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BACHELOR THESIS

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Statistical physics in games

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Title: Statistical physics in games

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Abstract: This thesis studies a way of connecting the notion of maximum cut from discrete mathematics with the Rosenthal Potential introduced by Rosenthal in 1973 and the ground state configuration on the Ising model, a theoretical model designed in 1920 to study macroscopic results of microscopic interaction in statistical physics. The underlying motivation for studying this problem stems from a publication of S. Torquato in 2011, where he proposes applying the Ising model as a tool for analyzing cancer growth. The thesis aims to understand the possible connection between dynamics from statistical physics and their application in games of multi-agent environments.

Keywords: Potential Games, Max-Cut, Ising model, Cancer modelling

Název práce: Statistická fyzika v hrách

Autor: Júlia Križanová

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Abstrakt: Táto práca skúma spôsob prepojenia problému maximálneho rezu z oblasti diskrétnej matematiky s Rosenthalovým potenciálom, ktorý zaviedol Rosenthal v roku 1973, a konfiguráciou stavu minimálnej energie v Isingovom modeli, teoretickom modeli navrhnutom v roku 1920 pre štúdium makroskopických výsledkov mikroskopických interakcií v štatistickej fyzike. Pôvodná motivácia pre štúdium tohto problému pochádza z publikácie od S. Torquata z roku 2011, v ktorej navrhuje použiť Isingov model ako nástroj na analýzu rastu rakoviny. Cieľom práce je pochopiť možné spojenie medzi dynamikou zo štatistickej fyziky a ich aplikáciou v hrách s multi-agentným prostredím.

Klíčová slova: Potenciálne hry, Maximálny rez, Isingov model, modelovanie rakoviny

Contents

Introduction

Scope of the thesis

Inspired by techniques that were proven successful in statistical physics, namely the Ising model, we are interested in studying game dynamics. This thesis explores the interdisciplinary connection between discrete mathematics, precisely the concept of maximum cut, statistical physics and the class of potential games. Potential applications could lie in the research of prediction of cancer evolution.

Organization of the thesis

The thesis is organized as follows: At the beginning, there are the preliminaries and general introduction to topics that need to be understood to follow the central part of the thesis. The preliminary chapter consists of basics from discrete math and graph theory. In this section, the notion of cuts is being introduced. It is followed by an introduction to algorithmic game theory, which is the core part of the thesis. In this section, the solution concept of Nash equilibrium is presented together with the notion of a normal-form game. The last introduction section is about computational complexity, namely the PLS class and its characteristics. After the introduction, the chapter about potential games follows. Chapter 3 gives a basic introduction to statistical physics and presents the Ising model, which is the crucial model that originated in statistical mechanics and is further used in our considerations. Chapter 4 is the main part of the thesis, where the core ideas are being put together. Here, the connection between statistical physics and algorithmic game theory is established by mapping one area onto the other. At the end, there is a conclusion chapter stating possible future work in this area.

1 Preliminaries

1.1 Introduction to Graph Theory

This section introduces reader to the basic notions from discrete mathematics and graph theory. The definitions presented here are sourced from [1]. For further details, the reader is referred to this source.

1.1.1 Graphs

Definition 1 (Graph)**.** *A graph G is a pair (V, E), where V is a set of vertices of G and E is a set of 2-element subsets of V called the set edges of G.*

Below are examples of graphs, that will be used further in the thesis:

Definition 2. *The path P_n is defined by the set of vertices $V = \{0, 1, ..., n\}$ and by $E = \{\{i-1, i\} : i = 1, 2, ..., n\}$, the set of edges connecting each pair of successive vertices.*

Definition 3 (Bipartite graph)**.** *Graph G is* bipartite *if it is possible to divide the set V into two disjoint sets V_1 and V_2 such that each edge $e \in E$ connects a vertex from partition V_1 to a vertex from partition V_2. I.e. $E \subseteq \{\{v, v'\} : v \in V_1, v' \in V_2\}$.*

Definition 4 (Subgraph)**.** *Let G and H be graphs. If $V(G) \subseteq V(H)$ and $E(G) \subseteq E(H)$, then G is a subgraph of H.*

Definition 5 (Directed graph)**.** *A graph $G = (V, E)$, is a directed graph defined by the set of vertices V and by the set of edges E being a subset of the Cartesian product $V \times V$. The ordered pairs $(x, y) \in E$ are called the directed edges. We say that a directed edge $e = (x, y)$ is an edge from x to y.*

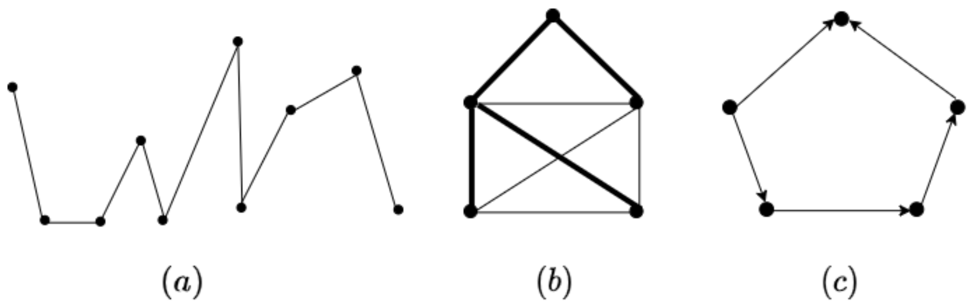

Figure 1.1 (a) Path P_9. (b) Subgraph. (c) Directed graph.

1.1.2 Cuts

In this section, max-cut is going to be described. The definitions are sourced from [2], [3] and [4].

