frac{1}{q} - \frac{1}{p}) = \frac{q(p-1)}{p(q-1)} - 1$ and $\frac{q(p-1)}{p(q-1)} > 0$. If $q \in [1, \infty)$, then we have

$$\left\| \tau^{\frac{1}{p}-\frac{1}{q}-1} b(\tau) \right\|_{L^{q}(t,\infty)} \approx t^{\frac{1}{p}-1} b(t) \tag{3.19}$$

on $(0, \infty)$ owing to (1.28) since $q(\frac{1}{p} - \frac{1}{q} - 1) = q(\frac{1}{p} - 1) - 1$ and $q(\frac{1}{p} - 1) < 0$. If $q = 1$, then we have

$$\sup_{\tau\in(0,t)} \tau^{1-\frac{1}{p}} b^{-1}(\tau) \approx t^{1-\frac{1}{p}} b^{-1}(t) \tag{3.20}$$

on $(0, \infty)$ by virtue of (1.29). If $q = \infty$, then we have

$$\sup_{\tau\in(t,\infty)} \tau^{\frac{1}{p}-1} b(\tau) \approx t^{\frac{1}{p}-1} b(t) \tag{3.21}$$

on $(0, \infty)$ owing to (1.30). If we combine (3.18)–(3.21), we obtain the fact that the function

$$t \mapsto \left\| \tau^{\frac{1}{q}-\frac{1}{p}} b^{-1}(\tau) \right\|_{L^{q'}(0,t)} \left\| \tau^{\frac{1}{p}-\frac{1}{q}-1} b(\tau) \right\|_{L^{q}(t,\infty)}, \quad t \in (0, \infty)$$

is equivalent to a constant on $(0, \infty)$ for every $q \in [1, \infty]$. It means that the inequality (3.17) holds for every $q \in [1, \infty]$.

Now we prove that the space $Z'_m(\mathbb{R}^n, \mu_D)$ is equivalent to $L^{\frac{Dp}{D-mp},q,b}(\mathbb{R}^n, \mu_D)$ under the assumption that $p = q = 1$ and that b is equivalent to a nonincreasing function on $(0, \infty)$. It means that we want to prove that the space $Z'_m(\mathbb{R}^n, \mu_D)$ is equivalent to the space $L^{\frac{D}{D-m},1,b}(\mathbb{R}^n, \mu_D)$. Thanks to (1.33) we have $\left(L^{\frac{D}{D-m},1,b}(\mathbb{R}^n, \mu_D)\right)' = L^{\frac{D}{m},\infty,b^{-1}}(\mathbb{R}^n, \mu_D)$. Owing to (1.5) it is sufficient to prove that the space $Z_m(\mathbb{R}^n, \mu_D)$ is equivalent to the space $L^{\frac{D}{m},\infty,b^{-1}}(\mathbb{R}^n, \mu_D)$. Thanks to (1.32) the space $L^{\frac{D}{m},\infty,b^{-1}}(\mathbb{R}^n, \mu_D)$ is equivalent to $L^{(\frac{D}{m},\infty,b^{-1})}(\mathbb{R}^n, \mu_D)$, so, it is sufficient to prove that $Z_m(\mathbb{R}^n, \mu_D)$ is equivalent to $L^{(\frac{D}{m},\infty,b^{-1})}(\mathbb{R}^n, \mu_D)$. By virtue of (1.33) we obtain

$$\|u\|_{Z_m(\mathbb{R}^n,\mu_D)} = \left\| t^{\frac{m}{D}} u^{**}_{\mu_D}(t) \right\|_{L^{\infty,\infty,b^{-1}}(0,\infty)} \tag{3.22}$$

for each $u \in \mathcal{M}^+(\mathbb{R}^n, \mu_D)$. For every $u \in \mathcal{M}^+(\mathbb{R}^n, \mu_D)$ we have

$$\begin{aligned}
\|u\|_{Z_m(\mathbb{R}^n,\mu_D)} &\leq \left\| \sup_{\tau \in [t,\infty)} \tau^{\frac{m}{D}} u^{**}_{\mu_D}(\tau) \right\|_{L^{\infty,\infty,b^{-1}}(0,\infty)} = \sup_{t\in(0,\infty)} b^{-1}(t) \sup_{\tau\in[t,\infty)} \tau^{\frac{m}{D}} u^{**}_{\mu_D}(\tau) \\
&= \sup_{\tau\in(0,\infty)} \tau^{\frac{m}{D}} u^{**}_{\mu_D}(\tau) \sup_{t\in(0,\tau]} b^{-1}(t) \leq K_2 \sup_{t\in(0,\infty)} b^{-1}(t) t^{\frac{m}{D}} u^{**}_{\mu_D}(t) \\
&= K_2 \|u\|_{L^{(\frac{D}{m},\infty,b^{-1})}(\mathbb{R}^n,\mu_D)},
\end{aligned}$$

