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Proving combinatorial identities via formal power series

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Abstract: In the thesis we study formal power series, an extension of the algebraic notion of a polynomial with a canonical metric. We define all the necessary tools, including said metric, composition and formal derivative. We prove basic theorems concerning these notions, such as the characterization of infinite sum convergence, proving some basic properties of composition and proving some analytical formulas about the derivative. We then focus on the Lagrange-Jacobi four-square theorem and define and use Gauss' coefficients to prove some more complex theorems, such as Jacobi's triple product formula and Roger-Ramanujan identities. We then use most of them to prove the mentioned four-square theorem.

Keywords: formal power series, null sequence, Gauss coefficient, triple product formula, four-square theorem

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Introduction

In the most basic algebra course, a student learns about the ring of polynomials over a certain field. They learn of it as an algebraic structure, without analytic notions such as convergence. They also learn, in their analysis courses, functional metric spaces, where there can be convergence, but little attention is paid to its algebraic structure. Some "nice" functions possess a convergent Taylor power series equal to them. The topic of this thesis, formal power series, combines the two: It extends our polynomial ring to "infinite power polynomials" and defines a metric. This allows us to combine our knowledge of both analysis and algebra. This thesis will allow the reader to learn more about the topic with minimal prerequisite knowledge, since only the basic algebra course will be enough.

We begin Chapter 1 by formalising the notion of a formal power series. We first show the set of formal power series forms a vector space with term-wise addition, then define multiplication (Definition 1.3) and show the set of power series also forms an integral domain (Lemma 1.5). We characterize the invertible elements (Lemma 1.9). We define a norm (Definition 1.10), prove the set of formal power series with the norm forms an ultrametric space (Lemma 1.12), then also define a canonical metric using the norm and show with this the set of formal power series forms a complete metric space (Theorem 1.13). We prove an important theorem, saying a sum converges if and only if the sequence we are summing converges to zero (unlike for example in $\mathbb{R}^n$) in Theorem 1.16. We also provide examples throughout the chapter to ensure understanding for any reader.

We continue with Chapter 2 where we define the more difficult, yet still general concepts about formal power series. We start by defining composition (Definition 2.1) and proving its order is interchangeable with summation, multiplication and that it is also associative (Lemma 2.3). We continue by defining the formal exponential (Definition 2.4) and proving the functional equation for the exponential (Lemma 2.5). We prove that all "re-composable" formal power series form a group with composition (2.6), which helps us define the formal logarithm (Definition 2.11) and prove its functional equation also (Theorem 2.12). We also define the formal derivative (Definition 2.8) and show that the usual identities concerning derivatives still hold in the context of formal power series (Lemma 2.9). We end the chapter with multiple extra examples to deepen the reader's understanding of the newly defined notions (Examples 2.13-2.17).

In Chapter 3 our goal is to prove the Lagrange-Jacobi four-square theorem. We start by introducing the Gauss binomial coefficient (Definition 3.2) and proving Gauss' binomial theorem (Theorem 3.5) and its infinite version, Euler's corollary (Lemma 3.7). We use these to prove Jacobi's triple product formula (Theorem 3.8).We then prove the Roger-Ramanujan identities (Theorem 3.12), applying Jacobi's triple product formula. Then we establish a technical result known as Jacobi's cubic formula (Theorem 3.14). Finally, we are ready to prove the four-square theorem (Theorem 3.1), which occupies the end of the chapter.

We are using the mentioned theorems and proofs from [1] throughout the thesis and we are expanding them, giving details and adding examples making the topic more accessible to an inexperienced reader.

The specific contribution of this thesis begins already on the first page, where

we include intuitive explanations of the established notions to help the reader, the best concrete example is the derivation of a suitable multiplication before its definition 1.3. Concerning Lemma 1.5, stating that the set of formal power series with the operations of addition and multiplication forms an integral domain, we write it in great detail, unlike in the source material, where most of the proof is deemed trivial and not given. We follow it up with an expanded example 1.6, where we show two different ways of computing a product to incentivise searching for tricks. In Lemma 1.9 about invertible elements, in the first part, we add the derivation of the formula for the inverse. In the second part, we greatly expand the proof, giving all the details, while in the source material the proof was presented inaccurately and with very few details. We add another example, 1.14, focusing again on the intuitive picture for our reader. In Lemma 1.16 we expand the estimate computation for greater clarity.
In Chapter 2, we have added an explanation of why composition (Definition 2.1) needs the assumptions it needs. We have greatly expanded the proof of Theorem 2.3 (three facts about composition), giving more details in the first two steps and adding the proof of (iii) which was only outlined in the source material. In the proof of Lemma 2.6, we have again added the derivation of the formula. In the proof of Lemma 2.9 about properties of the derivative, we give much more details, while in the source there are mainly outlines for the proofs. In the third and most difficult part of the proof, about the derivative of an infinite product, we have greatly expanded the proof to be fully detailed. We added Examples 2.13-2.16, which were left as an exercise in the source material. We also showed an example, 2.17, of using generating functions outlined in the applied part of the source material.
In Chapter 3 we add omitted details to proofs of Lemma 3.3 about Gauss polynomials, Gauss' binomial theorem (Theorem 3.5) and Euler corollary (Lemma 3.7). We clarify and detail the proof of Jacobi's triple product identity, Theorem 3.8. We add an explanation to the proof of Lemma 3.11, as well as give more intermediate results for maximum clarity. In Lemma 3.13 We fix a small problem in the source material at the beginning of the induction and again give more intermediate results for clarity. We give a more detailed computation in Jacobi's cubic formula, Theorem 3.14, mainly concerning the limit. In the end, we prove the four square theorem, Theorem 3.1 in much greater detail.

1. Definitions

We begin by defining the most important notions used in the thesis.

Definition 1.1. Let K be a commutative field. We define a formal power series over K as an infinite sequence $(a_0, a_1, a_2, ...)$ of terms from K. The set of all power series over K is denoted $K[[X]]$.

For most applications, we shall take $K = \mathbb{C}$, nevertheless, most of the statements apply to any field K. We define the addition term-wise. For formal power series $\alpha = (a_0, a_1, a_2, ...) \in K[[X]]$ and $\beta = (b_0, b_1, b_2, ...) \in K[[X]]$ we define

$$\alpha + \beta = (a_0 + b_0, a_1 + b_1, ...).$$

For $\lambda \in K$, we define

$$\lambda\alpha = (\lambda a_0, \lambda a_1, ...).$$

It is now clear that the set of the power series over K forms a vector space. We also identify each element a of K with the series $(a, 0, ...)$, calling it the constant series. Next, we want to define a multiplicative structure, which will turn the vector space into a ring of "infinite polynomials" over K. To do that, we define a notation similar to the one used with polynomials.

Definition 1.2. We introduce an indeterminant X and associate powers of X with formal power series in the following way:

$$\begin{aligned} X^0 &= (1, 0, 0, 0, ...) \\ X^1 &= (0, 1, 0, 0, ...) \\ X^2 &= (0, 0, 1, 0, ...) \\ X^3 &= (0, 0, 0, 1, ...). \end{aligned}$$

The last definition also shows the idea of defining multiplication everywhere on $K[[X]]$. The introduction of the formal X allows us to write series in the way we are used to with power series in analysis. For $\alpha \in K[[X]]$,

$$\alpha = (a_0, a_1, a_2, ...) = \sum_{n=0}^{\infty} a_n X^n.$$

We now only have to define the multiplication formally in such a way, that all of the above rules are satisfied, namely we need it to be linear and to keep the power property of X, that $X^n \cdot X^m = X^{n+m}$. A formula we can use that meets both is the multiplication formula for the polynomials (also known as the Cauchy formula) and simply extend it to infinity. Intuitively, it is the same process of extending polynomials into power series.

Definition 1.3. For any $\alpha, \beta \in K[[X]]$, we define

$$\alpha \cdot \beta = \sum_{k=0}^{\infty} \left(\sum_{n=0}^{k} a_n b_{k-n} \right) X^k.$$

We will soon prove that $K[[X]]$ forms an integral domain under the defined operations. As we intuitively understand, the integral domain of polynomials $K[X]$ is embedded into the domain of the power series $K[[X]]$ (which is also why we use such notation for the power series) since every polynomial $a_n X^n + ... + a_0$ can be extended into a series by adding zero coefficients: $\sum_{n=0}^{\infty} a_n X^n$ where $a_i = 0$ for every $i > n$, and the operations are identical. Just before proving the lemma, we will define a useful tool to describe formal power series.

Definition 1.4. For $\alpha \in K[[X]]$, we define

$$\inf(\alpha) := \inf\{n \in \mathbb{N}_0 | a_n \neq 0\}.$$

The infimum of a series is therefore simply the index of the first non-zero coefficient, and for 0 we have $\inf(\emptyset) = \infty$. Now we can easily prove that $K[[X]]$ is indeed an integral domain.

Lemma 1.5. Let K be a commutative field, then $K[[X]]$ with operations $(+, \cdot)$ is an integral domain.

