FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

BACHELOR THESIS

Magdaléna Mišinová

Permutohedral varieties as Chow quotients

Department of Algebra

Supervisor of the bachelor thesis: doc. RNDr. Jan Šťovíček, PhD.
Study programme: General mathematics

Prague 2024

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I would like to thank my consultant Prof. Dr. Mateusz Michałek for introducing me to the topic and helping me throughout the writing of the thesis, and to my advisor doc. RNDr. Jan Šťovíček, PhD. for his valuable remarks.

Title: Permutohedral varieties as Chow quotients

Author: Magdaléna Mišinová

Department: Department of Algebra

Supervisor: doc. RNDr. Jan Šťovíček, PhD., Department of Algebra

Abstract: We fix a specific action of the multiplicative group of complex numbers on a product of projective lines and examine the structure of its orbits. It turns out that the Chow quotient is isomorphic to permutohedral variety. We do not show this in the full extent, but we find a set-theoretical bijection and describe the isomorphism for a product of two lines. In the introduction, we sum up the necessary definitions and theorems from both toric geometry and the theory of Grassmannians and Chow varieties.

Keywords: Chow quotient, Chow varieties, permutohedral variety, toric geometry

Název práce: Permutohedrální variety jakožto Chowovy kvocienty

Autor: Magdaléna Mišinová

Katedra: Katedra algebry

Vedoucí bakalářské práce: doc. RNDr. Jan Šťovíček, PhD., Katedra algebry

Abstrakt: Vezmeme si konkrétní akci multiplikativní grupy komplexních čísel na součinu projektivních přímek a budeme zkoumat strukturu jejích orbit. Ukazuje se, že Chowův kvocient této akce je izomorfní permutohedrální varietě. Toto nedokážeme v plné šíři, ale najdeme množinovou bijekci a popíšeme izomorfismus pro součin dvou přímek. V úvodu shrneme potřebné definice a věty jak z torické geometrie, tak z teorie týkající se Grassmannianů a Chowových variet.

Klíčová slova: Chowův kvocient, Chowovy variety, permutohedrální varieta, torická geometrie

Contents

Introduction

We start by introducing some notation. We denote the multiplicative group of complex numbers by $\mathbb{C}^\star$, and the projectivisation of $\mathbb{C}^n$ by $\mathbb{P}^n$.

We study a specific action of $\mathbb{C}^\star$ on $(\mathbb{P}^1)^n$. A point in $(\mathbb{P}^1)^n$ is given by $((a_1 : b_1), \dots, (a_n : b_n))$, where for all i, $a_i, b_i \in \mathbb{C}$ and a_i, b_i are not both zero. The considered action is given by

$$t \cdot ((a_1 : b_1), \dots, (a_n : b_n)) = ((a_1 : t \cdot b_1), \dots, (a_n : t \cdot b_n)).$$

Informally, our goal is to study the variety of orbits of this action. It turns out that it is isomorphic to the so-called *permutohedral variety*. In the introduction we briefly sum up the theory necessary to define "variety of orbits" and the permutohedral variety. Then we apply the general theory to our case. In the first chapter, we describe the isomorphism set-theoretically by decomposing our objects of study into smaller pieces and constructing the bijection on them. We do not show that it is an isomorphism of varieties. However, in the second chapter we explicitly compute the isomorphism for $(\mathbb{P}^1)^2$.

0.1 Permutohedral variety

Toric geometry studies varieties on which a group acts, more specifically, on which a group called algebraic torus acts. In this section, we give a brief overview of basic definitions and tools in toric geometry, following Cox, Little, Schenck [1]. This enables us to define the projective toric variety we are interested in, the permutohedral variety.

The strength of toric geometry is that varieties correspond to polytopes in some lattice or so-called fans in a dual lattice. We first discuss in general those two lattices and correspondence between varieties and polytopes. Then we define a polytope called a permutohedron and use the general correspondence to define the permutohedral variety.

A *torus* is a group isomorphic to $(\mathbb{C}^\star)^n$ for some n. Every torus has two important lattices associated with it, the lattice of *characters* and the lattice of *one-parameter subgroups*, denoted by M and N. Here *lattice* is a free abelian group of a finite rank, i.e. group isomorphic to $\mathbb{Z}^n$ for some $n \in \mathbb{N}$.

Characters, respectively one-parameter subgroups, of a torus $T \simeq (\mathbb{C}^\star)^n$ are morphisms of algebraic varieties $T \to \mathbb{C}^\star$, respectively $\mathbb{C}^\star \to T$, which are also group homomorphisms. It can be shown that all characters are of the form χ^m for some $m = (a_1, \dots, a_n) \in \mathbb{Z}^n$ ([2], §16), where

$$\begin{aligned} \chi^m : T &\to \mathbb{C}^\star \\ (t_1, \dots, t_n) &\mapsto t_1^{a_1} \dots t_n^{a_n} \end{aligned}$$

Similarly, all one-parameter subgroups are of the form

$$\begin{aligned} \lambda^u : \mathbb{C}^\star &\to T \\ t &\mapsto \left(t^{b_1}, \dots, t^{b_n}\right) \end{aligned}$$

for some $u = (b_1, \dots, b_n) \in \mathbb{Z}^n$.

From this we can see that the characters and one-parameter subgroups indeed form a lattice- addition in $\mathbb{Z}^n$ translates to multiplication of characters and one-parameter subgroups. Using the usual dot product, those two lattices are dual to each other.

For a torus with a lattice of one-parameter subgroups N we usually write T_N. The reason is that the torus is determined by N, since there is an isomorphism between $N \otimes_{\mathbb{Z}} \mathbb{C}^\star$ and T_N given by $u \otimes t \mapsto \lambda^u(t)$ ([1],§1.1).

Now we can discuss how affine and projective toric varieties arise.

If we are given a set $\mathcal{A} = (m_1, \dots, m_k)$ in M, we can define an affine toric variety $Y_{\mathcal{A}}$ as the Zariski closure of the image of the map

$$\begin{aligned} \Phi_{\mathcal{A}} : T_N &\to \mathbb{C}^k \\ t &\mapsto (\chi^{m_1}(t), \dots, \chi^{m_k}(t)) \end{aligned}$$

To obtain projective toric varieties, we consider $\Phi_{\mathcal{A}}$ as a map to $(\mathbb{C}^\star)^k$, compose it with $\pi : (\mathbb{C}^\star)^k \to \mathbb{P}^{k-1}$ and then take the Zariski closure.

We can consider two vector spaces over $\mathbb{R}$, $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. We do this to use the concepts of polytopes and cones from combinatorial geometry. Given a sufficiently nice polytope in $M_{\mathbb{R}}$, the projective variety associated with it is obtained by taking $\mathcal{A}$ to be the lattice points of this polytope ([1], §2.3). This gives us a variety with its embedding into some projective space. To get rid of this embedding and to define an abstract toric variety, we consider a *normal fan* of the given polytope. Normal fan is a collection of convex polyhedral cones, where a *convex polyhedral cone* is a set of the form $\mathrm{Cone}(S) = \{\sum_{u \in S} \lambda_u u \mid \lambda_u \in \mathbb{R}_0^+\}$ for some finite set $S \subset N$. Given a polytope in $M_{\mathbb{R}}$, its normal fan is constructed as follows:

For each facet F of the polytope (i.e., codimension one face), we denote by u_F the dual vector in $N_{\mathbb{R}}$. For every face τ we put $\mathrm{Cone}(u_F \mid \tau \subset F)$ into our fan. So for every face of the polytope, there is a cone in the normal fan and vice versa. This correspondence reverses inclusion and dimension (to the face of dimension d corresponds a cone of dimension $n - d$).

Each cone in $N_{\mathbb{R}}$ is associated with an affine toric variety ([1], §1.2). A fan Σ is associated with an abstract variety X_Σ, obtained by gluing the varieties associated with its cones ([1], §3.1).

There is one important theorem we will need:

Theorem 1 (Orbit-cone correspondence, Theorem 3.2.6 in [1])**.** *Let X_Σ be the toric variety of the fan Σ in $N_{\mathbb{R}}$. Then there is a bijective correspondence*

$$\{\text{cones in } \Sigma\} \longleftrightarrow \{T_N\text{-orbits in } X_\Sigma\}$$

We proceed to define our main objects of study.