where $K_2 > 0$ is a constant that does not depend on u. The first inequality follows from (3.22). The last inequality holds since b^{-1} is equivalent to a nondecreasing function on $(0,\infty)$. So, we have proved the first inequality. To prove the opposite inequality, we exploit [17, Lemma 4.10]. This lemma works with the concept of a quasiconcave function. A function $\varphi\colon [0,\infty) \to [0,\infty)$ is quasiconcave if $\varphi(0) = 0$, φ is nondecreasing on $[0,\infty)$ and $\frac{\varphi(t)}{t}$ is nonincreasing on $(0,\infty)$. Owing to the lemma we have the fact that if $X(0,\infty)$ is a rearrangement-invariant space and φ is a quasiconcave function, then there exists a constant $K_3 > 0$ such that

$$\left\| \sup_{\tau\in[t,\infty)} \varphi(\tau) u^{**}_{\mu_D}(\tau) \right\|_{X(0,\infty)} \leq K_3 \left\| \varphi(t) u^{**}_{\mu_D}(t) \right\|_{X(0,\infty)} \tag{3.23}$$

for every $u \in \mathcal{M}^+(\mathbb{R}^n, \mu_D)$. For every $u \in \mathcal{M}^+(\mathbb{R}^n, \mu_D)$ we obtain

$$\begin{aligned}
\|u\|_{L^{(\frac{D}{m},\infty,b^{-1})}(\mathbb{R}^n,\mu_D)} &= \sup_{t\in(0,\infty)} b^{-1}(t) t^{\frac{m}{D}} u^{**}_{\mu_D}(t) \leq \sup_{t\in(0,\infty)} b^{-1}(t) \sup_{\tau\in[t,\infty)} \tau^{\frac{m}{D}} u^{**}_{\mu_D}(\tau) \\
&= \left\| \sup_{\tau\in[t,\infty)} \tau^{\frac{m}{D}} u^{**}_{\mu_D}(\tau) \right\|_{L^{\infty,\infty,b^{-1}}(0,\infty)} \leq K_4 \left\| t^{\frac{m}{D}} u^{**}_{\mu_D}(t) \right\|_{L^{\infty,\infty,b^{-1}}(0,\infty)} \\
&= K_4 \left\| u^{**}_{\mu_D} \right\|_{Z_m(\mathbb{R}^n,\mu_D)},
\end{aligned}$$

where $K_4 > 0$ is a constant that is independent of u. The last inequality holds owing to (3.23) since the function $t \mapsto t^{\frac{m}{D}}, t \in [0,\infty)$, is quasiconcave. The last equality is valid thanks to (3.22).

□

In the rest of this chapter, we consider the space $L^{\frac{D}{m},q,b}(\mathbb{R}^n, \mu_D)$.

Proposition 3.4. *Let $q \in [1,\infty]$ and let b be a slowly varying function. If*

$$\left\| t^{-\frac{1}{q'}} b^{-1}(t) \right\|_{L^{q'}(1,\infty)} < \infty, \tag{3.24}$$

then the optimal space in (3.2) for $X(\mathbb{R}^n, \mu_D)$ equal to the Lorentz–Karamata space $L^{\frac{D}{m},q,b}(\mathbb{R}^n, \mu_D)$ exists and it is equal to the space $Z'_m(\mathbb{R}^n, \mu_D)$ defined by (2.52). On the other hand, if (3.24) does not hold, then the inequality (3.2) for $X(\mathbb{R}^n, \mu_D)$ equal to $L^{\frac{D}{m},q,b}(\mathbb{R}^n, \mu_D)$ does not hold for any rearrangement-invariant space $Y(\mathbb{R}^n, \mu_D)$.