Proof. We need to show the validity of the axioms. The axioms for $+$ are clearly fulfilled, which follows from the addition being the same as the addition in K. $1 = (1, 0, 0, ...)$ is obviously the element neutral towards multiplication, simply by

$$\alpha \cdot 1 = \sum_{k=0}^{\infty} \left(\sum_{n=0}^{k} a_k \delta_0(n-k) \right) X^k = \sum_{k=0}^{\infty} a_k X^k = \alpha,$$

where

$$\delta_0(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{otherwise.} \end{cases}$$

For the associativity, we just need to compare the results for $\alpha, \beta, \gamma \in K[[X]]$.

$$\begin{aligned}
(\alpha\beta)\gamma &= \left(\sum_{k=0}^{\infty} a_k X^k \sum_{k=0}^{\infty} b_k X^k \right) \sum_{k=0}^{\infty} c_k X^k \\
&= \left(\sum_{k=0}^{\infty} \left(\sum_{n=0}^{k} a_n b_{k-n} \right) X^k \right) \sum_{k=0}^{\infty} c_k X^k \\
&= \sum_{k=0}^{\infty} \left(\sum_{\ell=0}^{k} \left(\sum_{n=0}^{\ell} a_n b_{\ell-n} \right) c_{k-\ell} \right) X^k \\
&= \sum_{k=0}^{\infty} \left(\sum_{\substack{\ell,m,n \in \mathbb{N}_0 \\ \ell+m+n=k}} a_\ell b_m c_n \right) X^k, \\
\alpha(\beta\gamma) &= \sum_{k=0}^{\infty} a_k X^k \left(\sum_{k=0}^{\infty} b_k X^k \sum_{k=0}^{\infty} c_k X^k \right) \\
&= \sum_{k=0}^{\infty} a_k X^k \left(\sum_{k=0}^{\infty} \left(\sum_{n=0}^{k} b_n c_{k-n} \right) X^k \right) \\
&= \sum_{k=0}^{\infty} \left(\sum_{\ell=0}^{k} a_\ell \left(\sum_{n=0}^{k-\ell} b_n c_{k-\ell-n} \right) \right) X^k
\end{aligned}$$

$$= \sum_{k=0}^{\infty} \left(\sum_{\substack{\ell,m,n \in \mathbb{N}_0 \\ \ell+m+n=k}} a_\ell b_m c_n \right) X^k.$$

About commutativity of multiplication, the formula for $\alpha\beta$ and $\beta\alpha$ differs only in the order of finite summation and the order of multiplication in K, both of which are commutative, meaning our new multiplication is commutative as well:

$$\alpha\beta = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) X^n,$$
$$\beta\alpha = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} b_k a_{n-k} \right) X^n.$$

The inner sums are clearly equal.
The distributive property is also fulfilled because for $\alpha, \beta, \gamma \in K[[X]]$ we have

$$\begin{aligned}
\gamma(\alpha+\beta) &= \left(\sum_{n=0}^{\infty} c_n X^n \right) \left(\sum_{n=0}^{\infty} (a_n + b_n) X^n \right) \\
&= \left(\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} c_k (a_{n-k} + b_{n-k}) \right) X^n \right) \\
&= \left(\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} c_k a_{n-k} \right) X^n \right) + \left(\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} c_k b_{n-k} \right) X^n \right) \\
&= \gamma\alpha + \gamma\beta.
\end{aligned}$$

And lastly, to prove $\alpha, \beta \neq 0$ implies $\alpha\beta \neq 0$, we simply compute the first non-zero term. If $n = \inf(\alpha), m = \inf(\beta)$ (both α, β are non-zero so both n and m are finite), then

$$(\alpha\beta)_{n+m} = \left(\sum_{k=0}^{n+m} a_k b_{n+m-k} \right) = a_n b_m \neq 0,$$

since all the other coefficients are zero ($a_i = 0$ for all $i < n$, while $b_i = 0$ for all $i < m$). □

Example 1.6. All of the following are formal power series:

$$X^k \text{ for any k}, \qquad \sum_{n=0}^{\infty} X^n, \qquad 1, \qquad \sum_{n=0}^{\infty} (-1)^n n X^n, \qquad \sum_{n=0}^{\infty} X^{kn},$$

and we can compute the results of our defined operations. Often, there are ways to make our computations much easier:

$$(1-X) \sum_{n=0}^{\infty} X^n = \sum_{n=0}^{\infty} X^n - \sum_{n=0}^{\infty} X^{n-1} = 1,$$

$$\begin{aligned}
\left(\sum_{n=0}^{\infty} X^n \right) \left(\sum_{n=0}^{\infty} (-1)^n n X^n \right) &\overset{(!)}{=} \left(\sum_{n=0}^{\infty} X^n \right) \left((1-X)(-1+X) \left(\sum_{n=1}^{\infty} n X^{2n-1} \right) \right) \\
&= (-1+X) \left(\sum_{n=1}^{\infty} n X^{2n-1} \right) \\
&= \sum_{n=1}^{\infty} -n X^{2n-1} + \sum_{n=0}^{\infty} n X^{2n}
\end{aligned}$$

$$= -X + X^2 - 2X^3 + 2X^4 - ...$$

We find a clever way of writing one of the series, marked with (!). To check this is true, our reader is invited to distribute the factors. This trick makes the computation trivial — we will look for something similar whenever possible. Without the trick, we need to multiply using the Cauchy formula and might get a more difficult expression to work with. Recall the formula

$$\alpha \cdot \beta = \sum_{k=0}^{\infty} \left(\sum_{n=0}^{k} a_n b_{k-n} \right) X^k$$

and we plug our series in:

$$\left(\sum_{n=0}^{\infty} X^n \right) \left(\sum_{n=0}^{\infty} (-1)^n n X^n \right) = \sum_{k=0}^{\infty} \left(\sum_{n=0}^{k} 1 \cdot (-1)^{k-n} (k-n) \right) X^k.$$

It requires more calculation and often a complete inductive argument to simplify the inner sum to get the same result.
We can often use the Taylor series for certain known functions, but while working with formal power series we cannot substitute X with a number (it is only our way of writing the series!). We will however be able to substitute one series for X in the other, and there we could use certain functions for their analytical properties. A good example is the exponential and the logarithm, which will be defined in Chapter 2 since they turn sums into products and back.

Notation. In case we work with a sequence of formal power series α_n for $n \in \mathbb{N}$, we will be using $(\alpha_n)_k$ to denote the k-th coefficient of α_n.

We will continue with understanding some properties of invertible elements.

Definition 1.7. We call $\alpha \in K[[X]]$ invertible if there exists $\beta \in K[[X]]$ such that $\alpha\beta = 1$. More generally, we will also define $\gamma = \frac{\alpha}{\beta}$ if $\alpha = \beta\gamma$.

Example 1.8. From the previous example, we have immediately

$$\frac{1}{1-X} = \sum_{n=1}^{\infty} X^n,$$

$$\frac{\sum_{n=0}^{\infty} (-1)^n n X^n}{(1-X)(-1+X)} = \sum_{n=0}^{\infty} n X^{2n-1}.$$

We can also use some well-known formulas. For example, for $\alpha \in K[[X]] \setminus \{1\}$:

$$\frac{1-\alpha^n}{1-\alpha} = \sum_{k=0}^{n-1} \alpha^k.$$

The inverse β is uniquely determined. Indeed, if we had $\alpha\beta_1 = \alpha\beta_2 = 1$, it would also be true that $\alpha(\beta_1 - \beta_2) = 0$ and we know formal power series form an integral domain, so $\beta_1 - \beta_2 = 0$ and $\beta_1 = \beta_2$.
We will keep using the typical notation for ideals, namely for $\alpha \in K[[X]]$ we have (α) denoting the principal ideal generated by α. All of the invertible elements form a group — a result easy to prove since we have already had all of the axioms (from Lemma 1.5, showing $K[[X]]$ is an integral domain) except for inverses, which were just added. The fact it is closed over inverses is clear from $(\alpha\beta)^{-1} = \beta^{-1}\alpha^{-1}$. This group will be denoted $K[[X]]^{\times}$. Let us follow with a lemma concerning the properties of the invertible elements.

Lemma 1.9. For $\alpha = \sum_{n=0}^{\infty} a_n X^n \in K[[X]]$, α is invertible if and only if $a_0 \neq 0$. That gives us $K[[X]]^\times = K[[X]] \setminus (X)$. Moreover, if there exists such $m \in \mathbb{N}$ that $\alpha^m = 1$ then $\alpha \in K$.