Definition 6 (Max-Cut problem)**.** *Let $G = (V, E)$ be an undirected graph, where each edge $e \in E$ is carries a weight $w_e \in \mathbb{N}^+$. A cut of G is defined as a partition of the vertices into two subsets $(U, V\backslash U)$ where $\emptyset \neq U \neq V$, typically being represented by the subset U. Let $\delta(U) = \{\{u, v\} \in E : u \in U \wedge v \notin U\}$ denote the set of edges crossing the cut U. The weight of the cut U is given by $w(U) = \sum_{e\in\delta(U)} w_e$. The objective of the* Max-Cut *problem, is to find a cut U^* such that the weight $w(U^*)$ is maximal.*

As optimization problem, the version of minimization (Min-Cut) can be solved in polynomial time while the Max-Cut is classified as NP-complete which means that finding the Max-Cut is hard. Hence, several heuristics were designed for this problem. We concentrate on one of them called local search under the *Flip-neighborhood.*

Definition 7 (Flip-neighborhood)**.** *Let U' be a cut obtained from the cut U by transferring a vertex from one partition of the cut to the other. The flip neighborhood $N(U)$ of a cut $(U, V\backslash U)$ includes all the cuts $(U', V\backslash U')$ where U and U' differ by exactly one vertex.*

Definition 8 (Local Max-Cut problem)**.** *A cut U is locally optimal if for all $U' \in N(U), w(U) \geq w(U')$, where $N(U)$ represents the set of all cuts obtainable by flipping any single vertex from U to $V \setminus U$ or from $V \setminus U$ to U. The goal in* Local Max-Cut *is to find a locally optimal cut U^*.*

1.2 Introduction to Game theory

In this section, the basic terms and definitions from the area of algorithmic game theory are being introduced.

The terms and concepts discussed in this section are partially derived from [5] and the lectures on algorithmic game theory by doc. RNDr. Martin Balko, Ph.D. (lecture notes).

Definition 9 (Normal Form Game)**.** *A (finite, n-player) normal form game is* $\Gamma(N, (S_i)_{i\in\mathbb{N}}, (u_i)_{i\in\mathbb{N}})$ *where*

- $N = \{1, 2, ..., n\}$ *is a finite set of n players,*
- $S = S_1 \times ... \times S_n$ *is a set of strategy profiles, where* S_i *is a set of pure strategies available to player i:* $1 \leq i \leq n$,
- *and* $u_i : S \to \mathbb{R}$ *is the utility function for each player i:* $1 \leq i \leq n$

Note: S_i are finite abstract sets.

The strategy set S_i represents the set of actions player i can play in the game. In normal form game, knowing the utility function, all players choose their strategies simultaneously. When each player chooses its action from their own strategy set, the resulting strategy profile is obtained: $s = (s_1, s_2, ..., s_n) \in S$ and is then evaluated using the utility function. During the game, players receive respective payoff by applying their utility function on the current strategy profile.

Notation: throughout the thesis s_{-i} will denote all strategies except for the strategy of player i. In such case strategy profile can be rewritten as $s = (s_i, s_{-i})$.

Payoffs of a normal form game can be represented by the payoff matrix:

Definition 10 (Payoff matrix)**.** *Every normal form game* Γ *can be represented by a real n-dimensional matrix* $M = (M_s)_{s\in S}$*, where* $M_s = u(s) = (u_1(s), ..., u_n(s))$. *Such matrix is called the payoff matrix.*

1.2.1 Basic solution concepts

The objective of a player in a game is to maximize the player's payoff using appropriate strategies. However, the complexity of optimizing player's payoff increases with the increasing number of players as the best strategy depends on the choices of the other players. That's why the notions of solution concepts were introduced. Solution concept is a mapping from the set of all normal-form games, mapping each game Γ to a set of strategy profiles of Γ.

Definition 11 (Pure Nash equilibrium (PNE))**.** *A strategy* s_i *is called a best response for player* $i \in \mathbb{N}$ *against a collection of strategies* s_{-i} *if* $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$ *for all* $s'_i \in S_i$. *A state* $s \in S$ *is called a pure Nash equilibrium if* s_i *is a best response against the other strategies* s_{-i} *for every player* $i \in \mathbb{N}$.

Remark. Pure Nash equilibria are not necessarily unique.

From the definition it can be seen that pure Nash equilibrium is a state in which no player can unilaterally increase its utility by taking a different strategy.

1.2.2 Normal form game example: Prisonner's dilemma

Prisonner's dilemma is one of the classic examples of algorithmic game theory. There are two players (criminals) which are being imprisoned and that cannot communicate with each other.
Each of them has two options: either betray the other criminal by testifying that the other prisonner commited the crime, or to remain silent. If they both decide to testify, they will both remain in prison for five years. If Player 1 testifies, but the other player remains silent, then Player 1 is set free and the Player 2 stays in prison for ten years, and vice versa. In case that they both decide to remain silent, both of them will stay in prison for only two years.

The corresponding payoff matrix of such game would then look as follows:

	Testify	Remain silent
Testify	(-5, -5)	(0, -10)
Remain silent	(-10, 0)	(-2, -2)

Table 1.1 A normal form of the Prisoner's dilemma game.

Despite the seeming advantage for both prisoners to remain silent, the game's equilibrium is reached when both players choose to testify against each other. The reason it that otherwise one of them can change his action to testify and improve his payoff.

1.3 Introduction to Computational Complexity

1.3.1 Complexity basics

The notions and concepts such as polynomial time, P, NP, and NP-completeness are central to the field of computational complexity. Their purpose is to categorize decision problems based on the resources (i.e. time) required to solve them. This section provides formal definitions and explanations of these terms, introducing the reader to the necessary vocabulary in order to comprehend the problematic of the thesis.

For further details on the discussed notions, the reader is referred to [2] and [6].

Definition 12 (Decision problem)**.** *Given a finite alphabet* Σ*, a decision problem is a language* $L \subseteq \Sigma^*$*, where* Σ^* *denotes the set of all possible strings (including empty string) that can be formed from symbols in* Σ*. The decision problem entails determining for any given string* $l \in \Sigma^*$*, whether* $l \in L$*. It is nontrivial if* $L \neq \emptyset$ *and* $L \neq \Sigma^*$*.*

Definition 13. *An algorithm is said to run in* Polynomial Time *if its running time can be expressed as* $O(n^c)$ *for some constant* c*, with* n *representing the size of the input.*

In simple words: When we say an algorithm runs in polynomial time, we mean the amount of time it takes to run the algorithm is proportional to a polynomial function of the size of the input, that can be for example the number of vertices in a graph or the number of digits in a number. Polynomial time algorithms are considered to be efficient because their running time grows at a manageable rate as the size of the input increases.