Definition 1. *Let $\mathcal{S}_{n+1}$ be the group of permutations of $\{0, 1, \dots n\}$ and denote by w_σ the point $(\sigma^{-1}(0), \dots, \sigma^{-1}(n)) \in \mathbb{R}^{n+1}$. The n-*dimensional permutohedron *is the polytope in $\mathbb{R}^{n+1}$ with vertices $\{w_\sigma \mid \sigma \in \mathcal{S}_{n+1}\}$.*

Definition 2 ([3], Definition 3.3.)**.** *The permutohedral variety of dimension n, denoted by Π_n, is the toric variety associated with the normal fan of the n-dimensional permutohedron with respect to the lattice $\mathbb{Z}^{n+1}/\langle(1, 1, \dots, 1)\rangle$.*

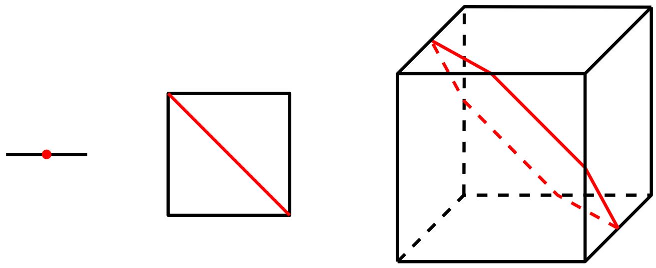

Figure 1 The 0-, 1- and 2-dimensional permutohedron

We study the permutohedral variety by the above mentioned correspondences between its orbits and the cones of the normal fan and between the cones of the normal fan and faces of the polytope. So, the key is to understand the permutohedron. Notice that the n-dimensional permutohedron is contained in the hyperplane with the sum of the coordinates $\frac{n(n+1)}{2}$. For $n \leq 3$, the permutohedron can be obtained by intersecting this hyperplane with the $(n+1)$-dimensional hypercube with vertices $(x_1, \dots x_n)$, where each x_i is either 0 or n (Figure 1).

Two vertices w_σ and $w_{\sigma'}$ of the permutohedron are connected with an edge if and only if $\sigma = \sigma' \circ (i, i+1)$ for some i, where $(i, i+1)$ is a transposition. This is an instance of a correspondence between the faces of the permutohedron and so-called *flags*, which we define later.

We conclude this section by finding the polytope of $(\mathbb{P}^1)^n$. It can be checked that the polytope of $\mathbb{P}^n$ is the n-dimensional simplex. So, the polytope of $\mathbb{P}^1$ is a segment. By Theorem 2.4.7 in Cox, Little, Schenck [1], taking products of varieties translates to taking products of polytopes. Hence, the polytope of $(\mathbb{P}^1)^n$ is the n-dimensional cube.

Note that the 1-dimensional permutohedron is the 1-dimensional simplex, so Π_1 is just $\mathbb{P}^1$. For higher dimensions, the straightforward analogue is obviously not true. However, Π_n can be obtained from $\mathbb{P}^n$ by consecutive blow-ups ([3], p. 19).

0.2 Background on Chow quotients

A Chow quotient is a variety for which general points correspond to general orbits. To define it, we need to interpret curves of some fixed degree as points of a variety. Then we take a correct subvariety of it.

The first step can be done even more generally. We will define a variety of subvarieties of $\mathbb{P}^n$ of fixed dimension and fixed degree, the so-called Chow variety. If we think of curves in an affine plane, we can see that if we take only irreducible curves, our variety would not be compact. Therefore, we also need to consider their limits, the *algebraic cycles*.

All this requires developing some technical tools. This is very well done in Gelfand, Kapranov, Zelevinsky [4]. Following this book, we sum up the needed definitions and results without proofs.

First, we establish what we mean by degree. The degree of a projective variety X in $\mathbb{P}^n$ is usually defined via its *Hilbert polynomial*, that is the Hilbert polynomial of its coordinate ring. The Hilbert polynomial of a finitely generated graded $\mathbb{C}[x_0, \dots, x_n]$-module is defined to be the unique polynomial P_M such that $P_M(s) = \dim_{\mathbb{C}} M_s$ for all s large enough. If the dimension of X is k, then k is the degree of its Hilbert polynomial P_X. The *degree* of X is defined as the leading coefficient of P_X divided by $k!$. The *arithmetic genus* is defined as $(-1)^k(P_X(0) - 1)$. (For details, see Hartshorne [5], §I.7.)

The degree can be computed without the Hilbert polynomial. Let k be the dimension of X. Then a general $(n-k)$-dimensional projective subspace of $\mathbb{P}^n$ intersects X in finitely many points. For a general subspace, the number of points in the intersection equals the degree of X ([6], p. 26).

0.2.1 Grassmannians

We start by describing the situation in degree one, i.e. for linear subspaces. Though it is a special case, it has the advantage that a general linear subspace is easily parameterised. The developed theory is later used to study the higher-degree subvarieties.

Definition 3. *A* Grassmannian *is a space of k-dimensional linear subspaces of $\mathbb{C}^n$. We denote it by $G(k, n)$.*

There is more than one way to define coordinates on a Grassmannian. The most useful for us is via the exterior algebra. Consider the *Plücker embedding*

$$G(k,n) \to \mathbb{P}\left(\bigwedge^k \mathbb{C}^n\right)$$
$$L \mapsto \bigwedge^k L \subset \bigwedge^k \mathbb{C}^n$$

Fixing a basis $x_1, \ldots, x_n$ of $\mathbb{C}^n$ induces a basis of $\bigwedge^k \mathbb{C}^n$. If

$$\bigwedge^k L = \sum_{1 \le i_1 < \cdots < i_k \le n} p_{i_1,\ldots,i_k}(x_{i_1} \wedge \cdots \wedge x_{i_k}),$$

the *Plücker coordinates* of L are

$$(p_{i_1,\ldots,i_k})_{1 \le i_1 < \cdots < i_k \le n} \in \mathbb{P}\left(\bigwedge^k \mathbb{C}^n\right) \simeq \mathbb{P}^{\binom{n}{k}-1}.$$

For the computations, we use the approach of Dalbec, Sturmfels [7], using the Plücker coordinates dual to those already described. For $L \in G(k,n)$, we consider its orthogonal complement $L^\perp \in G(n-k, n)$, take its basis, and write it out in $(n-k) \times n$ matrix. For any $1 \le j_1 < \cdots < j_{n-k} \le n$, we denote by $q_{j_1,\ldots,j_{n-k}}$ the $(n-k)$-minor corresponding to columns $j_1, \ldots, j_{n-k}$. The vector $(q_{j_1,\ldots,j_{n-k}})$ considered as an element of a projective space is independent of the choice of the basis of $L^\perp$. If we denote by s the sign of the permutation $(i_1, \ldots, i_k, j_1, \ldots, j_{n-k})$, then $q_{j_1,\ldots,j_{n-k}} = (-1)^s p_{i_1,\ldots,i_k}$. For more details, see Gelfand, Kapranov, Zelevinsky [4], page 94.

Though the Plücker embedding is injective, it is not surjective. Thus, the Grassmannian $G(k,n)$ is a subvariety of $\mathbb{P}(\bigwedge^k \mathbb{C}^n)$. It is given by *Plücker relations*, described in the following theorem.

Theorem 2 ([4], Chapter 3, Theorem 1.3.)**.**

(a) *For any two sequences* $1 \leq i_1 < \cdots < i_{k-1} \leq n$ *and* $1 \leq j_1 < \cdots < j_{k+1} \leq n$*, the Plücker coordinates on* $G(k,n)$ *satisfy the* Plücker relation

$$\sum_{a=1}^{k+1} (-1)^a p_{i_1,\ldots,i_{k-1},j_a} p_{j_1,\ldots,\hat{j}_a,\ldots j_{k+1}} = 0$$

(here the symbol $\hat{j}_a$ *means that the index* j_a *is omitted). Any vector* $(p_{i_1,\ldots,i_k}) \in \bigwedge^k \mathbb{C}^n$ *satisfying all such relations is a vector of the Plücker coordinates of some vector subspace* $L \in G(k,n)$

(b) *Moreover, the graded ideal of all polynomials in* $p_{i_1,\ldots,i_k}$ *vanishing on the image of* $G(k,n)$ *is generated by the left hand sides of the Plücker relations.*

The Plücker coordinates and the Plücker embedding give us a description of the coordinate ring of a Grassmannian, which we denote by $\mathcal{B}$. By this description, $\mathcal{B}$ is the coordinate ring of $\mathbb{P}(\bigwedge^k \mathbb{C}^n)$ modulo the ideal generated by the Plücker relations. It is generated by the $\binom{n}{k}$ variables corresponding to the coordinates of $\mathbb{P}(\bigwedge^k \mathbb{C}^n)$, i.e. $p_{i_1,\ldots,i_k}$ above, modulo the ideal. We call those generators *brackets*. The dual generators corresponding to $q_{j_1,\ldots,j_{n-k}}$ are *dual brackets*, and we denote the one corresponding to $j_1 < \cdots < j_{n-k}$ by $[j_1 \ldots j_{n-k}]$.

