FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

MASTER THESIS

Tomáš Raunig

Lipschitz-free spaces and actions of groups

Department of Mathematical Analysis

Supervisor of the master thesis: doc. Mgr. Marek Cúth, Ph.D.
Consultant: Mgr. Michal Doucha, Ph.D.
Study programme: Mathematical Analysis

Prague 2024

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I would like to express my gratitude to both my supervisor, doc. Mgr. Marek Cúth, Ph.D., and my consultant, Mgr. Michal Doucha, Ph.D.

Large part of this thesis is the culmination of three years of work under the supervision of doc. Cúth, during which he introduced me to the concept of Lipschitz-free spaces, guided me through writing this thesis and the previous one, shared with me much knowledge both theoretical and practical and introduced me to the world of academia. I am grateful for the copious amounts of time he spent on the hundreds of emails we have exchanged, dozens of consults he has provided and all the things he has done, which he did not need to do, but which made him a great supervisor. I am honoured to have had him as my supervisor.

Great thanks also belongs to dr. Doucha who has piqued my interest with the topics of the second part of this thesis and without whose help and guidance it would never come to be. I am looking forward to continuing the work on this and much else under his supervision in the future.

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Author: Tomáš Raunig

Department: Department of Mathematical Analysis

Supervisor: doc. Mgr. Marek Cúth, Ph.D., Department of Mathematical Analysis

Consultant: Mgr. Michal Doucha, Ph.D., Institute of Mathematics of the Czech Academy of Sciences

Abstract: The work is split into two parts. In the first one, we give detailed recount of some recent results from [7]. In particular, we describe an algorithm for computation of the Lipschitz-free p-norm and then apply it to show that, for Lipschitz-free p-spaces constructed over metric spaces, the canonical embeddings are isomorphisms. In the second part, we study actions of groups on the spaces Lip_0 which arise as dual actions to actions induced on Lipschitz-free spaces. We focus on a related question recently asked by Kazhdan and Yom Din. After laying out some groundwork, we give new positive results for some special cases.

Keywords: p-Banach space, Lipschitz-free p-space, p-amenability, group action, invariant vector

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Introduction

In the paper [11] published in 2003, G. Godefroy and N. Kalton introduced[1] the concept of Lipschitz-free spaces which are now, two decades later, a very popular topic of study in the field of Banach spaces. Soon thereafter, in 2009, F. Albiac and N. Kalton generalised[2] the notion to the setting of p-Banach spaces ($p \in (0,1)$), using it to construct two separable p-Banach spaces which are Lipschitz isomorphic, but not linearly isomorphic in their paper [2]. The question whether this is possible for two Banach spaces remains open to this day.

The generalisation has not enjoyed as much attention, possibly because of the problems inherent to working with p-Banach spaces – the topology of these spaces is not (necessarily) locally convex. Without methods and tools relying on the dual, as, most notably, the Hahn-Banach theorem, many things one takes for granted in the world of Banach spaces either do not hold for p-Banach spaces, or one has to be creative and find a different method of proof (the benefit of which is that this new method is often applicable to standard Banach spaces, resulting in new proofs).

Some progress was made in [1], where the authors asked the following question regarding canonical embeddings of Lipschitz-free p-spaces.

Question. Let $(M,d,0)$ be a p-metric space and $0 \in N \subset M$ be its subspace. Is the canonical embedding $\iota : \mathcal{F}_p(N) \to \mathcal{F}_p(M)$ always an isomorphism?

In [1], there is an example showing that the canonical embedding are not always isometries.

A seed for a new method to calculate the values of the p-norm in finite-dimensional spaces appeared in the bachelor thesis [19] and this idea was more fully developed in the paper [7], where it was applied to the problem of canonical embeddings. While the authors were unable to use this method to solve the problem fully, it was used to show that the question has positive answer for the arguably most important special case, Lipschitz-free p-spaces constructed over metric spaces:

Theorem (see Theorem 1.29)**.** *Let $(M,d,0)$ be a pointed metric space and $0 \in N \subset M$ be its subspace. Then the canonical embedding $\iota : \mathcal{F}_p(N) \to \mathcal{F}_p(M)$ is an isomorphism.*

The first chapter of the thesis is primarily dedicated to this result. We start with Section 1.1 where we recall all the necessary definitions and basic well-known facts. Section 1.2 reflects [7, Section 3] in its focus on the proof of the aforementioned algorithm for calculating the p-norm. Finally, Section 1.3 concerns the canonical embeddings with the ultimate goal of proving the theorem formulated above.

The second chapter takes us in a somewhat different direction to the first one. We will switch to the more popular setting of Banach spaces and consider group

[1]The history is more nuanced, but in the context this thesis is set in, the standard reference is [11].

[2]Again, history is more convoluted.

actions on Lipschitz-free spaces. This topic is also not completely quiet, with a recent paper concerning this being [8].

The main motivation for the second chapter this question asked by D. Kahzdan and A. Yom Din coming from their work in [12]:

Question. Let $\delta > 0$, X be a Banach space and let G be a discrete group acting on X by linear isometries. Suppose that $x^* \in S_{X^*}$ satisfies that $\|gx^* - x^*\| \leq \frac{\delta}{10}$ for all $g \in G$. Must there exist G-invariant $y^* \in X^*$ with $\|x^* - y^*\| \leq \delta$?

There are easy examples for which the answer is positive and in [10], E. Glasner and N. Monod found examples for which the answer is negative. The relation between the existence of so-called almost-invariant vectors and invariant vectors is often studied in more general context, one example is Kahzdan's property (T) (for reference see, e.g., [4]).

In Section 2.1 we again introduce the required definitions and collect some needed results, mainly from algebra. Section 2.2 mentions some results for actions of groups on Banach spaces in general and gives a proof of one of the examples in the positive directions, namely actions of discrete amenable groups. Section 2.3 contains new results concerning this question applied to actions of groups induced on Lipschitz-free spaces. We will lay out some groundwork, characterising properties related to the question in the particular case of Lipschitz-free spaces and then show that the question (or some weakening of it) has positive answer in some special cases, namely

Theorem (see Theorem 2.43)**.** *Let $\delta > 0$ and F_S be a free group with generating set S equipped with the word metric and action by left-translations. Let $f \in \mathrm{Lip}_0(F_S)$ be $\delta/3$-invariant. Then the mapping $\overline{f} : F_S \to \mathbb{R}$ defined by*

$$\overline{f}(g) = \sum_{i=1}^{n} a_i f(s_i),$$

where $n \in \mathbb{N}, a \in \{-1, 1\}^n$ and $s \in S^n$ are such that $g = s_1^{a_1} \cdots s_n^{a_n}$ is the reduced word representing g, is an invariant element of $\mathrm{Lip}_0(F_S)$ with $\left\|f - \overline{f}\right\| \leq \delta$.

and

Theorem (see Theorem 2.47)**.** *Let G be a finitely presented group equipped with the word metric and action by left-translations. Then there exists a constant $C > 0$ depending on G such that for any $\delta > 0$ and $f \in \mathrm{Lip}_0(G)$ δ-invariant there is $\overline{f} \in \mathrm{Inv}_G(G)$ with $\left\|f - \overline{f}\right\| \leq C\delta$.*

Both these results use the standard representation of the dual of the Lipschitz-free space as the space of Lipschitz functions vanishing at the base point.

Note that the goal of the thesis is not to necessarily provide the most efficient and straightforward proofs of said results. Take this as an stroll through the Forest of all knowledge; we shall make no haste and at times stray from the main path just to explore where other paths may lead, for, as a wise man once said, "not all those who wander are lost."[3]

[3]Here, the words "man" and "said" have to be understood quite broadly. The quoted text is, of course, a verse from a poem delivered to Bilbo Baggins in a letter written by Gandalf the Grey, one of the Istari. The content of this poem describes Aragorn, the heir of Isildur, at that time generally known as the Strider. All the aforementioned are fictional characters and events from the novel The Lord of the Rings written by J. R. R. Tolkien. The poem appears in the first book, The Fellowship of the Ring, in chapter 10 entitled "Strider".

Before jumping in, let us fix some notation and conventions which are common for both chapters. For $n, m \in \mathbb{Z}$ denote $[n..m] = \{k \in \mathbb{Z} \colon n \leq k \leq m\}$. We consider all ($p$-)Banach spaces over the field $\mathbb{R}$. The sum over empty sets is always considered to be equal to 0.

1. Lipschitz-free p-spaces

After some preliminaries, we will spend this chapter recounting two of the recent results from [7]. Section 1.2 will be dedicated to [7, Theorem 2.2] and Section 1.3 to [7, Theorem 3.21].

1.1 Preliminaries

Before tackling Lipschitz-free p-spaces, let us first introduce some required background machinery and notation. Unless stated otherwise, throughout this chapter we will assume that $p \in (0, 1]$. Let us mention a fact concerning the function $t \mapsto |t|^p$, proven e.g. in [19, Lemma 1.7].

Fact 1.1. *Let $0 < p < 1$ and $x, y \in \mathbb{R}$. Then $|x+y|^p \leq |x|^p + |y|^p$ and equality holds if and only if $xy = 0$.*

We recall the definitions of p-metric spaces and p-Banach spaces and quickly mention some of their properties. For more comprehensive overview see e.g. [14] or [19].

Definition 1.2. Let M be a set and $d : M^2 \to [0, \infty)$. We say (M, d) is a *p-metric space* if (M, d^p) is a metric space; that is,

(M1) $\forall x, y \in M \colon d(x, y) = 0 \iff x = y$,

(M2) $\forall x, y \in M \colon d(x, y) = d(y, x)$,

(M3) $\forall x, y, z \in M \colon d(x, y)^p \leq d(x, z)^p + d(z, y)^p$.

A triple $(M, d, 0)$ is a *pointed p-metric space* if (M, d) is a p-metric space and $0 \in M$ is some distinguished base point.

It is the case that the family $\{U(x, r) \colon x \in M, r > 0\}$, where $U(x, r) = \{y \in M \colon d(x, y) < r\}$, is a basis of a topology metrisable by the metric d^p. From the concavity of $[0, \infty) \ni t \mapsto t^r$ for any $r \in (0, 1]$ follows that if (M, d) is a p-metric space, then it is also q-metric for any $q \in (0, p]$. To make our lives easier, we will always assume all p-metric spaces have at least two points, which saves us from having to deal with some uninteresting degenerate cases.

To illustrate, we give

Example 1.3. The pair $(\mathbb{R}, d)$, where $d(x, y) = |x - y|^{1/p}$, $x, y \in \mathbb{R}$, is a p-metric space denoted in what follows as $(\mathbb{R}, |\cdot|^{1/p})$. Furthermore, this space is not q-metric for any $q \in (p, 1]$.

Proof. The space is p-metric by definition, as $d^p(x, y) = |x - y|$ is a metric on $\mathbb{R}$. To see it is not q-metric for $q > p$ it is enough to calculate

$$d^q(0, 2) = 2^{q/p} > 2 = 1^{q/p} + 1^{q/p} = d(0, 1)^q + d(1, 2)^q.$$

□

Lipschitz functions on p-metric spaces can be defined verbatim as in the metric case:

Definition 1.4. Let $(M,d),(N,e)$ be a p-metric spaces and $f: M \to N$. Define

$$L(f) = \sup_{x,y\in M, x\neq y} \frac{e(f(x),f(y))}{d(x,y)}.$$

We say f is *Lipschitz* if $L(f) < \infty$. If $(M,d,0_M)$ and $(N,e,0_N)$ are pointed p-metric spaces, denote

$$\mathrm{Lip}_0(M,N) = \{f \in N^M \colon L(f) < \infty, f(0_M) = 0_N\}.$$

If $N = (\mathbb{R}, |\cdot|, 0)$, we will write only $\mathrm{Lip}_0(M)$ instead of $\mathrm{Lip}_0(M,\mathbb{R})$.

Definition 1.5. Let X be a linear space. A map $\|\cdot\| : X \to [0,\infty)$ is called a *p-norm* on X if the following conditions are satisfied:

(N1) $\forall x \in X\colon\ \|x\| = 0 \iff x = 0$,

(N2) $\forall x \in X\ \forall \alpha \in \mathbb{R}\colon\ \|\alpha x\| = |\alpha|\,\|x\|$,

(N3) $\forall x,y \in X\colon\ \|x+y\|^p \leq \|x\|^p + \|y\|^p$.

The pair $(X,\|\cdot\|)$ is called a *p-normed linear space*. If $(X,\|\cdot\|)$ is complete (i.e. when endowed with the metric $(x,y) \mapsto \|x-y\|^p$, it is a complete metric space), we say that $(X,\|\cdot\|)$ is a p-Banach space.

Every p-Banach space $(X,\|\cdot\|)$ is also a p-metric space when equipped with the p-metric $d(x,y) = \|x-y\|$. Unless specified otherwise, we will consider 0 (the zero element of the linear space X) as the base point of this p-metric space. As in the case of p-metric spaces, if X is a p-Banach space, then it is also q-Banach for all $q \in (0,p]$. If we take $p = 1$, then these definitions are exactly the definitions of metric and Banach spaces. p-convexity and p-convex hulls are defined, mutatis mutandis, as usual:

Definition 1.6. Let X be a linear space. We say that $A \subseteq X$ is *p-convex* if for any $x,y \in A$ and $\lambda,\mu \in [0,1]$ such that $\lambda^p + \mu^p = 1$ holds $\lambda x + \mu y \in A$. The *p-convex hull* of the set A, denoted $\mathrm{co}_p(A)$, is defined as

$$\mathrm{co}_p(A) = \bigcap\{B \subseteq X\colon A \subseteq B, B \text{ is } p\text{-convex}\}.$$

The set $\mathrm{co}_p(A)$ is the smallest p-convex set containing A (in the sense of inclusion).

While these p-spaces share many properties with their metric counterparts, there are also some crucial differences. One that is important in the context of Lipschitz-free spaces, is that it may happen that the only real-valued Lipschitz functions are constants, as witnessed by $(\mathbb{R}, |\cdot|^{1/p})$:

Example 1.7. Let $p < 1$ and $f : (\mathbb{R}, |\cdot|^{1/p}) \to \mathbb{R}$ be Lipschitz. Then $L(f) = 0$.

Proof. Estimate, as per Definition 1.4,

$$L(f) = \sup \frac{|f(x) - f(y)|}{|x-y|^{1/p}} \leq \sup \frac{|f(x) - f((y+x)/2)| + |f((y+x)/2) + f(y)|}{|x-y|^{1/p}}$$
$$\leq \sup \frac{2L(f)\,|(x-y)/2|^{1/p}}{|x-y|^{1/p}} = 2^{1-1/p} L(f).$$

Since $2^{1-1/p} < 1$ and $L(f) \geq 0$, it must be that $L(f) = 0$. $\square$

An immediate consequence is that Lipschitz extension theorems, such as McShane's, cannot hold. Indeed, the function $f : (\{0,1\}, |\cdot|^{1/p}) \to \mathbb{R}$ defined as $f(0) = 0, f(1) = 1$ is 1-Lipschitz, but by the previous example, there is no Lipschitz extension to $(\mathbb{R}, |\cdot|^{1/p})$.

On the p-Banach side of things, causing a lot of trouble is the fact that for $p < 1$ the Hahn-Banach theorem does not hold and p-Banach spaces may have trivial dual. Classical examples of such spaces are L^p. Another example which requires some knowledge of Lipschitz-free p-spaces follows again from Example 1.7.

Now we are ready to define Lipschitz-free p-spaces. We will again omit details and proofs which can be found in [2], [1] or in the Bachelor thesis [19]. In this thesis, we will not go through the construction of these spaces and instead opt for a definition by theorem approach:

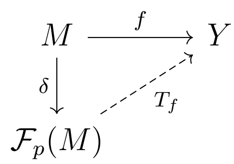
Theorem 1.8 ([19, Theorem 2.9])**.** *Let $(M, d, 0)$ be a pointed p-metric space. Then there exists an up to linear isometry unique p-Banach space $\mathcal{F}_p(M)$ and $\delta : M \to \mathcal{F}_p(M)$ satisfying*

(i) $\delta(0) = 0$*;*

(ii) $\delta(M \setminus \{0\})$ *is linearly independent and* $\mathcal{F}_p(M) = \overline{\operatorname{span}}\, \delta(M)$*;*

(iii) δ *is an isometry;*

(iv) if Y is a p-Banach space and $f \in \mathrm{Lip}_0(M, Y)$, then there exists a linear map $T_f : \mathcal{F}_p(M) \to Y$ satisfying $f = T_f \circ \delta$ and $\|T_f\| = L(f)$, i.e. the following diagram commutes.



The linear mapping T_f from property *(iv)* is called the *linearisation* of the mapping f.

Definition 1.9. The space $\mathcal{F}_p(M)$ from Theorem 1.8 is called the *Lipschitz-free p-space* (over M).

Some authors instead use the name Arens-Eells p-space and notation $\text{Æ}_p(M)$. For a different, less functional analysis based approach, the interested reader may look up the Wasserstein distance, also known as the Kantorovich-Rubenstein metric (for example in the paper [18, Section 1.6]).

Finite-dimensional subspaces of p-Banach spaces are always closed, hence from Theorem 1.8 (ii) follows that $\dim \mathcal{F}_p(M) < \infty$ if and only if $|M| < \infty$ and in this case $\mathcal{F}_p(M) = \operatorname{span} \delta(M)$; this observation will be of use later. Recall that by $c_{00}(M)$ we denote the space $c_{00}(M) = \{f \in \mathbb{R}^M \colon |\{x \in M \colon f(x) \neq 0\}| < \infty\}$. There are two well known ways to express the value of the p-norm:

Proposition 1.10. *Let $(M, d, 0)$ be a pointed p-metric space and $a \in c_{00}(M)$. Then*

$$\left\| \sum_{x \in M} a_x \delta(x) \right\|_{\mathcal{F}_p(M)} = \sup_{Y, f} \left\| \sum_{x \in M} a_x f(x) \right\|_Y ,$$

where the supremum [1] *is taken over all p-Banach spaces Y and mappings $f \in \mathrm{Lip}_0(M, Y)$ with $L(f) \leq 1$.*

The Proposition above is a consequence of the construction of Lipschitz-free p-spaces chosen in the thesis [19], see [19, Definition 2.6].

Proposition 1.11 ([19, Proposition 3.2])**.** *Let $(M, d, 0)$ be a pointed p-metric space and $a \in c_{00}(M)$. Then*

$$\left\| \sum_{x \in M} a_x \delta(x) \right\|_{\mathcal{F}_p} = \inf_{n, y, z, b} \left\{ \left(\sum_{i=1}^{n} |b_i|^p \right)^{1/p} : \sum_{x \in M} a_x \delta(x) = \sum_{i=1}^{n} b_i \frac{\delta(y_i) - \delta(z_i)}{d(y_i, z_i)} \right\},$$

where the infimum is taken over all $n \in \mathbb{N}$, $b \in \mathbb{R}^n$ and $y, z \in M^n$ such that $y_i \neq z_i, i \in [1..n]$.

In both cases, the value of the norm is given only on (finite) linear combinations of elements of the form $\delta(x), x \in M$, but by Theorem 1.8 (ii) and continuity of p-norms, this extends uniquely to the whole space.

Later on, the relation of the Lipschitz-free p-spaces of a p-metric space and its subspace will be of interest. If $(M, d, 0)$ is a p-metric space, and $0 \in N \subset M$, then technically $\mathcal{F}_p(N)$ is not a subset of $\mathcal{F}_p(M)$, but there is a natural way to identify elements of $\mathcal{F}_p(N)$ with elements of $\mathcal{F}_p(M)$:

Definition 1.12. Let $(M, d, 0)$ be a p-metric space, and $0 \in N \subset M$. Denote $i : N \to M$ the inclusion map (i.e. $i(x) = x, x \in N$). The *canonical embedding* of $\mathcal{F}_p(N)$ into $\mathcal{F}_p(M)$ is defined as the linearisation of $\delta_M \circ i$, where δ_M is the embedding of M into $\mathcal{F}_p(M)$.

If we denote the canonical embedding as ι, the definition above says that the diagram

$$\begin{array}{ccc} N & \xrightarrow{\;i\;} & M \\ {\scriptstyle \delta_N}\downarrow & & \downarrow{\scriptstyle \delta_M} \\ \mathcal{F}_p(N) & \xrightarrow[\iota]{} & \mathcal{F}_p(M) \end{array}$$

[1] At first glance, one might have doubts whether this is valid in ZF. The supremum is to be understood as the supremum over all real numbers $r \geq 0$ such that there exist a p-Banach space Y and $f \in \mathrm{Lip}_0(M, Y)$ with $L(f) = 1$ such that r equals the norm on the right hand side. By the axiom schema of specification, this is a subset of the real numbers.

commutes, or equivalently that $\iota : \mathcal{F}_p(N) \to \mathcal{F}_p(M)$ is the linear mapping such that for any $x \in N$ holds $\iota(\delta_N(x)) = \delta_M(x)$.

This definition is correct since $\mathcal{F}_p(M)$ is a p-Banach space and $L(\delta_M \circ i) \leq L(\delta_M)L(i) = 1$ as both mappings are isometries. From now on, we will identify the elements $\mu \in \mathcal{F}_p(N)$ and $\iota(\mu) \in \mathcal{F}_p(M)$. Immediate consequence of the definition and Theorem 1.8 (iv) is

Fact 1.13. *Let $(M, d, 0)$ be a p-metric space, $0 \in N \subset M$ be its subspace and $\iota : \mathcal{F}_p(N) \to \mathcal{F}_p(M)$ be the canonical embedding. Then $\|\iota\| \leq 1$, i.e. for any $\mu \in \mathcal{F}_p(N)$ holds $\|\mu\|_{\mathcal{F}_p(M)} \leq \|\mu\|_{\mathcal{F}_p(N)}$.*

Last useful piece of information is

Fact 1.14. *Let $(M, d, 0)$ be a pointed p-metric space. Then for a finite set $0 \in F \subset M$ and $a \in \mathbb{R}^F$ holds*

$$\left\|\sum_{x\in F} a_x\delta(x)\right\|_{\mathcal{F}_p(M)} = \inf\left\{\left\|\sum_{x\in F} a_x\delta(x)\right\|_{\mathcal{F}_p(N)} : F \subset N \subset M, |N| < \infty\right\}.$$

Proof. By Fact 1.13, the norm is not larger than the infimum. Let $\varepsilon > 0$, from Proposition 1.11 follows the existence of $n \in \mathbb{N}$, $b \in \mathbb{R}^n$ and $y, z \in M^n$ such that $\sum_{x\in M} a_x\delta(x) = \sum_{i=1}^n b_i \frac{\delta(y_i)-\delta(z_i)}{d(y_i,z_i)}$ and $\sum_{i=1}^n |b_i|^p \leq \|\sum_{x\in F} a_x\delta(x)\|^p_{\mathcal{F}_p(M)} + \varepsilon$. Put

$$N = \{0\} \cup \{y_i : i \in [1..n]\} \cup \{z_i : i \in [1..n]\}.$$

Since the set $\delta(M \setminus \{0\})$ is linearly independent in $\mathcal{F}_p(M)$, necessarily $F \subset N$ and again by Proposition 1.11 we have

$$\left\|\sum_{x\in F} a_x\delta(x)\right\|^p_{\mathcal{F}_p(N)} \leq \sum_{i=1}^n |b_i|^p \leq \left\|\sum_{x\in F} a_x\delta(x)\right\|^p_{\mathcal{F}_p(M)} + \varepsilon.$$

□

1.2 Computation of the Lipschitz-free p-norm

First problem we will concern ourselves with is computation of the Lipschitz-free p-norm. Propositions 1.10 and 1.11 can be used for some ad-hoc computation of the p-norm, but a reliable method is missing even for finite p-metric spaces. The starting point for us will be [7, Lemma 2.4], which is an adaptation of [19, Theorem 3.8] (in the Bachelor thesis [19], it was proven only for the case of Lipschitz-free p-spaces).

Lemma 1.15. *Let $X \neq \{0\}$ be a finite-dimensional p-Banach space and $Z \subseteq B_X$ be a symmetric set such that $\operatorname{co}_p(Z) = B_X$. Let us denote by $\mathfrak{B}$ the set of all the algebraic bases of X consisting of points from Z. Then for $x \in X$ we have*

$$\|x\| = \min\left\{\left(\sum_{b\in\mathfrak{b}} |c_b|^p\right)^{1/p} : \mathfrak{b} \in \mathfrak{B}, c \in \mathbb{R}^{\mathfrak{b}} \text{ is such that } x = \sum_{b\in\mathfrak{b}} c_b b\right\}.$$

While this result already allows us to compute the norm in finitely-dimensional Lipschitz-free spaces in finitely many steps, it has two drawbacks. It may not be immediately clear, but most of the terms over which the minimum is taken are redundant. For ease of computation, it is desirable to remove these redundancies. Second, sets of bases consisting of points from some set are not the most natural way to capture this problem. Both these can be addressed at once - by translating the previous lemma to the language of trees.

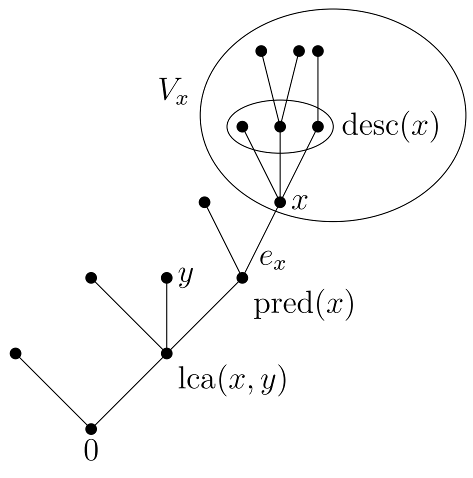
To be able to translate to a language, one must first learn it, which we do in

Definition 1.16. Let V be a finite set and $0 \in V$. A *rooted tree* with vertices V, edges E and root 0 is a triple $(V, E, 0)$ where (V, E) is an acyclic connected (unoriented) graph and $0 \in V$ is an arbitrary distinguished point of V. By $\mathcal{T}(V, 0)$ we denote the set of all rooted trees $T = (V, E, 0)$ with vertices V, edges E and root 0.