Proof. We have the fact that $\left(L^{\frac{D}{m},q,b}(\mathbb{R}^n, \mu_D)\right)' = L^{\frac{D}{D-m},q',b^{-1}}(\mathbb{R}^n, \mu_D)$ thanks to (1.33). In view of Theorem 3.1 it is enough to prove that (3.24) is true if and only if

$$\left\| t^{\frac{m}{D}-1} \chi_{(1,\infty)}(t) \right\|_{L^{\frac{D}{D-m},q',b^{-1}}(0,\infty)} < \infty. \tag{3.25}$$

We have

$$\left\| t^{\frac{m}{D}-1}\chi_{(1,\infty)}(t) \right\|_{L^{\frac{D}{D-m},q',b^{-1}}(0,\infty)} = \left\| t^{1-\frac{m}{D}-\frac{1}{q'}}(t+1)^{\frac{m}{D}-1}b^{-1}(t) \right\|_{L^{q'}(0,\infty)} \tag{3.26}$$

by virtue of (3.7). Now, we prove that

$$\left\| t^{1-\frac{m}{D}-\frac{1}{q'}}(t+1)^{\frac{m}{D}-1}b^{-1}(t) \right\|_{L^{q'}(0,1)} < \infty. \tag{3.27}$$

Recall that b^r is a slowly varying function for every $r \in \mathbb{R}$. If $q \in (1,\infty]$, then we obtain

$$\int_0^1 \tau^{q'(1-\frac{m}{D})-1}(\tau+1)^{q'(\frac{m}{D}-1)}b^{-q'}(\tau)\,d\tau < \infty$$

owing to (1.27) since $1-\frac{m}{D} > 0$. If $q = 1$, then we have

$$\sup_{t\in(0,1)} t^{1-\frac{m}{D}}(t+1)^{\frac{m}{D}-1}b^{-1}(t) < \infty$$

by virtue of (1.29) since $1-\frac{m}{D} > 0$. It means that (3.27) is valid. Now, combining (3.26) with (3.27), we obtain the fact that the inequality (3.24) is valid if and only if the inequality (3.25) is valid since the function $(t+1)^{\frac{m}{D}-1}$ is equivalent to the function $t^{\frac{m}{D}-1}$ on $(1,\infty)$.

$\square$

In the following theorem we describe the optimal space $Z'_m(\mathbb{R}^n,\mu_D)$ for the space $L^{\frac{D}{m},q,b}(\mathbb{R}^n,\mu_D)$ if $q \in (1,\infty]$.

Theorem 3.5. *Let $q \in (1,\infty]$ and let b be a slowly varying function. Assume that (3.24) is true. Define the function $\tilde{b}\colon (0,\infty) \to (0,\infty)$ by*

$$\tilde{b}(t) = \frac{b^{1-q'}(t)}{\int_t^\infty \tau^{-1}b^{-q'}(\tau)\,d\tau}, \quad t \in (0,\infty).$$

Then the optimal space $Z'_m(\mathbb{R}^n,\mu_D)$ for the space $L^{\frac{D}{m},q,b}(\mathbb{R}^n,\mu_D)$ is equivalent to $L^{\infty,q,\tilde{b}}(\mathbb{R}^n,\mu_D)$.

Remark. Owing to [31, Lemma 2.16] we obtain the fact that $\tilde{b}$ is a slowly varying function. So, $L^{\infty,q,\tilde{b}}(\mathbb{R}^n,\mu_D)$ is indeed a Lorentz–Karamata space.

Proof. The existence of the optimal space $Z'_m(\mathbb{R}^n,\mu_D)$ follows from Proposition 3.4. We have the fact that $L^{\infty,q,\tilde{b}}(\mathbb{R}^n,\mu_D)$ is equivalent to $L^{(\infty,q,\tilde{b})}(\mathbb{R}^n,\mu_D)$ thanks to (1.32). By virtue of [31, Theorem 3.32] we obtain the fact that the space $L^{(\infty,q,\tilde{b})}(\mathbb{R}^n,\mu_D)$ is equivalent to $\left(L^{(1,q',b^{-1})}(\mathbb{R}^n,\mu_D)\right)'$. So, it is sufficient to prove that $Z_m(\mathbb{R}^n,\mu_D)$ is equivalent to $L^{(1,q',b^{-1})}(\mathbb{R}^n,\mu_D)$ owing to (1.5). For every $u \in \mathcal{M}^+(\mathbb{R}^n,\mu_D)$ we have