Definition 14 (P class)**.** *P is the set of decision problems solvable in polynomial time.*

Definition 15 (NP class)**.** *NP refers to the class of decision problems for which polynomial-time verifiers exist, i.e the problem* $L \in \Sigma^*$ *is in NP class if there is a polynomial-time algorithm* $V(l, X)$*, that satisfies the following:*

- *If* $l \in L$*, then* $\exists X$*, s.t.* $V(l, X) = YES$*.*
- *If* $l \notin L$*, then* $\forall X$*,* $V(l, X) = NO$*.*

Additionally, the length of X *must be polynomial in the size of* I*.*

Note that the input X to the verifier V is often called a *witness.*

To illustrate, for an NP problem, no known polynomial-time algorithms exist for solving it directly. However if one is provided with a proposed solution, then verifying its correctness can be done easily. The class of problems where a YES answer can be supported by a proof of polynomial length that is verifiable in polynomial time, is called NP.

Naturally, any problem that belongs to the class P is also considered to be in the NP class, since the verifier V could simply disregard the provided solution X and solve the problem instance I directly. Hence, $P \subseteq NP$ holds.

Definition 16 (Polynomial reduction). *Let $L_1, L_2 \subseteq \Sigma^*$. We say that L_1 is polynomially reducible to L_2, denoted as $L_1 \leq_p L_2$ if there exists a polynomial-time computable function $f : \Sigma^* \to \Sigma^*$, s.t. $x \in L_1$ iff $f(x) \in L_2$.*

Stating that $L_1 \leq_p L_2$ means that L_2 is at least as difficult as L_1. To illustrate, L_1 is not more difficult than L_2, meaning that if one solves the decision problem for L_2, it is possible to solve the decision problem for L_1 by employing a function f.

Definition 17 (NP-hardness). *A decision problem $L \subseteq \Sigma^*$ is NP-hard if for every $L' \in NP$: $L' \leq_p L$.*

Remark. In the case of NP-hardness, the problem doesn't necessarily need to be in NP, meaning that possibly L can be some language that is harder than NP.

Definition 18 (NP-completeness). *Problem $L \subseteq \Sigma^*$ is NP-complete if:*

1. *L is in NP, and*
2. *For any other problem L' in NP, $L' \leq_p L$.*

Example: MAX cut is NP-complete.

If $P \neq NP$ (Conjecture), then the mentioned complexity classes are visualized in 1.2

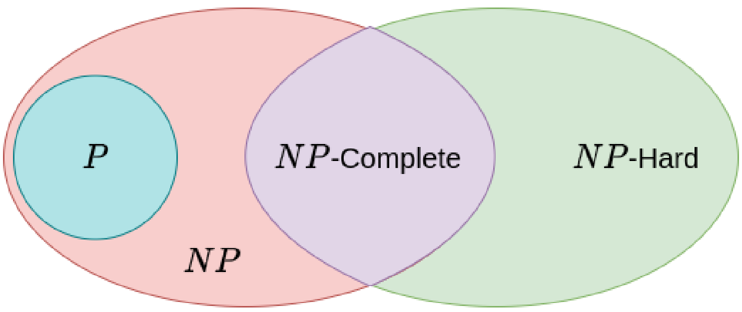

Figure 1.2 Complexity Class schema if $P \neq NP$ holds.

1.3.2 Polynomial-Time Local Search (PLS)

Building on the previously presented basic concepts of computational complexity, following subsection introduces the class of Polynomial Local Search (PLS) problems being a branch of complexity theory designed to reason about local search problems.

In simple words, PLS is a complexity class that encompasses heuristics. Problems in PLS are defined by local search procedures where one starts with an initial solution and iteratively improves it by moving to better neighboring solutions according to some local criterion, until no more improvements can be made. The set of possible solutions is typically being exponential in size, making exhaustive search impractical.

The PLS problem can be imagined as attempting to climb some mountain top in a landscape by walking. One climbs up to the top of a hill (i.e. a local optimum), but that hill might not be the tallest mountain in the entire range (i.e. the global optimum). Being at the top of one hill does not necessarily indicate that there isn't a higher peak elsewhere. Finding the highest peak requires a broader search algorithm that might not be practical.

PLS (Polynomial Local Search) class, defined by [7], formalises the problems of searching for local optima (i.e. heuristics). Formally:

Definition 19 (PLS class)**.** *PLS class describes a problem that is either a minimization or maximization problem, where:*

- I *is the set of instances recognizable in polynomial time,*
- $\forall i \in I$*:* S_i *is a finite set of solutions,*
- $\forall i \in I$*:* $cost_i : S_i \to \mathbb{N}$ *is a cost function on the solutions, and*
- $\forall s \in S_i$*:* $N_i(s)$ *is the set of "neighbouring" solutions to* s

The notion of "neighbouring solutions" can be imagined as such solutions that can be obtained by minor alteration of the solution s.

Moreover, for the problem to be in PLS class, the following three algorithms must exist:

1. Algorithm A_1, generates an initial solution $s \in S_i$ for any instance $i \in I$.
2. Algorithm A_2 confirms whether the proposed solution s is valid for given instance $i \in I$ and computes its respective cost, i.e. $\text{cost}_i(s)$.
3. Algorithm A_3, for given instance $i \in I$ and a solution $s \in S_i$, explores neighboring solutions of s and returns solution $s' \in N(i, s)$ with better valuation, meaning a lower cost for a minimization problem or higher cost for a maximization problem, or confirms that s is locally optimal.

Many well-known problem belong to the PLS class, such as the local max-cut.

Local max-cut

A standard form of local search for the Max-Cut problem is the FLIP approach defined in 7 which moves vertices across the two cut partitions until the local optimum is reached.

Let's consider an arbitrary cut U on a graph $G = (V, E)$, where for simplicity all edges have weight $w(e) = 1$. We then choose a vertex $v \in V$ for which moving it to the opposite partition would result in a cut with larger value, meaning that more edges would cross the cut U. This process is being repeated until the flip of any vertex wouldn't increase the resulting weight of the cut.