We denote the d-th degree of $\mathcal{B}$ by $\mathcal{B}_d$. The ring $\mathcal{B}$ has the following useful properties.

Proposition 3. *([4], Chapter 3, Proposition 2.1.)*

1. *The ring* $\mathcal{B}$ *is factorial.*

2. *Let* Z *be an irreducible hypersurface in* $G(k,n)$ *of degree* d*. Then there is an element* $f \in \mathcal{B}_d$ *defined uniquely up to a constant factor such that* Z *is given by the equation* $f = 0$*.*

0.2.2 Higher degrees

Linear subspaces have the great advantage of being described easily by linear algebra. This enables us to define coordinates on $G(k,n)$. The higher-degree subvarieties do not have such nice properties, so we do not even attempt to imitate the definitions and properties given above. Instead, we make use of all the tools for degree one. The key to doing it is a definition of the associated hypersurface.

Fix an irreducible subvariety X in $\mathbb{P}^{n-1}$ of degree d and dimension $k-1$. Note that while a general $(n-k)$-dimensional projective subspace of $\mathbb{P}^{n-1}$ intersects X, this is not true for $(n-k-1)$-dimensional projective subspaces. This leads us to the following definition:

Definition 4. *For an irreducible* $(k-1)$*-dimensional subvariety* $X \subset \mathbb{P}^{n-1}$ *we define its* associated hypersurface*, denoted by* $\mathcal{Z}(X)$*, as a subvariety of* $G(n-k,n)$ *consisting of those* $(n-k-1)$*-dimensional projective subspaces intersecting* X*.*

The following theorem gives a justification for the term “associated hypersurface”.

Theorem 4 ([4], Chapter 3, Theorem 2.2)**.** *The subvariety $\mathcal{Z}(X)$ is an irreducible hypersurface of degree d in $G(n-k,n)$.*

By Proposition 3, there is an irreducible polynomial $R_X \in \mathcal{B}_d$ that defines $\mathcal{Z}(X)$, where $\mathcal{B}$ is the coordinate ring of $G(n-k,n)$. The polynomial R_X is called *Chow form* of X. Since $\mathcal{Z}(X)$ is of degree d, the degree of R_X is d too. The coordinates of R_X in $\mathcal{B}_d$ are called *Chow coordinates* of X. We can reconstruct X from its Chow coordinates ([4], Chapter 3, §2.C). The Chow coordinates induced by the Plücker coordinates are those that we will work with.

So far, we have worked only with irreducible varieties and irreducible polynomials. We now extend the theory of Chow forms and Chow coordinates to the reducible ones, obtaining the Chow variety.

Definition 5. *An* algebraic cycle *in $\mathbb{P}^n$ is a formal sum $\sum_{i=1}^r m_i X_i$, where $m_i \in \mathbb{N}_0$, and X_i is an irreducible variety in $\mathbb{P}^n$. The* degree *of this algebraic cycle is $\sum_{i=1}^r m_i \cdot \deg(X_i)$.*

The set of algebraic cycles of dimension $k-1$ of degree d in $\mathbb{P}^{n-1}$ is denoted by $G(k,d,n)$. For example, $G(k,1,n)$ is the usual Grassmannian $G(k,n)$.

The Chow form of an algebraic cycle $\sum_{i=1}^r m_i X_i$ is $\prod_{i=1}^r R_{X_i}^{m_i}$, its Chow coordinates are the coordinates of its Chow form. By the definition of a degree of an algebraic cycle, the degree of an algebraic cycle is equal to the degree of its Chow form. The theorem of Chow and van der Waerden says that considering Chow forms makes $G(k,d,n)$ in a variety, called a Chow variety.

Theorem 5 (Theorem 1.1 in the fourth chapter of [4])**.** *The map $X \mapsto R_X$ defines an embedding of $G(k,d,n)$ into the projective space $\mathbb{P}(\mathcal{B}_d)$ as a projective variety.*

Consider a projective variety X and an algebraic group G acting on X. The *Chow quotient* of X by the action of G is defined by choosing an open subset of X, taking a family of closures of orbits of points in that subset, mapping this family into a proper Chow variety, and taking its closure there (see [8] for details). We need some conditions on the family of orbits, so that this process makes sense. For example, their closures must have the same degree as varieties, else we can not map them into one Chow variety. This is ensured by taking a *flat family.* A *family* of projective varieties is a map $\varphi : \mathcal{X} \to \mathcal{Y}$ of projective varieties, where individual varieties in the family are the fibers over points of Y ([9], p. 6). The family is flat if φ is a flat morphism:

Definition 6 ([9], Definition A)**.** *A morphism of projective varieties $f : X \to Y$ is* flat *if the induced map on stalks is a flat map of rings.*

This definition is equivalent to the following one ([9], Theorem 4.1.), which we use later:

Definition 7 ([9], Definition B)**.** *A morphism of projective varieties $f : X \to Y$ is* flat *if the the fibers of f have the same Hilbert polynomial.*

0.3 Orbits in $(\mathbb{P}^1)^n$

In this section we study the orbits of the fixed action of $\mathbb{C}^\star$ on $(\mathbb{P}^1)^n$. To do this, we need to embed $(\mathbb{P}^1)^n$ into some projective space. We do this using the Segre embedding.

Definition 8. *The* Segre embedding *of* $\mathbb{P}^a \times \mathbb{P}^b$ *into* $\mathbb{P}^{(a+1)(b+1)-1}$ *is the map*

$$((x_0 : \dots : x_a), (y_0 : \dots : y_b)) \mapsto (x_0y_0 : x_0y_1 : \dots : x_ny_n),$$

where on the right-hand side appears x_iy_j *for every pair* $(i,j) \in \{0,\dots,a\} \times \{0,\dots,b\}$.

Using the Segre embedding, we obtain an embedding of $(\mathbb{P}^1)^n$ into $\mathbb{P}^{2^n-1}$. We denote this map by $\mathcal{E}$.

Recall that we consider the action of $\mathbb{C}^\star$ on $(\mathbb{P}^1)^n$, where $t \cdot ((a_1 : b_1), \dots (a_n : b_n)) = ((a_1 : t \cdot b_1), \dots (a_n : t \cdot b_n))$.

Notation. For a point x in $(\mathbb{P}^1)^n$, we denote by $O(x)$ the orbit of x.

We start by studying the action on a single coordinate. If $(a_i : b_i)$ is $(0:1)$ or $(1:0)$, $(a_i : t \cdot b_i) = (a_i : b_i)$ for all $t \in \mathbb{C}^\star$ by the definition of projective space. If both a_i and b_i are non-zero, we consider two limits, $\lim_{t\to 0}(a_i : t\cdot b_i)$ and $\lim_{t\to\infty}(a_i : t \cdot b_i)$. Since a_i and b_i are fixed,

$$\lim_{t\to 0}(a_i : t\cdot b_i) = (1:0),$$

$$\lim_{t\to\infty}(a_i : t\cdot b_i) = \lim_{t\to\infty}(t^{-1}\cdot a_i : b_i) = (0:1).$$

For $S \subset \{1,\dots,n\}$, we denote by v_S the point $((a_i : b_i))_{i=1}^n \in (\mathbb{P}^1)^n$ such that $(a_i : b_i) = (1:0)$ for $i \notin S$ and $(a_i : b_i) = (0:1)$ for $i \in S$.

By the above, we see that a point is fixed point of our action if and only if it is v_S for some $S \subset \{1,\dots,n\}$. Moreover, we have the following lemma.