Proof. We need to find $\beta = \sum_{n=0}^{\infty} b_n X^n$ such that $\alpha\beta = 1$. We will evaluate $\alpha \sum_{n=0}^{\infty} b_n X^n$ for variables $b_n, n \in \mathbb{N}$, which will give us a way to calculate the coefficients and in the end, the whole series β. It is clear in advance, that $b_0 = \frac{1}{a_0}$ in order to have the correct constant term. We follow with

$$\begin{aligned} \alpha \sum_{n=0}^{\infty} b_n X^n &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) X^n \\ &= \sum_{n=0}^{\infty} \delta_0(n) X^n. \end{aligned}$$

For this equality to hold, we need for every $n > 0$:

$$\begin{aligned} 0 &= \left(\sum_{k=0}^{n} a_k b_{n-k} \right) \\ &= a_0 b_n + \left(\sum_{k=1}^{n} a_k b_{n-k} \right) \end{aligned}$$

and expressing b_n gives us

$$b_n = \frac{-1}{a_0} \sum_{k=1}^{n} a_k b_{n-k}.$$

We have calculated b_0 ahead, and the previous calculation gives us a formula to inductively construct the whole series β, so we have found the inverse.
We will now show $\alpha^m = 1$ is equivalent to $\alpha = 1$. The main body of the proof shows that for any p prime, if $\alpha^p \in K$, then $\alpha \in K$. We will now show that this is indeed all we need. Let $n = p_1^{k_1} p_2^{k_2} \ldots p_\ell^{k_\ell}$ be the prime factorisation of n. Induction on $\sum_{i=0}^{\ell} k_i$, the sum of exponents of the primes, will give us the same theorem for all $m \in \mathbb{N}$.
We continue by showing that if $\alpha^n = m_n \in K$ for some $n \in \mathbb{N}$, then for some $q \in \mathbb{N}$ we will have $\alpha_m^q = 1$. Assume $\alpha^m = 1$ for some $m \in \mathbb{N}$. If any m_n has a finite order q, then $(\alpha^n)^q = 1$. If m_n has infinite order for every n, it contradicts the assumption.
We now only have to prove that $\alpha^p \in K$ implies $\alpha \in K$, which we will do by contradiction. Let $\alpha^p = 1$ and $\alpha \notin K$ and denote by $n = \inf(\alpha - a_0)$, the lowest non-zero power of X with a non-zero coefficient. We will use the n-th coefficient of α^p for contradiction. The coefficient will be $p a_0^{n-1} a_n$, since it is the only term from the binomial theorem which does not get cancelled out by $a_1 = a_2 = a_3 = \ldots = a_{n-1} = 0$ and we will aim to prove $(\alpha^p)_n$ can't be zero. We realise, that $1 = \alpha^p = \alpha\alpha^{p-1}$ so α is invertible and so a_0 is therefore non-zero by the first part of the theorem. a_n is also non-zero by definition of n. That leaves the only option to be $p = 0$, so the characteristic of K is p.
With this knowledge, we move to the coefficient $(\alpha^p)_{pn}$. Since all of the binomial coefficients of $(a + bX)^p$ are divisible by p except for the first and the last, we can use the simplified formula $(a + bX)^p = a^p + b^p X^p$. This simplifies even more for our zero coefficients and gives us inductively

$$(a_0 + a_n X^n + \ldots a_{pn} X^{pn})^p = \left(\ldots ((a_0 + a_n X^n) + a_{n+1} X^{n+1}) + \ldots) + a_{pn} X^{pn} \right)^p$$

$$= \Big(...((a_0 + a_n X^n) + a_{n+1} X^{n+1}) + ...) + a_{pn-1} X^{pn-1}\Big)^p + (a_{pn} X^{pn})^p$$
$$= ... = a_0^p + a_n^p X^{pn} + ... + a_{pn}^p (X^{pn})^p.$$

The coefficient of $(\alpha^p)_{pn} = a_n^p \neq 0$, so $\alpha^p \neq 1$ and the proof is complete. □

We follow up with defining an analytic structure on $K[[X]]$. Let us start by defining the norm.

Definition 1.10 (Norm)**.** For any $\alpha \in K[[X]]$ we define

$$|\alpha| = 2^{-\inf(\alpha)}$$

and we define $|0| = 0$.

Example 1.11. Here are some examples of the norms of power series in K[[X]] for $k \neq 0 \in K$.

$$|k| = 1, \qquad |k + X + X^2| = 1, \qquad |X^5 + kX^6 + ...| = 2^{-5}.$$

The norm simply takes the index of the first non-zero coefficient and returns the value of a chosen strictly decreasing function with [0,1] being its domain (here we chose $f(n) = 2^{-n}$) in that index. Intuitively, comparing the absolute values of two sums is similar to comparing their values after substituting very small ϵ for X. We will now prove two lemmas about the structure defined by this new norm.

Lemma 1.12 (Formal power series form an ultrametric space)**.** For $\alpha, \beta \in K[[X]]$, it holds:

(i) $|\alpha| \geq 0$, and if $|\alpha| = 0$ then $\alpha = 0$.

(ii) $|\alpha\beta| = |\alpha||\beta|$.

(iii) $|\alpha + \beta| \leq \max\{|\alpha|, |\beta|\}$, with equality if $|\alpha| \neq |\beta|$.

Proof. (i) The definition clearly states $|\alpha| \in \mathbb{R}^+$ for any $\alpha \neq 0$.

(ii) We have already proven $|\alpha\beta| \geq |\alpha||\beta|$ in Lemma 1.5. On the other hand, with $\inf(\alpha) = n, \inf(\beta) = m$, it is clear that $(\alpha\beta)_i = 0$ for any $i < m + n$, since the coefficient will be a product of a binomial coefficient, α_k and β_ℓ where $k + \ell = i$, but then either $k < m$ or $\ell < n$ and so one of the factors will be zero.

(iii) We only need to realise that if $a_n + b_n \neq 0$, then either $a_n \neq 0$ or $b_n \neq 0$. Therefore $\inf(\alpha + \beta) \geq \max\{\inf(\alpha), \inf(\beta)\}$. Rewriting in terms of norms, this already gives us $|\alpha + \beta| \leq \max\{|\alpha|, |\beta|\}$. If $|\alpha| > |\beta|$, then $a_{\inf(\alpha)} + b_{\inf(\alpha)} = a_{\inf(\alpha)} + 0 \neq 0$, so $\inf(\alpha + \beta) = \inf(\alpha)$ and rewritten with norms we get $|\alpha + \beta| = |\alpha|$. □

Theorem 1.13 (Formal power series form a complete metric space)**.** $(K[[X]], d)$ *is a complete metric space for a metric d given by $d(\alpha, \beta) = |\alpha - \beta|$.*

Proof. Using the previous lemma, it is easy to see that for any $\alpha, \beta \in K[[X]]$ we have $d(\alpha,\beta) = |\alpha - \beta| \geq 0$ with equality for $\alpha = \beta$. To prove d is a norm we are only left with the triangle inequality, for $\alpha, \beta, \gamma \in K[[X]]$ we have

$$\begin{aligned} d(\alpha,\gamma) &= |\alpha - \gamma| \\ &= |\alpha - \beta + \beta - \gamma| \\ &\overset{1.12(iii)}{\leq} \max(|\alpha - \beta|, |\beta - \gamma|) \\ &\leq |\alpha - \beta| + |\beta - \gamma| \\ &= d(\alpha,\beta) + d(\beta,\gamma). \end{aligned}$$

We are only using the ultrametric identity, which we have proven in the last lemma. All that is left for us to prove is the completeness.
Let $\alpha_m = \sum_{n=0}^{\infty} (\alpha_m)_n X^n$ be a Cauchy sequence. In our norm, this means that for any index k there exists m_k, such that for every $m \geq m_k$ it holds $(\alpha_m)_k = (\alpha_{m_k})_k$ since otherwise it would mean $d(\alpha_{m_k}, \alpha_m) \geq 2^{-k}$. If there was no such m_k, this would be a contradiction to α_m being a Cauchy sequence.
We set $m'_k = \max\limits_{i \leq k}(m_i)$, since then for all series for $i, j > m'_k$ it is clear that $d(\alpha_i, \alpha_j) < 2^{-k}$, because the first k coefficients must all be equal. We continue by defining a new series β with coefficients $b_k = (\alpha_{m_k})_k$. We then have $|\beta - \alpha_{m_k}| < 2^{-k}$ for any $k \in \mathbb{N}$, which proves β is the limit we were looking for. □

Example 1.14. Our metric works only with the first nonzero term of the series, the term with the lowest power of X. This is why we can easily work with series that have infinite power of X and/or are divergent as real power series, and try to have their "ugly parts" cancel out. Lemma 1.9 gives that $\alpha \in K[[X]]$ is invertible if and only if its constant term is nonzero. We now know it is equivalent to its norm being one, so our norm meets this basic algebraic connection of norm and invertibility.

This shows us what the basic structure of the formal power series is - it is an analytic completion of the algebraic structure of polynomials. Since we are in a metric space, we immediately have for $\alpha, \beta, \alpha_n \in K[[X]]$:

$$\lim_{n\to\infty}(\alpha + \beta) = \lim_{n\to\infty} \alpha + \lim_{n\to\infty} \beta, \qquad \lim_{n\to\infty}(\alpha \cdot \beta) = \lim_{n\to\infty} \alpha \cdot \lim_{n\to\infty} \beta,$$

$$\sum_{n=1}^{\infty} \alpha_n = \alpha \implies \alpha_n \to 0,$$

as given in [2]. Since we work with such sequences often, we name them.

Definition 1.15 (Null-sequence)**.** Let $\alpha_1, \alpha_2, ...$ be a sequence in K[[X]]. We call $\{\alpha_n\}$ a null sequence if we have

$$\lim_{n\to\infty} \alpha_n = 0.$$

However, with the metric we are using we can prove a stronger theorem about sums of series, because the implication above turns out to be an equivalence in $K[[X]]$.

Lemma 1.16 (Convergence of sums of power series)**.** For $\alpha_1, \alpha_2, ... \in K[[X]]$ the following are equivalent:

(i) The sum $\sum_{n=1}^{\infty} \alpha_n$ is well-defined and converges to $\alpha \in K[[X]]$.

