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OF MATHEMATICS
AND PHYSICS
Charles University

BACHELOR THESIS

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Conway's topograph

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Study programme: Mathematics
Study branch: Mathematics for Information Technologies

Prague 2024

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I would like to express my gratitude to my supervisor, Vítězslav Kala, for his guidance throughout the process of writing this thesis and for providing me with valuable advice. I am also grateful to all my friends who took the time to read my thesis and provided me with their valuable feedback. Lastly, I would like to thank my family for their unwavering support and patience, especially for listening to me talk about my thesis during our family dinners.

Title: Conway's topograph

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Abstract: John H. Conway has introduced the topograph, a graph containing all possible bases of $\mathbb{Z}^2$ as its edges. This work aims to properly define this topograph and prove its key properties. In addition, we will delve into the attributes of continued fractions of negative rational numbers and conjugates of quadratic irrationals. Ultimately, we will demonstrate how these properties can be leveraged to determine paths in the topograph.

Keywords: continued fractions, paths in a graph, topograph, Farey tree, lax bases

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Introduction

In 1997, John H. Conway published a book called *The Sensual (quadratic) form.* In this work, he introduces a new structure called *the topograph.* It is an infinite graph that allows us to explore quadratic forms through the language of graphs.

The topograph's vertices are known as *lax superbases.* A lax superbase is a set of three lattice points $\{\pm u, \pm v, \pm w\}$ in $\mathbb{Z}^2$, where each point u, v, w is determined up to sign, and any two of them are linearly independent. Additionally, it satisfies the equation $\pm u \pm v \pm w = 0$ for some choice of signs. We connect vertices by edges when two lax superbases have two points in common.

In their article *Growth of values of binary quadratic forms and Conway rivers* from 2018, Kathryn Spalding and Alexander P. Veselov describe how we can use continued fractions to describe paths in this topograph. They rely on the book from Allen Hatcher *Topology of Numbers* to describe the paths in a graph called the Farey tree.

In this thesis, we aim to connect these two works. We look at propositions given by Veselov and Spalding in [1] on continued fractions and then provide abstract definitions for the topograph and the Farey tree. Ultimately, we describe how the paths in a topograph can be described using the continued fractions and provide new illustrations.

In Chapter 1, we explore continued fractions. We prove what the continued fraction of a negative real number and conjugates of quadratic irrationals look like. The main results of this chapter are Proposition 1.16 and 1.17, which state how the conjugates of quadratic irrationals with periodic continued fractions look like for one and more elements in the pre-period. In Section 1.1, we recall some definitions of continued fractions and quadratic irrationals, and we recall two important theorems from Lagrange and Galois, which describe periodic continued fractions. In Section 1.2, we prove some necessary lemmata which help us in proving Propositions 1.16 and 1.17 found in Section 1.3.

In Chapter 2, we aim to construct the topograph introduced by Conway in [2]. In Section 2.1, we recall some basic facts from graph theory. In Section 2.2, we introduce definitions given by Conway in [2], and we give new definitions relating to the work with navigators. We also prove the fundamental Lemma 2.22 provided by Conway. In Section 2.3, we describe the algorithm given by Conway. In Section 2.4, we provide the abstract definition of the topograph and prove its fundamental properties. Lastly, in Section 2.5, we define Conway's algorithm, proving that this definition is equivalent to the abstract definition provided in Section 2.4.

In Chapter 3, we focus on two structures introduced by Hatcher in [3] - the Farey diagram and the Farey tree, which we describe and define in Sections 3.1 and 3.2 respectively. In Section 3.3, we explain and prove how we can connect the Farey tree and the topograph, and lastly, in Section 3.4, we sketch a structure with which we could prove the tree property of the topograph.

In the last Chapter 4, we explain how the paths in a topograph can be described using continued fractions. In Section 4.1, we explain how we use matrices to signify turning left and right in the Farey tree and the topograph. In Section 4.2, we prove how a continued fraction determines a path in a topograph, which leads to an edge containing the number corresponding to the continued fraction, and in Section 4.3, we provide some illustrations of paths determined by a different continued fraction which properties are proved in Chapter 1.

This thesis takes primarily the three sources [1, 2, 3] and combines their observations in one cohesive work while also providing different ways of working with the topograph and more precise definitions of the topograph, Farey diagram and the Farey tree.

In Chapter 1, we stick to the article [1]. We prove Lemmata 1.2, 1.12, 1.13, 1.14; and 1.15, which were provided in the article without proof. We also provide more detailed proofs of Propositions 1.16 and 1.17.

In Chapter 2, we introduce crucial observations from [2], and we provide the proof of Lemma 2.22, which Conway offers without proof. While the things in the rest of the chapter are implicit in [2], they are our addition. We provide a new way of looking at the topograph through matrices and sets of navigators. We introduce the abstract definition and then prove the key properties of the topograph.

In Chapter 3, we look at the works of Hatcher in [3] through the lens of [1]. We explain the construction of the graph given by Hatcher, provide more precise definitions of the Farey tree and Farey diagram, and prove that the Farey tree is related to the topograph.

In Chapter 4, we circle back to the article [1], where we properly define left and right turns. We prove the observation in Propositions 4.9 and 4.10, which were not proved in [1]. We then provide our own illustrations and examples in the last Section 4.3.

1. Continued fractions

The primary objective of the first chapter is to present our proofs of Proposition 1.16 and Proposition 1.17. These propositions sheds light on the structure of the continued fraction of a conjugate of a quadratic irrational. To achieve this, we will examine negative continued fractions and continued fractions of quadratic irrationals and their conjugates.

1.1 Preliminaries

To begin, we shall revisit the definitions of continued fractions. They are commonly used to represent real numbers through a sequence of natural numbers. In this regard, we will use a particular definition of a continued fraction, although it is worth noting that variations of this definition can also be found in some texts.

Definition 1.1. *We define a* finite continued fraction *as*

$$[c_0, c_1, \ldots, c_k] = c_0 + \frac{1}{c_1 + \frac{1}{\cdots + \frac{1}{c_k}}}$$

for $c_i \in \mathbb{R}$ such that we never divide by 0.

An infinite continued fraction *is defined as $[c_0, c_1, \ldots] = \lim_{k\to\infty}[c_0, \ldots, c_k]$ if the limit does exist.*

We can see that, for example, the continued fraction $[c_0, c_1, \ldots, 0]$ is not defined since the last fraction would be $\frac{1}{0}$.

Unless stated otherwise, we will use the following expression for continued fractions of negative real numbers.

Lemma 1.2. *For $-\xi = \eta < 0$ we have the continued fraction:*

$$\eta = -[c_0, c_1, c_2, \ldots] = [-c_0, -c_1, -c_2, \ldots],$$

where $[c_0, c_1, c_2, \ldots]$ is the continued fraction of ξ.

Proof. From Definition 1.1 and arithmetic of limits of sequences we know that

$$\begin{aligned} -[c_0, c_1, c_2, \ldots] &= -\lim_{k\to\infty}[c_0, c_1, c_2, \ldots, c_k] = \lim_{k\to\infty} -[c_0, c_1, c_2, \ldots, c_k] = \\ &= \lim_{k\to\infty} -\left(c_0 + \frac{1}{c_1 + \frac{1}{\cdots + \frac{1}{c_k}}}\right) = \lim_{k\to\infty} -c_0 + \frac{1}{-c_1 + \frac{1}{\cdots + \frac{1}{-c_k}}} = \\ &= \lim_{k\to\infty}[-c_0, -c_1, -c_2, \ldots, -c_k] = [-c_0, -c_1, -c_2, \ldots]. \end{aligned}$$

□

It is worth noting that a particular type of continued fraction is known as a periodic continued fraction. This property is utilized in Theorem 1.6, which is pivotal in establishing a correlation between the periodicity of a continued fraction of a given number and its rationality.

Definition 1.3. *If α is an irrational number and $[c_0, c_1, c_2, \dots]$ is its infinite continued fraction, then we say that α is* periodic with period $(c_k, \dots, c_l)$ *if there exist two non-negative integers k and l such that $k \leq l$ and $\forall i \in \{0, 1, \dots, l-k\}, \forall j \in \mathbb{N}$, $c_{k+i} = c_{k+i+jp}$, where $p = l - k + 1$ is the length of the period. We denote it $[c_0, c_1, \dots, c_{k-1}, \overline{c_k, c_{k+1}, \dots, c_l}]$. A continued fraction is called* purely periodic *if $k = 0$.*

Before introducing Lagrange's Theorem 1.6, we must define a quadratic irrational.

Definition 1.4. *A* quadratic irrational *is a number $\alpha \in \mathbb{R}$ that can be written as $\alpha = \frac{A+\sqrt{D}}{B}$, where $A, B, D \in \mathbb{Z}$ for $D, B \neq 0$ and D is non-negative and is not a square.*

As is customary, we shall denote the number field $\mathbb{Q}(\sqrt{D})$, where D is not a square, by K. Then, we can define the following function.

Definition 1.5. *We will define* conjugation *for $\alpha = \frac{A+\sqrt{D}}{B}$ as a $\mathbb{Q}$-homomorphism*

$$' : K \to K$$
$$\frac{A+\sqrt{D}}{B} \mapsto \frac{A-\sqrt{D}}{B}$$

We can now note some results from Lagrange and Galois.

Theorem 1.6 (Lagrange)**.** *Let ξ be irrational. Then its continued fraction $\xi = [a_0, a_1, \dots]$, $a_i \in \mathbb{Z}$ is periodic from a certain a_k if and only if ξ is a quadratic irrational number.*

Proof. The proof of this theorem has been demonstrated in various ways. One of the most common proofs can be found in Hardy and Wright's book, "An Introduction to the Theory of Numbers" [4], in section 10.12, which starts on page 143. □

In addition, Galois proved a related theorem about purely periodic continued fractions in [5]:

Theorem 1.7 (Galois)**.** *A quadratic irrational $\alpha = \frac{A+\sqrt{D}}{B}$ has a purely periodic continued fraction expansion $\alpha = [\overline{b_1, \dots, b_l}]$, $b_i \in \mathbb{N}$ if and only if its conjugate $\alpha' = \frac{A-\sqrt{D}}{B}$ satisfies the inequality $-1 < \alpha' < 0$. Moreover, in that case $\alpha' = -[0, \overline{b_l, \dots, b_1}]$.*

Convention. We use $r_n = \frac{p_n}{q_n} = [c_0, \dots, c_n]$, where $p_n, q_n \in \mathbb{N},\ r_n \in \mathbb{R}$ for writing finite continued fraction, $\xi = [c_0, c_1, \dots]$ to denote the infinite continued fraction of a positive real number, $\eta = -[c_0, c_1, \dots]$ the continued fraction of a negative real number and we will describe the elements of a continued fraction as follows:

$$\alpha = [\underbrace{a_0, a_1, \dots, a_k}_{\text{pre-period}}, \underbrace{\overline{b_1, \dots, b_l}}_{\text{period}}].$$

We will also mention this definition of continued fractions used, for example, in [6] and will be used in the proof of Lemma 1.11:

Definition 1.8. *We define a sequence $\{c_i\}$, where c_i is a non-negative integer, for a real number $\xi \in \mathbb{R}$ as follows:*

$$\begin{aligned} \xi_0 &:= \xi, \\ c_i &:= \lfloor \xi_i \rfloor, \\ \xi_{i+1} &:= \frac{1}{\xi_i - c_i}, \quad \text{if } \xi_i \neq c_i. \end{aligned}$$

We call the sequence $\{c_0, c_1, \ldots, c_l\}$ a finite continued fraction *of ξ if it has a finite length l and denote it as $\xi = [c_0, c_1, \ldots, c_l]$. We call the sequence $\{c_0, c_1, \ldots\}$ an* infinite continued fraction *of ξ if it has an infinite length and denote it as $\xi = [c_0, c_1, \ldots]$.*

Lemma 1.9. *Definitions 1.8 and 1.1 are equivalent.*

Proof. The proof can be found in [6] on page 17. □

Lemma 1.10. *Let $c_0 \in \mathbb{Z},\ c_i \in \mathbb{N}$, then $[c_0, c_1, \ldots]$ is defined.*

Proof. It can be found in [6], Theorem 2.9. a), for $\xi = [c_0, c_1, \ldots] \in \mathbb{R}$. □

Lemma 1.11. *For every $\xi \in \mathbb{R} \setminus \mathbb{Q}$ there exist $c_i \in \mathbb{N},\ c_0 \in \mathbb{N} \cup \{0\}$ such that $\xi = [c_0, c_1, \ldots]$ or $\xi = -[c_0, c_1, \ldots]$.*

Proof. We can use an alternative definition of the continued fractions from Definition 1.8, which, as we know from Lemma 1.9, is equivalent to Definition 1.1.

Let us suppose $\xi > 0$, then from the aforementioned definition we know that there exist $c_0 \geq 0, c_i > 0$ for $i > 0$ such that $\xi = [c_0, c_1, \ldots]$. If $\xi < 0$ then we can apply Lemma 1.2 and we can find $-\xi = [c_0, c_1, \ldots]$ and from the Lemma we have $\xi = -[c_0, c_1, \ldots]$. □

1.2 Necessary lemmata

We want to examine the behaviour of quadratic irrationals that do not have a completely periodic continued fraction. However, we must first establish some significant observations that will aid us in demonstrating Proposition 1.16.

Lemma 1.12. *Let $c_i \in \mathbb{Z}$, then if at least one of the sides is defined, we can write:*

1. $[c_0, \ldots, c_{i-1}, c_i, 0, c_{i+1}, c_{i+2}, c_{i+3}, \ldots] = [c_0, \ldots, c_{i-1}, c_i + c_{i+1}, c_{i+2}, c_{i+3}, \ldots]$,
2. $[c_0, \ldots, c_{i-1}, c_i, 0, 0, c_{i+1}, c_{i+2}, c_{i+3}, \ldots] = [c_0, \ldots, c_{i-1}, c_i, c_{i+1}, c_{i+2}, c_{i+3}, \ldots]$.

Proof. Our approach to proving this lemma will involve conducting simple microscopical work.

1. Let us set $r_k = [c_0, \ldots, c_i, 0, c_{i+1}, c_{i+2} \ldots, c_k]$ and $s_k = [c_0, \ldots, c_i + c_{i+1}, c_{i+2} \ldots, c_k]$. Then we want to prove that for every $t \geq i + 1:\ r_{t+1} = s_{t-1}$.

By utilizing the substitution $\beta_t = [c_{i+2}, c_{i+3}, \dots, c_t]$ we effectively rewrite $[c_0, \dots, c_i, 0, c_{i+1}, c_{i+2} \dots, c_t]$ as $[c_0, \dots, c_{i-1}, c_i, 0, c_{i+1}, \beta_t]$. We can conclude that the following is true:

$$\begin{aligned}
[c_i, 0, c_{i+1}, \beta_t] &= c_i + \cfrac{1}{0 + \frac{1}{c_{i+1} + \frac{1}{\beta_t}}} \\
&= c_i + \cfrac{1}{\frac{1}{c_{i+1} + \frac{1}{\beta_t}}} = c_i + c_{i+1} + \frac{1}{\beta_t} = \\
&= [c_i + c_{i+1}, \beta_t],
\end{aligned}$$

therefore $r_{t+1} = [c_0, \dots, c_i, 0, c_{i+1}, \beta_t] = [c_0, \dots, c_i + c_{i+1}, \beta_t] = s_{t-1}$. Moreover, since this holds for every $t \geq i + 1$, the limits of r_k and s_k are equal.