A *path (from x to y)* in a graph $G = (V, E)$ is a one-to-one sequence of vertices $x = x_0, \dots, x_n = y$ such that $\{x_{i-1}, x_i\} \in E$ for $i \in [1..n]$. Given $T = (V, E, 0) \in \mathcal{T}(V, 0)$ and $x, y \in V$, we denote

- $\mathrm{desc}(x)$ the immediate descendants of x, that is vertices $v \in V$ such that there is a path $0 = x_0, \dots, x_n = v$ from 0 to v in T with $x_{n-1} = x$;
- if $x \neq 0$, then $\mathrm{pred}(x)$ denotes the unique vertex $v \in V$ such that if $0 = x_0, \dots, x_n = x$ is the unique path from 0 to x in T then $x_{n-1} = v$; we denote $e_x := \{\mathrm{pred}(x), x\}$;
- by $\mathrm{lca}(x, y) \in V$ we denote the lowest common ancestor of x and y, that is if $0 = x_0, \dots, x_n = x$ and $0 = y_0, \dots y_m = y$ are the paths from 0 to x and y and $k \in [1..n]$ is maximal such that $x_k = y_k$, then $\mathrm{lca}(x, y) = x_k$;
- *leaves of* T are vertices from the set $\mathrm{leaf}(T) := \{x \in V\colon\ \mathrm{desc}(x) = \emptyset\}$;
- by V_x we denote vertices in the maximal connected subgraph of $(V, E \setminus \{e_x\})$ containing x as a vertex (that is, vertices in the subtree of T "rooted at the point x").

A picture to illustrate:



If we wish to emphasize to which $T \in \mathcal{T}(V,0)$ the notions correspond, we write E^T, $\operatorname{desc}^T(x)$, $\operatorname{pred}^T(x)$, $\operatorname{lca}^T(x,y)$, e_x^T, and V_x^T instead of E, $\operatorname{desc}(x)$, $\operatorname{pred}(x)$, $\operatorname{lca}(x,y)$, e_x and V_x, respectively.

If $(M,d,0)$ is a pointed p-metric space, we write $\mathcal{T}(M)$ instead of $\mathcal{T}(M,0)$. For $T=(M,E,0) \in \mathcal{T}(M)$ and $e=\{x,y\} \in E$ we put $d(e) := d(x,y)$.

Now we must for a moment leave the notion of a distance behind and venture into the world of linear algebra. First, we describe the correspondence between trees and bases:

Lemma 1.17. *Let X be a linear space, $n \in \mathbb{N}$ and $\{e_1,\dots,e_n\}$ be a basis of X. Let $c_{i,j} \in \mathbb{R}\setminus\{0\}, i,j \in [0..n]$. Denote $e_0 = 0$, $V=\{e_i : i \in [0..n]\}$ and $\mathfrak{B}$ the set of all bases of X consisting of elements from*

$$W = \left\{ \frac{e_i - e_j}{c_{i,j}} : i,j \in [0..n], i \neq j \right\}.$$

Then for each basis $\mathfrak{b} \in \mathfrak{B}$ there is a tree $T \in \mathcal{T}(V,e_0)$ and $\alpha \in \{-1,1\}^n$ such that

$$\mathfrak{b} = \left\{ \alpha_i \frac{e_i - e_j}{c_{i,j}} : i \in [1..n], e_j = \operatorname{pred}^T(e_i) \right\} \tag{1.1}$$

Moreover, for every $T \in \mathcal{T}(V,e_0)$ there is $\mathfrak{b} \in \mathfrak{B}$ such that (1.1) holds with $\alpha \equiv 1$.

Proof. First, let $\mathfrak{b} \in \mathfrak{B}$. Define $T_{\mathfrak{b}} = (V, E_{\mathfrak{b}}, e_0)$, where

$$E_{\mathfrak{b}} = \left\{ \{e_i, e_j\} : \frac{e_i - e_j}{c_{i,j}} \in \mathfrak{b} \right\}.$$

We need to verify that $T_{\mathfrak{b}}$ is in fact a tree. To show this, it is enough to check that it has $|V|-1$ edges and contains no cycles (see [17, Section 3-2]). If $\{e_i,e_j\},\{e_{i'},e_{j'}\} \in E_{\mathfrak{b}}$ for some $i,j,i',j' \in [0..n]$ then $(e_i-e_j)/c_{i,j}$ and $(e_{i'}-e_{j'})/c_{i',j'}$ are elements of $\mathfrak{b}$ and hence $\{e_i,e_j\} \neq \{e_{i'},e_{j'}\}$ for otherwise one of the vectors would be a multiple of the other, which is impossible since $\mathfrak{b}$ is a basis. Hence $|E_{\mathfrak{b}}| = |\mathfrak{b}| = n = |V|-1$.

Now assume that $T_{\mathfrak{b}}$ contains a cycle. By renumbering the vectors $e_0,\dots,e_n$, we may without loss of generality assume that there is $k \in [3..n]$ such that $\{e_{i-1},e_i\} \in E_{\mathfrak{b}}, i \in [1..k]$ and $\{e_k,e_0\} \in E_{\mathfrak{b}}$. By definition of $E_{\mathfrak{b}}$, there are $b_i \in \mathfrak{b}, i \in [0..k]$ such that $e_{i-1}-e_i \in \operatorname{span}\{b_i\}, i \in [1..k]$ and $e_k - e_0 \in \operatorname{span}\{b_0\}$. We have that

$$e_k - e_0 = \sum_{i=1}^{k} e_i - e_{i-1} \in \operatorname{span}\{b_1,\dots,b_k\}$$

and hence also $b_0 \in \operatorname{span}\{b_1,\dots,b_k\}$. But this is in contradiction with the assumption that $\mathfrak{b}$ is a basis. Thus we have shown that $T_{\mathfrak{b}}$ is indeed a tree.

Since $\mathfrak{b} \subset W$, each element of $\mathfrak{b}$ can be written as $(e_i-e_j)/c_{i,j}$ and by definition of $E_{\mathfrak{b}}$, either $e_i = \operatorname{pred}(e_j)$ or $e_j = \operatorname{pred}(e_i)$. By appropriately choosing the value of α, one can guarantee that the vector with negative coefficient is the predecessor of the other. Taking into account that $E_{\mathfrak{b}} = \{\{e_i, \operatorname{pred}^{T_{\mathfrak{b}}}(e_i)\} : i \in [1..n]\}$ and that if $(e_i-e_j)/c_{i,j} \in \mathfrak{b}$, then $(e_j-e_i)/c_{j,i} \notin \mathfrak{b}$, we see that (1.1) holds.

For the moreover part, assume $T = (V, E, e_0) \in \mathcal{T}(V, e_0)$ and denote

$$\mathfrak{b}_T = \left\{ \frac{e_i - e_j}{c_{i,j}} : i \in [1..n], j \in [0..n], e_j = \mathrm{pred}^T(e_i) \right\} \subset W.$$

Once we verify that $\mathfrak{b}_T$ is a basis, we will be done, because then (1.1) clearly holds. This we do by induction. For $n = 1$ it is clear: in this case $W = \{e_1/c_{1,0}, -e_1/c_{0,1}\}$ and $\mathcal{T}(V, e_0)$ consists of only one tree, given by $E = \{\{e_0, e_1\}\}$. Now assume the claim holds for bases of size $n-1$. Let $T = (V, E, e_0) \in \mathcal{T}(V, e_0)$. Pick $k \in [1..n]$ such that $e_k \in \mathrm{leaf}(T)$. Then $T' = (V \setminus \{e_k\}, E \setminus \{e(e_k)\}, e_0) \in \mathcal{T}(V \setminus \{e_k\}, e_0)$. By induction assumption, $\mathfrak{b}_{T'}$ is a basis of $\mathrm{span}(V \setminus \{e_k\})$. It is the case that $\mathfrak{b}_T = \mathfrak{b}_{T'} \cup \{(e_k - e_{k'}/c_{k,k'})\}$, where $e_{k'} = \mathrm{pred}(e_k)$. Hence $\mathrm{span}(\mathfrak{b}_T) = \mathrm{span}\{e_1, \ldots e_n\} = X$ and $|\mathfrak{b}_T| = |\mathfrak{b}_{T'}| + 1 = (n-1) + 1 = n$, i.e. $\mathfrak{b}_T$ is a generating set of cardinality equal to the dimension of the space. □

In fact, for each tree T and $\alpha \in \{-1, 1\}^n$ there is a basis such that (1.1) holds: replacing $(e_i - e_j)/c_{i,j}$ by $(e_j - e_i)/c_{j,i}$ in a basis produces a different basis, but they both induce the same tree. This is where the promised reduction of number of expressions over which we minimize happens.

Now that we have the correspondence between bases and trees established, we calculate how to change coordinates between these bases.

Lemma 1.18. *Let $n \geq 1$ and $\{e_1, \ldots, e_n\}$ be a basis in a linear space X, $V = \{0, e_1, \ldots, e_n\}$ and $T = (V, E, 0) \in \mathcal{T}(V, 0)$. Suppose that for some $a, b \in \mathbb{R}^n$ and $c \in (\mathbb{R} \setminus \{0\})^n$ we have*

$$\sum_{i=1}^{n} a_i e_i = \sum_{i=1}^{n} b_i \frac{e_i - \mathrm{pred}^T(e_i)}{c_i}. \tag{1.2}$$

Then

$$b_i = c_i \sum_{\substack{j \in [1..n]: \\ e_j \in V_{e_i}^T}} a_j \quad \textit{for every } i \in [1..n].$$

Proof. The proof will be conducted by induction with respect to $\dim X = n$. For $n = 1$ the claim holds, because $\mathcal{T}(V, 0) = \{(V, \{\{e_1, 0\}\}, 0)\}$ and thus we have $a_1 e_1 = b_1 \frac{e_1 - 0}{c_1}$, implying that $b_1 = c_1 a_1$.

Suppose the claim holds for some $n \in \mathbb{N}$. For convenience's sake assume $e_{n+1} \in \mathrm{leaf}(T)$ and $e_n = \mathrm{pred}(e_{n+1})$, which we may do without loss of generality as we can simply renumber the vectors of our basis. Because $e_{n+1} \in \mathrm{leaf}(T)$, e_{n+1} only appears once in (1.2), in particular in the term $b_{n+1}(e_{n+1} - e_n)/c_{n+1}$. Hence, as before, $b_{n+1} = c_{n+1} a_{n+1}$. We have shown the desired equality for $i = n+1$ as $V_{e_{n+1}}^T = \{e_{n+1}\}$. For $i \leq n$, put $V' = V \setminus \{e_{n+1}\}$, $E' = E \setminus \{\{e_{n+1}, e_n\}\}$ and consider the tree $T' = (V', E', 0)$. If we put $a_i' = a_i, i \in [1..n-1]$ and

$a'_n = a_n + a_{n+1}$, then we have the equalities

$$\begin{aligned}\sum_{i=1}^{n} a'_i e_i &= \sum_{i=1}^{n} a_i e_i + a_{n+1} e_n = \sum_{i=1}^{n+1} a_i e_i + a_{n+1}(e_n - e_{n+1}) \\ &= \sum_{i=1}^{n+1} b_i \frac{e_i - \operatorname{pred}^T(e_i)}{c_i} + b_{n+1} \frac{e_n - e_{n+1}}{c_{n+1}} \\ &= \sum_{i=1}^{n+1} b_i \frac{e_i - \operatorname{pred}^T(e_i)}{c_i} - b_{n+1} \frac{e_{n+1} - \operatorname{pred}^T(e_{n+1})}{c_{n+1}} \\ &= \sum_{i=1}^{n} b_i \frac{e_i - \operatorname{pred}^T(e_i)}{c_i} = \sum_{i=1}^{n} b_i \frac{e_i - \operatorname{pred}^{T'}(e_i)}{c_i},\end{aligned}$$

where the last equality follows from the fact that removing a leaf does not change the predecessor of any other vertex. The induction assumption now guarantees that for $i \in [1..n]$ the first equality in

$$b_i = c_i \sum_{\substack{j \in [1..n]: \\ e_j \in V_{e_i}^{T'}}} a'_j = c_i \sum_{\substack{j \in [1..n+1]: \\ e_j \in V_{e_i}^{T}}} a_j$$

holds. The second equality follows from the following consideration: if for some $i \in [1..n]$ holds $e_{n+1} \notin V_{e_i}^T$, then $V_{e_i}^T = V_{e_i}^{T'}$ and for all relevant $j \in [1..n]$ holds $a_j = a'_j$. On the other hand, if $e_{n+1} \in V_{e_i}^T$, then $V_{e_i}^T = V_{e_i}^{T'} \cup \{e_{n+1}\}$ and since $i \le n$, it must be the case that $e_n \in V_{e_i}^{T'}$. Hence

$$\begin{aligned}c_i \sum_{\substack{j \in [1..n]: \\ e_j \in V_{e_i}^{T'}}} a'_j &= c_i a'_n + c_i \sum_{\substack{j \in [1..n-1]: \\ e_j \in V_{e_i}^{T'}}} a'_j \\ &= c_i(a_n + a_{n+1}) + c_i \sum_{\substack{j \in [1..n-1]: \\ e_j \in V_{e_i}^{T}}} a_j = c_i \sum_{\substack{j \in [1..n+1]: \\ e_j \in V_{e_i}^{T}}} a_j.\end{aligned}$$

in the first sum appears a'_n instead of $a_n + a_{n+1}$ in the second sum, but by definition of a'_n, those are equal. This yields the desired equality for $i < n+1$ and thus finishes the induction step and also the proof. □

At this point, we finally have all the tools ready to prove the main result of this section. Before doing so, let us define some notation.

Notation 1.19. Let $(M, d, 0)$ be a pointed p-metric space. We define the set of *molecules* in $\mathcal{F}_p(M)$ as

$$\mathcal{A}(M) = \left\{ \frac{\delta(x) - \delta(y)}{d(x,y)} : x, y \in M, x \neq y \right\} \subseteq \mathcal{F}_p(M).$$

Note that $\mathcal{A}(M)$ is symmetric and for M finite, $\mathcal{A}(M)$ is also finite.

Notation 1.20. Let $(M, d, 0)$ be a finite pointed p-metric space, $a \in \mathbb{R}^M$, $x \in M$ and $T \in \mathcal{T}(M)$. Denote

$$c^T(x,a) = \sum_{y \in V_x^T} a_y \quad \text{and} \quad T(a) = \left(\sum_{x \in M \setminus \{0\}} \left| c^T(x,a) d(e_x^T) \right|^p \right)^{1/p}.$$

Theorem 1.21 ([7, Theorem 2.2])**.** *Let $(M,d,0)$ be a finite pointed p-metric space. Then for any $a \in \mathbb{R}^M$ holds*

$$\left\| \sum_{x\in M} a_x \delta(x) \right\|_{\mathcal{F}_p} = \min_{T\in\mathcal{T}(M)} T(a).$$

Proof. Let $a \in \mathbb{R}^M$ and put $\mu = \sum_{x\in M} a_x\delta(x)$. It holds that $\mathrm{co}_p(\mathcal{A}(M)) = \overline{\mathrm{co}_p}(\mathcal{A}(M)) = B_{\mathcal{F}_p(M)}$. The first equality is a consequence of M, and thus also $\mathcal{A}(M)$, being finite. The second can be found e.g. in [19, Proposition 3.7]. By invoking our staring point, Lemma 1.15, with $X = \mathcal{F}_p(M)$ and $Z = \mathcal{A}(M)$ we obtain the equality

$$\|\mu\| = \min\left\{ \left(\sum_{b\in\mathfrak{b}} |c_b|^p\right)^{1/p} : \mathfrak{b}\in\mathfrak{B}, c\in\mathbb{R}^{\mathfrak{b}} \text{ is such that } \mu = \sum_{b\in\mathfrak{b}} c_b b \right\},$$

where $\mathfrak{B}$ is the set of all bases of $\mathcal{F}_p(M)$ consisting of points from $\mathcal{A}(M)$. Enumerate $M = \{0, x_1, \ldots, x_n\}$ and pick some $\mathfrak{b}\in\mathfrak{B}$. By Lemma 1.17 applied to $X = \mathcal{F}_p(M)$, $\{e_1,\ldots,e_n\} = \{\delta(x_1),\ldots,\delta(x_n)\}$ and $c_{i,j} = d(x_i,x_j)$ and using that $V = \{0,\delta(x_1),\ldots,\delta(x_n)\}$ is bijective to M, we infer that there is $T\in\mathcal{T}(M)$ such that

$$\mathfrak{b} = \left\{ \alpha_x \frac{\delta(x) - \delta(\mathrm{pred}^T(x))}{d(e_x^T)} : x \in M\setminus\{0\} \right\}$$

for some $\alpha \in \{-1,1\}^M$. Since the values the minimum is taken over depends only on the absolute value of the coefficients c_b, we may, by switch the sign of the appropriate coefficients, without loss of generality assume that

$$\mathfrak{b} = \left\{ \frac{\delta(x) - \delta(\mathrm{pred}^T(x))}{d(e_x^T)} : x \in M\setminus\{0\} \right\}.$$

If we denote $b_x = (\delta(x) - \delta(\mathrm{pred}^T(x)))/d(e_x^T)$, then the condition in the minimum guarantees that

$$\sum_{x\in M\setminus\{0\}} a_x\delta(x) = \sum_{x\in M} a_x\delta(x) = \sum_{b\in\mathfrak{b}} c_b b = \sum_{x\in M\setminus\{0\}} c_{b_x} \frac{\delta(x) - \delta(\mathrm{pred}^T(x))}{d(e_x^T)}.$$

By Lemma 1.18, $c_{b_x} = d(e_x^T)\sum_{y\in V_x^T} a_y = d(e_x^T)c^T(x,a)$ for $x\in M\setminus\{0\}$ and hence

$$\sum_{b\in\mathfrak{b}} |c_b|^p = \sum_{x\in M\setminus\{0\}} |c_{b_x}|^p = \sum_{x\in M\setminus\{0\}} \left| d(e_x^T)c^T(x,a)\right|^p = T(a)^p.$$

To summarize, for every basis $\mathfrak{b}\in\mathfrak{B}$ we have found $T\in\mathcal{T}(M)$ such that $(\sum_{b\in\mathfrak{b}}|c_b|^p)^{1/p} = T(a)$. This implies that $\|\mu\| \geq \min_{T\in\mathcal{T}(M)} T(a)$. The other inequality follows from the fact that every tree arises in this way for some basis, which we have shown in Lemma 1.17 as well. □

The significance of this Theorem is the fact that, albeit in finite dimension, it enables us to calculate the norm of any element of the Lipschitz-free space as a minimum over finitely many expressions, in particular over $|\mathcal{T}(M)|$ expressions. Cayley's formula (see e.g. [17, Theorem 3.10]) states that $|\mathcal{T}(M)| = |M|^{|M|-2}$.

The following Corollary has been proven in the thesis [19, Corollary 3.10]. To show the algorithm from Theorem 1.21 at work, we will use it to give a somewhat simpler proof.

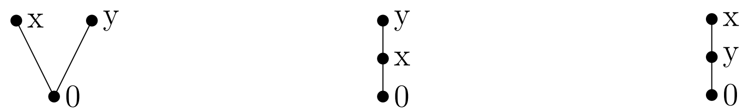
Corollary 1.22. *Let $(M, d, 0)$ be a pointed p-metric space with $M = \{0, x, y\}$ and denote $d_x = d(x, 0)$, $d_y = d(y, 0)$ and $d_{xy} = d(x, y)$. Then for $a, b \in \mathbb{R}$ holds*

$$\|a\delta(x) + b\delta(y)\|_{\mathcal{F}_p}^p = \min \left\{ \begin{array}{l} |ad_x|^p + |bd_y|^p \,, \\ |(a+b)d_x|^p + |bd_{xy}|^p \,, \\ |(a+b)d_y|^p + |ad_{xy}|^p \,. \end{array} \right\}$$

Proof. We begin by realising that $\mathcal{T}(M) = \{(M, E_i, 0) \colon i \in [1..3]\}$, where

$$E_1 = \{\{0, x\}, \{0, y\}\}, \quad E_2 = \{\{0, x\}, \{x, y\}\}, \quad E_1 = \{\{0, y\}, \{y, x\}\}.$$

Pictorially:



$(M, E_1, 0)$ $(M, E_2, 0)$ $(M, E_3, 0)$

We will work through the second tree in detail, the rest can be done in similar fashion. Let $T = (M, E_2, 0)$. Then $V_x^T = \{x, y\}$ and $V_y^T = \{y\}$. Calculate

$$c^T(x, (a, b)) = a + b, \quad c^T(y, (a, b)) = b$$

and

$$T((a, b))^p = \left|c^T(x, (a, b))d(e_x^T)\right|^p + \left|c^T(y, (a, b))d(e_y^T)\right|^p = |(a+b)d_x|^p + |bd_{xy}|^p \,.$$

For $T = (V, E_1, 0)$ we would obtain $T((a, b))^p = |ad_x|^p + |bd_y|^p$ and for $T = (V, E_3, 0)$ we would get $T((a, b))^p = |(a+b)d_y|^p + |ad_{xy}|^p$. By Theorem 1.21, the p-th power of the p-norm is the minimum of these values. □

This formula was derived for the case $p = 1$ in [9], where the authors were, in addition, able to recognize which of these three expressions was equal to the minimum based purely on the values of the coefficients a, b. This we cannot do, but it is not a fault of our method, but rather a property of these spaces when $p < 1$ as is demonstrated by the following theorem, which is the last result of the section dedicated to calculating the p-norm.

Theorem 1.23. *Let $0 < p < 1$ and $(M, d, 0)$ be a pointed p-metric space with $|M| \geq 3$. Then the following assertions are equivalent:*

(i) For any $a \in [0, \infty)^M$ such that $\sum_{x \in M} \big(a_x d(x, 0)\big)^p < \infty$ we have

$$\left\| \sum_{x \in M} a_x \delta(x) \right\|_{\mathcal{F}_p(M)}^p = \sum_{x \in M} \big(a_x d(x, 0)\big)^p.$$

(ii) For every $x \in M \setminus \{0\}$ we have $d(x, 0) = d(x, M \setminus \{x\})$.

We omit the proof of the theorem, as it has been proven in the Bachelor thesis [19, Theorem 3.4]. (For a shorter proof using Theorem 1.21 see [7, Theorem 2.8].)

This theorem can also be used to calculate the p-norm. The formula from (i) is very simple, but the trade-off is that two quite restrictive conditions have to be satisfied for it to hold - namely (iii) and non-negativity of coefficients. But as we will see in the beginning of the next section, it can still be used to obtain some interesting results.

We will end this section with some examples comparing the Lipschitz-free p-norm p-norm to a weighted ℓ_p p-norm.

Example 1.24. Let $0 < p < 1$ and $(M, d, 0)$ be a p-metric space with $M = \{0, x, y\}$. Denote d_x, d_y and d_{xy} as in Corollary 1.22. Define $\|\cdot\|_{\ell_{p,d}} : \mathcal{F}_p(M) \to \mathbb{R}$ as the weighted ℓ_p norm $\|a\delta(x) + b\delta(y)\|^p_{\ell_{p,d}} = |ad_x|^p + |bd_y|^p$, $a, b \in \mathbb{R}$. Denote $\mathcal{F}^+_p(M) = \{a\delta(x) + b\delta(y) \colon a, b \geq 0\}$. It may happen that

- $\|\cdot\|_{\mathcal{F}_p(M)} = \|\cdot\|_{\ell_{p,d}}$ on $\mathcal{F}_p(M)$:

  Let $a, b \in \mathbb{R}$. We have $|a|^p = |a + b - b|^p \leq |a + b|^p + |b|^p$. So, if $d^p_x + d^p_y = d^p_{xy}$, we get

  $$|ad_x|^p + |bd_y|^p \leq |(a + b)d_x|^p + |b|^p \left(d^p_x + d_y\right)^p = |(a + b)d_x|^p + |bd_{xy}|^p .$$

  Similarly we deduce

  $$|ad_x|^p + |bd_y|^p \leq |(a + b)d_y|^p + |ad_{xy}|^p$$

  and hence, by Corollary 1.22, $\|a\delta(x) + b\delta(y)\|_{\mathcal{F}_p(M)} = \|a\delta(x) + b\delta(y)\|_{\ell_{p,d}}$.

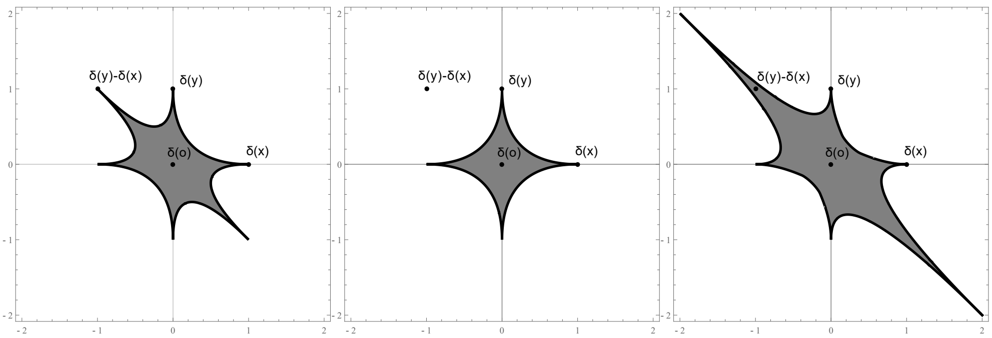
- $\|\cdot\|_{\mathcal{F}_p(M)} = \|\cdot\|_{\ell_{p,d}}$ on $\mathcal{F}^+_p(M)$ but not on $\mathcal{F}_p(M)$:

  Assume $\max\{d^p_x, d^p_y\} \leq d^p_{xy} < d^p_x + d^p_y$. Then Theorem 1.23 guarantees that $\|\cdot\|_{\mathcal{F}_p(M)} = \|\cdot\|_{\ell_{p,d}}$ on $\mathcal{F}^+_p(M)$, but, using the fact that δ is an isometry, we have $\|\delta(x) - \delta(y)\|^p_{\mathcal{F}_p(M)} = d^p_{xy} < d^p_x + d^p_y$.

