Evaluation of a Bachelor Thesis by Supervisor

Title: Permutohedral varieties as Chow quotients
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Summary of the contents

The thesis focuses on certain quotients in algebraic geometry—the so called Chow quotients. The idea is to start with a variety with a group action (in this thesis it is $(\mathbb{P}^1_{\mathbb{C}})^{\times n}$ with a natural componentwise action of $\mathbb{C}^*$) and construct a variety of algebraic cycles (=formal sums of algebraic subvarieties) which contains the set of closures of orbits.

Constructing quotients in algebraic geometry and giving them again a structure of a variety is usually technically involved. Quotients are typically well-behaved only on certain open subsets (at least if one wishes to avoid going to the realm of stacks where technicalities explode) and the resulting coordinates are everything but explicit. So the author decided to describe the quotient only set-theoretically, i.e. as a bijection from the set of suitable algebraic cycles in $(\mathbb{P}^1_{\mathbb{C}})^{\times n}$ to the target variety predicted by theory. At the end she was, however, more specific in the case $n = 2$. This has resulted in a relatively condensed, but nicely written thesis.

Evaluation of the thesis

Topic of the thesis. The topic of the thesis was suggested by Prof. Mateusz Michalek from University of Konstanz, whom the author visited in the course of her study and who acted as a consultant for the thesis. On one hand the topic is fairly involved (one needs a good background in algebraic geometry and especially in the theory of toric varieties). On the other it is very cleverly chosen in the sense that it focuses on a specific instance of the theory which, with some effort, can be described rather directly.

Contribution of the student. The text quickly and efficiently summarizes the necessary theory from several sources and then constructs the set-theoretic correspondence between algebraic cycles in the Chow quotient and a subvariety of a certain Grassmannian rather directly, using certain facts about pointwise limits of families of varieties developed by the author herself. The contribution is definitely well above what is expected from a Bachelor thesis at our faculty.

Mathematical quality. The text is well written and rigorous, but at certain places a more detailed treatment would help. Specifically:

1. Below the statement of Prop. 10, the text says "Since we are working over $\mathbb{C}$, it does not matter whether we use the Zariski or Euclidean closure." This is not obvious, the Zariski topology is way courser than the Euclidean one, so it would be nice to justify why here it indeed does not matter.
2. Some genericity arguments in the proof of Lemma 11 are a bit too vague to my taste. For example, the statement that "If $x \notin X$, there is an $(n-k-1)$-dimensional projective subspace s that contains x, which is not in H when considered as a point of the Grassmannian $G(n-k,n)$ (this follows from the fact that a general $(n-k-1)$-dimensional subspace does not intersect X)." Or "Every element of $G(n-k,n)$ containing x is in H by the definition of an associated hypersurface. Moreover, a general such element intersects X only in x."

3. Later in the same proof: What exactly does one mean by "a projective line $L \in G(n-k, n)$"?
4. One more detail: It would be more convenient to define a point-wise limit of a family of varieties as a set, not necessarily as a variety. This would be more compatible with statements like "There is some point-wise limit H' of $(H_i)_{i=1}^{\infty}$," where it is not a priori clear that H' is a variety. It would also help to explicitly write what one means by a point-wise limit of algebraic cycles, to give a good meaning to the sentence "Note that, due to multiplicities, point-wise limit is not uniquely defined." just above Lemma 11.

Usage of the literature. The literature is used and cited appropriately.

Formal aspects. The thesis is well written and meets usual formal standards.

I would just suggest that references in the text be more descriptive. For example, the sentences "By 6, an orbit of a point from U_n has a source .... Thus, by 7, the degree of its closure is n." in the proof of Lemma 8 should better read "By Lemma 6, an orbit of a point .... Thus, by Lemma 7, the degree of its closure is n." This is especially recommendable since the text also contains Definition 6 and Definition 7 and if hypertext links are not available (e.g. in a printed version or alternative PDF readers), references consisting only of a number become ambiguous.

Also, for some reason page numbers are missing on most pages.

Conclusion

I recommend to recognize this nice piece work as a Bachelor thesis.

The suggested grading will be communicated directly to the head of the examination (sub)committee.

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