Lemma 6. *Take a point* $((a_i : b_i))_{i=1}^n \in (\mathbb{P}^1)^n$ *and define* $A \subset \{1,\dots,n\}$, $B \subset \{1,\dots,n\}$, *such that* $(a_i : b_i) = (1:0)$ *if and only if* $i \in A$ *and* $(a_i : b_i) = (0:1)$ *if and only if* $i \in B$. *Then*

$$\lim_{t\to 0}((a_1 : t\cdot b_1), \dots, (a_n : t\cdot b_n)) = v_B,$$

$$\lim_{t\to\infty}((a_1 : t\cdot b_1), \dots, (a_n : t\cdot b_n)) = v_{\{1,\dots,n\}\setminus A},$$

We call v_B *and* $v_{\{1,\dots,n\}\setminus A}$ *respectively the* source *and the* sink *of the orbit* $O(((a_i : b_i))_{i=1}^n)$.

Lemma 7. *Consider* $S, T \subset \{1,\dots,n\}$, *and* $((a_1 : b_1), \dots (a_n : b_n)) \in (\mathbb{P}^1)^n$ *such that its orbit has a source* v_S *and a sink* v_T. *Then the closure of* $\mathcal{E}(O((a_1 : b_1), \dots, (a_n : b_n)))$ *has degree* n.

Proof. Labelling the coordinates of $\mathbb{P}^{2^n-1}$ by the subset of $\{1,\dots,n\}$, we obtain

$$\mathcal{E}(O((a_1 : b_1), \dots, (a_n : b_n))) = \left(\left(t^{|R|} \cdot \prod_{i\notin R} a_i \cdot \prod_{i\in R} b_i\right)_{R\subset\{1,\dots,n\}}, t \in \mathbb{C}^\star\right).$$

For $i \in S$, $(a_i : b_i) = (0:1)$. For $i \in \{1,\dots,n\} \setminus T$, $(a_i : b_i) = (1:0)$. So $\prod_{i\notin R} a_i \cdot \prod_{i\in R} b_i$ is non-zero if and only if $S \subset R \subset T$.

Denote the closure of $\mathcal{E}(O((a_1 : b_1), \dots, (a_n : b_n)))$ by Q.

We denote the coordinates of $\mathbb{P}^{2^n-1}$ by $(x_R)_{R\subset\{1,\dots,n\}}$. Consider a general hyperplane H given by $\sum_{R\subset\{1,\dots,n\}} c_R x_R = 0$, where $c_R \in \mathbb{C}$, such that the number of points in $H \cap Q$ is equal to the degree of Q. We may assume that $c_S \neq 0 \neq c_T$. If we plug into the equation of H the parameterisation of $\mathcal{E}(O((a_1 : b_1), \dots, (a_n : b_n)))$ and divide by $t^{|S|}$, we obtain a polynomial of degree $|T| - |S|$, with a non-zero constant term. Counting multiplicities, this polynomial has $|S| - |T|$ solutions in t over $\mathbb{C}$, neither of them 0. Therefore, the degree of Q is indeed $|S| - |T|$. ☐

0.4 Chow quotient of $(\mathbb{P}^1)^n$

To see what the Chow quotient of $(\mathbb{P}^1)^n$ is, we need to identify the open subset of $(\mathbb{P}^1)^n$ mentioned earlier, so that the closures of the orbits of those points form a flat family. There is a very natural choice:

Notation. We denote by U_n the set of those points $(\mathbb{P}^1)^n$ such that none of their coordinates is zero, and by Q_n the set of the closures of the orbits of points from U_n.

To prove that Q_n indeed form a flat family, we use Definition 7.

Lemma 8. *The elements of Q_n form a flat family.*

Proof. It is enough to prove that the Hilbert polynomial of elements of Q_n after the Segre embedding is the same.

Each of the orbits of points from U_n has dimension 1, so Hilbert polynomial of its closure is linear. By 6, an orbit of a point from U_n has a source $((1:0),\dots,(1:0))$ and a sink $((0:1),\dots,(0:1))$. Thus, by 7, the degree of its closure is n. So the leading coefficient of the Hilbert polynomial of the closure of all the orbits agrees.

The closure of each of those orbits is isomorphic to $\mathbb{P}^1$, so they have the same genus ([5], p. 230). By this we obtain the equality of the free term. □

Now we are finally ready to say what is the other side of our isomorphism, the Chow quotient of $(\mathbb{P}^1)^n$ by the fixed action of $\mathbb{C}^\star$, which we denote by $(\mathbb{P}^1)^n/\mathbb{C}^\star$. Take the elements of Q_n. They are of dimension 1 and degree n in $\mathbb{P}^{2^n-1}$ after the Segre embedding, so we can regard them as elements of $G(2,n,2^n)$. Hence we obtained a map from U_n to $G(2,n,2^n)$. The closure of its image is $(\mathbb{P}^1)^n/\mathbb{C}^\star$.

1 The set-theoretical bijection

In this chapter, we construct a bijection between $(\mathbb{P}^1)^n/\mathbb{C}^\star$ and Π_{n-1}. First, we decompose both of them into disjoint subsets, each subset corresponding to one *flag*, which is defined below. Second, for each flag we construct a bijection between the corresponding subset of the Chow quotient and the corresponding subset of the permutohedral variety.

Definition 9. *A* flag *is a strictly increasing chain of proper subsets ordered by inclusion. We denote by $\mathcal{F}_n$ the system of flags on the set $\{1, \dots, n\}$ and by $\emptyset^n$ the empty flag in $\mathcal{F}_n$.*

Here we give an informal illustration for the case $n = 3$. As discussed in the introduction, if we cut the cube, the polytope of $(\mathbb{P}^1)^3$, by a plane with correct constant sum of coordinates, we obtain a two dimensional permutohedron. A general orbit intersects the permutohedron somewhere in the interior, corresponding to the empty flag, while the broken cycles intersect it on the boundary. The most broken one, consisting of degree one orbits, intersects the permutohedron in a vertex, corresponding to the largest flag. This is very informal, but it might help to not get lost in the computations.

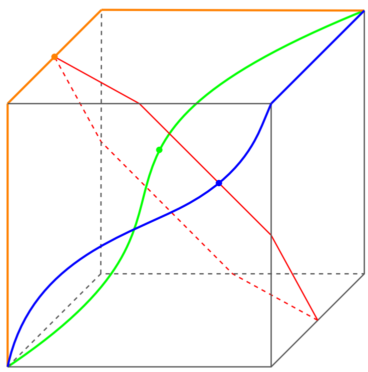

Figure 1.1 $(\mathbb{P}^1)^3$ with permutohedron and some cycles.

1.1 Decomposing the permutohedral variety

As we have mentioned in the Introduction, by the orbit-cone correspondence (Theorem 1), there is a bijection between the orbits of Π_{n-1} and the cones of the $(n-1)$-dimensional permutohedral fan, which in turn correspond to the faces of the permutohedron.

The set of the faces of the $(n-1)$-dimensional permutohedron is in bijection with $\mathcal{F}_n$ ([3], p. 18). Take a flag $F = (S_1 \subset \dots \subset S_k)$ and recall that we define the $(n-1)$-dimensional permutohedron as a convex hull of points $w_\sigma = (\sigma^{-1}(0), \dots \sigma^{-1}(n-1))$, where σ is a permutation of $\{0, 1, \dots, n-1\}$. The face corresponding to F is a convex hull of vertices w_σ such that the permutation σ satisfies the following: We denote $\emptyset$ by S_0, and $\{1, \dots, n\}$ by S_{k+1}. Then for each i and $a \in \{|S_{i-1}|, \dots, |S_i| - 1\}$, $\sigma(a) \in S_i \setminus S_{i-1}$.

Composing all those bijections, we denote by O_F the orbit of Π_{n-1} corresponding to $F \in \mathcal{F}_n$. For $F \neq F'$, O_F and $O_{F'}$ are disjoint open sets. Thus set-theoretically is Π_{n-1} a disjoint union of O_F for $F \in \mathcal{F}_n$. It remains to determine the structure of these orbits.

Lemma 9. *For every $F = (S_1 \subset \cdots \subset S_k) \in \mathcal{F}_n$, $O_F \subset \Pi_{n-1}$ is isomorphic to $(\mathbb{C}^\star)^{n-k-1}$.*

Proof. Denote by N the lattice of one-parameter subgroups associated with Π_{n-1} and by σ the cone of the normal fan of the n-dimensional permutohedron corresponding to F, by N_σ the lattice spanned by it in N. We know $O_F \simeq T_{N/N_\sigma}$ ([1], Lemma 3.2.5), so it suffices to prove $N/N_\sigma \simeq \mathbb{Z}^{n-k-1}$.