$$\begin{aligned}
\|u\|_{L^{(1,q',b^{-1})}(\mathbb{R}^n,\mu_D)} &= \left\| t^{1-\frac{1}{q'}}b^{-1}(t)u^{**}_{\mu_D}(t) \right\|_{L^{q'}(0,\infty)} \\
&\le \left\| t^{\frac{D-m}{D}-\frac{1}{q'}}b^{-1}(t)\sup_{\tau\in[t,\infty)} \tau^{\frac{m}{D}}u^{**}_{\mu_D}(\tau) \right\|_{L^{q'}(0,\infty)} \\
&= \left\| \sup_{\tau\in[t,\infty)} \tau^{\frac{m}{D}}u^{**}_{\mu_D}(\tau) \right\|_{L^{\frac{D}{D-m},q',b^{-1}}(0,\infty)} \le C_1 \left\| t^{\frac{m}{D}}u^{**}_{\mu_D}(t) \right\|_{L^{\frac{D}{D-m},q',b^{-1}}(0,\infty)} \\
&= C_1 \|u\|_{Z_m(\mathbb{R}^n,\mu_D)},
\end{aligned}$$

where $C_1 > 0$ is a constant that does not depend on u. The second inequality is true owing to [17, Lemma 4.10] (see (3.23)). The last equality is valid since

$$\left(L^{\frac{D}{m},q,b}(\mathbb{R}^n,\mu_D)\right)' = L^{\frac{D}{D-m},q',b^{-1}}(\mathbb{R}^n,\mu_D) \tag{3.28}$$

up to the equivalence of norms by virtue of (1.33).

On the other hand, for every $u \in \mathcal{M}^+(\mathbb{R}^n,\mu_D)$, we obtain

$$\begin{aligned}
\|u\|_{Z_m(\mathbb{R}^n,\mu_D)} &= \left\|t^{\frac{m}{D}}u^{**}_{\mu_D}(t)\right\|_{L^{\frac{D}{D-m},q',b^{-1}}(0,\infty)} \le \left\|\sup_{\tau\in[t,\infty)} \tau^{\frac{m}{D}}u^{**}_{\mu_D}(\tau)\right\|_{L^{\frac{D}{D-m},q',b^{-1}}(0,\infty)} \\
&= \left\|t^{\frac{D-m}{D}-\frac{1}{q'}}b^{-1}(t)\sup_{\tau\in[t,\infty)} \tau^{\frac{m}{D}}u^{**}_{\mu_D}(\tau)\right\|_{L^{q'}(0,\infty)} \le C_2\left\|t^{1-\frac{1}{q'}}b^{-1}(t)u^{**}_{\mu_D}(t)\right\|_{L^{q'}(0,\infty)} \\
&= \|u\|_{L^{(1,q',b^{-1})}(\mathbb{R}^n,\mu_D)},
\end{aligned} \tag{3.29}$$

where $C_2 > 0$ is a constant independent of u. The first equality is true thanks to (3.28). To prove the second inequality, we use [18, Theorem 3.2]. Owing to this theorem the second inequality in (3.29) is valid if for every $t \in (0,\infty)$ it is true that

$$t^{\frac{m}{D}}\left\|\tau^{\frac{D-m}{D}-\frac{1}{q'}}b^{-1}(\tau)\right\|_{L^{q'}(0,t)} \le K\left\|\tau^{1-\frac{1}{q'}}b^{-1}(\tau)\right\|_{L^{q'}(0,t)}, \tag{3.30}$$

where $K > 0$ is a constant that does not depend on t. The inequality (3.30) is true since by virtue of (1.27) we obtain the fact that

$$t^{\frac{m}{D}}\left\|\tau^{\frac{D-m}{D}-\frac{1}{q'}}b^{-1}(\tau)\right\|_{L^{q'}(0,t)} \approx tb^{-1}(t) \approx \left\|\tau^{1-\frac{1}{q'}}b^{-1}(\tau)\right\|_{L^{q'}(0,t)}$$

on $(0,\infty)$. It means that the second inequality in (3.29) is valid.

$\square$

In the last theorem of this thesis, we show the equivalent expression of the norm of the optimal space $Z'_m(\mathbb{R}^n,\mu_D)$ for the space $L^{\frac{D}{m},1,b}(\mathbb{R}^n,\mu_D)$.

Theorem 3.6. *Let b be a slowly varying function. Assume that*

$$\inf_{t\in(1,\infty)} b(t) > 0.$$

Furthermore, assume that b^{-1} is a locally Lipschitz function on $(0,\infty)$. Define the function $\hat{b}\colon (0,\infty)\to(0,\infty)$ by

$$\hat{b}(t) = \inf_{\tau\in[t,\infty)} b(\tau), \quad t\in(0,\infty).$$

Then the following two statements are true.