The process is visualized on the following figure:

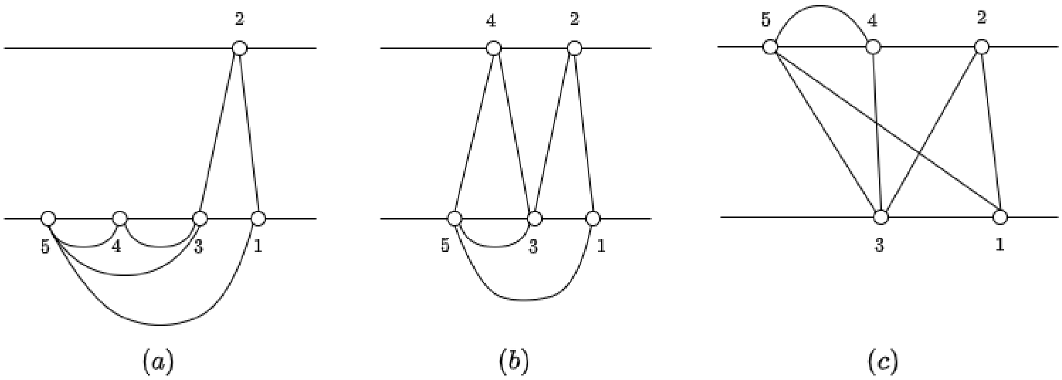

Figure 1.3 Local search for local-max-cut. (a) denotes an arbitrary cut. In (b), the cut value is increased as a result of moving vertex 4 from one partition to the other. In (c), the best possible local result is obtained.

PLS Reductions and Completeness

A local search problem L from PLS is said to be *PLS-reducible* to another local search problem L', when there are polynomial-time algorithms A_1 and A_2 such that:

1. A_1 maps instances of L to instances of L',
2. A_2 maps pairs of (solution, instance) for instances of L' which are produced by A_1 back into solutions of L, and
3. for all instances i of L, if s is a local optimum for instance $A_1(i)$ of L', then $A_2(s, A_1(i))$ is a local optimum for i.

In simpler terms, if you can convert one local search problem into another using a polynomial process, and if solving the second problem gives you a solution to the first, then the first problem is reducible to the second.

By definition, if a local search problem L is PLS-reducible to another local search problem L', then finding a local optimum for L' using a polynomial-time algorithm guarantees a polynomial-time algorithm for finding a local optimum for L. Moreover, a PLS-reduction is transitive.

We say that a problem L in PLS is *PLS-complete* if any problem in PLS is reducible to L. Being *PLS-complete* means that the problem is one of the most difficult in the PLS class, such that if one finds a polynomial-time solution to this problem, he effectively has a method to solve all PLS problems efficiently.

The notions from this subsection were sourced from [8] and from the original publication of [7], where is also stated that PLS-complete problems exist.

2 Potential games

2.1 Potential games

A *potential game* is a game for which there exists a *potential function* Φ such that, for any unilateral deviation by any of the players, the change in Φ is equal to the change in cost incurred by the deviating player.

In this chapter, the class of potential games is going to be defined. Information in the chapter consists of findings from the following publications: [9], [10], [11], [12], [13], [14] and the course notes on Algorithmic Game Theory by Prof. Dr. Thomas Kesselheim (lecture notes).

Definition 20 (Potential function of a game)**.** *Let $\Gamma(N, S, u_1, u_2, ..., u_n)$ be a game with a finite number of players $N = \{1, 2, ..., n\}$, the set of strategies of Player i being S_i and the utility function of Player i being $u_i : S \to \mathbb{R}$, where $S = S_1 \times S_2 \times ... \times S_n$ is the set of strategy profiles. A function $\Phi : S \to \mathbb{R}$ is a* potential function *for Γ, if for every $i \in N$ and for every $s_{-i} \in S_{-i}$*

$$u_i(s_i, s_{-i}) - u_i(s_i', s_{-i}) > 0 \text{ iff}$$

$$\Phi(s_i, s_{-i}) - \Phi(s_i', s_{-i}) > 0$$

where s_{-i} denotes the strategies of all other players except the player i, that is $s_{-i} = (s_1, ..., s_{i-1}, s_{i+1}, ..., s_n)$.

Note that when no confusion may arise $\Gamma(N, S, u^1, u^2, ..., u^n)$ will be denoted by Γ. Also note that the notion of a potential function is sometimes referred to in the literature as *ordinal potential.*

Definition 21 (Ordinal Potential game)**.** *Γ is called an ordinal potential game if there is a function $\Phi : S \to \mathbb{R}$ such that for all players $i \in N$ and strategy profile $s = (s_1, ..., s_n) \in S$,*

$$u_i(s_i, s_{-i}) - u_i(s_i', s_{-i}) > 0 \text{ iff}$$

$$\Phi(s_i, s_{-i}) - \Phi(s_i', s_{-i}) > 0$$

Definition 22 ((Exact) Potential game)**.** *A Game is a potential game if there exists a potential function $\Phi : S \to \mathbb{R}$ such that for all players $i \in N$ with strategy s_i,*

$$\Phi(s_i, s_{-i}) - \Phi(s_i', s_{-i}) = u_i(s_i, s_{-i}) - u_i(s_i', s_{-i})$$

Definition 23 (Weighted Potential game)**.** *Let $w = (w_i)_{i \in N}$ be a vector of positive numbers, i.e. weights. A function $\Phi : S \to \mathbb{R}$ is a w-potential for a game Γ if for every player $i \in N$ with strategy s_i,*

$$\Phi(s_i, s_{-i}) - \Phi(s_i', s_{-i}) = w_i \cdot (u_i(s_i, s_{-i}) - u_i(s_i', s_{-i}))$$

Observation 2.1.1. Potential game and Weighted potential game are subsets of ordinal potential games.

In summary, in order to determine whether a given game is a potential game, it is necessary to verify whether the game possesses a potential function. To illustrate, this involves determining whether there exist a function that aligns with every unilateral deviation by any player. If such function exists, the game is classified as a potential game, otherwise, it is not. It is thus important to realize, that *potential may not exist.*

2.1.1 Determining if a game is a potential game

Consider a Rock-Paper-Scissors game with the following payoff matrix:

	Rock	Paper	Scissors
Rock	(0, 0)	(-1, 1)	(1, -1)
Paper	(1, -1)	(0, 0)	(-1, 1)
Scissors	(-1, 1)	(1, -1)	(0, 0)

Table 2.1 A payoff matrix for Rock-Paper-Scissors game.

