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Spectrum of the density operator

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Abstract: This thesis investigates the properties of the spectrum of an operator defined by a density matrix in the context of quantum statistical physics. The focus is on the operator T_μ, given by

$$T_\mu = \int_{S_H} x \otimes x \, d\mu(x),$$

where μ is a probability measure on the unit sphere in a complex Hilbert space. The study demonstrates that T_μ is a positive nuclear operator with a trace of one. Two examples illustrate the operator's spectral properties under different measures. The thesis primarily covers known properties and examples involving nuclear operators.

Keywords: Nuclear operators, The trace of an operator, Hilbert space, Spectrum of an operator, Bochner integration, Probability measure

Název práce: Spetrum operátoru daného maticí hustoty.

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Abstrakt: Tato práce zkoumá vlastnosti spektra operátoru definovaného pomocí hustotní matice v kontextu kvantové statistické fyziky. Zaměřuje se na operátor T_μ, daný jako

$$T_\mu = \int_{S_H} x \otimes x \, d\mu(x),$$

kde μ je pravděpodobnostní míra na jednotkové sféře v komplexním Hilbertově prostoru. Studie ukazuje, že T_μ je pozitivní nukleární operátor s stopou rovnající se jedné. Dva příklady ilustrují spektrální vlastnosti operátoru pro různé míry. Práce se primárně zabývá známými vlastnostmi a příklady zahrnujícími nukleární operátory.

Klíčová slova: Jaderní operátoři, Stopa operátoru, Hilbertův prostor, Spektrum operátoru, Bochnerova integrace, Pravděpodobnostní míra

Contents

Introduction

This thesis aims to investigate the properties of the operator T_μ, defined as

$$T_\mu = \int_{S_H} x \otimes x \, d\mu(x),$$

where μ is a probability measure on the unit sphere with respect to weak Borel sets. Specifically, we will show that T_μ is a positive nuclear operator with trace equal to 1, and explore various examples for different measures μ.

Understanding the properties of T_μ is crucial for advancing our knowledge in quantum mechanics, as it provides a statistical description of quantum states.

We will employ advanced techniques in operator theory and functional analysis to examine the properties of T_μ. Various examples will be considered to illustrate the properties of its spectrum under different measures.

This thesis is structured as follows:

Chapter 1 to Chapter 4 provide the necessary background and theoretical framework. Chapter 5 delves into the properties of the operator T_μ. Finally, Chapter 6 presents examples of T_μ for different measures.

With this foundation laid, we now turn our attention to a detailed exploration of the operator T_μ and its intriguing properties.

1 Positive Operators on a Hilbert Space

Throughout this work, we will use H to denote a separable Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ over the field of complex numbers.

Definition 1.0.1. Let $A \in L(H)$. The adjoint of A, denoted as $A^* \in L(H)$, is an operator satisfying $\forall x, y \in H$, $\langle Ax, y \rangle = \langle x, A^* y \rangle$. The proof of existence and uniqueness can be found in [1, p. 31].

Definition 1.0.2. The operator $A \in L(H)$ is called self-adjoint, if $A^* = A$.

Definition 1.0.3. A self-adjoint operator on H is called a positive operator, denoted as $A \geq 0$, if for all $x \in H$, we have $\langle Ax, x \rangle \geq 0$.

Definition 1.0.4. For every $A \in L(H)$, there exists a unique positive operator $|A| \in L(H)$, also denoted as $\sqrt{A^* A}$ and called the absolute value of the operator A, that satisfies $|A|^2 = A^* A$. The proof of existence and uniqueness can be found in [1]. As a consequence of these definitions, for positive operators, we have $|A| = A$.

Lemma 1.0.5. From the definitions, it is easy to obtain the following useful facts:

1. For A and B positive operators in $L(H)$, the operator $A + B$ is also positive.

2. For $A \geq 0$ and λ a positive real number, we have $\lambda A \geq 0$.

2 Rank-one operators

Notation 2.0.1. A rank-one outer product operator $x \otimes y : H \to H$, when $x, y \in H$, is defined as $(x \otimes y)(z) = \langle z, y\rangle x, z \in H$.

Now we are able to formulate some properties of outer product operators, which we will need later.

Lemma 2.0.2. Let x and y be non-zero elements of the Hilbert space H. Then the following properties hold.

1. The adjoint of the rank-one outer product operator $x \otimes y$ is $y \otimes x$.
2. The operator $x \otimes x$ is a positive operator.
3. The square root of the operator $x \otimes x$ is $\frac{1}{\|x\|} x \otimes x$.
4. For a non-zero element z from H, the composition of the operators $y \otimes x$ and $x \otimes z$ is equal to the operator $\|x\|^2 y \otimes z$.
5. The absolute value of the operator $x \otimes y$ is $\frac{\|x\|}{\|y\|} y \otimes y$.

Proof. The proof consists of simply checking the definitions.

1. From the definition of the rank-one outer product, we obtain for $T = x \otimes y$:

$$\begin{aligned} \langle (x \otimes y)z, t\rangle &= \langle \langle z, y\rangle x, t\rangle = \langle z, y\rangle \langle x, t\rangle \\ &= \langle z, \langle t, x\rangle y\rangle = \langle z, (y \otimes x)t\rangle, \quad z, t \in H. \end{aligned}$$

   Thus, $T^* = y \otimes x$.

2. The fact that $x \otimes x$ is self-adjoint is a consequence of the first property. Let us check that it is also positive. For every non-zero element z in H the following holds:

$$\langle (x \otimes x)z, z\rangle = \langle \langle z, x\rangle x, z\rangle = \langle x, z\rangle \langle z, x\rangle = \langle x, z\rangle \overline{\langle x, z\rangle} \geq 0.$$

3. We will check that the operators $\sqrt{x \otimes x}$ and $\frac{1}{\|x\|} x \otimes x$ coincide on all elements of H. For an arbitrary $z \in H$, we have:

$$\begin{aligned} \left(\frac{1}{\|x\|} x \otimes x\right)\left(\frac{1}{\|x\|} x \otimes x\right) z &= \frac{1}{\|x\|^2}(x \otimes x)\langle z, x\rangle x \\ &= \frac{1}{\|x\|^2}\langle z, x\rangle \langle x, x\rangle x = \langle z, x\rangle x = (x \otimes x)z. \end{aligned}$$

   Since $\frac{1}{\|x\|} x \otimes x \geq 0$, we can conclude, that $\sqrt{x \otimes x} = \frac{1}{\|x\|} x \otimes x$.

