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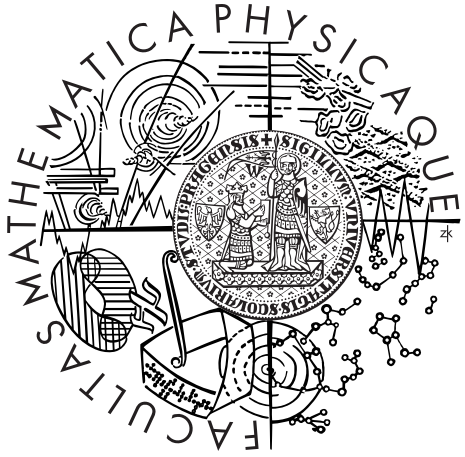

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Interacting spatial particle systems

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Abstrakt: Práce se zabývá několika typy náhodných sjednocení interagujících částic. Jsou definovány procesy interagujících úseček v $\mathbb{R}^2$ a interagujících destiček v $\mathbb{R}^3$ jako modely s hustotou vzhledem k Poissonovu procesu. Jsou odvozeny vzorce pro geometrické charakteristiky těchto modelů a je zkoumáno limitní chování pro intenzitu jdoucí do nekonečna. Pro časové rozšíření modelu je uveden simulační algoritmus a v rámci simulační studie jsou porovnávány různé druhy odhadů parametrů hustoty p, zejména se zaměřením na sekvenční Monte Carlo metody.

Klíčová slova: Boolovský model, proces interagujících částic, $U-$statistiky, exponenciální rodina rozdělení, germ-grain model, interakce, Markovská vlastnost, bodový process, náhodná uzavřená množina, Markov chain Monte Carlo.

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Abstract: Several kinds of random union of interacting particles is studied. We define line segment process of interacting particles in $\mathbb{R}^2$ and process of interacting surfaces in $\mathbb{R}^3$ as the models with density function p with respect to some Poisson point process. The formulas for moments of the geometrical characteristics of these models are derived and the limit behaviour when the intensity tends to infinity is investigated. For time extension of such models a simulation algorithm is developed. Various estimations of parameters of density p, among them those based on sequential Monte Carlo, are studied and compare in a simulation study.

Keywords: Boolean model, process with interacting particles, $U-$statistics, exponential family, germ-grain model, interaction, Markov properties, point process, random closed set, Markov chain Monte Carlo.

Chapter 1

Introduction

Spatio-temporal modeling in stochastic geometry is of great interest in both theory and applications. A large group of various models is based on spatio-temporal point processes, see e.g. [29] with applications in statistical seismology, [5] in modeling weeds, [4] in modeling a plant disease, [11], [13] in neurophysiology, [10] in urban development, [33], [25] in modeling forest fires, etc. Classical assumption of stationarity is relaxed in finite point process models, tools such as stochastic differential equations or sequential Monte Carlo help to deal with the temporal dynamics. More general models of spatio-temporal random sets appear e.g. in [37] with applications in geophysics, [12] in image analysis, [36] in turbulence, [30], [19] in tumour growth dynamics, etc.

While most frequently in the literature the temporal development concerns the volume of a set, we try to consider also other integral-geometrical characteristics, especially $U-$statistics which are fully given by all $k-$tuples of points.

This work addresses two main topics. One of them is theoretical part regarding the $U-$statistics and the second one is a simulation study in space–time. Both parts are linked to the choice of a class of random sets. In the theoretical probabilistic part the problem of analytical expression of moments is solved originally in the case of point processes with density with respect to a Poisson process, [3]. It presents an alternative derivation technique based on L^2-expansion of a functional of a Poisson process, [22], in comparison with [7], who use Georgii-Nguyen-Zessin formula. The central limit theorem when the intensity of the reference point process tends to infinity is discussed.

A second contribution is the development of MCMC and sequentional Monte Carlo to the case of space-time processes of interacting particles. The particle filter approach was published in [40], while particle marginal Metropolis–Hastings algorithm and comparison of all three parameter estimation procedures (including maximum likelihood method) is presented in [41]. Large simulation studies and long-run of numerical solutions were

extremely computationaly demanding.

Chapter 2 summarizes some basic facts from point process theory. The next chapter is dedicated to $U-$statistics of a point process with a density function. The first moment of $U-$statistics is introduced (Theorem 3.5) and further the second and the second mixed moment are computed (Theorem 3.7). The general formula for product of $U-$statistics is formulated in Theorem 3.8 and Theorem 3.10 solves its moment evaluation using diagrams [31]. In Chapter 4 there is specified the model of interacting particles and its properties. Three particular cases are studied - process of interacting discs in $\mathbb{R}^2$, process of interacting line segments on $\mathbb{R}^2$ and interacting surfaces in $\mathbb{R}^3$. For segment process and process of interacting surfaces the first, second and mixed moments of characteristics are introduced (Propositions 4.3, 4.5 and 4.6). At the end of this chapter the limit behavior of Poisson process with increasing intensity of reference points is discussed. For process with density with respect to such Poisson process the first and second moments of standardized characteristics are deduced. One particular example with analytically derived expectation of Papangelou conditional intensity is computed. Chapter 5 involves the space-time part. Section 5.1 creates preliminaries for simulation study - it summarizes particle filtering and finally the last two sections deal with simulation study. All results are discussed in Chapter 6 and the work is provided with an appendix where the simulation and computation programs are described.

Chapter 2

Point processes

Since the whole thesis deals with spatial point processes, their properties and applications, this chapter provides the readers with some basic facts and definitions from the theory of point processes. For more details on point procesess see [6] and [39]. For the basics of finite point procesess see [2] and [28].

2.1 Basic definitions

Definition 2.1 *Let E be separable, locally compact complete metric space equipped with Borel σ-field $\mathcal{B} = \mathcal{B}(E)$.* Locally finite measure *on E is a measure which is finite on all bounded Borel sets of E. We will denote $\mathcal{M}$ the set of all locally finite measures on $(E, \mathcal{B}(E))$.*

Definition 2.2 *On the space $(E, \mathcal{B}(E))$ define a set $\mathcal{N}$ of all locally finite measures taking the non-negative integer values or infinity*

$$\mathcal{N} \equiv \{\mu \in \mathcal{M};\ \mu(B) \in \mathbb{N} \cup \{0, \infty\} \text{ for all } B \in \mathcal{B}\}.$$

On spaces $\mathcal{M}, \mathcal{N}$ define σ-fields

$$\begin{aligned} \mathfrak{M} &= \sigma\{\mu \mapsto \mu(B) \text{ measurable},\ B \in \mathcal{B}\}, \\ \mathfrak{N} &= \{M \cap \mathcal{N} : M \in \mathfrak{M}\}. \end{aligned}$$

$\mathfrak{M}$ is the smallest σ-field on $\mathcal{M}$ for which the mapping $\mathcal{M} \longrightarrow \mathbb{R}$, $\mu \mapsto \mu(B)$ is measurable for all $B \in \mathcal{B}$.

In the thesis we deal with $(\Omega, \mathcal{A}, \mathbb{P})$ a probability space.

Definition 2.3 Point process *on E is a measurable mapping $\mu : (\Omega, \mathcal{A}, \mathbb{P}) \longrightarrow (\mathcal{N}, \mathfrak{N})$.* Distribution of a point process *μ is a probability measure $\mathbb{P}_\mu$ given by*

$$\mathbb{P}_\mu(B) = \mathbb{P}(\mu \in B),\ B \in \mathfrak{N}.$$

We say that point process is simple *if $\mathbb{P}(\mu \in \mathcal{N}^*) = 1$, where*

$$\mathcal{N}^* = \{\gamma \in \mathcal{N} :\ \gamma(\{y\}) \leq 1;\ \forall\, y \in E\}.$$

Remark 2.1 *Each $\mathbf{y} \in \mathcal{N}$ can be represented as a sum of at most countable number of Dirac measures, $\mathbf{y} = \sum_i \delta_{y_i}$. Thus if μ is a simple point process on E then there exists a sequence of measurable mappings $Y_i : \Omega \longrightarrow E$ such that*

$$\mu = \sum_{i=1}^{\mu(E)} \delta_{Y_i}, \tag{2.1}$$

where $\mu(E)$ denotes the total number of points Y_i in (2.1) *which is at most countable due to local finiteness.*

Further we assume that a point process is simple. Thus we can consider a point process alternatively as a locally finite set of random points Y_i, we write $Y_i \in \mu$. In the following both approaches are used, i.e. $\mathbf{y} \in \mathcal{N}$ means either a counting measure or a locally finite set of points.

Definition 2.4 *Point process μ on $\mathbb{R}^d$ is* stationary *if its distribution is invariant under the translation, i.e. if the process $\mu \oplus u = \{y + u;\ y \in \mu\}$ has the same distribution as μ for all $u \in \mathbb{R}^d$.*
μ is called isotropic *if its distribution is invariant under the rotations.*

Definition 2.5 *A locally finite measure Γ on $\mathcal{B}$ satisfying $\Gamma(B) = \mathbb{E}\mu(B)$ for all $B \in \mathcal{B}$ is called* the intensity measure.

Definition 2.6 *Let $\Lambda \in \mathcal{M}$ be diffuse and let η be a point process on E such that for all $n \in \mathbb{N}$ and bounded $B_1, \ldots, B_n \in \mathcal{B}$ pairwise disjoint the random variables $\eta(B_1), \ldots, \eta(B_n)$ are independent and for all $i \in \mathbb{N}$ $\eta(B_i)$ has Poisson distribution $Poiss(\Lambda(B_i))$ with parameter $\Lambda(B_i)$. Then η is called a* Poisson point process on E with intensity measure Λ.

Theorem 2.1 Campbell's Theorem. *For a point process μ on E with intensity measure Γ and nonnegative measurable function h we have*

$$\mathbb{E}\sum_{y\in\mu} h(y) = \int_E h(y)\Gamma(\mathrm{d}y). \tag{2.2}$$

Definition 2.7 *Let Γ be the intensity measure of a point process such that there exists its density with respect to Lebesgue measure, i.e. $\Gamma(A) = \int_A \gamma(y)\mathrm{d}y$, $A \in \mathcal{B}$, then nonnegative function γ is called* intensity function.

Remark 2.2 *Since Lebesgue measure is the only one (up to a multiplicative constant) locally finite measure invariant under the translation, the intensity function for any arbitrary stationary process in $E = \mathbb{R}^d$ is a constant called* intensity.

Definition 2.8 A homogeneous Poisson process *is a Poisson process with constant intensity function.*

Definition 2.9 A marked point process *on a space B with marks in a measurable space $(Z, \mathcal{Z})$ is a point process μ on $E = B \times Z$ such that $\mu(K \times Z) < \infty$ a.s. for all compact $K \subset B$. That is, the corresponding projected process (of points without marks) is locally finite.*

Definition 2.10 *Let $\mathcal{C}$ be a system of all closed sets in E and*

$$\mathfrak{C} = \sigma\{\mathcal{C}^K : K \text{ is a compact subset of } E\},$$

where $\mathcal{C}^K = \{C \in \mathcal{C} : C \cap K \neq \emptyset\}$. Then a random closed set $\boldsymbol{\Xi}$ *in E is a measurable mapping from $(\Omega, \mathcal{A})$ to $(\mathcal{C}, \mathfrak{C})$.*

Example 2.1 Boolean model in $\mathbb{R}^d$. *Suppose $\eta = \{y_1, y_2, \dots\}$ is a stationary Poisson process in $\mathbb{R}^d$ with intensity λ. Let $\Xi_1, \Xi_2 \dots$ be a sequence of i.i.d. random compact sets in $\mathbb{R}^d$ independent of the process η. Then the random set*

$$\Xi = \bigcup_{i=1}^{\infty} (y_i + \Xi_i)$$

is called a Boolean model.

2.2 Point processes with density function

Since all models considered in this work are given by a density function with respect to Poisson point process we introduce here theory of finite point processes.

Definition 2.11 *Point process μ on E is* finite *if $\mu(E) < \infty$ almost surely.*

Recall that $\mathcal{N}$ denotes a set of locally finite counting measures on E. By $\mathcal{N}^*$ we will denote a space of finite, simple counting measures on E. This space can be decomposed according to the total number of points, i.e.

$$\mathcal{N}^* = \mathcal{N}_0 \cup \mathcal{N}_1 \cup \dots,$$

where

$$\mathcal{N}_k = \{\mathbf{y} \in \mathcal{N}^*;\ \mathbf{y}(E) = k\}, \qquad k = 0, 1, \dots.$$

In fact we decompose the system of all locally finite sets of points into sets of all configurations of k points. This can be represented by ordered spaces (see [2])

$$E^{!k} = \{(y_1, \dots, y_k) :\ y_i \in E, y_i \neq y_j \text{ for } i \neq j\}.$$

Further if we define mapping $I_k : E^{!k} \longrightarrow \mathcal{N}_k$ by

$$I_k(y_1, \dots, y_k) = \delta_{y_1} + \dots + \delta_{y_k},$$

we get factorization of the space $E^{!k}$ to $\mathcal{N}_k$. In the other words,

$$\mathcal{N}_k \equiv E^{!k}/\sim$$

where $\sim$ is relation given by

$$(u_1, \dots, u_k) \sim (y_1, \dots, y_k) \iff \{u_1, \dots, u_k\} = \{y_1, \dots, y_k\}.$$

Example 2.2 Poisson point process. *For fixed $n \in \mathbb{N}$ a point process of n i.i.d. points with distribution Q in E is called* binomial process. *Let η be the Poisson point process on E with totally finite intensity measure Λ, i.e. $\Lambda(E) < \infty$. Then $\eta(E) \sim Poiss(\Lambda(E))$. Given $\eta(E) = n$ we can consider $\eta = (Y_1, \dots, Y_n)$ as a binomial process with $Q(B) = \Lambda(B)/\Lambda(E)$, $B \in \mathcal{B}$. Thus for each $A \in \mathfrak{N}$ we have*

$$\begin{aligned}
\mathbb{P}_\eta(A) &= \sum_{n=0}^{\infty} \mathbb{P}(\eta(E) = n)\, \mathbb{P}(I_n(Y_1, \dots, Y_n) \in A) \\
&= \sum_{n=0}^{\infty} e^{-\Lambda(E)} \frac{\Lambda(E)^n}{n!} \int_E \dots \int_E \mathbf{I}_{[I_n(y_1,\dots,y_n)\in A]}\, Q(\mathrm{d}y_1) \dots Q(\mathrm{d}y_n) \\
&= e^{-\Lambda(E)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_E \dots \int_E \mathbf{I}_{[I_n(y_1,\dots,y_n)\in A]}\, \Lambda(\mathrm{d}y_1) \dots \Lambda(\mathrm{d}y_n).
\end{aligned}$$

Definition 2.12 *Let η be the Poisson point process with an intensity measure Λ. Let $p : \mathcal{N} \longrightarrow \mathbb{R}_+$, $\int_{\mathcal{N}} p(\mathbf{y})\mathbb{P}_\eta(\mathrm{d}\mathbf{y}) = 1$. A point process μ is* given by a density with respect to the Poisson point process η *if*

$$\mathbb{P}_\mu(B) = \mathbb{P}(\mu \in B) = \int_B p(\mathbf{y})\mathbb{P}_\eta(\mathrm{d}\mathbf{y}), \qquad B \in \mathfrak{N}.$$

For the remaining text we use notation $\Lambda(\mathrm{d}y_1)\dots\Lambda(\mathrm{d}y_n) = \Lambda(\mathrm{d}(y_1,\dots,y_n))$, $n \in \mathbb{N}$, the multiplicity of Λ can be seen from the number of variables.

The following lemma is a straightforward consequence.

Lemma 2.2 *For a point process μ with probability density p with respect to a Poisson process η we have*

$$\begin{aligned}\mathbb{P}(\mu \in &A) = \\ &= e^{-\Lambda(E)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_E \dots \int_E \mathbf{I}_{[I_n(y_1,\dots,y_n)\in A]}\, p(I_n(y_1,\dots,y_n))\, \Lambda(\mathrm{d}(y_1,\dots,y_n)),\end{aligned}$$

for all $A \in \mathfrak{N}$, and

$$\mathbb{E}[g(\mu)] = e^{-\Lambda(E)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_E \dots \int_E g(I_n(y_1,\dots,y_n))\, p(I_n(y_1,\dots,y_n))\, \Lambda(\mathrm{d}(y_1,\dots,y_n)), \tag{2.3}$$

for any function $g : \mathcal{N} \longrightarrow \mathbb{R}^+$ integrable with respect to a product measure Λ^n.

Remark 2.3 *We may write* (2.3) *shortly as*

$$\mathbb{E}[g(\mu)] = \mathbb{E}[g(\eta)p(\eta)]. \tag{2.4}$$

Definition 2.13 *A function $f : \mathcal{N} \longrightarrow \mathbb{R}$ is called* hereditary *if for all finite configurations $\mathbf{y}, \tilde{\mathbf{y}} \in \mathcal{N}$ such that $\tilde{\mathbf{y}} \subset \mathbf{y}$, it holds that $f(\tilde{\mathbf{y}}) > 0$ whenever $f(\mathbf{y}) > 0$.*

Definition 2.14 *Let μ be a point process with a hereditary density p with respect to a Poisson point process η then*

$$\lambda^*(u;\mathbf{y}) = \frac{p(\mathbf{y}\cup\{u\})}{p(\mathbf{y})},\ \mathbf{y} \in \mathcal{N},\ u \in E \setminus \mathbf{y}, \tag{2.5}$$

$$= 0 \qquad \textit{otherwise} \tag{2.6}$$

is called a Papangelou conditional intensity *of the point process μ. We take $\frac{0}{0} = 0$.*

For $n > 1$ analogously

$$\lambda_n^*(u_1, \ldots, u_n; \mathbf{y}) = \frac{p(\mathbf{y} \cup \{u_1, \ldots, u_n\})}{p(\mathbf{y})}, \; u_1, \ldots, u_n \in E \setminus \mathbf{y} \text{ disjoint points}, \quad (2.7)$$

$$= 0 \qquad \text{otherwise} \quad (2.8)$$

is the Papangelou conditional intensity of n-th order*, we define* $\lambda_0^* = 1$.

Remark 2.4 *The event when some of fixed $u_1, \ldots, u_n \in E$ are points of μ has probability equal to zero. Therefore the random variable $\lambda_n^*(u_1, \ldots, u_n; \mu)$ is almost surely well defined.*

Also note that λ_n^* is a symmetric function in variables $u_1, \ldots, u_n$, that means invariant w.r.t. permutations of variables.

Chapter 3

Non–Poisson point processes

This chapter focuses on U–statistics of point processes with density function with respect to a Poisson process. The appeal of this theory is that moments of functionals of such processes can be expressed through moments of functionals of Poisson processes as suggested in (2.4). There was developed a lot of theory in the field of functionals of Poisson processes using Wiener–Itô chaos expansion, see [22], [34] and [23]. We use these tools as first in a broader field of point processes with density.

3.1 Functionals of non-Poisson point processes

Consider finite point processes in a bounded measurable set $E \subset \mathbb{R}^d$ with $|E| > 0$, where $|\,.\,|$ is the Lebesgue measure of a set in $\mathbb{R}^d$. Let η be a Poisson point process on E with intensity measure Λ and distribution $\mathbb{P}_\eta$, which is a probability measure on the measurable space $(\mathcal{N}, \mathfrak{N})$ of integer-valued finite measures. Assume that a point process μ on E is given by a hereditary density p w.r.t. $\mathbb{P}_\eta$, i.e.

$$\mathrm{d}\mathbb{P}_\mu(\mathbf{y}) = p(\mathbf{y})\mathrm{d}\mathbb{P}_\eta(\mathbf{y}),\ \mathbf{y} \in \mathcal{N}.$$

Denote $L^m(\mathbb{P}_\mu) = \{F : \mathcal{N} \to \mathbb{R} \text{ measurable}, \mathbb{E}[|F(\mu)|^m] < \infty\}$, $1 \le m < \infty$.

Lemma 3.1 *For $F \in L^m(\mathbb{P}_\mu)$ set $G_m = F^m p,\ m = 1, 2, \ldots$.*
Then

$$\mathbb{E}F^m(\mu) = \mathbb{E}G_m(\eta),\ m = 1, 2, \ldots,$$

especially

$$\mathbb{E}F(\mu) = \mathbb{E}G_1(\eta),\ varF(\mu) = \mathbb{E}G_2(\eta) - [\mathbb{E}G_1(\eta)]^2.$$

Proof. $\mathbb{E}F^m(\mu) = \int F^m(\mathbf{y})\mathrm{d}\mathbb{P}_\mu(\mathbf{y}) = \int F^m(\mathbf{y})p(\mathbf{y})\mathrm{d}\mathbb{P}_\eta(\mathbf{y}) = \mathbb{E}G_m(\eta)$, hence $\mathbb{E}F(\mu) = \mathbb{E}G_1(\eta)$.

$$\mathrm{var}F(\mu) = \mathbb{E}F^2(\mu) - (\mathbb{E}F(\mu))^2 = \int F^2(\mathbf{y})p(\mathbf{y})\mathrm{d}\mathbb{P}_\eta(\mathbf{y}) - [\mathbb{E}G_1(\eta)]^2.$$

□

Definition 3.1 *For a measurable function $F : \mathcal{N} \longrightarrow \mathbb{R}$ and $u \in E$ we define the* difference operator *as*

$$D_u F(\mathbf{y}) = F(\mathbf{y} \cup \{u\}) - F(\mathbf{y}).$$

*The i -*th order difference operator *is given by*

$$D_{u_1,\ldots,u_i}F = D_{u_1}D_{u_2,\ldots,u_i}F, \qquad u_1, \ldots u_i \in E.$$

For this iterated operator the following equality holds, cf. [22]

$$D_{u_1,\ldots,u_n}F(\mathbf{y}) = \sum_{J\subset\{1,\ldots,n\}} (-1)^{n-|J|}F\left(\mathbf{y} \cup \{u_j,\, j \in J\}\right), \tag{3.1}$$

where $|J|$ is the cardinality of J.

Operator $D_{u_1,\ldots,u_n}$ is symmetric in $u_1, \ldots, u_n$ and symmetric functions T_nF on E^n are defined as

$$\begin{aligned} &T_n^\mu F(u_1, \ldots, u_n) = \mathbb{E}D_{u_1,\ldots,u_n}F(\mu), \\ &T_0^\mu F = \mathbb{E}F(\mu). \end{aligned}$$

We write T_nF for $T_n^\eta F$.

For the functionals of a Poisson process Theorem 1.1 in [22] says that for given $F, \tilde{F} \in L^2(\mathbb{P}_\eta)$ it holds

$$\mathbb{E}[F(\eta)\tilde{F}(\eta)] = \mathbb{E}F(\eta)\mathbb{E}\tilde{F}(\eta) + \sum_{n=1}^{\infty}\frac{1}{n!}\langle T_nF, T_n\tilde{F}\rangle_n, \tag{3.2}$$

where $\langle\,.\,,.\rangle_n$ is the scalar product in $L^2(\Lambda^n)$, i.e.

$$\langle f, g\rangle_n = \int_{E^n} f(x_1, \ldots, x_n)g(x_1, \ldots, x_n)\Lambda(\mathrm{d}(x_1, \ldots, x_n)).$$

The formula (3.2) is our main tool in the following.

3.2 Explicit formulas for $U-$statistics

Symbol $L^p_s(\Lambda^k)$ denotes the subset of symmetric functions in $L^p(\Lambda^k), 1 \leq p < \infty$.