To determine whether the game has a potential, we will apply the definition of a potential game 22. Let's assume there exists a potential function Φ and that $\Phi(R, R) = 0$, then:

- When Player 1 unilaterally changes its strategy from R to P, while Player 2 keeps playing strategy R, then Player 1's payoff increases by 1. Hence
  $\Phi(P, R) - \Phi(R, R) = u_1(P, R) - u_1(R, R) = 1 - 0 = 1.$
  $\Phi(P, R) = 1.$

- Similarly, when Player 2 switches from R to P while Player 1 keeps playing R, the Player 2's payoff increases by 1. Hence:
  $\Phi(R, P) - \Phi(R, R) = u_2(R, P) - u_2(R, R) = 1 - 0 = 1.$
  $\Phi(R, P) = 1.$

$\Phi(P, R) = \Phi(R, P) = 1$, thus so far it seems that a potential function might exist.

- Now, consider Player 1 switching from P to S against Player 2's R. Player 1's payoff decreases by 2:
  $\Phi(S, R) - \Phi(P, R) = u_1(S, R) - u_1(P, R) = -1 - 1 = -2$
  $\Phi(S, R) = 1 - 2 = -1$

- Similarly, consider Player 2 switching from P to S against Player 1's R. Player 2's payoff decreases by 2:
  $\Phi(R, S) - \Phi(R, P) = u_2(R, S) - u_2(R, P) = -1 - 1 = -2$
  $\Phi(S, R) = 1 - 2 = -1$

Now we will check for consistency by considering Player 1's and Player 2's strategies (P, S) and (S, P):

- For $\Phi(P, S)$, we would expect Player 1's unilateral deviation from R to P, against Player 2's strategy S:
  $\Phi(P, S) - \Phi(R, S) = u_1(P, S) - u_1(R, S) = -1 + 1 = 0$
  $\Phi(P, S) = \Phi(R, S) = -1$

- For $\Phi(S, P)$, from Player 2's unilateral deviation from R to P, against Player 1's strategy S:
  $\Phi(S, P) - \Phi(S, R) = u_2(S, P) - u_2(S, R) = -1 + 1 = 0$
  $\Phi(S, P) = \Phi(S, R) = -1$

The problem arises when considering the strategy (S, S). Based on the previous results, $\Phi(S, S)$ needs to satisfy
$\Phi(S, S) - \Phi(R, S) = u_1(S, S) - u_1(R, S) = 0 - (-1) = 1$
$\Phi(S, S) - \Phi(S, R) = u_2(S, S) - u_2(S, R) = 0 - (-1) = 1$

The contradiction arises from the sequence of strategies R to P to S to R:
$\Phi(P, R) - \Phi(R, R) = 1$
$\Phi(S, R) - \Phi(P, R) = -2$
$\Phi(R, R) - \Phi(S, R) = 2$

As the latter sequence is a cycle, the result should be 0, but $1 - 2 + 2 \neq 0$.

Thus, we reach a contradiction, as it is impossible to assign a potential function Φ, such that the unilateral change in strategy for both players always corresponds to the change in their payoffs. Therefore, the Rock-Paper-Scissors game does not have a potential function and is not a potential game.

2.1.2 Existence of PNE in Potential Games

Theorem 2.1.2 (Existence of PNE in Potential Games)**.** Every potential game Γ has at least one Pure Nash equilibrium.

Proof. Let s^* be a pure strategy maximizing the potential function Φ. Note that a potential always has the maximum as there is a finite set of strategies. Thus:

$$\Phi(s_i^*, s_{-i}^*) \geq \Phi(s_i', s_{-i}^*)$$

for all $i \in \mathbb{N}, s_i' \in S_i$.
From the definition of the potential function 20:

$$\Phi(s_i^*, s_{-i}^*) - \Phi(s_i', s_{-i}^*) = u_i(s_i^*, s_{-i}^*) - u_i(s_i', s_{-i}^*) \geq 0$$

Hence $u_i(s_i^*, s_{-i}^*) - u_i(s_i', s_{-i}^*) \geq 0$ and s_i^* is a pure Nash equilibrium. □

2.2 Congestion Games

Congestion games represent a class of games, that have been studied mainly in the area of computer science, since their introduction by [15]. They were often studied from the point of analyzing the price of anarchy, i.e. the ration between solution achieved by the worst-case Nash equilibrium and the optimal solution. Congestion games being the special example of potential games, belong to the most extensively studied classes of games in the area of algorithmic game theory.

2.2.1 Definitions and characterizations

Definition 24 (Congestion game)**.** *A congestion game is a tuple* $\Gamma(N, R, (S_i)_{i \in N}), (c_r)_{r \in R}, (c_i(s))_{i \in N})$, *where:*

- $N = \{1, ..., n\}$ *is a set of players,*
- $R = \{1, ..., m\}$ *is a set of resources,*
- $\forall i \in N,\ S_i \subseteq 2^R$ *is a set of strategies of player* i*,*
- $\forall r \in R,\ c^r : \{1, ..., n\} \to \mathbb{R}$ *is a cost function being non-negative and non-decreasing,*
- $\forall i \in N,\ c_i(s) = \sum_{r \in s_i} c^r(|\{i; r \in s_i\}|)$ *is a cost of i-th player for given strategy profile* s*.*

In the thesis we will be interested mainly in the congestion games on graphs.

Definition 25 (Network congestion game)**.** *Let's consider a graph* $G = (V, E)$*. The set of resources* R *corresponds to the set of edges* E*. For each player* $i \in N$*, there exists an origin-destination pair* (o_i, d_i)*, s.t. the strategy set* S_i *corresponds to the set of paths from* o_i *to* d_i*.*

The goal of each player is to choose a strategy that minimizes its total cost. Note, that network congestion games are sometimes in the literature also being referred to as *routing games*.

Example of a congestion game

As example, consider the network congestion game with two players. The origin-destination path for both players is to get from point A to D and the cost function for each edge is defined as $c_e(x) = x$, where x is the number of players that are using the edge e.

Let the strategy profiles S_1 and S_2 for players 1 and 2 be the following:
$S_1 = S_2 = \{s_1 : A \to B \to D, s_2 : A \to B \to C \to D, s_3 : A \to C \to D\}$.