4. For an arbitrary $t \in H$ we have:

$$(y \otimes x)(x \otimes z)t = (y \otimes x)\langle t, z\rangle x = \langle x, x\rangle \langle t, z\rangle y = \|x\|^2 (y \otimes z)t.$$

   This proves the equality of the operators.

5. The last thing we need to verify is that $|x \otimes y| = \frac{\|x\|}{\|y\|} y \otimes y$. This holds because:

$$|x \otimes y| = \sqrt{(x \otimes y)^*(x \otimes y)} = \sqrt{(y \otimes x)(x \otimes y)} = \sqrt{\|x\|^2 y \otimes y} = \frac{\|x\|}{\|y\|} y \otimes y.$$

Hence, we have verified all the stated properties. □

3 Nuclear operators

Theorem 3.0.1. The operator $A \in L(H)$ is compact, if there exists $1 \leq N_A \leq +\infty$, a non-increasing sequence of non-negative numbers $(s_n : 1 \leq n \leq N_A)$ with zero limit in case where $N_A = +\infty$ and two orthonormal basis $\{\phi_n : 1 \leq n \leq +\infty\}, \{\psi_n : 1 \leq n \leq +\infty\}$ in H, such that $A = \sum_{n=1}^{N_A} s_n \psi_n \otimes \phi_n$, with the sum being in $L(H)$.

Lemma 3.0.2. The set of all compact operators in H is closed in norm operator topology.

Lemma 3.0.3. For $A = \sum_{n=1}^{N_A} s_n \psi_n \otimes \phi_n$ from **Theorem 3.0.1**, we have $\|A\| = s_1$.

Proof. Let $x \in H$. Since $\{\phi_n : 1 \leq n \leq +\infty\}$ is an orthonormal basis, there exists a sequence $\{a_n : 1 \leq n \leq \infty\}$ of numbers, such that $x = \sum_{n=1}^{\infty} a_n \phi_n$. Then we have:

$$Ax = \sum_{n=1}^{\infty} s_n \psi_n \otimes \phi_n x = \sum_{n=1}^{\infty} s_n \langle x, \phi_n \rangle \psi_n = \sum_{n=1}^{\infty} s_n a_n \psi_n,$$

$$\|Ax\| = \sqrt{\sum_{n=1}^{\infty} s_n^2 a_n^2} \leq s_1 \sqrt{\sum_{n=1}^{\infty} a_n^2} = s_1 \|x\|.$$

So, $\|A\| \leq s_1$. For $x = \phi_1$ we have $Ax = s_1 \psi_1$ so $\|Ax\| = s_1$. Thus $\|A\| = s_1$. $\square$

Notation 3.0.4. Denote the trace of a positive operator $A \in L(H)$ as $\operatorname{Tr} A = \sum_{i=1}^{\dim(H)} \langle Ae_i, e_i \rangle$, where $\{e_n : 1 \leq n \leq \dim(H)\}$ is an orthonormal basis. It can be shown that $\operatorname{Tr} A$ does not depend on the choice of the orthonormal basis.

Lemma 3.0.5. The trace Tr of an operator A does not depend on the choice of orthonormal basis.

Proof. Let $N := \dim(H) \in \mathbb{N} \cup \{\infty\}$. To show that the trace is independent of the choice of basis, we will first show that $\operatorname{Tr}(T^*T) = \operatorname{Tr}(TT^*)$ for every fixed orthonormal basis $\{e_n : 1 \leq n \leq N\}$.

$$\begin{aligned}
\operatorname{Tr}(T^*T) &= \sum_{n=1}^{N} \langle T^*Te_n, e_n \rangle = \sum_{n=1}^{N} \langle Te_n, Te_n \rangle = \sum_{n=1}^{N} \|Te_n\|^2 \\
&= \sum_{n=1}^{N} \sum_{m=1}^{N} |\langle Te_n, e_m \rangle|^2 = \sum_{n=1}^{N} \sum_{m=1}^{N} |\langle e_n, T^*e_m \rangle|^2 \\
&= \sum_{n=1}^{N} \sum_{m=1}^{N} |\langle T^*e_m, e_n \rangle|^2 = \sum_{m=1}^{N} \sum_{n=1}^{N} |\langle T^*e_m, e_n \rangle|^2 \\
&= \sum_{m=1}^{N} \|T^*e_m\|^2 = \operatorname{Tr}(TT^*).
\end{aligned}$$

Since all elements in the sums are nonnegative, we can freely change the order of summation. Thus, we have shown that $\operatorname{Tr}(T^*T) = \operatorname{Tr}(TT^*)$.

Now, let us consider two orthonormal bases $\{e_n : 1 \leq n \leq N\}$ and $\{f_n : 1 \leq n \leq N\}$. Define an operator $U : H \to H$ on the elements of the basis $\{e_n\}$ as

$Ue_n = f_n$. It is easy to see that $U^*f_n = e_n$ for all $n \in \mathbb{N}$, and thus the operator U is unitary ($U^*U = UU^* = I$).