2. Using the same argument, if we rewrite $[c_0, \dots, c_i, 0, 0, c_{i+1}, c_{i+2} \dots]$ as $[c_0, \dots, c_{i-1}, 0, 0, c_{i+1}, \beta_t]$ using the substitution $\beta_t = [c_{i+2}, c_{i+3}, \dots, c_t]$, then we can see that

$$\begin{aligned}
[c_i, 0, 0, c_{i+1}, \beta_t] &= c_i + \cfrac{1}{0 + \frac{1}{0 + \frac{1}{c_{i+1} + \frac{1}{\beta_t}}}} = \\
&= c_i + \cfrac{1}{\frac{1}{\frac{1}{c_{i+1} + \frac{1}{\beta_t}}}} = c_i + \cfrac{1}{c_{i+1} + \frac{1}{\beta_t}} = [c_i, c_{i+1}, \beta_t].
\end{aligned}$$

□

Remark. In the following text, we will be less formal, and we understand that two continued fractions, as in Lemma 1.12, are equal if at least one of them is defined.

It is easy to predict the behaviour of the conjugate of a quadratic irrational with only two elements in the pre-period.

Lemma 1.13. *Let $\alpha = [a_0, a_1, \beta]$, where β is a quadratic irrational, then α is also a quadratic irrational and $\alpha' = [a_0, a_1, \beta']$.*

Proof. We know that α is a quadratic irrational from Theorem 1.6 because if β is a quadratic irrational, then it means it has a periodic continued fraction $[b_0, b_1, \dots, \overline{b_l, \dots, b_k}]$, therefore, the continued fraction of α is also periodic, since we can write it as $[a_0, a_1, b_0, b_1, \dots, \overline{b_l, \dots, b_k}]$ and from Theorem 1.6 it is also a quadratic irrational. So, we know that α, β have well-defined conjugates.

From the definition, we can write $\alpha = a_0 + \frac{1}{a_1 + \frac{1}{\beta}}$. Then, since conjugation is a $\mathbb{Q}$-homomorphism, we have the property that $(a + b)' = a' + b'$ and $\left(\frac{1}{a}\right)' = \frac{1}{a'}$ which translates to

$$\begin{aligned}
\alpha' &= \left(a_0 + \frac{1}{a_1 + \frac{1}{\beta}} \right)' = \\
&= a_0' + \left(\frac{1}{a_1 + \frac{1}{\beta}} \right)' = a_0' + \frac{1}{\left(a_1 + \frac{1}{\beta}\right)'} = \\
&= a_0' + \frac{1}{a_1' + \left(\frac{1}{\beta}\right)'} = a_0' + \frac{1}{a_1' + \frac{1}{\beta'}},
\end{aligned}$$

but since $a_0, a_1 \in \mathbb{Q}$ then $a_0' = a_0$, $a_1' = a_1$ so

$$\alpha' = a_0 + \frac{1}{a_1 + \frac{1}{\beta'}} = [a_0, a_1, \beta'],$$

as we wanted. $\square$

Remark. When writing proofs with continued fractions, we understand they can be hard to follow. Therefore, we will use the following notation to help with orientation in these proofs. We write

- $[c_0, c_1, c_2] \overset{*}{=} \left[c_0, c_1 + \frac{1}{c_2}\right]$,
- $\left[c_0, c_1 + \frac{1}{c_2}\right] \doteq [c_0, c_1, c_2]$,

when we use these computation during our proofs.

The lemma below helps us understand negative continued fractions and how we can modify them to have only the first element negative. As discussed in the section after Definition 1.1, when referring to the continued fraction of a negative value ξ, we will always mean the notation $-[c_0, c_1, \dots]$; however, we will also show that the following formulation is equally valid.

Lemma 1.14. *Let $c_i \in \mathbb{N}$, then $-[c_0, c_1, \dots] = [-c_0 - 1, 1, c_1 - 1, c_2, c_3, \dots]$.*

Proof. Analogously to Lemma 1.12 we want to prove that for $r_k = [c_0, c_1, \dots, c_k]$, $s_k = [-c_0 - 1, 1, c_1 - 1, c_2, c_3, \dots, c_k]$ for every $t \geq 2$: $r_t = s_{t+1}$.

To proceed, we will substitute $\gamma_t = [c_2, c_3, \dots, c_t]$ from which we get

$$\begin{aligned}
-[c_0, c_1, \gamma_t] &= [-c_0 - 1 + 1, -c_1, -\gamma_t] \overset{*}{=} \left[-c_0 - 1 + 1, -c_1 - \frac{1}{\gamma_t}\right] = \\
&= \left[-c_0 - 1 + 1, -\frac{c_1\gamma_t + 1}{\gamma_t}\right] \overset{*}{=} \left[-c_0 - 1 + 1 - \frac{\gamma_t}{c_1\gamma_t + 1}\right] = \\
&= \left[-c_0 - 1 + \frac{c_1\gamma_t + 1 - \gamma_t}{c_1\gamma_t + 1}\right] \doteq \left[-c_0 - 1, \frac{c_1\gamma_t + 1}{c_1\gamma_t + 1 - \gamma_t}\right] = \\
&= \left[-c_0 - 1, 1 + \frac{\gamma_t(c_1 - 1) + 1}{\gamma_t}\right] = \left[-c_0 - 1, 1, c_1 - 1 + \frac{1}{\gamma_t}\right] \doteq \\
&\doteq \left[-c_0 - 1, 1 + \frac{\gamma_t}{c_1\gamma_t + 1 - \gamma_t}\right] \doteq \left[-c_0 - 1, 1, \frac{c_1\gamma_t + 1 - \gamma_t}{\gamma_t}\right] = \\
&= [-c_0 - 1, 1, c_1 - 1, \gamma_t]\,,
\end{aligned}$$

and since this is true for every $t \geq 2$ then the limits are equal. $\square$

The following observation simplifies the proof of Proposition 1.16. We recommend reading it after its reference in the proposition to comprehend the need for this particular formulation thoroughly.

Lemma 1.15. *For $\gamma = [c_0, c_1, \dots]$, $c_i, a_i, b_i \in \mathbb{Z}$ and $a_1 < b_l$ we can write the continued fractions*

(i) $[a_0, a_1, \gamma] = [a_0 - 1, 1, -a_1 - 1, -\gamma]$,

(ii) $[a_1, -1, 1, b_l - 1, \gamma] = [-b_l + a_1, -\gamma]$,

if at least one of the continued fractions in each point is defined.

Proof. In this proof, for simplicity, in both points, we consider the proof similar to previous proofs - we understand that we are proving that the equations are equal for every $\gamma_t = [c_0, c_1, \ldots, c_t]$ for $t \geq 0$. Then, the limits are equal.

(i) To prove this part, we add and subtract 1 from a_0 and adjust the continued fraction accordingly:

$$[a_0, a_1, \gamma_t] = [a_0 - 1 + 1, a_1, \gamma_t] \overset{*}{=} \left[a_0 - 1 + 1, a_1 + \frac{1}{\gamma_t}\right] =$$
$$= \left[a_0 - 1, \frac{a_1 + \frac{1}{\gamma_t}}{a_1 + 1 + \frac{1}{\gamma_t}}\right] = \left[a_0 - 1, \frac{a_1 + \frac{1}{\gamma_t} + 1 - 1}{a_1 + 1 + \frac{1}{\gamma_t}}\right] \doteq$$
$$\doteq \left[a_0 - 1, 1, -a_1 - 1 - \frac{1}{\gamma_t}\right] \doteq [a_0 - 1, 1, -a_1 - 1, -\gamma_t].$$

(ii) Our goal is to simplify the continued fraction to a single fraction, which we can then split into a continued fraction with only two elements:

$$[a_1, -1, 1, b_l - 1, \gamma_t] \overset{*}{=} \left[a_1, -1, 1, b_l - 1 + \frac{1}{\gamma_t}\right] = \left[a_1, -1, 1, \frac{\gamma_t b_l - \gamma_t + 1}{\gamma_t}\right] \overset{*}{=}$$
$$\overset{*}{=} \left[a_1, -1, \frac{\gamma_t b_l + 1}{\gamma_t b_l - \gamma_t + 1}\right] \overset{*}{=} \left[a_1, -1 + \frac{\gamma_t b_l - \gamma_t + 1}{\gamma_t b_l + 1}\right] =$$
$$= \left[a_1, \frac{\gamma_t}{-\gamma_t b_l - 1}\right] \overset{*}{=} \left[a_1 + \frac{-\gamma_t b_l - 1}{\gamma_t}\right]$$
$$= \left[\frac{\gamma_t(a_1 - b_l) - 1}{\gamma_t}\right] = \left[a_1 - b_l - \frac{1}{\gamma_t}\right] \doteq$$
$$\doteq [-b_l + a_1, -\gamma_t].$$

□

1.3 Continued fractions of conjugated quadratic irrationals

Now, we can finally prove the central proposition of this chapter. We want to show how to obtain the continued fraction of a quadratic irrational from its conjugate when the conjugate is not purely periodic without using negative numbers in the continued fraction. Firstly, in Proposition 1.16, we will prove it for a continued fraction with at least two elements in the pre-period, and then, in Proposition 1.17, we will demonstrate what happens when the continued fraction has a pre-period of length one.

Proposition 1.16. *Let* $\alpha = [a_0, a_1, \ldots, a_k, \overline{b_1, \ldots, b_l}]$, $\forall i, j \in \mathbb{N}: \ a_i, b_j > 0$, α *be the continued fraction expansion of a quadratic irrational with* $a_k < b_l$, $k \geq 1$. *Then the continued fraction expansion of its conjugate is*

$$\alpha' = [a_0, a_1, \ldots, a_{k-1} - 1, 1, b_l - a_k - 1, \overline{b_{l-1}, b_{l-2}, \ldots, b_1, b_l}].$$

If $a_k > b_l$, $k \geq 1$, *then*

$$\alpha' = [a_0, a_1, \ldots, a_{k-1}, a_k - b_l - 1, 1, b_{l-1} - 1, \overline{b_{l-2}, b_{l-3}, \ldots, b_1, b_l, b_{l-1}}].$$

Proof. Firstly, let us deal with the case $a_k < b_l$ for $k = 1$.

From Lemma 1.13 we know that $\alpha' = [a_0, a_1, \beta']$ for $\beta = [\overline{b_1, \ldots, b_l}]$ and from Theorem 1.7 we have $\beta' = -[0, \overline{b_l, \ldots, b_1}]$. That is, from Lemma 1.14, equal to $[-0 - 1, 1, b_l - 1, \overline{b_{l-1}, \ldots, b_1, b_l}]$. We can manipulate the period by writing $[\overline{b_l, \ldots, b_1}] = [b_l, \overline{b_{l-1}, \ldots, b_l}]$ and we will denote it as $\hat{\beta} = [\overline{b_{l-1}, \ldots, b_l}]$.

By putting all these steps together we get

$$\alpha' = [a_0, a_1, -1, 1, b_l - 1, \overline{b_{l-1}, \ldots, b_1, b_l}] = [a_0, a_1, -1, 1, b_l - 1, \hat{\beta}].$$

Now we want to prove that $[a_0, a_1, -1, 1, b_l - 1, \hat{\beta}] = [a_0 - 1, 1, b_l - a_1 - 1, \hat{\beta}]$. Using Lemma 1.15, we firstly apply point (ii) and then (i), and we get:

$$[a_0, a_1, -1, 1, b_l - 1, \hat{\beta}] \stackrel{\text{ii}}{=} [a_0, -b_l + a_1, -\hat{\beta}] \stackrel{\text{i}}{=} [a_0 - 1, 1, b_l - a_1 - 1, \hat{\beta}].$$

For $k > 1$ we have $\alpha = [a_0, a_1, \ldots, a_{k-1}, a_k, \beta]$ which we can express as $[a_0, a_1, \ldots, a_{k-2}, \gamma]$, where $\gamma = [a_{k-1}, a_k, \beta]$. From the previous case then γ' is equal to $[a_{k-1} - 1, 1, b_l - a_k - 1, \beta]$ and adhering to Lemma 1.13

$$\begin{aligned}\alpha' &= [a_0, a_1, \ldots, a_{k-1}, a_k, \beta]' = [a_0, a_1, \ldots, a_{k-2}, \gamma'] = \\ &= [a_0, a_1, \ldots, a_{k-1} - 1, 1, b_l - a_k - 1, \beta'].\end{aligned}$$

By applying Galois' Theorem 1.7 and adjusting the period in the same manner as above, we get $\alpha' = [a_0, a_1, \ldots, a_{k-1} - 1, 1, b_l - a_k - 1, \overline{b_{l-1}, \ldots, b_1, b_l}]$, which is what we wanted.

For $a_k > b_l$, we consider $\alpha \geq 0$ since if it were not, we could write it as $-[a_0, a_1, \ldots]$ and proceed to work with positive values in the continued fraction. We can use Lemma 1.12 and write

$$\begin{aligned}\alpha &= [a_0, a_1, \ldots, a_k, \overline{b_1, \ldots, b_l}] = [a_0, a_1, \ldots, a_{k-1}, a_k - b_l, 0, b_l, \overline{b_1 \ldots, b_l}] = \\ &= [a_0, a_1, \ldots, a_k - b_l, 0, \overline{b_l \ldots, b_1, b_{l-1}}].\end{aligned}$$

We now have a new continued fraction $[a_0, a_1, \ldots, a_k - b_l, 0, \overline{b_l \ldots, b_1, b_{l-1}}] = [a'_0, a'_1, , \ldots, a'_{k-1}, a'_k, \overline{b'_1, \ldots, b'_l}]$ and we get $a'_k = 0 < b'_l$, and $a'_{k-1} = a_k - b_l$ so by applying the result from the case for $a_k < b_l$ we get

$$\begin{aligned}\alpha &= [a_0, a_1, \ldots, a_k - b_l, 0, \overline{b_l \ldots, b_1, b_{l-1}}] = \\ &= [a_0, a_1, \ldots, a_k - b_l - 1, 1, b_l - 1, \overline{b_{l-2}, b_{l-3}, \ldots, b_1, b_l, b_{l-1}}],\end{aligned}$$

which concludes our proof. □

Remark. We have not covered the case $a_k = b_l$ because for this case we can rewrite it as $\alpha = [a_0, a_1, \ldots, a_k, \overline{b_1, \ldots, b_l}] = [a_0, a_1, \ldots, a_{k-1}, \overline{b_l, b_1, \ldots, b_{l-1}}]$.

All that is left to do in this chapter is to prove Proposition 1.16 for $k = 0$.

Proposition 1.17. *Let* $\alpha = [a_0, \overline{b_1, \dots, b_l}]$. *Then for* $a_0 < b_l$ *the conjugate* α' *can be given as the negative continued fraction expansion*

$$\alpha' = -[b_l - a_0, \overline{b_{l-1}, \dots, b_1, b_l}].$$

For $a_0 > b_l$ *we have*

$$\alpha' = [a_0 - b_l - 1, 1, b_l - 1, \overline{b_{l-1}, b_{l-2}, \dots, b_1, b_l}].$$

Proof. For $a_0 < b_l$ we can write $\alpha = [a_0, \beta]$ for $\beta = [\overline{b_1, \dots, b_l}]$ and $\overline{\beta} = [\overline{b_l, \dots, b_1}]$. Then, with the aid from Theorem 1.7 and Lemma 1.12

$$\begin{aligned} \alpha' = [a_0, \beta'] &\overset{1.7}{=} [a_0, 0, -\overline{\beta}] \overset{1.12}{=} [a_0 - b_l, \overline{-b_{l-1}, \dots, -b_1, -b_l}] = \\ &= -[b_l - a_0, \overline{b_{l-1}, \dots, b_1, b_l}]. \end{aligned}$$

For $a_0 > b_l$ we can again use the help of Lemma 1.12 and rewrite $\alpha = [a_0 - b_l, 0, \overline{b_1, \dots, b_l}]$ but here we have two elements in the pre-period of the continued fraction which we have already covered in Proposition 1.16. □

Remark. For the case $a_0 = b_l$, we would get the case of purely periodic continued fraction, which is already covered in Theorem 1.7.