Suppose that at first, players choose their strategies arbitrarily, i.e. Player 1 chooses s_1 and Player 2 chooses s_2. Then costs of the resources will be the following:

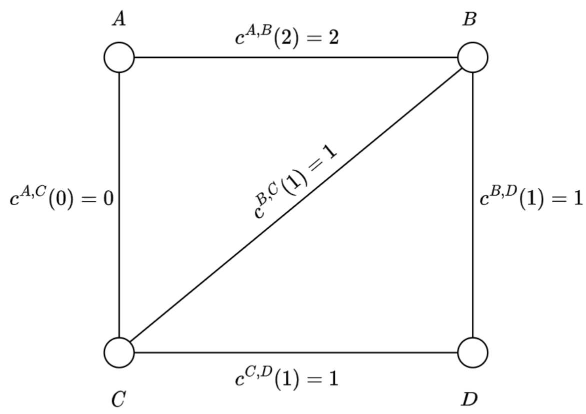

Figure 2.1 Network congestion game.

Thus the respective costs for each player are: $c_1(s_1) = 3$, $c_2(s_2) = 4$.

Note however, that if Player 2 would switch its strategy from s_2 to s_3, the costs of both players would decrease. In such setting: $c_1(s_1) = 2$, $c_2(s_2) = 2$.

Hence, the player's strategy choice impacts his final cost. The goal of the players is thus to generate a sequence of improvement steps to decrease their costs. The latter proceeds as follows: game starts from an arbitrary state s_0 and generate an improvement sequence of states $s_0, s_1, ...$ etc. If there is no improvement step (s_t, s') from state s_t to s', then s_t is a pure Nash equilibrium. Otherwise, there is an improvement step (s_t, s') and it would be possible to set $s_{t+1} = s'$. Improvement step equivalently means that as long as there is a player not playing a best response strategy, he is encouraged to switch to a strategy that is a better response. After finitely many steps, a Pure Nash equilibrium is reached: this holds due to the following theorem of [15];

Theorem 2.2.1 (Rosenthal 1973)**.** For every congestion game, every sequence of improvement steps is finite.

The proof of Rosenthal's theorem is based on the argument of a potential function, called *Rosenthal's potential function*, Theorem 2.1.2 and Theorem 2.3.1:

Definition 26 (Rosenthal's potential)**.** *For every state s, let*

$$\Phi(s) = \sum_{r \in R} \sum_{k=1}^{c^r(s)} c^r(k)$$

2.3 Congestion games are potential games

For the thesis, we have chosen the class of congestion games because of the nice properties they have and that we will further use in other areas of science. Moreover, congestion games are a specific example of potential games, which as seen before always posses a pure Nash equilibrium 2.1.2.

Theorem 2.3.1 (Rosenthal 1973)**.** Every congestion game is a potential game with Φ potential.

Proof. Let s_i and s_i' be strategies for player i. Let s be the strategy profile for all players with strategy s_i and s' the strategy profile with $s_i = s_i'$. Let $\Phi(s) = \sum_{r\in R}\sum_{k=1}^{c^r(s)} c^r(k)$ be defined as in Definition 26. For player i's strategy s_i and s_i' we have:

$$\Phi(s) = \sum_{r\in s\cap s'}\sum_{k=1}^{c^r(s)} c^r(k) + \sum_{r\in s_i\setminus s_i'}\sum_{k=1}^{c^r(s)} c^r(k) + \sum_{r\in s_i'\setminus s_i}\sum_{k=1}^{c^r(s)} c^r(k) + \sum_{r\in R\setminus(s\cup s')}\sum_{k=1}^{c^r(s)} c^r(k)$$

$$\Phi(s') = \sum_{r\in s\cap s'}\sum_{k=1}^{c^r(s')} c^r(k) + \sum_{r\in s_i\setminus s_i'}\sum_{k=1}^{c^r(s')} c^r(k) + \sum_{r\in s_i'\setminus s_i}\sum_{k=1}^{c^r(s')} c^r(k) + \sum_{r\in R\setminus(s\cup s')}\sum_{k=1}^{c^r(s')} c^r(k)$$

As only a single player i's strategy is changed, for $r \in s_i'\setminus s_i, c^r(s') = c^r(s) + 1$ and for $r \in s_i\setminus s_i', c^r(s) = c^r(s') + 1$, where $c^r(s)$ is the number of players using resource r.

The first and fourth term in $\Phi(s)$ and in $\Phi(s')$ are the same, hence:

$$\begin{aligned}\Phi(s) - \Phi(s') =& \Big(\sum_{r\in s_i\setminus s_i'}\sum_{k=1}^{c^r(s)} c^r(k) + \sum_{r\in s_i'\setminus s_i}\sum_{k=1}^{c^r(s)} c^r(k)\Big) - \\ & \Big(\sum_{r\in s_i\setminus s_i'}\sum_{k=1}^{c^r(s')} c^r(k) + \sum_{r\in s_i'\setminus s_i}\sum_{k=1}^{c^r(s')} c^r(k)\Big) \\ =& \sum_{r\in s_i\setminus s_i'} c^r(s) - \sum_{r\in s_i'\setminus s_i} c^r(s')\end{aligned}$$

Let the utility $u_i(s_i)$ be the sum of player i's costs when playing s_i:

$$\begin{aligned} u_i(s_i) = \sum_{r\in s_i} c^r(s) =& \sum_{r\in s_i\cap s_i'} c^r(s) + \sum_{r\in s_i\setminus s_i'} c^r(s) \\ u_i(s_i') =& \sum_{r\in s_i\cap s_i'} c^r(s') + \sum_{r\in s_i\setminus s_i'} c^r(s') \\ \Rightarrow u_i(s_i) - u_i(s_i') =& \sum_{r\in s_i\setminus s_i'} c^r(s) - \sum_{r\in s_i'\setminus s_i} c^r(s') \\ =& \Phi(s) - \Phi(s')\end{aligned}$$

□

2.3.1 Pure Nash equilibria in Congestion Games and PLS

In this part, we explain by example that effective equilibria calculations belong to PLS class and thus it makes good sense to ask if they are PLS-complete. The following example has been obtained thanks to the discussions with my supervisor M. Loebl.