- $\|\cdot\|_{\mathcal{F}_p(M)} = \|\cdot\|_{\ell_{p,d}}$ on some proper subset of $\mathcal{F}^+_p(M)$ but not on $\mathcal{F}^+_p(M)$:

  One can take e.g. $d_x = d_y = 1$ and $d_{xy} = 1/2$. By Theorem 1.23, $\|\cdot\|_{\mathcal{F}_p(M)} = \|\cdot\|_{\ell_{p,d}}$ cannot hold on $\mathcal{F}^+_p(M)$, but direct calculation using Corollary 1.22 verifies that the equality holds for $\delta(x) + \delta(y)$.

Note that this list is not exhaustive. For illustration we include pictures of the unit ball for $p = 1/2$ and three different combinations of values d_x, d_y and d_{xy}:



$d_x = d_y = d_{xy} = 1$ $\quad$ $d_x = d_y = 1, d_{xy} = 4$ $\quad$ $d_x = d_y = 1, d_{xy} = 1/2$

1.3 Canonical embeddings and amenability

This section is dedicated to the study of the canonical embeddings of Lipschitz-free p-spaces, which were defined in Definition 1.12. Throughout this section, we will often work with Lipschitz-free p-spaces over q-metric spaces for $q \geq p$ (this makes sense since for these values q-metric spaces are also p-metric). The reason for this is that the stronger condition on the q-metric space affects the Lipschitz-free p-space as well. To make statements less repetitive, we will implicitly assume that $0 < p \leq q \leq 1$ and ι will always denote canonical embeddings (we will specify between which spaces if it is not clear from the context). People acquainted with the standard theory might be at first sight surprised with all this fuss. Rightly so, as the canonical embeddings are very well-behaved in the case $p = 1$:

Proposition 1.25. *Let $(M, d, 0)$ be a pointed metric space and $0 \in N \subset M$ be its subspace. Then $\iota : \mathcal{F}_1(N) \to \mathcal{F}_1(M)$ is a linear isometry into.*

Proof. It is by definition linear and from Fact 1.13 we know that $\|\iota\| \leq 1$. Let $\mu = \sum_{i=1}^n a_i \delta(x_i)$ for some $n \in \mathbb{N}, a \in \mathbb{R}^n, x \in N^n$. Furthermore, let Y be a Banach space and $f \in \mathrm{Lip}_0(N, Y)$ satisfy $L(f) = 1$. By Hahn-Banach theorem, there is $y^* \in S_{Y^*}$ such that

$$\left\| \sum_{i=1}^n a_i f(x_i) \right\|_Y = y^* \left(\sum_{i=1}^n a_i f(x_i) \right) = \sum_{i=1}^n a_i y^*(f(x_i)).$$

The mapping $y^* \circ f : N \to \mathbb{R}$ is 1-Lipschitz and $y^*(f(0)) = 0$, so by McShane's extension theorem, there is $F \in \mathrm{Lip}_0(M, \mathbb{R})$, $L(F) = 1$ extending $y^* \circ f$. By Proposition 1.10, we have

$$\|\mu\|_{\mathcal{F}_1(M)} \geq \left| \sum_{i=1}^n a_i F(x_i) \right| = \left| \sum_{i=1}^n a_i y^*(f(x_i)) \right| = \left\| \sum_{i=1}^n a_i f(x_i) \right\|_Y .$$

Passing to the supremum on the right-hand side and using Proposition 1.10, we obtain $\|\mu\|_{F_1(M)} \geq \|\mu\|_{\mathcal{F}_1(N)}$. □

This is far from the case when $p < 1$.

Example 1.26. Define $N = \mathbb{N}_0$ and $M = N \cup \{z\}$, where $z \notin N$. Equip M with the q-metric d defined as $d(z, n) = d(n, z) = 2^{-1/q}$ for $n \in \mathbb{N}_0$ and $d(n, m) = 1$ for $n, m \in \mathbb{N}_0, n \neq m$. Then $\iota : \mathcal{F}_p(N) \to \mathcal{F}_p(M)$ is an isomorphism with $\|\iota^{-1}\| = 2^{1/q}$.

The example is based on [1, Theorem 6.1], where it was only shown that $\|\iota^{-1}\| \geq 2^{1/q}$. The other inequality was shown in [19, Theorem 3.12]. We will include the proof for both completeness' sake and the reason that the version presented here is technically cleaner than in [19]. As part of the proof of the example, let us prove

Proposition 1.27. *Let $(M, d, 0)$ be a q-metric space and $0 \in N \subset M$ be its subspace. If there is a Lipschitz retraction $r : M \to N$, then $\iota : \mathcal{F}_p(N) \to \mathcal{F}_p(M)$ is an isomorphism with $\|\iota^{-1}\| \leq L(r)$.*

In particular, if $M \setminus N = \{z\}$ and there exists $y \in N$ such that $d(z, N) = d(z, y)$, then $\iota : \mathcal{F}_p(N) \to \mathcal{F}_p(M)$ is an isomorphism with $\|\iota^{-1}\| \leq 2^{1/q}$.

Proof. We need to show that for $\mu = \sum_{i=1}^n a_i \delta(x_i)$, $n \in \mathbb{N}$, $a \in \mathbb{R}^n$, $x_1, \dots, x_n \in N$ holds $\|\mu\|_{\mathcal{F}_p(N)} \leq L(r) \|\mu\|_{\mathcal{F}_p(M)}$. The mapping $\delta_N \circ r : M \to \mathcal{F}_p(N)$ is $L(r)$-Lipschitz as δ_N is an isometry. Hence $(\delta_N \circ r)/L(r) \in \mathrm{Lip}_0(M, \mathcal{F}_p(N))$ is 1-Lipschitz and subsequently

$$\|\mu\|_{\mathcal{F}_p(M)} \geq \left\| \sum_{i=1}^n a_i \frac{(\delta_N \circ r)}{L(r)}(x_i) \right\|_{\mathcal{F}_p(N)} = \frac{1}{L(r)} \left\| \sum_{i=1}^n a_i \delta(x_i) \right\|_{\mathcal{F}_p(N)} = \frac{1}{L(r)} \|\mu\|_{\mathcal{F}_p(N)}.$$

For the "in particular" part, it is now enough to find a $2^{1/q}$-Lipschitz retraction $r : M \to N$. Define r by $r(z) = y$ and $r(x) = x$ for $x \in N$. Then indeed $L(r) \leq 2^{1/q}$, because for $x, x' \in N$ holds

$$\begin{aligned} &d(r(z), r(x))^q = d(y, x)^q \leq d(y, z)^q + d(z, x)^q \leq 2d(z, x)^q, \\ &d(r(x), r(x')) = d(x, x'). \end{aligned}$$

□

Proof of example 1.26. The mapping d is in fact a q-metric: properties (M1) and (M2) follow from definition and for (M3) we use the fact that $1^q = 1 = 1/2 + 1/2 = (2^{-1/q})^q + (2^{-1/q})^q$. The fact that $\|\iota^{-1}\| \leq 2^{1/q}$ follows from the "in particular" part of Proposition 1.27.

It remains to show that $\|\iota^{-1}\| \geq 2^{1/q}$. Let $n \in \mathbb{N}$ and put $\mu_n = \sum_{i=1}^n \delta(i) \in \mathcal{F}_p(N)$. By Theorem 1.23, we have $\|\mu_n\|_{\mathcal{F}_p(N)}^p = \sum_{i=1}^n d(0, i)^p = n$. In $\mathcal{F}_p(M)$ we can write

$$\mu_n = n\delta(z) + \sum_{i=1}^n \delta(i) - \delta(z).$$

A simple use of p-triangle inequality and the fact that δ is an isometry yields

$$\begin{aligned} \|\mu_n\|_{\mathcal{F}_p(M)}^p &\leq \|n\delta(z)\|^p + \sum_{i=1}^n \|\delta(i) - \delta(z)\|^p \\ &= (nd(z, 0))^p + \sum_{i=1}^n d(i, z)^p = 2^{-p/q} n^p + 2^{-p/q} n. \end{aligned}$$

So we have for any $n \in \mathbb{N}$

$$\left\|\iota^{-1}\right\|^p \geq \frac{\|\mu_n\|_{\mathcal{F}_p(N)}^p}{\|\mu_n\|_{\mathcal{F}_p(M)}^p} \geq \frac{n}{2^{-p/q} n^p + 2^{-p/q} n}.$$

The right-hand side converges to $2^{p/q}$ as $n \to \infty$, so we have $\|\iota^{-1}\| \geq 2^{1/q}$. □

Natural follow-up to this example is the

Question 1.28. Let $(M, d, 0)$ be a p-metric space and $0 \in N \subset M$ be its subspace. Is the canonical embedding $\iota : \mathcal{F}_p(N) \to \mathcal{F}_p(M)$ always an isomorphism?

This question is still open. However, the major result of this section gives at least a partial answer:

Theorem 1.29. *Let $(M, d, 0)$ be a pointed metric space and $0 \in N \subset M$ be its subspace. Then $\iota : \mathcal{F}_p(N) \to \mathcal{F}_p(M)$ is an isomorphism. Moreover, there is a constant C_p dependant only on p such that $\|\iota^{-1}\| \leq C_p$ regardless of N and M.*

The proof of the theorem will be given at the end of the chapter when we will have prepared all the required machinery. To shorten the statements, we introduce the concept of amenability.

Definition 1.30. Let $(M, d, 0)$ be a pointed q-metric space and $0 \in N \subset M$ be its subspace. We say that N is *p-amenable* in M with constant $C > 0$ if the canonical embedding $\iota : \mathcal{F}_p(N) \to \mathcal{F}_p(M)$ is an isomorphism and $\|\iota\|^{-1} \le C$. We say that N is p-amenable in M if there is some $C > 0$ such that the above holds.

The first step to proving Theorem 1.29 is to change the infinite-dimensional qualitative problem of Question 1.28 to a finite-dimensional quantitative one.

Proposition 1.31. *Let $(M, d, 0)$ be a pointed q-metric space and $0 \in N \subset M$. If $C > 0$ is such that $\|\iota^{-1}\| > C$, then there are finite subsets $0 \in N' \subset N$ and $N' \subset M' \subset M$ such that the canonical embedding $\iota' : \mathcal{F}_p(N') \to \mathcal{F}_p(M')$ satisfies $\|(\iota')^{-1}\| > C$.*

Proof. If $\|\iota^{-1}\| > C$, then there must be $\mu = \sum_{i=1}^n a_i \delta(x_i) \in \mathcal{F}_p(N)$, such that $\|\mu\|_{\mathcal{F}_p(N)} / \|\mu\|_{\mathcal{F}_p(M)} > C$. Let $\varepsilon > 0$ satisfy $\|\mu\|_{\mathcal{F}_p(N)} / (\|\mu\|_{\mathcal{F}_p(M)} + \varepsilon) > C$. Using Proposition 1.11 we can find $m \in \mathbb{N}$, $b \in M^m$ and $y, z \in M^m$ so that

$$\mu = \sum_{i=1}^m b_i \frac{\delta(y_i) - \delta(z_i)}{d(y_i, z_i)} \quad \text{and} \quad \left(\sum_{i=1}^m |b_i|^p \right)^{1/p} \le \|\mu\|_{\mathcal{F}_p(M)} + \varepsilon. \tag{1.3}$$

Put $N' = \{0\} \cup \{x_i \colon i \in [1..n]\}$ and $M' = N' \cup \{y_i, z_i \colon i \in [1..m]\}$. Because $N' \subset N$, we have $\|\mu\|_{\mathcal{F}_p(N)} \le \|\mu\|_{\mathcal{F}_p(N')}$. Since the decomposition (1.3) is also valid in $\mathcal{F}_p(M')$, we have $\|\mu\|_{\mathcal{F}_p(M')} \le (\sum_{i=1}^m |b_i|^p)^{1/p} \le \|\mu\|_{\mathcal{F}_p(M)} + \varepsilon$. Hence we have

$$\left\|(\iota')^{-1}\right\| \ge \frac{\|\mu\|_{\mathcal{F}_p(N')}}{\|\mu\|_{\mathcal{F}_p(M')}} \ge \frac{\|\mu\|_{\mathcal{F}_p(N)}}{\|\mu\|_{\mathcal{F}_p(M)} + \varepsilon} > C.$$

□

The significance of this proposition is twofold. If there is a q-metric space $(M, d, 0)$ and its subspace $0 \in N \subset M$ which is not p-amenable in M, then for any $C > 0$ there must be a finite q-metric space $(M', d, 0)$ and its subspace $0 \in N' \subset M'$ which is not p-amenable in M' with constant C. On the other hand, if there is $C > 0$ such that for every finite q-metric space $(M, d, 0)$ and subspace $0 \in N \subset M$ it is the case that N is p-amenable in M with constant C, then this must also hold for spaces of infinite cardinality. Of course, for finite M the embedding ι is always an isomorphism, because all injective linear mappings between finite-dimensional p-Banach spaces are. So the question is whether there is a universal bound for the norm of the inverses.

In light of this, we will focus on finite p-metric spaces.

Definition 1.32. Let $(M, d, 0)$ be a finite pointed q-metric space, $0 \in N \subset M$

be its subspace and $n, k \in \mathbb{N}$, $n \leq k$. Define[2]

$$\begin{aligned}
\mathfrak{a}_p(N, M) &:= \min\{C\colon N \text{ is } p\text{-amenable in } M \text{ with constant } C\},\\
\mathfrak{aa}_p^q(N) &:= \sup\{\mathfrak{a}_p(N, M')\colon M' \text{ is a } q\text{-metric space such that } N \subset M'\},\\
\mathfrak{aa}_p^q(n, k) &:= \sup\{\mathfrak{a}_p(N', M')\colon 0 \in N' \subset M' \text{ are } q\text{-metric spaces}, |N' \setminus \{0\}| \leq n\\
&\qquad \text{and } |M' \setminus \{0\}| \leq k\},\\
\mathfrak{aa}_p^q(n) &:= \sup\{\mathfrak{aa}_p^q(N')\colon N' \text{ is a } q\text{-metric space and } |N' \setminus \{0\}| \leq n\}.
\end{aligned}$$

For simplicity's sake, we write $\mathfrak{aa}_p(N)$, $\mathfrak{aa}_p(n, k)$ and $\mathfrak{aa}_p(n)$ instead of $\mathfrak{aa}_p^p(N)$, $\mathfrak{aa}_p^p(n, k)$ and $\mathfrak{aa}_p^p(n)$.

Note that $\mathfrak{a}_p(N, M) = \|\iota^{-1}\|$, where $\iota : \mathcal{F}_p(N) \to \mathcal{F}_p(M)$. We will often use

Fact 1.33. *Let $(M, d, 0)$ be a finite p-metric space and $0 \in N \subset M$. Then there is $a \in \mathbb{R}^N$ and $T \in \mathcal{T}(M)$ such that*

$$\left\|\sum_{x\in N} a_x\delta(x)\right\|_{\mathcal{F}_p(M)} = T(a) = 1 \quad \text{and} \quad \left\|\sum_{x\in N} a_x\delta(x)\right\|_{\mathcal{F}_p(N)} = \mathfrak{a}_p(N, M).$$

Proof. Denote $S = S_{\mathcal{F}_p(M)} \cap \operatorname{span}\delta(N) = S_{\mathcal{F}_p(M)} \cap \iota(\mathcal{F}_p(N))$. By definition of the operator norm we have

$$\mathfrak{a}_p(N, M) = \left\|\iota^{-1}\right\| = \sup\left\{\|\mu\|_{\mathcal{F}_p(N)} : \mu \in S\right\}.$$

So there is a sequence $(\mu_n) \in S^{\mathbb{N}}$ such that $\|\mu_n\|_{\mathcal{F}_p(N)} \to \mathfrak{a}_p(N, M)$. As we are working in a finite-dimensional space, S is compact and hence there is $\mu \in S$ such that $\|\mu\|_{\mathcal{F}_p(N)} = \mathfrak{a}_p(N, M)$. We can find $a \in \mathbb{R}^N$ so that $\mu = \sum_{x\in N} a_x\delta(x)$. This a satisfies our requirements; all that is left to do is to find the tree T. Its existence follows immediately from Theorem 1.21. □

It will be convenient to understand $a \in \mathbb{R}^N$ not only as coefficients in the space $\mathcal{F}_p(N)$, but also in $\mathcal{F}_p(M)$. To allow this, we identify every $a \in \mathbb{R}^N$ with the element $a' \in \mathbb{R}^M$ which is given by $a_x = a'_x$ for $x \in N$ and $a'_y = 0$ for $y \in M \setminus N$ (i.e. we identify $\mathbb{R}^N$ with $\mathbb{R}^N \times \{0\}^{M\setminus N}$).

In Proposition 1.27 we saw that by adding merely one point, one can only achieve $\|\iota^{-1}\| \leq 2^{1/p}$. So, in order to find a counterexample to Question 1.28, it would be necessary to add more points. But how many points to add? Does the size of the subspace matter or is it enough to make the superspace large? The following results shed some light on the situation.

Theorem 1.34. *Let $(M, d, 0)$ be a finite q-metric space and $0 \in N \subset M$. Then for $\mu \in \mathcal{F}_p(N)$ holds*

$$\|\mu\|_{\mathcal{F}_p(M)} = \min\left\{\|\mu\|_{\mathcal{F}_p(F)} : N \subset F \subset M, |F \setminus N| \leq |N| - 2\right\}.$$

Proof. As for any $F \subset M$ holds $\|\mu\|_{\mathcal{F}_p(M)} \leq \|\mu\|_{\mathcal{F}_p(F)}$ by Fact 1.13, it is enough to show that the value $\|\mu\|_{\mathcal{F}_p(M)}$ appears among the values we are taking the

[2]The same set-theoretic remark as for Proposition 1.10 applies to the suprema in this definition.

minimum over. If $|M \setminus N| \leq |N| - 2$, then this is clearly the case since we can just take $F = M$. So assume that $|M \setminus N| > |N - 2|$.

Let $a \in \mathbb{R}^N$ be such that $\mu = \sum_{x \in M} a_x \delta(x)$. Using Theorem 1.21 find $T \in \mathcal{T}(M)$ such that $\|\mu\|_{\mathcal{F}_p(M)} = T(a)$. We shall inductively find sets F_i and trees T_i, $i \in [1..n]$ for some $n \in \mathbb{N}$ satisfying

(i) $M = F_1 \supset F_2 \supset \cdots F_n \supset N$,

(ii) $\forall z \in F_n \setminus N\colon \left|\operatorname{desc}^{T_n}(z)\right| \geq 2$,

(iii) $\forall i \in [1..n]\colon T_i \in \mathcal{T}(F_i)$,

(iv) $\forall i \in [1..n-1]\colon |F_i \setminus F_{i+1}| = 1$ and $T_{i+1}(a) \leq T_i(a)$.

Put $F_1 = M$ and $T_1 = T$. Assume we have constructed F_i, T_i for $i \in [1..m]$. If (ii) holds for F_m and T_m, put $n = m$ and end the construction, otherwise continue. By this assumption, there is $z \in F_m \setminus N$ such that $\left|\operatorname{desc}^{T_m}(z)\right| \leq 1$. Put $F_{m+1} = F_m \setminus \{z\}$. We discern two cases.

If $\left|\operatorname{desc}^{T_m}(z)\right| = 0$, define $T_{m+1} \in \mathcal{T}(F_{m+1})$ by $E^{T_{m+1}} = E^{T_m} \setminus \{\{z, \operatorname{pred}(z)\}\}$. This definition is correct, because by our assumption, z is a leaf, i.e. $\{z, \operatorname{pred}(z)\}$ is the only edge connecting to z. Let $x \in F_{m+1}$. Then $c^{T_m}(x,a) = c^{T_{m+1}}(x,a)$: if $z \notin V_x^{T_m}$, then $V_x^{T_m} = V_x^{T_{m+1}}$ and

$$c^{T_m}(x,a) = \sum_{y \in V_x^{T_m}} a_y = \sum_{y \in V_x^{T_{m+1}}} a_y = c^{T_{m+1}}(x,a).$$

In case $z \in V_x^{T_m}$, we have $V_x^{T_m} = V_x^{T_{m+1}} \cup \{z\}$ and since $a_z = 0$,

$$c^{T_m}(x,a) = \sum_{y \in V_x^{T_m}} a_y = a_z + \sum_{y \in V_x^{T_{m+1}}} a_y = \sum_{y \in V_x^{T_{m+1}}} a_y = c^{T_{m+1}}(x,a).$$

Hence we have the equality

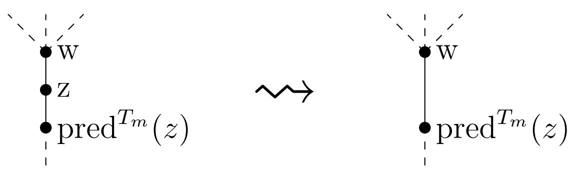
$$\begin{aligned} T_m(a)^p - \left|c^{T_m}(z,a) d(e_z^{T_m})\right| &= \sum_{x \in F_m \setminus \{0\}} \left|c^{T_m}(x,a) d(e_x^{T_m})\right|^p - \left|c^{T_m}(z,a) d(e_z^{T_m})\right| \\ &= \sum_{x \in F_{m+1} \setminus \{0\}} \left|c^{T_{m+1}}(x,a) d(e_x^{T_{m+1}})\right|^p = T_{m+1}(a)^p. \end{aligned}$$

As z is a leaf, $c^{T_m}(z,a) = a_z = 0$ and hence $T_m(a)^p = T_{m+1}(a)^p$.

Now assume $\left|\operatorname{desc}^{T_m}(z)\right| = 1$. Let $w \in F_m$ be the element such that $z = \operatorname{pred}^{T_m}(w)$. Define $T_{m+1} \in \mathcal{T}(F_{m+1})$ by

$$E^{T_{m+1}} = \left(E^{T_m} \setminus \{\{z, \operatorname{pred}(z)\}, \{w, z\}\}\right) \cup \{w, \operatorname{pred}(z)\}.$$

T_{m+1} is the tree which "skips" the vertex z, graphically:



One can deduce $c^{T_m}(x,a) = c^{T_{m+1}}(x,a)$ for every $x \in F_{m+1}$ exactly as in the previous case. Furthermore, we have

$$\begin{aligned} T_m(a)^p - \left|c^{T_m}(z,a)d(e_z^{T_m})\right| - \left|c^{T_m}(w,a)d(e_w^{T_m})\right| \\ &= \sum_{x\in F_m\setminus\{0,z,w\}} \left|c^{T_m}(x,a)d(e_x^{T_m})\right|^p \\ &= \sum_{x\in F_{m+1}\setminus\{0\}} \left|c^{T_{m+1}}(x,a)d(e_x^{T_{m+1}})\right|^p - \left|c^{T_{m+1}}(w,a)d(e_w^{T_{m+1}})\right|^p \\ &= T_{m+1}(a)^p - \left|c^{T_{m+1}}(w,a)d(e_w^{T_{m+1}})\right|^p. \end{aligned}$$

After rearranging the terms we have

$$T_m(a)^p - T_{m+1}(a)^p = \left|c^{T_m}(z,a)d(e_z^{T_m})\right| + \left|c^{T_m}(w,a)d(e_w^{T_m})\right| - \left|c^{T_{m+1}}(w,a)d(e_w^{T_{m+1}})\right|^p.$$

So we need to show that the right-hand side is non-negative. First, we realize that

$$c^{T_m}(z,a) = c^{T_m}(w,a) + a_z = c^{T_m}(w,a) = c^{T_{m+1}}(w,a),$$

because $a_z = 0$ and $V_z^{T_m} = V_w^{T_m} \cup \{z\}$. Furthermore, $d(e_z^{T_m}) = d(z, \mathrm{pred}^{T_m}(z))$, $d(e_w^{T_m}) = d(w,z)$ and $d(e_w^{T_{m+1}}) = d(w, \mathrm{pred}^{T_m}(z))$. Thus, our right-hand side is equal to

$$\left|c^{T_m}(z,a)\right|^p \left(d(z,\mathrm{pred}^{T_m}(z))^p + d(w,z)^p - d(w,\mathrm{pred}^{T_m}(z))^p\right),$$

which is non-negative by p-triangle inequality. This finishes the construction. Note that if for some $m \in \mathbb{N}$ holds $F_m = N$, then (ii) is satisfied and the construction ends. Since M is finite and always $F_i \supset N$, the construction has to end after at most $|M \setminus N|$ steps.