Facets of the permutohedron correspond to flags consisting of only one subset. If that subset is S, denote by u_S dual vector in N of the corresponding facet. Then $u_S = \sum_{i\in S} u_i$, where u_i is the i-th vector of the canonical basis modulo $\langle(1, 1, \ldots, 1)\rangle$ ([3], p. 18). We need to take the modulo because we defined the $(n-1)$-permutohedron as a polytope in $\mathbb{Z}^n$, contained in a hyperplane with a constant sum of coordinates, so its character lattice is not $\mathbb{Z}^n$, but points in that hyperplane. So $N = \mathbb{Z}^n/\langle(1, 1, \ldots, 1)\rangle$

By the construction of the normal fan, $\sigma = \mathrm{Cone}(u_{S_1}, \ldots, u_{S_k})$. Thus the lattice spanned by σ is the lattice spanned by $u_{S_1}, \ldots, u_{S_k}$.

We now find a basis of $N/N_\sigma = \mathbb{Z}^n/\langle u_{S_1}, \ldots, u_{S_k}, (1, 1, \ldots 1)\rangle$. Let $(e_i)_{i=1}^n$ be the canonical basis of $\mathbb{Z}^n$ and denote $\emptyset$ by S_0, and $\{1, \ldots, n\}$ by S_{k+1}. We claim that to obtain a basis of N/N_σ, it suffices to remove some vectors from the canonical basis. More specifically, for each $j \in \{1, \ldots, k + 1\}$ we choose one $a_j \in S_j \setminus S_{j-1}$ and we remove e_{a_j}. Since $e_{a_j} = -\sum_{i\in S_j, i\neq a_j} e_i$ in N/N_σ, the remaining vectors are generators of N/N_σ. It remains to prove their independence. We tensor with $\mathbb{R}$ and consider a short exact sequence $0 \to (N_\sigma) \otimes_{\mathbb{Z}} \mathbb{R} \to N_{\mathbb{R}} \to (N/N_\sigma) \otimes_{\mathbb{Z}} \mathbb{R} \to 0$ of vector spaces over $\mathbb{R}$ (we can do this because $\mathbb{R}$ is torsion-free, and hence flat over $\mathbb{Z}$). The set $\{u_{S_1} \otimes 1, \ldots, u_{S_{k+1}} \otimes\}$ is the basis over $\mathbb{R}$ of $(N_\sigma) \otimes_{\mathbb{Z}} \mathbb{R}$, so the dimension of $(N/N_\sigma) \otimes_{\mathbb{Z}} \mathbb{R}$ over $\mathbb{R}$ is equal to $n - k - 1$. Hence, the above generators $\{e_i \mid 1 \leq i \leq n, \forall j : i \neq a_j\}$ are independent over $\mathbb{R}$, so they have to be independent over $\mathbb{Z}$. Thus, $N/N_\sigma \simeq \mathbb{Z}^{n-(k+1)}$ as desired. □

1.2 Decomposing the Chow quotient

In this section, we show the correspondence between subsets of the Chow quotient of $(\mathbb{P}^1)^n$ and flags. This is done in Proposition 10. Before we state it, we introduce some notation.

Notation. For $S \subset T \subset \{1, \ldots, n\}$, denote by $\mathcal{C}_{S,T}$ the set of closures of orbits of $(\mathbb{P}^1)^n$ with a source v_S and a sink v_T.

Given $F = (S_1 \subset \cdots \subset S_k) \in \mathcal{F}_n$, let $\mathcal{C}_F \subset G(k, d, n)$ be the set of cycles of the form $\sum_{i=1}^{k+1} X_i$, where $X_i \in \mathcal{C}_{S_{i-1},S_i}$, $S_0 = \emptyset$ and $S_{k+1} = \{1, \ldots, n\}$.

Proposition 10. *The Chow quotient of $(\mathbb{P}^1)^n$ is set-theoretically $\bigcup_{F\in\mathcal{F}_n} \mathcal{C}_F$.*

Since we are working over $\mathbb{C}$, it does not matter whether we use the Zariski or Euclidean closure. Thus, we want to prove that there exists a sequence of algebraic cycles in $G(k, d, n)$, $(X_i)_{i=1}^\infty \to X$ in Chow coordinates, if and only if $X \in \mathcal{C}_F$ for some $F \in \mathcal{F}_n$. Since we do not know anything about the Chow coordinates of elements of $\mathcal{C}_F$, we first show an auxiliary lemma that translates the limits in the Chow coordinates into the limits in $(\mathbb{P}^1)^n$.

Definition 10. *Consider a variety X and a sequence of varieties $(X_i)_{i=1}^{\infty}$ in $\mathbb{P}^n$. We say that X is a* point-wise limit *of $(X_i)_{i=1}^{\infty}$ if every point $x \in \mathbb{P}^n$ is in X if and only if there is a sequence $(x_i)_{i=1}^{\infty}$ such that $x_i \in X_{j_i}$ for some subsequence $(X_{j_i})_{i=1}^{\infty}$ and $x_i \xrightarrow{i\to\infty} x$.*

Note that, due to multiplicities, point-wise limit is not uniquely defined. Because of this issue, there is not an equivalence in the following lemma.

Lemma 11. *Let X, $X_1, X_2, \dots$ be algebraic cycles of degree d and dimension $k-1$ in $\mathbb{P}^{n-1}$.*

1. *If R_X is a limit of $(R_{X_i})_{i=1}^{\infty}$ as Chow coordinates, then X is a point-wise limit of $(X_i)_{i=1}^{\infty}$.*
2. *If X is a point-wise limit of $(X_i)_{i=1}^{\infty}$ and $(R_{X_i})_{i=1}^{\infty}$ converges to R_Y as Chow coordinates for some algebraic cycle Y, then Y is supported on X (i.e. if $x \in Y$, then $x \in X$).*

Proof. Let H, H_1, H_2, ... be the associated collections of hypersurfaces in $G(n-k, n)$ of X, X_1, X_2, ... First we show that H is a point-wise limit of $(H_i)_{i=1}^{\infty}$ if and only if X is a point-wise limit of $(X_i)_{i=1}^{\infty}$.

We start by showing that if H is a point-wise limit of $(H_i)_{i=1}^{\infty}$, then X is a point-wise limit of $(X_i)_{i=1}^{\infty}$. Consider x in the point-wise limit of $(X_i)_{i=1}^{\infty}$ and the sequence $x_i \xrightarrow{i\to\infty} x$, where $x_i \in X_{j_i}$. By contradiction, we show that x is in X. If not, there is an $(n-k-1)$-dimensional projective subspace s that contains x, which is not in H when considered as a point of $G(n-k, n)$ (this follows from the fact that a general $(n-k-1)$-dimensional subspace does not intersect X). We construct a sequence $s_i \in H_{j_i}$ such that $s_i \to s$, obtaining the desired contradiction. We can extend x to the basis of s. Replacing x by x_i in this basis, we obtain a sequence $(s_i)_{i=1}^{\infty}$, whose point-wise limit is s. The point-wise limit of $(s_i)_{i=1}^{\infty}$ is s if and only if $s_i \xrightarrow{i\to\infty} s$ also in the Grassmannian, as desired.

To prove the opposite inclusion, take $x \in X$. Every element of $G(n-k, n)$ containing x is in H by the definition of an associated hypersurface. Moreover, a general such element intersects X only in x. Choose one such element and denote it by s_0. Since it is in H and H is a point-wise limit of $(H_i)_{i=1}^{\infty}$, there is a sequence of $s_i \in H_{j_i}$, $s_i \xrightarrow{i\to\infty} s_0$ both in the Grassmanian and point-wise. Let $x_i = s_i \cap X_{j_i}$. Since the projective space is compact, the sequence $(x_i)_{i=1}^{\infty}$ has a convergent subsequence with a limit x_0. Necessarily $x_0 \in s_0$. We have already proved that the point-wise limit of $(X_i)_{i=1}^{\infty}$ is a subset of X, so $x_0 \in s_0 \cap X = \{x\}$. Hence we have found the sequence converging to x.