Now for a fixed function $f \in L^1_s(\Lambda^k)$ we can define $U-$statistic F. Recall that $\lambda^*_j(u_1, \ldots, u_j; \mu)$ denotes the Papangelou conditional intensity of order j of the point process μ and denote

$$\varrho_n(u_1, \ldots, u_n) = \mathbb{E}\lambda^*_n(u_1, \ldots, u_n; \mu), \qquad u_1, \ldots, u_n \in E, \tag{3.3}$$

where $\mathbb{E}$ is the expectation with respect to the distribution $\mathbb{P}_\mu$.

Definition 3.2 *Let μ be a finite point process and $k \in \mathbb{N}$. Then a random variable $F(\mu)$ defined as*

$$F(\mu) = \sum_{(y_1,\ldots y_k)\in\mu^k_{\neq}} f(y_1, \ldots, y_k), \tag{3.4}$$

where $f \in L^1_s(\Lambda^k)$, is called $U-$statistic of order k. *Symbol $\mu^k_{\neq}$ denotes $k-$tuples of pairwise different points from realization of μ. We say that F is* driven by function f.

Note that by the Campbell's theorem we have

$$\mathbb{E}F(\eta) = \int_E \ldots \int_E f(y_1, \ldots, y_k)\Lambda(\mathrm{d}(y_1, \ldots, y_k)).$$

Following [34] we state some basic equalities for the difference operator of $U-$statistic F which hold for a finite point process μ.

Lemma 3.2 *Let $F \in L^2(\mathbb{P}_\mu)$ be a $U-$statistic of order k driven by f. Then*

$$D_{u_1}F(\mu) = k \sum_{(y_1,\ldots,y_{k-1})\in\mu^{k-1}_{\neq}} f(u_1, y_1, \ldots, y_{k-1}), \tag{3.5}$$

$$D_{u_1,\ldots,u_n}F(\mu) = \frac{k!}{(k-n)!} \sum_{(y_1,\ldots,y_{k-n})\in\mu^{k-n}_{\neq}} f(u_1, \ldots, u_n, y_1, \ldots, y_{k-n}), \quad n \leq k, \tag{3.6}$$

$$D_{u_1,\ldots,u_n}F(\mu) = 0, \quad n > k. \tag{3.7}$$

Proof. This follows from the symmetry of the function f. For illustration we will prove the first equality. For the difference of order 1 we have

$$
\begin{aligned}
D_{u_1}F(\mu) &= F(\mu \cup \{u_1\}) - F(\mu) = \\
&= \sum_{(y_1,\ldots,y_k)\in(\mu\cup\{u_1\})^k_{\neq}} f(y_1,\ldots,y_k) - \sum_{(y_1,\ldots,y_k)\in\mu^k_{\neq}} f(u_1,y_1,\ldots,y_k) \\
&= \sum_{(y_1,\ldots,y_{k-1})\in\mu^{k-1}_{\neq}} (f(u_1,y_1,\ldots,y_{k-1}) + \cdots + f(y_1,\ldots,y_{k-1},u_1)) \\
&= k \sum_{(y_1,\ldots,y_{k-1})\in\mu^{k-1}_{\neq}} f(u_1,y_1,\ldots,y_{k-1}).
\end{aligned}
$$

□

From the Campbell's theorem we have for the Poisson process η :

$$
T_nF(u_1,\ldots,u_n) = \binom{k}{n} \int_{E^{k-n}} f(u_1,\ldots,u_n,y_1,\ldots,y_{k-n})\Lambda(\mathrm{d}(y_1,\ldots,y_{k-n})), \tag{3.8}
$$

$n \leq k,\ T_nF(u_1,\ldots,u_n) = 0, n > k.$

Lemma 3.3 *For $p \in L^2(\mathbb{P}_\eta)$ it holds*

$$
T_np(u_1,\ldots,u_n) = \sum_{J\subset\{1,\ldots,n\}} (-1)^{n-|J|}\varrho_{|J|}(\{u_j,\ j\in J\};\mu). \tag{3.9}
$$

Proof. From (3.1) $D_{u_1,\ldots,u_n}p(\eta) = \sum_{J\subset\{1,\ldots,n\}}(-1)^{n-|J|}p(\eta\cup\{u_j,\ j\in J\})$. Then

$$
\begin{aligned}
T_np(u_1,\ldots,u_n) &= \mathbb{E}D_{u_1,\ldots,u_n}p(\eta) \\
&= \int \sum_{J\subset\{1,\ldots,n\}} (-1)^{n-|J|}p(\mathbf{y}\cup\{u_j,\ j\in J\})\mathrm{d}\mathbb{P}_\eta(\mathbf{y}) \\
&= \int \sum_{J\subset\{1,\ldots,n\}} (-1)^{n-|J|}p(\mathbf{y}\cup\{u_j,\ j\in J\})\frac{\mathrm{d}\mathbb{P}_\mu(\mathbf{y})}{p(\mathbf{y})}
\end{aligned}
$$

and (3.9) follows.

□

Theorem 3.4 *For $m \in \mathbb{N}$, the $U-$statistics F of order k driven by f such that $F^m \in L^2(\mathbb{P}_\eta)$ and μ a point process with density $p \in L^2(\mathbb{P}_\eta)$ we have*

$$
\mathbb{E}F^m(\mu) = \mathbb{E}F^m(\eta) + \sum_{n=1}^{mk} \frac{1}{n!}\langle T_nF^m, T_np\rangle_n, \tag{3.10}
$$

where T_np are given by (3.9).

Proof. For $m = 1$ we have from (3.2) and Lemma 3.2 (since $\mathbb{E}p(\eta) = 1$)

$$\mathbb{E}F(\mu) = \mathbb{E}F(\eta)p(\eta) = \mathbb{E}F(\eta) + \sum_{n=1}^{k} \frac{1}{n!}\langle T_n F, T_n p\rangle_n.$$

For $m = 2$ we have

$$\begin{aligned} D_u F^2(\eta) &= \left(\sum_{(y_1,\dots,y_k)\in(\eta\cup\{u\})^k_{\neq}} f(y_1,\dots,y_k) \right)^2 - \left(\sum_{(y_1,\dots,y_k)\in\eta^k_{\neq}} f(y_1,\dots,y_k) \right)^2 \\ &= \sum_{(y_1,\dots,y_k)\in(\eta\cup\{u\})^k_{\neq}} \sum_{(z_1,\dots,z_k)\in(\eta\cup\{z\})^k_{\neq}} f(y_1,\dots,y_k)f(z_1,\dots,z_k) \\ &\quad - \sum_{(y_1,\dots,y_k)\in\eta^k_{\neq}} \sum_{(z_1,\dots,z_k)\in\eta^k_{\neq}} f(y_1,\dots,y_k)f(z_1,\dots,z_k), \end{aligned}$$

here all terms cancel with the exception of those where among the variables in the left double sum there is u (either in one sum or in both). That means that idea of the proof of (3.5) applies here, so that $D^{2k}_{u_1,\dots,u_{2k}} F^2(\eta)$ is constant and $D^{2k+1}_{u_1,\dots,u_{2k}} F^2(\eta) = 0$. Analogously for general m we get the result.

□

Example 3.1 *Consider $k = 1$ and $U-$statistic*

$$F(\mu) = \mu(C),\ C \subset E,\ f(y) = \mathbf{I}_{[y\in C]}.$$

To show that $F \in L^2(\mathbb{P}_\eta)$ we need to compute

$$\begin{aligned} \mathbb{E}F^2(\eta) &= \sum_{n=0}^{\infty} e^{-\Lambda(E)} \frac{\Lambda^n(E)}{n!} \int_E \dots \int_E F^2(\{u_1,\dots,u_n\})\Lambda(\mathrm{d}(u_1,\dots,u_n)) \\ &\le \sum_{n=0}^{\infty} e^{-\Lambda(E)} \frac{\Lambda^n(E)}{n!} n^2 \\ &= e^{-\Lambda(E)}\Lambda(E) + \sum_{n=2}^{\infty} e^{-\Lambda(E)} \frac{\Lambda^n(E)}{(n-2)!} \frac{n}{n-1}\frac{n}{n} \\ &< e^{-\Lambda(E)}\Lambda(E) + 2\sum_{n=2}^{\infty} e^{-\Lambda(E)} \frac{\Lambda^n(E)}{(n-2)!} < \infty. \end{aligned}$$

Using Lemma 3.3 we have

$$\langle T_1 F, T_1 p\rangle = \int f(y)(\varrho_1(y;\mu) - 1)\Lambda(\mathrm{d}y),$$

$$\mathbb{E}F(\mu) = \Lambda(C) + \int_C (\varrho_1(y;\mu) - 1)\Lambda(\mathrm{d}y) = \int_C \varrho_1(y;\mu)\Lambda(\mathrm{d}y).$$

Next we evaluate the second moment.

$$D_u F^2(\eta) = 2f(u)\sum_{z\in\eta} f(z) + f^2(u) = \mathbf{I}_{[u\in C]}\big(2\cdot\sum_{z\in\eta}\mathbf{I}_{[z\in C]} + 1\big),$$

$$D_{u_2u_1}F^2(\eta) = 2\cdot\mathbf{I}_{[\{u_1,u_2\}\in C]}.$$

Thus $T_1F^2 = \mathbf{I}_{[u\in C]}(1+2\Lambda(C))$ *and* $T_2F^2 = 2\cdot\mathbf{I}_{[\{u_1,u_2\}\in C]}$.
Further

$$\begin{aligned}\langle T_1 p, T_1F^2\rangle &= (2\Lambda(C)+1)\left(\int_C \varrho_1(y;\mu)\Lambda(\mathrm{d}y) - \Lambda(C)\right)\\ \langle T_2 p, T_2F^2\rangle &= 2\int_{C\times C}\varrho_2(y_1,y_2;\mu)\Lambda(\mathrm{d}(y_1,y_2)) - 4\Lambda(C)\int_c \varrho_1(y;\mu)\Lambda(\mathrm{d}y) + 2\Lambda^2(C).\end{aligned}$$

Since $F(\eta)$ *has Poisson distribution with parameter* $\Lambda(C)$ *we have*

$$\mathbb{E}F^2(\eta) = \Lambda(C) + \Lambda^2(C).$$

A combination of these formulas gives us

$$\mathbb{E}F^2(\mu) = \int_{C\times C}\varrho_2(y_1,y_2;\mu)\Lambda(\mathrm{d}(y_1,y_2)) + \int_C \varrho_1(y;\mu)\Lambda(\mathrm{d}y).$$

Let μ *be a Strauss point process on* $E\subset\mathbb{R}^d$ *bounded, parameters* $\beta>0,\ 0\le\gamma\le 1,\ r>0$ *with density*

$$p(\mathbf{y}) = \alpha\beta^{n(\mathbf{y})}\gamma^{s(\mathbf{y})},\quad s(\mathbf{y}) = \sum_{(y,z)\in\mathbf{y}^2_{\neq}}\mathbf{I}_{[||z-y||\le r]},$$

w.r.t. a Poisson point process with Lebesgue intensity measure Λ, α *is a normalising constant. Here the conditional intensity is*

$$\lambda^*(u;\mathbf{y}) = \beta\gamma^{t(u,\mathbf{y})},\quad t(u,\mathbf{y}) = \sum_{y\in\mathbf{y}}\mathbf{I}_{[||u-y||\le r]}.$$

For density function p *it holds*

$$\begin{aligned}\int p^2(\mathbf{y})\Lambda(\mathrm{d}\mathbf{y}) &= \int \alpha^2\beta^{2n(\mathbf{y})}\gamma^{2s(\mathbf{y})}\Lambda(\mathrm{d}\mathbf{y}) \le \sum_{n=0}^{\infty} e^{-\Lambda(E)}\frac{\Lambda^n(E)}{n!}\alpha^2\beta^{2n}\gamma\\ &= \alpha^2\gamma\sum_{n=0}^{\infty}e^{-\Lambda(E)}\frac{(\Lambda(E)\beta^2)^n}{n!} < \infty,\end{aligned}$$

and thus $p\in L^2(\mathbb{P}_\eta)$. *For the Strauss process*

$$\mathbb{E}\mu(C) = \beta\int_C \mathbb{E}[\gamma^{t(y,\mu)}]\mathrm{d}y,$$

$$\mathbb{E}\mu^2(C) = \beta \int_C \mathbb{E}[\gamma^{t(y,\mu)}]\mathrm{d}y +$$

$$+\beta \int_C \int_C \mathbb{E}[\beta\gamma^{s(\mu\cup\{y_1,y_2\})-s(x)} - \gamma^{t(y_1,\mu)} - \gamma^{t(y_2,\mu)}]\Lambda(\mathrm{d}(y_1,y_2)).$$

3.2.1 Moments of $U-$statistics

In this section we derive general formulas for computing the moments of $U-$statistics. Note that moments of functionals of point processes could be studied also by means of Georgii–Nguyen–Zessin formula, cf.[7], which for processes with Papangelou conditional intensity λ^* can be expressed in its simplest form as

$$\mathbb{E}\left[\sum_{y\in\mu} h(y,\mu\setminus\{y\})\right] = \mathbb{E}\left[\int_E h(y,\mu)\lambda^*(y;\mu)\mathrm{d}y\right], \tag{3.11}$$

for any nonnegative measurable function h on $E\times\mathcal{N}$. We developed an alternative approach directed to $U-$statistics and their application in stochastic geometry.

Theorem 3.5 *Let $F\in L^2(\mathbb{P}_\eta)$ be a $U-$statistic of order k driven by f, $p\in L^2(\mathbb{P}_\eta)$, it holds*

$$\mathbb{E}F(\mu) = \int_{E^k} f(y_1,\ldots,y_k)\varrho_k(y_1,\ldots,y_k;\mu)\Lambda(\mathrm{d}(y_1,\ldots,y_k)).$$

Proof. From (3.8), (3.9) and (3.10) we have

$$\begin{aligned}
\mathbb{E}F(\mu) &= \sum_{n=0}^{k}\frac{1}{n!}\langle T_nF, T_np\rangle_n \\
&= \sum_{n=0}^{k}\frac{1}{n!}\int_{E^n}\frac{k!}{(k-n)!}\int_{E^{k-n}} f(u_1,\ldots,u_n,y_1,\ldots,y_{k-n})\Lambda(\mathrm{d}(y_1,\ldots,y_{k-n})) \\
&\quad\times\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\varrho_j(y_1,\ldots,y_j;\mu)\Lambda(\mathrm{d}(u_1,\ldots,u_j)) \\
&= \sum_{n=0}^{k}\sum_{j=0}^{n}\binom{k}{n}\binom{n}{j}(-1)^{n-j}\int_{E^k} f(u_1,\ldots,u_n,y_1,\ldots,y_{k-n}) \\
&\quad\times\varrho_j(y_1,\ldots,y_j;\mu)\Lambda(\mathrm{d}(u_1,\ldots,u_j,y_1,\ldots,y_{k-n}))
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{k}\sum_{n=j}^{k}\binom{k}{n}\binom{n}{j}(-1)^{n-j}\int_{E^k} f(u_1,\dots,u_n,y_1,\dots,y_{k-n})\\
&\quad\times\varrho_j(y_1,\dots,y_j;\mu)\Lambda(\mathrm{d}(u_1,\dots,u_j,y_1,\dots,y_{k-n})).\\
&= \sum_{j=0}^{k}\sum_{n=j}^{k}\binom{k}{n}\binom{n}{j}(-1)^{n-j}\int_{E^k} f(y_1,\dots,y_k)\varrho_j(y_1,\dots,y_j;\mu)\Lambda(\mathrm{d}(y_1,\dots,y_k)).
\end{aligned}
$$

Now from "symmetry" of variables in the integral and the equality

$$
\sum_{n=j}^{k}(-1)^{n-j}\binom{k}{n}\binom{n}{j}=0,\qquad j<k,
$$

see [20], p.39, identity 11, we obtain the only nonzero term for $j=k$, which is

$$
\int_{E^k} f(y_1,\dots,y_k)\varrho_k(y_1,\dots,y_k;\mu)\Lambda(\mathrm{d}(y_1,\dots,y_k)).
$$

□

Lemma 3.6 *Let F be a $U-$statistics of order k driven by f and G be a $U-$statistic of order l driven by g, $l\geq k$, $F,G\in L^2(\mathbb{P}_\eta)$. Then*

$$
FG(\mu)=\sum_{j=0}^{k}\binom{k}{j}\frac{l!}{(l-k+j)!}\sum_{(y_1,\dots,y_{l+j})\in\mu_{\neq}^{l+j}} f(y_1,\dots,y_k)g(y_1,\dots,y_{k-j},y_{k+1},\dots,y_{l+j}). \tag{3.12}
$$

Especially

$$
F^2(\mu)=\sum_{j=0}^{k}\binom{k}{j}\frac{k!}{j!}\sum_{(y_1,\dots,y_{k+j})\in\mu_{\neq}^{k+j}} f(y_1,\dots,y_k)f(y_1,\dots,y_{k-j},y_{k+1},\dots,y_{k+j}). \tag{3.13}
$$

Proof. Since $FG(\mu)=\left(\sum_{(y_1,\dots,y_k)\in\mu_{\neq}^{k}} f(y_1,\dots,y_k)\right)\left(\sum_{(z_1,\dots,z_l)\in\mu_{\neq}^{l}} g(z_1,\dots,z_l)\right)$ we can consider product of F and G as a sum whose members are double sums of product of functions f and g with $k-j$, $j=0,\dots,k$ equal variables. Each of this double sum can be rewritten as a single sum over $(l+j)$ tuples:

$$
\sum_{(y_1,\dots,y_{l+j})\in\mu_{\neq}^{l+j}} f(y_1,\dots,y_k)g(y_1,\dots,y_{k-j},y_{k+1},\dots,y_{l+j}),
$$

where for $j=k$ we understand $g(y_1,\dots,y_{k-j},y_{k+1},\dots,y_{l+j})=g(y_{k+1},\dots,y_{k+l})$. We have $\binom{k}{k-j}=\binom{k}{j}$ options how to choose these $k-j$ variables. Their order in the first factor is given through the argument of sum (sum is over all $(k+j)-$tuples so it determines different order of variables) but in the second factor we have $\frac{l!}{(l-k+j)!}$ options to locate them. The choice $g=f$ and $k=l$ gives us a formula for the second power of F. □

Theorem 3.7 *Let F, G be $U-$statistics of order k, l driven by f, g respectively, $l \geq k$. For $j = 0, 1, \ldots, k$ denote*

$$h(y_1, \ldots, y_{l+j}) = f(y_1, \ldots, y_k) g(y_1, \ldots, y_{k-j}, y_{k+1}, \ldots, y_{l+j}).$$

Then if $FG \in L^2(\mathbb{P}_\eta)$, $p \in L^2(\mathbb{P}_\eta)$ we have

$$\begin{aligned}\mathbb{E}FG(\mu) &= \\ &= \sum_{j=0}^{k} \binom{k}{j} \frac{l!}{(l-k+j)!} \int_{E^{l+j}} h(y_1, \ldots, y_{l+j}) \varrho_{l+j}(y_1, \ldots, y_{l+j}; \mu) \Lambda(\mathrm{d}(y_1, \ldots, y_{l+j})).\end{aligned}$$

Especially

$$\mathbb{E}F^2(\mu) = \sum_{j=0}^{k} \binom{k}{j} \frac{k!}{j!} \int_{E^{k+j}} h(y_1, \ldots, y_{k+j}) \varrho_{k+j}(y_1, \ldots, y_{k+j}; \mu) \Lambda(\mathrm{d}(y_1, \ldots, y_{k+j})). \tag{3.14}$$

Proof. Consider the symmetrization of the function $h : E^{l+j} \longrightarrow \mathbb{R}$ given by

$$\overline{h}(y_1, \ldots, y_{l+j}) = \frac{1}{(l+j)!} \sum_{\pi \in \Pi_{l+j}} h(y_{\pi(1)}, \ldots, y_{\pi(l+j)}),$$

where Π_{l+j} denotes all permutations of the set $\{1, \ldots, l+j\}$. Then the functional

$$H_{l+j}(\mu) = \sum_{(y_1, \ldots, y_{l+j}) \in \mu^{l+j}_{\neq}} \overline{h}(y_1, \ldots, y_{l+j}) \tag{3.15}$$

is a $U-$statistic of order $l+j$ and according to Lemma 3.6 and Theorem 3.5 we have $\mathbb{E}FG(\mu) = \sum_{j=0}^{k} \binom{k}{j} \frac{l!}{(l-k+j)!} \mathbb{E}H_{l+j}(\mu)$, since

$$\begin{aligned}&\mathbb{E}H_{l+j}(\mu) \\ &\quad = \int_{E^{l+j}} \overline{h}(y_1, \ldots, y_{l+j}) \varrho_{l+j}(y_1, \ldots, y_{l+j}; \mu) \Lambda(\mathrm{d}(y_1, \ldots, y_{l+j})) \\ &\quad = \int_{E^{l+j}} \frac{1}{(l+j)!} \sum_{\pi \in \Pi_{l+j}} h(y_{\pi(1)}, \ldots, y_{\pi(l+j)}) \varrho_{l+j}(y_1, \ldots, y_{l+j}; \mu) \Lambda(\mathrm{d}(y_1, \ldots, y_{l+j})) \\ &\quad = \frac{1}{(l+j)!} \sum_{\pi \in \Pi_{l+j}} \int_{E^{l+j}} h(y_{\pi(1)}, \ldots, y_{\pi(l+j)}) \varrho_{l+j}(y_1, \ldots, y_{l+j}; \mu) \Lambda(\mathrm{d}(y_1, \ldots, y_{l+j})) \\ &\quad = \int_{E^{l+j}} h(y_1, \ldots, y_{l+j}) \varrho_{l+j}(y_1, \ldots, y_{l+j}; \mu) \Lambda(\mathrm{d}(y_1, \ldots, y_{l+j})).\end{aligned}$$

The second equality is a special case of the first one.

□

Using techniques from Lemma 3.6, Theorems 3.5 and 3.7 we can generalize Theorem 3.4.

Theorem 3.8 *Let $\prod_{i=1}^m F_i \in L^2(\mathbb{P}_\eta), m \in \mathbb{N}$, $p \in L^2(\mathbb{P}_\eta)$, where F_i are $U-$statistics of orders k_i, driven by nonnegative functions f_i, $i = 1, \dots, m$. Denote $q = \sum_{i=1}^m k_i$, then*

$$\mathbb{E}\left[\prod_{i=1}^m F_i(\mu)\right] = \mathbb{E}\left[\prod_{i=1}^m F_i(\eta)\right] + \sum_{n=1}^q \frac{1}{n!}\left\langle T_n \prod_{i=1}^m F_i, T_n p\right\rangle_n. \tag{3.16}$$

Assume that $k_1 \geq k_2 \geq \dots \geq k_m$. Then furthermore the product $\prod_{i=1}^m F_i(\mu)$ is a finite linear combination of $U-$statistics with orders $k_1, k_1 + 1, \dots, q$.

Proof. In detail we will analyze the proof for $m = 2, 3$. For $m > 3$ the procedure is similar and we will just show how to express the product in a proper way.