Put $F = F_n$ and $T' = T_n$. We have

$$\|\mu\|^p_{\mathcal{F}_p(M)} \le \|\mu\|^p_{\mathcal{F}_p(F)} \le T'(a)^p \le T(a)^p = \|\mu\|^p_{\mathcal{F}_p(M)}$$

and hence the equality $\|\mu\|_{\mathcal{F}_p(M)} = \|\mu\|_{\mathcal{F}_p(F)}$ holds. Now it only remains to show that $|F \setminus N| \le |N| - 2$. Because each vertex has a unique predecessor, the sets $\mathrm{desc}^{T'}(z), z \in (F\setminus N)\cup\{0\}$ are pairwise disjoint. Combined with $\mathrm{desc}^{T'}(0) \neq \emptyset$, we obtain the following estimate

$$|F\setminus N| + |N\setminus\{0\}| = |F\setminus\{0\}| \ge \sum_{z\in F\setminus N} \left|\mathrm{desc}^{T'}(z)\right| + \left|\mathrm{desc}^{T'}(0)\right| \ge 2|F\setminus N| + 1.$$

Hence $|F\setminus N| \le |N\setminus\{0\}| - 1 = |N| - 2$. □

Corollary 1.35. $\mathfrak{aa}_p^q(n) = \mathfrak{aa}_p^q(n, 2n-1)$ *for* $n \in \mathbb{N}$.

Proof. By definition, $\mathfrak{aa}_p^q(n) \ge \mathfrak{aa}_p^q(n, 2n-1)$. Let $0 < C < \mathfrak{aa}_p^q(n)$. Then there is a q-metric space $(M,d,0)$ and $0 \in N \subset M$ such that $|N\setminus\{0\}| \le n$ and $\mathfrak{a}_p(M,N) > C$. From Proposition 1.31 follows the existence of $N' \subset N$ and $N' \subset M' \subset M$ such that M' is finite and again $\mathfrak{a}_p(N',M') > C$. Using Fact 1.33 we may find $\mu \in \mathcal{F}_p(N')$ such that $\|\mu\|_{\mathcal{F}_p(N')} = \mathfrak{a}_p(N',M')$ and $\|\mu\|_{\mathcal{F}_p(M')} = 1$. By Theorem 1.34, there is $F \subset M'$ with $N' \subset F$ and

$$|F\setminus\{0\}| = |F\setminus N'| + |N'\setminus\{0\}| \le |N'| - 2 + |N'\setminus\{0\}| = 2|N'\setminus\{0\}| - 1 \le 2n-1$$

such that $\|\mu\|_{\mathcal{F}_p(F)} = \|\mu\|_{\mathcal{F}_p(M')} = 1$. Hence $\mathfrak{a}_p(N',F') \ge \|\mu\|_{\mathcal{F}_p(N')} / \|\mu\|_{\mathcal{F}_p(F)} > C$. This implies $\mathfrak{aa}_p^q(n,2n-1) > C$ and the claim follows. □

The corollary gives us a bound on the size of the superspace. It is possible to use Proposition 1.27 to obtain a very crude estimate on $\mathfrak{aa}_p^q(N)$ for p-metric space $(N, d, 0)$ based on the size of N: From Theorem 1.34 we know it is enough to get a bound on $\mathfrak{aa}_p^q(N, M)$ where $|M| \leq 2n - 1$. Applying Proposition 1.27 $n - 1$ times yields the estimate $\mathfrak{a}_p^q(N) \leq 2^{(n-1)/q}$.

However, with some more work, this bound can be significantly improved. To do this we use a result on extending Lipschitz functions by Basso. We will state and prove the theorem only for finite spaces as that is the setting we are working in, which allows us to obtain a mildly stronger result which is also easier to state. The proof is virtually unchanged from the one given in [3].

Lemma 1.36 ([3, Theorem 1.1])**.** *Let (X, d) be a metric space, $m \in \mathbb{N}$, $S, T \subset X$ satisfy that S is finite, $X = S \cup T$ and $|T| \leq m$. Then there is a retraction $R : X \to S$ with $L(R) \leq (m + 1)$.*

Proof. Assume that S, T are as in the statement. Of course, we can assume that they are disjoint (switch to S and $T \setminus S$). Define

$$E = \{\{u, v\} \in T \times (T \cup S) : u \neq v\}.$$

The set E is finite (for both T and S are). Let $G = (X, E)$ be the graph with vertices X and edges E. Define $w : E \to \mathbb{R}$ by $w(\{u, v\}) = d(u, v)$. Denote $n = |E|$ and let $e : [1..n] \to E$ be an enumeration of E such that $w \circ e$ is non-decreasing. Subset $E' \subset E$ is said to be admissible if the graph $G' = (X, E')$ has no cycles and no two distinct points of S are connected by a path.

We construct a subset of E by starting with an empty set and looking at the edges in E in non-decreasing order of length, adding each edge that does not violate the condition of admissibility. Formally, we inductively construct a sequence $(E_i)_{i=0}^n$ of subsets of E by setting

$$E_0 = \emptyset \quad \text{and for } i \in [1..n] \quad E_i = \begin{cases} E_{i-1} \cup \{e(i)\} & \text{if } E_{i-1} \cup \{e(i)\} \text{ is admissible,} \\ E_{i-1} & \text{otherwise.} \end{cases}$$

By the construction (and the fact that E_0 is admissible), E_i is admissible for all $i \in [1..n]$. We start by showing that for all $z \in T$ there is a unique $x_z \in S$ and a unique path from z to x_z in (X, E_n). The uniqueness follows from admissibility of E_n: if there was a path from z to two distinct points x_z, y_z of S, then there would also be a path between x_z and y_z, which there cannot be, so x_z is unique. If there were two different paths from z to x_z, then (X, E_n) would contain a cycle, which is also impossible. To show the existence, assume that $z \in T$ and choose $x \in S$ arbitrarily. If $\{z, x\} \in E_n$, then $x = x_z$ and we are done. So assume that $\{z, x\} \notin E_n$. By the construction, that means that $E_i \cup \{\{x, z\}\}$, where $e(i + 1) = \{x, z\}$ is not admissible. There are two possibilities why this could happen. One possibility is that $(X, E_i \cup \{\{x, z\}\})$ contains a cycle, which implies that in (X, E_i) there is already a path connecting x to z; in this case we again get $x = x_z$ and are done. The second is that in $(X, E_i \cup \{\{x, z\}\})$ there is a path connecting two points of S, which means that in (X, E_i) there is a path connecting z to some other $y \in S$. In this case, put $x_z = y$.

Now we define the retraction $R : X \to S$ by

$$R(z) = \begin{cases} z, & z \in S, \\ x_z, & z \in T. \end{cases}$$

For $x, y \in S$ obviously holds $d(R(x), R(y)) = d(x, y)$.

Let $z \in T$ and $x \in S$. If $x = x_z$, then $d(R(x), R(z)) = d(x, x_z) = 0$, so suppose $x \neq x_z$. Let $z = z_0, \dots, z_k = x_z$ be the path from z to x_z in (X, E_n). Using the triangle inequality, we obtain

$$d(R(x), R(z)) = d(x, x_z) \leq d(x, z) + \sum_{i=1}^{k} d(z_{i-1}, z_i).$$

Let $j \in [1..n]$ be such that $e(j+1) = \{x, z\}$. Since $x \neq x_z$, $\{x, z\} \notin E_n$, so $E_j \cup \{\{x, z\}\}$ is not admissible. If $(X, E_j \cup \{\{x, z\}\})$ contained a cycle, then there would be a path from x to z, implying $x = x_z$. So it must be that $(X, E_j \cup \{\{x, z\}\})$ already contains the path from z to x_z. Hence, by the construction, we have for all $i \in [1..k]$ that $d(z_{i-1}, z_i) = w(\{z_{i-1}, z\}) \leq w(\{z, x\}) = d(z, x)$. Thus,

$$d(R(x), R(z)) \leq d(x, z) + \sum_{i=1}^{k} d(z_{i-1}, z_i) \leq (1 + k) d(x, z).$$

By admissibility, $z_0, \dots, z_{k-1} \in T$. As all points of a path are by definition pairwise distinct, we have $k \leq |T|$ and consequently $d(R(x), R(z)) \leq (1 + m) d(x, z)$.

Now, let $z, z' \in T$. If $x_z = x_{z'}$, then $d(R(z), R(z')) = d(x_z, x_{z'}) = 0$, so suppose that $x_z \neq x_{z'}$. Let $z = z_0, \dots, z_k = x_z$ be the path from z to x_z and $z' = z'_0, \dots, z'_{k'} = x_{z'}$ be the path from z' to $x_{z'}$. From the triangle inequality follows that

$$\begin{aligned} d(R(z), R(z')) = d(x_z, x_{z'}) &\leq d(x_z, z) + d(z, z') + d(z', x_{z'}) \\ &\leq d(z, z') + \sum_{i=1}^{k} d(z_{i-1}, z_i) + \sum_{i=1}^{k'} d(z'_{i-1}, z'_i). \end{aligned} \tag{1.4}$$

As above, let $j \in [1..n]$ be such that $e(j+1) = \{z', z\}$. We have that $\{z, z'\} \notin E_n$, because otherwise there would be a path from z to z' implying that there is a path from x_z to z', but this is not possible by uniqueness of $x_{z'}$ and the assumption that $x_z \neq x_{z'}$. This means that $E_j \cup \{\{z, z'\}\}$ is not admissible. If $E_j \cup \{\{z, z'\}\}$ contained a cycle, then (X, E_j) would contain a path from z to z' and we would arrive to the same contradiction as a moment ago. So the only option left is that (X, E_j) already contains the paths from z to x_z and from z' to $x_{z'}$ (and so adding the edge $\{z, z'\}$ would create a path between two points of S). As in the previous case, by the construction we obtain

$$\forall i \in [1..k]\colon d(z_{i-1}, z_i) \leq d(z, z') \quad \text{and} \quad \forall i \in [1..k']\colon d(z'_{i-1}, z'_i) \leq d(z, z').$$

Returning to (1.4), we have $d(R(z), R(z')) \leq (1 + k + k') d(z, z')$. Finally, as x_z and $x_{z'}$ are not connected by a path, the sets $\{z_i \colon i \in [0..k]\}$ and $\{z'_i \colon i \in [0..k']\}$ are disjoint subsets of T and so $k + k' \leq |T|$, implying that

$$d(R(z), R(z')) \leq (1 + m) d(z, z').$$

We have considered all the possible cases and thus we conclude that $L(R) \leq 1 + m$, finishing the proof. □

Corollary 1.37. *For all $n, k \in \mathbb{N}$ holds $\mathfrak{aa}_p^q(n, k) \leq (k - n + 1)^{1/q}$. In particular $\mathfrak{aa}_p^q(n) \leq n^{1/q}$.*

Proof. Let $(M,d,0)$ be a q-metric space and $0 \in N \subset M$ with $|N \setminus \{0\}| = n \leq k = |M \setminus \{0\}|$. Then by Lemma 1.36 there is a retraction $R : (M, d^q) \to (N, d^q)$ with $L(R) \leq |M \setminus N| + 1 = (k - n + 1)$. Then R is a $(k-n+1)^{1/q}$-retraction when viewed as a mapping from (M,d) to (N,d). Appealing to Proposition 1.27 yields the claim. For the "in particular" part, it is enough to again realize that it suffices to estimate $\mathfrak{aa}_p^q(n, 2n-1)$ by Corollary 1.35. □

We shall now start reducing the problem of amenability to special classes of q-metric spaces with the goal of proving Theorem 1.29.

Definition 1.38. Let M be a finite set, $0 \in M$, $T \in \mathcal{T}(M,0)$ and $w : E^T \to (0,\infty)$. We define the *weighted tree q-metric* (generated by T and w) on M, denoted $d_{T,w}$, as follows. Let $x, y \in M$. Obviously, if $x = y$ put $d_{T,w}(x,y) = 0$. Otherwise, let $x = x_0, \dots, x_n = y$ be the path from x to y in T and define

$$d_{T,w}(x,y)^q = \sum_{i=1}^{n} w(\{x_{i-1}, x_i\})^q.$$

Fact 1.39. *Using the notation from the definition above, $d_{T,w}$ is indeed a q-metric.*

Proof. One implication of (M1) holds by definition, the other follows from the fact that w has strictly positive values. (M2) is easy, since our tree is unoriented and if $x = x_0, \dots x_n = y$ is the path from x to y, then $y = x_{n-0}, \dots x_{n-n} = x$ is the path from y to x. It remains to verify that q-triangle inequality holds. So let $x, y, z \in M$ and $y = x_0, \dots, x_n = x$ be the path from y to x and $y = z_0, \dots, z_m = z$ be the path from y to z. Since $x_n, \dots x_1, y, z_1, \dots, z_m$ is a sequence of vertices starting at x and ending at z, the path from x to z must be its subsequence and moreover if we denote $x = s_0, \dots, s_k = z$ the path from x to z, we have

$$\begin{aligned} d_{T,w}(x,z)^q &= \sum_{i=1}^{k} w(\{s_{i-1}, s_i\})^q \\ &\leq \sum_{i=1}^{n} w(\{x_{i-1}, x_i\})^q + \sum_{i=1}^{m} w(\{z_{i-1}, z_i\})^q \\ &= d_{T,w}(x,y)^q + d_{T,w}(y,z)^q, \end{aligned}$$

thus proving (M3). □

Proposition 1.40. *If we start with a q-metric space $(M,d,0)$ and take a tree $T \in \mathcal{T}(M)$, then this pair induces a weight w on the tree by setting $w(\{x, \operatorname{pred}^T(x)\}) = d(x, \operatorname{pred}^T(x))$. The two q-metrics are related as follows:*

(i) $\forall x \in M \setminus \{0\} \colon d(x, \operatorname{pred}^T(x)) = d_{T,w}(x, \operatorname{pred}^T(x))$*;*

(ii) $\forall x, y \in M \colon d(x,y) \leq d_{T,w}(x,y)$*.*

Moreover, there are also the following relations between the appropriate Lipschitz-free p-spaces:

(iii) for $a \in \mathbb{R}^M$ holds $T(a,d) = T(a, d_{T,w})$, where $T(a,d)$ $(T(a, d_{T,w}))$ is the value $T(a)$ calculated with respect to the metric d $(d_{T,w})$;

(iv) $\|\cdot\|_{\mathcal{F}_p(M,d)} \leq \|\cdot\|_{\mathcal{F}_p(M,d_{T,w})}$ *if we identify the spaces as linear spaces.*

Proof. The weight is positive because no edge connects a vertex to itself and d is positive for a pair of distinct points. (i) follows easily from the definition as the path from x to $\mathrm{pred}^T(x)$ is the sequence consisting of the two points, hence $d_{T,w}(x, \mathrm{pred}^T(x))^q = d(x, \mathrm{pred}^T(x))^q$. To show (ii), consider $x, y \in M$ and let the path between them be $x = x_0, \dots x_n = y$. Then the inequality

$$d(x,y)^q \leq \sum_{i=1}^{n} d(x_{i-1}, x_i)^q = d_{T,w}(x,y)^q$$

is enforced by q-triangle inequality.

Property (iii) follows straight from the definition of $T(a)$ as it only depends on distances of points to their predecessors with respect to T, where both metrics coincide as was shown in (i).

Finally, (iv) can be deduced from (ii) as follows: Let $a \in \mathbb{R}^M$. If $T' \in \mathcal{T}(M)$, then as a consequence of the definition and (ii) we obtain

$$\begin{aligned} T'(a,d) &= \left(\sum_{x \in M \setminus \{0\}} \left| c^{T'}(x,a) d(e_x^{T'}) \right|^p \right)^{1/p} \\ &\leq \left(\sum_{x \in M \setminus \{0\}} \left| c^{T'}(x,a) d_{T,w}(e_x^{T'}) \right|^p \right)^{1/p} = T'(a, d_{T,w}). \end{aligned}$$

Hence $\min_{T' \in \mathcal{T}(M)} T'(a,d) \leq \min_{T' \in \mathcal{T}(M)} T'(a, d_{T,w})$ and Theorem 1.21 finishes the proof. □

Definition 1.41. Let $n, k \in \mathbb{N}$, $n \leq k$. Define $\mathcal{M}_T^q(k)$ as the set of all pointed q-metric spaces $(M, d, 0)$ satisfying that $|M \setminus \{0\}| \leq k$ and $d = d_{T,w}$ for some $T \in \mathcal{T}(M)$ and $w : E^T \to (0, \infty)$. Next, define

$${}^T\mathfrak{aa}_p^q(n,k) = \sup\{\mathfrak{a}_p(N,M) \colon 0 \in N \subset M \text{ are } q\text{-metric spaces, } |N \setminus \{0\}| \leq n \text{ and } M \in \mathcal{M}_T^q(k)\}.$$

As this set is smaller than the one considered in the definition of $\mathfrak{aa}_p^q(n,k)$, clearly ${}^T\mathfrak{aa}_p^q(n,k) \leq \mathfrak{aa}_p^q(n,k)$. What is not immediately clear is

Proposition 1.42. *Let $n, k \in \mathbb{N}$, $n \leq k$. Then ${}^T\mathfrak{aa}_p^q(n,k) = \mathfrak{aa}_p^q(n,k)$.*

Proof. Let $(M, d, 0)$ be a q-metric space, $0 \in N \subset M$, $|N \setminus \{0\}| \leq n$ and $|M \setminus \{0\}| \leq k$. Fact 1.33 ensures us that there is $a \in \mathbb{R}^N$ and $T \in \mathcal{T}(M)$ such that for $\mu = \sum_{x \in N} a_x \delta(x)$ holds $\|\mu\|_{\mathcal{F}_p(N)} = \mathfrak{a}_p(N,M)$ and $\|\mu\|_{\mathcal{F}_p(M)} = T(a) = 1$.

Consider the weight w induced by d and M as described in Proposition 1.40. Then by double application of Proposition 1.40 we are graced with the equality $\|\mu\|_{\mathcal{F}_p(M,d)} = \|\mu\|_{\mathcal{F}_p(M,d_{T,w})}$: from (iii) we have $\|\mu\|_{\mathcal{F}_p(M,d_{T,w})} \leq T(a, d_{T,w}) = T(a,d) = \|\mu\|_{\mathcal{F}_p(M,d)}$ and the other inequality is exactly (iv).

Appealing to Proposition 1.40 one last time, we get $\|\mu\|_{\mathcal{F}_p(N,d)} \leq \|\mu\|_{\mathcal{F}_p(N,d_{T,w})}$. Put together,

$$\mathfrak{a}_p^q(N,M) = \frac{\|\mu\|_{\mathcal{F}_p(N,d)}}{\|\mu\|_{\mathcal{F}_p(M,d)}} \leq \frac{\|\mu\|_{\mathcal{F}_p(N,d_{T,w})}}{\|\mu\|_{\mathcal{F}_p(M,d_{T,w})}} \leq {}^T\mathfrak{aa}_p^q(n,k).$$

Taking the supremum over all appropriate N and M yields the non-trivial inequality. □

In fact, we can do one better. We can only consider superspaces generated by trees such that $N \subset \operatorname{leaf}(T) \cup \{0\}$.

Notation 1.43. Denote ${}^T_l\mathfrak{aa}^q_p(n,k) = \sup_{N,M} \mathfrak{a}_p(N,M)$, where $(M,d,0)$ is a q-metric space, $0 \in N \subset M$, $|N \setminus \{0\}| \le n$, $|M \setminus \{0\}| \le k$, there are $T \in \mathcal{T}(M)$ and $w : E^T \to (0,\infty)$ such that $d = d_{T,w}$ and $N \subset \operatorname{leaf}(T) \cup \{0\}$.

The crux of the matter is the following "tree-ripping" lemma.

Lemma 1.44. *Let $(M,d,0)$ be a finite p-metric space, $T \in \mathcal{T}(M)$, $0 \neq z \in M \setminus \operatorname{leaf}(T)$ and $a \in \mathbb{R}^M$. Denote*

$$\begin{aligned} M^1 &= V_z^T, & a^1 &= a|_{M_1} \\ M^2 &= M \setminus (V_z^T \setminus \{z\}), & a^2 &= (a_x^2)_{x\in M^2}, \end{aligned}$$

where

$$a_x^2 = \begin{cases} a_x, & x \neq z, \\ c^T(z,a), & x = z. \end{cases}$$

Let $T^1 \in \mathcal{T}(M^1)$ and $T^2 \in \mathcal{T}(M^2)$ be given by

$$E^{T_1} = E^T \cap \{\{u,v\} \colon u,v \in M^1\} \quad \textit{and} \quad E^{T_2} = E^T \cap \{\{u,v\} \colon u,v \in M^2\}.$$

Then $T(a)^p = T^1(a^1)^p + T^2(a^2)$.

Proof. Directly from these definitions follows that for $x \in M^1$ holds $V_x^T = V_x^{T^1}$ and $c^T(x,a) = c^{T^1}(x,a^1)$. For $x \in M^2$ with $z \in V_x^{T^2}$ we obtain the same equality by calculating

$$\begin{aligned} c^T(x,a) &= \sum_{y\in V_x^T} a_y = \sum_{y \in V_x^T \setminus V_z^T} a_y + \sum_{y\in V_z^T} a_y = \sum_{y\in V_x^T\setminus V_z^T} a_y + c^T(z,a) \\ &= \sum_{y \in V_x^{T^2}\setminus\{z\}} a_y^2 + a_z^2 = c^{T^2}(x,a^2). \end{aligned}$$

If $z \notin V_x^{T^2}$ the equality holds as well since $c^T(x,a) = \sum_{y\in V_x^T} a_y = \sum_{y\in V_x^{T^2}} a_y^2 = c^{T^2}(x,a^2)$. Finally, we calculate

$$\begin{aligned} T(a)^p &= \sum_{x\in M\setminus\{0\}} \left|c^T(x,a) d(e_x^T)\right|^p \\ &= \sum_{x\in M^1\setminus\{z\}} \left|c^{T^1}(x,a^1) d(e_x^{T^1})\right|^p + \sum_{x\in M^2\setminus\{0\}} \left|c^{T^2}(x,a^2) d(e_x^{T^2})\right|^p \\ &= T^1(a^1)^p + T^2(a^2)^p. \end{aligned}$$

□

Proposition 1.45. *Let $n,k \in \mathbb{N}$, $n < k$. Then ${}^T_l\mathfrak{aa}^q_p(n,k) = \mathfrak{aa}^q_p(n,k)$.*

Proof. In light of Proposition 1.42 and the trivial fact ${}^T_l\mathfrak{aa}^q_p(n,k) \le {}^T\mathfrak{aa}^q_p(n,k)$, it is enough to verify that ${}^T_l\mathfrak{aa}^q_p(n,k) \ge {}^T\mathfrak{aa}^q_p(n,k)$. Let N,M be a pair contributing to the supremum defining ${}^T\mathfrak{aa}^q_p(n,k)$ with $d = d_{T,w}$ for some tree T and weights w. If $N \subset \operatorname{leaf}(T) \cup \{0\}$, then clearly ${}^T_l\mathfrak{aa}^q_p(n,k) \ge \mathfrak{a}_p(N,M)$. So suppose that there is $z \in N \setminus (\operatorname{leaf}(T) \cup \{0\})$.

As per usual, we use Fact 1.33 to find $a \in \mathbb{R}^N$ and $T \in \mathcal{T}(M)$ such that for $\mu = \sum_{x\in N} a_x \delta(x)$ holds $\mathfrak{a}_p(N, M) = \|\mu\|_{\mathcal{F}_p(N)}$ and $\|\mu\|_{\mathcal{F}_p(M)} = T(a) = 1$. The idea is to split everything into two parts. Define

$$\begin{aligned} &N^1 = N \cap V_z^T, && M^1 = V_z^T, && a^1 = a|_{N_1}, \\ &N^2 = N \setminus (V_z^T \setminus \{z\}), && M^2 = M \setminus (V_z^T \setminus \{z\}), && a^2 = (a_x^2)_{x\in N_2}, \end{aligned}$$

where

$$a_x^2 = \begin{cases} a_x, & x \neq z, \\ c^T(z, a), & x = z. \end{cases}$$

Consider the pointed q-metric spaces $(M^1, d|_{M^1\times M^1}, z)$ and $(M^2, d|_{M^2\times M^2}, 0)$. Suppose that neither of $\mu_1 = \sum_{x\in N^1} a_x^1 \delta(x)$ and $\mu_2 = \sum_{x\in N^2} a_x^2 \delta(x)$ are zero. Let $T^1 \in \mathcal{T}(M^1)$ and $T^2 \in \mathcal{T}(M^2)$ be given by

$$E^{T_1} = E^T \cap \{\{u, v\} \colon u, v \in M^1\} \quad \text{and} \quad E^{T_2} = E^T \cap \{\{u, v\} \colon u, v \in M^2\}.$$

We have essentially ripped the tree T into two parts at the vertex z. From Lemma 1.44 we know that $T(a)^p = T^1(a^1)^p + T^2(a^2)^p$. It is the case that $\|\mu^1\|_{\mathcal{F}_p(M^1)} = T^1(a^1)$: If there was $\widetilde{T}^1 \in \mathcal{T}(M^1)$ such that $\widetilde{T}^1(a^1) < T^1(a^1)$, then for the tree $\widetilde{T} \in \mathcal{T}(M)$ created by gluing $\widetilde{T}^1$ and T^2, i.e. $E^{\widetilde{T}} = E^{\widetilde{T}^1} \cup E^{T^2}$, we would have (by Lemma 1.44)

$$\widetilde{T}(a)^p = \widetilde{T}^1(a^1)^p + T^2(a^2)^p < T^1(a^1)^p + T^2(a^2)^p = T(a)^p.$$

But this is in contradiction with the assumption that $\|\mu\|_{\mathcal{F}_p(M)} = T(a)$ and the fact that by Theorem 1.21 we have $T(a) \leq \widetilde{T}(a)$. By an analogous consideration, we arrive to the equality $\|\mu^2\|_{\mathcal{F}_p(M^2)} = T^2(a^2)$.