For all $n \in N$, we have $\langle UTU^*f_n, f_n\rangle = \langle Te_n, e_n\rangle$. From the fact that $\mathrm{Tr}(V^*V) = \mathrm{Tr}(VV^*)$ for every positive operator V, with respect to the $\{f_n : 1 \leq n \leq N\}$, we have:

$$\mathrm{Tr}(UTU^*) = \mathrm{Tr}(UT^{1/2}T^{1/2}U^*) = \mathrm{Tr}(T^{1/2}U^*UT^{1/2}) = \mathrm{Tr}(T).$$

Combining all together, we have shown that the trace Tr is indeed independent of the choice of orthonormal basis. □

Lemma 3.0.6. For a compact operator $A = \sum_{n=1}^{N_A} s_n\psi_n \otimes \phi_n$, where s_n, ϕ_n, and ψ_n are from **Theorem 3.0.1**, we have $|A| = \sum_{n=1}^{N_A} s_n\phi_n \otimes \phi_n$.

Proof. Firstly, because the adjoint operator $^* : L(H) \to L(H)$ is linear and bounded ($\|A^*\| = \|A\|$ for all operators in $L(H)$), we also have that it is continuous, and we are allowed to swap the sum and the operator. Thus, from **Lemma 2.0.2**, we obtain $A^* = \sum_{n=1}^{N_A} s_n\phi_n \otimes \psi_n$.

It can be easily checked that $A^*A = \sum_{n=1}^{N_A} s_n^2\phi_n \otimes \phi_n$. To prove that $|A| = T := \sum_{n=1}^{N_A} s_n\phi_n \otimes \phi_n$, it is enough to verify that T is a positive operator and that $T^2 = A^*A$.

1. Positive definiteness. From **Lemma 2.0.2** and the fact that for all $n \in \mathbb{N}$, $s_n \geq 0$, we have that $s_n\phi_n \otimes \phi_n$ is positive for all $n \in \mathbb{N}$. Thus, $T \geq 0$ because it is the sum of positive operators.

2. We will check that $T^2 = A^*A$ on elements of H. For all $x \in H$,

$$T^2x = \sum_{n=1}^{N_A} s_n \left\langle \sum_{k=1}^{N_A} s_k\langle x, \phi_k\rangle\phi_k, \phi_n \right\rangle \phi_n = \sum_{n=1}^{N_A} s_n^2\langle x, \phi_n\rangle\phi_n = A^*Ax.$$

And we are done.

□

Lemma 3.0.7. For a compact operator $A = \sum_{n=1}^{N_A} s_n\psi_n \otimes \phi_n$, where s_n, ϕ_n, and ψ_n are from **Theorem 3.0.1**, we have $\mathrm{Tr}\,|A| = \sum_{n=1}^{N_A} s_n$.

Proof. From **Lemma 3.0.6** we obtain $|A| = \sum_{n=1}^{N_A} s_n\phi_n \otimes \phi_n$. Thus, computing the trace with the orthonormal basis $\{\phi_n\}_{n=1}^{\infty}$, we have

$$\mathrm{Tr}\,|A| = \sum_{n=1}^{N_A}\langle |A|\phi_n, \phi_n\rangle = \sum_{n=1}^{N_A}\langle\sum_{k=1}^{N_A} s_k\langle\phi_n, \phi_k\rangle\phi_k, \phi_n\rangle = \sum_{n=1}^{N_A} s_n\langle\phi_n, \phi_n\rangle = \sum_{n=1}^{N_A} s_n.$$

□

Now we are ready to introduce the main concept of this section — Nuclear operators.

Definition 3.0.8. The operator $A \in L(H)$ is nuclear, when $\|\cdot\|_1 := \mathrm{Tr}\,|A| < \infty$. The set of all nuclear operators we will denote by $N_1(H)$.

Lemma 3.0.9. Every nuclear operator is compact

Proof. Proof can be found in [2, p 5-6]. $\square$

Lemma 3.0.10. For $A \in N_1(H)$, the following inequality holds: $\|A\| \leq \|A\|_1$.

Proof. For $A \in N_1(H)$, we know that A is compact. Therefore, there exists a sequence of non-increasing non-negative numbers $(s_n)_{n=1}^{+\infty}$ such that $\|A\| = s_1$ and $\|A\|_1 = \sum_{n=1}^{+\infty} s_n$. Thus, the inequality $\|A\| \leq \|A\|_1$ is clear. $\square$

To prove that the space of nuclear operators is a normed vector space, we need the following lemma.

Lemma 3.0.11. For a compact operator $A \in L(H)$, the following holds:

$$\mathrm{Tr}(|A|) = \max\left\{\sum_{n=1}^{\infty} |\langle Ae_n, g_n\rangle|\right\},$$

where the maximum is taken over all orthonormal bases $\{e_n : n \geq 1\}$ and $\{g_n : n \geq 1\}$.

Proof. Since A is compact, from **Theorem 3.0.1**, it can be written in the form $A = \sum_{n=1}^{N_A} s_n \psi_n \otimes \phi_n$ for some orthonormal bases $\{\psi_n\}$, $\{\phi_n\}$ and positive real numbers $s_n, 1 \leq n \leq N_A \leq +\infty$.

Consider any two orthonormal bases $\{e_n : n \geq 1\}$ and $\{g_n : n \geq 1\}$. Thus, for every k we have the following inequality:

$$\begin{aligned}
|\langle Ae_k, g_k\rangle| &= \left|\left\langle \sum_{n=1}^{N_A} s_n \langle e_k, \phi_n\rangle \psi_n, g_k \right\rangle\right| = \left|\sum_{n=1}^{N_A} s_n \langle e_k, \phi_n\rangle\langle \psi_n, g_k\rangle\right| \\
&\leq \sum_{n=1}^{N_A} s_n |\langle e_k, \phi_n\rangle||\langle \psi_n, g_k\rangle|.
\end{aligned}$$

Using the AM-GM inequality, we obtain:

$$\begin{aligned}
\sum_{n=1}^{\infty} |\langle Ae_n, g_n\rangle| &\leq \sum_{n=1}^{\infty}\sum_{k=1}^{N_A} s_n |\langle e_n, \phi_k\rangle||\langle \psi_k, g_n\rangle| \\
&= \sum_{k=1}^{N_A} s_k \sum_{n=1}^{\infty} |\langle e_n, \phi_k\rangle||\langle \psi_k, g_n\rangle| \\
&\leq \sum_{k=1}^{N_A} s_k \sum_{n=1}^{\infty} \frac{|\langle e_n, \phi_k\rangle|^2 + |\langle \psi_k, g_n\rangle|^2}{2} \\
&= \sum_{k=1}^{N_A} s_k \frac{\|\phi_k\|^2 + \|\psi_k\|^2}{2} \\
&= \sum_{k=1}^{N_A} s_k.
\end{aligned}$$

Equality holds, for example, when $\{e_n : n \geq 1\} = \{\phi_n : n \geq 1\}$ and $\{g_n : n \geq 1\} = \{\psi_n : n \geq 1\}$. $\square$

Lemma 3.0.12. $(N_1(H), \|\cdot\|_1)$ is a normed vector space.