(ii) The sequence $\{\alpha_n\}_{n=1}^{\infty}$ is a null sequence.

Furthermore, (ii) also implies $\prod_{n=1}^{\infty}(1+\alpha_n)$ is well defined and converges in $K[[X]]$.

Proof. $(i) \implies (ii)$ is proven in any metric space, for example in [2] on page 132 (the more general version is completely analogous).
$(ii) \implies (i)$ We will show that the partial sums form a Cauchy sequence since completeness will immediately give us the existence of the limit. For a fixed $\epsilon > 0$ we find such $N \geq 0$, that $|\alpha_k| < \epsilon$ for all $k \geq N$. Then for $k > \ell \geq N$, with the use of the ultrametric identity from Lemma 1.12, we have

$$\left|\sum_{n=1}^{k} \alpha_n - \sum_{n=1}^{\ell} \alpha_n\right| = \left|\sum_{n=\ell+1}^{k} \alpha_n\right| \overset{1.12(iii)}{\leq} \max_{l<i\leq k}|a_i| < \epsilon.$$

The product convergence is shown similarly:

$$\begin{aligned}\left|\prod_{n=1}^{k}(1+\alpha_n) - \prod_{n=1}^{\ell}(1+\alpha_n)\right| &= \left|\prod_{n=1}^{\ell}(1+\alpha_n)\right|\left|\prod_{n=\ell+1}^{k}(1+\alpha_n) - 1\right| \\ &= \prod_{n=1}^{\ell}|1+\alpha_n|\left|\prod_{n=\ell+1}^{k}(1+\alpha_n) - 1\right| \\ &\leq \left|\prod_{n=\ell+1}^{k}(1+\alpha_n) - 1\right|.\end{aligned}$$

We can eliminate the first product since its norm is ≤ 1. We can now rewrite the finite product as the sum of products of α_n and use the ultrametric triangle inequality again.

$$\begin{aligned}\left|\prod_{n=\ell+1}^{k}(1+\alpha_n) - 1\right| &= |(1+\alpha_{\ell+1})(1+\alpha_{\ell+2})...(1+\alpha_k) - 1| \\ &= |1 + (\alpha_{\ell+1} + ... + \alpha_k) + (\alpha_{\ell+1}\alpha_{\ell+2} + ... + \alpha_{k-1}\alpha_k) \\ &\quad + ... + (\alpha_{\ell+1}\cdots\alpha_k) - 1| \\ &= \left|1 + \left(\sum_{i_{\ell+1}=\ell+1}^{k} \alpha_{i_{\ell+1}}\right) + \left(\sum_{\substack{i_{\ell+1},i_{\ell+2}=\ell+1 \\ i_{\ell+1}<i_{\ell+2}}}^{k} \alpha_{i_{\ell+1}}\alpha_{i_{\ell+2}}\right)\right. \\ &\quad \left. + ... + \left(\sum_{\substack{i_{\ell+1},...,i_k=\ell+1 \\ i_{\ell+1}<...<i_k}}^{k} \alpha_{i_{\ell+1}}\cdots\alpha_{i_{\ell+1}}\right) - 1\right| \\ &= \left|\sum_{\emptyset\neq I\subseteq\{\ell+1,...,k\}} \prod_{i\in I}\alpha_i\right|\end{aligned}$$

$$\overset{1.12(iii)}{\leq} \max_{\emptyset \neq I \subseteq \{\ell+1,\ldots,k\}} \left| \prod_{i\in I} \alpha_i \right| \tag{1.1}$$
$$\overset{1.12(ii)}{\leq} \max_{\ell < i \leq k} |\alpha_i| < \epsilon.$$

We can simplify the expression by removing the product, since $|\alpha_i \alpha_j| = |\alpha_i||\alpha_j|$ (by Lemma 1.12(ii)) and all the series' norms are less than or equal to 1, so $|\alpha_i \alpha_j| \leq \max\{|\alpha_i|, |\alpha_j|\}$. ☐

For null sequences $\{\alpha_n\}, \{\beta_n\} \in K[[X]]$, and $\gamma \in K$, we also have $\sum_{n=1}^{\infty} \alpha_n + \sum_{n=1}^{\infty} \beta_n = \sum_{n=1}^{\infty}(\alpha_n + \beta_n)$. Then $\{\alpha_n + \beta_n\}$ is a null sequence, because $|\alpha_i + \alpha_j| \leq \max\{|\alpha_i|, |\alpha_j|\}$. We also have $\gamma \sum_{n=1}^{\infty} \alpha_n = \sum_{n=1}^{\infty} \gamma\alpha_n$, where the sum clearly converges, because scalar multiplication does not change the norm.
Moreover, for any null sequence $\{\alpha_n\}$ in $K[[X]]$, the convergent sum of its terms does not depend on the order of summation. For any fixed $\epsilon > 0$, there are only finitely many terms with a norm greater than ϵ. For any bijection $\pi : \mathbb{N} \to \mathbb{N}$, there will exist such N that $\alpha_{\pi(n)} \leq \epsilon$ for every $n > N$. We will simply find the highest index of the images of the terms with a norm higher than ϵ and choose a higher N. Therefore $\{\alpha_{\pi(n)}\}$ will be a null sequence for any such bijection, and by the last theorem (Theorem 1.16) it will be convergent.
The result also has to be the same, since for the k-th coefficient of the sum there is only a finite number of summands, they come from the terms $|\alpha_i| > 2^{-k}$. We will be using this to great effect by often swapping the order of summation and/or reindexing of sums.

Example 1.17. From this moment on we will be working with infinite sums and products of our power series. We would like to warn the reader, that both the sum of power series and the sum denoting a single power series are written the same way, the only difference is whether the index belongs to a whole new power series or just to some power of X. It is therefore important to always understand, which sum we are dealing with.
For any α in $K[[X]]$ such that $a_0 = 0$ we have $|\alpha^n| \leq 2^{-n}$, so the sequence $\{\alpha^n\}_{n=0}^{\infty}$ is a null sequence. Therefore the following sum converges and we can write

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}. \tag{1.2}$$

We can also understand the value of certain sums or products by understanding what they mean, for example, the product

$$\prod_{n=0}^{\infty} (1 + X^{2^n}).$$

Intuitively, we understand that this will give those powers of X, which can be written as a sum of powers of 2. That, however, is every number, exactly once! So this product will be equal to

$$\prod_{n=0}^{\infty} (1 + X^{2^n}) = \sum_{n=0}^{\infty} X^n = \frac{1}{1-X}.$$

Fully formally, the same result can also be reached from the following calculation using a trick multiplying both the numerator and the denominator with the right term:

$$\begin{aligned}\prod_{n=0}^{\infty}(1+X^{2^n}) &= \prod_{n=0}^{\infty}\frac{(1+X^{2^n})(1-X^{2^n})}{1-X^{2^n}} \\ &= \prod_{n=0}^{\infty}\frac{1-X^{2^{n+1}}}{1-X^{2^n}} \\ &= \frac{1}{1-X}.\end{aligned}$$

because this is a telescopic product — the n-th numerator cancels out with $n+1$-st denominator.

2. Useful notions concerning formal power series

In this chapter, we will be defining other useful notions concerning formal power series and proving certain simpler theorems and lemmas about them. Namely, we will define the composition of power series and prove it "behaves well". We define the formal exponential and the formal logarithm and prove some of their important properties. We will also define the formal derivative and prove it meets all of the properties we would expect. Even though these notions and their connected lemmas have their own value, we will be using them mainly as tools in the final chapter in the more complex theorems. The chapter ends with more examples, showing the use of all newly defined notions.
First of all, we will formalize "plugging a polynomial into a series". The idea that, such as with finite polynomials, we can substitute and reverse-substitute a power of X for X. We will also define plugging a power series into a polynomial, or even into another power series if it meets the required criteria. This allows for a much more diverse set of operations we can compute with the formal power series.

Definition 2.1. For power series α, β, denoted $\alpha = \sum_{n=0}^{\infty} a_n X^n \in K[[X]]$ and $\beta = \sum_{n=0}^{\infty} b_n X^n \in K[[X]]$, where either $\alpha \in K[X]$ or $\beta \in (X)$, we define

$$\alpha \circ \beta = \alpha(\beta) = \sum_{n=0}^{\infty} a_n \beta^n.$$

The first case uses α as a polynomial, meaning we are only using operations of finite multiplication and addition of power series, which are well-defined. The second case is β being a series without a constant term. In that case, our new definition uses the Lemma (1.16), speaking about the convergence of sums of power series. Since β lacks a constant term, we know

$$\begin{aligned}
||\beta^n|| &= 2^{-\inf(\beta^n)} \\
\inf(\beta^n) &= \inf\left(\left(\sum_{k=1}^{\infty} b_k X^k\right)^n\right) \\
&= \inf\left(\left(b_{\inf(\beta)} X^{\inf(\beta)}\right)^n\right) \\
&= \inf\left(b_{\inf(\beta)}^n X^{n\inf(\beta)}\right) \\
&\geq n.
\end{aligned}$$

Therefore the sequence $\alpha_n \beta^n$ is a null sequence, and Lemma 1.16 gives the sum of these terms is well-defined.