2. Topograph

In his book "The Sensual (Quadratic) Form", John H. Conway discusses a unique graph called the "topograph," which can be used to visualise binary integral quadratic forms.

This chapter will in Section 2.3 explain how Conway created the topograph, in Section 2.4 provide an abstract definition of the graph and its properties, and then in Section 2.5 define the algorithm for constructing the topograph. Finally, we will prove that the output generated by the algorithm is equivalent to the abstract definition.

Before we begin, we will mention some basic properties of graphs, as we will define the topograph as a graph.

2.1 Notions on graphs

We will start by setting the definition of a graph and its sets of edges and vertices:

Definition 2.1. *A* graph *is a pair* (V, E) *where* V *is a set and* $E \subseteq \{\{u, v\} \mid u, v \in V,\ u \neq v\}$.

Definition 2.2. *Let* $G = (V, E)$ *be a graph. Then, the set* V *is called* the vertex set *of* G *and is denoted by* $V(G)$. *The set* E *is called* the set of edges *of* G *and is denoted by* $E(G)$.

The following notation is used heavily in the following chapters, so let us remember what being adjacent and neighbours mean in the language of graphs. We will provide a similar definition for the language of lax bases and superbases in Definition 2.23.

Definition 2.3. *Let* $G = (V, E)$ *be a graph. Vertex* $u \in V(G)$ *and an edge* $e \in E(G)$ *are said to be* adjacent *(to each other) if* $v \in e$.

Definition 2.4. *Let* $G = (V, E)$ *be a graph. Let* $u, v \in V(G)$ *and* $e, f \in E(G)$.
The neighbors *of* v *(in* G*) are the vertices* u *of* G *that satisfy* $\{u, v\} \in E(G)$.
The neighbors *of* e *(in* G*) are the edges* f *of* G *that satisfy* $|e \cap f| = 1$.

In Lemma 2.26, we will prove that all the vertices of a topograph are of degree 3; for that purpose, let us define the degree of a vertex.

Definition 2.5. *Let* $G = (V, E)$ *be a graph. Let* $v \in V$ *be a vertex. Then, the* degree *of* v *is defined to be*

$$\begin{aligned}
\deg v :&= (\textit{the number of edges } e \in E \textit{ that contain } v) \\
&= (\textit{the number of neighbors of } v) \\
&= |u \in V \mid \{u, v\} \in E| \\
&= |e \in E \mid v \in e|.
\end{aligned}$$

We will, slightly unorthodoxly, in this thesis, work more with paths on edges rather than vertices and distance between edges.

Definition 2.6. *A* walk *in a graph $G = (V, E)$ is a finite sequence $(g_1, g_2, \ldots, g_n)$ of edges in G such that $\forall\ 1 \leq i, j \leq n$:*

- *$g_i \in E$,*
- *g_i, g_{i+1} are neighbours.*

Accordingly we would define an infinite walk $\gamma = (g_i)_{i\in\mathbb{Z}}$ as well as a semi-infinite walk $\gamma = (g_i)_{i\in\mathbb{N}}$.

Definition 2.7. *A* path *in graph G is a walk $\gamma = (g_1, g_2, \ldots, g_n)$ such that for all $i, j \leq n,\ i \neq j :\ g_i \neq g_j$.*

Definition 2.8. *Let $\gamma = (g_1, g_2, \ldots, g_n)$ be a walk in the graph G. Then we say that the edge g_1 is* the root *of this walk.*

Lastly, we will provide definitions of a connected graph and a tree:

Definition 2.9. *A graph $G = (V, E)$ is* connected *if for every $u, v \in V$ there exists a path $(g_1, g_2, \ldots, g_n)$ such that $u \in g_1,\ v \in g_n$.*

Definition 2.10. *A graph $G = (V, E)$ is a* tree *if any two vertices in G can be connected by exactly one path.*

2.2 The lax bases and superbases

Before commencing the construction of the topograph, we must first introduce several definitions. Our objective is to establish some bases that are flexible in terms of their sign, which Conway refers to as lax bases. To begin with, we will define primitive and lax vectors. Note that we work with vectors from $\mathbb{Z}^2$ in the whole thesis.

Definition 2.11. *A* primitive vector *is a vector $v \in \mathbb{Z}^2$ for which there does not exist $k \in \mathbb{Z} \setminus \{\pm 1\}$ such that $ku = v$ for any vector $u \in \mathbb{Z}^2$.*

Definition 2.12. *A* lax vector *is defined as a set $\pm v = \{+v, -v\}$ for a primitive vector v.*

Having established what a lax vector is, we can now proceed to define a lax base. The definition of a strict base is the same as that of a regular base, which the reader might already be familiar with. However, for clarity in our discussion, we will refer to it as a strict base.

Definition 2.13. *A* strict base *is defined as an ordered pair (e_1, e_2) of vectors whose integral linear combinations are all the lattice vectors.*

Definition 2.14. *A* lax base *is a set $\{\pm e_1, \pm e_2\}$ for a strict base (e_1, e_2).*

Finally, we need to introduce superbases. We can imagine them as friends to our defined lax bases. The superbase is a triple where two-thirds form a strict base, and the last third is derived from the first two.

Definition 2.15. *A* strict superbase *is an ordered triple (e_1, e_2, e_3) for which $e_1 + e_2 + e_3 = 0$ and (e_1, e_2) is a strict base.*

Definition 2.16. *A* lax superbase *is a set* $\{\pm e_1, \pm e_2, \pm e_3\}$ *where* (e_1, e_2, e_3) *is a strict superbase.*

Working with bases and superbases can be quite challenging. To simplify the process, we have developed a concept called the "set of navigators". Although the term "set of navigators" is not used in Conway's text, we have adopted his naming conventions that remind the reader of a pleasant afternoon hike.

Definition 2.17. *A* navigator of a lax base $\{\pm e_1, \pm e_2\}$ *is a strict base* (e_1, e_2) *such that* $\{\pm e_1, \pm e_2\}$ *is obtained form* (e_1, e_2)*.*

Definition 2.18. *A* navigator of a lax superbase $\{\pm e_1, \pm e_2, \pm e_3\}$ *is a strict base* (e_1, e_2, e_3) *such that* $\{\pm e_1, \pm e_2, \pm e_3\}$ *is obtained form* (e_1, e_2, e_3)*.*

Definition 2.19. *Let us define the* set of navigators *for a lax base e (resp. lax superbase v) as the set of all its navigators, which we denote* $\mathcal{N}_e$ *(resp.* $\mathcal{N}_v$*).*

Lastly, we work extensively with sets of lax bases and lax superbases, so we have developed a notation for these two sets.

Definition 2.20. *We will define* $\mathcal{L}$ *as the set of all the lax bases in* $\mathbb{Z}^2$*,* $\mathcal{S}$ *as the set of all the lax superbases in* $\mathbb{Z}^2$ *and* $\mathcal{V}$ *as the set of all the lax vectors in* $\mathbb{Z}^2$*.*

To determine the navigators of a lax superbase, we will prove the following lemma.

Lemma 2.21. *If* (e_1, e_2, e_3) *is a strict superbase then also* (e_2, e_1, e_3)*,* (e_1, e_3, e_2)*,* (e_2, e_3, e_1)*,* (e_3, e_2, e_1) *and* (e_3, e_1, e_2) *are strict superbases.*

Proof. For an ordered triple (a, b, c) to be the superbase we need $c = -a - b$ and (a, b) to be a strict base.

We will only provide proof for (e_3, e_1, e_2), but it holds for all variations.

If (e_1, e_2, e_3) is a strict superbase, then we have $e_3 = -e_1 - e_2$. For (e_3, e_1, e_2), we need to verify that $e_3 + e_1 + e_2 = -e_1 - e_2 + e_1 + e_2 = 0$ and that (e_3, e_1) is a strict base. We can see that (e_3, e_1) generates $\mathbb{Z}$ since we can express $e_2 = -e_1 - e_3$, and we already know that (e_1, e_2) is a strict base.

All that is left to prove is that e_3 and e_1 are primitive vectors. e_1 is already primitive from (e_1, e_2) being a strict base. We see that e_3 is primitive, we need $-e_1 - e_2$ to be primitive. Since in the definition of a primitive vector, we allow it to be divisible by -1, we can check that $e_1 + e_2$ is primitive. We know that e_1 and e_2 form a strict base, so we can write $e_1 = (a, b)$, $e_2 = (c, d)$ for some $a, b, c, d \in \mathbb{Z}$ and since we know that e_1 and e_3 form a base of $\mathbb{Z}^2$ we get that

$$\det \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \pm 1.$$

From this, we have $ad - cb = \pm 1$, and we can write

$$ad - cb = ad - ab + ab - cb = a(d + b) - b(a + c) = \pm 1.$$

This implies that $a + c$ and $b + d$ are coprime. If they were not, there would exist $k \neq \pm 1$ such that $k|(a+c)$ and $k|(b+d)$. Then we would get $a+c = kx,\ b+d = ky$ for some $x, y \in \mathbb{Z}$ and the equation would be equal to

$$akx - bky = k(ax - by) = \pm 1.$$

But then $k | \pm 1$ but for that $k = \pm 1$ which is in contradiction with our choice of k. Therefore, the vector $(a + c, b + d)$ is primitive, and this vector is equal to $e_1 + e_2$, so e_3 is indeed a primitive vector. $\square$

The definitions are new and might be somewhat difficult to comprehend. To assist with understanding, the reader can take a look at the following examples:

Example. Let us take two vectors $(1, 0)$ and $(0, 1)$. Then the corresponding *lax vectors* would be $\pm(1, 0) = \{(1, 0), (-1, 0)\}$ and $\pm(0, 1) = \{(0, 1), (0, -1)\}$.

A *strict base* formed from $(1, 0)$ and $(0, 1)$ is $((1, 0), (0, 1))$ and a corresponding *lax base* would be $\{\pm(1, 0), \pm(0, 1)\} = \{\{(1, 0), (-1, 0)\}, \{(0, 1), (0, -1)\}\}$.

A *strict superbase* for $((1, 0), (0, 1))$ would be $((1, 0), (0, 1), (-1, -1))$ for which the lax superbase would be

$$\begin{aligned}\{\pm(1,0), \pm(0,1), \pm(-1,-1)\} &= \{\pm(1,0), \pm(0,1), \pm(1,1)\} = \\ &= \{\{(1,0), (-1,0)\}, \{(0,1), (0,-1)\}, \{(1,1), (-1,-1)\}\}.\end{aligned}$$

If we get the *set of navigators* for $e = \{\pm(1, 0), \pm(0, 1)\}$ we get

$$\begin{aligned}\mathcal{N}_e = \{&((1,0),(0,1)),\ ((-1,0),(0,-1)),\ ((-1,0),(0,1)),\ ((1,0),(0,-1)),\\ &((0,1),(1,0)),\ ((0,-1),(-1,0)),\ ((0,-1),(1,0)),\ ((0,1),(-1,0))\}.\end{aligned}$$

For the lax superbase $v = \{\pm(1, 0), \pm(0, 1), \pm(1, 1)\}$ we have

$$\begin{aligned}\mathcal{N}_v = \{&((1,0),(0,1),(-1,-1)),\ ((-1,0),(0,-1),(1,1))\\ &((0,1),(1,0),(-1,-1)),\ ((0,-1),(-1,0),(1,1))\\ &((1,0),(-1,-1),(0,1)),\ ((-1,0),(1,1),(0,-1))\\ &((-1,-1),(1,0),(0,1)),\ ((1,1),(-1,0),(0,-1))\\ &((0,1),(-1,-1),(1,0)),\ ((0,-1),(1,1),(-1,0))\\ &((-1,-1),(0,1),(1,0)),\ ((1,1),(0,-1),(-1,0))\}.\end{aligned}$$

Convention. In the following chapter, we will adhere to the following notation:

- e, f, g, h, i will denote lax bases,
- u, v, w will denote lax superbases,
- $e_1, e_2, \ldots, f_1, f_2, \ldots$ will denote primitive vectors and $\pm e_1, \pm e_2, \ldots$ will denote lax vectors,
- n, m, l will denote navigators.

Before we construct the topograph, we must first prove the following lemma formulated by Conway in [2]. We will use it extensively in the upcoming sections.

Lemma 2.22 (On bases)**.** *Each lax superbase* $\{\pm e_1, \pm e_2, \pm e_3\}$ *contains exactly three lax bases*

$$\{\pm e_1, \pm e_2\}, \{\pm e_1, \pm e_3\}, \{\pm e_2, \pm e_3\}$$

and each lax base $\{\pm e_1, \pm e_2\}$ *is in exactly two lax superbases*

$$\{\pm e_1, \pm e_2, \pm(e_1 + e_2)\}, \{\pm e_1, \pm e_2, \pm(e_1 - e_2)\}$$

Proof. It is evident that

$$\{\pm e_1, \pm e_2\}, \{\pm e_1, \pm e_3\}, \{\pm e_2, \pm e_3\} \subset \{\pm e_1, \pm e_2, \pm e_3\},$$

we need to check that all $\{\pm e_1, \pm e_2\}, \{\pm e_1, \pm e_3\}, \{\pm e_2, \pm e_3\}$ are lax bases.

We need both vectors to generate all the lattice vectors from the definition. Since $\{\pm e_1, \pm e_2, \pm e_3\}$ is a lax superbase, then again from the definition (e_1, e_2, e_3) is a strict superbase, and (e_1, e_2) is a strict base, which induces a lax base $\{\pm e_1, \pm e_2\}$. From Lemma 2.21 we get strict superbases (e_1, e_3, e_2) and (e_3, e_2, e_1) that, following the same pattern, give us lax bases $\{\pm e_1, \pm e_3\}$ and $\{\pm e_2, \pm e_3\}$.

For any other subset of $\{\pm e_1, \pm e_2, \pm e_3\}$, we get only $\{\pm e_1\}, \{\pm e_2\}, \{\pm e_3\}$ and the empty set, which are obviously (from the definition) not lax bases.

For the second part, we can see that $\{\pm e_1, \pm e_2\} \subset \{\pm e_1, \pm e_2, \pm(e_1 + e_2)\}$ and $\{\pm e_1, \pm e_2\} \subset \{\pm e_1, \pm e_2, \pm(e_1 - e_2)\}$. We need to check that they are both superbases and that for any other superbase v, we get $\{\pm e_1, \pm e_2\} \not\subseteq v$.

We have a lax base $\{\pm e_1, \pm e_2\}$ with the set of navigators

$$\begin{aligned}\mathcal{N}_e = \{&(e_1, e_2), (-e_1, -e_2), (e_1, -e_2), (-e_1, e_2),\\ &(e_2, e_1), (-e_2, -e_1), (e_2, -e_1), (-e_2, e_1)\}.\end{aligned}$$

we can generate strict superbases for them. For (e_1, e_2), we can obtain a strict superbase as $(e_1, e_2, -(e_1 + e_2))$ to ensure that $e_1 + e_2 - (e_1 + e_2) = 0$ and we can do that for all the other strict bases as well, and we will get these following eight strict superbases:

$$\begin{gathered}(e_1, e_2, (-e_1 - e_2)), (-e_1, -e_2, (e_1 + e_2)), (e_2, e_1, (-e_2 - e_1)), (-e_2, -e_1, (e_2 + e_1)),\\ (e_1, -e_2, (-e_1 + e_2)), (-e_1, e_2, (e_1 - e_2)), (e_2, -e_1, (-e_2 + e_1)), (-e_2, e_1, (e_2 - e_1))\end{gathered}$$

The first four superbases form the set of navigators for $\{\pm e_1, \pm e_2 \pm (e_1 + e_2)\}$ and the last four strict superbases form the set of navigators for the lax superbase $\{\pm e_1, \pm e_2 \pm (e_1 - e_2)\}$.