Proof. To prove that $(N_1(H), \|\cdot\|_1)$ is a normed vector space, we need to verify its four axioms.

1. Non-negativity. It follows from $\|A\|_1 \geq \|A\| \geq 0$.

2. Positive definiteness: We need to check that for $A \in N_1(H)$, $\|A\|_1 = 0 \iff A = 0$:

$$\|A\|_1 = 0 \implies 0 \leq \|A\| \leq \|A\|_1 = 0 \implies \|A\| = 0 \implies A = 0.$$

   The converse is straightforward from the definitions.

3. Absolute homogeneity: For all $\lambda \in \mathbb{C}$ and $A \in N_1(H)$, $\|\lambda A\|_1 = |\lambda|\|A\|_1$. This follows directly from the definitions.

4. Triangle inequality: Let A and B be nuclear operators. From **Lemma 3.0.9** we have that these operators are compact. From the triangle inequality and the previous lemma, we obtain:

$$\begin{aligned}
\|A+B\|_1 &= \max_{\{e_n\},\{g_n\}} \left\{ \sum_{n=1}^{\infty} |\langle (A+B)e_n, g_n\rangle| \right\} \\
&\leq \max_{\{e_n\},\{g_n\}} \left\{ \sum_{n=1}^{\infty} (|\langle Ae_n, g_n\rangle| + |\langle Be_n, g_n\rangle|) \right\} \\
&\leq \max_{\{e_n\},\{g_n\}} \left\{ \sum_{n=1}^{\infty} |\langle Ae_n, g_n\rangle| \right\} + \max_{\{e_n\},\{g_n\}} \left\{ \sum_{n=1}^{\infty} |\langle Be_n, g_n\rangle| \right\} \\
&= \|A\|_1 + \|B\|_1.
\end{aligned}$$

Therefore, all axioms hold, and $N_1(H)$ is indeed a normed vector space.

□

Lemma 3.0.13. For compact operators $A, B \in L(H)$, we have that for all positive integers n:

$$|s_n(A) - s_n(B)| \leq \|A - B\|.$$

Proof. Proof can be found in [3, ex. 12.37]. □

Theorem 3.0.14. $(N_1(H), \|\cdot\|_1)$ is a Banach space.

Proof. From **Lemma 3.0.11** we have that $(N_1(H), \|\cdot\|)$ is a normed vector space. It remains to prove that it is complete.

Let $\{A_n : n \geq 1\}$ be a fundamental sequence in $N_1(H)$. Thus, from $\|A\| \leq \|A\|_1$, we obtain that $\{A_n : n \in \mathbb{N}\}$ is fundamental in the operator norm. Since the operator norm is complete and the set of compact operators is closed, there exists a compact operator $A \in L(H)$ such that $A_n \to A$ in $L(H)$.

1. We need to check that $A \in N_1(H)$. Using the triangle inequality, for every positive integers n and m, we have:

$$
\begin{aligned}
s_1(A) + \ldots + s_n(A) &\leq \sum_{k=1}^{n} |s_k(A) - s_k(A_m) + s_k(A_m)| \\
&\leq \sum_{k=1}^{n} |s_k(A) - s_k(A_m)| + \sum_{k=1}^{n} s_k(A_m) \\
&\leq n\|A - A_m\| + \|A_m\|_1.
\end{aligned}
$$

In the last inequality, we used the previous lemma and **Lemma 3.0.6**, which states that $\|A_m\|_1 = \sum_{k=1}^{\infty} s_k(A_m) \geq \sum_{k=1}^{n} s_k(A_m)$.

By assumption, $\{A_n : n \geq 1\}$ is fundamental in $N_1(H)$. From the norm inequality $|\|A\|_1 - \|B\|_1| \leq \|A - B\|_1$, we obtain that $\{\|A_n\|_1 : n \geq 1\}$ is fundamental in $\mathbb{R}$. Since $\mathbb{R}$ is complete, we conclude that this sequence has a limit and is therefore bounded, which means that there exists a constant $C > 0$ such that for all $m \geq 1$, $\|A_m\|_1 \leq C$. Since $A_n \to A$ in operator norm as $n \to \infty$, we have that for all $n \geq 1$, there exists $m \geq 1$ such that $n\|A - A_m\| \leq 1$. Combining all of this, we obtain that for all $n \geq 1$, the following holds: $s_1(A) + \ldots + s_n(A) \leq 1 + C$. Taking the limit as n approaches infinity, we obtain $\|A\|_1 \leq 1 + C$, and therefore $A \in N_1(H)$.

2. It remains to show that the sequence $\{A_n : n \geq 1\}$ converges to the operator A in the $\|\cdot\|_1$ norm.

   Let $\epsilon > 0$ and $n \in \mathbb{N}$ be fixed. Using the same inequalities as above, we obtain that for all m and k being positive integers:

$$
s_1(A - A_m) + \ldots + s_n(A - A_m) \leq n\|A - A_k\| + \|A_m - A_k\|_1.
$$

   From the fundamentality of the sequence $\{A_n : n \geq 1\}$ in $N_1(H)$, there exists $n_0 \in \mathbb{N}$ such that for all positive integers $k, m \geq n_0$, we have $\|A_m - A_k\|_1 \leq \epsilon$. By the same argument used in the proof that $A \in N_1(H)$, we have that for all $n \in \mathbb{N}$, there exists $k \geq n_0$ satisfying $n\|A - A_k\| \leq \epsilon$. Therefore, for all $m \geq n_0$, we obtain:

$$
s_1(A - A_m) + \ldots + s_n(A - A_m) \leq n\|A - A_k\| + \|A_m - A_k\|_1 \leq 2\epsilon.
$$

   Thus, $\|A - A_m\|_1 \leq 2\epsilon$, and finally $A_n \to A$ in $N_1(H)$.

