Advisor's report on Natália Bátorová's Bachelor's thesis

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This thesis describes the arithmetic-geometric mean sequences over finite fields and briefly discusses their connection with elliptic curves. Chapter 1 contains the introduction of the arithmetic-geometric mean sequences over real numbers, the origin of these sequences. Chapter 2 starts with a theoretical background on finite fields and graph theory so that later, the arithmetic-geometric mean sequences over finite fields and their corresponding graphs are properly introduced. It already contains some new results compared to the paper [Gri+23] which served as the main literature for this thesis, concretely, Lemma 23 and Theorems 26 and 27. Chapter 3 follows the paper, and describes the results when finite fields are $\mathbb{F}_q$, where $q \equiv 3 \pmod 4$. Still, the results are explained in greater detail than in the paper, and some nice illustrations are made. Chapter 4 is the main part of the thesis because it contains new and original results. In this chapter, the definition of the arithmetic-geometric mean sequences is generalised to finite fields $\mathbb{F}_q$, where $q \equiv 5 \pmod 8$. The graphs corresponding to these sequences are precisely described in Theorem 41, which is one of the two most important results of the thesis. The other is Theorem 43, which explains that for $q > 13$, the arithmetic-geometric mean sequences are well-defined and non-trivial. Chapter 5 briefly and informally explains the connection between elliptic curves and the arithmetic-geometric mean sequences over finite fields. This was used to give a lower bound on the number of the components of the corresponding graph, which is the content of Theorem 52.

The thesis is well-written, the objects are carefully defined, and the statements and proofs are clear. Also, there are several very helpful illustrations. The results are correct and well-explained, and the thesis contains very nice and illustrative figures (for which the author wrote a Python code) which help in understanding the main results.

There are fairly frequent stylistic and English imperfections. Also, some formulations are hard to understand (e.g., first bullet in Lemma 7). However, I am not aware of any serious mistakes.

This thesis was motivated by the paper [Gri+23] about the arithmetic-geometric mean sequences over $\mathbb{F}_q$, where $q \equiv 3 \pmod 4$. However, this thesis goes further and develops similar results for $\mathbb{F}_q$, where $q \equiv 5 \pmod 8$. It is worth mentioning that this generalisation is non-trivial and that the definition in this case is less obvious than for $q \equiv 3 \pmod 4$. All results in Chapter 4, especially Theorems 41 and 43, are more difficult than the corresponding results in the cited paper. The author also gives general results about the arithmetic-geometric mean sequences over finite fields, i.e., without any assumption on their cardinality. These results are Lemma 23, Theorem 27, and partially Theorem 26.

In conclusion, the known parts of the thesis are still explained in more detail than in the cited sources, and there are several new results that will be submitted for publication as a joint paper with the consultant of the thesis, Stevan Gajović (who also prepared a draft of this report).

Overall, I recommend the thesis to be defended as a Bachelor's thesis with the grade 1. Given the original results that will be soon submitted for publication, I suggest that the committee considers nominating the thesis for *Dean's prize*.

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