Using Theorem 1.21 find trees $S^i \in \mathcal{T}(N^i)$ such that $\|\mu^i\|_{\mathcal{F}_p(N^i)} = S^i(a^i)$, $i = 1, 2$. In the same spirit as above, glue them together to form a tree $S \in \mathcal{T}(N)$ (with $E^S = E^{S^1} \cup E^{S^2}$). Lemma 1.44 guarantees that $S(a)^p = S^1(a^1)^p + S^2(a^2)^p$. One last use of Theorem 1.21 yields the inequality

$$\begin{aligned} \|\mu\|_{\mathcal{F}_p(N)}^p &= \min_{S'\in\mathcal{T}(N)} S'(a)^p \leq S(a)^p = S^1(a^1)^p + S^2(a^2)^p \\ &= \left\|\mu^1\right\|_{\mathcal{F}_p(N^1)}^p + \left\|\mu^2\right\|_{\mathcal{F}_p(N^2)}^p. \end{aligned}$$

Using the fact that for $A, B, C, D > 0$ the inequality

$$\frac{A + B}{C + D} \leq \max\left\{\frac{A}{C}, \frac{B}{D}\right\}$$

holds[3], we obtain that for some $i \in \{1, 2\}$ holds

$$\begin{aligned} \mathfrak{a}_p(N, M)^p = \frac{\|\mu\|_{\mathcal{F}_p(N)}^p}{\|\mu\|_{\mathcal{F}_p(M)}^p} &\leq \frac{\|\mu^1\|_{\mathcal{F}_p(N^1)}^p + \|\mu^2\|_{\mathcal{F}_p(N^2)}^p}{\|\mu^1\|_{\mathcal{F}_p(M^1)}^p + \|\mu^2\|_{\mathcal{F}_p(M^2)}^p} \\ &\leq \max\left\{\frac{\|\mu^1\|_{\mathcal{F}_p(N^1)}^p}{\|\mu^1\|_{\mathcal{F}_p(M^1)}^p}, \frac{\|\mu^2\|_{\mathcal{F}_p(N^2)}^p}{\|\mu^2\|_{\mathcal{F}_p(M^2)}^p}\right\} = \frac{\|\mu^i\|_{\mathcal{F}_p(N^i)}^p}{\|\mu^i\|_{\mathcal{F}_p(M^i)}^p}. \end{aligned}$$

[3] $\frac{A+B}{C+D} = \frac{A}{C+D} + \frac{B}{C+D} = \frac{A}{C} \cdot \frac{C}{C+D} + \frac{B}{D} \cdot \frac{D}{C+D} \leq \max\left\{\frac{A}{C}, \frac{B}{D}\right\}\left(\frac{C}{C+D} + \frac{D}{C+D}\right) = \max\left\{\frac{A}{C}, \frac{B}{D}\right\}$

Put $\widetilde{N} = N^i$, $\widetilde{M} = M^i$ and $\widetilde{T} = T^i$. Then $\widetilde{N} \subsetneq N$, the metric on $\widetilde{M}$ is again generated by the tree $\widetilde{T}$ and the restriction of the original weight function (this follows from the fact that for two points belonging to $\widetilde{T}$, the whole path between them must also belong to $\widetilde{T}$) and $\mathfrak{a}_p(\widetilde{N}, \widetilde{M}) \geq \mathfrak{a}_p(N, M)$. If it happened that $\mu_j = 0$ for either of $j = 1, 2$, we could simply take $i \in \{1, 2\} \setminus \{j\}$ and arrive to the same conclusion. We may repeat this process with $\widetilde{N}$ and $\widetilde{M}$ and $\widetilde{T}$ as long as there is $z \in \widetilde{N} \setminus (\operatorname{leaf}(\widetilde{T}) \cup \{0\})$. But since each time we remove some points from N and we start with N finite, this process must stop after finitely many iterations. This implies that we must arrive to a pair $\widetilde{N} \subset \widetilde{M}$ such that $\widetilde{N} \subset \operatorname{leaf}(T) \cup \{0\}$ and $\mathfrak{a}_p(\widetilde{N}, \widetilde{M}) \geq \mathfrak{a}_p(N, M)$. This means that

$$\mathfrak{a}_p(N, M) \leq \mathfrak{a}_p(\widetilde{N}, \widetilde{M}) \leq {}^T_l\mathfrak{aa}_p^q(n, k).$$

Taking the supremum over all admissible $N \subset M$ we obtain the desired inequality ${}^T\mathfrak{aa}_p^q(n, k) \leq {}^T_l\mathfrak{aa}_p^q(n, k)$. □

The last missing ingredient is a result on extending Lipschitz functions from subsets of metric spaces whose metric is generated by a tree to p-Banach spaces. One possibility is to modify a result of Matoušek ([16]). The benefit of this approach is that it yields a better constant (for our special case we would obtain that the Lipschitz constant of the extended function grows at most by a factor of $7 \cdot 6^{1/p}$). The drawback is that we would need to embed the spaces in so-called metric trees, which would require a rather technical construction. Instead, we appeal to a recent result by Bíma ([5]) which, while resulting in a worse constant, only requires us to check that the metrics arising from trees have finite Nagata dimension, which is defined in

Definition 1.46. Let (N, d) be a metric space, $\gamma \geq 1$ and $d \in \mathbb{N}_0$. We say that N has *Nagata dimension at most d with constant γ* if for every $s > 0$ there is a family $\mathcal{C}$ of non-empty subsets of N satisfying

(Na1) $\mathcal{C}$ is a covering, that is $\bigcup \mathcal{C} = N$;

(Na2) $\forall C \in \mathcal{C}\colon\ \operatorname{diam} C \leq s\gamma$ and

(Na3) $\forall A \subset N, \operatorname{diam} A \leq s\colon\ |\{C \in \mathcal{C} \colon C \cap A \neq \emptyset\}| \leq d + 1$.

Definition 1.47. Let (M, d) be a metric space and $N \subset M$. Define $\mathfrak{t}_p(N, M)$ as the infimum over all values $C > 0$ such that for any p-Banach space X and any Lipschitz function $f : N \to X$ there exists $\overline{f} : M \to X$ extending f with $L(\overline{f}) \leq CL(f)$. If (N, d) is a metric space, define $\mathfrak{ae}_p(N)$ as the supremum[4] of $\mathfrak{t}_p(N, M)$ over all metric superspaces M of N.

Theorem 1.48 ([5, Theorem 14])**.** *Let (N, d) be a metric space. If N has Nagata dimension at most d with constant γ, then $\mathfrak{ae}_p(N) \leq C(p, d, \gamma)$, where C is some constant depending only on the listed parameters.*

This chapter's penultimate result will be the bridge that connects the problem of p-amenability and Theorem 1.48.

[4]The same set-theoretic remark as for Proposition 1.10 applies here.

Proposition 1.49. *Let $(M, d, 0)$ be a finite pointed metric space, $T \in \mathcal{T}(M)$ and $w : E^T \to (0, \infty)$ be such that $d = d_{T,w}$. Then any $N \subset M$ has Nagata dimension at most* 1 *with constant* 4*.*

The proof is more-or-less a special case of the proof given in [15, Proposition 3.2].

Proof. First we show this for $N = M$. Let $s > 0$. We split M into, in some sense, level sets. For $n \in \mathbb{N}_0$ define

$$A_n = \{x \in M : ns \leq d(x, 0) < (n + 1)s\}.$$

Clearly $A = \bigcup_{n\in\mathbb{N}_0} A_n$. If $A \subset M$ has $\operatorname{diam} A \leq s$, then A can only intersect two of these sets: indeed, if $x \in A \cap A_n$ and $y \in A \cap A_m$ for some $m, n \in \mathbb{N}_0$ satisfying $|n - m| > 1$, then, without loss of generality assuming $n > m + 1$, we would have

$$\operatorname{diam} A \geq d(x, y) \geq d(x, 0) - d(y, 0) > ns - (m + 1)s \geq s,$$

which is a contradiction. However, we cannot simply take $(A_n)_{n\in\mathbb{N}_0}$ to be our cover, as the best possible bound for $\operatorname{diam} A_n$ in general is $2(n + 1)s$. The trick is to break the sets A_n apart into pieces of just the right size.

To do this, fix $n \in \mathbb{N}_0$. For convenience of notation, define $A_{-1} = \emptyset$. We define an equivalence $\sim_n$ on A_n by saying that for $x \sim_n y$ if and only if $\operatorname{lca}^T(x, y) \in A_n \cup A_{n-1}$.

This is in fact an equivalence: reflexivity and symmetry follow from the fact that $\operatorname{lca}(x, x) = x$ and $\operatorname{lca}(x, y) = \operatorname{lca}(y, x)$. To show transitivity, assume that $x \sim_n y$ and $y \sim_n z$ for some $x, y, z \in A_n$. Let $0 = y_0, \dots, y_m = y$ be the path from 0 to y and $k, l \in [0..m]$ be such that $\operatorname{lca}(x, y) = y_k$ and $\operatorname{lca}(y, z) = y_l$. Take $w = y_{\min\{k,l\}}$. Then $w \in A_n \cup A_{n-1}$ and $x, y, z \in V_w$. This implies $\operatorname{lca}(x, z) \in V_w$ and consequently $\operatorname{lca}(x, z) \in A_n \cup A_{n-1}$ as

$$(n + 1)s > d(0, x) \geq d(0, \operatorname{lca}(x, z)) \geq d(0, w) \geq (n - 1)s.$$

We will show that if $A \subset M$ has $\operatorname{diam} A \leq s$ then $A \cap A_n$ intersects at most one of these equivalence classes. Assume the contrary, i.e. there are $x, y \in A_n \cap A$ with $x \not\sim_n y$. Then, denoting $z = \operatorname{lca}(x, y)$, we have $d(x, z) > s$ and $d(y, z) > s$: since $x \not\sim_n y$, $z \in A_k$ for some $k \in [0..n - 2]$ and hence

$$d(x, z) \geq d(x, 0) - d(z, 0) > ns - ((n - 2) + 1)s = s.$$

The inequality $d(y, z) > s$ can be deduced in the same way. By definition of tree metrics, we have $d(x, y) = d(x, z) + d(z, y) > 2s$, a contradiction with the fact that $\operatorname{diam} A \leq s$.

Finally, we show that the diameter of the equivalence classes is bounded by $4s$. Let $x, y \in A_n$ and $x \sim_n y$. Again, by definition of the tree metric, $d(x, 0) = d(x, \operatorname{lca}(x, y)) + d(\operatorname{lca}(x, y), 0)$ and hence

$$d(x, \operatorname{lca}(x, y)) = d(x, 0) - d(\operatorname{lca}(x, y), 0) \leq (n + 1)s - (n - 1)s = 2s.$$

Similarly we deduce $d(y, \operatorname{lca}(x, y)) \leq 2s$ and finish by using the triangle inequality: $d(x, y) \leq d(x, \operatorname{lca}(x, y)) + d(\operatorname{lca}(x, y), y) \leq 4s$.

We have shown that the family $\mathcal{C} = \bigcup_{n\in\mathbb{N}_0} A_n/\sim_n$, where $A_n/\sim_n$ is the set of classes of equivalence of the relation $\sim_n$, i.e.

$$A_n/\sim_n = \{[x]_{\sim_n} : x \in A_n\},$$

satisfies the conditions (Na1) - (Na3).

The property of having Nagata dimension at most d with constant γ is clearly inherited by subspaces as we can take the covering $\{C \cap N : C \in \mathcal{C}\} \setminus \{\emptyset\}$. □

With this result complete, we are now ready to finally prove the crown jewel of this chapter.

Proof of Theorem 1.29. By Proposition 1.31, it is enough to get a bound on $\mathfrak{aa}_p^1(n)$ independent of n. By Corollary 1.35, it suffices[5] to get a bound on $\mathfrak{aa}_p^1(n, 2n-1)$ independent of n. By Proposition 1.42 we only need to bound ${}^T\mathfrak{aa}_p^1(n, 2n-1)$.

So let $0 \in N \subset M$, $|M \setminus \{0\}| \leq 2n-1$, $|N \setminus \{0\}| \leq n$ and let the metric on M be generated by some tree from $\mathcal{T}(M)$ and some weight. By Proposition 1.49, N has Nagata dimension at most 1 with constant 4. By Theorem 1.48 there is a constant C_p depending only on p such that $\mathfrak{ae}_p(N) \leq C_p$. The mapping $\delta : N \to \mathcal{F}_p(N)$ is 1-Lipschitz and so it can be extended to $f : M \to \mathcal{F}_p(N)$ with $L(f) \leq C_p$. Using Fact 1.33, find $a \in \mathbb{R}^n$ such that for $\mu = \sum_{x\in N} a_x \delta(x)$ holds $\|\mu\|_{\mathcal{F}_p(N)} = \mathfrak{a}_p(N, M)$ and $\|\mu\|_{\mathcal{F}_p(M)} = 1$. By Proposition 1.10 we have

$$1 = \|\mu\|_{\mathcal{F}_p(M)} \geq \left\| \sum_{x\in N} a_x \frac{f}{L(f)}(x) \right\|_{\mathcal{F}_p(N)} \geq \frac{1}{C_p} \left\| \sum_{x\in N} a_x \delta(x) \right\|_{\mathcal{F}_p(N)} = \frac{1}{C_p} \|\mu\|_{\mathcal{F}_p(N)} .$$

Hence $\mathfrak{a}_p(N, M) = \|\mu\|_{\mathcal{F}_p(N)} \leq C_p$. As this estimate is independent of N, M, and even n, we get $\mathfrak{aa}_p^1(n, 2n-1) \leq C_p$ and even $\mathfrak{aa}_p^1(n) \leq C_p$. □

Before moving on to the next chapter, let us mention that this has been by no means a complete recount of the results included in [7]. For example, we have skipped the whole part concerning the study of p-amenability for spaces consisting of three points, in which the following characterisation has been given.

Theorem 1.50 ([7, Theorem 3.9])**.** *Let $0 < p < 1$ and (N, d, x_0) be a pointed p-metric space, where $N = \{x_0, x_1, x_2\}$. Then the following two conditions are equivalent:*

(i) one of the p-triangle inequalities between points of N is an equality;

(ii) $\mathfrak{aa}_p(N) = 1$.

If these conditions are not met, then $\mathfrak{aa}_p(N) = \mathfrak{a}_p(N, M)$, where $M = N \cup \{z\}$ and the p-metric on M is given by

$$d(x_{\pi(0)}, z)^p = \frac{d(x_{\pi(1)}, x_{\pi(0)})^p + d(x_{\pi(2)}, x_{\pi(0)})^p + d(x_{\pi(1)}, x_{\pi(2)})^p}{2}$$

for any permutation π of $\{0, 1, 2\}$.

[5] It would be possible to skip this step. Theorem 1.48 does not require finiteness. Proposition 1.49 could be proven in exactly the same way if M was infinite, we would just need to extend our definition of tree metrics and work with infinite trees.

Some of the questions still remaining are

Question 1.51. Is it true that $\sup_{n\in\mathbb{N}} \mathfrak{aa}_p^q(n) \leq 2^{1/q}$? If not, is it the case for $p = q$ or $p = 1$?

Question 1.52. For which $p \in (0, 1)$ is it true that $\mathfrak{aa}_p^1(2) > 1$?

In [7], there is a sufficient condition for the statement of Question 1.52, but not a necessary one. Also, the authors have shown that $\mathfrak{aa}_p(2) \geq \frac{4}{4+2^p(2^p-2)}$, but still unknown is the answer to

Question 1.53. Is it true that $\mathfrak{aa}_p(2) = \frac{4}{4+2^p(2^p-2)}$?

Finally, there is a question oriented more towards the computer science side of things.

Question 1.54. Is there an algorithm of polynomial time complexity (in the number of points of the p-metric space) for the computation of the Lipschitz-free p-norm for some/any $p \in (0, 1)$?

2. Group actions on Lipschitz-free spaces

In the second chapter, we will go in a different direction. First of all, we return to the standard setting of Banach spaces; in terms of the previous chapter, we will always have $p = 1$.

The pivotal concept of this chapter is the one of actions of groups on metric or Banach spaces. Note that for groups other than $\mathbb{Z}$ we will use the multiplicative notation, that is $G = (G, \cdot, ^{-1}, e)$.

Definition 2.1. Let (M, d) be a metric space and G be a group. A mapping $\alpha : G \times M \to M$ is said to be an (left) *action of G on M* if the conditions

(A1) $\forall x \in M\colon \alpha(e, x) = x$ and

(A2) $\forall g, h \in G\ \forall x \in M\colon \alpha(g, \alpha(h, x)) = \alpha(gh, x)$

are satisfied. For $g \in G$ we denote by $\alpha_g : M \to M$ the mapping $\alpha_g(x) = \alpha(g, x)$. If the action is clear from the context, we will often use the even shorter notation $gx = \alpha_g(x) = \alpha(g, x)$ instead of $\alpha(g, x)$ for $g \in G, x \in M$.

We say that α is an action by isometries if for every $g \in G$ the mapping α_g is an isometry. If, instead of a metric space (M, d), G acts on a Banach space $(X, \|\cdot\|)$, we say that the action α is linear/affine if for each $g \in G$ the mapping α_g is linear/affine.

Definition 2.2. Let G be a group, X be a Banach space and let $\alpha : G \times X \to X$ be an action by linear isometries. We define the *dual action* $\alpha^* : G \times X^* \to X^*$ by

$$\alpha^*(g, x^*)(x) = x^*(\alpha(g^{-1}, x)), \quad x \in X, x^* \in X^*, g \in G.$$

Or, written using the shortened notation, $gx^*(x) = x^*(g^{-1}x)$. We say that $x^* \in X^*$ is *G-invariant* if $gx^* = x^*$ for all $g \in G$.

The dual action is in fact an action: for any $x^* \in X^*$ and $x \in X$ we have

$$\alpha^*(e, x^*)(x) = x^*(\alpha(e^{-1}, x)) = x^*(x)$$

and for $x^* \in X^*$, $x \in X$ and $g, h \in G$ holds

$$\begin{aligned}\alpha^*_g(\alpha^*_h(x^*))(x) &= \alpha^*_h(x^*)(\alpha_{g^{-1}}(x)) = x^*(\alpha_{h^{-1}}(\alpha_{g^{-1}}(x))) \\ &= x^*(\alpha_{h^{-1}g^{-1}}(x)) = \alpha^*_{gh}(x^*)(x).\end{aligned}$$

Now we can state a question asked by Kazhdan and Yom Din, which was the motivation for this chapter.

Question 2.3. Let $\delta > 0$, X be a Banach space and let G be a discrete[1] group acting on X by linear isometries. Suppose that $x^* \in S_{X^*}$ satisfies that $\|gx^* - x^*\| \leq \frac{\delta}{10}$ for all $g \in G$. Must there exist G-invariant $y^* \in X^*$ with $\|x^* - y^*\| \leq \delta$?

[1] By discrete group we mean a group equipped with the discrete topology.

The question is related to their work in [12] and is very similar to the well known and widely studied Kazhdan's property (T) (for a book concerning this property see, e.g., [4]). Some counterexamples were published in [10]. An easy and elementary example in the positive direction is $\ell_\infty(G)$.

Example 2.4. Let G be a group acting on $\ell_1(G)$ by translations, i.e. $\alpha(g,x)(h) = x(g^{-1}h)$ for $g,h \in G$ and $x \in \ell_1(G)$. This action is by linear isometries and the dual action satisfies the statement in Question 2.3.

Proof. The formula $gx(h) = x(g^{-1}h)$ indeed prescribes an action: for $x \in \ell_1(G)$ and $h \in G$ holds $ex(h) = x(e^{-1}h) = x(h)$ and for $x \in \ell_1(G)$, $g, g' \in G$ and $h \in G$ holds

$$\alpha_{g'}(\alpha_g(x))(h) = \alpha_g(x)(g'^{-1}h) = x(g^{-1}g'^{-1}h) = x((g'g)^{-1}h) = \alpha_{g'g}(x)(h).$$

Linearity is clear: for $g,h \in G$, $x,y \in \ell_1(G)$, and $t \in \mathbb{R}$ holds

$$\begin{aligned}\alpha_g(x+ty)(h) &= (x+ty)(g^{-1}h) = x(g^{-1}h) + ty(g^{-1}h)\\ &= \alpha_g(x)(h) + t\alpha_g(y)(h) = (\alpha_g(x) + t\alpha_g(y))(h);\end{aligned}$$

and so is the fact that the action is by isometries: for $g \in G$ and $x \in \ell_1(G)$ we have

$$\|x\|_1 = \sum_{h\in G} |x(h)| = \sum_{h\in G} \left|x(g^{-1}h)\right| = \sum_{h\in G} |gx(h)| = \|gx\|_1,$$

where we used that $h \mapsto g^{-1}h$ is a bijection on G.

Of course, $\ell_1(G)^* \cong \ell_\infty(G)$ and the dual action is given by

$$gx^*(x) = x^*(g^{-1}x) = \sum_{h\in G} x^*(h)g^{-1}x(h) = \sum_{h\in G} x^*(h)x(gh) \tag{2.1}$$

for $x \in \ell_1(G), x^* \in \ell_\infty(G)$.

Denote

$$\delta_{g,h} = \begin{cases} 1, & g = h \\ 0, & g \neq h \end{cases} \quad \text{for } g,h \in G$$

and $e_g \in \ell_1(G)$ the element given by $e_g(h) = \delta_{g,h}$. Let $k,k' \in G$. For $g = k'k^{-1}$ and any $h \in G$ we get

$$ge_k(h) = e_k(g^{-1}h) = \delta_{k,g^{-1}h} = \delta_{gk,h} = e_{gk}(h) = e_{k'}(h), \tag{2.2}$$

i.e. $ge_k = e_{k'}$.

We will show that G-invariant functions in $\ell_\infty(G) \cong \ell_1(G)^*$ are exactly the constants. If $x^* = (c)_{h\in G}$ for some $c \in \mathbb{R}$, then for any $g \in G$ and $x \in \ell_1(G)$ we have

$$\begin{aligned}gx^*(x) &\overset{(2.1)}{=} \sum_{h\in G} x^*(h)x(gh) = \sum_{h\in G} cx(gh) = \sum_{h\in G} cx(h)\\ &= \sum_{h\in G} x^*(h)x(h) = x^*(x).\end{aligned}$$

On the other hand, let $x^* \in \ell_\infty(G)$ be G-invariant. Let $k,k' \in G$ be arbitrary and denote $g = k'k^{-1}$. Using (2.2) we deduce

$$x^*(k') = \sum_{h\in G} x^*(h)\delta_{k',h} = x^*(e_{k'}) = x^*(ge_k) = g^{-1}x^*(e_k) = x^*(e_k) = x^*(k),$$

i.e. x^* is constant when understood as an element of $\ell_\infty(G)$.

Finally, assume that $x^* \in S_{\ell_\infty(G)}$ and for any $g \in G$ holds $\|gx^* - x^*\|_\infty \leq \delta/10$. Since $x^* \in S_{\ell_\infty(G)}$, there must be $a \in \{-1, 1\}$ and some $k \in G$ with $|a - x^*(k)| < \delta/10$. Let $k' \in G$ be arbitrary. Denoting $g = k'k^{-1}$ and again using (2.2), we get

$$\begin{aligned}|x^*(k) - x^*(k')| &= |x^*(e_k) - x^*(e_{k'})| = |x^*(e_k) - x^*(ge_k)| \\ &= \left|x^*(e_k) - g^{-1}x^*(e_k)\right| \leq \left\|x^* - g^{-1}x^*\right\|_\infty \leq \frac{\delta}{10}\end{aligned}$$

and so $|a - x^*(k')| \leq |a - x^*(k)| + |x^*(k) - x^*(k')| \leq \delta/5$. We have shown that for the G-invariant function $\overline{x^*} = (a)_{h\in G} \in \ell_\infty(G)$ holds $\|x^* - \overline{x^*}\|_\infty \leq \delta$. □

2.1 Preliminaries

Seeing that Question 2.3 pertains to dual spaces, the following piece of information will be fundamental for this chapter.

Theorem 2.5. *Let $(M, d, 0)$ be a pointed metric space. Equip the space $\mathrm{Lip}_0(M)$ with the norm $\|f\| = L(f), f \in \mathrm{Lip}_0(M)$. Then $\mathcal{F}(M)^* \cong \mathrm{Lip}_0(M)$ via the mapping*

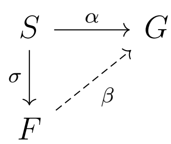
$$\varphi : \mathrm{Lip}_0(M) \to \mathcal{F}(M)^*, \quad \varphi(f)\left(\sum_{x\in M} a_x\delta(x)\right) = \sum_{x\in M} a_x f(x), \quad a \in c_{00}(M).$$

The proof can be found e.g. in [21, Chapter 3]. In the statement above, we only specify how $\varphi(f)$ behaves on $\operatorname{span}\delta(M)$, but from Theorem 1.8 we know that for continuous functions this is enough.

Before returning to group actions, we give a quick primer on some required topics from group theory. For more detail see e.g. [20, Chapter 2].

Notation 2.6. Let G be a group and $A \subset G$. Denote $A^{-1} = \{a^{-1} : a \in A\}$ and $A^{\pm 1} = A \cup A^{-1}$. By $\langle A\rangle$ we denote the subgroup generated by A, which is $\langle A\rangle = \{a_1 \cdots a_n \in G : n \in \mathbb{N}, a_i \in A^{\pm 1}, i \in [1..n]\}$. To symbolize that H is a subgroup of G we write $H \leq G$.

Definition 2.7. Let F be a group, $S \neq \emptyset$ a set and $\sigma : S \to F$ be any mapping. We say F (or more precisely (F, σ)) is *free* on S if for every group G and mapping $\alpha : S \to G$ there exists a unique homomorphism $\beta : F \to G$ with $\alpha = \beta \circ \sigma$, that is, the following diagram commutes.