Any other superbase v would have to take the shape of $v = \{\pm e_1, \pm e_2, \pm k\}$ for such vector k that $e_1 + e_2 + k = 0$ (and accordingly for other combinations from the set of navigators) but we have already shown that all the possibilities are taken into account in the part above. $\square$

The following definition sums up this lemma, whereby *containing* we mean the same relation as in the lemma above.

Definition 2.23. *Let e, f be lax bases and u, v lax superbases. Then if*

- *e is contained in u we say that e and u are* adjacent*.*
- *there exists a lax superbase w such that e and f are contained in w, then we say that e and f are* neighbours*.*
- *there exists a lax base e such that e is in u and e is in v, then we say that u, v are* neighbours*.*

We will also note the following corollary and remark, which we will use in the following text without further explanation. The corollary will be expanded in Proposition 2.27.

Corollary. Every subset of cardinality 2 of a lax superbase is a lax base.

Remark. Every lax (super)base is determined by its set of navigators.

It is evident, as each strict (super)base corresponds to a singular lax (super)base, all encapsulated within the navigators.

2.3 How we get the topograph

Now, we can finally start constructing the topograph. For this, we can choose any base of $\mathbb{Z}^2$; in this case, we will use (e_1, e_2). We find the corresponding lax base $\{\pm e_1, \pm e_2\}$ for this strict base.

The initial step is to visualise Lemma 2.22 in the following graph,

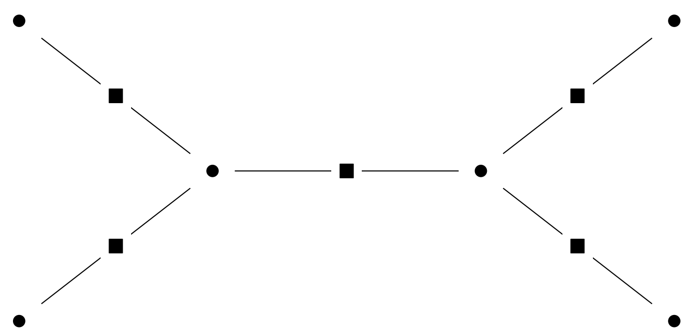

Figure 2.1: The visualisation of Lemma 2.22

where the dots • represent lax superbases, the squares ■ represent lax bases, and edges connect two lax superbases if they are neighbours.

If we have the vectors e_1, e_2 we assign their lax bases and lax superbases to the graph above. For better understanding, we will describe the whole process:

1. To begin with, we assign the lax base to the centre square:
2. Next, we know that the lax base $\{\pm e_1, \pm e_2\}$ is in two superbases $\{\pm e_1, \pm e_2, \pm(e_1 + e_2)\}, \{\pm e_1, \pm e_2, \pm(e_1 - e_2)\}$ so we assign them to the two adjacent vertices.

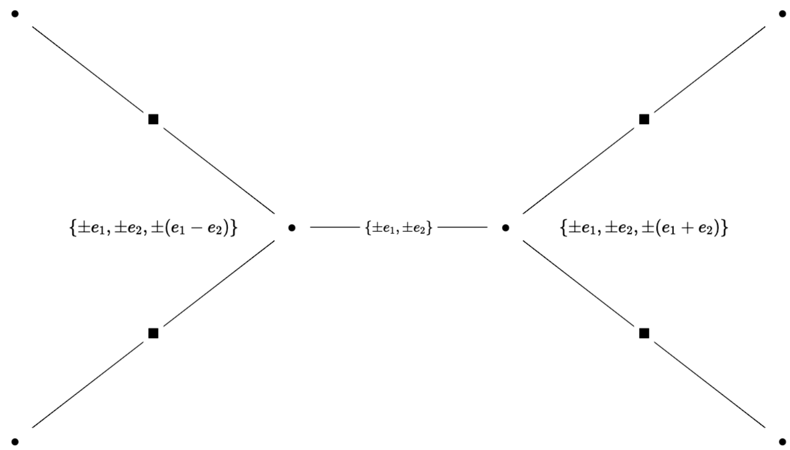

Figure 2.2: Topograph with one edge and two vertices denoted

3. To identify the remaining black squares that represent the edges, we need to find the lax bases that belong to the lax superbase assigned to the adjacent vertex. We then assign one of these bases to each edge. One of the bases will be $\{\pm e_1, \pm e_2\}$, already assigned to the first adjacent edge we marked.

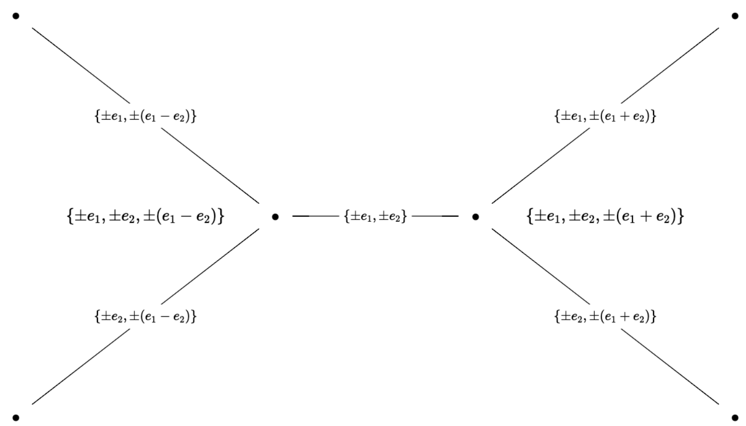

Figure 2.3: Topograph with assigned lax bases and superbases

We can proceed similarly and construct an infinite graph.

2.4 Abstract definition of the topograph

A precise formulation is necessary to establish the properties of our topograph. Therefore, we propose the following definition:

Definition 2.24. *The* topograph *is a pair $T = (V, E)$ where V is a set of all the lax superbases in $\mathbb{Z}^2$ and $E = \{\{v_1, v_2\} \mid v_1, v_2 \in V, |v_1 \cap v_2| = 2\}$.*

The edges are defined according to Lemma 2.22 and Conway's algorithm, where we have established that two vertices are connected by an edge e if and only if they are both adjacent to e. That from Lemma 2.22 means that e is contained in both vertices. Since edges are in Conway's algorithm denoted by lax bases (which are sets of cardinality 2) and vertices are denoted as lax superbases (sets of cardinality 3), then we can conclude that their intersection has to be 2; otherwise, the lax superbases are equal.

Theorem 2.25. *A topograph is a tree.*

Proof can be accessed and read in [2] on page 10, but we will sketch the proof in Section 3.4.

This Theorem provides some interesting observations. Mainly, we know that T is connected and that there is exactly one path between any two vertices. Another important property is given in the upcoming lemma:

Lemma 2.26. *Let $T = (V, E)$ be a topograph. Then $\forall v \in V : \deg(v) = 3$.*

Proof. From Definition 2.24, we know that every vertex v_1 is a set of three elements. We need to find all the other sets v_i in V such that $|v_1 \cap v_i| = 2$. We wish to find exactly three.

Since v_1 has three elements, it has three different subsets of cardinality 2. Since a two-element subset of a lax superbase will always be a lax base, for every one of those subsets from Lemma 2.22, we have precisely two lax superbases in which this subset lies. One of them has to be v_1, the other is neighbouring v_1. Since we have three different subsets, we get three different neighbouring vertices.

They are different because all the three lax vectors in the lax superbase have to be different - the first two e_1, e_2 have to be different as they form the lax base and the third one is equal to $e_1 + e_2$ which would be equal to e_1 or e_2 if and only if the other was a linear combination of the first which is in contradiction with them being basis vectors. ☐

We understand that the topograph contains all the lax superbases as its vertices, but we also want it to include all the lax bases. To achieve this, we define a function that maps the edges of the topograph to the lax bases. We shall recall that $\mathcal{L}$ denotes the set of all lax bases.

Proposition 2.27. *Let T be a topograph. Then the mapping*

$$\begin{aligned} \varphi : E &\to \mathcal{L} \\ \{v_1, v_2\} &\mapsto v_1 \cap v_2 \end{aligned}$$

is a bijection.

Proof. Let $e = \{v_1, v_2\}$ be an edge from E. Then, from the definition of the topograph, the intersection of v_1 and v_2 is a set of cardinality 2; let us denote it $\{\pm e_1, \pm e_2\}$ where (e_1, e_2) is a strict base. Then we can find vectors f, g distinct from each other and from e_1, e_2 such that WLOG $v_1 = \{\pm e_1, \pm e_2, \pm f\}$ and $v_2 = \{\pm e_1, \pm e_2, \pm g\}$. Then $\varphi(e)$ is a set of the cardinality two, which is a subset of a lax superbase, which from Lemma 2.22 means that $\varphi(e)$ is a lax base.

Furthermore, since $f \neq g$, each lax base has exactly one corresponding edge in the topograph under φ. ☐

Remark. From this Proposition 2.27 we can see that the topograph contains all the lax bases.

2.5 Algorithmic construction of a topograph

This section will define Conway's algorithm properly and demonstrate how it is equivalent to our abstract definition of a topograph from Definition 2.24.

A crucial aspect of creating the topograph involves demonstrating that its edges are produced utilising matrices from $\mathrm{GL}_2(\mathbb{Z})$. To accomplish this, we will rely on the following theorems. The proof for Theorem 2.28 can be found, with minor modifications, in [7], specifically on page 78 in Theorem 2. Theorem 2.29 follows naturally from Theorem 2.28.

Theorem 2.28. *$SL_2(\mathbb{Z})$ is generated by matrices*

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Theorem 2.29. *$GL_2(\mathbb{Z})$ is generated by matrices*

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Remark. In the following section, we will multiply navigators by matrices. We understand that for a matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and a navigator of a lax base $(e, f) = ((e_1, e_2), (f_1, f_2))$ we have

$$\begin{aligned} M(e,f) = \begin{pmatrix} a & b \\ c & d \\ h & i \end{pmatrix} \begin{pmatrix} e_1 & e_2 \\ f_1 & f_2 \end{pmatrix} &= \begin{pmatrix} ae_1 + bf_1 & ae_2 + bf_2 \\ ce_1 + df_1 & ce_2 + df_2 \\ he_1 + if_1 & he_2 + if_2 \end{pmatrix} = \\ &= ((ae_1 + bf_1, ae_2 + bf_2), (ce_1 + df_1, ce_2 + df_2), (he_1 + if_1, he_2 + if_2)) = \\ &= (ae + bf, ce + df, he + if). \end{aligned}$$

For a lax superbase, we get the matrix $M = \begin{pmatrix} a & c & e \\ b & d & f \end{pmatrix}$ and the navigator of a lax superbase $(u, v, w) = ((u_1, u_2), (v_1, v_2), (w_1, w_2))$. Then

$$\begin{aligned} M(e,f) = \begin{pmatrix} a & c & e \\ b & d & f \end{pmatrix} \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \\ w_1 & w_2 \end{pmatrix} &= \begin{pmatrix} au_1 + cv_1 + ew_1 & au_2 + cv_2 + ew_2 \\ bu_1 + dv_1 + fw_1 & bu_2 + dv_2 + fw_2 \end{pmatrix} = \\ &= ((au_1 + cv_1 + ew_1, au_2 + cv_2 + ew_2), \\ &\quad (bu_1 + dv_1 + fw_1, bu_2 + dv_2 + fw_2) = \\ &= (au + cv + ew, bu + dv + fw). \end{aligned}$$

To describe travelling in the topograph between lax bases and superbases, we can use matrices and sets of navigators. Let us consider a lax base $e = \{\pm e_1, \pm e_2\}$ and its neighboring lax superbases $u = \{\pm e_1, \pm e_2, \pm(e_1 + e_2)\}$ and $v = \{\pm e_1, \pm e_2, \pm(e_1 - e_2)\}$. Then, we define a set of matrices

$$G_v = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & -1 \\ 0 & 1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Moreover, we want to prove the following:

Proposition 2.30. *Let e be a lax base and u, v lax superbases. Then e is adjacent to both of them if and only if $\{X \cdot n \mid X \in G_v, n \in \mathcal{N}_e\} = \mathcal{N}_u \cup \mathcal{N}_v$*

Proof. For brevity, in this proof, we will show the full expansion only for the first two matrices; for the next four, we provide only the result of multiplying with the strict base (e_1, e_2), and we denote it using an overline. The result will be analogical to the expansion with the first two matrices.

Let us write $e = \{\pm e_1, \pm e_2\}$, then through expansion we get that the set $\{X \cdot n \mid X \in G_v, n \in \mathcal{N}_e\}$ is equal to

$$\begin{aligned}
&\{(e_1, e_2, -e_1 - e_2), (-e_1, -e_2, e_1 + e_2), (e_2, e_1, -e_2 - e_1), (-e_2, -e_1, e_2 + e_1), \\
&\quad (e_1, -e_2, -e_1 + e_2), (-e_1, e_2, e_1 - e_2), (e_2, -e_1, -e_2 + e_1), (-e_2, e_1, e_2 - e_1) \\
&\quad \overline{(e_1, -e_1 - e_2, e_2)}, \overline{(-e_1, e_1 - e_2, e_2)}\}, \overline{(-e_1 - e_2, e_1, e_2)}, \overline{(e_1 - e_2, -e_1, e_2)} = \\
&= \{\overline{(e_1, e_2, -e_1 - e_2)}, \overline{(e_1, -e_1 - e_2, e_2)}, \overline{(-e_1 - e_2, e_1, e_2)}\} \cup \\
&\cup \{\overline{(e_1, -e_2, -e_1 + e_2)}, \overline{(-e_1, e_1 - e_2, e_2)}\}, \overline{(e_1 - e_2, -e_1, e_2)}\} = \\
&= \mathcal{N}_u \ \cup \ \mathcal{N}_v.
\end{aligned}$$

We can observe that if e is adjacent to u, v then WLOG from Lemma 2.22 $u = \{\pm e_1, \pm e_2, \pm(e_1 + e_2)\}$ and $v = \{\pm e_1, \pm e_2, \pm(e_1 - e_2)\}$ and the set is indeed equal to their sets of navigators. The other implication shows that if the equality holds for e, then the sets of navigators have to belong to $u = \{\pm e_1, \pm e_2, \pm(e_1 + e_2)\}$ and $v = \{\pm e_1, \pm e_2, \pm(e_1 - e_2)\}$ and again from Lemma 2.22 that means that they are adjacent. $\square$

In the same way, we can define the adjacency relation between lax bases.

Proposition 2.31. *Let v be a lax superbase and e, f, g lax bases. Then v is adjacent to all of them if and only if $\{X \cdot n \mid n \in \mathcal{N}_v\} = \mathcal{N}_e \cup \mathcal{N}_f \cup \mathcal{N}_g$, where $X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.*

Proof. We write $v = \{\pm e_1, \pm e_2, \pm(e_1 + e_2)\}$ and through expansion, we get that the set is equal to

$$\begin{aligned}
&\{(e_1, e_2), (e_1, -e_1 - e_2), (e_2, -e_1 - e_2), (-e_1, -e_2), (-e_1, e_1 + e_2), (-e_2, e_1 + e_2), \\
&\ (e_2, e_1), (e_2, -e_2 - e_1), (e_1, -e_2 - e_1), (-e_2, -e_1), (-e_2, e_2 + e_1), (-e_1, e_2 + e_1)\} = \\
&= \{(e_1, e_2), (-e_1, -e_2), (e_2, e_1), (-e_2, -e_1)\} \cup \\
&\cup \{(e_1, -e_1 - e_2), (-e_1, e_1 + e_2), (-e_1, -e_1 - e_2), (-e_2, e_2 + e_1)\} \cup \\
&\cup \{(e_2, -e_1 - e_2), (-e_2, e_1 + e_2), (e_1, -e_2 - e_1), (-e_1, e_2 + e_1)\} \cup \\
&= \mathcal{N}_e \ \cup \ \mathcal{N}_f \ \cup \ \mathcal{N}_g.
\end{aligned}$$

We can observe that if v is adjacent to e, f, g then WLOG from Lemma 2.22 $e = \{\pm e_1, \pm e_2\}$, $f = \{\pm e_1, \pm(e_1 + e_2)\}$ and $g = \{\pm e_2, \pm(e_1 + e_2)\}$ and the set is indeed equal to their sets of navigators. The other implication shows that if the equality holds for v then the sets of navigators have to belong to $e = \{\pm e_1, \pm e_2\}$, $f = \{\pm e_1, \pm(e_1 + e_2)\}$ and $g = \{\pm e_2, \pm(e_1 + e_2)\}$ and again from Lemma 2.22 that means that they are adjacent. $\square$

We want to explain how to get between neighbouring lax bases. For that purpose, let us denote the set

$$G = \{XB \mid B \in G_v\} = \\ = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} \right\}$$

and prove the following lemma.