□

Lemma 3.0.15. The operator $x \otimes y$ is a nuclear operator and $\|x \otimes y\|_1 = \|x\|\|y\|$.

Proof. If $y = 0$, then $x \otimes y = 0$, and thus $\|x \otimes y\|_1 = 0 = \|x\|\|y\|$.

Now, consider the case $y \neq 0$. From **Lemma 2.0.2**, we have $|x \otimes y| = \frac{\|x\|}{\|y\|} y \otimes y$. Let us calculate $\mathrm{Tr}(y \otimes y)$. Let $\{e_n : n \geq 1\}$ be an orthonormal basis in H, and in this basis, $y = \sum_{i=1}^{\infty} y_i e_i$, where the coefficients y_i are complex numbers.

$$
\mathrm{Tr}(y \otimes y) = \sum_{i=1}^{\infty} \langle (y \otimes y) e_i, e_i \rangle = \sum_{i=1}^{\infty} \langle \langle e_i, y \rangle y, e_i \rangle = \sum_{i=1}^{\infty} |\langle y, e_i \rangle|^2 = \|y\|^2 < \infty.
$$

In the last equality, we used Parseval's identity. Now, we are able to compute $\mathrm{Tr}(|x \otimes y|)$:

$$\|x \otimes y\|_1 = \frac{\|x\|}{\|y\|} \operatorname{Tr}(y \otimes y) = \frac{\|x\|}{\|y\|}\|y\|^2 = \|x\|\|y\| < \infty.$$

This implies $x \otimes y \in N_1(H)$. $\square$

For simplicity of notation, series with a lower bound greater than the upper bound will be defined to be equal to zero.

Theorem 3.0.16. The Banach space $N_1(H)$ is separable.

Proof. Consider any nuclear operator A. Since it is compact, it can be written in the form $A = \sum_{n=1}^{N_A} s_n \phi_n \otimes \psi_n$, where $\sum_{n=1}^{N_A} s_n < \infty$. It can be approximated by $A_k = \sum_{n=1}^{k} s_n \psi_n \otimes \phi_n$.

$$\|A_k - A\|_1 = \left\| \sum_{n=k+1}^{N_A} s_n \psi_n \otimes \phi_n \right\|_1 = \sum_{n=k+1}^{N_A} s_n \to 0 \text{ as } k \to \infty.$$

This means that the set of all such operators A_k for all k and for all $A \in N_1(H)$, which we will denote as $F(H)$, is dense.

By the initial assumption, H is separable, so there exists a countable dense set X in it. We want to approximate the operators in $F(H)$ by elements of a countable subset of $N_1(H)$. Let $\epsilon > 0$ be a fixed positive number. Consider any operator $A = \sum_{n=1}^{k} s_n \psi_n \otimes \phi_n \in F(H)$, where $\{\psi_n : 1 \leq n \leq k\}$ and $\{\phi_n : 1 \leq n \leq k\}$ form orthonormal sets in H. For every $n \in \{1, \ldots, k\}$, there exist $x_n, y_n \in X$ satisfying $\|\phi_n - y_n\| < \frac{\epsilon}{4}$, $\|\psi_n - x_n\| < \frac{\epsilon}{4(\|y_n\|+1)}$, and $q_n \in \mathbb{Q}$ such that $|s_n - q_n| < \frac{\epsilon}{2}$. Consequently, $q_n < s_n + \frac{\epsilon}{2} \leq s_1 + \frac{\epsilon}{2} =: C$. For every n, we can write:

$$\begin{aligned}
\|\psi_n \otimes \phi_n - x_n \otimes y_n\|_1 &= \|\psi_n \otimes \phi_n - \psi_n \otimes y_n + \psi_n \otimes y_n - x_n \otimes y_n\|_1 \\
&\leq \|\psi_n \otimes (\phi_n - y_n)\|_1 + \|(\psi_n - x_n) \otimes y_n\|_1 \\
&= \|\psi_n\|\|\phi_n - y_n\| + \|\psi_n - x_n\|\|y_n\| \\
&= \|\phi_n - y_n\| + \|\psi_n - x_n\|\|y_n\| \\
&< \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}.
\end{aligned}$$

Thus, we have:

$$\begin{aligned}
\|s_n \psi_n \otimes \phi_n - q_n x_n \otimes y_n\|_1 &< \|s_n \psi_n \otimes \phi_n - q_n \psi_n \otimes \phi_n\|_1 \\
&\quad + \|q_n \psi_n \otimes \phi_n - q_n x_n \otimes y_n\|_1 \\
&= |s_n - q_n|\|\psi_n \otimes \phi_n\|_1 + |q_n|\|\psi_n \otimes \phi_n - x_n \otimes y_n\|_1 \\
&< \epsilon + C\|\psi_n \otimes \phi_n - x_n \otimes y_n\|_1 \\
&< \epsilon + C\epsilon \\
&= (C+1)\epsilon
\end{aligned}$$

And finally, from the subadditivity of the norm, we obtain:

$$\begin{aligned}
\left\| \sum_{n=1}^{k} (s_n \psi_n \otimes \phi_n - q_n x_n \otimes y_n) \right\|_1 &\leq \sum_{n=1}^{k} \|s_n \psi_n \otimes \phi_n - q_n x_n \otimes y_n\|_1 \\
&< k(C+1)\epsilon \to 0 \text{ as } \epsilon \to 0.
\end{aligned}$$

So, we have shown that every operator from $F(H)$ can be arbitrarily closely approximated by an operator from the set

$$U(H) := \left\{ \sum_{n=1}^{k} q_n x_n \otimes y_n : k \in \mathbb{N},\, x_n, y_n \in X \right\}.$$

Since $F(H)$ is dense in $N_1(H)$, we conclude that $U(H)$ is also dense. It is also easy to see that $U(H)$ is countable, so finally, $N_1(H)$ is separable.