1. *If $\lim_{t\to 0_+}\hat{b}(t) = 0$, then the norm of the optimal space $Z'_m(\mathbb{R}^n,\mu_D)$ for the space $L^{\frac{D}{m},1,b}(\mathbb{R}^n,\mu_D)$ is equivalent to the functional*

$$u \mapsto \left\|\hat{b}'(t)u^*_{\mu_D}(t)\right\|_{L^1(0,\infty)}, \quad u\in\mathcal{M}(\mathbb{R}^n,\mu_D).$$

2. *If* $\lim_{t\to 0_+}\hat{b}(t) > 0$, *then the norm of the optimal space* $Z'_m(\mathbb{R}^n,\mu_D)$ *for the space* $L^{\frac{D}{m},1,b}(\mathbb{R}^n,\mu_D)$ *is equivalent to the functional*

$$u \mapsto \left\|\hat{b}'(t)u^*_{\mu_D}(t)\right\|_{L^1(0,\infty)} + \|u\|_{L^\infty(\mathbb{R}^n,\mu_D)}, \quad u \in \mathcal{M}(\mathbb{R}^n,\mu_D).$$

Proof. The optimality of the space $Z'_m(\mathbb{R}^n,\mu_D)$ follows from Proposition 3.4. Note that

$$\hat{b}(t) = \frac{1}{\sup\limits_{\tau\in[t,\infty)} b^{-1}(\tau)}, \quad t \in (0,\infty). \tag{3.31}$$

Owing to [31, Lemma 2.16] we have the fact that $\hat{b}$ is a slowly varying function. By virtue of [31, Proposition 3.7] we obtain the fact that

$$L^{(1,\infty,\hat{b}^{-1})}(\mathbb{R}^n,\mu_D) = L^{(1,\infty,b^{-1})}(\mathbb{R}^n,\mu_D) \tag{3.32}$$

up to the equivalence of norms. Now, we prove that

$$Z_m(\mathbb{R}^n,\mu_D) = L^{(1,\infty,b^{-1})}(\mathbb{R}^n,\mu_D) \tag{3.33}$$

up to the equivalence of norms. For each $u \in \mathcal{M}^+(\mathbb{R}^n,\mu_D)$ we have

$$\begin{aligned}
&\|u\|_{Z_m(\mathbb{R}^n,\mu_D)} = \left\|t^{\frac{m}{D}}u^{**}_{\mu_D}(t)\right\|_{L^{\frac{D}{D-m},\infty,b^{-1}}(0,\infty)} \le \left\|\sup_{\tau\in[t,\infty)} \tau^{\frac{m}{D}}u^{**}_{\mu_D}(\tau)\right\|_{L^{\frac{D}{D-m},\infty,b^{-1}}(0,\infty)}\\
&= \sup_{t\in(0,\infty)} t^{\frac{D-m}{D}}b^{-1}(t)\sup_{\tau\in[t,\infty)}\tau^{\frac{m}{D}}u^{**}_{\mu_D}(\tau) = \sup_{\tau\in(0,\infty)}\tau^{\frac{m}{D}}u^{**}_{\mu_D}(\tau)\sup_{t\in(0,\tau]} t^{1-\frac{m}{D}}b^{-1}(t)\\
&\le C_1 \sup_{t\in(0,\infty)} tb^{-1}(t)u^{**}_{\mu_D}(t) = C_1\|u\|_{L^{(1,\infty,b^{-1})}(\mathbb{R}^n,\mu_D)},
\end{aligned}$$

where $C_1 > 0$ is a constant that does not depend on u. The first equality is valid since

$$\left(L^{\frac{D}{m},1,b}(\mathbb{R}^n,\mu_D)\right)' = L^{\frac{D}{D-m},\infty,b^{-1}}(\mathbb{R}^n,\mu_D) \tag{3.34}$$

up to the equivalence of norms owing to (1.33). The second inequality follows from the fact that the function $t^{1-\frac{m}{D}}b^{-1}(t)$ is equivalent to a nondecreasing function on $(0,\infty)$.