Lemma 2.32. *e, f, g, h are neighbouring lax bases to i, if and only if*

$$\{X \cdot n \mid X \in G, n \in \mathcal{N}_i\} = \mathcal{N}_e \cup \mathcal{N}_f \cup \mathcal{N}_g \cup \mathcal{N}_h \cup \mathcal{N}_i$$

Proof. Once again, let us write $i = \{\pm e_1, \pm e_2\}$ then if we apply the matrices from G to elements of $\mathcal{N}_i$ then we get that the set $\{X \cdot n \mid X \in G, n \in \mathcal{N}_i\}$ is equal to

$$\begin{aligned} &\left\{ \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot n \mid n \in \mathcal{N}_i \right\} \cup \left\{ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \cdot n \mid n \in \mathcal{N}_i \right\} \right\} \cup \\ \cup &\left\{ \left\{ \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} \cdot n \mid n \in \mathcal{N}_i \right\} \cup \left\{ \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \cdot n \mid n \in \mathcal{N}_i \right\} \right\} \cup \\ \cup &\left\{ \left\{ \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \cdot n \mid n \in \mathcal{N}_i \right\} \cup \left\{ \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} \cdot n \mid n \in \mathcal{N}_i \right\} \right\} = \\ &= \mathcal{N}_i \cup \left\{ \mathcal{N}_{\{\pm e_1, \pm(e_1+e_2)\}} \cup \mathcal{N}_{\{\pm e_2, \pm(e_1+e_2)\}} \right\} \cup \\ &\cup \left\{ \mathcal{N}_{\{\pm e_1, \pm(e_1-e_2)\}} \cup \mathcal{N}_{\{\pm e_2, \pm(e_1-e_2)\}} \right\}, \end{aligned}$$

and from Lemma 2.22 we know that the neighbouring edges to $i = \{\pm e_1, \pm e_2\}$ are exactly $\{\pm e_1, \pm(e_1 + e_2)\}, \{\pm e_2, \pm(e_1 + e_2)\}, \{\pm e_1, \pm(e_1 - e_2)\}, \{\pm e_2, \pm(e_1 - e_2)\}$ from which the equivalence is obvious. $\square$

We can observe that transitioning between lax bases is similar to changing basis vectors. As all change of basis matrices must be invertible, we are interested in the group $\mathrm{GL}_2(\mathbb{Z})$.

Lemma 2.33. *G generates $GL_2(\mathbb{Z})$.*

Proof. Firstly, we can see that all the matrices in G are in $\mathrm{GL}_2(\mathbb{Z})$.

From Theorem 2.29 we know that all we need is for G to generate the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.1}$$

By simple computations, we can see that

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}.$$

Using this matrix, we can express

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and again, by applying the newly obtained matrices, we see that

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

and the last matrix is already can be obtained as

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$\square$

In the upcoming lemma, we aim to demonstrate that it is possible to obtain any lax base in $\mathcal{L}$ from another lax base in $\mathcal{L}$ solely by utilising matrices from $\mathrm{GL}_2(\mathbb{Z})$. We will do so by analysing the set of navigators, as we understand that this set determines any lax base.

Lemma 2.34. *Let $e \in \mathcal{L}$, then for any lax base $f \in \mathcal{L}$ it holds that*

$$\forall n \in \mathcal{N}_f \ \forall m \in \mathcal{N}_e \ \exists M \in GL_2(\mathbb{Z}): \ n = Mm.$$

Proof. Let us recall that a navigator n is a strict base, meaning it is a base of $\mathbb{Z}^2$. We aim to switch from a basis given by the pair of vectors n to one given by the pair of vectors m. Therefore, M must be a change of the basis matrix, and hence, it has to be invertible with integer entries. Therefore, M must be an element of $\mathrm{GL}_2(\mathbb{Z})$. $\square$

Let us remind us that

$$G_v = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & -1 \\ 0 & 1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Using our sets of matrices, we can now properly formulate the algorithm described by Conway:

Definition 2.35 (Algorithmic construction)**.** *Let $e = \{\pm e_1, \pm e_2\}$ be a lax base, $\mathcal{L}$ the set of all lax bases, and $\mathcal{S}$ the set of all the lax superbases. We define $V_0 = \{\}$, $L_0 = \{e\}$ and for $i \geq 1$ we recursively define*

$$\begin{aligned} V_{i+1} &= \{v \in \mathcal{S} \mid \exists e \in L_i : \mathcal{N}_v \subseteq \{X \cdot n \mid n \in \mathcal{N}_e, \ X \in G_v\}\}, \\ L_{i+1} &= \{f \in \mathcal{L} \mid \exists e \in L_i : \mathcal{N}_f \subseteq \{X \cdot n \mid X \in G, \ n \in \mathcal{N}_e\}\}. \end{aligned}$$

Then we set $\widetilde{V} = \bigcup_i V_i, \ \widetilde{L} = \bigcup_i L_i$.

Our task is to demonstrate that the algorithm aligns with our abstract definition of the topograph. To do so, we will establish that $\widetilde{L}$ equals the set of all lax bases we represent as $\mathcal{L}$. By virtue of Lemma 2.27, this set is bijective to all edges. Following this, we will prove that $\widetilde{V}$ corresponds to the set of all lax superbases, which we defined as V in Definition 2.24.

Theorem 2.36. *Let $T = (V, E)$ be a topograph and $\widetilde{V}$ and $\widetilde{L}$ from Definition 2.35. Then $V = \widetilde{V}$ and $\varphi(E) = \widetilde{L}$.*

Proof. We need to prove four inclusions: $\widetilde{L} \subseteq \mathcal{L}$, $\widetilde{V} \subseteq V$, $\mathcal{L} \subseteq \widetilde{L}$ and $V \subseteq \widetilde{V}$. Then the rest follows from Proposition 2.27, which tells us that $\varphi(E) = \mathcal{L}$. The first two inclusions come from Definition 2.35.

"$\mathcal{L} \subseteq \widetilde{L}$" We want to prove that the algorithm outputs all the lax bases.

Lemma 2.34 tells us that any base $f = \{\pm f_1, \pm f_2\} \in \mathcal{L}$ can be obtained from the starting base $e = \{\pm e_1, \pm e_2\}$ using only matrices $M \in \mathrm{GL}_2(\mathbb{Z})$. By looking at the Definition 2.35, we can see that every element in $\widetilde{L}$ is a product of matrices from G and the initial base. For any $f \in L_i$, we can obtain $\mathcal{N}_f$ as a subset of $\{X \cdot n \mid X \in G, n \in \mathcal{N}_g\}$, where $g \in L_{i-1}$. Through induction, we can prove that $\mathcal{N}_f$ is a subset of $\{X_i X_{i-1} \dots X_0 \cdot n \mid X_i \in G, n \in \mathcal{N}_e\}$. Therefore, we can obtain $\mathcal{N}_f$ as a subset $\{Mn \mid n \in \mathcal{N}_e\}$, where M is generated by matrices in G.

Moreover, Lemma 2.33 yields that G generates $\mathrm{GL}_2(\mathbb{Z})$. Since the topograph is a tree, there exists exactly one path between any two edges, which we can describe as a product of matrices from G. The final matrix will be called M, in $\mathrm{GL}_2(\mathbb{Z})$ from Lemma 2.33. Therefore using Lemma 2.34 we can get any lax base in $\widetilde{L}$.

"$V \subseteq \widetilde{V}$" We will prove that the algorithm outputs all the lax superbases.

V_i consists of all the lax superbases adjacent to the lax bases in L_{i-1} based on its definition. This conclusion is derived from Proposition 2.31. We know that every lax base e is present in some L_i. However, we need to prove that every element $u \in V$ can be obtained from some L_j.

One can observe this by referring to Lemma 2.22. It states that each lax superbase is connected to precisely three lax bases, and every lax base is connected to precisely two lax superbases. Thus, for any vertex $v \in V$, there are three adjacent lax bases: $e \in L_k$, $f \in L_l$, $g \in L_m$ and Proposition 2.31 implies that v belongs to $V_{min\{k,l,m\}+1}$.

We have now proved that every lax base is in $\widetilde{L}$ and every lax superbase is in $\widetilde{V}$.

The remaining two inequities follow the definition where we only allow elements of L_i and V_i to be elements of $\mathcal{L}$ and V respectively. $\square$

3. Farey tree

In this chapter, we will explain what a Farey diagram is and how it produces the Farey tree and provide more precise definitions of both structures. Our work on the Farey diagram is mainly based on Hatcher's Topology of Numbers [3]. He devotes the entire first chapter to the Farey diagram. Additionally, we will define the Farey tree and prove that it is isomorphic to our topograph, to which we have dedicated the previous chapter.

Please note that this chapter is less formal than the previous two as we work with two structures that were only explained intuitively in the source materials.

3.1 Farey diagram

Convention. We will introduce a new set which we will call $\mathbb{Q}^* = \mathbb{Q} \cup \left\{\frac{1}{0}\right\}$. We understand the elements of $\mathbb{Q}^*$ as $\frac{a}{b} \in \mathbb{Q}^*$, such that $b \geq 0$ and $\gcd(a, b) = 1$.

The following definition is established to aid in a better understanding of the construction of the Farey diagram.

Definition 3.1. *We define the* Farey sequence *of order n, denoted F_n, as the sequence of rational numbers of the form $\frac{a}{b}$, where a and b are positive integers such that $1 \leq a \leq b \leq n$ and $\gcd(a, b) = 1$ (in other words $\frac{a}{b}$ is in reduced form). We write the Farey sequence in increasing order, starting from $\frac{0}{1}$ and ending with $\frac{1}{1}$.*

We will also define a modified Farey sequence, visualised in Figure 3.1.

Definition 3.2. *Let $F_n = (a_1, a_2, \ldots, a_m)$ be a Farey sequence of order m. A* modified Farey sequence *of order n is the following sequence*

$$\widetilde{F}_n := \Big(a_1, a_2, \ldots, a_{m-1}, a_m, a_{m-1}^{-1}, \ldots, a_2^{-1}, a_1^{-1}, \\ -a_2^{-1}, \ldots, -a_{m-1}^{-1}, -a_m, -a_{m-1}, \ldots, -a_2\Big).$$

We can now start constructing the Farey diagram. We will be working with one of its versions used by Hatcher in [3]. He visualises it as a disk with inscribed arcs, where we start by splitting the disk in half and then iteratively split all the existing sections in half and connect them using an arc until we get a diagram resembling 3.1 (let us note that we will always only draw a part of the diagram as it is infinite). We assign fractions to each node on a given diagram during each iteration.

At the start, when we split the disk in half, we assign the fractions $\frac{1}{0}, \frac{0}{1}$. In the second iteration we assign fractions $\frac{1}{1}, \frac{-1}{1}$.

Now, we will explain what happens in the top right quarter of the circle.

We assign fractions from a set F_n for every other iteration, excluding those previously assigned in F_{n-1}. We want to assign these fractions in ascending order to the nodes from $\frac{0}{1}$ to $\frac{1}{0}$. To do this, we follow the mediant rule.

Definition 3.3. *We define* addition under the mediant rule *as $\frac{a}{c} \oplus \frac{b}{d} = \frac{a+b}{c+d}$.*

Lemma 3.4. The mediant rule *states that if we have three fractions in the Farey sequence, which follow one after another,* $\frac{a}{c} < \frac{e}{f} < \frac{b}{d}$*, then* $\frac{e}{f} = \frac{a+b}{c+d}$.

Proof. For example, the proof of this can be accessed in [8]. $\square$

Then we mirror the new sequence on the other three arcs between $\frac{1}{0}, \frac{0}{1}, \frac{1}{1}, \frac{-1}{1}$ with the elements from $\widetilde{F}_n \setminus \widetilde{F}_{n-1}$.

The mediant rule translates to the diagram as if we have two adjacent fractions $\frac{a}{c}$ and $\frac{b}{d}$ on the diagram; we can find a new point in between them, and connect it with arcs. We assign the value $\frac{a+b}{c+d}$ to this new point. Note that the fraction $\frac{1}{0}$ acts as $\frac{-1}{0}$ when added on the lower side of the diagram.

Ultimately, we can get a part of the Farey diagram; for example, [3] provides us with beautiful pictures just as Figure 3.1.

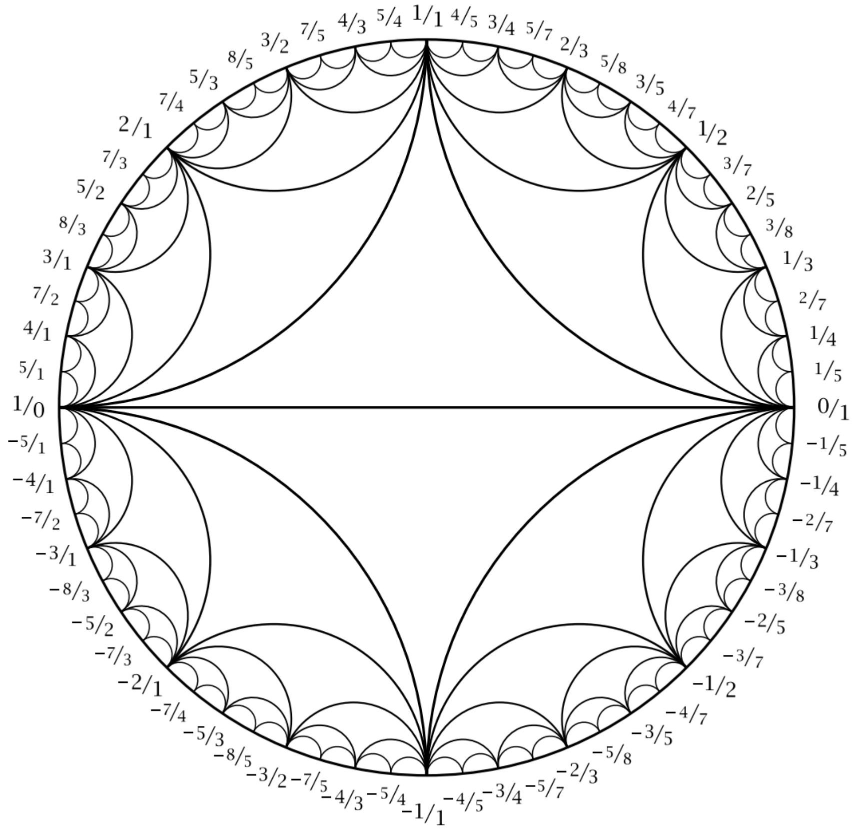

Figure 3.1: Farey Diagram by Allen Hatcher from [3], page 20

More observations about this visualisation of the Farey diagram can be found in Section 3.4.