Example: Routing game as PLS problem

Let's consider $(G, o_i, d_i, i \in N)$ as input instance. In this case, the feasible solutions are equivalent to the strategy profiles $s = (s_1, ..., s_n)$, where for each strategy profile s, neighboring feasible solutions are all unilateral deviations. The cost of e is defined as $c^e(k) = k$ for $k \in N$, i.e. cost is identity based on the number of players using the edge e. Cost of a solution s will be defined as Rosenthal potential $\Phi(s)$. Note that the latter is related to cost c^e through Theorem 2.3.1.

For congestion game to be in PLS class, we define the following three algorithms:

- A_1: Initial solution are arbitrary paths o_i, d_i for each $i \in N$.
- A_2: It is straightforward to verify if a set is a strategy profile. We calculate its potential by formula in 26.
- A_3: Given a stratey profile s, we need to find out if there is a unilateral deviation of s of lower potential. By the defining property of a potential game 22(i.e. in a unilateral deviation step, the change in the Rosenthal potential is equal to the corresponding change in the cost of the deviating player.) it is equivalent to check if for some player i there is s'_i so that the cost $c_i(s'_i, s_{-i}) < c_i(s_i, s_{-i})$.

This is done as follows:
Define weights $w(s, i) : E \to \mathbb{N}$ by $w(s, i)(e) = |\{j \neq i; e \in s_j\}| + 1$.
Let P be the shortest path $o_i d_i$. If $w(s, i)(P) < w(s, i)(s_i)$ then we switch s to (P, s_{-i}). Otherwise s is the local optimum.

Observation 2.3.2. The set of local optima is the same as the set of Pure Nash equilibria.

2.4 Future work with congestion games

Note, that for the thesis, we chose specifically congestion games for their nice and natural properties applicable in other fields of science. Notably, there are some special cases of congestion games, such as max-cut games, that happened to have appeared also in the area of statistical physics. This relation will be discussed in the next chapters of the thesis.

3 Statistical Physics

3.1 Introduction

Statistical physics is a branch of physics that uses statistical methods in order to explain and predict the behavior of systems that are composed of a large number of particles. Its is to understand and provide an explanation on how complex behaviours can emerge from the interactions of large numbers of identical elementary components.

The principles of statistical physics rely on two steps: starting with a probabilistic description of the microscopic system and only then reinstating the determinism at the macroscopic level. The information presented in this chapter is sourced from [16].

3.2 Ising Model

Ising model is one of the standard models used in statistical mechanics. It represents a powerful tool to study magnetic systems (sometimes also refered to as a *ferromagnetic Ising model*) and was proven successful in describing a large spectrum of different problems in various fields of physics.

3.2.1 2D Ising Model

As an example, let's consider a two-dimensional lattice $\mathbb{L}$, which is a square grid of side length L: $\mathbb{L} = \{(i, j) \mid 1 \leq i, j \leq L\}$. At every site $(i, j) \in \mathbb{L}$, there is an *Ising spin* $\sigma_{i,j} \in \{-1, +1\}$. Collection $\sigma : V(\mathbb{L}) \to \{-1, +1\}$ of all spins is called a *state*.

As illustrated in figure 3.1, a spin σ_i on each vertex i, is typically being visualized by an upward-pointing arrow if $\sigma_i = +1$ (colored red) while a downward-pointing arrow signifies $\sigma_i = -1$ (colored blue).

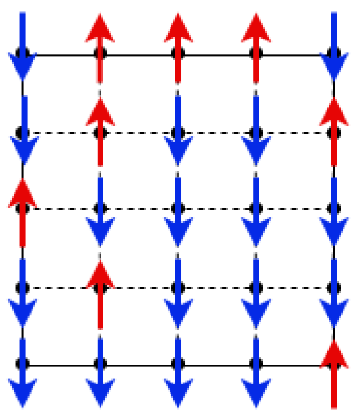

Figure 3.1 Configuration of a 2D Ising model with the lattice size $L = 5$.

In the Ising model, the total energy of the system is determined by the interaction term $-\sigma_i\sigma_j$ for each pair (i, j) of nearest neighboring sites, where the energy is being minimized when the spins σ_i and σ_j at these sites align in the same direction.

Definition 27 (Energy of state σ)**.** *The energy of σ is defined as sum over all pairs of nearest sites:*

$$H(\sigma) = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j$$

, where

- *J, coupling constant, represents the interaction strength between neighboring sites .*

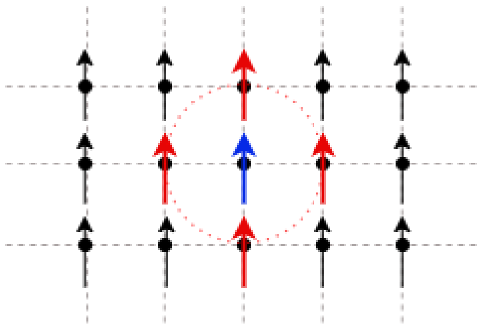

Figure 3.2 Spin interaction with its nearest neighbors: red spins indicate all the spins that the blue spin interacts with

3.2.2 Role of temperature in the Ising model

Definition 28 (Gibbs distribution)**.** *The probability $\pi(s)$ for the system to be found in the state s is given by the* Gibbs distribution*:*

$$\pi(s) = \frac{e^{-\beta H(s)}}{Z(\beta)}$$

where $\beta = 1/(k_B T)$ is referred to as the inverse temperature, with k_B being the Boltzmann's constant. This favours states with minimal energy which are called ground states*.*

Definition 29 (Partition function)**.** *The normalization constant $Z(\beta)$ of the Gibbs distribution is called the* partition function*, defined as*

$$Z(\beta) = \sum_{s:V \to \{1,-1\}} e^{-\beta H(s)}$$

The Gibbs distribution is important for understanding how energy is distributed among the states of a system and for predicting the system's macroscopic properties. The partition function Z serves as the normalization factor in the

Gibbs distribution and determines the critical temperature β_C^{-1}, that can be interpreted as temperature of a macroscopic change in the model. Existence of the critical temperature for the 2D Ising model was proved by [17] who for these findings received the Nobel Prize in 1968. The crucial importance of the finding was that he was the first to predict macroscopic physical phenomena called the *Phase transition*, using microscopic mathematical model, namely $Z(\beta)$.