Assuming that X is a point-wise limit of $(X_i)_{i=1}^{\infty}$, we now prove that H is a point-wise limit of $(H_i)_{i=1}^{\infty}$. There is some point-wise limit H' of $(H_i)_{i=1}^{\infty}$. We denote its associated cycle by X'. We already know that X' is a point-wise limit of $(X_i)_{i=1}^{\infty}$. Hence, X and X' must have the same support. Therefore, H and H' have the same support too and H is a point-wise limit of $(H_i)_{i=1}^{\infty}$.

Now we prove that if R_X is a limit of $(R_{X_i})_{i=1}^\infty$, then H is a point-wise limit of $(H_i)_{i=1}^\infty$. We again denote the point-wise limit of $(H_i)_{i=1}^\infty$ by H'. First, we show $H' \subset H$. Take a sequence $x_i \in H_{j_i}$ converging to some $x \in H'$. Then $R_{X_{j_i}}(x_i) = 0$. We prove that $R_X(x_i) \xrightarrow{i\to\infty} 0$. Take a real number $\varepsilon > 0$. Since $(R_{X_{j_i}})_{i=1}^\infty$ converges to R_X, for i large enough, we can bound the absolute value of the sum of coefficients $R_{X_{j_i}} - R_X$ by ε. Since $x_i \to x$, we can bound the absolute value of the entries of x_i by some constant K for i large enough. Also both $R_{X_{j_i}}$ for all i and R_X are homogeneous polynomials of degree d. Altogether, for i large enough,

$$|(R_{X_{j_i}} - R_X)(x_i)| \leq \varepsilon \cdot K^d \xrightarrow{\varepsilon\to 0} 0.$$

Therefore, $R_X(x_i) \xrightarrow{i\to\infty} 0$. Polynomials are continuous, so $R_X(x) = 0$, hence $x \in H$.

Now we take $x \in H$ and we want to show that $x \in H'$, i.e. there is a sequence $x_i \in H_{j_i}$ such that $x_i \xrightarrow{i\to\infty} x$. Since H is a collection of hypersurfaces with sum of degrees d, a general projective line intersects H in d points. We proceed by contradiction, showing that if $x \notin H'$, a general line intersects H in infinitely many points. Take a general projective line $L \in G(n-k, n)$ that contains x. It is isomorphic to $\mathbb{P}^1$, with restrictions of R_{X_i} and R_X being holomorphic functions on it. Fix a real number $\varepsilon_0 > 0$, such that the intersection of each H_i with the ε_0-neighbourhood of x in L is empty. Such an ε_0 exists by our assumption that x is not a limit point of $(H_i)_{i=1}^\infty$. Take any $\varepsilon \in (0, \varepsilon_0)$. We will show that there is $x_\varepsilon \in H \cap L$, such that $|x, x_\varepsilon| = \varepsilon$, obtaining a contradiction. We know that $R_{x_i}(y) \neq 0$ for every $y \in L$ in the ε-neighbourhood of x. By the minimum modulus principle, R_{X_i} attains the minimal absolute value on the ε-neighbourhood of x in some y_i such that $|x, y_i| = \varepsilon$. Since $R_{X_i} \to R_X$ and $R_X(x) = 0$, $R_{X_i}(x) \to 0$. As $R_{X_i}(y_i) \leq R_{X_i}(x)$, also $R_{X_i}(y_i) \to 0$. Choose a converging subsequence of y_i, denote its limit by y. Since $|x, y_i| = \varepsilon$, $|x, y| = \varepsilon$ too. By $R_{X_i} \to R_X$ and $R_{X_i}(y_i) \to 0$, using similar arguments as above, we obtain $R_X(y) = 0$. So, we can set $x_\varepsilon = y$.

At last we finish the proof of the second statement. Let G be a zero locus of R_Y, which is a hypersurface in $G(n-k, n)$. We already know that G is a point-wise limit of $(H_i)_{i=1}^\infty$. Hence, H and G have the same support, as desired. ☐

When using this lemma for $(\mathbb{P}^1)^n$, we first apply the Segre embedding to $(\mathbb{P}^1)^n$. Since the Segre embedding is continuous on $\mathbb{P}^{2^n-1}$ and is an isomorphism on its image, we can consider limits in $(\mathbb{P}^1)^n$ instead of in $\mathbb{P}^{2^n-1}$.

Proof of Proposition 10. "$\subset$":

Take $X \in (\mathbb{P}^1)^n / \mathbb{C}^\star$. We will use following facts about X:

1. Since $(\mathbb{P}^1)^n / \mathbb{C}^\star$ is defined to be a subset of $G(2, n, 2^n)$, X is an algebraic cycle of degree n and dimension 1. So, in the following, we can use Lemma 11. For example, we know that there is a sequence $(X_i)_{i=1}^\infty$ of elements of Q_n, such that X is its point-wise limit.

2. The $\mathbb{C}^\star$ action on $(\mathbb{P}^1)^n$ induces a $\mathbb{C}^\star$ action on $G(2, n, 2^n)$. The torus fixed points form a closed subvariety and $\overline{O(x)}$ is a torus fixed point in $G(2, n, 2^n)$ for every $x \in U_n$. So, the points in $(\mathbb{P}^1)^n / \mathbb{C}^\star$ are also torus fixed. Hence X is a formal sum of orbit closures.

3. One of the summands of X has a source $v_\emptyset$ and one has a sink $v_{\{1,\dots,n\}}$. This follows from Lemma 11 and the fact that $v_\emptyset$ and $v_{\{1,\dots,n\}}$ are in $\overline{O(x)}$ for every $x \in U_n$.

4. If there is a summand of X with a sink v_S, where S is a proper subset of $\{1,\dots,n\}$, there is a summand of X with a source v_S. Vice versa, if there is a summand of X with a source v_S, where S is a proper subset of $\{1,\dots,n\}$, there is a summand of X with a sink v_S. Here, we extend the notion of a sink and a source of an orbit to its closure. To show this fact, we take $x = (a_j : b_j)_{j=1}^n$ in the orbit with a sink S. Hence $(a_j : b_j) = (1:0)$ if and only if $j \notin S$ according to Lemma 6. Since X is a point-wise limit of $(X_i)_{i=1}^\infty$, there is a sequence $(x_i)_{i=1}^\infty$ that converges to x, where $x_i \in X_{j_i}$. Since X is a formal sum of orbits, it suffices to show that there is a point in it such that is also in an orbit with source v_S. For each i, let $x_i = (a_j^i : b_j^i)_{j=1}^n$ and denote $t_i = \max_{j\notin S} \left|\frac{b_j^i}{a_j^i}\right|$. Since x_i is in U_n, t_i is well-defined and non-zero. Hence, we can consider a sequence $\left(\frac{1}{t_i}\cdot x_i\right)_{i=1}^\infty$. Because $(\mathbb{P}^1)^n$ is compact, we can take its converging subsequence and denote its limit by $y = (a'_j : b'_j)_{j=1}^\infty$. Since X is a point-wise limit of $(X_i)_{i=1}^\infty$ and $\frac{1}{t_i}\cdot x_i \in X_{j_i}$, y is in X. We claim that $O(y)$ has a source v_S. Since we have chosen t_i maximal, $(a'_j : b'_j) \neq (0:1)$ for $j \notin S$. Since $|\frac{b_j^i}{a_j^i}| \xrightarrow{i\to\infty} 0$ for $j \notin S$, $t_i \xrightarrow{i\to\infty} 0$. Moreover, $(a_j : b_j) \neq (1:0)$ for $j \in S$. Thus, $(a'_j : b'_j) = (0:1)$ for $j \in S$. Therefore, the orbit of y has source v_S, as desired. The other direction can be proved analogously.

Denote the summand of X with the source $v_\emptyset$ by X_1. Take the longest sequence of summands of X, denoting it by $(X_1, X_2, \dots, X_k)$, such that the source of X_i is the sink of X_{i-1}. By (4), the sink of X_k cannot be v_S for a proper subset S of $\{1,\dots,n\}$, so is $v_{\{1,\dots,n\}}$. By 7, $\sum_{i=1}^k X_i$ has degree n, just as X. Since the coefficients in an algebraic cycle are non-negative, there cannot be anything more in X, as desired.

"$\supset$":

We proceed by induction on the size of the flag F. If F is the empty flag, $\mathcal{C}_F$ is the flat family we used to define the Chow quotient, so it is contained in the Chow quotient.