We will formally define the Farey diagram as follows:

Definition 3.5. *We define the Farey diagram as a graph* $D = (V, E)$*, where* $V = \mathbb{Q}^*$ *and* $E = \left\{\left\{\frac{a}{c}, \frac{b}{d}\right\}, ad - bc = \pm 1\right\}$.

The provided illustration in Figure 3.1 shows how we can draw part of the graph on a disk (inside a circle). Understand that the circle is not a part of the graph but that the arcs converge towards the circle.

3.2 Farey tree

The section about the Farey tree will explain how we can find the Farey tree when we know what the Farey diagram looks like.

Hatcher draws the so-called dual tree superimposed inside the Farey diagram, as he visualises in Figure 3.2. Here, every closed part of the diagram contains precisely one vertex, and every edge crosses exactly one arc.

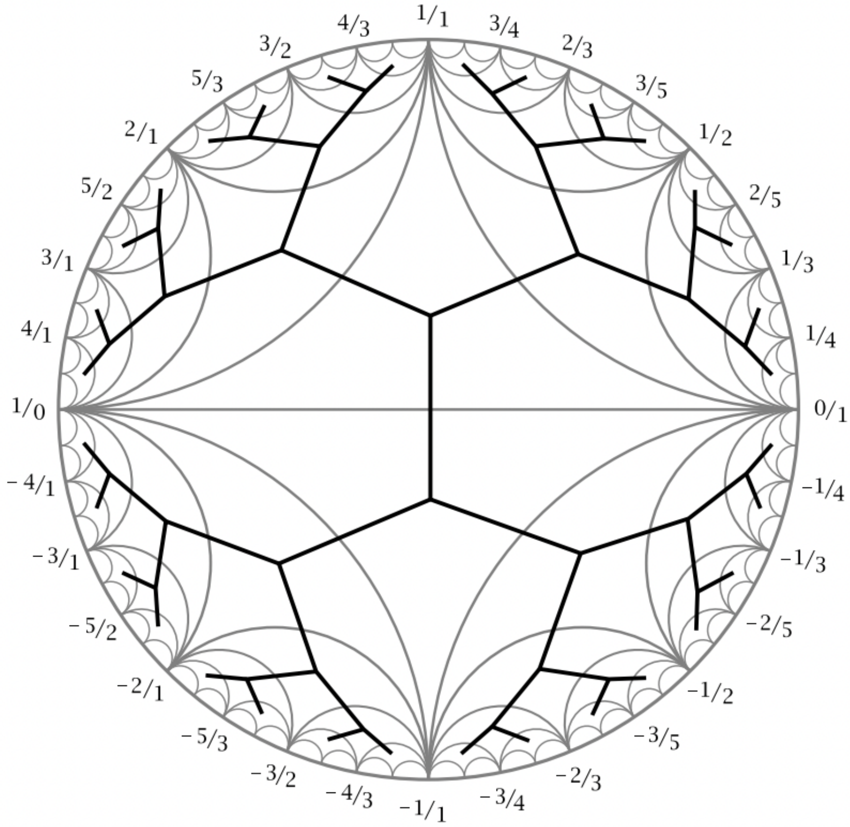

Figure 3.2: Farey tree by Allen Hatcher from [3], page 88

We will understand this graph as a tree with a root edge that crosses the arc between ($\frac{1}{0}$ and $\frac{0}{1}$ and with edges with the distance n from this root edge as sets of elements from the n-th modified Farey sequence.

Before formally defining this tree, we will prove the following theorem, which will help us establish the relationship between vertices in the Farey tree:

Theorem 3.6 (Cauchy)**.** *Let $\frac{a}{c} < \frac{b}{d}$ be neighboring elements of the Farey sequence F_n. Then $cb - ad = 1$.*

Proof. We know that $\frac{a}{c}$ is in the reduced form, so $\gcd(a, c) = 1$ and also $\gcd(-a, c) = 1$. Since $-a, c$ are integers, we get them from Bézout's identity that there exists $x_0, y_0 \in \mathbb{Z}$ such that they solve the equation $cx - ay = 1$. Also it holds for all the possible pairs that $(x, y) = (x_0 + ra, y_0 + rc)$ for $r \in \mathbb{Z}$. By the right choice of r, we can find such a pair for which

$$0 < n - c \leq y < n. \tag{3.1}$$

Therefore $\gcd(x, y) = 1$. Consider the fraction $\frac{x}{y}$. We want to show that it equals $\frac{b}{d}$.

We can see that $\frac{x}{y} > \frac{a}{c}$ from the following inequality:

$$\frac{x}{y} - \frac{a}{c} = \frac{xc - ay}{yc} = \frac{1}{yc} > 0.$$

Now $\frac{x}{y}$ is in the reduced form and $y < n$, so it can either be bigger than one or it could belong to the Farey sequence. We must show that it is not bigger than $\frac{b}{d}$.

Let us prove this by contradiction: Let us have $\frac{x}{y} > \frac{b}{d}$, then $xd - yb \geq 1$ and the same holds for $cb - ad \geq 1$ we get the following equation:

$$\frac{x}{y} - \frac{b}{d} + \frac{b}{d} - \frac{a}{c} = \frac{xd - by}{yd} + \frac{cb - ad}{cd} \geq \frac{1}{yd} + \frac{1}{cd} = \frac{c + y}{cyd}.$$

At the same time

$$\frac{x}{y} - \frac{b}{d} + \frac{b}{d} - \frac{a}{c} = \frac{x}{y} - \frac{a}{c} = \frac{1}{yc.}$$

So we get that

$$\begin{aligned} \frac{1}{yc} \geq \frac{c+y}{cyd} &= \frac{1}{yc}\frac{c+y}{d} \\ 1 &\geq \frac{c+y}{d} \\ d \geq c + y &\overset{3.1}{\geq} n, \end{aligned}$$

which contradicts that $\frac{b}{d} \in F_n$ so $\frac{x}{y} = \frac{b}{d}$. □

Remark. Theorem 3.6 provides properties for only one-quarter of the diagram. It is evident that on the right side of the diagram, we obtain this equality, while on the other side, we get $cb - ad = -1$. However, this issue will not affect us since we will use this Theorem when we define edges and vertices which we do for unordered sets.

For the purpose of formulating Definition 3.8 we will prove the following observation:

Proposition 3.7. *Let $\frac{a}{c}, \frac{b}{d} \in F_n,\ n \in \mathbb{N}$, then $\frac{a+c}{b+d}$ is in the reduced form.*

Proof. From Theorem 3.6 we know that $cb - ad = 1$. We can then write

$$cb - ad = cb + ab - ab - ad = b(a + c) - a(b + d) = 1.$$

If there existed $k \in \mathbb{N} \setminus \{\pm 1\}$ such that $kx = (a + c)$ and $ky = (b + d)$ then we would have $bkx - aky = k(bx - ay) = 1$ then k would have to be ± 1 which is in contradiction with our choice of k. □

Now, we can provide the abstract definition of the Farey tree:

Definition 3.8. *The* Farey tree *is defined as* $F = (V, E)$ *where*

$$V = \left\{ \left\{ \frac{a}{c}, \frac{b}{d}, \frac{a}{c} \oplus \frac{b}{d} \right\} \mid \left(\frac{a}{c}, \frac{b}{d}, \frac{a}{c} \oplus \frac{b}{d} \in \mathbb{Q}^* \right) \wedge (ad - bc = \pm 1) \right\},$$

$$E = \left\{ \{v_1, v_2\} \;\middle|\; v_1, v_2 \in V,\ |v_1 \cap v_2| = 2 \right\}.$$

As we can see, we assign every edge to the two fractions connected by the arc. The edge intersects, and the vertices add to this pair a third fraction, which we get using the mediant rule.

3.3 Linking the topograph and the Farey tree

In Chapter 4, we aim to demonstrate how continued fractions can be used to determine paths in the topograph. However, to achieve this goal, we must first establish where we can find fractions on the topograph. To do so, we want to establish a connection between the Farey tree and topograph. This section will explain how we can easily connect lax vectors and fractions from $\mathbb{Q}^*$ and then extend this connection to both graphs.

Let us recall the Definition 2.24 which states that a *topograph* is a pair $T = (V, E)$, where V is a set of all the lax superbases in $\mathbb{Z}^2$ and $E = \{\{v_1, v_2\} \mid v_1, v_2 \in V, |v_1 \cap v_2| = 2\}$.

Let us also remind the reader that in Definition 2.20, we have set $\mathcal{V}$ as being the set of all the lax vectors in $\mathbb{Z}^2$.

Definition 3.9. *We will define a function*

$$\begin{aligned} \tau : \mathbb{Q}^* &\to \mathcal{V} \\ \frac{a}{b} &\mapsto \pm(a, b). \end{aligned}$$

Lemma 3.10. *The function* τ *is well-defined and is a bijection.*

Proof. We know that all the elements $\frac{a}{b} \in \mathbb{Q}^*$ have $\gcd(a, b) = 1$. On the other hand a lax vector is derived from a primitive vector $u = (a, b)$ for which we know that we cannot find such $k \in \mathbb{Z} \setminus \{\pm 1\}$ and $v = (e, f) \in \mathbb{Z}^2$ such that $u = kv$, but that is equivalent to saying that $(a, b) \neq (ke, kf)$ which is equivalent to $\gcd(a, b) \neq k$ therefore $\gcd(a, b) = 1$.

Due to the lax notation, we can only obtain lax vectors in the form of (a, b) and $(-a, b)$, where $b \geq 0$.

We understand that every lax vector has one corresponding fraction from $\mathbb{Q}^*$. That is because, for a lax vector $\pm(u, v)$, we have only one of its elements, such that $v > 0$, which we need for $\frac{u}{v}$ to be in $\mathbb{Q}^*$. Similarly, we obtain exactly one lax vector for every element of $\mathbb{Q}^*$. $\square$

From this lemma, we can also map lax bases and superbases onto sets of fractions. As an example, we provide how applying τ can look on parts of the graphs: on the left side, we can see a part of the Farey tree, and on the right side, we can see the Farey tree where every fraction $\frac{a}{b}$ is assigned the value $\tau\left(\frac{a}{b}\right)$

which, as we can see, is equal to a part of the topograph containing the edge $\{\pm(1,0), \pm(0,1)\}$:

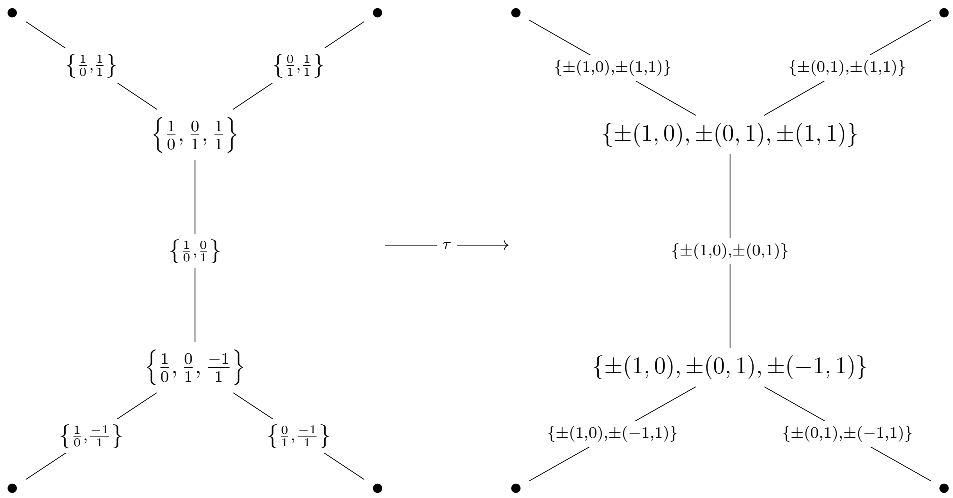

Figure 3.3: Applying τ on the Farey tree

We can then apply τ to the entire Farey tree and want it to be equal to the topograph.

Proposition 3.11. *Let $F = (V, E)$ be the Farey tree. Then $\tau(F) = (\tau(V), \tau(E))$ is the topograph.*

Proof. From Lemma 3.10, we can see that if we apply τ to the set of all vertices of F, we get all the lax superbases. Therefore $\tau(V) = V(T)$. We must also understand that the mediant rule acts on F as the relationships between lax bases act on T. If we have a vertex $\left\{\frac{e_1}{e_2}, \frac{f_1}{f_2}, \frac{e_1+f_1}{e_2+f_2}\right\}$ in F, then if we apply τ we get a lax base $\{\pm(e_1, e_2), \pm(f_1, f_2), \pm(e_1 + f_1, e_2 + f_2)\}$. We can see that the mediant rule gives the same property as the rule for lax bases, which tells us that for some choice of signs, we need to be able to sum everything to a zero, which is the same as summing under the mediant rule. From Lemma 2.22 we know that for lax vectors $\pm(e_1, e_2), \pm(f_1, f_2)$ we can get another lax base that they are contained in and that is $\{\pm(e_1, e_2), \pm(f_1, f_2), \pm(e_1 - f_1, e_2 - f_2)\}$. Nevertheless, here we can see that this would be the image of a set $\left\{\frac{e_1}{e_2}, \frac{f_1}{f_2}, \frac{e_1-f_1}{e_2-f_2}\right\}$ where $\frac{f_1}{f_2} \oplus \frac{e_1-f_1}{e_2-f_2} = \frac{e_1}{e_2}$ so it is a vertex in F.

We need to prove that the sets of edges will still be equal. For that, we will recall the definitions. We have set:

$$E(F) = \Big\{\{v_1, v_2\} \;\Big|\; v_1, v_2 \in V(F),\ |v_1 \cap v_2| = 2\Big\},$$

$$E(T) = \{\{v_1', v_2'\} \mid v_1', v_2' \in V(T), |v_1' \cap v_2'| = 2\}.$$

From this, we can see that

$$\tau(E) = \Big\{\{\tau(v_1), \tau(v_2)\} \;\Big|\; \tau(v_1), \tau(v_2) \in \tau(V),\ |v_1 \cap v_2| = 2\Big\} =$$
$$= \{\{v_1', v_2'\} \mid v_1', v_2' \in V(T), |v_1' \cap v_2'| = 2\} = E(T).$$

$\square$

Remark. We can also see that if $T = (V, E)$ is the topograph, then $\tau^{-1}(T) = (\tau^{-1}(V), \tau^{-1}(E))$ is the Farey tree.

3.4 A couple of observations

In this section, we will very informally sketch interesting observations about the Farey diagram seen in Figure 3.1, which can show us that the topograph is a tree. It would be an interesting topic for another work.

We get the Farey diagram by splitting a disk in half using arcs and then iteratively splitting the areas under the existing arcs using two new arcs.

We can define the arcs as follows, which will help us to define triangles in the diagram:

Definition 3.12. *An* arc *is a set of two fractions such that there exists a modified Farey sequence in which these fractions are adjacent. We denote it as* $A = \left\{\frac{p}{q}, \frac{r}{s}\right\}$*. This is equal to the fact that* $pq - rs = \pm 1$*.*

A Farey triangle is a section in the Farey diagram bound by exactly three arcs; more formally, we define it as follows.

Definition 3.13. *A* Farey triangle *is a set of three arcs* $\{A, B, C\}$ *such that there exists* n *for which* $A \subseteq \widetilde{F}_n,\ B, C \subseteq \widetilde{F}_{n+1}$ *and* $|A \cap B| = |A \cap C| = |C \cap B| = 1$ *and* $|A \cap B \cap C| = 0$*.*

Observation. Two Farey triangles T_1, T_2 are adjacent if and only if $|T_1 \cap T_2| = 1$.

Every Farey triangle can be assigned a set of fractions such that every fraction in this set is at the end of some of the arcs; we can define it as follows.

Definition 3.14. The set of fractions of a Farey triangle *is a set* $A \cup B \cup C$*.*

Observation. For every Farey triangle, its set of fractions has three elements.