Consider $U-$statistics F, G of orders k, l, driven by f, g, respectively, $k \leq l$. Denote

$$\overline{FG}(\mu) = \sum_{j=0}^k \binom{k}{j} \frac{l!}{(l-k+j)!} H_{l+j}(\mu),$$

where $H_{l+j} = \sum_{(y_1,\dots,y_{l+j}) \in \mu^{l+j}_{\neq}} \bar{h}(y_1, \dots, y_{l+j})$, $\bar{h} \in L^1_s(\Lambda^{l+j})$, is a $U-$statistic from (3.15). One can easily see that $\overline{FG} = FG$. It follows from symmetry of the functions f and g, respectively, and from the fact, that FG is a sum over all configurations of points $\{x_1, \dots, x_{l+j}\}$. Using linearity of T_n, (3.8) and Theorem 3.4 with $m = 1$

$$\begin{aligned}
\sum_{n=0}^{l+k} \frac{1}{n!}\langle T_n FG, T_n p\rangle_n &= \sum_{n=0}^{l+k} \frac{1}{n!}\left\langle T_n \sum_{j=0}^k \binom{k}{j} \frac{l!}{(l-k+j)!} H_{l+j}, T_n p\right\rangle_n \\
&= \sum_{n=0}^{l+k} \frac{1}{n!} \sum_{j=0}^k \binom{k}{j} \frac{l!}{(l-k+j)!} \langle T_n H_{l+j}, T_n p\rangle_n \\
&= \sum_{n=0}^{l+k} \sum_{j=0}^k \frac{1}{n!} \binom{k}{j} \frac{l!}{(l-k+j)!} \langle T_n H_{l+j}, T_n p\rangle_n \\
&= \sum_{j=0}^k \binom{k}{j} \frac{l!}{(l-k+j)!} \sum_{n=0}^{l+k} \frac{1}{n!} \langle T_n H_{l+j}, T_n p\rangle_n \\
&= \sum_{j=0}^k \binom{k}{j} \frac{l!}{(l-k+j)!} \sum_{n=0}^{l+j} \frac{1}{n!} \langle T_n H_{l+j}, T_n p\rangle_n \\
&= \sum_{j=0}^k \binom{k}{j} \frac{l!}{(l-k+j)!} \mathbb{E} H_{l+j}(\mu) \\
&= \mathbb{E} \sum_{j=0}^k \binom{k}{j} \frac{l!}{(l-k+j)!} H_{l+j}(\mu) = \mathbb{E} FG(\mu).
\end{aligned}$$

Here we use the fact that H_{l+j} is $U-$statistic of order $l+j$ and $T_n H_{l+j} = 0$ for $n > l+j$ and the statement for $m = 2$ holds.
For $m = 3$ and F_i driven by $f_i, i = 1, 2, 3$ take F_1F_2 and consider $(F_1F_2) \cdot F_3$. We have

$$(F_1F_2)F_3(\mu) = \left(\sum_{j_2=0}^{k_2} \binom{k_2}{j_2} \frac{k_1!}{(k_1 - k_2 + j_2)!} H_{k_1+j_2}(\mu)\right) \sum_{(y_1,\ldots,y_{k_3})\in\mu_{\neq}^{k_3}} f_3(y_1, \ldots, y_{k_3})$$

$$= \sum_{j_2=0}^{k_2} \binom{k_2}{j_2} \frac{k_1!}{(k_1 - k_2 + j_2)!} \left(H_{k_1+j_2}(\mu) \sum_{(y_1,\ldots,y_{k_3})\in\mu_{\neq}^{k_3}} f_3(y_1, \ldots, y_{k_3})\right), \tag{3.17}$$

where $H_{k_1+j_2} = \sum_{(y_1,\ldots,y_{k_1+j_2})\in\mu_{\neq}^{k_1+j_2}} \bar{h}(y_1, \ldots, y_{k_1+j_2})$, $\bar{h} \in L_s^1(\Lambda^{k_1+j_2})$ is $U-$statistic of order $k_1 + j_2 \geq k_3$ and thus from Lemma 3.6 the last expression in brackets is a product of two $U-$statistics and is equal to

$$\sum_{j_3=0}^{k_3} \binom{k_3}{j_3} \frac{(k_1 + j_2)!}{(k_1 + j_2 + j_3 - k_3)!} \sum_{(y_1,\ldots,y_{k_1+j_2+j_3})\in\mu_{\neq}^{k_1+j_2+j_3}} h_{k_1+j_2+j_3}(y_1, \ldots, y_{k_1+j_2+j_3}),$$

where

$$h_{k_1+j_2+j_3}(y_1, \ldots, y_{k_1+j_2+j_3}) = f_3(y_1, \ldots, y_{k_3})\bar{h}_{k_1+j_2}(y_1, \ldots, y_{k_3-j_3}, y_{k_3+1}, \ldots, y_{k_1+j_2+j_3}),$$

for a function $h_{k_1+j_2+j_3} : \mathbb{R}^{k_1+j_2+j_3} \longrightarrow [0, \infty)$ derived from f_1, f_2 and f_3. Taking the symmetrization of the functions $h_{k_1+j_2+j_3}$ and performing the same calculation to the expectation of (3.17) with $U-$statistic $H_{k_1+j_2+j_3}$ driven by $h_{k_1+j_2+j_3}$ as for $m = 2$ we get the proposition for $m = 3$.
For general $m \in \mathbb{N}$ consider that the scheme holds for $m - 1$. Then

$$\left(\prod_{i=1}^{m-1} F_i\right) \cdot F_m(\mu) =$$

$$= \sum_{j_2,\ldots,j_m} A(j_2 : j_m) \sum_{(y_1,\ldots,y_{k_1+\sum_{i=2}^m j_i})\in\mu_{\neq}^{k_1+\sum_{i=2}^m j_i}} h_{k_1+\sum_{i=2}^m j_i}(y_1, \ldots, y_{k_1+\sum_{i=2}^m j_i}),$$

where $j_i = 0, \ldots, k_i$, $h_{k_1+\sum_{i=2}^m j_i} : \mathbb{R}^{k_1+\sum_{i=2}^m j_i} \longrightarrow [0, \infty)$ are nonnegative (but not necessarily symmetric) functions

$$h_{k_1+\sum_{i=2}^m j_i}(y_1, \ldots, y_{k_1+\sum_{i=2}^m j_i})$$
$$= f_m(y_1, \ldots, y_{k_m})\bar{h}_{k_1+\sum_{i=2}^{m-1} j_i}(y_1, \ldots, y_{k_m-j_m}, y_{k_m+1}, \ldots, y_{k_1+\sum_{i=2}^m j_i})$$

and

$$A(j_2 : j_m) = \tag{3.18}$$
$$= \binom{k_2}{j_2} \ldots \binom{k_m}{j_m} \frac{k_1!(k_1 + j_2)! \ldots (k_1 + \sum_{i=2}^{m-1} j_i)!}{(k_1 + j_2 - k_2)!(k_1 + j_2 + j_3 - k_3)! \ldots (k_1 + \sum_{i=2}^m j_i - k_m)!}. \tag{3.19}$$

Again taking the symmetrization of $h_{k_1+\sum_{i=2}^m j_i}$ we can compute the expectation. The L^2 integrability of sub-products $\prod_{i=1}^j$, $j < m$ can be shown as in the proof of Theorem 3.10. □

A more condensed form of functions $h_{k_1+\sum j_i}$ can be derived by using techniques in [31]. Consider a set $[q] = \{1, \ldots, q\}$, $q \in \mathbb{N}$ and denote Π_q the set of all partitions $\{J_i\}$, where J_i are disjoint blocks and $\bigcup J_i = [q]$. $|J_i|$ denotes the cardinality of a block.

Definition 3.3 *Define* $q = k_1 + \cdots + k_l$ *and*

$$J_i = \{j : k_1 + \cdots + k_{i-1} < j \leq k_1 + \cdots + k_i\},\ i = 1, \ldots, m.$$

Let $\pi = \{J_i,\ 1 \leq i \leq m\}$ *be a partition of* $[q]$*, we define* $\Pi_{k_1,\ldots,k_m} \subset \Pi_q$ *the set of all partitions* $\sigma \in \Pi_q$ *such that* $|J \cap J'| \leq 1$ *for all* $J \in \pi$ *and all* $J' \in \sigma$*.*

Let $f_i : E^{k_i} \longrightarrow \mathbb{R},\ k_i \in \mathbb{N},\ i = 1, \ldots, m$. The tensor product $\otimes_{j=1}^m f_j : E^q \longrightarrow \mathbb{R}$ is defined as

$$\begin{aligned}\big(\otimes_{j=1}^m f_j\big)&(y_1, \ldots, y_{k_1}, y_{k_1+1}, \ldots, y_q)\\ &= f_1(y_1, \ldots, y_{k_1}) f_2(y_{k_1+1}, \ldots, y_{k_1+k_2}) \ldots f_m(y_{k_1+\cdots+k_{m-1}+1}, \ldots, y_q). \qquad (3.20)\end{aligned}$$

Definition 3.4 *Let* $f_i : E^{k_i} \longrightarrow \mathbb{R}$, $i = 1, \ldots, m$*. For a partition* $\sigma \in \Pi_{k_1,\ldots,k_m}$ *we define the function* $(\otimes_{j=1}^m f_j)_\sigma : E^{|\sigma|} \to \mathbb{R}$ *by replacing all variables of the tensor product* $\otimes_{j=1}^m f_j$ *that belong to the same block of* σ *by a new common variable,* $|\sigma|$ *is the number of blocks in the partition* σ*.*

Proposition 3.9 *For the* $U-$*statistics* F_i, $i = 1, \ldots, m$ *from Theorem 3.8 it holds*

$$\prod_{i=1}^m F_i(\mu) = \sum_{\sigma \in \Pi_{k_1,\ldots,k_m}} \sum_{(y_1,\ldots,y_{|\sigma|}) \in \mu_{\neq}^{|\sigma|}} (\otimes_{i=1}^m f_i)_\sigma(y_1, \ldots, y_{|\sigma|}). \qquad (3.21)$$

Proof. We show that the sum of coefficients $A(j_2 : j_m)$ from the proof of Theorem 3.8 sets the total number of partitions in $\Pi_{k_1,\ldots,k_m}$, i.e.

$$\sum_{j_2,\ldots,j_m} A(j_2 : j_m) = card \Pi_{k_1,\ldots,k_m}.$$

For $m = 1$ the equality holds since $\Pi_{k_1} = 1$. Consider that it holds for $m - 1$. Let J_m be the $m-$th block of partition π in Definition 3.3. For $0 \leq j_m \leq k_m$ the factor $\binom{k_m}{j_m}$ gives the number of combinations of j_m blocks J of partitions $\sigma \in \Pi_{k_1,\ldots,k_m}$ with $|J| = 1$

(subsets of $|J_m|$) and the factor $\frac{(k_1+\sum_{i=2}^{m-1} j_i)!}{(k_1+\sum_{i=2}^{m} j_i-k_m)!}$ contributes to the number of partitions $\sigma \in \Pi_{k_1,\dots,k_m}$ when the rest $k_m - j_m$ items in J_m participate in blocks with $|J| \geq 2$. □

Combining the Theorem 3.5 and Proposition 3.9 we obtain the following theorem.

Theorem 3.10 *Let $\prod_{i=1}^m F_i \in L^2(\mathbb{P}_\eta)$, $m \in \mathbb{N}$, $p \in L^2(\mathbb{P}_\eta)$, where F_i are $U-$statistics of orders k_i driven by nonnegative functions f_i, respectively, $i = 1, \dots, m$. Then*

$$\mathbb{E}\prod_{i=1}^m F_i(\mu) = \sum_{\sigma\in\Pi_{k_1,\dots,k_m}} \int_{E^{|\sigma|}} (\otimes_{i=1}^m f_i)_\sigma(y_1,\dots,y_{|\sigma|})\varrho_{|\sigma|}(y_1,\dots,y_{|\sigma|};\mu)\Lambda(\mathrm{d}(y_1,\dots,y_{|\sigma|})). \tag{3.22}$$

Proof. Each inner sum in (3.21) is a $U-$statistic of order $|\sigma|$ by symmetrization. To show that it is in $L^2(\mathbb{P}_\eta)$ we square the formula (3.21) with η instead of μ. On the right hand side all terms are nonnegative so the expectation of $\left(\sum_{(y_1,\dots,y_{|\sigma|})\in\eta^{|\sigma|}_{\neq}} (\otimes_{i=1}^m f_i)_\sigma(y_1,\dots,y_{|\sigma|})\right)^2$ is finite for all $\sigma \in \Pi_{k_1,\dots,k_m}$. The assumptions of Theorem 3.5 are fulfilled and (3.22) follows. □

3.2.2 Functionals in logarithmic form

In the previous section we used the relation

$$\mathbb{E}F^m(\mu) = \mathbb{E}[F^m(\eta)p(\eta)],\ m = 1, 2, \dots,$$

where η is a Poisson process and μ a point process with probability density p w.r.t. η. In the following we investigate the functional

$$H_m = \log(F^m p) = m \log F + \log p,\ m = 1, 2, \dots \tag{3.23}$$

under the assumption $H_m(\eta) \in L_m(P_\eta)$. From Jensen inequality we have

$$\log \mathbb{E}F^m(\mu) \geq \mathbb{E}H_m(\eta).$$

According to Theorem 4.3 in [2] $\lambda^*(u;\mathbf{y})$, $\mathbf{y} \in \mathcal{N}$, $u \in E$, is a conditional intensity of a finite point process μ if and only if it can be expressed in the form

$$\lambda^*(u;\mathbf{y}) = \exp\left[V_1(u) + \sum_{y\in\mathbf{y}} V_2(u,y) + \sum_{(y_1,y_2)\subset\mathbf{y}^2_{\neq}} V_3(u,y_1,y_2) + \dots\right], \tag{3.24}$$

where $V_k : E^k \to \mathbb{R} \cup \{-\infty\}$ is called the potential of order k. Then the density is that of a Gibbs process

$$p(\mathbf{y}) = \exp\left[V_0 + \sum_{y\in\mathbf{y}} V_1(y) + \sum_{(y_1,y_2)\subset\mathbf{y}^2_{\neq}} V_2(y_1,y_2) + \dots\right]. \tag{3.25}$$

Consequently

$$\log p(\mathbf{y}) = V_0 + \sum_{y\in\mathbf{y}} V_1(y) + \sum_{(y_1,y_2)\subset\mathbf{y}^2_{\neq}} V_2(y_1,y_2) + \dots$$

is a sum of a constant and $U-$statistics.

Assume that there is only a finite number l of terms $V_1,\dots,V_l$ on the right of (3.24) and further that

$$F(\eta) = \exp\left[\sum_{(y_1,\dots,y_k)\in\eta^k_{\neq}} f(y_1,\dots,y_k)\right]. \tag{3.26}$$

Then $\log F$ is a $U-$statistics of order k. Using (3.2) we can evaluate variance of H_m.

Example 3.2 *Consider again the Strauss process with parameters β,γ and density p as in the Example 3.1, here*

$$V_1(u) = \log\beta,\ V_2(u,v) = \log\gamma\mathbf{I}_{[||u-v||\le r]},$$
$$\lambda^*(u;\mathbf{y}) = \beta\gamma^{t(u,\mathbf{y})},$$
$$\log p(\mathbf{y}) = \log\alpha + n(\mathbf{y})\log\beta + s(\mathbf{y})\log\gamma.$$

We have

$$T_0(\log p) = \mathbb{E}\log p(\eta) = \log\alpha + \log\beta\mathbb{E}n(\eta) + \log\gamma\mathbb{E}s(\eta),$$
$$T_1(\log p) = \log\beta + \log\gamma\mathbb{E}t(y,\eta),$$
$$T_2(\log p) = \log\gamma\,\mathbf{I}_{[||y-z||\le r]}.$$

Then from the formula (3.2)

$$\mathrm{var}\,[\log p(\eta)] = \int_E T_1(\log p)^2\Lambda(\mathrm{d}y) + \frac{1}{2}\int\int_{||y-z||\le r}\Lambda(\mathrm{d}y)\Lambda(\mathrm{d}z).$$

Put $F(\mathbf{y}) = \exp(\mathbf{y}(C)),\ \mathbf{y}\in\mathcal{N},\ C\subset E$. We have

$$T_0(\log F) = \Lambda(C),\quad T_1(\log F) = \mathbf{I}_{[y\in C]},$$
$$\mathrm{var}\,[\log F(\eta)] = \Lambda(C),$$
$$\mathrm{cov}(\log F,\log p) = \Lambda(C)\log\beta + \log\gamma\int_C \mathbb{E}t(y,\eta)\Lambda(\mathrm{d}y).$$

Thus

$$\mathbb{E}H_m(\eta) = m\Lambda(C) + \log\alpha + \log\beta\mathbb{E}n(\eta) + \log\gamma\mathbb{E}s(\eta),$$
$$\mathrm{var}H_m(\eta) = m^2\Lambda(C) + 2m\,\mathrm{cov}(\log F(\eta),\log p(\eta)) + \mathrm{var}\,[\log p(\eta)].$$

Chapter 4

Random union of interacting particles

In this chapter we will consider $E = B \times Z$, where $B \subset \mathbb{R}^2$ or $\mathbb{R}^3$ is bounded and $Z \subset \mathbb{R}^l$, B, Z are Borel sets. Let η be a Poisson point process on E with intensity measure Λ. Let μ be a point process on E having a density $p_{\mathbf{x}}(\mathbf{y})$, $\mathbf{y} \in \mathcal{N}$ w.r.t. η, here $\mathbf{x} \in \mathbb{R}^d$ is a parameter vector. In the sense of Definition 2.9 both μ and η are marked point processes and we assume that each mark defines a particle, i.e. a random closed set. Typically the shape of the particle is prescribed and given $(o, z) \in \mu$ the particle is centered in o and its random size and orientation is determined by the mark. For configuration $\mathbf{y} \in \mathcal{N}$ denote $\mathcal{U}_{\mathbf{y}}$ the union of all particles in $\mathbf{y}$. Assume that the density is in the exponential form

$$p_{\mathbf{x}}(\mathbf{y}) = c_{\mathbf{x}} \exp\{\mathbf{x} \cdot G(\mathbf{y})\} \tag{4.1}$$

where $G(\mathbf{y})$ is a vector of some geometrical statistics of $\mathcal{U}_{\mathbf{y}}$, $\mathbf{x} \cdot G(\mathbf{y})$ is the inner product and $c_{\mathbf{x}}$ a normalising constant.

According to [28]

$$\mathcal{X} \;=\; \{\mathbf{x} \in \mathbb{R}^d : \mathbb{E}[\exp(\mathbf{x} \cdot G(\eta))] < \infty\} \tag{4.2}$$

is the largest set of such $\mathbf{x}$ that the exponential family density (4.1) is well defined.

For a vector of geometrical characteristics $G(\mathbf{y}) = (G_1(\mathbf{y}), \ldots, G_d(\mathbf{y})), d \in \mathbb{N}$, denote

$$D^m_{u_1,\ldots,u_m} G(\mathbf{y}) = (D^m_{u_1,\ldots,u_m} G_1(\mathbf{y}), \ldots, D^m_{u_1,\ldots,u_m} G_d(\mathbf{y}))^T$$

the vector of $m-$th differences, see Definition 3.1.

Theorem 4.1 *For the density $p_{\mathbf{x}}(\mathbf{y})$ in (4.1), $\mathbf{y} \in \mathcal{N}$ the Papangelou conditional intensity λ_m^* of order m of the point process μ is*

$$\lambda_m^*(u_m, \dots, u_1; \mathbf{y}) = e^{\mathbf{x} \cdot Q_m G(\mathbf{y})}, \qquad u_1, \dots, u_m \in E \setminus \mathbf{y}, \tag{4.3}$$

with

$$\begin{aligned} Q_m G(\mathbf{y}) &= D^m_{u_m,\dots,u_1} G(\mathbf{y}) \\ &\quad + \sum_{i_1,\dots,i_{m-1} \in \{1,\dots,m\}} D^{m-1}_{u_{i_{m-1}},\dots,u_{i_1}} G(\mathbf{y}) + \cdots + \sum_{1 \le i \le m} D_{u_i} G(\mathbf{y}), \end{aligned}$$

where the indices in the sums must be different.

Proof. For configuration $\mathbf{y} \in \mathcal{N}$ of the process μ according to the definition we have

$$\lambda_m^*(u_m, \dots, u_1; \mathbf{y}) = \frac{p_{\mathbf{x}}(\mathbf{y} \cup \{u_m, \dots, u_1\})}{p_{\mathbf{x}}(\mathbf{y})} = e^{\mathbf{x} \cdot G(\mathbf{y} \cup \{u_m, \dots, u_1\}) - \mathbf{x} \cdot G(\mathbf{y})}.$$

We need to prove that

$$Q_m G(\mathbf{y}) = G(\mathbf{y} \cup \{u_m, \dots, u_1\}) - G(\mathbf{y}).$$

For $m = 1$ we have

$$\lambda_1^*(y; \mathbf{y}) = \frac{e^{\mathbf{x} \cdot G(\mathbf{y} \cup y)}}{e^{\mathbf{x} \cdot G(\mathbf{y})}} = e^{\mathbf{x} \cdot (G(\mathbf{y} \cup y) - G(\mathbf{y}))} = e^{\mathbf{x} \cdot D^1_y G(\mathbf{y})} = e^{\mathbf{x} \cdot Q_1 G(\mathbf{y})}.$$

Now assume that the proposition holds for $m-1$ and we shall prove it for m. Then

$$\begin{aligned} Q_m G(\mathbf{y}) &= D^1_{u_m} G(\mathbf{y}) \\ &\quad + \sum_{j=1}^{m-1} D^2_{u_m,u_j} G(\mathbf{y}) + \sum_{1 \le i < j \le m-1} D^3_{u_m,u_j,u_i} G(\mathbf{y}) + \cdots + D^m_{u_m,\dots,u_1} G(\mathbf{y}) \\ &\quad + \sum_{j=1}^{m-1} D^1_{u_j} G(\mathbf{y}) + \sum_{1 \le i < j \le m-1} D^2_{u_j,u_i} G(\mathbf{y}) + \cdots + D^{m-1}_{u_{m-1},\dots,u_1} G(\mathbf{y}). \end{aligned}$$

From the assumption the second line is equal to $Q_{m-1}G$ and further

$$\begin{aligned} Q_m G(\mathbf{y}) &= D^1_{u_m} G(\mathbf{y}) \\ &\quad + D^1_{u_m} \left(\sum_{j=1}^{m-1} D^1_{u_j} G(\mathbf{y}) + \sum_{1 \le i < j \le m-1} D^2_{u_j,u_i} G(\mathbf{y}) + \cdots + D^{m-1}_{u_{m-1},\dots,u_1} G(\mathbf{y}) \right) \\ &\quad + G(\mathbf{y} \cup \{u_{m-1}, \dots, u_1\}) - G(\mathbf{y}) \\ &= D^1_{u_m} G(\mathbf{y}) + D^1_{u_m} \big(G(\mathbf{y} \cup \{u_{m-1}, \dots, u_1\}) - G(\mathbf{y}) \big) \\ &\quad + G(\mathbf{y} \cup \{u_{m-1}, \dots, u_1\}) - G(\mathbf{y}) \\ &= D^1_{u_m} G(\mathbf{y}) + G(\mathbf{y} \cup \{u_m, \dots, u_1\}) - G(\mathbf{y} \cup \{u_{m-1}, \dots, u_1\}) - D^1_{u_m} G(\mathbf{y}) \\ &\quad + G(\mathbf{y} \cup \{u_{m-1}, \dots, u_1\}) - G(\mathbf{y}) \end{aligned}$$

$$= G(\mathbf{y} \cup \{u_m, \ldots, u_1\}) - G(\mathbf{y}).$$

□

Recall

$$\varrho_n(u_1, \ldots, u_n) = \mathbb{E}\lambda_n^*(u_1, \ldots, u_n; \mu), \qquad u_1, \ldots, u_n \in E. \tag{4.4}$$

4.1 Interacting discs

Consider a bounded region $B \subset \mathbb{R}^2$ and a finite marked point process μ with points o_i in B and marks $r_i > 0,\ i = 1, \ldots, m$. Each $(o_i, r_i) \in \mu$ defines a circular particle (disc) $b(o_i, r_i)$ with centre o_i and radius r_i. We assume that μ has the probability density

$$p_{\mathbf{x}}(\mathbf{y}) = c_{\mathbf{x}} \exp\{\mathbf{x} \cdot G(\mathbf{y})\}, \qquad \mathbf{y} \in \mathcal{N},\ \mathbf{x} \in \mathbb{R}^d \tag{4.5}$$

w.r.t. a given reference Poisson point process of discs η, i.e. a Poisson point process on $E = B \times [0, \infty)$ with intensity measure

$$\Lambda(\mathrm{d}(o, r)) = \rho(o)\mathrm{d}o\, Q(\mathrm{d}r). \tag{4.6}$$

Here ρ is an intensity function of centres of discs and Q is the distribution of radii. Here $\mathbf{x} \in \mathbb{R}^d$ is an unknown parameter and $c_{\mathbf{x}}$ a normalizing constant.