The inductive step is easily reduced to the case $|F| = 1$ in the following way. Take $F = (S_1 \subset \cdots \subset S_k)$, $k \geq 2$. It is enough to show that $\mathcal{C}_F$ is in the closure of $\mathcal{C}_{(S_2\subset\cdots\subset S_k)}$. Since $\mathcal{C}_{\emptyset,S_2}$ is isomorphic to $\mathcal{C}_{F'}$, where F' is the empty flag in $\mathcal{F}_{|S_2|}$, this indeed reduces to the case $|F| = 1$ in dimension $|S_2|$.

It remains to show that every $X \in \mathcal{C}_F$, where $F = (S)$, is a point-wise limit of some $(X_i)_{i=1}^\infty$, where $X_i \in \mathcal{C}_{\emptyset^n}$. We know that X is of the form

$$X' + X'' = \overline{O((a'_1 : b'_1), \dots (a'_n : b'_n))} + \overline{O((a''_1 : b''_1), \dots, (a''_n : b''_n))},$$

where $(a'_i : b'_i) = (1:0)$ if and only if $i \notin S$ and $(a''_i : b''_i) = (0:1)$ if and only if

$i \in S$. So, for every $k \in \mathbb{N}$, the following points are well-defined:

$$
\begin{aligned}
x'_k = ((a_1 : b_1), \ldots, (a_n : b_n)), &\text{ where} \\
&(a_i : b_i) = (a'_i : b'_i), && \text{if } i \in S, \\
&(a_i : b_i) = \left(a''_i : \frac{1}{k} \cdot b''_i\right), && \text{if } i \notin S, \\
x''_k = ((c_1 : d_1), \ldots, (c_n : d_n)), &\text{ where} \\
&(c_i : d_i) = \left(\frac{1}{k} \cdot a'_i : b'_i\right), && \text{if } i \in S, \\
&(c_i : d_i) = (a''_i : b''_i), && \text{if } i \notin S.
\end{aligned}
$$

Moreover, $O(x'_k) = O(x''_k)$. This can be seen by acting by $\frac{1}{k}$ on x''_k. We denote the closure of the common orbit of x'_k and x''_k by X_k and we show that X is the point-wise limit of $(X_k)_{k=1}^{\infty}$.

First, we note that

$$
\begin{aligned}
\lim_{k\to\infty} x'_k &= ((a'_1 : b'_1), \ldots, (a'_1 : b'_n)), \\
\lim_{k\to\infty} x''_k &= ((a''_1 : b''_1), \ldots, (a''_1 : b''_n)).
\end{aligned}
$$

We have used the fact that $(a''_i : b''_i) \neq (0 : 1)$ for $i \notin S$ and $(a'_i : b'_i) \neq (1 : 0)$ for $i \in S$.

As we have shown in the first part of the proof, $\lim_{k\to 0} X_k$ has to be in $\mathcal{C}_{F'}$ for some flag F'. Since $x'_k \in \lim_{k\to\infty} X_k$, the first summand of $\lim_{k\to\infty} X_k$ has to be X' and the smallest subset of F' has to be S. Similarly, since $x''_k \in \lim_{k\to\infty} X_k$, the last summand of $\lim_{k\to\infty} X_k$ must be X'' and the largest subset of F' has to be S. Thus, $F' = F$ and $\lim_{k\to\infty} X_k = X' + X'' = X$. So X is in the Chow quotient. □

1.3 Constructing the bijection

We have now the partition of both Π_{n-1} and $(\mathbb{P}^1)^n / \mathbb{C}^\star$, so it remains to construct the bijection of O_F and $\mathcal{C}_F$ for each flag F. We proceed by induction. The crucial part is to construct the bijection between $O_{\emptyset^n}$ and $\mathcal{C}_{\emptyset^n}$. We then finish the proof by passing to lower dimensions and putting the partial bijections together.

Lemma 12. *There is a bijection between $O_{\emptyset^n}$ and $\mathcal{C}_{\emptyset^n}$.*

Proof. By 9, $O_{\emptyset^n}$ is isomorphic to $(\mathbb{C}^\star)^{n-1}$, which is in bijection with general points of $\mathbb{P}^n$ (by general points we mean those points where no coordinate is zero). By 6, $\mathcal{C}_{\emptyset^n} = Q_n$. Consider a map

$$
\begin{aligned}
\varphi : U_n &\to \mathbb{P}^n \\
((a_1 : b_1), \ldots, (a_n : b_n)) &\mapsto \left(\frac{b_1}{a_1} : \frac{b_2}{a_2} : \cdots : \frac{b_n}{a_n}\right)
\end{aligned}
$$

This map is well-defined, its image are general points of $\mathbb{P}^n$ and it is constant on orbits. Moreover, only points of the form $((1 : t \cdot x_1), \ldots (1 : t \cdot x_n))$ can map on the point $(x_1 : \cdots : x_n)$. So, the map $\bar{\varphi} : Q_n \to \mathbb{P}^n$ induced by φ by mapping an element of Q_n to $\varphi(x)$ for its general point x is well-defined, and injective. The desired bijection is hence $\bar{\varphi}$ composed with the bijection of $O_{\emptyset^n}$ and general points of $\mathbb{P}^n$.

$\square$

Lemma 13. *There is a bijection between O_F and $\mathcal{C}_F$ for every $F \in \mathcal{F}_n$.*

Proof. By 9, $O_F \simeq (\mathbb{C}^\star)^{n-k-1}$ for $F = (S_1 \subset \cdots \subset S_k) \in \mathcal{F}_n$.

Since $\mathcal{C}_{S,T} \simeq \mathcal{C}_{\emptyset^{|S|-|T|}}$ is in bijection with $(\mathbb{C}^\star)^{|S|-|T|-1}$,

$$\mathcal{C}_F = \mathcal{C}_{\emptyset,S_1} \times \mathcal{C}_{S_1,S_2} \times \cdots \times \mathcal{C}_{S_{k-1},S_k} \times \mathcal{C}_{S_k,\{1,\ldots,n\}}$$

is in bijection with

$$(\mathbb{C}^\star)^{|S_1|-1} \times (\mathbb{C}^\star)^{|S_2|-|S_1|-1} \times \cdots \times (\mathbb{C}^\star)^{|S_k|-|S_{k-1}|-1} \times (\mathbb{C}^\star)^{n-|S_k|-1} = (\mathbb{C}^\star)^{n-(k+1)}.$$

$\square$

Proposition 14. *There is a bijection between Π_{n-1} and the Chow quotient of $(\mathbb{P}^1)^n$.*

Proof. We have seen that Π_{n-1} is a disjoint union of O_F for $F \in \mathcal{F}_n$. By 10, the Chow quotient of $(\mathbb{P}^1)^n$ is a disjoint union of $\mathcal{C}_F$ for $F \in \mathcal{F}_n$. Thus, putting together the bijections constructed in 13 for each $F \in \mathcal{F}_n$ separately, we obtain the required bijection between Π_{n-1} and $(\mathbb{P}^1)^n$. $\square$

2 Computations for $(\mathbb{P}^1)^2$

Our goal is to show by direct computation that the bijection defined in the previous chapter is a morphism for $n = 2$. We do this by explicitly writing down the morphism and showing that it is locally a quotient of polynomials, thus a morphism by definition.

To be able to write down the morphism, we first need to find the Chow coordinates of any element of the flat family we used to define the Chow quotient.

Claim 15. *Let $X = \overline{O(x)}$ for $x = ((a_1 : b_1), (a_2 : b_2)) \in U_2$. Then*

$$R_X = a_1b_1a_2b_2 \cdot [03]^2 - (a_1b_2 \cdot [01] + b_1a_2 \cdot [02])(a_1b_2 \cdot [13] + b_1a_2 \cdot [23]).$$

Proof. Denote by F the polynomial on the right-hand side. We want to show that the zero locus of F and R_X is the same.

The Chow form is an element of a projective space, so F is defined independently of the choice of $x \in X$ and its representation.