As mentioned in Section 3.2, we can get the dual-tree of the Farey diagram as the Farey tree, visible in Figure 3.2. We could prove other observations about this tree.

Observation. The vertices of the Farey tree correspond to the Farey triangles. More precisely, they are equal to the sets of fractions of these Farey triangles.

Observation. Two Farey triangles T_1, T_2 are adjacent if and only if an edge connects the vertices corresponding to them in the Farey tree.

Observation. We can see that the Farey tree is indeed a tree.

As we have explained in Proposition 3.11, we can map the Farey tree to the topograph, from which we could finally provide the proof of Theorem 2.25.

Observation. The topograph is a tree.

4. Walks and paths

4.1 Turning left and turning right

As promised, let us discuss how to represent continued fractions in a topograph. Our primary focus will be the Farey tree F. According to Proposition 3.11, we can assign values to F and get the topograph. In this chapter, we will talk about walks in the Farey tree, but we understand that under the power of τ, it also holds for the topograph.

Before anything, we will assign matrices to the edges of the Farey tree.

Definition 4.1. *For an edge $e = (v_1, v_2)$ in the Farey tree F we assign unimodular matrix*

$$A_e = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $\frac{a}{c}, \frac{b}{d}$ are in $v_1 \cap v_2$.

We can represent this in Figure 3.2 as a matrix containing the two fractions connected by the arc that intersects our edge.

We will also often work with addition over sets using the mediant rule. For this purpose, we introduce the following definition.

Definition 4.2. *We define the* mediant sum *of a set $H = \{h_1, h_2, \dots, h_k\}$, $h_i \in \mathbb{Q}^*$ as*

$$\sum^{\oplus} H = h_1 \oplus h_2 \oplus \cdots \oplus h_k,$$

where we understand $\oplus$ to be addition under the mediant rule.

Remark. The mediant sum is associative. We can see that for a set of cardinality 3:

$$\left(\frac{a}{b} \oplus \frac{c}{d}\right) \oplus \frac{e}{f} = \frac{a+c}{b+d} \oplus \frac{e}{f} = \frac{a+c+e}{b+d+f} = \frac{a}{b} \oplus \left(\frac{c}{d} \oplus \frac{e}{f}\right) \tag{4.1}$$

In the following section, we will work with the orientation of the graph.

Definition 4.3. *Let $F = (V, E)$ be the Farey tree. Then we say that it is* oriented *if every edge is an ordered pair (v_1, v_2) such that $\left|\sum^{\oplus} v_1\right| \geq \left|\sum^{\oplus} v_2\right|$.*

We can add a helpful remark to avoid confusion with left and right orientation, which can be problematic for the reader (or the author).

Remark. This is left, and this is right.

The graph can be drawn in different ways. To better understand our work, we will consider the graph drawn as shown in Figure 3.2, paying special attention to the left and right orientation. Formally we will define turning left and right as follows.

Definition 4.4. *We understand that path $\{g_1, g_2\} = \{(u, v), (v, w)\}$ turns* left *on the vertex $v = \left\{\frac{a}{c}, \frac{b}{d}, \frac{a+b}{c+d}\right\}$ and the path $\{g_1, g_3\} = \{(u, v), (v, y)\}$ turns* right *if $\sum^{\oplus} |v \cap w| > \sum^{\oplus} |v \cap y|$.*

We can then also express left and right turns as matrix multiplications.

Definition 4.5. *Let e be an edge in the Farey tree with the assigned matrix $A_e = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then the edge f on its right side has the assigned matrix*

$$A_f = A_e R = \begin{pmatrix} a+b & b \\ c+d & d \end{pmatrix}$$

and the edge g on its left side has the matrix

$$A_g = A_e L = \begin{pmatrix} a & a+b \\ c & c+d \end{pmatrix}$$

where

$$R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \ L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

From Definition 4.4, we can see that a walk in our topograph can be described as matrix multiplication.

For an oriented edge, which will be our root (v_0, v_1), we can then assign a matrix $A_{(v_0,v_1)}$, and the walk then can be described as a mix of right and left turns in the graph. Let us demonstrate in an example:

Example. In Figure 4.1, we have a part of the Farey tree with matrices assigned to its edges, and we have coloured and changed the structure of the edges contained on a walk γ.

We can see that the root of this walk is $\left(\frac{1}{0}, \frac{1}{1}\right)$ and the corresponding matrix is $A_{g_0} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

We also see that the walk meanders in the following manner - right, left, left, left, and it ends on the edge $\left(\frac{2}{1}, \frac{7}{4}\right)$ with corresponding matrix $A_{g_4} = \begin{pmatrix} 2 & 7 \\ 1 & 4 \end{pmatrix}$. As explained in the Definition 4.5 we can write that $A_{g_4} = A_{g_0} R L^3 = L R L^3$, since $A_{g_0} = L$. Now, if we look at the continued fraction of $\frac{7}{4}$, we see that it is $[1, 1, 3]$. Is it a coincidence?

4.2 The paths and walks on continued fractions

In this section, we will finally prove that the continued fraction shows the path for the corresponding number. We will recall the definition of continued polynomials and two well-known propositions from the course of Number Theory. Their proofs and further observations can be found in [6].

We will then prove Proposition 4.9, which gives us the algorithm described in Example 4.1 for rational numbers, and then we will generalise it for real numbers.

In this section, when we work with continued fractions we take as the root of the walk the root $\left(\left\{\frac{1}{0}, \frac{0}{1}, \frac{1}{1}\right\}, \left\{\frac{1}{0}, \frac{0}{1}, \frac{-1}{1}\right\}\right)$ which has the corresponding matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. We will leave it out as it does not change the result of matrix multiplications.

We will recall a definition and two propositions, which can be found in [6], on pages 15 and 16, alongside their proofs.

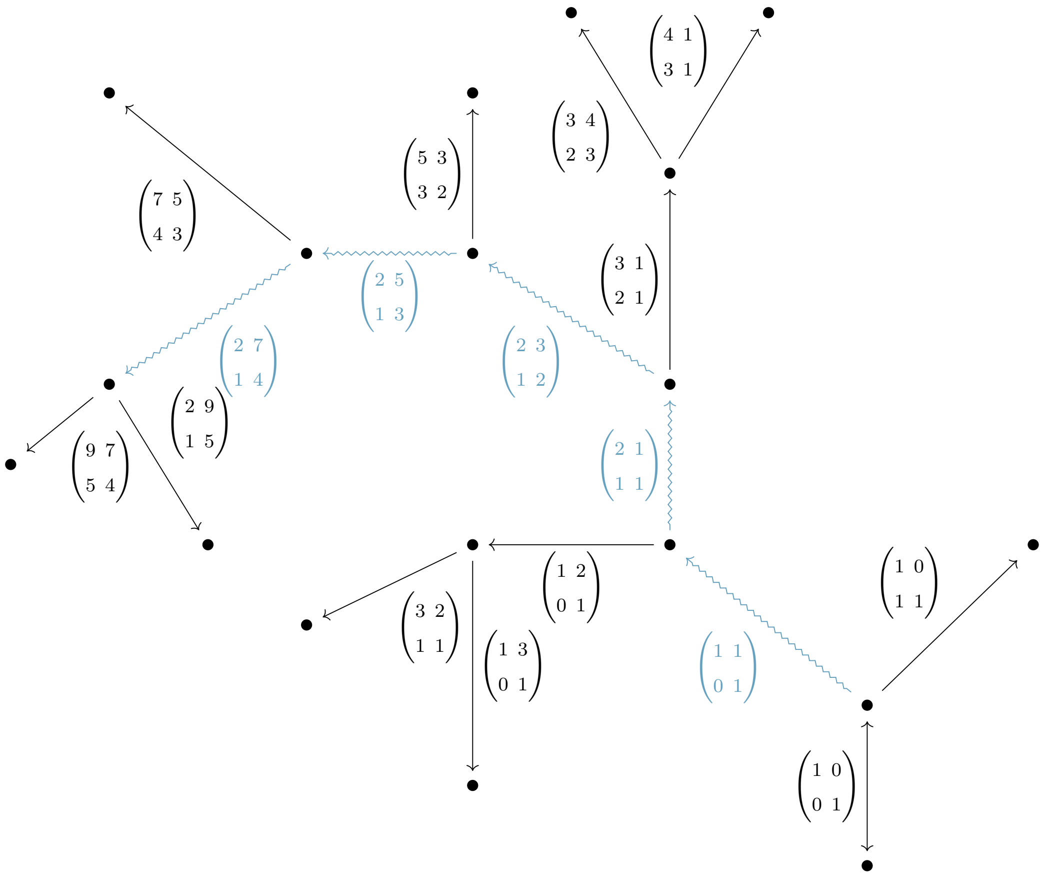

Figure 4.1: Path of $\frac{7}{4}$

Definition 4.6. *The* n-th continued polynomial *in variables $x_1, \dots, x_n$ is defined recurrently:* $K_{-1} := 0, K_0 := 1$,

$$K_n(x_1, \dots, x_n) := x_n \cdot K_{n-1}(x_1 \dots, x_{n-1}) + K_{n-2}(x_1, \dots, x_{n-2})$$

for $n \geq 1$.

Proposition 4.7. *For $a_0 \in \mathbb{R},\ a_i \in \mathbb{R}^+$ it holds*

$$[a_0, a_1, \dots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \frac{1}{a_n}}} = \frac{K_{n+1}(a_0, \dots, a_n)}{K_n(a_1, \dots, a_n)}.$$

Proposition 4.8. *For* $n \geq 1$

$$\begin{pmatrix} K_{n+1}(a_0, \dots, a_n) & K_n(a_0, \dots, a_{n-1}) \\ K_n(a_1, \dots, a_n) & K_{n-1}(a_1, \dots, a_{n-1}) \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}.$$

Now, let us prove the following proposition describing what we saw in the example above. It can be counter intuitive to see that for n even we get a matrix with switched columns, but it can be nicely seen in the example. On every vertex, if we turn left, we can see that the new fraction, not yet seen on the last edge, appears on the right side of the matrix, and when we turn right, it appears on the left side. It can also be visible from the matrix multiplications.

Proposition 4.9. *Let $\xi = [c_0, c_1, \dots, c_n] = \frac{a}{c}$ for $c_i \in \mathbb{Z}$, then we have a fraction $\frac{b}{d} = [c_0, \dots, c_{n-1}]$ such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = L^{c_0} R^{c_1} L^{c_2} \dots R^{c_n}$ for n odd and $\begin{pmatrix} b & a \\ d & c \end{pmatrix} = L^{c_0} R^{c_1} L^{c_2} \dots L^{c_n}$ for n even.*

Proof. From Proposition 4.7 we see that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} K_{n+1}(c_0, \dots, c_n) & K_n(c_0, \dots, c_{n-1}) \\ K_n(c_1, \dots, c_n) & K_{n-1}(c_1, \dots, c_{n-1}) \end{pmatrix},$$

which we know from Proposition 4.8 to be equal to

$$\begin{pmatrix} c_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_n & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.2}$$

But we want it to be equal to

$$\begin{pmatrix} 1 & c_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c_1 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & 0 \\ c_n & 1 \end{pmatrix} \tag{4.3}$$

if n is even and

$$\begin{pmatrix} 1 & c_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c_1 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & c_n \\ 0 & 1 \end{pmatrix} \tag{4.4}$$

if n is odd.

But that comes from the equality

$$\begin{pmatrix} c_i & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_j & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & c_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c_j & 1 \end{pmatrix} = \begin{pmatrix} c_i c_j + 1 & c_i \\ c_j & 1 \end{pmatrix}.$$

We can see that we can group (4.2) into pairs and (4.3) as well, and they will be equal. For (4.4) we have to multiply both sides of the equation by $\begin{pmatrix} 1 & c_n \\ 0 & 1 \end{pmatrix}$. We get

$$
\begin{aligned}
&\left(\begin{pmatrix} c_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} c_{n-1} & 1 \\ 1 & 0 \end{pmatrix}\right)\begin{pmatrix} 1 & c_n \\ 0 & 1 \end{pmatrix} = \\
&= \begin{pmatrix} K_n(c_0,\dots,c_{n-1}) & K_{n-1}(c_0,\dots,c_{n-2}) \\ K_{n-1}(c_1,\dots,c_{n-1}) & K_{n-2}(c_1,\dots,c_{n-2}) \end{pmatrix}\begin{pmatrix} 1 & c_n \\ 0 & 1 \end{pmatrix} = \\
&= \begin{pmatrix} K_n(c_0,\dots,c_{n-1}) & c_n\left(K_n(c_0,\dots,c_{n-1})\right) + K_{n-1}(c_0,\dots,c_{n-2}) \\ K_{n-1}(c_1,\dots,c_{n-1}) & c_n\left(K_{n-1}(c_1,\dots,c_{n-1})\right) + K_{n-2}(c_1,\dots,c_{n-2}) \end{pmatrix} = \\
&= \begin{pmatrix} K_n(c_0,\dots,c_{n-1}) & K_{n+1}(c_0,\dots,c_n) \\ K_{n-1}(c_1,\dots,c_{n-1}) & K_n(c_1,\dots,c_n) \end{pmatrix} = \\
&= \begin{pmatrix} b & a \\ d & c \end{pmatrix}.
\end{aligned}
$$

□

The following proposition explains that for any $\xi \in \mathbb{R}$, the path given by its continued fraction converges to ξ. We use the fact that if we get the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then we understand the fraction $\frac{a}{c}$ as

$$
\frac{\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}}{\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}}, \tag{4.5}
$$

where we understand that if we have the fraction with matrices $\frac{(a)}{(c)}$, then it is equal to $\frac{a}{c}$.

Proposition 4.10. *Let $\xi \in \mathbb{R}$ have the continued fraction $[c_0, c_1 \dots]$, $c_i \in \mathbb{N}$. Then*

$$
\begin{aligned}
&\lim_{k\to\infty} \frac{\begin{pmatrix} 1 & 0 \end{pmatrix}(L^{c_0}R^{c_1}\dots R^{c_{2k}})\begin{pmatrix} 1 \\ 0 \end{pmatrix}}{\begin{pmatrix} 0 & 1 \end{pmatrix}(L^{c_0}R^{c_1}\dots R^{c_{2k}})\begin{pmatrix} 1 \\ 0 \end{pmatrix}} = \\
&= \lim_{k\to\infty} \frac{\begin{pmatrix} 1 & 0 \end{pmatrix}(L^{c_0}R^{c_1}\dots L^{c_{2k+1}})\begin{pmatrix} 0 \\ 1 \end{pmatrix}}{\begin{pmatrix} 0 & 1 \end{pmatrix}(L^{c_0}R^{c_1}\dots L^{c_{2k+1}})\begin{pmatrix} 0 \\ 1 \end{pmatrix}} = \xi.
\end{aligned}
$$

Proof. From the basics of limits of sequences, we know that if $\{a_n\} \to A$, then for a subsequence, we get $\{a_{n_k}\} \to A$. So that means that $\lim_{k\to\infty}[c_0,\dots,c_{2k}] = \lim_{k\to\infty}[c_0,\dots,c_{2k+1}] = \lim_{k\to\infty}[c_0,\dots,c_k] = \xi$.