The model, simulations and maximum likelihood estimation of $\mathbf{x}$ for statistics

$$G_1(\mathbf{y}) = (A(\mathbf{y}), L(\mathbf{y}), N_{cc}(\mathbf{y}), N_h(\mathbf{y})) \tag{4.7}$$

$$G_2(\mathbf{y}) = (A(\mathbf{y}), L(\mathbf{y}), \chi(\mathbf{y})), \tag{4.8}$$

where $A(.)$ is the total area, $L(.)$ the perimeter, N_{cc} the number of connected components, N_h number of holes and $\chi(.)$ Euler–Poincare characteristic, were well described in [26] and [27]. We will discuss this topic in the time developing case in Chapter 5.4.

Note that condition (4.2) is equivalent (see [26]) to

$$\mathcal{X} = \{\mathbf{x} \in \mathbb{R}^d : \int_0^\infty \exp(\pi x_1 r^2 + 2\pi x_2 r) Q(\mathrm{d}r) < \infty\}, \qquad d = 3, 4,$$

$\mathbf{x} = (x_1, \ldots, x_d)$. This condition is satisfied e.g. for Q with a bounded support.

4.2 Interacting line segments in $\mathbb{R}^2$

Let $E = B \times Z$, $B \subset \mathbb{R}^2$ bounded, $|B| > 0$ and $Z = (0, D_0) \times [-\pi/2, \pi/2)$, $D_0 \in \overline{\mathbb{R}}_+$, where $\overline{\mathbb{R}}$ is extended real line. A marked point process μ has points o_i and marks (r_i, ϕ_i),

where o_i are lexicographical minima of segments (later o_i will alternatively be centres of segments) of length r_i and direction ϕ_i. Let us consider a Poisson segment process η which can be identified with the Poisson point process on

$$E = B \times (0, D_0) \times [-\pi/2, \pi/2) \tag{4.9}$$

with intensity measure $\Lambda(\mathrm{d}(o, r, \phi)) = \rho(o)\mathrm{d}o\, D(\mathrm{d}r)\nu(\mathrm{d}\phi)$, where ρ is an intensity function of Poisson process of lexicographical minima of segments, D and ν the distribution of lengths and directions, respectively. The process μ has a density (4.1) w.r.t. η with vector $G(\mathbf{y})$ of geometrical characteristics in the form

$$G(\mathbf{y}) = (L(\mathbf{y}), N(\mathbf{y}), N_{is}(\mathbf{y})), \tag{4.10}$$

where

$L(\mathbf{y})$... total length of the union of segments configuration $\mathbf{y}$

$N(\mathbf{y})$... the number of intersections

$N_{is}(\mathbf{y})$... the number of isolated segments.

Then (4.1) with the vector of statistics given by (4.10) and $\mathbf{x} \in \mathcal{X}$ is a density function.

Analogously define random variables

$L(\eta)$, $L(\mu)$... total length of the segments in η and μ, respectively

$N(\eta)$, $N(\mu)$... the total number of intersections

$N_{is}(\eta)$, $N_{is}(\mu)$... the number of isolated segments.

4.2.1 The parametric space $\mathcal{X}$

Proposition 4.2 *Let*

$$\mathcal{X} = \mathbb{R} \times (-\infty, 0] \times \mathbb{R}. \tag{4.11}$$

Then if length distribution has bounded support, i.e. $D_0 < \infty$, we have

(*i*) $p_{\mathbf{x}} \in L^2(\mathbb{P}_\eta)$, $\mathbf{x} \in \mathcal{X}$,

(*ii*) $L^p N^q N_{is}^r(\cdot) \in L^2(\mathbb{P}_\eta)$, $p, q, r \in \mathbb{N}_0$.

Proof. Due to (2.3)

$$\mathbb{E}[\exp^2(\mathbf{x}\cdot G(\eta))] \;=\; \sum_{n=0}^{\infty} e^{-\Lambda(E)}\frac{\Lambda^n(E)}{n!}\int_E\dots\int_E e^{2\mathbf{x}\cdot G(\{u_1,\dots,u_n\})}\Lambda(\mathrm{d}(u_1,\dots,u_n)).$$

Statistics L, N, N_{is} have upper bounds

$$\begin{aligned} L(\{u_1,\dots,u_n\}) &\le nD_o \\ N(\{u_1,\dots,u_n\}) &\le \binom{n}{2} \le n^2 \\ N_{is}(\{u_1,\dots,u_n\}) &\le n \end{aligned}$$

and thus

$$\mathbb{E}[\exp(2\mathbf{x}\cdot G(\mu))] \;\le\; \sum_{n=0}^{\infty} e^{-\Lambda(E)}\frac{\Lambda^n(E)}{n!}e^{2(x_1nD_o+x_2n^2+x_3n)}.$$

According to D'Alembert ratio criterion this serie converges for $x_1, x_3 \in \mathbb{R}$ and $x_2 \le 0$.

The second part of the proposition will be shown as follows.

$$\begin{aligned} \mathbb{E}(L^pN^qN_{is}^r)^2(\eta) \;&=\; \sum_{n=0}^{\infty} e^{-\Lambda(E)}\frac{\Lambda^n(E)}{n!}\int_E\dots\int_E (L^pN^qN_{is}^r)^2(\{u_1,\dots,u_n\})\Lambda(\mathrm{d}(u_1,\dots,u_n)) \\ &< \sum_{n=0}^{\infty} e^{-\Lambda(E)}\frac{\Lambda^n(E)}{n!}n^{2(p+2q+r)} < \infty. \end{aligned}$$

□

4.2.2 Simulation of the process

The realization of union of interacting particles can by simulated according to the Metropolis–Hastings birth and death algorithm ([28]). Denote $\mathcal{U}_Y$ the random set which is given by the union of interacting segments with the density (4.5) and $\mathcal{U}_y$ the realization of such random set. For segment $v = (o, r, \phi)$ and configuration $\mathbf{y}$ define the Hastings ratio by

$$H_{\mathbf{x}}(\mathbf{y}, v) = \lambda_1^*(v;\mathbf{y})\frac{\int_B \rho(l)\,\mathrm{d}l}{\rho(o)(n(\mathbf{y})+1)}, \tag{4.12}$$

where $n(\mathbf{y})$ is a number of segments in $\mathbf{y}$. If $\mathbf{y}^{(iter)}$ is the state at iteration *iter*, we generate a proposal which is either a "birth" $\mathbf{y}^{(iter)} \cup \{v\}$ of a new segment v with the reference point o, length r and direction ϕ or a "death" $\mathbf{y}^{(iter)} \setminus \{v_i\}$ of an old segment $v_i = (o_i, r_i, \phi_i) \in \mathbf{y}^{(iter)}$. In the case of a birth proposal, o, r, and ϕ are independent, s has

a density proportional to the intensity function $\rho(o)$, r and ϕ follows the distributions $D(r)$ and $\nu(\phi)$, respectively. In the case of a death proposal, v_i is a uniformly randomly selected point from $\mathbf{y}^{(iter)}$, and each of these two proposals may arrive with fixed probability α. Their acceptance depends on the Hastings ratios $H_{\mathbf{x}}(\mathbf{y}^{(iter)}, v)$, $H_{\mathbf{x}}(\mathbf{y}^{(iter)} \setminus \{v_i\}, v_i)$, respectively. The scheme of algorithm runs as follows.

1. Let $\mathbf{y}^{(0)}$ be the empty configuration.
2. Suppose that $\mathbf{y}^{(iter)}$ is a configuration in the $iter$-th iteration.
3. With probability α, we propose adding a segment v and with probability
$$\min(1, H_{\mathbf{x}}(\mathbf{y}^{(iter)}, v))$$
this proposal is accepted,
else we set $\mathbf{y}^{(iter+1)} = \mathbf{y}^{(iter)}$,
4. else we propose deleting a segment y_i and with probability
$$\min(1, 1/H_{\mathbf{x}}(\mathbf{y}^{(iter)} \setminus \{y_i\}, y_i))$$
the proposal is accepted, else we set $\mathbf{y}^{(iter+1)} = \mathbf{y}^{(iter)}$.
5. After a given number of iterations $ITER$, we set $\mathbf{y} = \mathbf{y}^{(ITER)}$.

Figure 4.1 draws realizations of process of interacting line segments with various choices of parameter vector $\mathbf{x}$. The observation window is a square of size 10×10, the distribution of directions is uniform on $[-\pi/2, \pi/2)$ and the length distribution is log-normal distribution with parameters $(-0.5, 0.5)$ restricted to the interval $[0, 15]$. The intensity of lexicographical minima is a constant equal to 1.5.

4.2.3 Moments of characteristics

Statistics $L(\mu)$ and $N(\mu)$ can be expressed as follows:

$$\begin{aligned} L(\mu) &= \sum_{y \in \mu} l(y) \\ N(\mu) &= \frac{1}{2} \sum_{(y_1, y_2) \in \mu^2_{\neq}} \mathbf{I}_{[y_1 \cap y_2 \neq \emptyset]} \end{aligned}$$

and they present $U-$statistics of orders 1 and 2, respectively. On the other hand the statistic N_{is} cannot be expressed as a sum of finite $k-$tuples and it is not $U-$statistic. Using Theorems 3.5 and 3.7 we have following equalities.

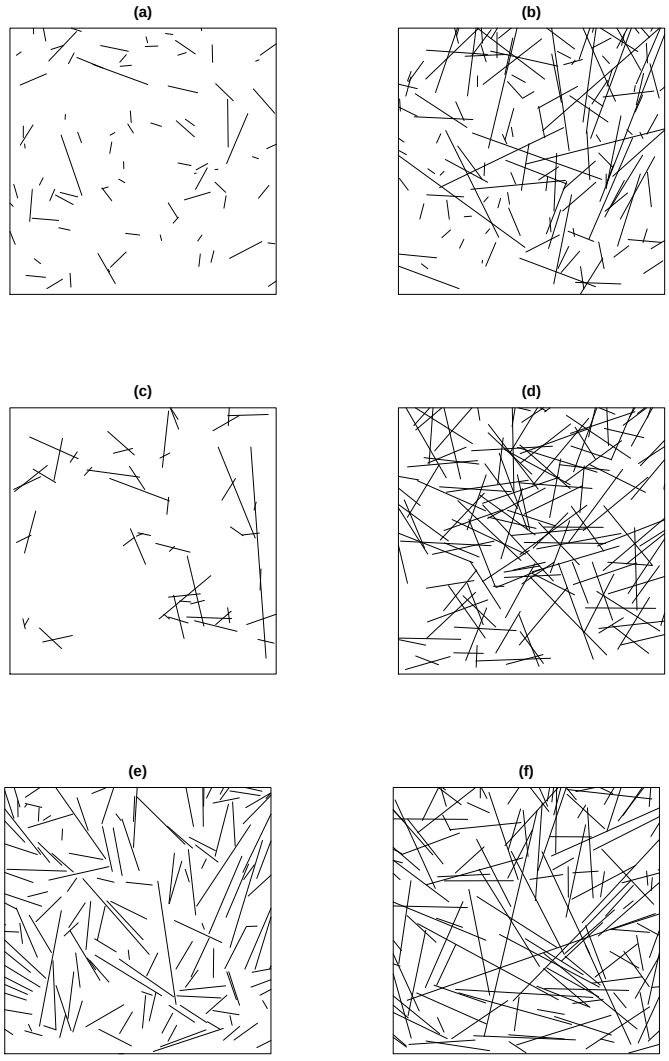

Figure 4.1: Simulated models with interacting line segments with various parameters $\mathbf{x}$, the observation window S is a square of size 10×10. Here D is log-normal distribution with parameters $(-0.5, 0.5)$ restricted to the interval $[0, 15]$, the ditribution of directions is uniform on $[-\pi/2, \pi/2)$ and $\rho = 1.5$ is constant. Number of iterations $ITER = 100\,000$. The parameter vector (a): $\mathbf{x} = (0, -1, 0)$, (b): $\mathbf{x} = (3, 0, 0)$, (c): $\mathbf{x} = (0, 0, -10)$, (d): $\mathbf{x} = (3, 0, -10)$, (e): $\mathbf{x} = (3, 0, 1)$, (f): $\mathbf{x} = (3, -1, -3)$.

Proposition 4.3 *For the density* (4.1) *with* $\mathcal{X}$ *as in* (4.11) *and the condition* $D_0 < \infty$ *we have*

$$\begin{aligned}
\mathbb{E}L(\mu) &= \int_E l(y)\varrho_1(y;\mu)\Lambda(\mathrm{d}y) \\
\mathbb{E}L^2(\mu) &= \int_E l^2(y)\varrho_1(y;\mu)\Lambda(\mathrm{d}y) + \int_{E^2} l(y_1)l(y_2)\varrho_2(y_1,y_2;\mu)\Lambda(\mathrm{d}(y_1,y_2)) \\
\mathbb{E}N(\mu) &= \frac{1}{2}\int_{E^2} \mathbf{I}_{[y_1\cap y_2\neq\emptyset]}\varrho_2(y_1,y_2;\mu)\Lambda(\mathrm{d}(y_1,y_2)) \\
\mathbb{E}N^2(\mu) &= \frac{1}{2}\int_{E^2} \mathbf{I}_{[y_1\cap y_2\neq\emptyset]}\varrho_2(y_1,y_2;\mu)\Lambda(\mathrm{d}(y_1,y_2)) \\
&\quad + \int_{E^3} \mathbf{I}_{[y_1\cap y_2\neq\emptyset]}\mathbf{I}_{[y_1\cap y_3\neq\emptyset]}\varrho_3(y_1,y_2,y_3;\mu)\Lambda(\mathrm{d}(y_1,y_2,y_3)) \\
&\quad + \frac{1}{4}\int_{E^4} \mathbf{I}_{[y_1\cap y_2\neq\emptyset]}\mathbf{I}_{[y_3\cap y_4\neq\emptyset]}\varrho_4(y_1,y_2,y_3,y_4;\mu)\Lambda(\mathrm{d}(y_1,y_2,y_3,y_4)) \\
\mathbb{E}LN(\mu) &= \int_{E^2} l(y_1)\mathbf{I}_{[y_1\cap y_2\neq\emptyset]}\varrho_2(y_1,y_2;\mu)\Lambda(\mathrm{d}(y_1,y_2)) \\
&\quad + \frac{1}{2}\int_{E^3} l(y_1)\mathbf{I}_{[y_2\cap y_3\neq\emptyset]}\varrho_3(y_1,y_2,y_3;\mu)\Lambda(\mathrm{d}(y_1,y_2,y_3)).
\end{aligned}$$

4.3 Interacting surfaces in $\mathbb{R}^3$

In this section we introduce two dimensional circular surfaces in $\mathbb{R}^3$. Symbol $\mathbb{S}^2$ denotes the unit hemisphere in $\mathbb{R}^3$. Let $E = B \times Z,\ B \subset \mathbb{R}^3$ bounded, $|B| > 0$, $Z = (0, R_0] \times \mathbb{S}^2$, $R_0 < \infty$. A marked point process μ has points o_i and marks (r_i, v_i), where o_i are centres of circular surfaces, r_i the radii and v_i normal orientations.

4.3.1 Process of interacting surfaces

Consider a Poisson surface process η which is identified with Poisson point process on E with intensity measure

$$\Lambda(\mathrm{d}(o,v,r)) = \rho(o)\mathrm{d}o\,V(\mathrm{d}v)R(\mathrm{d}r),$$

where $\rho(o)$ is an intensity of centers o, $V(\mathrm{d}v)$ distribution of normal vectors and $R(\mathrm{d}r)$ distribution of radii.

Further denote μ process of interacting surfaces with density (4.1)

$$p_{\mathbf{x}}(\mathbf{y}) = c_{\mathbf{x}}\exp(\mathbf{x}\cdot G(\mathbf{y})) \tag{4.13}$$

with respect to η. Vector G of characteristics of the union $\mathcal{U}_{\mathbf{y}}$ of configuration $\mathbf{y}$ of interacting surfaces is

$$G(\mathbf{y}) = (S(\mathbf{y}), L(\mathbf{y}), N(\mathbf{y})),$$

where

$S(\mathbf{y})$... total area of all surfaces

$L(\mathbf{y})$... total length of (one dimensional) intersection of all pairs of surfaces

$N(\mathbf{y})$... number of intersection points of all triples of surfaces.

Again, as in case of interacting line segments, define random variables

$S(\eta)$, $S(\mu)$... total area of all points in η and μ, respectively

$L(\eta)$, $L(\mu)$... total length of (one dimensional) intersection of all pairs of points

$N(\eta)$, $N(\mu)$... number of intersections of all triples of points.

As in the case of line segments we need to find out the parametric space $\mathcal{X}$.

Proposition 4.4 *Let*

$$\mathcal{X} = \mathbb{R} \times (-\infty, 0] \times (-\infty, 0]. \tag{4.14}$$

We have

(*i*) $p_{\mathbf{x}} \in L^2(\mathbb{P}_\eta)$, $\mathbf{x} \in \mathcal{X}$,

(*ii*) $S^p L^q N^r(\cdot) \in L^2(\mathbb{P}_\eta)$, $p, q, r \in \mathbb{N}_0$.

Proof. For the vector of characteristics $G(\{u_1, \ldots, u_n\})$ of given configuration $\{u_1, \ldots, u_n\}$ we can estimate

$$\begin{aligned} S(\{u_1, \ldots, u_n\}) &\leq n\pi R_0^2 \\ L(\{u_1, \ldots, u_n\}) &\leq \binom{n}{2} 2R_0 \leq 2R_0 n^2 \\ N(\{u_1, \ldots, u_n\}) &\leq \binom{n}{3} \leq n^3 \end{aligned}$$

and

$$\mathbb{E}[\exp(2\mathbf{x} \cdot G(\mu))] \leq \sum_{n=0}^{\infty} e^{-\Lambda(E)} \frac{\Lambda^n(E)}{n!} e^{2x_1 n\pi R_o^2 + x_2 4R_o n^2 + 2x_3 n^3}.$$

Similar to segments process according to D'Alembert criterion this serie converges for $x_1 \in \mathbb{R}$, $x_2 \le 0$, $x_3 \le 0$ and $\mathcal{X} = \mathbb{R} \times (-\infty, 0] \times (-\infty, 0]$. For $\mathbf{x} \in \mathcal{X}$ is $p_{\mathbf{x}} \in L^2(\mathbb{P}_\eta)$. The second part of proof is analogous to Proposition 4.2.

□

4.3.2 Moments of characteristics

For the statistics from the previous Subsection it holds

$$\begin{aligned} S(\mu) &= \sum_{y \in \mu} s(y) \\ L(\mu) &= \frac{1}{2} \sum_{(y_1, y_2) \in \mu^2_{\neq}} l(y_1 \cap y_2) \\ N(\mu) &= \frac{1}{6} \sum_{(y_1, y_2, y_3) \in \mu^3_{\neq}} \mathbf{I}_{[y_1 \cap y_2 \cap y_3 \neq \emptyset]}, \end{aligned} \tag{4.15}$$

where $s(y)$ is an area of a surface y and $l(y_1 \cap y_2)$ a length of intersection of two surfaces y_1 and y_2 and $\mathbf{I}_{[y_1 \cap y_2 \cap y_3 \neq \emptyset]}$ is the indicator of an event that surfaces y_1, y_2, y_3 intersect in a single point. One can see that all of these characteristic are $U-$statistics of orders $1, 2, 3$ respectively.

Recall a functional

$$H_m(\eta) = m \log F(\eta) + \log p(\eta), \ m = 1, 2, \dots$$

in (3.23) having in mind that the process μ with density $p_{\mathbf{x}}$ w.r.t. η is related by means of $\log \mathbb{E} F^m(\mu) \ge \mathbb{E} H_m(\eta)$. Now consider the density (4.13) where

$$\log p_{\mathbf{x}}(\mathbf{y}) = -\log c_{\mathbf{x}} + x_1 S(\mathbf{y}) + x_2 L(\mathbf{y}) + x_3 N(\mathbf{y})$$

which is a finite Gibbsian form, cf. (3.25) with $l = 3$ non-constant terms. For $F(\mathbf{y})$ consider one of the three choices: $F(\mathbf{y}) = e^{S(\mathbf{y})}$, $e^{L(\mathbf{y})}$, $e^{N(\mathbf{y})}$, accordingly we write H^1_m, H^2_m, H^3_m, respectively:

$$\begin{aligned} H^1_m(\eta) &= -\log c_{\mathbf{x}} + (m + x_1) S(\eta) + x_2 L(\eta) + x_3 N(\eta) \\ H^2_m(\eta) &= -\log c_{\mathbf{x}} + x_1 S(\eta) + (m + x_2) L(\eta) + x_3 N(\eta) \\ H^3_m(\eta) &= -\log c_{\mathbf{x}} + x_1 S(\eta) + x_2 L(\eta) + (m + x_3) N(\eta) \end{aligned}$$

In order to study the statistics H^p_m we need to investigate multivariate behavior of a vector of $U-$statistics, i.e. for the process of surfaces in $\mathbb{R}^3$

$$(S(\eta), L(\eta), N(\eta)).$$

Using results from Section 3.2.1 we obtain the following proposition.