For every non-empty set S there exists a group F_S and $\sigma : S \to F_S$ which is free on S and $F_S = \langle\sigma(S)\rangle$. For ease of notation, we will identify every element $s \in S$ with the corresponding element of its free group $\sigma(s) \in F_S$.

Definition 2.8. Let S be a non-empty set and let $S^{-1} = \{s^{-1} : s \in S\}$[2] be a set of the same cardinality as S disjoint with S. By a *word* in S we mean a finite sequence $s_1 \cdots s_n$ of elements of the set $S^{\pm 1} = S \cup S^{-1}$. A word $s_1, \cdots s_n$ in S is *reduced* if no two consecutive symbols are of the form ss^{-1} or $s^{-1}s$ for some $s \in S$. The *empty word* is the empty sequence.

To each word in S there is a single corresponding reduced word which can be obtained by removing all offending pairs ss^{-1} and $s^{-1}s$. This allows the following description of free groups.

Proposition 2.9. *Let S be a non-empty set. The set F of all reduced words in S together with*

- *the inverse operation defined as $(s_1 \cdots s_n)^{-1} = s_n^{-1} \cdots s_1^{-1}$,*
- *the group operation of two words $s_1 \cdots s_n$ and $t_1 \cdots t_m$ defined as the reduced word corresponding to the word $s_1 \cdots s_n t_1 \cdots t_m$,*
- *the unit defined as the empty word and*
- *the mapping $\sigma : S \to F$ defined as $\sigma(s) = s$ (where the s on the right is the word consisting of a single element s)*

is a free group on S.

This is in fact characteristic of the free groups:

Proposition 2.10. *Let F be a group and $S \subset F$. Then the following are equivalent:*

(i) for every $g \in G$ there are unique $n \in \mathbb{N}_0$, $s_1, \dots, s_n \in S$ with $s_i \neq s_{i+1}$ for $i \in [1..n-1]$ and $l_1, \dots, l_n \in \mathbb{Z} \setminus \{0\}$ such that $g = s_1^{l_1} \cdots s_n^{l_n}$;

(ii) F is free on S.

A simple yet very useful consequence of the definition is

Proposition 2.11. *Let F_S be a free group with generating set S and $x \in \mathbb{R}^S$. Then the mapping $f : F_S \to \mathbb{R}$ defined for $g \in F_S$ as*

$$f(g) = \sum_{i=1}^{n} a_i x_{s_i},$$

where $n \in \mathbb{N}, a \in \{-1, 1\}^n$ and $s \in S^n$ are such that $g = s_1^{a_1} \cdots s_n^{a_n}$ is the reduced word representing g, is a homomorphism.

Proof. By definition of a free group, there is a homomorphism $h : F_S \to \mathbb{R}$ with $h(s) = x_s$. Let $g = s_1^{a_1} \cdots s_n^{a_n}$ be an element of F_S and its representation as reduced word. Then we have

$$h(g) = h(s_1^{a_1} \cdots s_n^{a_n}) = \sum_{i=1}^{n} a_i h(s_i) = \sum_{i=1}^{n} a_i x_{s_i} = f(g).$$

So $h = f$ and thus f is a homomorphism. □

[2]Here, we understand s^{-1} to be just a symbol, not an inverse of s in any sense. The reason for this notation is that we will use the symbol s^{-1} to somehow represent the inverse of $\sigma(s)$ in the free group.

Finally we remark that for two sets S_1 and S_2 with $|S_1| = |S_2|$ the free groups F_{S_1} and F_{S_2} are isomorphic.

Next, we recall the definitions on normal groups and normal closures.

Definition 2.12. Let G be a group and $N \leq G$. We say N is a *normal subgroup* of G and write $N \trianglelefteq G$ if $\forall g \in G\colon gNg^{-1} \subset N$. If A is a subset of G, we define the *normal closure* of A, denoted $\operatorname{ncl} A$, to be the smallest (with respect to inclusion[3]) normal subgroup containing A.

Proposition 2.13. *Let G be a group and $A \subset G$. Then*

$$\operatorname{ncl} A = \langle A^G \rangle, \text{ where } A^G = \{g^{-1}ag\colon a \in A, g \in G\}.$$

Definition 2.14. Let G be a group and $N \trianglelefteq G$. For $g \in G$ define its *coset* as $[g] = gN = \{gn\colon n \in N\}$. Define the *quotient* G/N as the set $\{[g]\colon g \in G\}$ equipped with the unit $[e]$ and operations $[g]^{-1} = [g^{-1}]$ and $[g][h] = [gh]$. Then G/N is a group and the mapping $q : G \to G/N, q(g) = [g]$, called the *quotient map*, is a homomorphism.

Proposition 2.15. *Let G be a group and $S \subset G$ be such that $G = \langle S \rangle$. Then there is $N \trianglelefteq F_S$ such that $G \cong F_S/N$.*

Definition 2.16. Let S be a finite set and $R \subset F_S$ be also finite. We define the group $\langle S|R \rangle$ as the quotient

$$\langle S|R \rangle = F_S / \operatorname{ncl} R.$$

We call the elements of S the *generators* of $\langle S|R \rangle$ and elements of R its *relators*. The group $\langle S|R \rangle$ is said to be a *presentation* of a group G if G is isomorphic to $\langle S|R \rangle$. A group G is said to be *finitely presented* if there are a finite set S and finite $R \subset F_S$ with G isomorphic to $\langle S|R \rangle$.

For the sake of clarity of notation, when working with a finitely presented group G, we will assume $G = \langle S|R \rangle$ instead of just G being isomorphic to the presentation.

Last notion we introduce are so-called amenable groups. More detail can be found e.g. in [4, Appendix G].

Definition 2.17. Let G be a topological Hausdorff group. For $f \in \ell_\infty(G)$ and $g \in G$ define $fg \in \ell_\infty(G)$ by

$$(fg)(h) = f(hg^{-1}), h \in G.$$

Define

$$\mathcal{C}^b_{ru}(G) = \{f \in \ell_\infty(G)\colon g \mapsto fg^{-1} \in \ell_\infty(G) \text{ is } G\text{-} \|\cdot\|_\infty \text{ continuous}\}$$

A linear functional ᛗ $: \mathcal{C}^b_{ru}(G) \to \mathbb{R}$[4] is a *right-invariant mean* if it satisfies the conditions

[3]Talking about the smallest normal subgroup makes sense since an arbitrary intersection of normal subgroups is a normal subgroup and every normal subgroup contains the unit.

[4]Reserving M for metric spaces and m to be a natural number we arrive to a bit of a conundrum. Either don't use the letter m for denoting mean or choose some other version of it. Neither Greek or Cyrillic alphabets offer any satisfying substitute. Among the options, I went with ᛗ, the Anglo-Saxon ruinic version of M.

(Me1) $\mathfrak{M}(1_G) = 1$, where 1_G is the constant function $(1)_{g\in G}$;

(Me2) $\forall f \in \mathcal{C}^b_{ru}(G)\ \forall g \in G\colon \mathfrak{M}(fg) = \mathfrak{M}(f)$;

(Me3) $\forall f \in \mathcal{C}^b_{ru}(G), f \geq 0\colon \mathfrak{M}(f) \geq 0$.

Definition 2.18. A group G is *amenable* if there is a right-invariant mean $\mathfrak{M} : \mathcal{C}^b_{ru}(G) \to \mathbb{R}$.

2.2 Group actions on Banach spaces

Before moving to Lipschitz-free spaces, we will go through some basic results which hold for Banach spaces in general. In Definition 2.2, we only defined the dual action to actions by linear isometries in order to keep the introduction simple. We will start by generalising the definition to actions by affine isometries. To do so, we prove

Fact 2.19. *Let α be an action of a group G a Banach space X by affine isometries. Then there is an action L_α of G on X by linear isometries such that $\alpha(g,x) = L_\alpha(g,x) + \alpha(g,0)$ for $g \in G, x \in X$. L_α is the so-called* linear part *of α.*

Proof. For each $g \in G$ the mapping $\alpha_g : X \to X$ is an affine isometry. So it can be written as $\alpha_g(x) = (L_\alpha)_g(x) + \alpha_g(0)$ for some linear isometry $(L_\alpha)_g$. We need to show that $L_\alpha(g,x) = (L_\alpha)_g(x)$ is an action of G on X.

By (A1) for α we have that α_e is the identity mapping which is itself linear, so $(L_\alpha)_e = \mathrm{Id}$ and (A1) is satisfied for L_α as well.

Let $g, h \in G$ and $x \in X$. Then

$$\begin{aligned} L_\alpha(g, L_\alpha(h,x)) &= L_\alpha(g, \alpha_h(x) - \alpha_h(0)) \\ &= L_\alpha(g, \alpha_h(x)) - L_\alpha(g, \alpha_h(0)) \\ &= \alpha_g(\alpha_h(x)) - \alpha_g(0) - \big(\alpha_g(\alpha_h(0)) - \alpha_g(0)\big) \\ &= \alpha_g(\alpha_h(x)) - \alpha_g(\alpha_h(0)) \end{aligned}$$

and

$$L_\alpha(gh, x) = \alpha_{gh}(x) - \alpha_{gh}(0).$$

These equalities imply (A2) for L_α, because by (A2) for α, we have $\alpha_{gh}(x) = \alpha_g(\alpha_h(x))$ and $\alpha_{gh}(0) = \alpha_g(\alpha_h(0))$. □

Definition 2.20. Let α be an action of a group G on a Banach space X by affine isometries. We define the dual action $\alpha^* : G \times X^* \to X^*$ to α as the dual action to L_α. That is,

$$\alpha^*(g, x^*)(x) = x^*(L_\alpha(g^{-1}, x)),$$

or, using the shorter notation

$$gx^*(x) = x^*(g^{-1}x - g^{-1}0) = x^*(g^{-1}x) - x^*(g^{-1}0).$$

If α is linear, then for each $g \in G$ we have $g^{-1}0 = 0$ and so this definition is compatible with Definition 2.2.

Notation 2.21. Let α be an action of a group G on a Banach space X by affine isometries. We denote

$$\mathrm{Inv}_G(X) = \{x^* \in X^* \colon x^* \text{ is } G\text{-invariant}\}.$$

The following fact is essential in working with invariant functionals, so in the following text we will automatically use it without mention.

Fact 2.22. *Let X be a Banach space and G be a group acting on X by affine isometries. Then for $x^* \in X^*$ are the following equivalent:*

(i) x^* is G-invariant;

(ii) $\forall g \in G \colon x \mapsto x^*(gx) - x^*(g0) - x^*(x)$ is constant zero;

(iii) $\forall g \in G \colon x \mapsto x^*(gx) - x^*(x)$ is constant.

Proof. By definition, $x^* \in X^*$ is G-invariant if and only if for every $g \in G$ holds $gx^* = x^*$. This is equivalent to

$$\forall g \in G \; \forall x \in X \colon x^*(x) = g^{-1}x^*(x) = x^*(gx - g0) = x^*(gx) - x^*(g0).$$

This establishes the equivalence (i) $\iff$ (ii). The equivalence (ii) $\iff$ (iii) is clear, because the two functions differ only by the constant (with respect to x) $x^*(g0)$. □

Proposition 2.23. *Let X be a Banach space and G be a group acting on X by affine isometries. Then $\mathrm{Inv}_G(X)$ is a norm-closed subspace of X^*.*

Proof. Let $x^*, y^* \in \mathrm{Inv}_G(X)$. Then the functions $x^*(g\cdot) - x^*(\cdot)$ and $y^*(g\cdot) - y^*(\cdot)$ are constant. Let $\lambda \in \mathbb{R}$. Then also the functions

$$(\lambda x^*)(g\cdot) - (\lambda x^*)(\cdot) = \lambda(x^*(g\cdot) - x^*(\cdot))$$
$$(x^* + y^*)(g\cdot) - (x^* + y^*)(\cdot) = (x^*(g\cdot) - x^*(\cdot)) + (y^*(g\cdot) - y^*(\cdot))$$

are constant. Hence $\lambda x^*, x^* + y^* \in \mathrm{Inv}_G(X)$.

Now let $(x_n^*) \in \mathrm{Inv}_G(X)^{\mathbb{N}}$ and $x_n^* \to x^* \in X^*$. Fix $g \in G$. We want to show that $x^*(g\cdot) - x^*(\cdot) - x^*(g0)$ is constant zero. This holds for all $x_n^*, n \in \mathbb{N}$. Pick $x \in X$ arbitrarily. Then

$$\begin{aligned} &|x^*(gx) - x^*(x) - x^*(g0)| \\ &\qquad = |(x^*(gx) - x^*(x) - x^*(g0)) - (x_n^*(gx) - x_n^*(x) - x_n^*(g0))| \\ &\qquad = |(x^* - x_n^*)(gx - x - g0)| \\ &\qquad \le \|x^* - x_n^*\| \, \|gx - x - g0\| \, . \end{aligned}$$

We get $|x^*(gx) - x^*(x) - x^*(g0)| = 0$ for the right hand side converges to 0 as $n \to \infty$. □

Once again, to simplify notation we introduce

Definition 2.24. Let X be a Banach space, G a group acting on X by affine isometries and $\delta > 0$. We say that $x^* \in X^*$ is *δ-invariant* (for the action of G) if

$$\forall g \in G \colon \; \|gx^* - x^*\| \le \delta.$$

Akin to Fact 2.22, we have

Fact 2.25. *Let X be a Banach space, G a group acting on X by affine isometries and $\delta > 0$. Then for $x^* \in X^*$ are the following equivalent:*

(i) x^* *is δ-invariant;*

(ii) $\forall g \in G\ \forall x \in X\colon\ |x^*(x) - x^*(gx) + x^*(g0)| \leq \delta \|x\|$;

(iii) $\forall g \in G\ \forall x, y \in X\colon\ |x^*(x - gx) - x^*(y - gy)| \leq \delta \|x - y\|$.

Proof. Again, the equivalence (i) $\iff$ (ii) can be proved by simply writing out the definition: x^* is δ-invariant if and only if for all $g \in G$ holds $\|x^* - gx^*\| \leq \delta$ or, equivalently, for every $g \in G$ and $x \in X$ holds

$$\delta \|x\| \geq \left|(x^* - g^{-1}x^*)(x)\right| = \left|x^*(x) - g^{-1}x^*(x)\right| = |x^*(x) - x^*(gx - g0)|\,.$$

If x^* is δ-invariant, $g \in G$ and $x, y \in X$, we calculate in the same spirit

$$\begin{aligned}\delta \|x - y\| \geq \left|(x^* - g^{-1}x^*)(x - y)\right| &= \left|(x^* - g^{-1}x^*)(x) - (x^* - g^{-1}x^*)(y)\right| \\ &= |(x^*(x) - x^*(gx) - x^*(g0)) - (x^*(y) - x^*(gy) - x^*(g0))| \\ &= |x^*(x - gx) - x^*(y - gy)|\,.\end{aligned}$$

On the other hand, if (iii) holds, by taking $y = 0$ we obtain (ii). ☐

One more easy case, also briefly mentioned in [10], when the answer to Question 2.3 is positive are actions of amenable groups.

Proposition 2.26. *Let G be an amenable group acting by affine isometries on a Banach space X. Assume that the mapping $g \mapsto gx$ is continuous for every $x \in X$*[5]*. Let $\delta > 0$ and $x^* \in X^*$ be δ-invariant. Denote $\overline{x^*}(x) = \mathfrak{M}(g \mapsto g^{-1}x^*(x)), x \in X$, where $\mathfrak{M}$ is a right-invariant mean on G. Then $\overline{x^*} \in \mathrm{Inv}_G(X)$ and $\|x^* - \overline{x^*}\| \leq \delta$.*

Proof. During the proof we will use the notation "*variable* $\mapsto$ *expression*" to refer to functions without giving them names (often, for example, for elements of $\ell_\infty(G)$ which are function from G to $\mathbb{R}$).

We will start by showing that $\overline{x^*}$ is a well-defined function. For $x \in X$ define $x^*_x : G \to \mathbb{R}$ by

$$x^*_x(g) = g^{-1}x^*(x) = x^*(gx) - x^*(g0).$$

First, $x^*_x \in \ell_\infty(G)$ follows from x^*'s δ-invariance: for any $x \in X$ and $g \in G$ we have

$$\begin{aligned}|x^*_x(g)| &= |x^*(gx) - x^*(g0)| \\ &\leq |x^*(x - gx) - x^*(0 - g0)| + |x^*(x - 0)| \\ &\leq \delta \|x - 0\| + |x^*(x) - x^*(0)| \\ &\leq (\delta + \|x^*\|) \|x\| < \infty.\end{aligned} \tag{2.3}$$

We need to verify that $x^*_x \in \mathcal{C}^b_{ru}(G)$, that is, we need to check that $h \mapsto (g \mapsto x^*(ghx - gh0))$ is continuous from G to $\ell_\infty(G)$. Let $h \in G$ be arbitrary and

[5] This is automatically true for discrete groups.

$(h_i)_{i\in I}$ be a net in G indexed by I such that $h_i \to h$. Then, by the assumption, $h_i x \to hx$ and $h_i 0 \to 0$. From continuity of x^* and the fact that the action is by isometries we now get the desired convergence:

$$\begin{aligned}
&\|(g \mapsto x^*(ghx - gh0)) - (g \mapsto x^*(gh_i x - gh_i 0))\|_{\ell_\infty(G)} \\
&\qquad \le \|g \mapsto x^*(ghx - gh_i x)\|_{\ell_\infty(G)} + \|g \mapsto x^*(gh0 - gh_i 0)\|_{\ell_\infty(G)} \\
&\qquad \le \|x^*\| \sup_{g\in G} (\|ghx - gh_i x\| + \|gh0 - gh_i 0\|) \\
&\qquad = \|x^*\| \sup_{g\in G} (\|hx - h_i x\| + \|h0 - h_i 0\|) \\
&\qquad = \|x^*\| (\|hx - h_i x\| + \|h0 - h_i 0\|) \to 0.
\end{aligned}$$

So $x^*_x \in \mathcal{C}^b_{ru}(G)$ for all $x \in X$.

The mapping $\overline{x^*}$ is linear as we are composing linear functionals: for $x, y \in X$ and $t \in \mathbb{R}$ holds

$$\begin{aligned}
\overline{x^*}(x + ty) &= \mathfrak{M}\Big(g \mapsto g^{-1}x^*(x + ty)\Big) = \mathfrak{M}\Big(g \mapsto (g^{-1}x^*(x) + tg^{-1}x^*(y))\Big) \\
&= \mathfrak{M}\Big((g \mapsto g^{-1}x^*(x)) + t(g \mapsto g^{-1}x^*(y))\Big) \\
&= \mathfrak{M}\Big(g \mapsto g^{-1}x^*(x)) + t\mathfrak{M}(g \mapsto g^{-1}x^*(y)\Big) \\
&= \overline{x^*}(x) + t\overline{x^*}(y).
\end{aligned}$$

Using the estimate (2.3) and properties (Me1) and (Me3)[6] of the mean we obtain

$$|\overline{x^*}(x)| = |\mathfrak{M}(x^*_x)| \overset{\text{(Me3)}}{\le} \mathfrak{M}(|x^*_x|) \overset{\text{(Me3)}}{\le} \mathfrak{M}(g \mapsto (\|x^*\| + \delta)\,\|x\|) \overset{\text{(Me1)}}{=} (\|x^*\| + \delta)\,\|x\|\,.$$

So $\overline{x^*} \in X^*$ and $\|\overline{x^*}\| \le \|x^*\| + \delta$.

To show that $\overline{x^*} \in \mathrm{Inv}_G(X)$, we will show that for all $h \in G$ and $x \in X$ holds $\overline{x^*}(hx - x - h0) = 0$. Fix $h \in G, x \in X$. Then for any $g \in G$ holds

$$\begin{aligned}
(x^*_{hx} - x^*_x - x^*_{h0})(g) &= x^*(ghx) - x^*(g0) - x^*(gx) + x^*(g0) - x^*(gh0) + x^*(g0) \\
&= x^*(ghx - gh0) - x^*(gx - g0)
\end{aligned}$$

and using the invariance and linearity of the mean we obtain

$$\begin{aligned}
\overline{x^*}(hx - x - h0) &= \mathfrak{M}(x^*_{hx} - x^*_x - x^*_{h0}) \\
&= \mathfrak{M}\Big(g \mapsto x^*(ghx - gh0)\Big) - \mathfrak{M}\Big(g \mapsto x^*(gx - g0)\Big) \\
&= 0.
\end{aligned}$$

By Fact 2.22, $\overline{x^*}$ is invariant.

It remains to show that $\|x^* - \overline{x^*}\| < \delta$. Pick $x \in X$. By definition of x^*_x and δ-invariance of x^*, for any $g \in G$ holds

$$|x^*(x) - x^*_x(g)| = \Big|(x^* - g^{-1}x^*)(x)\Big| \le \delta\,\|x\|\,.$$

[6] For $f \in \mathcal{C}^b_{ru}(G)$ we have $|f| \in \mathcal{C}^b_{ru}(G)$ and $|f| - f \ge 0$, so by the property (Me3) we obtain $\mathfrak{M}(|f| - f) \ge 0$, which by linearity yields $\mathfrak{M}(|f|) \ge \mathfrak{M}(f)$. Analogously, $|f| + f \ge 0$ and so $\mathfrak{M}(|f|) \ge -\mathfrak{M}(f)$. Hence $|\mathfrak{M}(f)| \le \mathfrak{M}(|f|)$.

Using this estimate and the fact that by (Me1) we have $\mathfrak{M}(g \mapsto C) = C$ for any $C \in \mathbb{R}$, we get

$$\begin{aligned}|(x^* - \overline{x^*})(x)| = |x^*(x) - \mathfrak{M}(x^*_x)| &= |\mathfrak{M}(g \mapsto (x^*(x) - x^*_x(g))| \\ &\overset{(\text{Me3})}{\leq} \mathfrak{M}(g \mapsto \delta \|x\|) = \delta \|x\| .\end{aligned}$$

Hence $\|x^* - \overline{x^*}\| \leq \delta$. □

Notable examples of amenable groups, and hence groups for which the answer to Question 2.29 is positive are abelian and compact groups (see [4, Example G.1.5] and [4, Theorem G.2.1]). On the other hand, free groups with generating sets consisting of two elements are not amenable (see [4, Example G.2.4]).

The just proven Proposition already answers Question 2.3 for actions of discrete amenable groups, but if we furthermore assume that the action has bounded orbits, we can use this method to glean more insight into the situation.

Corollary 2.27. *Let G, X, δ, x^* and $\overline{x^*}$ be as in Proposition 2.26. Assume that G has bounded orbits. Then $\overline{x^*}$ is constant on each orbit of G. In particular, if the action of G is topologically transitive (that is, for any two open sets $U, V \subset X$ there is $g \in G$ such that $gU \cap V \neq \emptyset$), then $\|x^*\| \leq \delta$.*

Proof. Since G has bounded orbits, $(g \mapsto x^*(gx)) \in \ell^\infty(G)$ for any $x \in X$. We need to show that $\mathfrak{M}(g \mapsto x^*(gx))$ is well-defined for any $x \in X$. This means showing that $h \mapsto (g \mapsto x^*(ghx))$ is continuous from G to $\ell_\infty(G)$. Let $(h_i)_{i \in I}$ be a net in G indexed by I such that $h_i \to h$. Then, by the assumption, $h_i x \to hx$ and using the assumption that the action is by isometries we obtain

$$\begin{aligned}\|(g \mapsto x^*(gh_i x)) - (g \mapsto x^*(ghx))\|_\infty &= \sup_{g \in G} |x^*(gh_i x - ghx)| \\ &\leq \|x^*\| \sup_{g \in G} \|gh_i x - ghx\| \\ &= \|x^*\| \, \|h_i - hx\| \to 0.\end{aligned}$$

For $x \in X$ and $h \in G$ we calculate

$$\begin{aligned}&\overline{x^*}(x) - \overline{x^*}(hx) \\ &\quad = \mathfrak{M}(g \mapsto x^*(gx) - x^*(g0)) - \mathfrak{M}(g \mapsto x^*(ghx) - x^*(g0)) \\ &\quad = \mathfrak{M}(g \mapsto x^*(gx)) - \mathfrak{M}(g \mapsto x^*(g0)) - \mathfrak{M}(g \mapsto x^*(ghx)) + \mathfrak{M}(g \mapsto x^*(g0)) \\ &\quad = \mathfrak{M}(g \mapsto x^*(gx)) - \mathfrak{M}(g \mapsto x^*(ghx)) \overset{(\text{Me2})}{=} 0.\end{aligned}$$

That is, $\overline{x^*}$ is in fact constant on every orbit.

In particular, for any $g \in G$ we have $\overline{x^*}(g0) = \overline{x^*}(0) = 0$. We will show that if the action is topologically transitive, then $\overline{x^*} = 0$. This will conclude the proof because then $\|x^*\| = \|x^* - \overline{x^*}\| \leq \delta$. Let $y \in X$ and $\varepsilon > 0$. Since the action is topologically transitive, there is $g \in G$ and $y' \in U(y, \varepsilon)$ such that $y' \in U(y, \varepsilon) \cap gU(0, \varepsilon)$. We estimate

$$|\overline{x^*}(y)| \leq |\overline{x^*}(y - y')| + |\overline{x^*}(y' - g0)| + |\overline{x^*}(g0)| \leq 2 \|\overline{x^*}\| \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, $\overline{x^*}(y) = 0$. □

Notice that this in particular implies that the only invariant function for an action satisfying the assumptions of Corollary 2.27 is the constant zero. Indeed, if x^* is such an invariant function, then it is δ-invariant for any $\delta > 0$ and, by the Corollary, $\|x^*\| \leq \delta$ for any $\delta > 0$, i.e. $x^* = 0$.