□

Theorem 3.0.17. The operator $\phi : H \to N_1(H)$ defined by $\phi(x) = x \otimes x$ is continuous.

Proof. Let us consider an arbitrary $x_0 \in H$ and prove that ϕ is continuous at this point. Let $\epsilon > 0$. For some $\delta > 0$, which we will choose later, for all $x \in H$ such that $\|x - x_0\| < \delta$, we can write:

$$\begin{aligned} \|\phi(x) - \phi(x_0)\|_1 = \|x \otimes x - x_0 \otimes x_0\|_1 &= \|x \otimes x - x \otimes x_0 + x \otimes x_0 - x_0 \otimes x_0\|_1 \\ &= \|x \otimes (x - x_0) + (x - x_0) \otimes x_0\|_1 \\ &\leq \|x \otimes (x - x_0)\|_1 + \|(x - x_0) \otimes x_0\|_1. \end{aligned}$$

$$\|x \otimes (x - x_0)\|_1 + \|(x - x_0) \otimes x_0\|_1 = \|x\|\|x - x_0\| + \|x - x_0\|\|x_0\|.$$

From $\|x - x_0\| < \delta$, we have $\|x\| < \delta + \|x_0\|$. Therefore:

$$\begin{aligned} \|x\|\|x - x_0\| + \|x - x_0\|\|x_0\| &< (\delta + \|x_0\|)\delta + \delta\|x_0\| \\ &= \delta(\delta + 2\|x_0\|) < \epsilon \quad \text{for some } 0 < \delta < \|x_0\|. \end{aligned}$$

Thus, ϕ is continuous at x_0, and consequently, it is continuous on H.

□

4 Bochner integration and measures

Let (A, σ, μ) be a measurable space, and X a Banach space.

Definition 4.0.1. A function $f : A \to X$ is called simple if there exist pairwise disjoint measurable sets $A_1, \dots, A_N$ in A and $x_1, \dots, x_N \in X$ such that $f = \sum_{i=1}^{N} 1_{A_i} x_i$.

Definition 4.0.2. A function $f : A \to X$ is called **strongly measurable** if there exists a sequence $\{f_n\}_{n=1}^{\infty}$ of simple functions such that for all $x \in A$ holds $f(x) = \lim_{n\to\infty} f_n(x)$.

The function f is called **measurable** if, for every measurable set in X, the preimage of this set is also measurable.

The function f is called **weakly measurable** if $\forall x^* \in X^*$, the composition $x^* \circ f$ is measurable.

We will need the following well-known theorem and its consequence.

Theorem 4.0.3 (Pettis' Theorem)**.** A function $f : A \to X$ is strongly measurable if and only if f is weakly measurable and $f(A)$ is contained in a separable subspace of X. As a consequence, for every separable Banach space, these notions of measurability coincide.

Definition 4.0.4. For a simple function $f = \sum_{k=1}^{n} 1_{A_k} x_k$, define the Bochner integral of f as $\int_A f d\mu = \sum_{k=1}^{n} \mu(A_k) x_k \in X$. It is straightforward to show that this integral for a simple function does not depend on the particular representation of the function.

Definition 4.0.5. For a strongly measurable function $f : A \to X$ and $\{f_n\}_{n=1}^{\infty}$ - a sequence of simple functions from A to X such that $\forall x \in A$, $f(x) = \lim_{n\to\infty} f_n(x)$, we define the Bochner integral as

$$\int_A f \, d\mu = \lim_{n\to\infty} \int_A f_n \, d\mu$$

when

$$\int_A \|f(x) - f_n(x)\| \, d\mu(x) \to 0 \quad \text{as} \quad n \to \infty.$$

Further, we will need the following properties of Bochner integration.

Theorem 4.0.6 (Basic properties of the Bochner integral)**.**

1. For a Bochner integrable function $f : A \to X$, the value of the integral $\int_A f \, d\mu$ does not depend on the choice of the sequence $\{f_n : n \geq 1\}$ from the definition.

2. **Characterization of Bochner integrable functions**. A strongly measurable function $f : A \to X$ is Bochner integrable if, and only if,

$$\int_A \|f(x)\| \, d\mu(x) < \infty.$$

3. For a Bochner integrable function $f : A \to X$ and every $T \in L(X, Y)$, where Y is a Banach space, we have that Tf is also Bochner integrable and

$$T\left(\int_A f \, d\mu\right) = \int_A Tf \, d\mu.$$

Proof. The proofs for the first two statements can be found in [4, Chapter 1].

To prove the third statement, note that since for all $x \in H$, we have $\|Tf(x)\| \leq \|T\|\|f(x)\|$, we obtain:

$$\int_A \|Tf(x)\| \, d\mu(x) \leq \|T\| \int_A \|f(x)\| \, d\mu(x) < \infty.$$

Thus, Tf is also Bochner integrable.
Let $\{f_n : n \geq 1\}$ be a sequence of simple functions such that $f(x) = \lim_{n\to\infty} f_n(x)$. Then Tf is the limit of Tf_n, and Tf_n are also simple functions. For simple functions, it is straightforward to see that we can swap the integral and T. Therefore:

$$\begin{aligned} T\left(\int_A f(x) \, d\mu(x)\right) &= \lim_{n\to\infty} T\left(\int_A f_n(x) \, d\mu(x)\right) \\ &= \lim_{n\to\infty} \int_A Tf_n(x) \, d\mu(x) = \int_A Tf(x) \, d\mu(x). \end{aligned}$$

This completes the proof. □

5 Definition of T_μ and its properties

Definition 5.0.1. The weak topology on a Banach space X is the smallest topology τ_{weak} such that $\forall x^* \in X^*$, the map $x^* : (X, \tau_{\text{weak}}) \to \mathbb{K}$ is continuous, where $\mathbb{K}$ denotes the field of scalars (either $\mathbb{R}$ or $\mathbb{C}$).