Proposition 4.5 *For the density* (4.13) *with parametric space* (4.14) *we have*

$$
\begin{aligned}
\mathbb{E}S(\mu) &= \int_E s(y)\varrho_1(y;\mu)\Lambda(\mathrm{d}y)\\
\mathbb{E}S^2(\mu) &= \int_E s^2(y)\varrho_1(y;\mu)\Lambda(\mathrm{d}y) + \int_{E^2} s(y_1)s(y_2)\varrho_2(y_1,y_2;\mu)\Lambda(\mathrm{d}(y_1,y_2))\\
\mathbb{E}L(\mu) &= \frac{1}{2}\int_{E^2} l(y_1\cap y_2)\varrho_2(y_1,y_2;\mu)\Lambda(\mathrm{d}(y_1,y_2))\\
\mathbb{E}L^2(\mu) &= \frac{1}{2}\int_{E^2} l^2(y_1\cap y_2)\varrho_2(y_1,y_2;\mu)\Lambda(\mathrm{d}(y_1,y_2))\\
&\quad + \int_{E^3} l(y_1\cap y_2)l(y_1\cap y_3)\varrho_3(y_1,y_2,y_3;\mu)\Lambda(\mathrm{d}(y_1,y_2,y_3))\\
&\quad + \frac{1}{4}\int_{E^4} l(y_1\cap y_2)l(y_3\cap y_4)\varrho_4(y_1,\ldots,y_4;\mu)\Lambda(\mathrm{d}(y_1,\ldots,y_4))\\
\mathbb{E}N(\mu) &= \frac{1}{6}\int_{E^3} \mathbf{I}_{[y_1\cap y_2\cap y_3\neq\emptyset]}\varrho_3(y_1,y_2,y_3;\mu)\Lambda(\mathrm{d}(y_1,y_2,y_3))\\
\mathbb{E}N^2(\mu) &= \frac{1}{6}\int_{E^3} \mathbf{I}_{[y_1\cap y_2\cap y_3\neq\emptyset]}\varrho_3(y_1,y_2,y_3;\mu)\Lambda(\mathrm{d}(y_1,y_2,y_3))\\
&\quad + \frac{1}{2}\int_{E^4} \mathbf{I}_{[y_1\cap y_2\cap y_3\neq\emptyset]}\mathbf{I}_{[y_1\cap y_2\cap y_4\neq\emptyset]}\varrho_4(y_1,\ldots,y_4;\mu)\Lambda(\mathrm{d}(y_1,\ldots,y_4))\\
&\quad + \frac{1}{4}\int_{E^5} \mathbf{I}_{[y_1\cap y_2\cap y_3\neq\emptyset]}\mathbf{I}_{[y_1\cap y_4\cap y_5\neq\emptyset]}\varrho_5(y_1,\ldots,y_5;\mu)\Lambda(\mathrm{d}(y_1,\ldots,y_5))\\
&\quad + \frac{1}{36}\int_{E^6} \mathbf{I}_{[y_1\cap y_2\cap y_3\neq\emptyset]}\mathbf{I}_{[y_4\cap y_5\cap y_6\neq\emptyset]}\varrho_6(y_1,\ldots,y_6;\mu)\Lambda(\mathrm{d}(y_1,\ldots,y_6))\\
\mathbb{E}SL(\mu) &= \int_{E^2} s(y_1)l(y_1\cap y_2)\varrho_2(y_1,y_2;\mu)\Lambda(\mathrm{d}(y_1,y_2))\\
&\quad + \frac{1}{2}\int_{E^3} s(y_1)l(y_2\cap y_3)\varrho_3(y_1,y_2,y_3;\mu)\Lambda(\mathrm{d}(y_1,y_2,y_3))\\
\mathbb{E}SN(\mu) &= \frac{1}{2}\int_{E^3} s(y_1)\mathbf{I}_{[y_1\cap y_2\cap y_3\neq\emptyset]}\varrho_3(y_1,y_2,y_3;\mu)\Lambda(\mathrm{d}(y_1,y_2,y_3))\\
&\quad + \frac{1}{6}\int_{E^4} s(y_1)\mathbf{I}_{[y_2\cap y_3\cap y_4\neq\emptyset]}\varrho_4(y_1,\ldots,y_4;\mu)\Lambda(\mathrm{d}(y_1,\ldots,y_4))\\
\mathbb{E}LN(\mu) &= \frac{1}{2}\int_{E^3} l(y_1\cap y_2)\mathbf{I}_{[y_1\cap y_2\cap y_3\neq\emptyset]}\varrho_3(y_1,y_2,y_3;\mu)\Lambda(\mathrm{d}(y_1,y_2,y_3))\\
&\quad + \frac{1}{2}\int_{E^4} l(y_1\cap y_2)\mathbf{I}_{[y_1\cap y_3\cap y_4\neq\emptyset]}\varrho_4(y_1,\ldots,y_4;\mu)\Lambda(\mathrm{d}(y_1,\ldots,y_4))\\
&\quad + \frac{1}{2}\int_{E^5} l(y_1\cap y_2)\mathbf{I}_{[y_3\cap y_4\cap y_5\neq\emptyset]}\varrho_5(y_1,\ldots,y_5;\mu)\Lambda(\mathrm{d}(y_1,\ldots,y_5))
\end{aligned}
$$

Proof. Formulas for mean values follow from Theorem 3.5, formulas for second mo-

ments from Theorem 3.7.

Proposition 4.6 *Under the conditions of Proposition 4.5 we have*

$$\begin{aligned}
[\mathbb{E}LNS(\mu)] \;=\; & \frac{1}{12}\int_{E^6} s(y_1)l(y_2\cap y_3)\mathbf{I}_{[y_4\cap y_5\cap y_6\neq\emptyset]}\varrho_6(y_1,\dots,y_6;\mu)\Lambda(\mathrm{d}(y_1,\dots,y_6))\\
& +\frac{1}{6}\int_{E^5} s(y_1)l(y_1\cap y_2)\mathbf{I}_{[y_3\cap y_4\cap y_5\neq\emptyset]}\varrho_5(y_1,\dots,y_5;\mu)\Lambda(\mathrm{d}(y_1,\dots,y_5))\\
& +\frac{1}{4}\int_{E^5} s(y_1)l(y_2\cap y_3)\mathbf{I}_{[y_1\cap y_4\cap y_5\neq\emptyset]}\varrho_5(y_1,\dots,y_5;\mu)\Lambda(\mathrm{d}(y_1,\dots,y_5))\\
& +\frac{1}{2}\int_{E^5} s(y_1)l(y_2\cap y_3)\mathbf{I}_{[y_2\cap y_4\cap y_5\neq\emptyset]}\varrho_5(y_1,\dots,y_5;\mu)\Lambda(\mathrm{d}(y_1,\dots,y_5))\\
& +\frac{1}{2}\int_{E^4} s(y_1)l(y_1\cap y_2)\mathbf{I}_{[y_1\cap y_3\cap y_4\neq\emptyset]}\varrho_4(y_1,\dots,y_4;\mu)\Lambda(\mathrm{d}(y_1,\dots,y_4))\\
& +\frac{1}{2}\int_{E^4} s(y_1)l(y_1\cap y_2)\mathbf{I}_{[y_2\cap y_3\cap y_4\neq\emptyset]}\varrho_4(y_1,\dots,y_4;\mu)\Lambda(\mathrm{d}(y_1,\dots,y_4))\\
& +\int_{E^4} s(y_1)l(y_2\cap y_3)\mathbf{I}_{[y_1\cap y_2\cap y_4\neq\emptyset]}\varrho_4(y_1,\dots,y_4;\mu)\Lambda(\mathrm{d}(y_1,\dots,y_4))\\
& +\frac{1}{2}\int_{E^4} s(y_1)l(y_2\cap y_3)\mathbf{I}_{[y_2\cap y_3\cap y_4\neq\emptyset]}\varrho_4(y_1,\dots,y_4;\mu)\Lambda(\mathrm{d}(y_1,\dots,y_4))\\
& +\int_{E^3} s(y_1)l(y_1\cap y_2)\mathbf{I}_{[y_1\cap y_2\cap y_3\neq\emptyset]}\varrho_3(y_1,\dots,y_3;\mu)\Lambda(\mathrm{d}(y_1,\dots,y_3))\\
& +\frac{1}{2}\int_{E^3} s(y_1)l(y_2\cap y_3)\mathbf{I}_{[y_1\cap y_2\cap y_3\neq\emptyset]}\varrho_3(y_1,\dots,y_3;\mu)\Lambda(\mathrm{d}(y_1,\dots,y_3)).
\end{aligned}$$

Proof. Using Lemma 3.6 we can compute

$$\begin{aligned}
12LNS(\mu) \;=\; & 12(LN)S = \left(\sum_{(u_1,u_2)\in\mu^2_{\neq}} l(u_1\cap u_2)\sum_{(y_1,y_2,y_3)\in\mu^3_{\neq}}\mathbf{I}_{[y_1\cap y_2\cap y_3\neq\emptyset]}\right)\sum_{z_1\in\mu} s(z_1)\\
= \; & \left(\sum_{(y_1,\dots,y_5)\in\mu^5_{\neq}} l(y_1\cap y_2)\mathbf{I}_{[y_3\cap y_4\cap y_5\neq\emptyset]} + 6\sum_{(y_1,\dots,y_4)\in\mu^4_{\neq}} l(y_1\cap y_2)\mathbf{I}_{[y_1\cap y_3\cap y_4\neq\emptyset]}\right.\\
& \left. +6\sum_{(y_1,y_2,y_3)\in\mu^3_{\neq}} l(y_1\cap y_2)\mathbf{I}_{[y_1\cap y_2\cap y_3\neq\emptyset]}\right)\sum_{z_1\in\mu} s(z_1).
\end{aligned}$$

Now using the same consideration as in proof of Lemma 3.6 we obtain

$$
\begin{aligned}
12LNS(\mu) \;=\; & \sum_{(y_1,\dots,y_6)\in\mu^6_{\neq}} s(y_1)l(y_2\cap y_3)\mathbf{I}_{[y_4\cap y_5\cap y_6\neq\emptyset]} \\
& +2 \sum_{(y_1,\dots,y_5)\in\mu^5_{\neq}} s(y_1)l(y_1\cap y_2)\mathbf{I}_{[y_3\cap y_4\cap y_5\neq\emptyset]} \\
& +3 \sum_{(y_1,\dots,y_5)\in\mu^5_{\neq}} s(y_1)l(y_2\cap y_3)\mathbf{I}_{[y_1\cap y_4\cap y_5\neq\emptyset]} \\
& +6 \sum_{(y_1,\dots,y_4)\in\mu^4_{\neq}} s(y_1)l(y_1\cap y_2)\mathbf{I}_{[y_1\cap y_3\cap y_4\neq\emptyset]} \\
& +6 \sum_{(y_1,\dots,y_4)\in\mu^4_{\neq}} s(y_1)l(y_1\cap y_2)\mathbf{I}_{[y_2\cap y_3\cap y_4\neq\emptyset]} \\
& +12 \sum_{(y_1,\dots,y_4)\in\mu^4_{\neq}} s(y_1)l(y_2\cap y_3)\mathbf{I}_{[y_1\cap y_2\cap y_4\neq\emptyset]} \\
& +6 \sum_{(y_1,\dots,y_5)\in\mu^5_{\neq}} s(y_1)l(y_2\cap y_3)\mathbf{I}_{[y_2\cap y_4\cap y_5\neq\emptyset]} \\
& +6 \sum_{(y_1,\dots,y_4)\in\mu^4_{\neq}} s(y_1)l(y_2\cap y_3)\mathbf{I}_{[y_2\cap y_3\cap y_4\neq\emptyset]} \\
& +12 \sum_{(y_1,y_2,y_3)\in\mu^3_{\neq}} s(y_1)l(y_1\cap y_2)\mathbf{I}_{[y_1\cap y_2\cap y_3\neq\emptyset]} \\
& +6 \sum_{(y_1,y_2,y_3)\in\mu^3_{\neq}} s(y_1)l(y_2\cap y_3)\mathbf{I}_{[y_1\cap y_2\cap y_3\neq\emptyset]}.
\end{aligned}
$$

From Theorem 3.8 the statement follows.

Remark 4.1 *Higher–order moments can be expressed by means of partitions rather than in full detail. The product LNS in the sense of Proposition 3.9 corresponds to*

$$
12LNS(\mu) = \sum_{\sigma\in\Pi_{1,2,3}} \sum_{(y_1,\dots,y_{|\sigma|})} (l\otimes\mathbf{I}\otimes s)_\sigma(y_1,\dots,y_{|\sigma|})
$$

and thus

$$
12[\mathbb{E}LNS(\mu)] = \sum_{\sigma\in\Pi_{1,2,3}} \int_{E^{|\sigma|}} (l\otimes\mathbf{I}\otimes s)_\sigma(y_1,\dots,y_{|\sigma|})\varrho_{|\sigma|}(y_1,\dots,y_{|\sigma|};\mu)\Lambda^{|\sigma|}\mathrm{d}((y_1,\dots,y_{|\sigma|})),
$$

where $l\otimes\mathbf{I}\otimes s$ *involves functions* $s(\cdot)$, $l(\cdot\cap\cdot)$, $\mathbf{I}_{[\cdot\cap\cdot\cap\cdot\neq\emptyset]}$.

4.4 Limit behavior in Poisson and non–Poisson case

Let $E \subset \mathbb{R}^k$ and Λ as in Section 3.1. For $l \geq 1$ and $i = 1, \ldots, l$ let $k_i \in \mathbb{N}$, $f^{(i)} \in L_1(\Lambda^{k_i})$ be symmetric nonnegative functions,

$$F^{(i)}(\eta) = \sum_{(y_1,\ldots,y_{k_i}) \in \eta^{k_i}_{\neq}} f^{(i)}(y_1, \ldots, y_{k_i}).$$

Consider Poisson processes η_a with intensity measures $\Lambda_a = a\Lambda,\ a > 0$. Following [23] $U-$statistics

$$F^{(i)}_a(\eta_a) = \sum_{(y_1,\ldots,y_{k_i}) \in \eta^{k_i}_{a\neq}} f^{(i)}(y_1, \ldots, y_{k_i})$$

are transformed to

$$\hat{F}^{(i)}_a = a^{-(k_i - \frac{1}{2})}(F^{(i)}_a - \mathbb{E}F^{(i)}_a). \tag{4.16}$$

The asymptotic covariances are

$$C_{ij} = \lim_{a\to\infty} cov(\hat{F}^{(i)}_a, \hat{F}^{(j)}_a) = \int T_1F^{(i)}(x)T_1F^{(j)}(x)\Lambda(\mathrm{d}x),\ i, j \in \{1, \ldots, l\}. \tag{4.17}$$

The convergence under the distance between l-dimensional random vectors X, Y

$$d_3(X, Y) = \sup_{g\in\mathcal{H}} |\mathbb{E}g(X) - \mathbb{E}g(Y)|,$$

where $\mathcal{H}$ is the system of functions $h \in C^3(\mathbb{R}^l)$ with

$$\max_{1\leq i_1\leq i_2\leq l} \sup_{x\in\mathbb{R}^l} \Big|\frac{\partial^2 h(x)}{\partial x_{i_1}\partial x_{i_2}}\Big| \leq 1, \quad \max_{1\leq i_1\leq i_2\leq i_3\leq l} \sup_{x\in\mathbb{R}^l} \Big|\frac{\partial^3 h(x)}{\partial x_{i_1}\partial x_{i_2}\partial x_{i_3}}\Big| \leq 1$$

implies convergence in distribution. Based on the multi-dimensional Malliavin-Stein inequality derived in [32] for the distance d_3 of a random vector from a centered Gaussian random vector X with covariance matrix $C = (C_{ij})_{i,j=1,\ldots,l}$, [23] show that under the assumption

$$\int |T_1F^{(i)}|^3 \mathrm{d}\Lambda < \infty,\ i = 1, \ldots, l, \tag{4.18}$$

there exists a constant c such that

$$d_3((\hat{F}^{(1)}_a, \ldots, \hat{F}^{(l)}_a), X) \leq ca^{-\frac{1}{2}},\ a \geq 1. \tag{4.19}$$

Example 4.1 *Consider the Poisson segment process on E introduced in Section 4.2. In (4.17)*

$$C_{11} = \int_E l(s)^2\Lambda(\mathrm{d}s),\ \ C_{22} = \int_E \Lambda(\{s : s \cap t \neq \emptyset\})^2\Lambda(\mathrm{d}t),$$

$$C_{12} = 2\int_E l(y)\Lambda(\{s : s \cap y \neq \emptyset\})\Lambda(\mathrm{d}y).$$

The assumption (4.18) transforms to conditions:

$$\int_E l(s)^3\Lambda(\mathrm{d}s) < \infty,\ \int_E \Lambda(\{s; s \cap y \neq \emptyset\})^3\Lambda(\mathrm{d}y) < \infty.$$

The finiteness of the intensity measure Λ and the assumption $D_0 < \infty$ in (4.9) guarantee that all integrals are finite. Thus in this case for the random vector $(\hat{F}_a^{(1)}, \hat{F}_a^{(2)})$ obtained by transform (4.16) of

$$(L(\eta_a), N(\eta_a))$$

both the central limit theorem and the Berry-Esseen type inequality (4.19) hold.

Next we consider segment processes from Section 4.2 with $D_0 < \infty$ and o_i being centres of segments. Let us focus on the process μ_a with density

$$p_{\mathbf{x},a}(\mathbf{y}) = c_{\mathbf{x},a}e^{x_1L(\mathbf{y})+x_2N(\mathbf{y})}, \qquad \mathbf{y} \in \mathcal{N} \tag{4.20}$$

with respect to the Poisson segment processes η_a with intensity $a\Lambda$. For illustration we will derive second moments of $U-$statistics of segment point processes μ_a with $x_3 = 0$. From Proposition 4.3 we have

$$\begin{aligned}\mathrm{var}L(\mu_a) = a\int_E l^2(y)\varrho_1(y;\mu_a)\Lambda(\mathrm{d}y)\\ + a^2\int_{E^2} l(y_1)l(y_2)\Big(\varrho_2(y_1,y_2;\mu_a) - \varrho_1(y_1;\mu_a)\varrho_1(y_2;\mu_a)\Big)\Lambda(\mathrm{d}(y_1,y_2)),\end{aligned} \tag{4.21}$$

$$\begin{aligned}\mathrm{var}N(\mu_a) &= \frac{1}{2}a^2\int_{E^2}\mathbf{I}_{[y_1\cap y_2\neq\emptyset]}\varrho_2(y_1,y_2;\mu_a)\Lambda(\mathrm{d}(y_1,y_2))\\ &+ a^3\int_{E^3}\mathbf{I}_{[y_1\cap y_2\neq\emptyset]}\mathbf{I}_{[y_1\cap y_3\neq\emptyset]}\varrho_3(y_1,y_2,y_3;\mu_a)\Lambda(\mathrm{d}(y_1,y_2,y_3))\\ &+ \frac{1}{4}a^4\int_{E^4}\mathbf{I}_{[y_1\cap y_2\neq\emptyset]}\mathbf{I}_{[y_3\cap y_4\neq\emptyset]}\times\\ &\qquad\times\Big(\varrho_4(y_1,\ldots,y_4;\mu_a) - \varrho_2(y_1,y_2;\mu_a)\varrho(y_3,y_4;\mu_a)\Big)\Lambda(\mathrm{d}(y_1,\ldots,y_4))\end{aligned} \tag{4.22}$$

and

$$\begin{aligned}\mathrm{cov}(L(\mu_a),N(\mu_a)) &= a^2\int_{E^2} l(y_1)\mathbf{I}_{[y_1\cap y_2\neq\emptyset]}\varrho(y_1,y_2;\mu_a)\Lambda(\mathrm{d}(y_1,y_2))\\ &+ \frac{a^3}{2}\int_{E^3} l(y_1)\mathbf{I}_{[y_2\cap y_3\neq\emptyset]}\Big(\varrho_3(y_1,y_2,y_3;\mu_a) - \varrho_1(y_1;\mu_a)\varrho_2(y_2,y_3;\mu_a))\Big)\Lambda(\mathrm{d}(y_1,y_2,y_3)).\end{aligned} \tag{4.23}$$

We will also investigate

$$\begin{aligned}
&\mathbb{E}(L(\mu_a)-\mathbb{E}L(\mu_a))^3 \qquad (4.24)\\
&\quad=\mathbb{E}L^3(\mu_a)-3\mathbb{E}L(\mu_a)\mathbb{E}L^2(\mu_a)+2(\mathbb{E}L(\mu_a))^3\\
&\quad=a^3\int_{E^3}l(y_1)l(y_2)l(y_3)\Big(\varrho_3(y_1,y_2,y_3;\mu_a)-3\varrho_2(y_1,y_2;\mu_a)\varrho_1(y_3;\mu_a)+\\
&\qquad\qquad+2\varrho_1(y_1;\mu_a)\varrho_1(y_2;\mu_a)\varrho_1(y_3;\mu_a)\Big)\Lambda(\mathrm{d}(y_1,y_2,y_3))\\
&\qquad+3a^2\int_{E^2}l^2(y_1)l(y_2)\Big(\varrho_2(y_1,y_2;\mu_a)-\varrho_1(y_1;\mu_a)\varrho_1(y_2;\mu_a)\Big)\Lambda(\mathrm{d}(y_1,y_2))\\
&\qquad+a\int_E l^3(y)\Lambda(\mathrm{d}y). \qquad (4.25)
\end{aligned}$$

Recall that

$$\begin{aligned}
\varrho_1(y_1;\mu_a)&=\mathbb{E}\lambda_1^*(y_1;\mu_a)=\mathbb{E}e^{x_1(L(\mu_a\cup\{y_1\})-L(\mu_a))+x_2(N(\mu_a\cup\{y_1\})-N(\mu_a))}\\
&=e^{x_1l(y_1)}\mathbb{E}e^{x_2(N(\mu_a\cup\{y_1\})-N(\mu_a))}\\
\varrho_2(y_1,y_2;\mu_a)&=\mathbb{E}\lambda_2^*(y_1,y_2;\mu_a)=\mathbb{E}e^{x_1(L(\mu_a\cup\{y_1,y_2\})-L(\mu_a))+x_2(N(\mu_a\cup\{y_1,y_2\})-N(\mu_a))}\\
&=e^{x_1(l(y_1)+l(y_2))}\mathbb{E}e^{x_2(N(\mu_a\cup\{y_1,y_2\})-N(\mu_a))}\\
\varrho_3(y_1,y_2,y_3;\mu_a)&=\mathbb{E}\lambda_3^*(y_1,y_2,y_3;\mu_a)=\mathbb{E}e^{x_1(L(\mu_a\cup\{y_1,y_2,y_3\})-L(\mu_a))+x_2(N(\mu_a\cup\{y_1,y_2,y_3\})-N(\mu_a))}\\
&=e^{x_1(l(y_1)+l(y_2)+l(y_3))}\mathbb{E}e^{x_2(N(\mu_a\cup\{y_1,y_2,y_3\})-N(\mu_a))}.
\end{aligned}$$

We assume $x_2<0$, then ϱ_i are bounded nonincreasing in variable a for each $y_i,\ i=1,2,3$. Let us define following random variables:

$$\begin{aligned}
X_1(\mu_a)&=N(\mu_a\cup\{y_1\})-N(\mu_a)\\
X_{2,3}(\mu_a)&=N(\mu_a\cup\{y_2,y_3\})-N(\mu_a)\\
X_{1,2,3}(\mu_a)&=N(\mu_a\cup\{y_1,y_2,y_3\})-N(\mu_a).
\end{aligned}$$

We have

$$\begin{aligned}
\varrho_3(y_1,y_2,y_3;\mu_a)&=e^{x_1(l(y_1)+l(y_2)+l(y_3))}\mathbb{E}e^{x_2X_{1,2,3}}\\
\varrho_1(y_1;\mu_a)\varrho_2(y_2,y_3;\mu_a)&=e^{x_1(l(y_1)+l(y_2)+l(y_3))}\mathbb{E}e^{x_2X_1}\mathbb{E}e^{x_2X_{2,3}}.
\end{aligned}$$

The randomness of all variables $X_1(\mu_a), X_{2,3}(\mu_a), X_{1,2,3}(\mu_a)$ is given through the process μ_a and thus they are correlated and the only possibility of the equality

$$\varrho_3(y_1,y_2,y_3;\mu_a)=\varrho_1(y_1;\mu_a)\varrho_2(y_2,y_3;\mu_a)$$

is if x_2 is equal to zero and the process μ_a is Poisson.