We need to show that F vanishes at every point of $\mathcal{Z}(X) \subset G(2,4)$, the associated hypersurface of X. Points of $G(2,4)$ are given by a full-rank matrix

$$U = \begin{pmatrix} u_{0,0} & u_{0,1} & u_{0,2} & u_{0,3} \\ u_{1,0} & u_{1,1} & u_{1,2} & u_{1,3} \end{pmatrix}.$$

A point $u \in G(2,4)$ is in $\mathcal{Z}(X)$ if and only if there is $t \in \mathbb{C}^\star$ such that

$$\begin{pmatrix} a_1a_2 \\ t \cdot a_1b_2 \\ t \cdot a_2b_1 \\ t^2 \cdot b_1b_2 \end{pmatrix}$$

is in the kernel of the matrix that defines u. Hence if $u \in \mathcal{Z}(X)$, we have

$$a_1a_2 \cdot u_{0,0} = -t \cdot a_1b_2 \cdot u_{0,1} - t \cdot a_2b_1 \cdot u_{0,2} - t^2 \cdot b_1b_2 \cdot u_{0,3}, \tag{2.1}$$
$$a_1a_2 \cdot u_{1,0} = -t \cdot a_1b_2 \cdot u_{1,1} - t \cdot a_2b_1 \cdot u_{1,2} - t^2 \cdot b_1b_2 \cdot u_{1,3}. \tag{2.2}$$

We extend the notation of brackets by denoting $[ij]$ be the minor corresponding to columns i and j of U. Then $[ii] = 0$ and $[ij] = -[ji]$. Substituting for $u_{0,0}$ from (2.1) and for $u_{1,0}$ from (2.2), we obtain

$$\begin{aligned} a_1a_2 \cdot [01] &= -(t \cdot a_1b_2 \cdot [11] + t \cdot a_2b_1 \cdot [21] + t^2 \cdot b_1b_2 \cdot [31]) = \\ &= t \cdot a_2b_1 \cdot [12] + t^2 \cdot b_1b_2 \cdot [13], \\ a_1a_2 \cdot [02] &= -(t \cdot a_1b_2 \cdot [12] + t \cdot a_2b_1 \cdot [22] + t^2 \cdot b_1b_2 \cdot [32]) = \\ &= -t \cdot a_1b_2 \cdot [12] + t^2 \cdot b_1b_2 \cdot [23]), \\ a_1a_2 \cdot [03] &= -(t \cdot a_1b_2 \cdot [13] + t \cdot a_2b_1 \cdot [23] + t^2 \cdot b_1b_2 \cdot [33]) = \\ &= -(t \cdot a_1b_2 \cdot [13] + t \cdot a_2b_1 \cdot [23]). \end{aligned}$$

Putting this together, we get

$$\begin{aligned} a_1^2b_1a_2^2b_2 \cdot [03]^2 &= b_1b_2 \cdot (t \cdot (a_1b_2 \cdot [13] + b_1a_2 \cdot [23]))^2, \\ a_1^2b_2a_2 \cdot [01] + a_1b_1a_2^2 \cdot [02] &= t^2 \cdot a_1b_1b_2^2 \cdot [13] + t^2 \cdot b_1^2a_2b_2 \cdot [23]. \end{aligned}$$

Since a_1 and a_2 are non-zero, we can multiply F by a_1a_2 without changing its zero locus. Using the computations above, we have

$$a_1a_2 \cdot F =$$

$$b_1b_2\cdot(t\cdot(a_1b_2\cdot[13]+b_1a_2\cdot[23]))^2-t^2(\cdot a_1b_1b_2^2\cdot[13]+b_1^2a_2b_2\cdot[23])(a_1b_2\cdot[13]+b_1a_2\cdot[23]).$$

$$= 0$$

By Proposition 3, we know that R_X is of degree 2 and is unique up to constant factor. The polynomial F vanishes on $\mathcal{Z}(X)$, so it has to be divisible by R_X. Since both are of the same degree, $F = R_X$ as points of $\mathbb{P}(\mathcal{B}_2)$. $\square$

Claim 16. *All the Chow forms* $R \in (\mathbb{P}^1)^2/\mathbb{C}^\star$ *are of the form*

$$uv \cdot [03]^2 - u^2 \cdot [01][13] - uv[01][23] - uv[02][13] - v^2[02][23]$$

for some $u, v \in \mathbb{C}$.

Proof. The Chow forms of elements of Q_n are of this form, with the additional condition that u and v are both non-zero. The other points in the Chow quotient are limits of those, so we include exactly those where one of u and v is zero. $\square$

We used a representing point of the orbit to find the Chow form. However, the map from the permutohedral variety to the Chow variety can be defined without it. This is an essential step of the proof of the main result of this chapter.

Proposition 17. *The Chow quotient of* $(\mathbb{P}^1)^2$ *by the given action and* Π_1 *are isomorphic as varieties.*

Proof. Since Π_1 is isomorphic to $\mathbb{P}^1$, we can represent its point as $(x : y)$. Then we define the map

$$\begin{aligned}&\psi : \Pi_1 \to G(2,2,4)\\ &(x : y) \mapsto xy \cdot [03]^2 - (x \cdot [01] + y \cdot [02])(x \cdot [13] + y \cdot [23])\end{aligned}$$

Multiplying both x and y by $t \in \mathbb{C}$ does not change the image, so it is a well-defined map. By 16, it is clear that the image of ψ is in the Chow quotient of $(\mathbb{P}^1)^2$. Since it is defined by polynomials, it indeed is a morphism.

To show that ψ is an isomorphism, it suffices to find a well-defined inverse. We denote respectively U_1 and U_2 the open subsets of $(\mathbb{P}^1)^2/\mathbb{C}^\star$, where respectively $[01][13]$ and $[02][23]$ has a non-zero coefficient. Then we have maps

$$\begin{aligned}\psi_1^{-1} &: U_1 \to \Pi_2\\ f \cdot [03]^2 - g \cdot [01][13] - f[01][23] - f[02][13] - h[02][23] &\mapsto \left(1 : \frac{f}{g}\right)\end{aligned}$$

$$\begin{aligned}\psi_2^{-1} &: U_2 \to \Pi_2\\ f \cdot [03]^2 - g \cdot [01][13] - f[01][23] - f[02][13] - h[02][23] &\mapsto \left(\frac{f}{h} : 1\right)\end{aligned}$$

Multiplying the f, g and h by a common factor does not change the result, so the map is well-defined. Taking into account that $gh = f^2$ and that we work with projective spaces, it is easy to see that ψ and ψ_1^{-1} are inverse to on another on the proper open subsets, similarly for ψ and ψ_2^{-1}. $\square$

Conclusion

We have examined some aspects of the isomorphism between the Chow quotient of $(\mathbb{P}^1)^n$ and the $(n-1)$-dimensional permutohedral variety. We have seen how both of those objects decompose into subsets corresponding to flags and we explicitly constructed the set-theoretical bijection on those subsets.

To show for $(\mathbb{P}^1)^2$ that there is not only a set-theoretical bijection, but also an isomorphism of varieties, we computed the Chow coordinates of a general orbit closure. Since finding the Chow coordinates is computationally hard, extending our approach to higher dimensions would require some guesswork. By induction, we know what must be the Chow form of a degenerate element of the Chow quotient. This might help to make a correct guess.

Bibliography

1. Cox, D. A.; Little, J. B.; Schenck, H. K. *Toric Varieties*. American Mathematical Society, 2011.
2. Humphreys, J.E. *Linear Algebraic Groups*. Springer, 1975. Graduate Texts in Mathematics. isbn 9780387901084.
3. Huh, J. Tropical geometry of matroids. In: 2016. Available also from: `https://api.semanticscholar.org/CorpusID:21738928`.
4. Gelfand, I. M.; Kapranov, M. M.; Zelevinsky, A. V. *Discriminants, Resultants, and Multidimensional Determinants*. Birkhäuser, 1994.
5. Hartshorne, R. *Algebraic Geometry*. Springer, 1977. Graduate Texts in Mathematics. isbn 9780387902449.
6. Michałek, M.; Sturmfels, B. *Invitation to Nonlinear Algebra*. American Mathematical Society, 2021. Graduate Studies in Mathematics. isbn 9781470453671.
7. Dalbec, J.; Sturmfels, B. Introduction to Chow Forms. In: *Invariant Methods in Discrete and Computational Geometry: Proceedings of the Curaçao Conference, 13–17 June, 1994*. Ed. by White, Neil L. Dordrecht: Springer Netherlands, 1995, pp. 37–58.
8. Thaddeus, M. *Complete collineations revisited*. 2001. Available from arXiv: `math/9808114 [math.AG]`.
9. Tyrrel, B. *Flatness in algebraic geometry*. 2017. Available also from: `https://www.maths.tcd.ie/~btyrrel/flatness.pdf`.