Note that our mapping $\overline{x^*}$ can also be written as $\overline{x^*}(x) = R_G(x)(x^*)$, where $R_G : X \to X^{**}$ is defined as $R_G(x)(x^*) = \mathfrak{M}(g \mapsto g^{-1}x^*(x))$. This mapping R_G was studied in similar context in [8].

2.3 Group actions on Lipschitz-free spaces

In this section, we finally focus on Question 2.3 in Lipschitz-free spaces. By $\|\cdot\|$ on the space $\mathrm{Lip}_0(M)$ we will mean the norm given by the Lipschitz number, i.e. $\|f\| = L(f), f \in \mathrm{Lip}_0(M)$.

Let us establish what our actions look like. We begin with some action by isometries of a group G on a pointed metric space $(M, d, 0)$. To fit within the setting of Question 2.3, we need to, in some sense, extend this action to a linear action by isometries on $\mathcal{F}(M)$. For $g \in G$ define $f_g : M \to \mathcal{F}(M)$ by $f_g(x) = \delta(gx) - \delta(g0)$. We have for any $x, y \in M$

$$\|f_g(x) - f_g(y)\| = \|\delta(gx) - \delta(gy)\| = d(gx, gy) = d(x, y)$$

and $f_g(0) = 0$ so $f_g \in \mathrm{Lip}_0(M, \mathcal{F}(M))$ with $L(f_g) = 1$. Define $\alpha : G \times \mathcal{F}(M) \to \mathcal{F}(M)$ by setting α_g to be the linearization of the mapping f_g (which is given by Theorem 1.8 (iv)), i.e. for $\mu = \sum_{i=1}^n a_i\delta(x_i) \in \mathcal{F}(M)$

$$\alpha(g, \mu) = \sum_{i=1}^n a_i(\delta(gx_i) - \delta(g0)).$$

Then α is an action of G on $\mathcal{F}(M)$: for $\mu = \sum_{i=1}^n a_i\delta(x_i) \in \operatorname{span}\delta(M)$ we have

$$\alpha_e(\mu) = \sum_{i=1}^n a_i(\delta(ex) - \delta(e0)) = \sum_{i=1}^n a_i(\delta(x) - \delta(0)) = \mu$$

and for $g, h \in G$ we have

$$\begin{aligned}\alpha_g(\alpha_h(\mu)) &= \alpha_g\left(\sum_{i=1}^n a_i(\delta(hx) - \delta(h0))\right) \\ &= \sum_{i=1}^n a_i(\delta(ghx) - \delta(g0) - (\delta(gh0) - \delta(g0))) \\ &= \sum_{i=1}^n a_i(\delta(ghx) - \delta(gh0)) = \alpha_{gh}(\mu).\end{aligned}$$

If $\mu \in \mathcal{F}(M)$, there is a sequence $(\mu_n) \in (\operatorname{span}\delta(M))^{\mathbb{N}}$ such that $\mu_n \to \mu$. Using the fact that α_g are continuous for every $g \in G$, we deduce

$$\alpha_e(\mu) = \lim_{n\to\infty} \alpha_e(\mu_n) = \lim_{n\to\infty} \mu_e = \mu$$

and for $g, h \in G$

$$\alpha_g(\alpha_h(\mu)) = \lim_{n\to\infty} \alpha_g(\alpha_h(\mu_n)) = \lim_{n\to\infty} \alpha_{gh}(\mu_n) = \alpha_{gh}(\mu).$$

Moreover, this action is by isometries: since $L(f_g) \leq 1$ and $L(f_{g^{-1}}) \leq 1$, it holds that $\|\alpha_g\| \leq 1$ and $\|\alpha_{g^{-1}}\| \leq 1$; if $\mu \in \mathcal{F}(M)$, then $\|\alpha_g(\mu)\| \leq \|\mu\|$ and $\|\mu\| = \|\alpha_{g^{-1}}(\alpha_g(\mu))\| \leq \|\alpha_g(\mu)\|$.

Remark 2.28. It would seem more natural to instead consider an action such that $g\delta(x) = \delta(gx)$ for $x \in M$. However, if there is $g \in G$ such that $g0 \neq 0$, then the action cannot be by linear isometries (since it would not map 0 to 0). So, in general, the best one can hope for is an action by affine isometries. This is in fact achievable by the affine action α' which is constructed by taking our linear action α and offsetting it so that the condition $g\delta(x) = \delta(gx)$, $x \in M$ holds. That is,

$$\alpha'_g(\mu) = \delta(g0) + \alpha_g(\mu), \quad \mu \in \mathcal{F}(M), g \in G, \tag{2.4}$$

or, written out explicitly,

$$\alpha'_g(\mu) = \delta(g0) + \sum_{i=1}^{n} a_i(\delta(gx_i) - \delta(g0)), \text{where } \mu = \sum_{i=1}^{n} a_i\delta(x_i) \in \mathcal{F}(M).$$

However, since the linear part of α' is α, by definition we have $(\alpha')^* = \alpha^*$, and thus, for the purposes of Question 2.29, it is immaterial which action we choose. For this reason, we will not check that the formula (2.4) really defines an action and we shall work with the linear action α.

Denote the dual action to α by $\alpha^* : G \times \mathcal{F}(M)^* \to \mathcal{F}(M)^*$. To be able to use Theorem 2.5, we need to define an action $\beta : G \times \mathrm{Lip}_0(M) \to \mathrm{Lip}_0(M)$ so that, using the notation from said theorem, $\alpha^*_g(\varphi(f)) = \varphi(\beta_g f)$ for any $g \in G$ and $f \in \mathrm{Lip}_0(M)$. We calculate for $g \in G$, $f \in \mathrm{Lip}_0(M)$ and $x \in M$

$$\begin{aligned}\alpha^*_g(\varphi(f))(\delta(x)) &= \varphi(f)(\alpha_{g^{-1}}(\delta(x))) = \varphi(f)(\delta(g^{-1}x) - \delta(g^{-1}0)) \\ &= f(g^{-1}x) - f(g^{-1}0).\end{aligned}$$

Hence, the action of G on $\mathrm{Lip}_0(M)$ we seek must be defined as $\beta_g f(x) = f(g^{-1}x) - f(g^{-1}0)$. First we need to show that $x \mapsto f(g^{-1}x) - f(g^{-1}0)$ is in fact an element of $\mathrm{Lip}_0(M)$. Clearly $\beta_g(f)(0) = f(g^{-1}0) - f(g^{-1}0) = 0$. For $x, y \in M$ we calculate

$$|\beta_g(f)(x) - \beta_g(f)(y)| = \left|f(g^{-1}x) - f(g^{-1}y)\right| \leq L(f)d(g^{-1}x, g^{-1}y) = L(f)d(x, y).$$

So $\beta_g(f) \in \mathrm{Lip}_0(M)$. We can easily show that β satisfies (A1): for $x \in M$ holds

$$\beta_e(f)(x) = f(e^{-1}x) - f(e^{-1}0) = f(x) - f(0) = f(x).$$

The condition (A2) we can deduce from the fact that it holds for α^*: for $g, h \in G$ and $f \in \mathrm{Lip}_0(M)$ holds

$$\begin{aligned}\varphi(\beta(g, \beta(h, f))) &= \alpha^*(g, \varphi(\beta(h, f))) = \alpha^*(g, \alpha^*(h, \varphi(f))) \\ &= \alpha^*(gh, \varphi(f)) = \varphi(\beta(gh, f)).\end{aligned}$$

This implies $\beta(g, \beta(h, f)) = \beta(gh, f)$ since φ is injective. This way, we also show that β is by linear isometries: for $g \in G$, $f, f' \in \mathrm{Lip}_0(M)$ and $t \in \mathbb{R}$ we have

$$\varphi(\beta_g(f + tf')) = \alpha^*_g(\varphi(f + tf')) = \alpha^*_g(\varphi(f)) + t\alpha^*_g(\varphi(f')) = \varphi(\beta_g(f) + t\beta_g(f'))$$

and

$$\|\beta_g f\|_{\mathrm{Lip}_0(M)} = \|\varphi(\beta_g f)\|_{\mathcal{F}(M)^*} = \left\|\alpha^*_g\varphi(f)\right\|_{\mathcal{F}(M)^*} = \|\varphi(f)\|_{\mathcal{F}(M)^*} = \|f\|_{\mathrm{Lip}_0(M)}.$$

To simplify all of this, we restate Question 2.3 for Lipschitz-free spaces:

Question 2.29. Let $\delta > 0$, $(M, d, 0)$ be a pointed metric space and G be a discrete group acting on M by isometries. Consider the action of G by linear isometries $\alpha : G \times \mathrm{Lip}_0(M) \to \mathrm{Lip}_0(M)$ given by

$$\alpha(g, f)(x) = f(g^{-1}x) - f(g^{-1}0), \quad g \in G, f \in \mathrm{Lip}_0(M), x \in M.$$

Assume $f \in \mathrm{Lip}_0(M)$ with $L(f) = 1$ is $\delta/10$-invariant (with respect to the action α). Must there exist invariant $\overline{f} \in \mathrm{Lip}_0(M)$ with $\left\|f - \overline{f}\right\| \leq \delta$?

From now on, whenever we have a pointed metric space M equipped with an action of some group by isometries, we will always assume $\mathrm{Lip}_0(M)$ to be equipped with the action given above. First step is to characterise functions which are invariant for this action.

Theorem 2.30. *Let $(M, d, 0)$ be a pointed metric space, G be a group acting by isometries on M, $S \subset G$ satisfy $G = \langle S \rangle$ and assume $f \in \mathrm{Lip}_0(M)$. Then the following are equivalent:*

(i) f is G-invariant;

(ii) for every $g \in G$ and $x \in M$ holds $f(gx) = f(x) + \sum_{i=1}^n a_i f(s_i 0)$, where $n \in \mathbb{N}, a \in \{-1, 1\}^n$ and $s \in S^n$ are such that $g = s_1^{a_1} \cdots s_n^{a_n}$;

(iii) there is a homomorphism $H : G \to \mathbb{R}$ such that for any $x \in M$ and $g \in G$ holds $f(gx) = f(x) + H(g)$.

Proof. First, recall that by Fact 2.22, f is G-invariant if and only if $f(h\cdot) - f(\cdot)$ is a constant function.

$(i) \implies (ii)$: Taking values of the constant function $f(h\cdot) - f(\cdot)$ at $z \in M$ and 0 yields

$$\forall h \in G \colon f(hz) - f(z) = f(h0) - f(0) = f(h0).$$

Taking $z = y$ or $z = h^{-1}y$ results in

$$\forall y \in M \ \forall h \in G \colon f(hy) = f(y) + f(h0), \tag{2.5}$$
$$\forall y \in M \ \forall h \in G \colon f(h^{-1}y) = f(y) - f(h0). \tag{2.6}$$

Let $x \in M$ and $g \in G$. Write $g = s_1^{a_1} \cdots s_n^{a_n}$ for some $n \in \mathbb{N}$, $s \in S^n$ and $a \in \{-1, 1\}^n$. Applying either (2.5) if $a_1 = 1$, or (2.6) if $a_1 = -1$, to $h = s_1$ and $y = s_2^{a_2} \cdots s_n^{a_n} x$ we obtain

$$f(s_1^{a_1} \cdots s_n^{a_n} x) = f(s_2^{a_2} \cdots s_n^{a_n} x) + a_1 f(s_1 0).$$

Proceeding inductively, we arrive at the equality

$$f(gx) = f(s_1^{a_1} \cdots s_n^{a_n} x) = f(x) + \sum_{i=1}^n a_i f(s_i 0).$$

$(ii) \implies (iii)$: In light of (ii), we need to verify that the mapping $H : G \to \mathbb{R}$ defined by $H(g) = \sum_{i=1}^n a_i f(s_i 0)$, where $g = s_1^{a_1} \cdots s_n^{a_n}$, $s \in S^n$ and $a \in \{-1, 1\}^n$, is a well-defined homomorphism. It is well defined for if $s_1^{a_1} \cdots s_n^{a_n} =$

$g = t_1^{b_1} \cdots t_m^{b_m}$, where $n, m \in \mathbb{N}$, $s \in S^n$, $t \in S^m$, $a \in \{-1, 1\}^n$ and $b \in \{-1, 1\}^m$, then by (ii) holds

$$\sum_{i=1}^{n} a_i f(s_i 0) = f(gx) - f(x) = \sum_{i=1}^{m} b_i f(t_i 0).$$

Now, if $g, h \in G$, then

$$H(gh) = f(gh0) - f(0) = f(gh0) - f(h0) + f(h0) - f(0) = H(g) + H(h),$$

where in the last equality we used (ii) for $x = h0$ and $x = 0$.

$(iii) \implies (i)$: Let $x \in M$ and $g \in G$. Then

$$gf(x) = f(g^{-1}x) - f(g^{-1}0) = f(x) + H(g^{-1}) - f(0) - H(g^{-1}) = f(x).$$

□

The case we will consider primarily is the case when a group G acts on itself. To do this, we must equip G with some metric. We use the one given in

Definition 2.31. Let G be a group and $S \subset G \backslash \{e\}$ be generating (i.e. $\langle S \rangle = G$). The word metric on G (induced by S) is defined by

$$d(g, h) = \min \left\{ n \in \mathbb{N}_0 \colon (\exists s \in (S^{\pm 1})^n) \ g^{-1}h = s_1 \cdots s_n \right\}, \quad g, h \in G.$$

We interpret empty product as the unit.

When equipping a group with a metric, we will implicitly consider the unit to be the base point.

Definition 2.32. Let G be a group equipped with a metric d. We say that d is *left-invariant* if

$$\forall g, h, k \in G \colon d(g, h) = d(kg, kh).$$

We say that d is *right-invariant* if

$$\forall g, h, k \in G \colon d(g, h) = d(gk, hk).$$

Finally, we say that d is *invariant* if it is both left- and right-invariant.

Fact 2.33. *The word metric is a left-invariant metric.*

Proof. Let $g, h \in G$. If $g = h$, then by definition $d(g, h) = 0$. On the other hand, if $d(g, h) = 0$, then necessarily $g^{-1}h = e$ and thus $g = h$. If $g^{-1}h = s_1 \cdots s_n$, where $n = d(g, h)$ and $s \in (S^{\pm 1})^n$, then $h^{-1}g = s_n^{-1} \cdots s_1^{-1}$ and hence $d(h, g) \leq n = d(h, g)$. Switching the role of g and h, we have $d(g, h) \leq d(h, g)$ and thus $d(g, h) = d(h, g)$ for any $g, h \in G$. We have checked that (M1) and (M2) hold.

To verify (M3), let $g, h, k \in G$ and let $g^{-1}h = s_1 \cdots s_n$ and $h^{-1}k = t_1 \cdots t_m$, where $n = d(g, h)$, $m = d(h, k)$ and $s \in (S^{\pm 1})^n, t \in (S^{\pm 1})^m$. Then $g^{-1}k = g^{-1}hh^{-1}k = s_1 \cdots s_n t_1 \cdots t_m$ and hence $d(g, k) \leq n + m = d(g, h) + g(h, k)$.

Finally, if $g, h, k \in G$ and $g^{-1}h = s_1 \cdots s_n$, for some $n \in \mathbb{N}_0$ and $s \in (S^{\pm 1})^n$, then $(kg)^{-1}(kh) = g^{-1}k^{-1}kh = g^{-1}h = s_1 \cdots s_n$ and hence $d(kg, kh) \leq d(g, h)$. If $g', h', k' \in G$, then applying the previous to $k = (k')^{-1}, g = k'g'$ and $h = k'h'$ we get $d(g', h') = d((k')^{-1}k'g', (k')^{-1}k'h') \leq d(k'g', k'h')$. □

Definition 2.34. Let G be a group endowed with a metric d. We define the action by *left-translations* of G on itself as the action $(g, h) \mapsto gh : G \times G \to G$.

Notice that a direct consequence of the definitions is

Fact 2.35. *Let (G, d) be a group equipped with a metric and the action by left-translations on itself. Then the action is by isometries if and only if d is left-invariant.*

Proof. The metric is left invariant if and only if for all $g, h, k \in G$ holds $d(g, h) = d(kg, kh)$ and the action is by isometries if and only if for all $k \in G$ the mapping $g \mapsto kg$ is an isometry. □

Corollary 2.36. *Let G be a group endowed with a left-invariant metric. Then $f \in \mathrm{Lip}_0(G)$ is G-invariant with respect to left-translations if and only if $f : G \to \mathbb{R}$ is a homomorphism.*

Proof. By Theorem 2.30, f is invariant if and only if there is a homomorphism $H : G \to \mathbb{R}$ such that $f(gx) = f(x) + H(g)$ for every $x, g \in G$.

If f is a homomorphism, then for every $g, x \in G$ holds $f(gx) = f(x) + f(g)$, so we may take $H = f$.

On the other hand, if there is a homomorphism $H : G \to \mathbb{R}$ such that $f(gx) = f(x) + H(g)$ for every $x, g \in G$, then for any $g \in G$ holds $f(g) = f(ge) = f(e) + H(g) = H(g)$, so $f = H$ and thus f is a homomorphism. □

Next, we shall characterise δ-invariant functions.

Theorem 2.37. *Let $(M, d, 0)$ be a pointed metric space and G be a group acting on M by isometries. Let $\delta > 0$ and $f \in \mathrm{Lip}_0(M)$. Then the following conditions are equivalent:*

(i) f is δ-invariant;

(ii) $f - gf$ is δ-Lipschitz for every $g \in G$;

(iii) the mapping $x \mapsto f(gx) - f(x)$ is δ-Lipschitz for every $g \in G$;

(iv) for any $x, y \in M$ and $g_1, \ldots g_n \in G$ holds

$$\left| f(g_1 \cdots g_n x) - f(x) - \sum_{i=1}^{n} (f(g_i y) - f(y)) \right| \leq \delta \left(nd(x, y) + \sum_{i=2}^{n} d(g_i x, x) \right).$$

Proof. $(i) \iff (ii)$: Function $f \in \mathrm{Lip}_0(M)$ is δ-invariant if and only if $\|f - gf\| \leq \delta$ for all $g \in G$. Since the norm on $\mathrm{Lip}_0(M)$ we use is the Lipschitz number, the equivalence follows.

$(ii) \iff (iii)$: This equivalence follows from the equality

$$\begin{aligned} &\left|(g^{-1}f - f)(x) - (g^{-1}f - f)(y)\right| \\ &= \left|f(gx) - f(g0) - f(x) - \big(f(gy) - f(g0) - f(y)\big)\right| \\ &= \left|\big(f(gx) - f(x)\big) - \big(f(gy) - f(y)\big)\right| \end{aligned}$$

which is valid for any $x, y \in M$ and $g \in G$.

$(iii) \iff (iv)$: Notice that for $n = 1$, (iv) reduces to (iii). So $(iv) \implies (iii)$ and (iii) directly implies (iv) for $n = 1$. If $n > 1$, we expand $f(g_1 \cdots g_n x) - f(x)$ into the sum

$$f(g_1 \cdots g_n x) - f(x) = \sum_{i=1}^{n} f(g_1 \cdots g_i x) - f(g_1 \cdots g_{i-1} x),$$

where by $f(g_1 \cdots g_{i-1} x)$ for $i = 1$ we understand as just $f(x)$. Now we may write

$$\begin{aligned}
&\left| f(g_1 \cdots g_n x) - f(x) - \sum_{i=1}^{n} (f(g_i x) - f(x)) \right| \\
&\quad = \left| \sum_{i=1}^{n} \Big(f(g_1 \cdots g_i x) - f(g_1 \cdots g_{i-1} x) \Big) - \Big(f(g_i x) - f(x) \Big) \right| \\
&\quad \leq \sum_{i=1}^{n} \left| \Big(f(g_1 \cdots g_i x) - f(g_1 \cdots g_{i-1} x) \Big) - \Big(f(g_i x) - f(x) \Big) \right| \\
&\quad = \sum_{i=1}^{n} \left| \Big(f(g_1 \cdots g_{i-1} g_i x) - f(g_i x) \Big) - \Big(f(g_1 \cdots g_{i-1} x) - f(x) \Big) \right| \\
&\quad = \sum_{i=2}^{n} \left| \Big(f(g_1 \cdots g_{i-1} g_i x) - f(g_i x) \Big) - \Big(f(g_1 \cdots g_{i-1} x) - f(x) \Big) \right| \\
&\qquad\quad + |f(g_1 x) - f(g_1 x) - (f(x) - f(x))| \\
&\quad = \sum_{i=2}^{n} \left| \Big(f(g_1 \cdots g_{i-1} g_i x) - f(g_i x) \Big) - \Big(f(g_1 \cdots g_{i-1} x) - f(x) \Big) \right|.
\end{aligned}$$

For each summand, we use (iii) for $g' = g_1 \cdots g_{i-1}$, $x' = g_i x$ and $y' = x$ to obtain

$$\left| f(g_1 \cdots g_n x) - f(x) - \sum_{i=1}^{n} \Big(f(g_i x) - f(x) \Big) \right| \leq \sum_{i=2}^{n} \delta d(g_i x, x).$$

Finally,

$$\begin{aligned}
&\left| f(g_1 \cdots g_n x) - f(x) - \sum_{i=1}^{n} \Big(f(g_i y) - f(y) \Big) \right| \\
&= \left| f(g_1 \cdots g_n x) - f(x) - \sum_{i=1}^{n} \Big(f(g_i x) - f(x) - f(g_i x) + f(x) + f(g_i y) - f(y) \Big) \right| \\
&\leq \sum_{i=2}^{n} \delta d(g_i x, x) + \sum_{i=1}^{n} \left| \Big(f(g_i x) - f(x) \Big) - \Big(f(g_i y) - f(y) \Big) \right| \\
&\leq \delta \left(\sum_{i=2}^{n} d(g_i x, x) + n d(x, y) \right),
\end{aligned}$$

where in the last inequality we have again used (iii) for each summand. □

Before we proceed, let us make a remark on the relation of δ-invariance and (partial) quasimorphisms. For more details and uses for quasimorphisms see, e.g., [6].

Definition 2.38. Let G be a group and $f : G \to \mathbb{R}$. We say that f is a *quasi-morphism* if there is a constant $D \geq 0$ such that

$$\forall g, h \in G\colon \; |f(gh) - f(g) - f(h)| \leq D.$$

If d is an invariant metric on G, then the mapping f is said to be a *partial quasimorphism* (or D-partial quasimorphism) if there is a constant $D \geq 0$ such that

$$\forall g, h \in G\colon\ |f(gh) - f(g) - f(h)| \leq D \min\{d(g,e), d(h,e)\}.$$

Proposition 2.39. *Let (G, d) be a group equipped with an invariant metric d. Let $\delta > 0$ and $f \in \mathrm{Lip}_0(G)$. Consider the statements*

(i) f is $\frac{\delta}{2}$-partial quasimorphism;

(ii) the mappings $h \mapsto f(gh) - f(h)$ and $h \mapsto f(hg) - f(h)$ are δ-Lipschitz;

(iii) f is δ-partial quasimorphism.

Then the implications (i) $\implies$ (ii) $\implies$ (iii) hold.

Proof. (i) $\implies$ (ii): Assume that f is $\frac{\delta}{2}$-partial quasimorphism. Pick $g, h, k \in G$. We estimate

$$\begin{aligned}
&\Big|f(gh) - f(h) - \big(f(gk) - f(k)\big)\Big| \\
&\qquad = \Big|f(gh) - f(gk) - \big(f(h) - f(k)\big)\Big| \\
&\qquad = \Big|f(gh) - f(gk) - f(k^{-1}h) - \big(f(h) - f(k) - f(k^{-1}h)\big)\Big| \\
&\qquad \leq \frac{\delta}{2}\min\{d(gk,e), d(k^{-1}h,e)\} + \frac{\delta}{2}\min\{d(k,e), d(k^{-1}h,e)\} \\
&\qquad \leq \delta d(k^{-1}h, e) \\
&\qquad = \delta d(h,k)
\end{aligned}$$

and

$$\begin{aligned}
&\Big|f(hg) - f(h) - \big(f(kg) - f(k)\big)\Big| \\
&\qquad = \Big|f(hg) - f(kg) - \big(f(h) - f(k)\big)\Big| \\
&\qquad = \Big|f(hg) - f(hk^{-1}) - f(kg) - \big(f(h) - f(hk^{-1}) - f(k)\big)\Big| \\
&\qquad \leq \frac{\delta}{2}\min\{d(hk^{-1},e), d(kg,e)\} + \frac{\delta}{2}\min\{d(hk^{-1},e), d(k,e)\} \\
&\qquad \leq \delta d(hk^{-1}, e) \\
&\qquad = \delta d(h,k).
\end{aligned}$$

(ii) $\implies$ (iii): Using the fact that $f(e) = 0$ and the assumption (ii), we obtain for any $g, h \in G$

$$|f(gh) - f(g) - f(h)| = \Big|f(gh) - f(h) - \big(f(ge) - f(e)\big)\Big| \leq \delta d(h,e)$$

and

$$|f(gh) - f(g) - f(h)| = \Big|f(gh) - f(g) - \big(f(eh) - f(e)\big)\Big| \leq \delta d(g,e).$$

□

Notice that by Theorem 2.37 the first part of condition (*ii*) is equivalent to f being δ-invariant for the action of left-translations. The second part would be equivalent to f being δ-invariant for the (right) action by right-translations which we have not defined.