From the formulas of $\mathrm{var}L$, $\mathrm{var}N$ and $\mathrm{cov}(L,N)$ we choose the standardized characteristics

$$\begin{aligned}
\hat{L}(\mu_a)&=\frac{1}{a}(L(\mu_a)-\mathbb{E}L(\mu_a)) \qquad (4.26)\\
\hat{N}(\mu_a)&=\frac{1}{a^2}(N(\mu_a)-\mathbb{E}N(\mu_a)). \qquad (4.27)
\end{aligned}$$

We denote leading terms of variances, covariance and the third moment corresponding to (4.21)-(4.25) for standardized characteristics

$$\begin{aligned}
\hat{C}_{11} &= \int_{E^2} l(y_1)l(y_2)\Big(\varrho_2(y_1,y_2;\mu_a)-\varrho_1(y_1;\mu_a)\varrho_1(y_2;\mu_a)\Big)\Lambda(\mathrm{d}(y_1,y_2))\\
\hat{C}_{22} &= \frac{1}{4}\int_{E^4}\mathbf{I}_{[y_1\cap y_2\neq\emptyset]}\mathbf{I}_{[y_3\cap y_4\neq\emptyset]}\times\\
&\qquad\times\Big(\varrho_4(y_1,\dots,y_4;\mu_a)-\varrho_2(y_1,y_2;\mu_a)\varrho(y_3,y_4;\mu_a)\Big)\Lambda(\mathrm{d}(y_1,\dots,y_4))\\
\hat{C}_{12} &= \frac{1}{2}\int_{E^3} l(y_1)\mathbf{I}_{[y_2\cap y_3\neq\emptyset]}\Big(\varrho_3(y_1,y_2,y_3;\mu_a)-\varrho_1(y_1;\mu_a)\varrho_2(y_2,y_3;\mu_a))\Big)\Lambda(\mathrm{d}(y_1,y_2,y_3)).\\
M_3^L &= \int_{E^3} l(y_1)l(y_2)l(y_3)\Big(\varrho_3(y_1,y_2,y_3;\mu_a)-3\varrho_2(y_1,y_2;\mu_a)\varrho_1(y_3;\mu_a)+\\
&\qquad+2\varrho_1(y_1;\mu_a)\varrho_1(y_2;\mu_a)\varrho_1(y_3;\mu_a)\Big)\Lambda(\mathrm{d}(y_1,y_2,y_3)).
\end{aligned}$$

Example 4.2 *Let*

$$E=[0,1]^2\times\{2\}\times\left\{0,\frac{\pi}{2}\right\}, \tag{4.28}$$

$$\Lambda(\mathrm{d}(o,\phi))=\mathrm{d}o\,\frac{1}{2}\,(\delta_0+\delta_{\frac{\pi}{2}})(\mathrm{d}\phi), \tag{4.29}$$

where o denotes the centre of the segment. The space E is constructed such that all orthogonal segments have an intersection. We will compute $\hat{C}_{12}$ and M_3^L. Using the variables $X_1, X_{2,3}, X_{1,2,3}$, Lemma 2.2 and Lemma 3.1 for $y\in E$ we have

$$\begin{aligned}
&\mathbb{E}_\mu e^{x_2X_1}=\mathbb{E}_\eta e^{x_2X_1}p(\eta)=\\
&c_{\mathbf{x},a}e^{-a\Lambda(E)}\sum_{n=0}^{\infty}\frac{a^n}{n!}\int_E\dots\int_E e^{x_2(\#\{y\cap\{u_1,\dots,u_n\}\})}e^{x_1L(\{u_1,\dots,u_n\})}e^{x_2N(\{u_1,\dots,u_n\})}\Lambda(\mathrm{d}(u_1,\dots,u_n)).
\end{aligned} \tag{4.30}$$

The integration does not depend on locations of the fibres, for $u_i=(o_i,\phi_i)$ it holds in (4.30)

$$\Lambda(\mathrm{d}(u_1,\dots,u_n))=\frac{1}{2^n}\sum_{j=0}^{n}\binom{n}{j}\delta_0^{\otimes j}\otimes\delta_{\frac{\pi}{2}}^{\otimes(n-j)}(\mathrm{d}\phi_1,\dots,\mathrm{d}\phi_n), \tag{4.31}$$

here $\otimes$ means the product or a power of measures. If y_1 has an orientation $\frac{\pi}{2}$ then

$$\mathbb{E}_\mu e^{x_2X_1}=c_{\mathbf{x},a}e^{-a}\sum_{n=0}^{\infty}\frac{a^ne^{2nx_1}}{2^n}\sum_{j=0}^{n}\frac{e^{x_2(j+j(n-j))}}{j!(n-j)!}, \tag{4.32}$$

where j represents the number of segments perpendicular to the segment y_1. From the symmetry of the second sum one can see that the expectation does not depend on the orientation of y_1. Define $S(k,z)$, $z=\{z_j\}_{j=0}^{k}$, $k\in\mathbb{N}$, as

$$S(k,z)=\sum_{j=0}^{k}\frac{e^{x_2(j(k-j)+z_j)}}{j!(k-j)!}. \tag{4.33}$$

In the following we write briefly $S(k,z)=S(k,z_j)$. Here if z_j does not depend on j, then z is a constant vector. We have

$$\mathbb{E}_\mu e^{x_2X_1}=c_{\mathbf{x},a}e^{-a}\sum_{n=0}^{\infty}\frac{a^n e^{2nx_1}}{2^n}S(n,j). \tag{4.34}$$

For computing $\mathbb{E}_\mu e^{x_2X_{1,2}}$, $y_1,y_2\in E$, we need to distinguish between two following situations

1. *$y_1\cap y_2\neq\emptyset$. Then*

$$\mathbb{E}_\mu e^{x_2X_{1,2}}=c_{\mathbf{x},a}e^{-a}\sum_{n=0}^{\infty}\frac{a^n e^{2nx_1}}{2^n}S(n,n+1). \tag{4.35}$$

2. *$y_1\cap y_2=\emptyset$. Then*

$$\mathbb{E}_\mu e^{x_2X_{1,2}}=c_{\mathbf{x},a}e^{-a}\sum_{n=0}^{\infty}\frac{a^n e^{2nx_1}}{2^n}S(n,2j). \tag{4.36}$$

Both of these situations occur with probability $\frac{1}{2}$.

Finally, for each triple (the case when some segments overlap has probability zero) of points $y_1,y_2,y_3\in E$ we have two different situations.

1. *All three segments are parallel. Then*

$$\mathbb{E}_\mu e^{x_2X_{1,2,3}}=c_{\mathbf{x},a}e^{-a}\sum_{n=0}^{\infty}\frac{a^n e^{2nx_1}}{2^n}S(n,3j). \tag{4.37}$$

   The second argument in $S(n,3j)$ represents the number of intersections with segments orthogonal to the segments y_1,y_2,y_3. This situation has probability equal to $\frac{1}{4}$.

2. *Two of segments y_1, y_2, y_3 are parallel and the third one is perpendicular. Then*

$$\mathbb{E}_\mu e^{x_2 X_{1,2,3}} = c_{\mathbf{x},a} e^{-a} \sum_{n=0}^{\infty} \frac{a^n e^{2nx_1}}{2^n} S(n, n+j+2), \tag{4.38}$$

where $n+j+2 = n-j+j+j+2$ represents the number of intersections with the segments y_1, y_2, y_3. This has probability equal to $\frac{3}{4}$.

Using Cauchy product of two series we have

$$\begin{aligned}
&\varrho_1(y_1)\varrho_1(y_2)\varrho_1(y_3) = \qquad (4.39)\\
&\quad = c_{\mathbf{x},a}^3 e^{6x_1-3a} \left(\sum_{n=0}^{\infty} \frac{a^n e^{2nx_1}}{2^n} S(n,j) \right)^3 \\
&\quad = c_{\mathbf{x},a}^3 e^{6x_1-3a} \left(\sum_{n=0}^{\infty} \frac{a^n e^{2nx_1}}{2^n} \sum_{k=0}^{n} S(k,j) S(n-k,j) \right) \left(\sum_{n=0}^{\infty} \frac{a^n e^{2nx_1}}{2^n} S(n,j) \right) \\
&\quad = c_{\mathbf{x},a}^3 e^{6x_1-3a} \sum_{n=0}^{\infty} \frac{a^n e^{2nx_1}}{2^n} \sum_{k=0}^{n} \left(\sum_{m=0}^{k} S(m,j) S(k-m,j) \right) S(n-k,j). \qquad (4.40)
\end{aligned}$$

Analogously

$$\begin{aligned}
\varrho_1(y_3)\varrho_2(y_1,y_2) &= \frac{1}{2} c_{\mathbf{x},a}^2 e^{6x_1-2a} \sum_{n=0}^{\infty} \frac{a^n e^{2nx_1}}{2^n} \times \\
&\quad \times \left(\sum_{m=0}^{n} S(m,m+1) S(n-m,j) + S(m,2j) S(n-m,j) \right) \\
\varrho_3(y_1,y_2,y_3) &= \frac{1}{4} c_{\mathbf{x},a} e^{6x_1-a} \sum_{n=0}^{\infty} \frac{a^n e^{2nx_1}}{2^n} \Big(S(n,3j) + 3S(n,n+j+2) \Big). \qquad (4.41)
\end{aligned}$$

In order to compare the terms we need to have the same power of the unknown constant $c_{\mathbf{x},a}$. Therefore we use the equality

$$1 = \mathbb{E}p(\eta) = c_{\mathbf{x},a} e^{-a} \sum_{n=0}^{\infty} \frac{a^n e^{2nx_1}}{2^n} \sum_{j=0}^{n} \frac{e^{x_2 j(n-j)}}{j!(n-j)!} = c_{\mathbf{x},a} e^{-a} \sum_{n=0}^{\infty} \frac{a^n e^{2nx_1}}{2^n} S(n,0)$$

and we can express

$$\begin{aligned}
\varrho_1(y_3)\varrho_2(y_1,y_2) &= \frac{1}{2}c_{\mathbf{x},a}^3 e^{6x_1-3a}\sum_{n=0}^{\infty}\frac{a^n e^{2nx_1}}{2^n}\sum_{k=0}^{n}\Bigg(\sum_{m=0}^{k}S(m,2j)S(k-m,j)\\
&\qquad +S(m,m+1)S(k-m,j)\Bigg)S(n-k,0)\\
\varrho_3(y_1,y_2,y_3) &= \frac{1}{4}c_{\mathbf{x},a}^3 e^{6x_1-3a}\sum_{n=0}^{\infty}\frac{a^n e^{2nx_1}}{2^n}\sum_{k=0}^{n}\sum_{m=0}^{k}\Bigg(S(m,3j)+\\
&\qquad +3\,S(m,m+j+2)\Bigg)S(k-m,0)S(n-k,0). \qquad (4.42)
\end{aligned}$$

For the leading terms then it holds

$$\begin{aligned}
\hat{C}_{12} &= c_{\mathbf{x},a}^2 e^{6x_1-2a}\sum_{n=0}^{\infty}\frac{a^n e^{2nx_1}}{2^n}\sum_{k=0}^{n}\Big(S(k,k+j+2)S(n-k,0)-S(k,k+1)S(n-k,j)\Big)\\
M_3^L &= 8c_{\mathbf{x},a}^3 e^{6x_1-3a}\sum_{n=0}^{\infty}\frac{a^n e^{2nx_1}}{2^n}\sum_{k=0}^{n}\sum_{m=0}^{k}\Bigg(\frac{1}{4}S(m,3j)S(k-m,0)S(n-k,0)\\
&\quad +\frac{3}{4}S(m,m+j+2)S(k-m,0)S(n-k,0)\\
&\quad -\frac{3}{2}S(m,m+1)S(k-m,j)S(n-k,0)\\
&\quad -\frac{3}{2}S(m,2j)S(k-m,j)S(n-k,0)+2S(m,j)S(k-m,j)S(n-k,j)\Bigg).
\end{aligned}$$

Denote

$$\begin{aligned}
b_n &= \sum_{k=0}^{n}\Big(S(k,k+j+2)S(n-k,0)-S(k,k+1)S(n-k,j)\Big),\\
c_n &= \sum_{k=0}^{n}\sum_{m=0}^{k}\Bigg(\frac{1}{4}S(m,3j)S(k-m,0)S(n-k,0)\\
&\quad +\frac{3}{4}S(m,m+j+2)S(k-m,0)S(n-k,0)\\
&\quad -\frac{3}{2}S(m,m+1)S(k-m,j)S(n-k,0)\\
&\quad -\frac{3}{2}S(m,2j)S(k-m,j)S(n-k,0)+2S(m,j)S(k-m,j)S(n-k,j)\Bigg),\\
d_n &= \sum_{k=0}^{n}S(k,0)S(n-k,0),\\
e_n &= \sum_{k=0}^{n}\sum_{m=0}^{k}S(m,0)S(k-m,0)S(n-k,0).
\end{aligned}$$

Evaluating $c_{\mathbf{x},a}$ and rating a variable

$$x(a) = \frac{ae^{2x_1}}{2}$$

we obtain the desired quantities as ratios of two power series:

$$\hat{C}_{12} = e^{6x_1} \frac{\sum_{n=0}^{\infty} b_n x^n(a)}{\sum_{n=0}^{\infty} d_n x^n(a)}, \qquad M_3^L = 8e^{6x_1} \frac{\sum_{n=0}^{\infty} c_n x^n(a)}{\sum_{n=0}^{\infty} e_n x^n(a)},$$

which can be expressed as single power series. Their coefficients can be evaluated numerically. If the limit of $\hat{C}_{12}$ when $a \to \infty$ is finite nonzero, then the standardization (4.26), (4.27) *is proper for a limit theorem, but when moreover the limit of M_3^L is nonzero for some x_1, x_2, then the limit in distribution of $(\hat{L}(\mu_a), \hat{N}(\mu_a))$ for $a \to \infty$ would not be Gaussian.*

Chapter 5

Space–time models

In the next part of the thesis we return to random processes of interacting discs in $\mathbb{R}^2$, see Subsection 4.1. Since the analytical treatment analogous to line segments and circular surfaces is not available because of overlappings, we turn to the analysis based on simulations. Moreover we involve temporal dynamics in the modeling, that means space–time systems are investigated. To this purpose we apply some known algorithms in an original way. Their descriptions follow in Section 5.1.

5.1 Sequential Monte Carlo methods

Sequential Monte Carlo methods are attractive class of simulation algorithms. They serve to draw from the posterior distribution recursively and thus to evaluate long time data. These methods are very flexible, easy to implement and applicable. Among these methods we will use the particle filter (PF) which is described below as Algorithm I and particle marginal Metropolis–Hastings algorithm (PMMH) described in Algorithm II. More details, e.g. convergence theorems, generalisations of the model and applications one can find in [9] and [24].

5.1.1 State space model

Consider a state space model (also known as Hidden Markov model) with

$$X = \{X_t, t \in \mathbb{N}_0\} \tag{5.1}$$

being a Markov process in the state space $\mathbb{R}^d$, having initial distribution with density $p(x_0)$ and transition probability density $p(x_t|x_{t-1})$. The index t is interpreted as time.

Assume that instead of X we observe random variables $\{Y_t, t \in \mathbb{N}\}$ which are conditionally independent given $\{X_t, t \in \mathbb{N}_0\}$ with conditional density $g(\cdot|x)$. The aim is to draw from the posterior distribution $p(x_{0:t}|y_{1:t})$, and to evaluate expectations of a function f_t on $\mathbb{R}^{d(t+1)}$:

$$\mathbb{E}(f_t) = \int f_t(x_{0:t}) p(x_{0:t}|y_{1:t}) \mathrm{d}x_{0:t}.$$

The notation $X_{0:t} = \{X_0, \ldots, X_t\},\ Y_{1:t} = \{Y_1, \ldots, Y_t\}$ for the processes will be used and analogously $x_{0:t} = \{x_0, \ldots, x_t\},\ y_{1:t} = \{y_1, \ldots, y_t\}$ for their realizations.

From the Bayes theorem we have

$$p(x_{0:t}|y_{1:t}) = \frac{p(y_{1:t}|x_{0:t}) p(x_{0:t})}{\int p(y_{1:t}|x_{0:t}) p(x_{0:t}) \mathrm{d}x_{0:t}}.$$

Using this formula and Chapman–Kolmogorov theorem for Markov processes we obtain that the so called filtering distribution $p(x_t|y_{1:t})$ satisfies recursion equations

$$p(x_t|y_{1:t-1}) = \int p(x_t|x_{t-1}) p(x_{t-1}|y_{1:t-1}) \mathrm{d}x_{t-1}, \tag{5.2}$$

$$p(x_t|y_{1:t}) = \frac{p(y_t|x_t) p(x_t|y_{1:t-1})}{\int p(y_t|x_t) p(x_t|y_{1:t-1}) \mathrm{d}x_t},$$

since we have $p(x_{0:t}|y_t, y_{1:t-1}) \propto p(y_t|x_{0:t}) p(x_{0:t}|y_{1:t-1})$ from the conditional independence.

Analytical evaluation of the system (5.2) for large t is hardly possible. Therefore Monte Carlo methods were developed. Among them the importance sampling (see [1]) is the basic tool. Let $q(x_{0:t}|y_{1:t})$ be a proposal distribution such that for the supports it holds that $supp\, p(x_{0:t}|y_{1:t}) \subset supp\, q(x_{0:t}|y_{1:t})$. Then

$$\mathbb{E}(f_t) = \frac{\int f_t(x_{0:t}) w(x_{0:t}) q(x_{0:t}|y_{1:t}) \mathrm{d}x_{0:t}}{\int w(x_{0:t}) q(x_{0:t}|y_{1:t}) \mathrm{d}x_{0:t}},$$

where

$$w(x_{0:t}) = \frac{p(x_{0:t}|y_{1:t})}{q(x_{0:t}|y_{1:t})}$$

is the importance weight. Simulating N independent identically distributed particles $\{X_{0:t}^k,\ i = 1, \ldots, N\}$ according to $q(x_{0:t}|y_{1:t})$, we obtain a Monte Carlo estimate

$$\hat{\mathbb{E}}_N(f_t) = \sum_{i=1}^{N} f_t(X_{0:t}^k) \tilde{w}_t^k,\ t = 1, \ldots, T, \tag{5.3}$$

with normalized importance weights

$$\tilde{w}_t^k = \frac{w(X_{0:t}^k)}{\sum_{j=1}^{N} w(X_{0:t}^k)},\ i = 1, \ldots, N.$$

The convention is that whenever the index k is used, we mean for all $k = 1, \ldots, N$, where N is the number of particles (iid samples). Further $\tilde{w}_t = (\tilde{w}_t^1, \ldots, \tilde{w}_t^N)$ are normalized importance weights at time t, $\mathcal{F}(.|\tilde{w}_t)$ is a discrete probability distribution with atoms proportional to the weights and A_{t-1}^k represents the index of the parent of the particle $X_{1:t}^k$ at time $t-1$ for $t = 2, \ldots, T$.

5.1.2 Particle filter

Unlike classical importance sampling, in sequential case the distribution of weights is more and more skewed with increasing time t. The reason is that in common importance sampling used for generating $X_{0:t}^k$ at each time t we obtain a brand new sample, but in case of sequential method we generate only according to transition kernel and this new simulation is dependent on history of a given particle. That is why particles with higher weights are favored and after a few times we have only one particle with non-zero weight. This can be corrected by additional step. At the end of importance sampling all particles are resampled according to their weights. In the other words we make a new sample from existing particles according to the distribution given by the importance weights. It means that some particles with lower weights can disappear and the other with high weights will be replicated. Note that there are more methods used for resampling.

Algorithm I (PF):

1. Sample $x_0 \sim p(x)$.
2. At time $t = 1$:
   (a) sample $X_1^k \sim q_\theta(.|y_1)$,
   (b) compute and normalize weights

$$\begin{aligned} w_1(X_1^k) &= \frac{p_\theta(X_1^k|x_0)p(y_1|X_1^k)}{q_\theta(X_1^k|y_1)}, \\ \tilde{w}_1^k &= \frac{w_1(X_1^k)}{\sum_{m=1}^N w_1(X_1^m)}. \end{aligned}$$

3. At times $t = 2, \ldots, T$:
   (a) sample $A_{t-1}^k \sim \mathcal{F}(.|\tilde{w}_{t-1})$,
   (b) sample $X_t^k \sim q_\theta(.|y_t, X_{t-1}^{A_{t-1}^k})$ and set $X_{1:t}^k = (X_{1:t-1}^{A_{t-1}^k}, X_t^k)$,

(c) compute and normalize weights

$$w_t(X_{1:t}^k) = \frac{p_\theta(X_t^k|X_{t-1}^{A_{t-1}^k})p(y_t|X_t^k)}{q_\theta(X_t^k|y_t, X_{t-1}^{A_{t-1}^k})}, \tag{5.4}$$

$$\tilde{w}_t^k = \frac{w_t(X_{1:t}^k)}{\sum_{m=1}^N w_t(X_{1:t}^m)}.$$

5.1.3 Particle marginal Metropolis–Hastings algorithm

The previous algorithm is useful in case of the auxiliary (multidimensional) parameter θ known or if we are able to estimate it properly. Specifically in our simulation study in Chapter 5.4 we use maximum likelihood estimator for auxiliary parameters (this method was described in [27]). Using the combination of particle filter and Markov chain Monte Carlo algorithms (e.g. Metropolis–Hasting algorithm or Gibbs sampler plan) we can estimate auxiliary parameter and the process X_t simultaneously. The properties and applications of particle marginal Metropolis–Hastings algorithm one can find in [1].

Algorithm II (PMMH):

1. Initialization: $i = 0$,
   (a) set $\theta(0)$ arbitrarily,
   (b) run a sequential Monte Carlo (SMC) algorithm I targeting $p_{\theta(0)}(x_{1:T}|y_{1:T})$, sample $X_{1:T} \sim \hat{p}_{\theta(0)}(.|y_{1:T})$ and let $\hat{p}_{\theta(0)}(y_{1:T})$ denote marginal likelihood estimate.
2. For iteration $i \geq 1$:
   (a) sample $\theta^* \sim q(.|\theta(i-1))$,
   (b) run a sequential Monte Carlo (SMC) algorithm I targeting $p_{\theta^*}(x_{1:T}|y_{1:T})$,
   (c) sample $X_{1:T}^* \sim \hat{p}_{\theta^*}(.|y_{1:T})$ and let $\hat{p}_{\theta^*}(y_{1:T})$ denote marginal likelihood estimate,
   (d) with probability

$$1 \wedge \frac{\hat{p}_{\theta^*}(y_{1:T})p(\theta^*)}{\hat{p}_{\theta(i-1)}(y_{1:T})p(\theta(i-1))} \frac{q(\theta(i-1)|\theta^*)}{q(\theta^*|\theta(i-1))}$$

   set $\theta(i) = \theta^*$, $X_{1:T}(i) = X_{1:T}^*$ and $\hat{p}_{\theta(i)}(y_{1:T}) = \hat{p}_{\theta^*}(y_{1:T})$, otherwise set $\theta(i) = \theta(i-1)$, $X_{1:T}(i) = X_{1:T}(i-1)$ and $\hat{p}_{\theta(i)}(y_{1:T}) = \hat{p}_{\theta(i-1)}(y_{1:T})$.

The estimation $\hat{p}(y_{1:n})$ of marginal likelihood $p(y_{1:n})$ follows from the description of algorithm. For all $n \leq T$

$$\begin{aligned} p_\theta(x_{1:n}, y_{1:n}) &= p_\theta(x_1) \cdot \prod_{j=2}^{n} p_\theta(x_j|x_{j-1}) \cdot \prod_{j=1}^{n} g_\theta(y_j|x_j) \\ &= p_\theta(x_{1:n-1}, y_{1:n-1}) \cdot p_\theta(x_n|x_{n-1}) \cdot g_\theta(y_n|x_n) \end{aligned}$$

and

$$\begin{aligned} p(y_j|y_{1:j-1}) &= \int p(x_{1:j}, y_j|y_{1:j-1}) \, \mathrm{d}x_{1:j} = \\ &= \int \frac{p(x_{1:j}, y_{1:j})}{p(y_{1:j-1})} \, \mathrm{d}x_{1:j} = \\ &= \int p(x_{1:j}, y_{1:j}) \cdot \frac{p(x_{1:j-1}|y_{1:j-1})}{p(x_{1:j-1}, y_{1:j-1})} \, \mathrm{d}x_{1:j} = \\ &= \int p_\theta(x_j|x_{j-1}) g_\theta(y_j|x_j) p(x_{1:j-1}|y_{1:j-1}) \, \mathrm{d}x_{1:j}, \end{aligned}$$

where

$$p_\theta(x_j|x_{j-1}) g_\theta(y_j|x_j) = w_j(x_{1:j}) q_\theta(x_j|y_j, x_{j-1}) p(x_{1:j-1}|y_{1:j-1}).$$

So finally we have

$$p(y_j|y_{1:j-1}) = \int w_n(x_{1:j}) q_\theta(x_j|y_j, x_{j-1}) p(x_{1:j-1}|y_{1:j-1}) \, \mathrm{d}x_{1:j}.$$

Since particles in time $j = 1$ are obtained from the importance density $q_\theta(x_1|y_1)$ and in time $j \geq 2$ approximately according to $p_\theta(x_{1:j-1}|y_{1:j-1}) q_\theta(x_j|y_j, x_{j-1})$ (see [1], p. 272) it is clear to see the MCMC estimation of $p_\theta(y_j|y_{j-1})$ as

$$\hat{p}_\theta(y_j|y_{j-1}) = \frac{1}{N} \sum_{k=1}^{N} w_j(X_{1:j}^k)$$

An estimate of the marginal likelihood $p_\theta(y_{1:T})$ is given by

$$\hat{p}_\theta(y_{1:T}) = \hat{p}_\theta(y_1) \prod_{t=2}^{T} \hat{p}_\theta(y_t|y_{1:t-1}).$$

5.2 Simulation of space–time model of random union of interacting particles

For the union of interacting particles we developed a space–time generalization. The realization of such union of particles in time t is both spatially dependent on the previous configuration and also temporally dependent on the previous value of parameter x_{t-1}. Since our original motivation was to use particle filter to estimate parameters of such union, the notation of the model is in compliance with Section 5.1. The simulation algorithm is described for the union of interacting discs with statistics given by (4.7), but it can be easily implemented for the other models. This research was published in [40].