We can see that for every n we get

$$\frac{p_n}{q_n} = [c_0, \ldots, c_n] = \frac{\begin{pmatrix} 1 & 0 \end{pmatrix} (L^{c_0} R^{c_1} \ldots R^{c_{2n}}) \begin{pmatrix} 1 \\ 0 \end{pmatrix}}{\begin{pmatrix} 0 & 1 \end{pmatrix} (L^{c_0} R^{c_1} \ldots R^{c_{2n}}) \begin{pmatrix} 1 \\ 0 \end{pmatrix}},$$

and f

$$\frac{p_n}{q_n} = [c_0, \ldots, c_n] = \frac{\begin{pmatrix} 1 & 0 \end{pmatrix} (L^{c_0} R^{c_1} \ldots L^{c_{2n+1}}) \begin{pmatrix} 0 \\ 1 \end{pmatrix}}{\begin{pmatrix} 0 & 1 \end{pmatrix} (L^{c_0} R^{c_1} \ldots L^{c_{2n+1}}) \begin{pmatrix} 0 \\ 1 \end{pmatrix}}.$$

This means that we can write the limits in the proposition as

$$\lim_{k\to\infty} [c_0, \ldots, c_{2k}] = \lim_{k\to\infty} [c_0, \ldots, c_{2k+1}]$$

but that is from the properties of subsequences equal to $\lim_{k\to\infty}[c_0, \ldots, c_k] = \xi$. That concludes our proof. $\square$

4.3 Special cases of walks and paths for continued fractions

In Chapter 1, we discussed continued fractions and made some observations about them. Proposition 1.16 and Proposition 1.17 were particularly noteworthy. In the following section, we will provide visual representations of the paths that are represented by continued fractions using the topograph and the Farey tree.

4.3.1 Negative continued fractions

First, we will explore the implications of negative integers in a continued fraction. We will consider the negative continued fraction $0 > \eta = -\xi = -[c_0, c_1, \ldots] = [-c_0, -c_1, \ldots]$. Then, we will demonstrate the resulting walk as described in Proposition 4.9 as $(L^{-1})^{c_0}(R^{-1})^{c_1} \ldots$, where

$$L^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, R^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

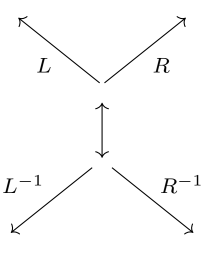

Figure 4.2: Turns from the root

In Figure 4.2, we can see the outcomes when starting with different matrices L, R, L^{-1}, and R^{-1} from the root, which is represented by the identity matrix. According to Definition 4.5, matrices L and L^{-1} turn left, while matrices R and R^{-1} turn right on their respective parts of the Farey tree.

In Lemma 1.14 we proved that $-[c_0, c_1, \dots] = [-c_0 - 1, 1, c_1 - 1, c_2, c_3, \dots]$. What it says in the language of walks in the topograph is that we can go in the wrong direction and then come back, as illustrated by the equation $(L^{-1})^{c_0}(R^{-1})^{c_1}(L^{-1})^{c_2}\cdots = (L^{-1})^{c_0+1}RL^{c_1-1}R^{c_2}\dots$ which can be visible in the Figures 4.3 and 4.4 on the example of $-\frac{13}{5} = -[2, 1, 1, 2] = [-3, 1, 0, 1, 2]$.

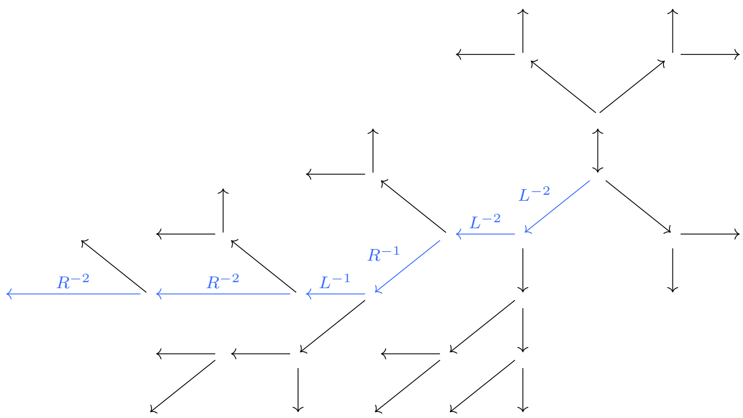

Figure 4.3: The walk of $-[2, 1, 1, 2]$

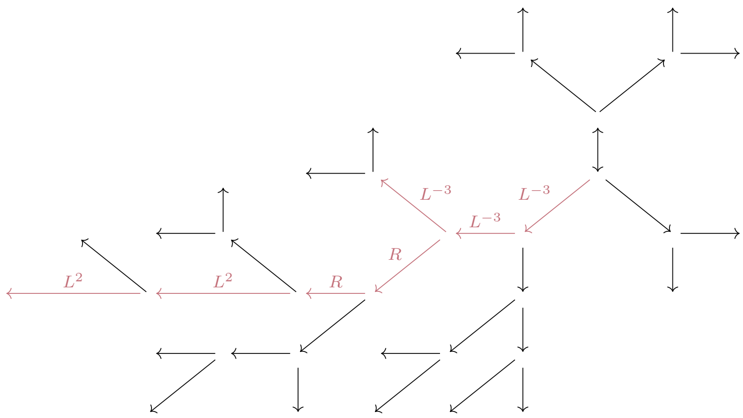

Figure 4.4: The walk of $[-3, 1, 0, 1, 2]$

4.3.2 Continued fractions of quadratic irrationals

We can use three findings from Chapter 1 to study periodic continued fractions for quadratic irrationals: Theorem 1.7, Proposition 1.16, and Proposition 1.17.

Theorem 1.7 states that for a quadratic irrational α with continued fraction $[\overline{b_1, \dots, b_l}]$, we have $\alpha' = -[0, \overline{b_l, \dots, b_1}]$. This means that the paths determined by such continued fractions are $L^{b_1}R^{b_2}\dots$ and $R^{-b_1}L^{-b_2}\dots$.

For example, for $\frac{2+\sqrt{10}}{3}$ the continued fraction is $[\overline{1,1,2}]$, for $\frac{2-\sqrt{10}}{3}$ it is from Theorem 1.7 $[0,\overline{-2,-1,-1}]$.

Figure 4.5 visualises this and shows two semi-infinite paths connected by the edge corresponding to the identity matrix.

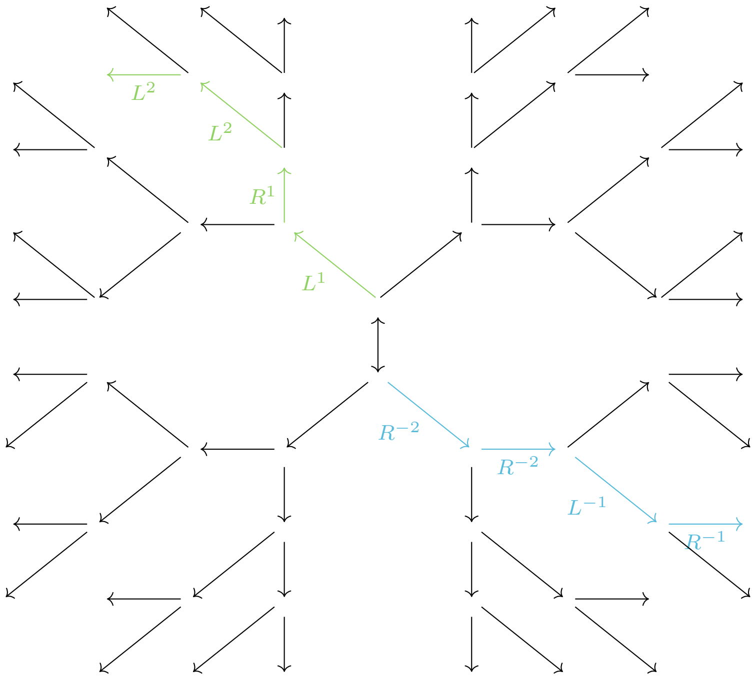

Figure 4.5: Path for $\frac{2+\sqrt{10}}{3}$ and $\frac{2-\sqrt{10}}{3}$

Proposition 1.17 tells us that if we have a continued fraction $\alpha = [a_0, \overline{b_1, \dots, b_l}]$, then the continued fraction of its conjugate α' is $[a_0 - b_l - 1, 1, b_l - 1, \overline{b_{l-1}, b_{l-2}, \dots, b_1, b_l}]$. To give an example, we can consider the continued fraction of $\sqrt{3}$, which is $[1,\overline{1,2}]$, and for $-\sqrt{3}$, it is $-[1,\overline{2,1}]$. Figure 4.6 illustrates this as two semi-infinite paths connected by the edge corresponding to the identity matrix.

The last case is the one mentioned in Proposition 1.16 which states that for $\alpha = [a_0, a_1, \dots, a_k, \overline{b_1, \dots, b_l}]$, $k \geq 1$ we get for $a_k < b_l$

$$\alpha' = [a_0, a_1, \dots, a_{k-1} - 1, 1, b_l - a_k - 1, \overline{b_{l-1}, b_{l-2}, \dots, b_1, b_l}]$$

and for $a_k > b_l$ we get

$$\alpha' = [a_0, a_1, \dots, a_{k-1}, a_k - b_l - 1, 1, b_{l-1} - 1, \overline{b_{l-2}, b_{l-3}, \dots, b_1, b_l, b_{l-1}}].$$

We can show examples if we take the numbers $\frac{4387+\sqrt{37}}{2764}$ and $\frac{4387-\sqrt{37}}{2764}$ with the continued fractions $[1,1,1,2,3,2,\overline{1,2,3}]$ and $[1,1,1,2,2,3,\overline{1,3,2}]$ respectively. In Figure 4.7, we can see that both paths given by these continued fractions begin on the same path given by the red arrow

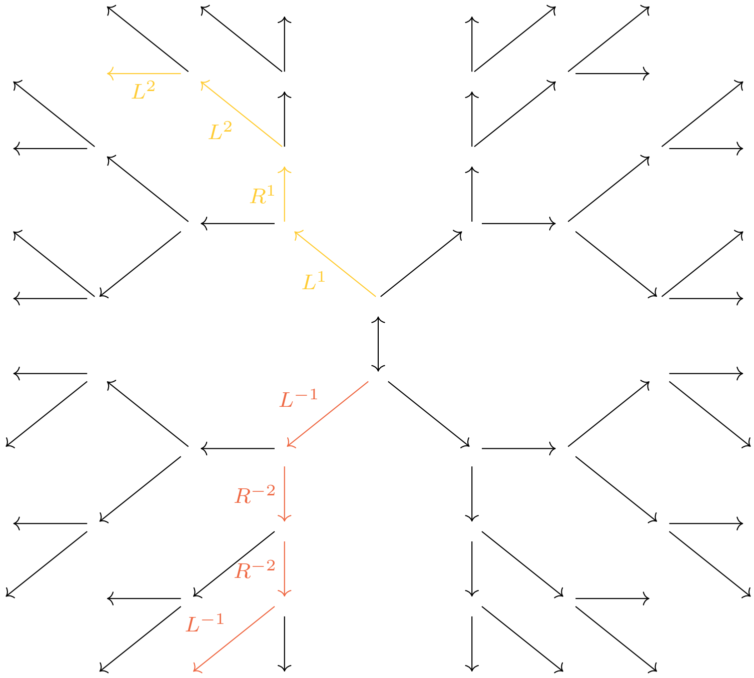

Figure 4.6: Paths for $\sqrt{3}$ and $-\sqrt{3}$

Moreover, their paths diverge at the vertex denoted by a bullet $\bullet$. For $\frac{4387+\sqrt{37}}{2764}$ we can continue to follow the path continuing on the left denoted by

which leads to the periodic part denoted by the arrow

Similarly, we can watch the path for $\frac{4387-\sqrt{37}}{2764}$ diverge in the other direction - going right from the bullet.

We can use this knowledge to study Conway Rivers, which has been the initial desire behind this thesis. We can use the observations made in this chapter to assign different quadratic forms to the topograph, which is explained very well in The Sensual (quadratic) form [2] from page 8. In this topograph with reassigned values, called *topograph of* Q, we can find paths called *Conway rivers* that always balance on the edges containing one positive and one negative value. For every quadratic form, we can find a quadratic irrational such that $(\alpha, 1)$ and $(\alpha', 1)$ are the roots of this quadratic form, and then the paths for α, α' diverge at the Conway river of the given quadratic form.

Working with our example illustrated in Figure 4.7 if we assigned the quadratic form

$$Q(x, y) = x^2 - \left(\frac{4387}{1382}\right) xy + \left(\frac{6963}{2764}\right) y^2$$

to the topograph, Figure 4.7 would show the Conway river of this quadratic form as the path that intersects the bullet and moves away from it. That is because $\left(\frac{4387\pm\sqrt{37}}{2764}, 1\right)$ are the roots of this quadratic form.

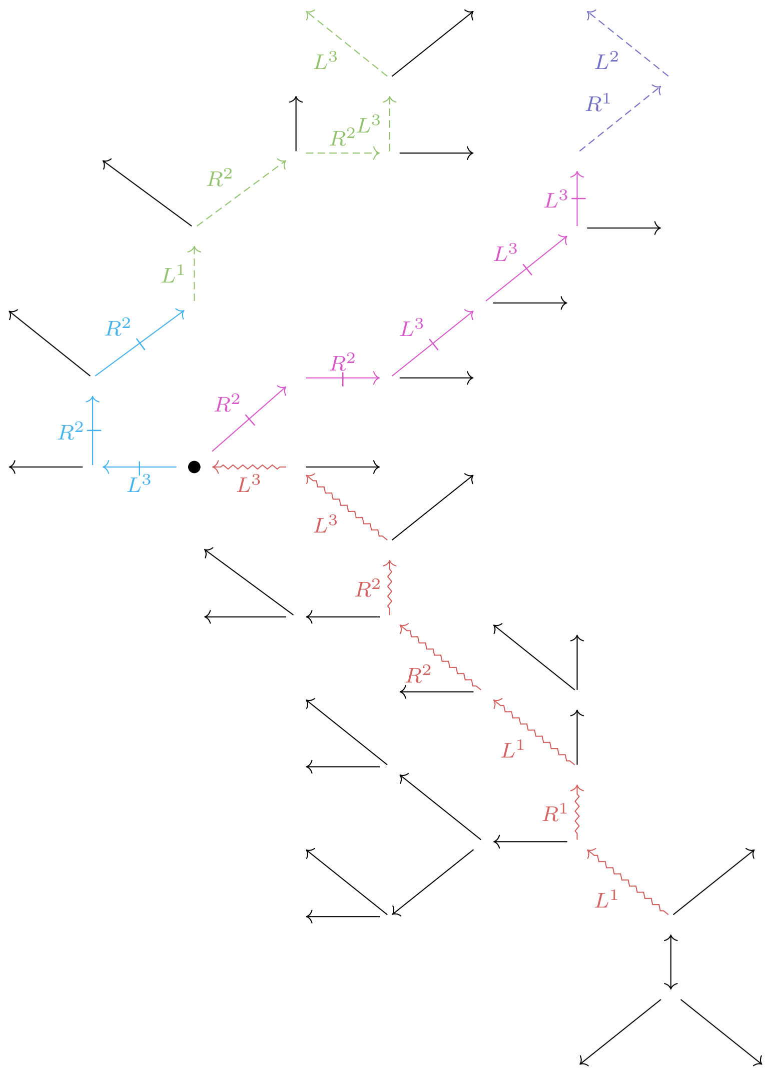

Figure 4.7: The continued fractions of $\frac{4387 \pm \sqrt{37}}{2764}$

Conclusion

In this thesis, we have explained how continued fractions can be used to draw paths in the topograph. To define our illustrations accurately, we proved several properties of continued fractions and formally defined the topograph. Additionally, we defined the structure of the Farey tree, which introduces fractions into the topograph. We also proved how continued fractions determine paths in both graphs.

Despite completing this thesis, there are still many properties we would be interested in studying. We would focus on Theorem 2.25, which states that the topograph is a tree, using our Definition 2.24 and expand the observations in Section 3.4. We would also work on properly defining how quadratic forms can be illustrated in the topograph and study the properties of Conway rivers, which are mentioned in the last section of our work.

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