Example 2.2. For our favourite sum $\sum_{n=0}^{\infty} X^n$ we can compose it with kX^2 to get $\sum_{n=0}^{\infty} X^n \circ kX^2 = \sum_{n=0}^{\infty} k^n X^{2n}$.
For any $\alpha \in K[[X]]$, we can also compose $\alpha(0)$, which will give us only the constant term (0 is clearly in (X), the operation is therefore well-defined).
We cannot however compute $\alpha(1)$ since 1 does not lie in (X) and so goes against our definition. If we want to use any number from K, we would have to use only a finite polynomial for this operation to make sense.

This is the formal reasoning, but what is the deeper reason we can use only these series? The first case is clear, we work with a polynomial, there everything we need is well-defined. The second case can be viewed as a sum of the sequence $\{a_n X^n\}$. They are polynomials, so we can compose any β with them — but then we need the resulting sum to be well-defined. This happens if and only if the sequence forms a null sequence. For it to be so however, we can't have any constant terms since $(1+\beta)^n = 1 + ...$ and the norm would remain one for every power of X. For any other $\beta \in (X)$, we have $|X| \geq |\beta|$ (the infimum of β is either one for the equality, or more for the inequality). That way the norm of each of the terms will be at most the same as the norm of the corresponding X^n. Those form a null sequence though, which means our own new sum's terms form a null sequence and the sum is well-defined.
We will continue by proving some simple facts about composition.

Lemma 2.3 (Facts about composition)**.** For $\{\alpha_n\}$ a null sequence of power series in $K[[X]]$ and for $\alpha, \beta, \gamma \in (X)$, it holds that

(i)
$$\left(\sum_{n=0}^{\infty} \alpha_n\right) \circ \beta = \sum_{n=0}^{\infty} (\alpha_n \circ \beta),$$

(ii)
$$\left(\prod_{n=0}^{\infty} (1+\alpha_n)\right) \circ \beta = \prod_{n=0}^{\infty} (1+\alpha_n(\beta)),$$

(iii)
$$(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma).$$

Proof. We first have to show all of the expressions are well-defined. The left-hand side of (i) is seen immediately from Lemma 1.16 because α_n is a null sequence. For the right-hand side, we need to realise that

$$|\alpha_n(\beta)| = |\alpha_n(b_1 X + b_2 X^2 + ...)| \leq |\alpha_n b_{\inf(\beta)} X^{\inf(\beta)}| \leq |\alpha_n| \to 0. \tag{2.1}$$

so $\alpha_n(\beta)$ is a null sequence if α_n is a null sequence, which we have from the assumption. Therefore, the right-hand side is well-defined again from Lemma 1.16 about sums of series.
For (ii), we will use the last part of the same lemma. The left-hand side is precisely the lemma, while the right-hand side uses again the calculation (2.1) before applying the lemma.
In (iii), the left-hand side is clearly well-defined ($\beta \in (X)$ and $\gamma \in (X)$). We need to show $(\beta \circ \gamma)_0 = 0$. Neither of the series has a non-zero constant term, so we have

$$\begin{aligned}\beta \circ \gamma = \sum_{n=0}^{\infty} b_n \gamma^n &= 0 + b_{\inf(\beta)} \gamma^{\inf(\beta)} + O(X^{\inf(\beta)\inf(\gamma)+1}) \\ &= b_{\inf(\beta)} c_{\inf(\gamma)} X^{\inf(\beta)\inf(\gamma)} + O(X^{\inf(\beta)\inf(\gamma)+1}),\end{aligned}$$

where $O(X^n)$ denotes any terms with the power of X greater than or equal to n. Since both $\inf(\beta) \geq 1$ and $\inf(\gamma) \geq 1$, $(\beta \circ \gamma)_0 = 0$ and $\alpha \circ (\beta \circ \gamma)$ is well-defined

as well.
We have now proven all expressions of all identities are well defined, so we can proceed to proving the identities.
(i) We will start by considering a rearrangement of the sum. Since α_n is a null sequence, for every $k \in \mathbb{N}$ the number of terms α_n with non-zero k-th coefficient must be finite (otherwise $|\alpha_n| \geq 2^{-k}$), so we can write the sum as

$$\left(\sum_{k=1}^{\infty} \alpha_k\right) \circ \beta = \left(\sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} (\alpha_k)_n\right) X^n\right) \circ \beta.$$

The inner sum, though indexed to infinity, is therefore finite — we can use the definition of composition and then change the order of summation back to the original one.

$$\begin{aligned}
\left(\sum_{k=1}^{\infty} \alpha_k\right) \circ \beta &= \left(\sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} (\alpha_k)_n\right) X^n\right) \circ \beta \\
&= \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} (\alpha_k)_n\right) \beta^n \\
&= \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} (\alpha_k)_n \beta^n\right) \\
&= \sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} (\alpha_k)_n \beta^n\right) \\
&= \sum_{k=1}^{\infty} (\alpha_k \circ \beta),
\end{aligned}$$

and we have proven (i).
(ii) We will now show the equality holds for two series.

$$\begin{aligned}
(\alpha_1 \alpha_2) \circ \beta &= \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} a_k b_{n-k}\right) X^n \circ \beta \\
&= \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} a_k b_{n-k}\right) \beta^n \\
&= \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} a_k b_{n-k} \beta^n\right) \\
&= \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} (a_k \beta^k)(b_{n-k} \beta^{n-k})\right) \\
&= \alpha_1(\beta) \alpha_2(\beta).
\end{aligned}$$

This establishes the equality for any finite amount by induction. Next, we use the assumption that $\{\alpha_n\}$ is a null sequence and follow up by taking the limit. For $m \in \mathbb{N}$ fixed we compute

$$\begin{aligned}
&\left| \left(\prod_{k=1}^{\infty} (1 + \alpha_k(\beta))\right) - \left(\prod_{k=1}^{m} (1 + \alpha_k)\right) \circ \beta \right| \\
&\qquad = \left| \left(\prod_{k=1}^{\infty} (1 + \alpha_k(\beta))\right) - \left(\prod_{k=1}^{m} (1 + \alpha_k(\beta))\right) \right|
\end{aligned}$$

$$= \left|\prod_{k=1}^{m}(1+\alpha_k(\beta))\right| \left|\prod_{k=m+1}^{\infty}\Big(1+\alpha_k(\beta)\Big) - 1\right|$$
$$\overset{(1.1)}{=} \left|\prod_{k=1}^{m}(1+\alpha_k(\beta))\right| \max_{k>m} |\alpha_k(\beta)| \overset{m\to\infty}{\longrightarrow} 0.$$

We used the same calculation as in the proof of Lemma 1.16. The limit for $m \to \infty$ gives us the theorem.
(iii) can be proven using the first two parts. We will write α as a sequence of terms $a_n X^n$, and we will write each of these terms as $a_n \prod_{k=1}^{n} X$. We use the previous parts, which say that we can compose first and sum (i) and multiply (the finite version of (ii)) second. We can calculate

$$\begin{aligned}
(\alpha \circ \beta) \circ \gamma &= \left(\sum_{n=1}^{\infty} a_n X^n \circ \beta\right) \circ \gamma \\
&= \left(\sum_{n=1}^{\infty} a_n \left(\prod_{k=1}^{n} X \circ \beta\right)\right) \circ \gamma \\
&= \sum_{n=1}^{\infty} a_n \prod_{k=1}^{n} \sum_{\ell=0}^{\infty} b_\ell X^\ell \circ \gamma \\
&= \sum_{n=1}^{\infty} a_n \prod_{k=1}^{n} \sum_{\ell=0}^{\infty} b_\ell \gamma^\ell, \\
\alpha \circ (\beta \circ \gamma) &= \sum_{n=1}^{\infty} a_n X^n \circ \left(\sum_{k=0}^{\infty} b_k \gamma^k\right) \\
&= \sum_{n=1}^{\infty} a_n \prod_{k=1}^{n} X \circ \left(\sum_{k=0}^{\infty} b_k \gamma^k\right) \\
&= \sum_{n=1}^{\infty} a_n \prod_{k=1}^{n} \sum_{\ell=0}^{\infty} b_\ell \gamma^\ell. \qquad \square
\end{aligned}$$

Now armed with this new knowledge we can define and use the exponential function, defined the same way as we know it.

Definition 2.4 (Exponential function)**.** For K a field of characteristic 0, we define
$$\exp(X) = \sum_{n=0}^{\infty} \frac{X^n}{n!}.$$

In the study of formal power series, it is useful mainly for its property of turning sums into products, which will be immediately proven in the next lemma.