Consider a process Y of interacting discs given by density (4.5) and vector of geometrical characteristics (4.7). The generalization of Y to a space-time random set $Y = \{Y_t,\ t = 0, 1, \ldots, T\}$ is based on state equations for the parameter vector. Let the parameter $\mathbf{x} \in \mathbb{R}^4$ develop in discrete time (upper index) as

$$\mathbf{x}^{(t)} = \mathbf{x}^{(t-1)} + \gamma^{(t)},\ t = 1, 2 \ldots, T, \tag{5.5}$$

where $\mathbf{x}^{(0)}$ fixed is given, $\gamma^{(t)}$ are i.i.d. Gaussian $\mathcal{N}(a, \sigma^2 I)$, random vectors with $a \in \mathbb{R}^4, \sigma > 0$.

Our basic assumption for the simulation study in Subsection 5.4.1 is the conditional independence of Y_t given $\mathbf{x}^{(t)}$, c.f. Subsection 5.1.1. However it is possible to involve more space-time dependence in the model within its simulation procedure. This partly heuristic approach is described in this subsection.

The temporal dependence in the random set is defined within its simulation algorithm as follows. We start the simulation of the time evolution of the process Y so that we choose a fixed $\mathbf{x}^{(0)}$ and according to (5.5) we simulate parameter vectors $\mathbf{x}^{(t)}, t = 1, 2, \ldots, T$.

Define Hastings ratio as in (4.12). Using the birth-death Metropolis-Hastings algorithm MCMC, we simulate a realization $\mathbf{y}_0$ of the random set Y_0 which is given by the density (4.5) where $\mathbf{x} = \mathbf{x}^{(0)}$. In this part of the simulation, if $\mathbf{y}_0^{(iter)}$ is the state at iteration $iter$, we generate a proposal which is either a "birth" $\mathbf{y}_0^{(iter)} \cup \{v\}$ of a new disc v with the centre s and radius r or a "death" $\mathbf{y}_0^{(iter)} \setminus \{v_i\}$ of an old disc $v_i = (o_i, r_i) \in \mathbf{y}_0^{(t)}$. In the case of a birth proposal, o and r are independent, o has a density proportional to the intensity function $\rho(o)$ and r follows the distribution $Q(r)$ of the reference process. In the case of a death proposal, v_i is a uniformly randomly selected disc from $\mathbf{y}_0^{(iter)}$, and each of these two proposals may arrive with probability α. Their acceptance depends on the Hastings ratios $H_{\mathbf{x}}(\mathbf{y}_0^{(iter)}, v)$ and $H_{\mathbf{x}}(\mathbf{y}_0^{(iter)} \setminus \{v_i\}, v_i)$, respectively (the details of the algorithm are described in [26]).

Then we simulate realizations $\mathbf{y}_t,\ t = 1, 2 \ldots, T$, of the random sets Y_t which have the

density (4.5) with $\mathbf{x} = \mathbf{x}^{(t)}$. The algorithm described in the previous paragraph is used again with a different way of adding a disc. Since Y_t, $t = 0, \ldots, T$, are aimed to be dependent, the choice of a newly added disc v cannot depend on ρ and Q only, but also on the previously simulated configuration $\mathbf{y}_{t-1}$. This dependence is ensured so that the proposal distribution $Prop_t$ at time t is a mixture

$$Prop_t = (1 - \beta) \cdot Prop^{(RP)} + \beta \cdot Prop_{t-1}^{(emp)}, \quad \beta \in (0, 1),$$

where $Prop^{(RP)}$ is a distribution of the reference process and $Prop_{t-1}^{(emp)}$ is the empirical distribution obtained from the configuration $\mathbf{y}_{t-1}$. It means that $(\beta \times 100)\%$ of the added discs are taken from the previous configuration and the remaining discs are simulated as described for Y_0, therefore the time dependence is stronger when β is bigger.

This method evokes a question how to determine the probability α of adding a disc for $t = 1, \ldots, T$. With probability β an added disc is taken from a finite set of discs and it may happen that it is already involved in the configuration $\mathbf{y}_t^{(iter)}$ for the *iter*-th iteration. Thus adding such a disc does not change the configuration and the death-part of the algorithm is dominating.

Proposition 5.1 *The choice of α so that the probabilities of deleting a disc and that of adding a disc are the same is*

$$\alpha^{(iter)} = \frac{1}{2 - (\beta \cdot \frac{n_{useddiscs}^{(iter)}}{n_{\mathbf{y}_{t-1}}})}, \tag{5.6}$$

where on the right side, $n_{useddiscs}^{(iter)}$ is the number of discs from the configuration $\mathbf{y}_{t-1}$ which are already obtained in the configuration $\mathbf{y}_t^{(iter)}$ in the iter-th iteration and $n_{\mathbf{y}_{t-1}}$ is the total number of discs in the configuration $\mathbf{y}_{t-1}$.

Proof. It is obvious that in order to fulfill the condition, α is changing with increasing iterations *iter* depending on how many discs from the configuration $\mathbf{y}_{t-1}$ have been already added. We get $\alpha^{(iter)}$, i.e. the parameter α for the *iter*-th iteration, by solving the equation

$$1 - \alpha^{(iter)} = \alpha^{(iter)}[(1 - \beta) + \beta(1 - \frac{n_{useddiscs}^{(iter)}}{n_{\mathbf{x}_{t-1}}})].$$

Its solution yields (5.6).

□

The scheme of the simulating algorithm is then the following:

1. Choose a fixed $\mathbf{x}^{(0)}$.

2. Parameter vectors $\mathbf{x}^{(t)}, t = 1, 2, \ldots, T$, are simulated according to (5.5).

3. A realization $\mathbf{y}_0$ of Y_0 is simulated.

4. For $t = 1, \ldots, T$, use the following steps:

   (a) Let $\mathbf{y}_t^{(0)}$ be the empty configuration.

   (b) Suppose that $\mathbf{y}_t^{(iter)}$ is a configuration in the $iter$-th iteration.

   (c) Calculate $\alpha^{(iter)}$ according to (5.6).

   (d) With probability $\alpha^{(iter)}$, we propose adding a disc v so that

      i. with probability β, v is chosen from the configuration $\mathbf{y}_{t-1}$, i.e. from the distribution $Prop_{t-1}^{(emp)}$,

      ii. else v is simulated from the distribution $Prop^{(RP)}$

   and with probability

   $$\min(1, H_{\mathbf{x}}(\mathbf{y}_t^{(iter)}, v))$$

   this proposal is accepted, i.e. $\mathbf{y}_t^{(iter+1)} = \mathbf{y}_t^{(iter)} \cup \{v\}$, else we set $\mathbf{y}_t^{(iter+1)} = \mathbf{y}_t^{(iter)}$,

   (e) else we propose deleting a disc y_i and with probability

   $$\min(1, 1/H_{\mathbf{x}}(\mathbf{y}_t^{(iter)} \setminus \{y_i\}, y_i))$$

   the proposal is accepted, i.e. $\mathbf{y}_t^{(iter+1)} = \mathbf{y}_t^{(iter)} \setminus \{y_i\}$, else we set $\mathbf{y}_t^{(iter+1)} = \mathbf{y}_t^{(iter)}$.

   (f) After a given number of simulations $ITER$, we set $\mathbf{y}_t = \mathbf{y}_t^{(ITER)}$.

Analogously to [14] we obtain that the generated Markov chain is for each t aperiodic and positive Harris recurrent and converges to the distribution of Y_t.

A simple characteristics of the temporal evolution given by this algorithm consists in the evaluation of the ratio of discs which remain at their place during some time interval. For this purpose define

$$\rho_t = \frac{2n_t^s}{\sum_{i=0}^{n-t}(n_i + n_{i+t})},\ t = 1, \ldots, T,$$

where n_t^s is the total number of the same circles present at times i and $i + t$ (for all i) and n_j is the total number of circles observed at time j.

Figure 5.1 draws one realization of the process $Y = \{Y_t,\ t = 0, \ldots, 25\}$ with $\mathbf{x}_0 = (1, -0.5, -1)$, $a = (-0.07, 0.035, 0.07)$ and $\sigma^2 = 0.001$. The window is a square of size 10×10. The distribution of radii Q is uniform on $[0.2, 0.7]$ and the intensity function $\rho = 1$ is a constant and the probability $\beta = 0.5$. The decrease of ρ_t for this simulated model is shown in Table 5.1.

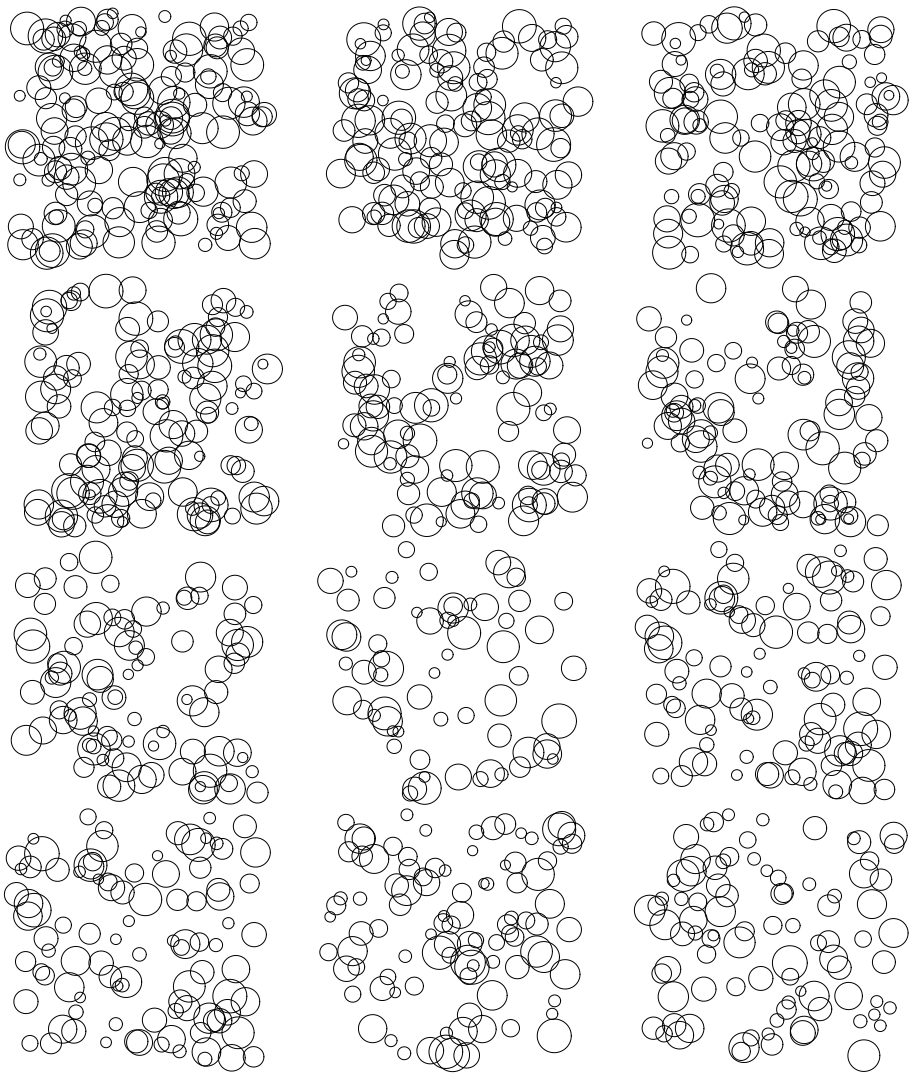

Figure 5.1: Simulated model with time–dependent interacting discs, evolution in the time $t = 0, 5, 10, 15, 20, 25$, $x_0 = (1, -0.5, -1)$, $a = (-0.07, 0.035, 0.07)$, $\sigma^2 = 0.001$, S is a square of size 10×10. Here Q is uniform distribution on $[0.2, 0.7]$ and $\rho = 1$ is constant.

k	1	2	3	4	5	6
ρ_k	0.38	0.15	0.06	0.02	0.01	0.00

Table 5.1: The characteristics ρ_t of the temporal evolution of the simulated germ-grain model from Fig.5.1.

5.3 Statistics of the model with interacting particles

5.3.1 Maximum likelihood

The MLE method using MCMC simulations (MCMC MLE) is based on finding $\hat{x} = \text{argmax}_{x\in\mathbb{R}^d}\ p(y|x)$, where the data y (i.e. realization of a planar random set $\tilde{Y}$) are represented by the vector $G(U_y)$. However, since c_x has no explicit expression, $p(y|x)/p(y|x^0)$ for fixed $x^0 \in \mathbb{R}^d$ is maximized instead, because in that case, we can use importance sampling for approximatio ratio of normalizing constants. Log-likelihood ratio is then given by

$$\begin{aligned} l_{x^0}(x) = \log \frac{p(y|x)}{p(y|x^0)} &= (x - x^0)\cdot G(y) - \log \frac{c_x}{c_{x^0}} \\ &\approx (x - x^0)\cdot G(y) - \log \frac{1}{n}\sum_{i=1}^{n} \exp\{(x - x^0)\cdot G(z_i)\}, \end{aligned} \tag{5.7}$$

where z_i, $i = 1, \ldots, n$, for a given n are realizations from $p(.|x^0)$ obtained by MCMC simulations. The function (5.7) has simple analytical form, so the maximum likelihood estimate is obtained as

$$\hat{x} = \text{argmax}\ l_{x^0}(x). \tag{5.8}$$

5.3.2 Model checking

A discussion on the model fit is based on spherical contact distribution function. Given a compact convex set $B \subset \mathbb{R}^2$ and a stationary planar random set $\tilde{Y}$ define

$$D = inf\{r \geq 0 :\ \tilde{Y} \cap rB \neq \emptyset\}.$$

Assuming $P(D > 0) > 0$ and that B is the unit disc, the spherical contact distribution function is defined as

$$H_B(r) = P(D \leq r | D > 0),\ r \geq 0.$$

A non-parametric estimator of H_B for stationary $\tilde{Y}$ including edge-effect correction is

$$\hat{H}_B(r) = \frac{\sum_{u\in L} \mathbf{I}_{[u\notin\tilde{Y},\, u+rB\subset S,\, (u+rB)\cap\tilde{Y}\neq\emptyset]}}{\sum_{u\in L} \mathbf{I}_{[u\notin\tilde{Y},\, u+rB\subset S]}}, \tag{5.9}$$

where L is a regular lattice of test points.

5.3.3 Envelopes

Consider a control function (statistic) $M(r)$, e.g. the spherical contact distribution function, of the process $\tilde{Y}$. According to [28] we introduce a graphical interpretation of a non-parametric estimate of $M(r)$. For each value r a confidence interval is constructed in the following way. Consider a simple hypothesis H_0 that the data correspond to the process Y. For a given $r > 0$, let $M_0(r) = M(Y, r)$ denote $\hat{M}(r)$ obtained from the point process $\tilde{Y}$ observed within the observation window S. Let $M_1(r) = M(\tilde{Y}^1, r), \ldots, M_n(r) = M(\tilde{Y}^n, r)$ be obtained from i.i.d. simulations of the process $\tilde{Y}^1, \ldots, \tilde{Y}^n$ under hypothesis H_0. Let

$$M_{min}(r) = min\{M_1(r), \ldots, M_n(r)\},\ M_{max}(r) = max\{M_1(r), \ldots, M_n(r)\}. \tag{5.10}$$

Under H_0, for each r holds

$$\mathbb{P}(M_0(r) < M_{min}(r)) = \mathbb{P}(M_0(r) > M_{max}(r)) \leq \frac{1}{(n+1)}$$

with equality if $M_0(r), M_1(r), \ldots, M_n(r)$ are all different. The bounds $M_{min}(r)$ and $M_{max}(r)$ are called the $100/(n+1)\%$-*lower and the* $100/(n+1)\%$-*upper envelope at the distance* $r > 0$. For example, if we let $n = 39$, we obtain a 2.5% lower and a 97.5% upper envelope, or we can also say we have 95% envelopes for the value $\hat{M}(r)$. The same techniques are often used when instead of a simple hypothesis the fit of an estimated model is checked. This is the so–called local test, recently in [15] global tests were developed. The tests are not used in the thesis.

5.4 Simulation study

5.4.1 Independent time extension

Recall a generalization of Y to a space-time random set in discrete time is

$$Y = \{Y_{0:T}\}, \tag{5.11}$$

where $Y_t,\ 0 \leq t \leq T$, are models with interacting particles having densities (4.5) with vector of geometrical characteristics $G = G_2$ in (4.8) and with the state vectors

$$\mathbf{x}_t = (x_t^{(1)}, \ldots, x_t^{(d)})$$

developing in time as realizations of a Markov process $X_{0:T}$ in $\mathbb{R}^d$, $d = 3$. More precisely, let X_t develop as a random walk

$$X_t = X_{t-1} + \gamma_t,\ t = 1, \ldots, T, \tag{5.12}$$

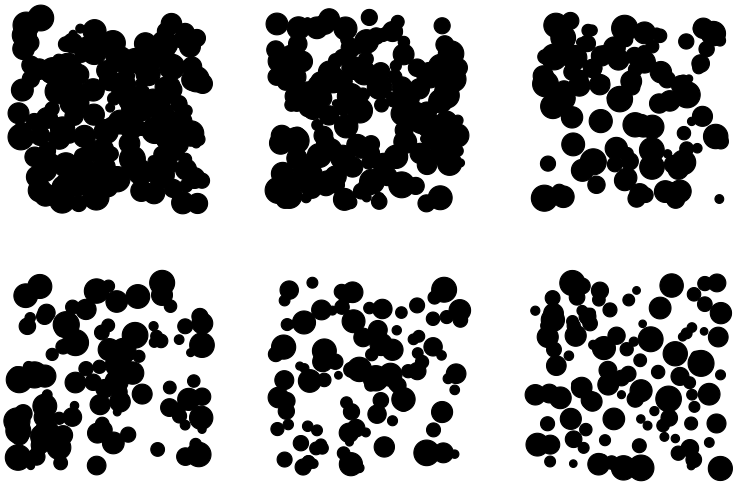

Figure 5.2: Simulated time–independent model with interacting discs, evolution in the time $t = 0, 5, 10, 15, 20, 25$, $x_0 = (1, -0.5, -1)$, $a = (-0.07, 0.035, 0.07)$, $\sigma^2 = 0.001$, B is a square of size 10×10. Here Q is uniform distribution on $[0.2, 0.7]$ and $\rho = 1$ is constant.

where γ_t are i.i.d. Gaussian $\mathcal{N}(a, \sigma^2 I_d)$ random variables, I_d is the unit matrix of size d. Here we consider $\theta = (x_0, a, \sigma) \in \mathbb{R}^{2d+1}$ as an unknown auxiliary parameter. Note that since for any $t = 1, \ldots, T$, X_t may acquire an arbitrary value from $\mathbb{R}^d$, the distribution of radii Q, see (4.6), must be chosen so that it fulfills the condition (4.2).

In this situation the only time dependence is given through the Markov chain (5.12), i.e. given $\{X_t\}$, $t = 1, \ldots, T$, the random sets Y_t are conditionally independent.

As a criterion for quality of any estimator $\hat{x}_{1:t}$ of $x_{1:t}$, we will use the mean integrated square error

$$MISE = \frac{1}{l} \sum_{j=1}^{l} \sum_{t=0}^{T} (\hat{x}_{t,j}^{(i)} - x_{t,true}^{(i)})^2, \ i = 1, 2, 3, \tag{5.13}$$

where $x_{t,true}$ is the true value at time t and l is the number of simulated realizations of Y, cf. (5.11).

5.4.2 Numerical results

We consider 19 simulated realizations of the process $\{Y_{0:25}\}$ as data for the statistical analysis. At each time t we used the classical Metropolis-Hastings birth-death algorithm (see e.g. [26]) to simulate the realizations from the true distribution (4.5). In this simulation study, B is a square window of size 10×10, $\rho = 1$ and the distribution Q is

uniform on $[0.2, 0.7]$. The parameter $\mathbf{x}_{0:25}$ is the same for all realizations generated as a Gaussian random walk with $\mathbf{x}_0 = (1, -0.5, -1)$, $a = (-0.07, 0.035, 0.07)$ and $\sigma^2 = 0.001$. One selected simulation is drawn in Fig. 5.2 and the evolutions of its characteristics are in Fig. 5.6. The characteristics of all 19 simulated realizations were used to the computing of maximum likelihood estimator according to (5.8) and throughout as an input to PMMH and PF.

Using the linear regression on MLE of x_t, cf. [27], we obtained estimations $\mathbf{x}_0^{MLE}$ and a^{MLE} of $\mathbf{x}_0$ and a, respectively (see Table 5.3). The estimation σ^{MLE} is then obtained from differences of MLE estimations as an empirical variance

$$s^2 = \frac{1}{3T-4} \sum_{i=1}^{3} \left(\sum_{t=1}^{T} \left(\tilde{x}_t^{(i)} \right)^2 - (T-1) \left(\bar{x}_t^{(i)} \right)^2 \right) \tag{5.14}$$

of the sample of differences $\tilde{x}_t^{(i)} = x_t^{(i)} - x_{t-1}^{(i)}$, $i = 1, 2, 3$, see [40]. The denominator responds to $3 \times (T-1)$ differences. The proposal density for PMMH is a Gaussian random walk

$$q(\cdot|\theta) \sim \mathcal{N}(\theta; 0.001)$$

and the prior distribution of θ is uniform with independent components on intervals with endpoints given by 0 and double size of $\mathbf{x}_0^{MLE}$, a^{MLE} respectively. The total number of iterations was 100 000, basic graphical diagnostics shows that this number is enough for approaching the target distribution. The means and variances of estimated auxiliary parameters are in Table 5.3.

In this section we review the PMMH estimator in two different ways. The first one is numerical comparing given by MISE (5.13) where also the results for the PF method and MLE are given (see Table 5.2). Here it is obvious that the best results are obtained by PMMH which gives the lowest MISE for all parameters $x^{(1)}$, $x^{(2)}$ and $x^{(3)}$.

The second way is graphical (plots of estimates, spherical distribution function described in Subsection 5.3.2 etc.). For simplicity denote R the realization from Fig. 5.2. The comparison of true parameters $\mathbf{x}_{0:25}$ and $\hat{\mathbf{x}_{0:25}}$ computed for R are in Fig. 5.3, where we can see that these two evolutions are very similar. The envelopes of all estimators based on 19 realizations are in Fig. 5.4, where we observe that except for a few cases ($x^{(2)}$ in later times), all the envelopes cover the true evolution of parameters and the envelopes given by PMMH are the narrowest. It means again that this method gives the best results in the sense of estimation variance.