Notation 5.0.2. B_X denotes the unit ball in X, and S_X denotes the unit sphere in X, where X is a metric space.

Theorem 5.0.3 (A consequence of Kakutani's theorem)**.** The unit ball B_H is weakly compact in the Hilbert space H.

Proof. Since Hilbert spaces are reflexive, by Kakutani's theorem, the unit ball B_H is weakly compact. □

Theorem 5.0.4. For every separable Hilbert space, the Borel sigma-algebras generated by the weak topology and by the norm topology coincide.

Proof. Let H be a separable Hilbert space. Since H is separable, every open set in the norm topology can be generated by no more than a countable union of open balls. Thus, the Borel sets with respect to the norm topology are generated by open balls. Since the weak topology is coarser (i.e., smaller) than the norm topology, it is enough to prove that every open ball in the norm topology can be generated by weak Borel sets.

From the previous theorem, it is easy to conclude that any closed ball $\overline{B}_r(x_0) := \{x \in H : \|x - x_0\| \leq r\}$, where $x_0 \in H$ and $r > 0$, is compact in the weak topology, and thus closed. Therefore, we can obtain that any open ball in the norm topology is in the Borel sets generated by the weak topology, since for every $r > 0$ and $x_0 \in H$, we have:

$$B_r(x_0) = \{x \in H : \|x - x_0\| < r\} = \bigcup_{n \in \mathbb{N}} \overline{B}_{r - \frac{1}{n}}(x_0).$$

Therefore, the Borel sigma-algebras indeed coincide. □

To define the density operator, we first need to define a probability measure on a Hilbert space. We will consider only so-called inner regular measures.

Definition 5.0.5. A measure μ on a topological space X is called **inner regular** if for every measurable set $A \subseteq X$,

$$\mu(A) = \sup\{\mu(K) : K \subseteq A, K \text{ is compact}\}.$$

In other words, μ is inner regular if the measure of any measurable set A can be approximated from within by the measures of compact subsets of A.

As topology, we choose the weak topology on the unit ball B_H, because in this topology the unit ball is compact, thus we are able to consider inner regular measures on it. Moreover, we want the measure to be concentrated on the sphere, which means that the measure of S_H is equal to 1. We will denote the set of all such probability measures by $M^1(S_H, \text{weak})$.

We already had that $\phi(x) = x \otimes x$ is continuous in the norm topology. As a consequence, this operator is measurable with respect to Borel sets generated by the norm topology. From the last theorem, the Borel sets generated by the weak and norm topologies coincide, which means that ϕ is also measurable with respect to weak Borel sets. Since $N_1(H)$ is separable, from Pettis's theorem we obtain that ϕ is strongly measurable.

Let μ be a probability measure from $M^1(B_H, \text{weak})$. Then we will denote the Bochner integral $\int_{x \in S_H} \phi(x)\, d\mu(x) = \int_{x \in S_H} x \otimes x\, d\mu(x)$ as T_μ or T, when the measure is clear from the context.

Lemma 5.0.6. For the measure μ defined above, the Bochner integral T_μ is well-defined, and moreover, it is a positive definite, nuclear operator with $\|T\|_1 = \operatorname{Tr} T = 1$.

Proof. Firstly, we will show that T_μ is well-defined. From **Lemma 3.0.15**, we have that $\forall x \in S_H : \|\phi(x)\|_1 = \|x \otimes x\|_1 = \|x\|^2 = 1$ and

$$\int_{x \in S_H} \|x \otimes x\|_1\, d\mu(x) = \mu(S_H) = 1 < \infty.$$

Since ϕ is strongly measurable, by the characterization theorem of Bochner integrable functions, we obtain that T_μ is a well-defined Bochner integral. Additionally, because the operator ϕ maps to $N_1(H)$, we have that $T_\mu \in N_1(H)$.

To prove that T is a positive operator, we need to show that for all $z \in S_H$, $\langle Tz, z\rangle \geq 0$.

Consider a bounded linear functional $\psi_z \in (N_1(H))^*$ defined as $\psi_z(S) = \langle Sz, z\rangle$ for $S \in N_1(H)$. The linearity of ψ_z is evident, and its boundedness follows from the Cauchy-Schwarz inequality and **Lemma 3.0.10**:

$$|\langle Sz, z\rangle| \leq \|Sz\|\|z\| \leq \|S\|\|z\|^2 = \|S\| \leq \|S\|_1.$$

Since for all $x \in S_H$, the operator $x \otimes x$ is positive, we have $\langle (x \otimes x)z, z\rangle \geq 0$ for all $x \in S_H$. Using the property that we can swap the operator and the Bochner integral, we obtain:

$$\langle Tz, z\rangle = \psi_z(T) = \int_{S_H} \psi_z(x \otimes x)\, d\mu(x) = \int_{S_H} \langle (x \otimes x)z, z\rangle\, d\mu(x) \geq 0.$$

Thus, T_μ is positive definite.

Finally, since $\mu(S_H) = 1$ and $\operatorname{Tr} \in N_1(H)^*$, we have:

$$\operatorname{Tr} T = \int_{x \in S_H} \operatorname{Tr}(x \otimes x)\, d\mu(x) = \int_{x \in S_H} 1\, d\mu(x) = 1.$$

Thus, $\|T\|_1 = \operatorname{Tr} T = 1$. ☐

So, we obtained that T is nuclear and thus compact. Because T is positive definite, there exists a sequence or a finite set of eigenvalues that are positive and sum to 1. Since T is compact, we have $\sigma = \{0\} \cup \sigma_p$, where σ is the spectrum and σ_p is the point spectrum of the operator T.