Lemma 2.5 (Functional equation)**.** For $\{\alpha_n\} \in (X) \subset \mathbb{C}[[X]]$ a null sequence we have
$$\exp\left(\sum_{k=0}^{\infty} \alpha_k\right) = \prod_{k=0}^{\infty} \exp(\alpha_k).$$

Proof. We first check both sides of the equation are well-defined. The left-hand side is simple since none of the series have a constant term, the sum won't either, and the exponential is then defined as $1 + \alpha + \frac{\alpha^2}{2} + ...$, a sum of a null sequence, so it is well-defined. As for the right-hand side, we have

$$\prod_{k=0}^{\infty} \exp(\alpha_k) = \prod_{k=0}^{\infty} \left(1 + \alpha_k + \frac{\alpha_k^2}{2} + ...\right) = \prod_{k=0}^{\infty} \left(1 + \left(\alpha_k + \frac{\alpha_k^2}{2} + ...\right)\right),$$

and the inner bracket in the product forms a null sequence since its norm is the same as the norm of α_k. By Lemma 1.16 the product is well-defined.
We will prove the equality in two steps, we first prove it for two summands, which by induction gives us the equality for finite numbers of terms and then we take the limit:

$$\begin{aligned}\exp(\alpha + \beta) = \sum_{n=0}^{\infty} \frac{(\alpha+\beta)^n}{n!} &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{k}{n} \frac{\alpha^k \beta^{n-k}}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{\alpha^k \beta^{n-k}}{k!(n-k)!} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{\alpha^k}{k!} \frac{\beta^{n-k}}{n-k!} \\ &= \exp\alpha \exp\beta.\end{aligned}$$

Finally, for a fixed $n \in \mathbb{N}$ we calculate

$$\begin{aligned}\left|\prod_{k=0}^{\infty} \exp(\alpha_k) - \exp\left(\sum_{k=0}^{n} \alpha_k\right)\right| &= \prod_{k=0}^{n} |\exp(\alpha_k)| \left|\prod_{k=n+1}^{\infty} \exp(\alpha_k) - 1\right| \\ &= \prod_{k=0}^{n} \left|1 + \alpha_k + \alpha_k^2 + ...\right| \left|\prod_{k=n+1}^{\infty} \alpha_k + \alpha_k^2 + ...\right| \\ &= \max_{k>n} |\alpha_k| \overset{n\to\infty}{\longrightarrow} 0.\end{aligned}$$

The norm of the product from 1 to n is 1 since all of the factors have a constant term, and for the remaining product we have used the same estimate for the norm as at the beginning, where we showed the product is well-defined. Taking the limit as n goes to infinity gives us the theorem. □

Specially, from this lemma we also get $\exp(kX) = \exp(X)^k$ for $k \in \mathbb{N}_0$, simply by writing $\exp(kX) = \exp(X + X + X + X + ...) = \exp(X)\exp(X)... = \exp(X)^k$.

Theorem 2.6. *The set of $K[[X]]^\circ$, defined as $K[[X]]^\circ = (X) \setminus (X^2)$, forms a group with the operation of composition.*

Proof. We begin with the realisation that $K[[X]]^\circ$ is closed under $\circ$ since for $\alpha, \beta \in K[[X]]^\circ$, $\alpha(\beta) = \sum_{n=0}^{\infty} \alpha_n \beta^n$ and the first coefficient will be $\alpha_1\beta_1 \neq 0$.
$\circ$ is also associative by Lemma 2.3(iii)
It is also simple to see that X is the identity element, as $\alpha \circ X = X \circ \alpha = \alpha$ for any $\alpha \in K[[X]]^\circ$. We will just have to find the inverses. Since we have already shown $K[[X]]^\circ$ is a monoid, we only need to find inverses from one side since they already have to be the unique inverse we're looking for (as proven in [3] on page 2).

We will be searching for $\beta = \sum b_n X^n$, such that $\beta \circ \alpha = X$. It is clear we need $b_0 = 0$, otherwise β would not be in $K[[X]]^\circ$. We also have immediately $b_1 = \frac{1}{\alpha_1}$ since $(\alpha \circ \beta)_1 = a_1 b_1 = 1$. Next, we express b_n in terms of $\alpha_{n,k}$, the k-th coefficient of α^n. With that, we compute

$$\begin{aligned}
\beta \circ \alpha &= \sum_{n=1}^{\infty} b_n \alpha^n \\
&= \sum_{n=1}^{\infty} b_n \left(\sum_{k=1}^{\infty} a_k X^k \right)^n \\
&= \sum_{n=1}^{\infty} b_n \left(\sum_{k=1}^{\infty} \alpha_{n,k} X^k \right) \\
&= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left(b_n \alpha_{n,k} \right) X^k.
\end{aligned}$$

We continue with changing the order of summation, after which we show the inner sum only has finitely many non-zero summands.

$$\begin{aligned}
\beta \circ \alpha &= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left(b_n \alpha_{n,k} \right) X^k \\
&= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \left(b_n \alpha_{n,k} \right) X^k \\
&= \sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} b_n \alpha_{n,k} \right) X^k \\
&= \sum_{k=1}^{\infty} \left(\sum_{n=1}^{k} b_n \alpha_{n,k} \right) X^k.
\end{aligned}$$

We realise $\alpha_{n,k} = 0$ for $n > k$ since by definition it would mean it is a product of more terms than what the power of X is — so one of the factors is a_0, which we know is zero. That allows us to only sum the inner sum up to k, all the other terms will be cancelled out by these zeroes. We want to show the final expression is equal to X, so we will set b_n in such a way, that the inner sum is zero for $k > 1$. We already have b_0, b_1 and we define b_k so that

$$\sum_{n=1}^{k} b_n \alpha_{n,k} = 0.$$

Concretely, that means

$$\sum_{n=1}^{k-1} b_n \alpha_{n,k} = -b_k \alpha_{k,k} \qquad \Longrightarrow \qquad b_k = -\frac{1}{\alpha_{k,k}} \sum_{n=1}^{k-1} b_n \alpha_{n,k},$$

and we define b_n inductively for every $n > 1$. For $\beta = \sum_{n=0}^{\infty} b_n$ we then really have

$$\beta \circ \alpha = \sum_{k=1}^{\infty} \left(\sum_{n=1}^{k} b_n \alpha_{n,k} \right) X^k = X + \sum_{k=2}^{\infty} 0 X^k = X. \qquad \square$$

Example 2.7. It is typically very difficult to find the precise expression for this inverse to composition, in [1] called the reverse. One of the simple examples is, again, the geometric series, this time without the constant term. A nice way to avoid the largest chunk of complicated calculations is to use the geometric formula

$$\sum_{n=1}^{\infty} X^n = \frac{X}{1-X},$$

since then we can calculate the reverse by swapping X and α and expressing α.

$$X = \frac{\alpha}{1-\alpha} \implies \alpha = \frac{X}{1+X}$$

Indeed,

$$\begin{aligned}
\sum_{n=1}^{\infty} X^n \circ \frac{X}{1+X} &= \frac{X}{1-X} \circ \frac{X}{1+X} \\
&= \frac{\frac{X}{1+X}}{1-\frac{X}{1+X}} \\
&= \frac{\frac{X}{1+X}}{\frac{1}{1+X}} \\
&= X.
\end{aligned}$$

We will further continue with defining another useful tool, the formal derivative.

Definition 2.8. For any $\alpha \in K[[X]]$ we define

$$\alpha' = \sum_{n=1}^{\infty} n a_n X^{n-1}.$$

as the formal derivative of α, and inductively the n-th derivative $\alpha^{(n)} = \alpha''''...$ n times.

This is an amazing way of getting the coefficients of the power series since $\alpha^{(n)}(0) = a_n n!$. Moreover, this gives us another option of finding and proving more difficult identities, when we prove the basic properties of the derivative.

Lemma 2.9 (Properties of the derivative)**.** For any $\alpha, \beta \in \mathbb{C}[[X]]$ and a null sequence $\{\alpha_k\}$ it holds

(I) $(\sum_{k=0}^{\infty} \alpha_k)' = (\sum_{k=0}^{\infty} \alpha_k')$,

(II) $(\alpha\beta)' = \alpha'\beta + \alpha\beta'$,

(III) $(\prod_{k=0}^{\infty}(1+\alpha_k))' = \prod_{k=0}^{\infty}(1+\alpha_k)\sum_{k=0}^{\infty}\frac{\alpha_k'}{1+\alpha_k}$,

(IV) $\left(\frac{\alpha}{\beta}\right)' = \frac{\alpha'\beta - \alpha\beta'}{\beta^2}$ if $\beta_0 \neq 0$,

(V) $(\alpha \circ \beta)' = \alpha'(\beta)\beta'$.

We will be using the lemma repeatedly throughout the thesis, which is why we have chosen capital Roman numerals to denote it. It is the only lemma using them to help the reader identify the use of this lemma.
To prove the lemma, we just need to write the power series as a sum $\sum a_n X^n$ and apply the definition of the formal derivative. Then the only part where we actually need to be clever is the infinite product one, (III).

Notation. Whenever we use induction in proofs, we will use the abbreviation IS for induction step over equal signs in equations.