In order to check the model we used the PMMH estimation of $x_{0:25}$ for R to simulate a set $\mathcal{R}_{new}$ of 19 new realizations of $\{Y_{0:25}\}$. One of these realizations is in Fig. 5.5. For R the estimator $\hat{H}_B(r)$ of spherical contact distribution function was computed in times $t = 0, 5, 10, 15, 20$ and 25. We avoided edge-effects by using subwindow $[0.7, 9.3] \times [0.7, 9.3]$. The envelopes of $\hat{H}_B(r)$, obtained from $\mathcal{R}_{new}$, given by pointwise maximum and minimum

are presented in Fig. 5.7. Until time $t = 15$ the envelopes cover $\hat{H}_B(r)$ for R while in later times $\hat{H}_B(r)$ for R lies on the lower envelope or slightly under the envelope. This small misfit in spherical contact distribution function correspond with Fig. 5.2 and Fig. 5.5 where it can be seen that the intensity of discs in the simulated model is a little bit higher than in the data in later times.

	$x^{(1)}$	$x^{(2)}$	$x^{(3)}$
MLE	2.645685	1.086167	1.550637
PF	4.010112	2.530336	2.503888
PMMH	1.0807717	0.9361609	0.5929558

Table 5.2: Square root of MISE obtained for all three methods used. The estimations are based on 19 characteristics obtained from simulated realizations.

	true	MLE		PMMH	
		mean	s^2	mean	s^2
x_0	1	1.08015	0.17412	1.07455	0.02527
	-0.5	-0.52289	0.06559	-0.57967	0.01067
	-1	-0.98794	0.09270	-0.99238	0.01490
a	-0.07	-0.08049	0.01445	-0.07106	0.00183
	0.035	0.05043	0.00544	0.06170	0.00066
	0.07	0.06751	0.00714	0.06446	0.00111
σ^2	0.001	0.09669	0.00366	0.01034	0.00015

Table 5.3: Mean and sample variance of estimations of auxiliary parameter $\theta = (x_0, a, \sigma^2)$ based on 19 realizations.

5.4.3 Some selected results for model with four parameters

At the beginning of our study of this topic we focused on a four parametric model of the union of interacting discs with vector of characteristics $G = G_1$ in (4.7), see [40]. We compared maximum likelihood estimator and particle filter. Since from the results of the section 5.4.2 a PF seems to be the worst estimator, we introduce here some results from this older study just to show that it is not a general fact.

Consider again the model from Section 5.4.1 with $d = 4$. The reference process is a Boolean model with intensity of the germs $\rho = 1$ and with radii of discs uniformly distributed in the interval $[0.2, 0.7]$. We simulated discrete time realizations for $t = 0, \ldots, T = 11$ with given $\mathbf{x}_0 = (0.5, -0.25, -0.5, 0.5)$, $a = (-0.1, 0.05, 0.1, -0.1)$, $\sigma^2 = 0.001$, number of MCMC iterations $ITER = 30\,000$ and a square window S of size 10×10.

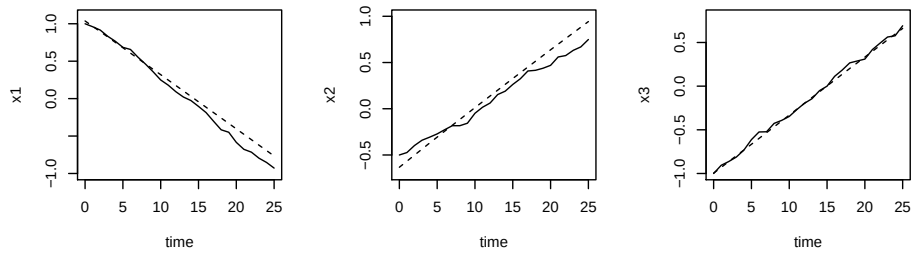

Figure 5.3: Comparison of true parameter $x_{0:25}$ (full line) and the estimation $\hat{x}_{0:25}$ (dashed line) obtained from the realization R using the PMMH method. The first plot corresponds to the parameter $x^{(1)}$, the second one to $x^{(2)}$ and the last one to $x^{(3)}$.

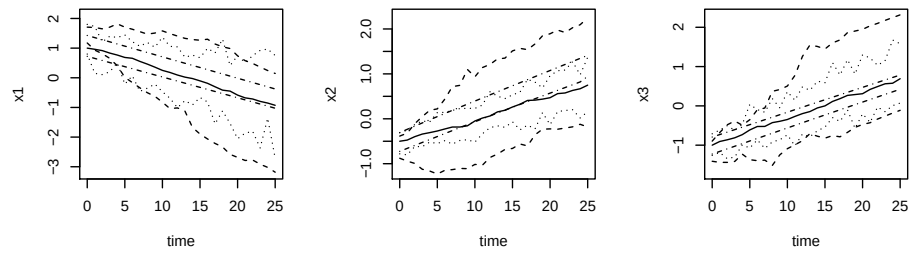

Figure 5.4: The envelopes for all estimators based on 19 realizations given by (5.10). Full line denotes the true evolution, dotted line denotes the envelopes for MLE, dashed line denotes the envelopes for PF and dot-dashed line denotes the envelopes for PMMH. The first plot corresponds to the parameter $x^{(1)}$, the second one to $x^{(2)}$ and the last one to $x^{(3)}$.

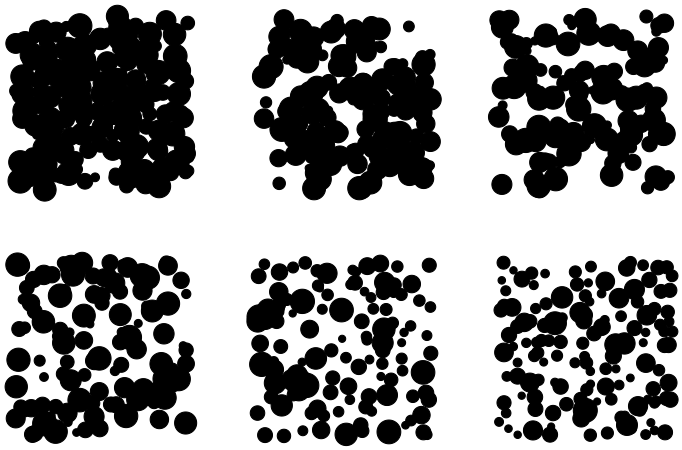

Figure 5.5: Simulated model with time–independent interacting discs, evolution at times $t = 0, 5, 10, 15, 20, 25$ with parameter $\hat{x}_{0:25}$ obtained from PMMH for the realization R.

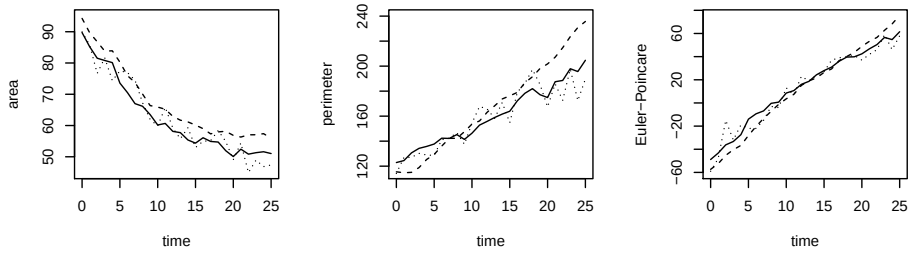

Figure 5.6: Comparison of averaged geometrical characteristics of original 19 realizations (full line) and those of $\mathcal{R}_{new}$ (dashed line). Dotted line denotes the evolution of geometrical characteristics of the realization from Fig. 5.2.

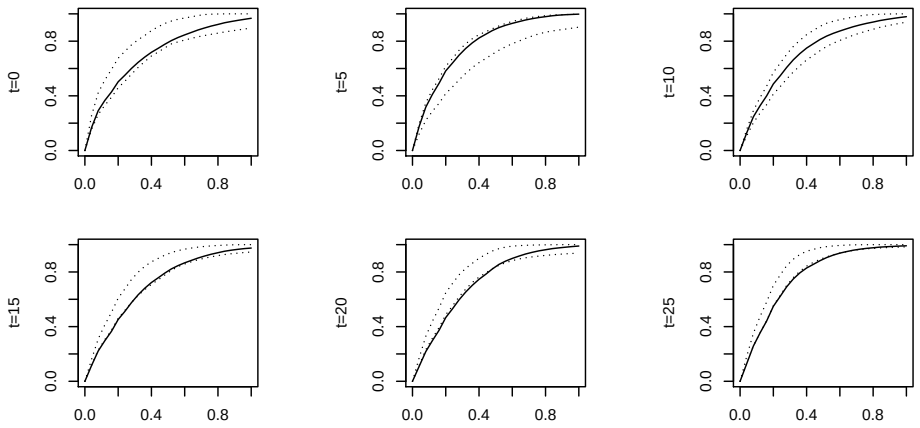

Figure 5.7: The envelopes for the spherical contact distribution function at times $t = 0, 5, 10, 15, 20, 25$ based on $\mathcal{R}_{new}$ and given by (5.10). Full line denotes the estimation of the spherical contact distribution function obtained from R.

The edge effects are not considered, the disc centres lie in S and the whole discs are drawn and evaluated. We simulated a varying number of simulations $l = 1, 5, 10, 39, 100$, where in each case the data of all simulations are summed. The estimation of auxiliary parameters is the same as for Quermass model in section 5.4.2. The evolution of estimated parameters for varying number of simluation are in Fig. 5.8, in Fig. 5.9 one can see envelopes based on $l = 39$ simulations for MLE and PF respectively and finally the results for MISE (5.13) with $i = 1, \ldots, 4$ are given in table 5.4.

no. of simul.	$l = 5$		$l = 39$	
parameter	PF	MLE	PF	MLE
$x_t^{(1)}$	3.031	3.735	11.47	30.16
$x_t^{(2)}$	3.892	1.529	32.10	6.439
$x_t^{(3)}$	3.524	3.733	14.97	19.41
$x_t^{(4)}$	5.132	7.424	31.76	74.72

Table 5.4: A comparison of MISE for the estimation of parameter x using the methods PF (particle filter) and MLE. Results for the number of simulations $l = 5$ and $l = 39$ are presented.

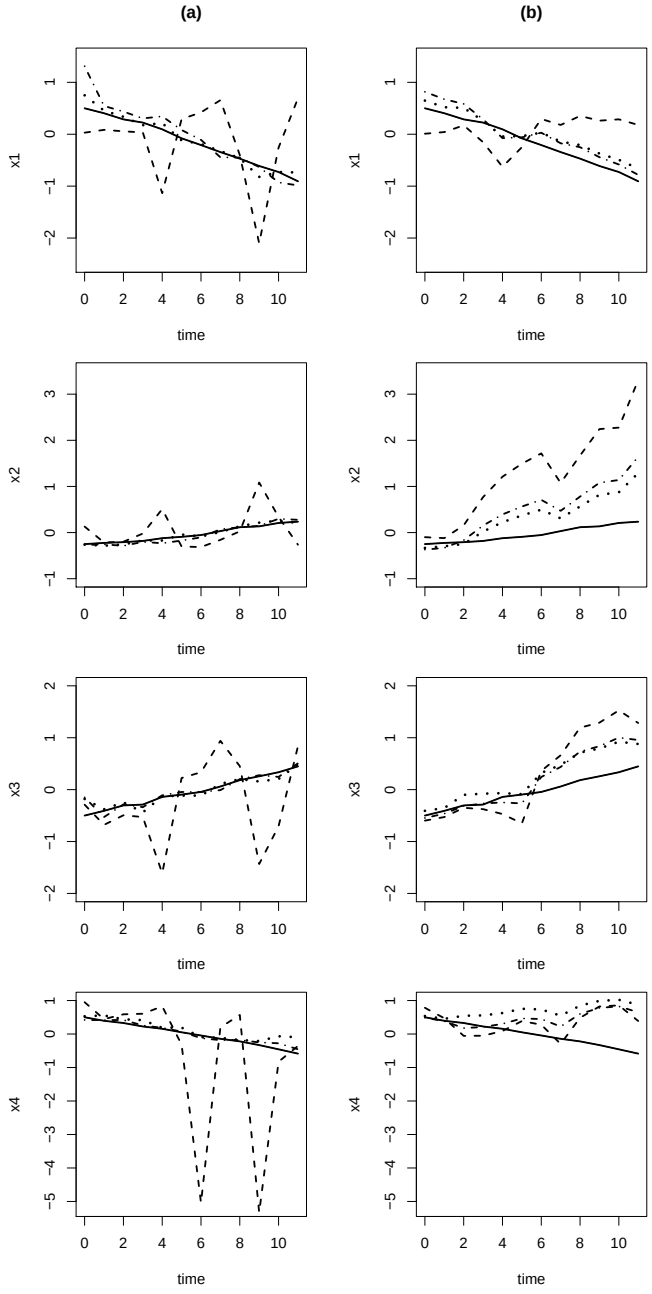

Figure 5.8: Comparison of true evolution of the parameter x_t (full line) at times $t = 0, 1, \ldots, 11$ with the estimations obtained from data added up of $l = 1$ (dashed), $l = 10$ (dotted) and $l = 100$ (dashed–dotted) simulated realizations using MLE (a) and PF (b). The plots correspond to the parameters $x_t^{(1)}$ assigned to the area, $x_t^{(2)}$ assigned to the perimeter, $x_t^{(3)}$ assigned to the number of connected components and $x_t^{(4)}$ assigned to the number of holes.

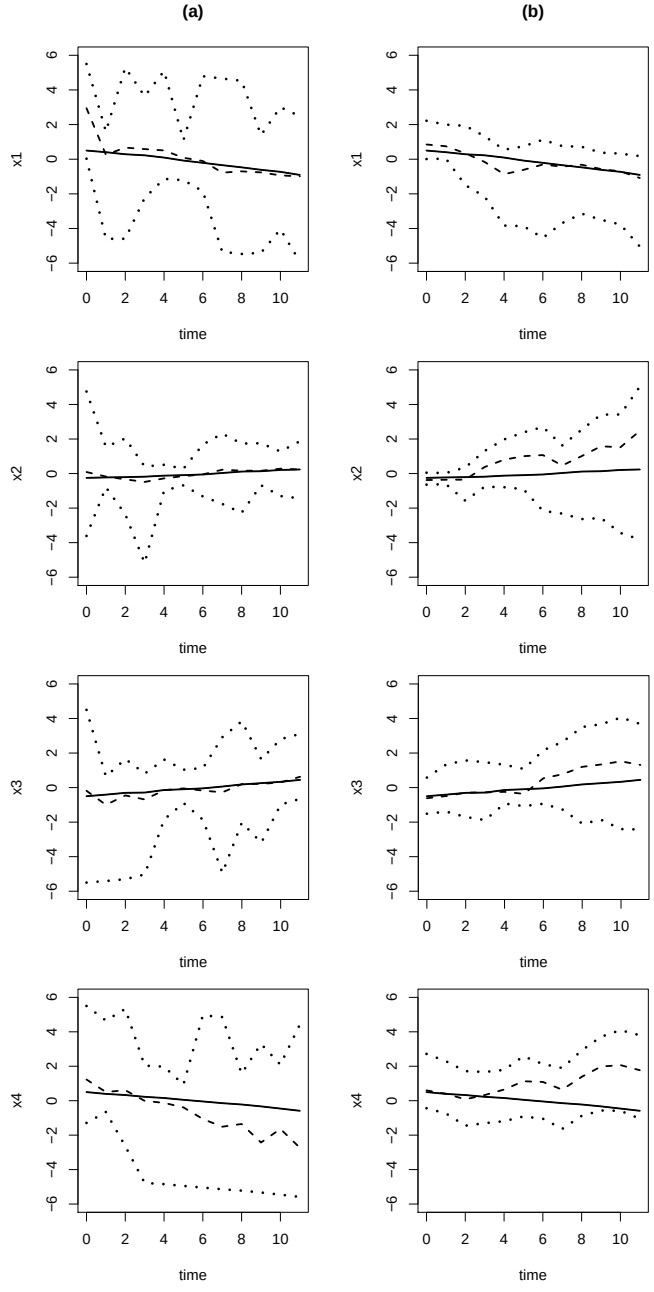

Figure 5.9: Comparison of true evolution of the parameter x_t (full line) at times $t = 0, 1, \ldots, 11$ with its estimations obtained from the $l = 39$ simulations (dashed line as their average, with dotted envelopes) using MLE (a) and PF (b). The plots correspond to the parameters $x_t^{(1)}$ assigned to the area, $x_t^{(2)}$ assigned to the perimeter, $x_t^{(3)}$ assigned to the number of connected components and $x_t^{(4)}$ assigned to the number of holes.

Chapter 6

Conclusion

In the theoretical part of the work we introduced $U-$statistics for point processes with probability density function with respect to a Poisson point process. We derived equalities for the first and the second moments and also for higher moments. Moreover formulas for mixed moments of two and more different $U-$statistics were proved. These results were aimed at explicit formulas for characteristics of union of interacting discs. Note that since Papangelou conditional intensity has complicated expression even in case of density in form (4.1) we are not able to compute it without using numerical methods.

The aim of the simulation study was to compare PMMH estimator of parameters of a given space-time model with interacting discs, see (5.11) and (5.12), with the particle filter estimator suggested in [40] and MLE studied in [27].

First note that in Section 5.4.3 we observed better performance of PF than of MLE (in the sense of the variance), while the results for Quermass model are different. The probable reason is that the four-parametric model was investigated in [40] considering number of components and number of holes separately instead of Euler-Poincare characteristic. Simulations in four-parametric model then often worked with realizations with only one component or zero holes, respectively, i.e. with the lowest possible values of those characteristics, which lead to the undervaluation of the estimates of parameters from the reasons described in [38].

Concerning PMMH estimator, it has smaller variability than both PF and MLE. Also the values of MISE show advantage of the PMMH estimator. We can observe that the tangent of PMMH estimators of $\hat{x}^{(2)}_{0:25}$ slightly differs from the true evolution of the parameters, so there is a small misfit in PMMH estimators at later times. Consequently, also the realizations simulated from estimated parameters may differ a little bit from the original ones. In order to quantify the presence of such a bias we used the spherical contact distribution functions provide a difference between simulations with true and estimated

parameters, especially at later times, but the misfit is again very small.

So the conclusion is that in the real data analysis when a single realization of a space-time random set is available, PMMH may be recommendable.

There still remain some open questions like how to choose the right estimation method in general or whether the properties of the introduced methods can be improved by using some additional method, e.g. Takacs–Fiksel estimator ([8]). These questions are going to be solve in further research.

Appendix–Program comments

This thesis includes simulation and computation programs. Both programs were produced in language $C++$.

Simulation of interacting line segments

The program (file **segment_interaction.cpp**) generates the union of interacting line segments with vector of geometrical characteristics (4.10). The intensity of the reference points of Poisson process is a constant ρ, the distribution of lengths is uniform on $[ex, var]$.

Inputs

Line 17 : maxs... maximum number of line segments (350 default)
Line 21 : IT... number of iteration of birth and death algorithm (100000 default).
Line 22 : ex... lower boundary of the uniform distribution of lengths (0 default).
Line 23 : var... upper boundary of the uniform distribution of lengths (0 default).
Line 314 : ρ... intensity of reference points (1.5 default).
Lines 315, 316 : the side lengths of rectangle observation window (10, 10 default).
Lines 323 – 325 : the parameter **x**.

Outputs

The outputs of programs are the text files including:

"lexmin.txt"... the coordinates of the reference points.
"length.txt"... the lengths of simulated segments.
"characteristics.txt"... the geometrical characteristics of the simulated union.

Remark 6.1 *For the other choice of length distribution it is sufficient to change the generating function (uniform default) in lines* **194, 213, 352**.

Particle Marginal Metropolis Hastings estimation

This program (file **PMMH.cpp**) was produced for Quermass interaction model of the union of interacting discs in $\mathbb{R}^2$, but its construction allows us to use this program for any three dimensional parametric model with intensity given by (4.1). The only problem is a normalising constant $c_{\mathbf{x}}$, which is needed for computing the weights. Since this normalising constant is, exept for the Poisson process, unknown we use the estimation of ratio of normalising constants of two processes of discs (or other particles)

$$\frac{c_{\mathbf{x}}}{c_{\mathbf{x}_0}}$$

with fixed process $\{X_0\}$. The weights in step $2(b)$ of Algorithm I are normalized and thus the ratios can be used instead of unknown normalizing constants. The auxiliary process X_0 was simulated using program from the page

`http://math.feld.cvut.cz/helisova/02progr.html`,

one realization is enclosed in CD. Note that suitable choice of auxiliary process has an essential importance for quality of the estimation. It should be as close to real distribution of the process as possible. In the remaining text of this section the inputs and outputs of the program are described.

Inputs

Line 15 : Kpart... number of particles (1024 default).
Line 17 : ITERATION... number of iterations of Metropolis–Hastings (100000 default).
Line 18 : MTime... time development (default 25, time 0 is not included, it is considered in auxiliary parameter).
Line 20 : Nint... the length of auxiliary chain X_0 (75000 default).
Lines 27 – 33 : p1... p7... boundaries of the prior distribution obtained from MLE.
Line 36 : prum... default 200, that means that the output is the average from each 200-th estimation.
Lines 143 – 156 : reading of input data: the realization of auxiliary process X_0. The files should have names "area.txt", "length.txt" and "ep.txt", respectively.

Lines 159 − 173 : the observed geometrical characteristics of the estimated process. The files should have names "data1.txt","data2.txt" and "data3.txt", respectively.
Lines 177 − 257 : the true values of parameter **x**. The program computes the integrated square error.
Lines 276 − 282 : the initial adjustment of auxuliary parameter θ. Based on MLE.

Outputs

The outputs of programs are the text files including:

"parameters_mean200.txt" ... the mean of estimations of auxiliary parameter.
"parameters_deviation200.txt"... the deviation of the estimations of auxiliary parameter.
"ISE200.txt"... writes ISE for each 200−th iteration of the algorithm.
"ISEmean.txt"... mean of ISE.
"final.txt"... final estimation is a mean of all iteration.

Remark 6.2 *To perform the common PF estimation it is sufficient to break (command "break") the process before the for-loop*

$$\text{for } (\text{IT} = 1; \text{IT} < \text{ITERATION}; \text{IT}++)$$

begins (line **477***), i.e. after finishing the initialisation of PMMH.*

List of Abbreviations

η	Poisson point process
μ	point process, usually with density with respect to η
Λ	the intensity measure of process η
λ_i^*	the Papangelou conditional intensity of order i
$\mathcal{B}(E)$	the Borel σ-field on E
$D^m_{u_1,\ldots,u_m}$	difference operator of order m
E	separable, locally compact space
$G(\mathcal{U}_\mathbf{y})$	vector of geometrical characteristics of $\mathcal{U}_\mathbf{y}$
$L^2(\cdot)$	set of L^2-integrable functions with respect to a measure $\cdot$
MCMC	Markov Chain Monte Carlo
MISE	mean integrated square error
MLE	maximum likelihood estimator
$\mathbb{P}_\eta$	distribution of the process η
PF	particle filter
PMMH	particle marginal Metropolis–Hastings algorithm
$\mathbb{R}^d$	d-dimensional Euclidean space
$T_m F(u_1,\ldots,u_m)$	the expectation of $D^m_{u_1,\ldots,u_m}F$ with respect to η
$T^\mu_m F(u_1,\ldots,u_m)$	the expectation of $D^m_{u_1,\ldots,u_m}F$ with respect to μ
$\mathcal{U}_\mathbf{y}$	the union of all particles of configuration $\mathbf{y}$

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