4 Putting it all together

4.1 Equivalence of Max-cut and Ground State

To see how these two problems are equivalent, consider mapping of Ising model onto a graph where vertices represent sites and edges represent interactions between the neighboring pairs of sites.

4.1.1 Ising model on a graph

The Ising model can be defined on a graph as follows: Let $G = (V, E)$ be an undirected graph, with $V = \{v_1, ..., v_n\}$ and E being a set of edges representing the interactions between pairs of vertices. Each vertex i is associated with a spin variable σ_i. These spins can interact with each other and are again, taking values $+1$ for up or -1 for down.

For each pair of interacting spins σ_i, σ_j, there exists a corresponding edge $(i, j) \in E$. The state of the model, denoted by s, represents assignment of all n variables σ_i. Energy of the state s is given by:

$$H(s) = -\sum_{e=\{i,j\}\in E} w(e)\sigma_i\sigma_j$$

The system prefers lower energy states, i.e., those s that minimise $H(s)$. These are the ground states.

4.1.2 Max-cut revisited:

The Max-Cut problem starts with an undirected graph $G = (V, E)$ with a set of vertices V and a set of edges E between the vertices. The weight $w(e)$ of an edge $e \in E$ is a positive real number. As seen before, a cut is a set of edges that separates the vertices V into two disjoint sets U and $U\backslash V$ and the value of a cut is defined as the sum of all weights of edges connecting vertices in U with vertices in $U\backslash V$. In MAX-CUT problem, the goal is to compute a cut of maximum total value.

4.1.3 Problem mapping:

Observation 4.1.1. States in Ising model $\equiv$ edge-cuts.

One can convert the cut problem to the Ising model by rewriting the weight of each cut $U \subseteq V$ in terms of the Ising energy function. If $s : V \to \{-1, 1\}$ and $U = \{v_i; s_i = 1\}$, then the weight of corresponding cut is:

$$\begin{aligned} U(s) &= \sum_{e=\{i,j\}\in E, s_i\neq s_j} w(e) = \frac{1}{2}\left(-\sum_{e=\{i,j\}\in E} w(e)s_is_j + \sum_{e=\{i,j\}\in E} w(e))\right) \\ &= \frac{1}{2}\sum_{e=\{i,j\}\in E} w(e) + \frac{1}{2}H(s) \end{aligned}$$

4.1.4 Summary:

The Max-Cut problem aims at partitioning the nodes so that the cost of the resulting cut is maximised. Therefore, the Max-Cut problem is essentially reduced to determining the minimum energy state of the Ising model, where $J_{ij} = w_{ij}$:

$$\max(s : U(s)) = \min(s : H(s))$$

The sources for this section are from the following publications, where the equivalence is explained mroe in detail: [18], [19] and in [20].

4.2 Max-Cut Game as a Potential Game

This part has been obtained thanks to the discussions with my supervisor M. Loebl. *Local-max-cut* can be perceived as cut game.

Definition 30 (Local-max-cut as cut game)**.** *Let players correspond to vertices of graph $G(V, E)$, let $\{-1, 1\}$ be possible individual strategies and let strategy profile be each state $s : V \to \{-1, 1\}$. Strategy profile s is a spin assignment and corresponds to a cut.*

4.2.1 Classical approach

Given strategy profile s, we define, for $v \in V$, the cost of v by

$$c_v(s) = -\sum_{e=\{u,v\}\in E} w(e)s_u s_v$$

Each player aims to minimize its cost $c_v(s)$. Equivalently each player v aims to maximize $w_v(s) = \sum_{e=\{u,v\}\in E: s_u \neq s_v} w(e)$, which is the total weight of his incident edges crossing the cut. In the game, players prefer strategy profiles s of minimal energy $H(s)$.

$$H(s) = -\sum_{e=\{u,v\}\in E} w(e)s_u s_v$$

Hence, Ising energy is the potential of the game. Consequently, local max-cuts correspond to the pure Nash equilibria of the defined cut game.

4.2.2 Dynamics approach

We consider potential games evolving using logit dynamics defined by the potential.

Definition 31 (Logit dynamics)**.** *Given strategy profile s, fixed rationality level $T > 0$, choose player v uniformly at random, have v choose strategy $s'_v \in S_v$ with probability:*

$$\frac{e^{-T\Phi(s'_v, s_{-v})}}{\sum_{t\in S_v} e^{-T\Phi(t, s_{-v})}}$$

, where Φ is the potential.

Logit dynamics induces an irreducible and time-reversible Markov chain on the se S with unique stationary distribution (called *Gibbs distribution*).

$$\pi(s) = \frac{e^{-T\Phi(s)}}{\sum_{s \in S} e^{-T\Phi(s)}}$$

If the potential game we study is the cut game, Φ is the energy potential and the Gibbs distribution is used to predict critical behaviour of the Ising model. This leads to the question:

Question 4.2.1. Does there exist critical T for some games analogous to criticality of the Ising model for which there is a natural interpretation?

Our main proposition is that it is possible to define the critical T of Ising model for general congestion games, where the potential Φ would substitute the energy.

The topic of this section is further described in [6], [21], [22].

Conclusion and Future Work

4.3 Application in cancer modelling

The interest in the particular application of cancer evolution, comes from the publication of [23], which was the original motivation for the topic of this thesis.

While there were many attempts to model the cancer growth once the cancer melanoma is formed and after cancer is already diagnosed by the professionals, not much attention was given to the period before the cancer formation.

Problem:
The question we are posing is whether there is some sequence of cell interactions on the microscopic level leading to the macroscopic result in the form of a tumor. Our Question 4.2.1 suggests a possible tool for a novel approach to this problem.

There are numerous papers on the cancer evolution after the macroscopic detection, using evolutionary game theory and coordination games such as [24], [25] and [26], but none examining what's happening before.

4.4 Future work

In the future, aside of cancer modelling, we would like to examine the potential games more deeply and begin to focus on the problem of minimum bisection. We believe that it would be interesting to study the potential equivalence of maximum cut and the problem of minimum bisection from the algorithmic game theory perspective. We can see many applications of min bisection in statistical physics (see. [27]), but none in the area of algorithmic game theory. Our proposition is to extend these ideas to this field.

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A Attachments

A.1 First Attachment