Proof. (I) We sum the n-th coefficients of k series. As we have shown in the proof of Lemma 2.3 (i), the sum $(\sum_{k=0}^{\infty}(\alpha_k)_n)$ actually only has a finite amount of non-zero terms and so we can take the formal derivative of the following sum:

$$\left(\sum_{k=0}^{\infty} \alpha_k\right)' = \left(\sum_{n=0}^{\infty}\left(\sum_{k=1}^{\infty}(\alpha_k)_n\right) X^n\right)' = \sum_{n=0}^{\infty}\left(\sum_{k=1}^{\infty} n(\alpha_k)_n\right) X^{n-1}$$
$$= \sum_{k=1}^{\infty}\left(\sum_{n=0}^{\infty} n(\alpha_k)_n X^{n-1}\right) = \sum_{k=0}^{\infty} \alpha_k'.$$

(II) We can use (I) to see that

$$\begin{aligned}
\left(\left(\sum_{n=0}^{\infty} a_n X^n\right)\left(\sum_{n=0}^{\infty} b_n X^n\right)\right)' &= \\
&= (a_0b_0 + a_0b_1X + a_1b_0X + a_0b_2X^2 + a_1b_1X^2 + a_2b_0X^2 + ...)' \\
&= (a_0b_0)' + (a_0b_1X)' + (a_1b_0X)' \\
&\quad + (a_0b_2X^2)' + (a_1b_1X^2)' + (a_2b_0X^2)' + ... \\
&= a_0b_0(X^0)' + a_0b_1(X)' + a_1b_0(X)' \\
&\quad + a_2b_0(X^2)' + a_1b_1(X^{1+1})' + a_2b_0(X^{2+0})' + ...
\end{aligned}$$

and so it suffices we prove the identity for α, β monomials.

$$\begin{aligned}
(\alpha\beta)' = (X^k X^\ell)' = (X^{k+\ell})' &= \\
&= (k+\ell)X^{k+\ell-1} = kX^{k-1}X^\ell + \ell X^k X^{\ell-1} = \alpha'\beta + \alpha\beta'.
\end{aligned}$$

(III) We first suppose $\alpha_k \neq -1$, in which case both sides of the equation are just reduced to 0. We then choose $N \in \mathbb{N}$ and find n such that $|\alpha_k| < 2^{-N-1}$ for every $k > n$. So, we chose a power N and such an index n that if a series a_k has a non-zero coefficient with a power N or lower, its index k will be lesser than n. We will show, that the formula holds for the first N powers of X and for that we only need to sum until n. After we prove it for finite products, we only need to take the limit $N \to \infty$ and the proof will be complete.
We use (II) and compute

$$\left(\prod_{k=1}^{n}(1+\alpha_k)\right)' \overset{(II)}{=} \left(\prod_{k=1}^{n-1}(1+\alpha_k)\right)'(1+\alpha_n) + \left(\prod_{k=1}^{n-1}(1+\alpha_k)\right)(1+\alpha_n)'$$

$$\overset{IS}{=} \left(\prod_{k=1}^{n-1}(1+\alpha_k)\right)\left(\sum_{k=1}^{n-1}\frac{\alpha_k'}{1+\alpha_k}\right)(1+\alpha_n) + \left(\prod_{k=1}^{n-1}(1+\alpha_k)\right)\alpha_n'\frac{1+\alpha_n}{1+\alpha_n}$$
$$= \left(\prod_{k=1}^{n}(1+\alpha_k)\right)\left(\sum_{k=1}^{n-1}\frac{\alpha_k'}{1+\alpha_k} + \frac{\alpha_n'}{1+\alpha_n}\right)$$
$$= \prod_{k=1}^{n}(1+\alpha_k)\sum_{k=1}^{n}\frac{\alpha_k'}{1+\alpha_k}.$$

Let us denote the left-hand side of the formula (III) $\sum_{n=0}^{\infty} p_n X^n$. We have shown, that

$$\left(\prod_{k=1}^{n}(1+\alpha_k)\right)' = \sum_{n=0}^{N} p_n X^n + \sum_{n=N+1}^{\infty} \tilde{p_n} X^n,$$

for some $\tilde{p_n} \in K$, so we can calculate

$$\left|\left(\prod_{k=1}^{\infty}(1+\alpha_k)\right)' - \prod_{k=1}^{n}(1+\alpha_k)\sum_{k=1}^{n}\frac{\alpha_k'}{1+\alpha_k}\right| =$$
$$= \left|\sum_{n=0}^{\infty} p_n X^n - \left(\sum_{n=0}^{N} p_n X^n + \sum_{n=N+1}^{\infty} \tilde{p_n} X^n\right)\right|$$
$$= \left|\sum_{n=N+1}^{\infty} (p_n - \tilde{p_n}) X^n\right|$$
$$\leq 2^{-N-1} \overset{N\to\infty}{\longrightarrow} 0,$$

and so the limit of $N \to \infty$ gives us the formula.

(IV) We use (II)

$$\alpha' = \left(\frac{\alpha}{\beta}\beta\right)' = \left(\frac{\alpha}{\beta}\right)'\beta + \frac{\alpha}{\beta}\beta' \implies \left(\frac{\alpha}{\beta}\right)' = \frac{\alpha'\beta + \alpha\beta'}{\beta^2}.$$

(V) We first realise, that we can use the power rule by plugging $-1+\alpha$ into (III), we get $(\alpha^n)' = n\alpha^{n-1}\alpha'$. We use this in the following

$$(\alpha \circ \beta)' = \left(\sum_{n=0}^{\infty} a_n \beta^n\right)'$$
$$= \sum_{n=0}^{\infty} a_n (\beta^n)'$$
$$= \sum_{n=0}^{\infty} n a_n \beta^{n-1}\beta'$$
$$= \alpha'(\beta)\beta'.$$

□

Example 2.10. It will surprise no one that this derivative is consistent with what we know, we have

$$k' = 0, \qquad (kX)' = k$$

$$
\begin{aligned}
(\exp X)' &= \left(\sum_{n=0}^{\infty} \frac{X^n}{n!}\right)' & (2.2)\\
&= \left(\sum_{n=0}^{\infty} \frac{(X^n)'}{n!}\right)\\
&= \left(\sum_{n=0}^{\infty} n\frac{X^{n-1}}{n!}\right)\\
&= \left(\sum_{n=0}^{\infty} \frac{X^n}{n!}\right) = \exp X. & (2.3)
\end{aligned}
$$

Definition 2.11. We define the formal logarithm as

$$\log(1+X) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} X^n = X - \frac{X^2}{2} + \frac{X^3}{3} - \frac{X^4}{4} + \frac{X^5}{5} - ...$$

We construct the logarithm to be the reverse of the exponential, which we will now prove. We now know the exponential has a reverse by Theorem 2.6 and we will show it is indeed this logarithm. We first realize the derivative of the logarithm using the last lemma,

$$
\begin{aligned}
\log(1+X)' &= \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n} X^n\right)'\\
&\overset{(I)}{=} \sum_{n=1}^{\infty} \frac{(-1)^n}{n}(X^n)'\\
&= \sum_{n=1}^{\infty}(-1)^n X^{n-1}\\
&= 1 - X + X^2 - X^3 + ... = \frac{1}{1+X}.
\end{aligned}
$$

and continue with the following derivative using the chain rule. Note that the composition is well-defined since $\exp(X) - 1 \in (X)$.

$$\log(1+(\exp X - 1))' \overset{2.9(V)}{=} \frac{1}{1+\exp X - 1}(\exp X - 1)' \overset{(2.3)}{=} \frac{\exp X}{\exp X} = 1.$$

Because the logarithm has no constant term, this uniquely determines $\log(1 + (\exp X - 1)) = X$ and that log is indeed the reverse of exp and Theorem 2.6 says it is also the same the other way around. We continue with a functional equation, where we prove the basic logarithm property, turning products into sums. With this proven, we can turn sums into products and products into sums at will.

Theorem 2.12 (Functional equation for the logarithm)**.** *For every* $\{\alpha_n\} \in (X) \subset \mathbb{C}[[X]]$ *a null sequence, it holds that*

$$\log\left(\prod_{n=1}^{\infty}(1+\alpha_n)\right) = \sum_{n=1}^{\infty} \log(1+\alpha_n).$$

Proof. We will insert a pair of $\exp(\log(-))$ inside the product, which we have already proven does not change the result. Note that it is well-defined since we always take the log of $1+\alpha_i$, where $\alpha_i \in (X)$, and $\log(X)$ has no constant term so

$\exp \circ \log(X)$ is well-defined too. Then we can already use the functional equation we have got for the exponential, Lemma 2.5:

$$\begin{aligned}
\log\left(\prod_{n=1}^{\infty}(1+\alpha_n)\right) &= \log\left(\prod_{n=1}^{\infty}\exp(\log(1+\alpha_n))\right) \\
&\overset{2.5}{=} \log\left(\exp\left(\sum_{n=1}^{\infty}\log(1+\alpha_n)\right)\right) \\
&= \sum_{n=1}^{\infty}\log(1+\alpha_n).
\end{aligned}$$
□

We will now end the chapter with a few examples to show practically how we work with formal power series in $\mathbb{C}[[X]]$. These show the use of formal series' addition, multiplication, reindexation, differentiation and composition. We will show certain identities every mathematician knows. We can define any function we use in analysis by its Taylor series, recall

$$\exp X = \sum_{n=0}^{\infty}\frac{1}{n!}X^n.$$

For these examples, we will also be using sine, cosine, tangent and arctangent.

$$\begin{aligned}
\sin X &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}X^{2n+1}, \\
\cos X &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}X^{2n}.
\end{aligned}$$

We define $\tan X = \frac{\sin X}{\cos X}$ and

$$\arctan X = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}X^{2n+1}.$$

Note that $\tan X$ is well-defined, because $\cos X$ has a constant term, and so by Lemma 1.9 it is invertible.