Every $f \in \mathrm{Lip}_0(M)$ is automatically $2\|f\|$-invariant for any action by isometries: for $x, y \in M$ with $x \neq y$ and $g \in G$ holds

$$\begin{aligned}\frac{|(f(x) - f(gx)) - (f(y) - f(gy))|}{d(x,y)} &\leq \frac{|f(x) - f(y)|}{d(x,y)} + \frac{|f(gx) - f(gy)|}{d(x,y)} \\ &= \frac{|f(x) - f(y)|}{d(x,y)} + \frac{|f(gx) - f(gy)|}{d(gx,gy)} \\ &\leq 2\|f\|.\end{aligned}$$

The computation above shows that, for a Lipschitz mapping f attaining zero at the base point on a group equipped with an invariant metric, the mapping $h \mapsto f(gh) - f(h)$ is $2\|f\|$-Lipschitz. Using the same argument, mutatis mutandis, one may also show that the second part of the condition (ii) holds for $\delta = 2\|f\|$, that is, $h \mapsto f(hg) - f(h)$ is $2\|f\|$-Lipschitz. Hence, on a group G equipped with an invariant metric, functions from $\mathrm{Lip}_0(G)$ are automatically partial quasimorphisms with a sufficiently large constant. The relation between Lipschitzness and partial quasimorphisms is described in detail in [13].

As one might expect, δ-invariant functions need not be quasimorphisms.

Example 2.40. Define a sequence of integers (a_n) recursively by setting $a_1 = 1$ and $a_{n+1} = 2a_n + 1$. Define $f : \mathbb{Z} \to \mathbb{R}$ by putting $f(k) = 0$ for $k \leq 0$ and $f(k) = n\delta$ for $a_n \leq k \leq 2a_n$. Then f is δ-invariant (with respect to the action by left-translations), but is not a quasimorphism.

Proof. If $k = a_n$ for some $n \in \mathbb{N}$, then $f(k) - f(k-1) = \delta$, otherwise $f(k) - f(k-1) = 0$. Since f is non-decreasing, for any $m, n \in \mathbb{Z}$ (and without loss of generality $m \leq n$) we have

$$0 \leq f(n) - f(m) = \sum_{i=m+1}^{n} f(i) - f(i-1) \leq (n-m)\delta.$$

It follows that f is δ-Lipschitz. Clearly $f \in \mathrm{Lip}_0(\mathbb{Z})$. Note that the action by (any) translations on $\mathbb{Z}$ is merely addition. Let $g, x, y \in \mathbb{Z}$. If $y \leq x$, we have

$$f(x) - f(y) \in [0, (x-y)\delta] \quad \text{and} \quad f(x+g) - f(y+g) \in [0, (x-y)\delta].$$

Since both values are within the same interval of length $(x-y)\delta$, it must hold that

$$\Big| f(x) - f(x+g) - \Big(f(y) - f(g+y)\Big)\Big| \leq |x-y|\,\delta.$$

If $y \leq x$, we proceed analogously. Combined with Theorem 2.37 (iii), this shows that f is δ-invariant. But f is not a quasimorphism: for $n \in \mathbb{N}$ we get

$$|f(2a_n) - f(a_n) - f(a_n)| = |n\delta - n\delta - n\delta| = n\delta \overset{n\to\infty}{\to} \infty.$$

□

Now we can finally get to Question 2.29. We will not answer the question in general, but we shall give positive results for some special cases. First, we look at actions by translations on free groups endowed with the word metric.

Lemma 2.41. *Let G be a group equipped with a left-invariant metric d and the action on itself by left-translations. If $h : G \to \mathbb{R}$ is a homomorphism, then*

$$L(h) = \sup_{e \neq g \in G} \frac{|h(g)|}{d(g,e)}.$$

Proof. By definition of the Lipschitz number, $\sup_{e \neq g \in G} \frac{|h(g)|}{d(g,e)} \leq L(h)$. To show the converse inequality, let $g, g' \in G$, $g \neq g'$ and calculate

$$\frac{|h(g) - h(g')|}{d(g,g')} = \frac{|h(g^{-1}g')|}{d(g,g')} = \frac{|h(g^{-1}g')|}{d(g^{-1}g',e)} \leq \sup_{e \neq k \in G} \frac{|h(k)|}{d(k,e)}.$$

Taking the supremum over all $g \neq g'$ yields the inequality. □

Lemma 2.42. *Let G be a group equipped with a left-invariant metric d and the action by left-translations on itself. If $\delta > 0$, $a \in \{-1, 1\}$ and $f \in \mathrm{Lip}_0(G)$ is δ-invariant, then for any $g \in G$ holds $|f(g^a) - af(g)| \leq \delta d(g,e)$.*

Proof. From the definition of the action by left-translations we obtain

$$\forall g \in G\colon\ f(g^{-1}) = -(f(e) - f(g^{-1}e)) = -gf(g)$$

and using Theorem 2.37 (ii)

$$\forall g \in G\colon\ \left|f(g^{-1}) + f(g)\right| = |gf(g) - f(g)| \leq \delta d(g,e) + |gf(e) - f(e)| = \delta d(g,e)$$

where we used that $f, gf \in \mathrm{Lip}_0(G)$ and so $gf(e) = f(e) = 0$. If $a = 1$ then $f(g^a) - af(g) = f(g) - f(g) = 0$ and if $a = -1$, then

$$|f(g^a) - af(g)| = \left|f(g^{-1}) - (-f(g))\right| = \left|f(g^{-1}) + f(g)\right| \leq \delta d(g,e)$$

Hence $|f(g^a) - af(g)| \leq \delta$. □

Theorem 2.43. *Let $\delta > 0$ and F_S be a free group with generating set S equipped with the word metric and action by left-translations. Let $f \in \mathrm{Lip}_0(F_S)$ be $\delta/3$-invariant. Then the mapping $\overline{f} : F_S \to \mathbb{R}$ defined by*

$$\overline{f}(g) = \sum_{i=1}^{n} a_i f(s_i),$$

where $n \in \mathbb{N}, a \in \{-1, 1\}^n$ and $s \in S^n$ are such that $g = s_1^{a_1} \cdots s_n^{a_n}$ is the reduced word representing g, is an invariant element of $\mathrm{Lip}_0(F_S)$ with $\left\|f - \overline{f}\right\| \leq \delta$.

Proof. The mapping $\overline{f}$ is a homomorphism by Proposition 2.11. If we show $\overline{f} \in \mathrm{Lip}_0(F_S)$, then by Corollary 2.36, it will be invariant. By definition (and the convention that empty sum is equal to zero), $\overline{f}(e) = 0$. Let $g = s_1^{a_1} \cdots s_n^{a_n}$ be an element of F_S and its representation as a reduced word. Then[7] $d(g,e) = n$ and, using the fact that $d(s,e) = 1$ for every $s \in S$, we estimate

$$\frac{\left|\overline{f}(g)\right|}{d(g,e)} = \frac{\left|\sum_{i=1}^n a_i f(s_i)\right|}{n} \leq \frac{1}{n}\sum_{i=1}^n |f(s_i)| = \frac{1}{n}\sum_{i=1}^n |f(s_i) - f(e)|$$
$$\leq \frac{1}{n}\sum_{i=1}^n \|f\|\, d(s_i,e) = \|f\|\,.$$

Use of Lemma 2.41 shows that $\overline{f}$ is $\|f\|$-Lipschitz.

It remains to show that $\left\|f - \overline{f}\right\| < \delta$. Let $g, h \in G$. Then $-\overline{f}(g) + \overline{f}(h) = \overline{f}(g^{-1}h)$ and using Theorem 2.37 (iii) we deduce

$$\left|\Big(f(g) - f(h)\Big) - \Big(f(e) - f(g^{-1}h)\Big)\right| = \left|\Big(f(g) - f(g^{-1}g)\Big) - \Big(f(h) - f(g^{-1}h)\Big)\right|$$
$$\leq \frac{\delta}{3}d(g,h).$$

Put together,

$$\left|(f - \overline{f})(g) - (f - \overline{f})(h)\right|$$
$$\leq \left|(f - \overline{f})(e) - (f - \overline{f})(g^{-1}h)\right| + \left|\Big(f(e) - f(g^{-1}h)\Big) - \Big(f(g) - f(h)\Big)\right|$$
$$\leq \left|(f - \overline{f})(e) - (f - \overline{f})(g^{-1}h)\right| + \frac{\delta}{3}d(g,h)$$
$$= \left|(f - \overline{f})(g^{-1}h)\right| + \frac{\delta}{3}d(g,h)$$

and hence it is enough to obtain the estimate $\left|(f - \overline{f})(g)\right| \leq \frac{2}{3}d(g,e)$ for any $g \in G$.

Let $g \in F_S$ and $g = s_1^{a_1} \cdots s_n^{a_n}$ be the reduced word representing g. Then we have

$$\begin{aligned}\left|(f - \overline{f})(g)\right| &= \left|f(s_1^{a_i} \cdots s_n^{a_n}) - \sum_{i=1}^n a_i f(s_i)\right| \\ &\leq \left|f(s_1^{a_i} \cdots s_n^{a_n}) - \sum_{i=1}^n f(s_i^{a_i})\right| + \left|\sum_{i=1}^n f(s_i^{a_i}) - a_i f(s_i)\right|\end{aligned} \tag{2.7}$$

To estimate the first term, we use Theorem 2.37 (iv) with $g_i = s_i^{a_i}$ and $x = y = e$:

$$\left|f(s_1^{a_i} \cdots s_n^{a_n}) - \sum_{i=1}^n f(s_i^{a_i})\right| = \left|f(s_1^{a_i} \cdots s_n^{a_n}) - f(e) - \sum_{i=1}^n \Big(f(s_i^{a_i}) - f(e)\Big)\right|$$
$$\leq \frac{\delta}{3}\sum_{i=2}^n d(s_i^{a_i}, e) \leq \frac{\delta}{3}n = \frac{\delta}{3}d(g,e).$$

We have used the fact that for $s \in S^{\pm 1}$ we have by definition $d(s,e) = 1$. Using Lemma 2.42 for each pair s_i and a_i we obtain that $|f(s_i^{a_i}) - a_i f(s_i)| \leq \delta/3$ for $i \in [1..n]$ and hence the second term in (2.7) is estimated by $n\delta/3 = d(g,e)\delta/3$. Altogether, we have shown that $\left|(f - \overline{f})(g)\right| \leq d(g,e)2\delta/3$ which finishes the proof. □

[7]This follows from the fact that $d(g,e) = d(g^{-1},e)$ which is the minimal length of a word representing g. Of these, the (unique) reduced word is the shortest since all others contain some extra pairs of elements which can be cancelled out and thus the length of the word shortened.

Later on we will need finer control over the exact values of the invariant function by which we approximate. To enable this, we have

Corollary 2.44. *Let $\delta > 0$ and F_S be a free group with generating set S. Equip F_S with the word metric and the action on itself by left-translations. Let $f \in \mathrm{Lip}_0(F_S)$ be $\delta/3$-invariant. Let $u \in \ell_\infty(S)$ and $\eta \geq 0$ satisfy $|f(s) - u(s)| \leq \eta, s \in S$. Define $\overline{f} : F_S \to \mathbb{R}$ by*

$$\overline{f}(s_1^{a_1} \dots s_n^{a_n}) = \sum_{i=1}^{n} a_i u(s_i), \quad n \in \mathbb{N}, s \in S^n, a \in \{-1, 1\}^n.$$

Then $\overline{f} \in \mathrm{Inv}_{F_S}(F_S)$ and $\left\| f - \overline{f} \right\| \leq \delta + \eta$.

Proof. The mapping $\overline{f}$ is a homomorphism by Proposition 2.11. Denote $\widetilde{f}(g) = \sum_{i=1}^{n} a_i f(s_i)$ the function from Theorem 2.43. For $g \in G$ and its representation as a reduced word $g = s_1^{a_1} \cdots s_n^{a_n}$ we calculate

$$\frac{\left| \widetilde{f}(g) - \overline{f}(g) \right|}{d(g, e)} \leq \frac{1}{n} \sum_{i=1}^{n} |a_i(u(s_i) - f(s_i))| \leq \frac{1}{n} \sum_{i=1}^{n} \eta = \eta.$$

From Lemma 2.41 we obtain that $\left\| \widetilde{f} - \overline{f} \right\| \leq \eta$. This means that $\overline{f} \in \mathrm{Lip}_0(F_S)$ and, as it is a homomorphism, by Corollary 2.36 we have $\overline{f} \in \mathrm{Inv}_{F_S}(F_S)$. Moreover,

$$\left\| f - \overline{f} \right\| \leq \left\| f - \widetilde{f} \right\| + \left\| \widetilde{f} - \overline{f} \right\| \leq \delta + \eta.$$

□

As we have seen in Proposition 2.15, every group can be written as a quotient of a free group. Since we already have a result for free groups, we would like to transfer it to the quotients.

Proposition 2.45. *Let S be any set, F_S a free group over S equipped with the word metric, $N \trianglelefteq F_S$ and $\delta > 0$. Denote $G = F_S/N$, $q : F_S \to G$ the quotient map and equip G with the word metric induced by the set $q(S) = \{q(s) \colon s \in S\}$. Equip both F_S and G with actions by left-translations. Let $f \in \mathrm{Lip}_0(G)$ be δ-invariant. Denote $F : F_S \to \mathbb{R}$, $F = f \circ q$. Then $F \in \mathrm{Lip}_0(F_S)$, F is δ-invariant and for $\eta > 0$ the following conditions are equivalent*

(i) there exists $\overline{f} \in \mathrm{Inv}_G(G)$ with $\left\| f - \overline{f} \right\| < \eta$;

(ii) there exists $\overline{F} \in \mathrm{Inv}_{F_S}(F_S)$ with $\left\| F - \overline{F} \right\| < \eta$ satisfying $N \subset \ker \overline{F}$.

Proof. We will denote both metrics as d as it will always be obvious which metric we refer to. First, note that $d(q(g), q(h)) \leq d(g, h)$ for any $g, h \in F_S$: if $d(g, h) = n$ and $g^{-1}h = s_1 \cdots s_n$ for some $s \in (S^{\pm 1})^n$, then $q(g^{-1}h) = q(s_1) \cdots q(s_n)$ and hence $d(q(g), q(h)) \leq n$. Clearly $F(e) = f(q(e)) = 0$ and for any $g, h \in G$ holds

$$|F(g) - F(h)| = |f(q(g)) - f(q(h))| \leq \|f\| \, d(q(g), q(h)) \leq \|f\| \, d(g, h).$$

So, $F \in \mathrm{Lip}_0(F_S)$. Using Theorem 2.37, we show that F is δ-invariant: for any $g, h, k \in G$ we have

$$\begin{aligned}&\left|F(gh) - F(h) - \big(F(gk) - F(k)\big)\right| \\ &\qquad = \left|f(q(g)q(h)) - f(q(h)) - \big(f(q(g)q(k)) - f(q(k))\big)\right| \\ &\qquad \leq \delta d(q(h), q(k)) \leq \delta d(h, k).\end{aligned}$$

$(i) \implies (ii)$: Put $\overline{F} = \overline{f} \circ q$. Then for $g \in N$ holds $\overline{F}(g) = \overline{f}(q(g)) = \overline{f}(e_G) = 0$. This shows $N \subset \ker \overline{F}$ and in particular $\overline{F}(e) = 0$. As before, for any $g, h \in F_S$ we get

$$\begin{aligned}\left|(F - \overline{F})(g) - (F - \overline{F})(h)\right| &= \left|(f - \overline{f})(q(g)) - (f - \overline{f})(q(h))\right| \\ &< \eta d(q(g), q(h)) \leq \eta d(g, h),\end{aligned}$$

i.e. $\left\|F - \overline{F}\right\| < \eta$ and hence also $\overline{F} \in \mathrm{Lip}_0(F_S)$. For $g, h \in G$ holds $\overline{F}(gh) = \overline{f}(q(gh)) = \overline{f}(q(g)q(h))$. Since $\overline{f}$ is invariant, by Corollary 2.36 it is a homomorphism and so $\overline{F}(gh) = \overline{f}(q(g)) + \overline{f}(q(h)) = \overline{F}(g) + \overline{F}(h)$. Corollary 2.36 now implies that $\overline{F}$ is invariant, because it is a homomorphism.

$(ii) \implies (i)$: $\overline{F}$ is invariant and hence a homomorphism by Corollary 2.36. Since $\ker q = N \subseteq \ker \overline{F}$, $\overline{F}$ factors through q to a homomorphism $\overline{f} : G \to \mathbb{R}$ with $\overline{F} = \overline{f} \circ q$. To show $\overline{f} \in \mathrm{Lip}_0(G)$, choose $g, h \in F_S$ and $s_1, \ldots, s_n \in S^{\pm 1}$ such that $q(g^{-1}h) = q(s_1) \cdots q(s_n)$ and $d(q(g), q(h)) = n$. Then

$$\begin{aligned}\left|\overline{f}(q(g)) - \overline{f}(q(h))\right| = \left|\overline{f}(q(g^{-1}h))\right| &= \left|\sum_{i=1}^{n} \overline{f}(q(s_i))\right| = \left|\sum_{i=1}^{n} \overline{F}(s_i)\right| \\ &\leq n \left\|\overline{F}\right\| = \left\|\overline{F}\right\| d(q(g), q(h)).\end{aligned}$$

So we have $\overline{f} \in \mathrm{Lip}_0(G)$ and since it is a homomorphism, by Corollary 2.36 we also have $\overline{f} \in \mathrm{Inv}_G(G)$. It remains to show that $\left\|f - \overline{f}\right\| < \eta$. Choose $q(g), q(h) \in G$ and $s_1, \ldots, s_n \in S^{\pm 1}$ such that $q(g^{-1}h) = q(s_1) \cdots q(s_n)$ and $d(q(g), q(h)) = n$. This implies that the word $s_1 \cdots s_n$ must be reduced. We may assume $h = gg^{-1}h = gs_1 \cdots s_n$ and hence by the left-invariance of the word metric holds

$$d(g, h) = d(g, gs_1 \cdots s_n) = d(e, s_1 \cdots s_n) = n = d(q(g), q(h)).$$

We conclude by estimating for $g, h \in F_S$:

$$\begin{aligned}\left|(f - \overline{f})(q(g)) - (f - \overline{f})(q(h))\right| &= \left|f(q(g)) - \overline{f}(q(g)) - f(q(h)) + \overline{f}(q(h))\right| \\ &= \left|F(g) - \overline{F}(g) - F(h) + \overline{F}(h)\right| \\ &\leq \left\|F - \overline{F}\right\| d(g, h) < \eta d(q(g), q(h)).\end{aligned}$$

□

We have distilled the problem down to the question of whether there exists a homomorphism on a free group which is close to our δ-invariant function on the generating set while being zero on the corresponding normal subgroup. We will demonstrate how this can be done, for the price of worsening the constant, for finitely presented groups.

Lemma 2.46. *Let X, Y be Banach spaces and $A : X \to Y$ be a finite-dimensional linear operator. Then there is a constant $C > 0$ such that for any $x \in X$ there exists $u \in \ker A$ with $\|x - u\| \leq C \|Ax\|$.*

Proof. Put $\widetilde{X} = X/\ker A$ and $\widetilde{Y} = \operatorname{Im} A$. Denote $q : X \to \widetilde{X}$ the quotient map and $B : \widetilde{X} \to \widetilde{Y}$ the unique linear operator satisfying $A = Bq$. The operator B is bijective, so its inverse exists. Since the domain of B^{-1} is the finite-dimensional space $\widetilde{Y}$, it is continuous.

Fix $x \in X$. Then $B^{-1}Ax = q(x)$ and we can estimate

$$\|q(x)\| = \left\|B^{-1}Ax\right\| \leq \left\|B^{-1}\right\| \|Ax\| .$$

By the definition of the quotient norm, there is $u \in \ker A$ such that $\|x - u\| \leq \|q(x)\| + \|Ax\|$. Putting these together we obtain

$$\|x - u\| \leq \|q(x)\| + \|Ax\| \leq (\left\|B^{-1}\right\| + 1) \|Ax\| .$$

We have shown the desired inequality with $C = \|B^{-1}\| + 1$. □

While the following theorem does not reflect Question 2.29 exactly, it states that if we allow the constant to depend on the group, the answer is positive for finitely-presented groups equipped with word metrics acting on themselves by translations.

Theorem 2.47. *Let G be a finitely presented group equipped with the word metric and action by left-translations. Then there exists a constant $C > 0$ depending on G such that for any $\delta > 0$ and $f \in \operatorname{Lip}_0(G)$ δ-invariant there is $\overline{f} \in \operatorname{Inv}_G(G)$ with $\left\|f - \overline{f}\right\| \leq C\delta$.*

Proof. Let $f \in \operatorname{Lip}_0(G)$ be δ-invariant. Let S be the generators of G and R be the relators of G. Denote $n = |S|$.

We start by defining a mapping $\# : F_S \times S \to \mathbb{R}^S$. Fix $g = s_1^{a_1} \cdots s_m^{a_m}$, where $m \in \mathbb{N}$ and $s \in S^m$, $a \in \mathbb{Z}^m$, an element of F_S and its representation as a reduced word. For $s \in S$ put $I_{g,s} = \{j \in [1..m]\colon s_j = s\}$ and define $\#(g, s) = \sum_{j \in I_{g,s}} a_j$. This mapping is well-defined, because the representation as a reduced word is unique.

By Proposition 2.45, there is a δ-invariant function $F \in \operatorname{Lip}_0(F_S)$ satisfying $F = f \circ q$. Let $r \in R$ and $r = s_1^{a_1} \cdots s_k^{a_k}$, where $d(r, e) = k$, $s \in S^k$ and $a \in \{-1, 1\}^k$. Notice that $\sum_{i=1}^k a_i F(s_i) = \sum_{s \in S} \#(r, s)F(s)$. Again employing Theorem 2.37 (iv) in conjunction with Lemma 2.42 and the fact that $F(r) = F(e) = 0$, we obtain the inequality

$$\begin{aligned} \left|\sum_{s \in S} \#(r, s)F(s)\right| &= \left|F(r) - \sum_{s \in S} \#(r, s)F(s)\right| \\ &= \left|F(r) - F(e) - \sum_{i=1}^{k}(a_i F(s_i) - F(e))\right| \\ &\leq \left|F(r) - F(e) - \sum_{i=1}^{k}(F(s_i^{a_i}) - F(e))\right| + n\delta \\ &\leq 2\delta n = 2\delta d(r, e) \leq C\delta, \end{aligned} \tag{2.8}$$

where $C = 2\max_{r\in R} d(r,e)$. Define $A : \ell_\infty(S) \to \ell_\infty(R)$ as the (unique) linear operator satisfying $A(e_s)(r) = \#(r,s)$, $s \in S, r \in R$ and $x = (F(s))_{s\in S}$. From (2.8) follows that $\|Ax\|_\infty \leq C\delta$. By Lemma 2.46, there is a constant $D > 0$ depending only on A (and hence only on G) and $u \in \ell_\infty(S)$ such that $Au = 0$ and $\|x-u\|_\infty \leq D\|Ax\|_\infty \leq CD\delta$.

Corollary 2.44 applied to the function F guarantees the existence of $\overline{F} \in \mathrm{Inv}_{F_S}(F_S)$ with $\overline{F}(s) = u(s)$, $s \in S$ and $\left\|F - \overline{F}\right\| \leq 3\delta + \|x-u\|_\infty \leq (3+CD)\delta$. Thanks to Proposition 2.45, it is enough to verify that $\overline{F}|_N \equiv 0$ where $N = \mathrm{ncl}(R)$. Let $g \in F_S$ and $r \in R$. $\overline{F}$ is invariant, thus a homomorphism by Corollary 2.36 and hence

$$\overline{F}(grg^{-1}) = \overline{F}(g) + \overline{F}(r) + \overline{F}(g^{-1}) = \overline{F}(r).$$

Finally, we have

$$\overline{F}(r) = \sum_{s\in S} \#(r,s)\overline{F}(s) = \sum_{s\in S} \#(r,s)u(s) = \sum_{s\in S} u(s)A(e_s)(r) = (Au)(r) = 0.$$

By Proposition 2.13, $\overline{F}|_N \equiv 0$. □

We end here, but this has been but the tip of the iceberg. While we have laid out some groundwork for understanding (δ-)invariant functions for actions induced on Lipschitz-free spaces, Question 2.29 still stands. There are, of course, questions regarding the optimality of constants for Theorems 2.43 and 2.47, namely

Question 2.48. Let $\mathcal{C}$ be the set of all constants $C \geq 0$ such that the following statement is true: Let $\delta > 0$ and F_S be a free group with generating set S equipped with the word metric and action by left-translations. Let $f \in \mathrm{Lip}_0(F_S)$ be δ-invariant. Then there is $\overline{f} \in \mathrm{Inv}_{F_S}(F_S)$ with $\left\|f - \overline{f}\right\| \leq C\delta$. What does $\inf \mathcal{C}$ equal to? Is $\inf \mathcal{C} \in \mathcal{C}$?

and

Question 2.49. Let G be a finitely presented group equipped with the word metric and action by left-translations. Does there exists a constant $C > 0$ *independent* of G such that for any $\delta > 0$ and $f \in \mathrm{Lip}_0(G)$ δ-invariant there is $\overline{f} \in \mathrm{Inv}_G(G)$ with $\left\|f - \overline{f}\right\| \leq C\delta$?

We have restrained ourselves only to the study of actions by (left-)translations with respect to word metrics. If we replace the word metric by some arbitrary left-invariant metric, matters complicate.

Finally, there is the question of actions on general metric spaces. One possible approach is captured in

Question 2.50. Let G be a discrete group and $C > 0$ such that for any left-invariant metric d on G, $\delta > 0$ and δ-invariant $f \in \mathrm{Lip}_0(G,d)$ there exists $\overline{f} \in \mathrm{Inv}_G(G,d)$ with $\left\|f - \overline{f}\right\| \leq C\delta$. Is it true that for any action of G on a metric space (M,ρ) and δ-invariant $f \in \mathrm{Lip}_0(M)$ there is $\overline{f} \in \mathrm{Inv}_G(M)$ with $\left\|f - \overline{f}\right\| \leq C\delta$?

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