Lemma 5.0.7. For an operator T_μ defined above, where $\mu \in M^1(S_H, \text{weak})$:

$$T_\mu z = \int_{S_H} x \otimes x \, d\mu(x), \quad z \in H.$$

Proof. Let $z \in H$ be fixed. Consider a bounded linear operator $\chi : N_1(H) \to H$ defined as $\chi(T) = Tz$ for $T \in N_1(H)$. The proof that the operator χ is well-defined is the same as above, and by the same argument, we are allowed to swap χ and the integral. Finally, we obtain:

$$T_\mu z = \chi(T_\mu) = \int_{S_H} \chi(x \otimes x) \, d\mu(x) = \int_{S_H} (x \otimes x) z \, d\mu(x).$$

□

6 Examples of T_μ

6.1 Discrete measures

Let n be a positive integer, $x_1, \ldots, x_n \in S_H$ are linearly independent and $\alpha_1, \ldots, \alpha_n$ be positive real numbers such that $\sum_{k=1}^n \alpha_k = 1$. Consider a measure $\mu = \alpha_1\delta_{x_1} + \cdots + \alpha_n\delta_{x_n}$, where $\delta.$ is a Dirac measure.

Then we can easily compute the corresponding T:

$$T = \int_{x \in S_H} x \otimes x d\mu(x) = \sum_{k=1}^{n} \alpha_k x_k \otimes x_k$$

Then for every $z \in H$:

$$Tz = \sum_{k=1}^{n} \alpha_k \langle z, x_k \rangle x_k \in \operatorname{span}\{x_1, \ldots, x_n\}.$$

So, we will look at eigenvectors in the form $\beta_1 x_1 + \ldots + \beta_n x_n$. Let $\lambda \in \mathbb{R}$, $\lambda \geq 0$, be an eigenvalue of the operator T and $z = \beta_1 x_1 + \ldots + \beta_n x_n$ be the corresponding eigenvector.

$$\begin{aligned} \lambda \left(\sum_{k=1}^{n} \beta_k x_k \right) &= \lambda z = Tz = \sum_{k=1}^{n} \alpha_k \langle z, x_k \rangle x_k \\ &= \sum_{k=1}^{n} \alpha_k \left\langle \sum_{j=1}^{n} \beta_j x_j, x_k \right\rangle x_k = \sum_{k=1}^{n} \alpha_k \sum_{j=1}^{n} \beta_j G_{kj} x_k, \end{aligned}$$

where $G_{kj} = \langle x_k, x_j \rangle$ form the Gram matrix of vectors $x_1, x_2, \ldots, x_n$.

So, for every $k \in \{1, 2, \ldots, n\}$:

$$\lambda \beta_k = \alpha_k \sum_{j=1}^{n} \beta_j G_{kj}.$$

This can be rewritten in a more compact form as $\lambda\beta = \operatorname{diag}(\alpha)G\beta$, or equivalently as $(\operatorname{diag}(\alpha)G - \lambda I)\beta = \overline{0}$. This means that we need to find the eigenvectors and eigenvalues of the matrix $\operatorname{diag}(\alpha)G$.

In the special case where all x are pairwise orthogonal, we have $G = I$, and thus the eigenvalues in this case are equal to the α's, which is true also when $n = \infty$.

6.2 Continuous measures

Let H be an infinite-dimensional separable Hilbert space with basis $\{e_n : n \geq 1\}$. Consider the set $S_{e_1,e_2} = \{\cos(t)e_1 + \sin(t)e_2 : t \in [0, 2\pi]\}$. Then, the measure of a set in $B(H)$ is defined as the measure (length) of the projection of the set onto the circle S_{e_1,e_2}. We will denote this measure as μ.

Firstly, let us consider an arbitrary vector $z \in H$ orthogonal to both e_1 and e_2. From **Lemma 5.0.7**, we have:

$$T_\mu z = \int_{S_H} (x \otimes x) z \, d\mu(x) = \int_{S_{e_1,e_2}} \langle z, x \rangle x \, d\mu(x) = 0 = 0 \cdot z.$$

Hence, 0 is in the point spectrum.

All other vectors can be written in the form $z = x_1 e_1 + x_2 e_2 + z'$, where z' is orthogonal to S_{e_1,e_2}. Suppose such a vector is an eigenvector with eigenvalue $\lambda \geq 0$.

$$Tz = x_1 T e_1 + x_2 T e_2 = \lambda (x_1 e_1 + x_2 e_2 + z').$$

If there are no eigenvectors inS_{e_1,e_2}, then

$$x_1 T e_1 + x_2 T e_2 \neq \lambda (x_1 e_1 + x_2 e_2), \forall x_1, x_2 \in \mathbb{R}.$$

If we fix $x_1, x_2 \in \mathbb{R}$ and modify $z' \in H$, we obtain that every real number is an eigenvalue. This contradicts the fact that the number of eigenvalues is no more than countable.

So, there is at least one eigenvector that lies in S_{e_1,e_2}. Due to symmetry, any vector from S_{e_1,e_2} is an eigenvector with the same unknown eigenvalue λ. Therefore, for any vector $z \in H$ written in the form $z = x_1 e_1 + x_2 e_2 + z'$, where $z' \neq 0$, we have that it cannot be an eigenvector because $z' \neq 0$.

Since the trace of the operator T_μ is 1, and the dimension of S_{e_1,e_2} is 2, we have $2\lambda = 1$, which implies $\lambda = \frac{1}{2}$. Therefore, the spectrum of this operator is $\sigma = \{0, \frac{1}{2}\}$.

Conclusion

This thesis investigated the properties of the operator T_μ, defined as

$$T_\mu = \int_{S_H} x \otimes x \, d\mu(x),$$

where μ is a probability measure on the unit sphere with respect to weak Borel sets. We demonstrated that T_μ is a positive nuclear operator with a trace equal to 1.

While the analysis focused on specific probability measures, extending this work to more general measures on Hilbert spaces would be a valuable direction for future research.

In conclusion, this thesis provides foundational insights into the operator T_μ, contributing to the broader field of quantum mechanics.

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