On the other hand, for every $u \in \mathcal{M}^+(\mathbb{R}^n,\mu_D)$, we obtain

$$\begin{aligned}
&\|u\|_{L^{(1,\infty,b^{-1})}(\mathbb{R}^n,\mu_D)} = \sup_{t\in(0,\infty)} tb^{-1}(t)u^{**}_{\mu_D}(t) \le \sup_{t\in(0,\infty)} t^{1-\frac{m}{D}}b^{-1}(t)\sup_{\tau\in[t,\infty)}\tau^{\frac{m}{D}}u^{**}_{\mu_D}(\tau)\\
&= \left\|\sup_{\tau\in[t,\infty)}\tau^{\frac{m}{D}}u^{**}_{\mu_D}(\tau)\right\|_{L^{\frac{D}{D-m},\infty,b^{-1}}(0,\infty)} \le C_2\left\|t^{\frac{m}{D}}u^{**}_{\mu_D}(t)\right\|_{L^{\frac{D}{D-m},\infty,b^{-1}}(0,\infty)}\\
&= C_2\|u\|_{Z_m(\mathbb{R}^n,\mu_D)},
\end{aligned}$$

where $C_2 > 0$ is a constant that is independent of u. The second inequality is valid by virtue of [17, Lemma 4.10] (see (3.23)). The last equality is true owing to (3.34). So, we have proved (3.33).

Owing to (3.31) we have the fact that $\hat{b}^{-1}(t) = \sup_{\tau\in[t,\infty)} b^{-1}(\tau)$ for every $t \in (0,\infty)$. Now, we prove that this function is locally absolutely continuous on $(0,\infty)$. We know that b^{-1} is a locally Lipschitz function on $(0,\infty)$. We show that $\hat{b}^{-1}$ is also locally Lipschitz on $(0,\infty)$. So, choose an arbitrary closed interval $[t_1,t_2] \subseteq (0,\infty)$. For arbitrary $s_1, s_2 \in [t_1,t_2]$, $s_1 \le s_2$, it is true that $|b^{-1}(s_1) - b^{-1}(s_2)| \le L(s_2 - s_1)$ for some $L > 0$, which depends on the interval $[t_1,t_2]$. If $\sup_{\tau\in[s_1,\infty)} b^{-1}(\tau) = \sup_{\tau\in[s_2,\infty)} b^{-1}(\tau)$, then we obtain $\hat{b}^{-1}(s_1) - \hat{b}^{-1}(s_2) = 0$. If $\sup_{\tau\in[s_1,\infty)} b^{-1}(\tau) > \sup_{\tau\in[s_2,\infty)} b^{-1}(\tau)$, then we have

$$\begin{aligned}\hat{b}^{-1}(s_1) - \hat{b}^{-1}(s_2) &= \max_{\tau\in[s_1,s_2]} b^{-1}(\tau) - \sup_{\tau\in[s_2,\infty)} b^{-1}(\tau) \le \max_{\tau\in[s_1,s_2]} b^{-1}(\tau) - b^{-1}(s_2)\\ &\le L(s_2 - s_1).\end{aligned}$$

So, the function $\hat{b}^{-1}$ is locally Lipschitz on $(0,\infty)$, and so it is also locally absolutely continuous on $(0,\infty)$. It means that we can use [31, Theorem 3.32] with the space $L^{(1,\infty,\hat{b}^{-1})}(\mathbb{R}^n,\mu_D)$ to obtain the fact that the statements (1.) and (2.) are true since (3.32) and (3.33) are valid.

$\square$

Remark. The assumption that b^{-1} is locally Lipschitz on $(0,\infty)$ does not entail significant loss of generality since common slowly varying functions satisfy this condition.

Remark. If the function b is in addition nonincreasing on $(0,1]$ and constant on $[1,\infty)$, then we obtain the fact that $\hat{b}$ is constant on $(0,\infty)$, so $\hat{b}' \equiv 0$ on $(0,\infty)$. It means that the optimal space $Z'_m(\mathbb{R}^n,\mu_D)$ is equivalent to $L^\infty(\mathbb{R}^n,\mu_D)$.

Remark. The optimal space $Z'_m(\mathbb{R}^n,\mu_D)$ from the preceding theorem is equivalent to the function space $\Lambda^1(\hat{b}')(\mathbb{R}^n,\mu_D)$ if $\lim_{t\to 0_+} \hat{b}(t) = 0$ and to $\Lambda^1(\hat{b}')(\mathbb{R}^n,\mu_D) \cap L^\infty(\mathbb{R}^n,\mu_D)$ if $\lim_{t\to 0_+} \hat{b}(t) > 0$. The function space $\Lambda^1(\hat{b}')(\mathbb{R}^n,\mu_D)$ is an example of a so-called *classical Lorentz space* (e.g., see [33, Chapter 10]). It is defined as the collection of all functions $f \in \mathcal{M}(\mathbb{R}^n,\mu_D)$ such that $\|\hat{b}'(t) f^*_{\mu_D}(t)\|_{L^1(0,\infty)} < \infty$.

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