Example 2.13. We begin with Euler's identity, $\exp(iX) = \cos X + i\sin X$. This is surprisingly easy, we need to do little more than write the definitions. We can compute $\exp(iX)$, because $iX \in (X)$, so the composition is well-defined. The calculation itself is simple:

$$\begin{aligned}
\exp(iX) &= \sum_{n=0}^{\infty}\frac{1}{n!}(iX)^n \\
&= \sum_{n=0}^{\infty}\frac{i^n}{n!}X^n \\
&= \sum_{\substack{n=0 \\ n\ \text{even}}}^{\infty}\frac{i^n}{n!}X^n + \sum_{\substack{n=1 \\ n\ \text{odd}}}^{\infty}\frac{i^n}{n!}X^n \\
&= \sum_{n=0}^{\infty}\frac{i^{2n}}{(2n)!}X^{2n} + \sum_{n=0}^{\infty}\frac{i^{2n+1}}{(2n+1)!}X^{2n+1} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}X^{2n} + i\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}X^{2n+1} \\
&= \cos(X) + i\sin(X).
\end{aligned}$$

Example 2.14. The next formula we show is $\sin^2(X) + \cos^2(X) = 1$. First of all we compute the squares by definition of multiplication. We denote the k-th coefficient of $\sin(X)$ $\sin_k$.

$$\sin^2(X) = \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} X^{2n+1}\right) \cdot \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} X^{2n+1}\right)$$
$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sin_k \sin_{n-k} X^n.$$

We now analyze the product. Immediately we see that $\sin_k = 0$ for k even, so we know all non-zero coefficients will come with odd powers of X. Therefore, we can substitute $2k+1$ instead of k since the rest will be zero. We can also change the indexing of the outer sum with $2n \to n$ since the product of two odd powers of X will be even and therefore every term in the product will have even power. We can also start with $n = 1$ because the 0-th coefficient is a product of 0-th (even) coefficients, which are 0.
Realising this, we need to adjust the inner sum's indexing adequately. The highest k allowed is going to be $n-1$. This reindexation can be done because $k = n$ in the old indexing gives a product of the constant and the n-th term, which is zero. We now only have to realise all we have done is formally correct. The second factor will be $\sin_{2n-2k+1}$. Our new inner series sums the products of all odd-indexed coefficients starting with 1 and ending with $2n-1$ multiplied with the corresponding second term, while in the outer series the lowest power of X is two. We have, therefore, done nothing else then rename the non-zero coefficients and omit the zero ones. All of this changes the coefficients in the following way:

$$\sin^2(X) = \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \sin_{2k+1} \sin_{2(n-1-k)+1} X^{2n}$$
$$= \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \frac{(-1)^k}{(2k+1)!} \frac{(-1)^{n-1-k}}{(2(n-1-k)+1)!} X^{2n}$$
$$= \sum_{n=1}^{\infty} (-1)^{n-1} \sum_{k=0}^{n-1} \frac{1}{(2k+1)!(2n-2k-1)!} X^{2n}$$
$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n)!} \sum_{k=0}^{n-1} \frac{(2n)!}{(2k+1)!(2n-2k-1)!} X^{2n}$$
$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n)!} \sum_{k=0}^{n-1} \binom{2n}{2k+1} X^{2n}.$$

We will now compute the same product with cosine.

$$\cos^2(X) = (\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} X^{2n})(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} X^{2n})$$
$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \cos_k \cos_{n-k} X^n.$$

We will reindex the inner sum again, but now it will be much simpler. We know $\cos_k = 0$ for k odd, so we can write $2k$ instead of k. Same as before, a product of two even powers of X will be even, so we can substitute $2n \to n$. That transforms our product in the following way:

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \cos_{2k} \cos_{2n-2k} X^{2n} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} \frac{(-1)^{(n-k)}}{(2n-2k)!} X^{2n} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \sum_{k=0}^{n} \frac{(2n)!}{(2k)!(2n-2k)!} X^{2n} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \sum_{k=0}^{n} \binom{2n}{2k} X^{2n}.
\end{aligned}
$$

Now we only need to use a simple lemma from discrete mathematics, proven for example in [4] on page 64, which says for $n \in \mathbb{N}$

$$
\sum_{\substack{k=0 \\ k \text{ even}}}^{n} \binom{n}{k} = 2^{n-1} = \sum_{\substack{k=0 \\ k \text{ odd}}}^{n} \binom{n}{k}.
$$

Using this we can finish the calculation:

$$
\begin{aligned}
\sin^2(X) + \cos^2(X) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n)!} \sum_{k=0}^{n-1} \binom{2n}{2k+1} X^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \sum_{k=0}^{n} \binom{2n}{2k} X^{2n} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{n-1}}{(2n)!} X^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n 2^{n-1}}{(2n)!} X^{2n} \\
&= -\sum_{n=1}^{\infty} \frac{(-1)^n 2^{n-1}}{(2n)!} X^{2n} + 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{n-1}}{(2n)!} X^{2n} \\
&= 1.
\end{aligned}
$$

Example 2.15. We will compute the derivative of $\sin X$, $\cos X$ and $\arctan X$. Let us calculate using the lemma about properties of the derivative Lemma 2.9.

$$
\begin{aligned}
\sin'(X) &= \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} X^{2n+1} \right)' \\
&\overset{2.9(I)}{=} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (X^{2n+1})' \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} X^{2n} = \cos X.
\end{aligned}
$$

With cosine, we will first split away the constant and shift the indexes $n \to n-1$ and rewrite the series accordingly to simplify the coming calculation:

$$
\begin{aligned}
\cos'(X) &= \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} X^{2n} \right)' \\
&= \left(1 + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+2)!} X^{2n+2} \right)' \\
&\overset{2.9(I)}{=} 0 + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+2)!} (X^{2n+2})'
\end{aligned}
$$

$$= -\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} X^{2n+1} = -\sin X.$$

The calculation for the arctangent is straightforward.

$$\begin{aligned}
\arctan'(X) &= \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} X^{2n+1} \right)' \\
&\overset{2.9(I)}{=} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} (X^{2n+1})' \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} (2n+1) X^{2n} \\
&= \sum_{n=0}^{\infty} (-X^2)^n \overset{1.2}{=} \frac{1}{1-(-X^2)} = \frac{1}{1+X^2},
\end{aligned}$$

where in the end we use the geometric sum formula.

Example 2.16. Now we will show that $\arctan \circ \tan(X) = X$. We have stated earlier that these computations are usually extremely difficult and technical since they require computing the n-th power of the inner series. In case the derivative can be written in a simpler expression, however, there is a trick to be used. If the composition is equal to X, the derivative of the composition is equal to one, which is what we will show. Then it suffices to show the composition has no constant term, then the composition is uniquely determined to be X.
First of all, however, we need to show our composition is well-defined. The outer series is not finite, therefore we have to show the inner is lacking its constant term. A simple way to do so, without the need to show anything about the series $\frac{1}{\cos X}$ is to use norms — We have

$$|\tan X| = \left| \frac{\sin X}{\cos X} \right| = \frac{|\sin X|}{|\cos X|} = \frac{2^{-1}}{2^0} = \frac{1}{2},$$

so the series $\tan X$ indeed has no constant term. We can therefore use the derivative trick, using Lemma 2.9 about the properties of the derivative and the preceding example:

$$\begin{aligned}
\arctan(\tan(X))' &\overset{2.9(V)}{=} \arctan'(\tan^2(X)) \left(\frac{\sin X}{\cos X} \right)' \\
&\overset{2.15}{=} \frac{1}{1+\tan^2(X)} \left(\frac{\sin X}{\cos X} \right)' \\
&\overset{2.9(IV)}{=} \frac{1}{1+\frac{\sin^2(X)}{\cos^2(X)}} \frac{\sin'(X)\cos(X) - \sin(X)\cos'(X)}{\cos^2(X)} \\
&\overset{2.15}{=} \frac{1}{\frac{\cos^2(X)+\sin^2(X)}{\cos^2(X)}} \frac{\cos^2(X)+\sin^2(X)}{\cos^2(X)} \\
&= 1.
\end{aligned}$$

It shows indeed $\arctan \circ \tan(X) = X$ since both lack a constant term — arctan by definition, and about tan this has been shown earlier in this example.

Even though we will not be using it in this thesis, what we already know has a big value in combinatorics. When we are looking for so-called generating functions of a naturally occurring sequence, we use an equation assembled from known properties of the sequence. There we can express this generating function and finding such an equation is the only difficult step of finding the precise formula for the n-th term of the sequence.

Example 2.17. We will show this with the Fibonacci sequence. We want to find a formula for n-th term. The sequence is defined in the following way

$$f_0 = 0, \qquad f_1 = 1, \qquad f_{n+1} = f_n + f_{n-1}.$$

We will work with the power series α, where the coefficients are the Fibonacci numbers — we call it the generating function of the sequence. From the definition, we realise it holds $\alpha = X + X^2\alpha + X\alpha$; every Fibonacci number is used to calculate the following two, which in the power series is equivalent to multiplying by X and X^2 respectively. The X in the front is needed to keep $f_1 = 1$ since $f_0 X^0 \cdot X \neq f_1 X$, it is the only exception we have to manually cover. With this equation, we only have to compute α.
We express α from this equation and see

$$\alpha = \frac{X}{1 - X - X^2},$$

which gives us the precise generating function. Now we will use the partial fraction decomposition and the geometric series formula to get the precise formula for any Fibonacci number (We write $\phi_\pm = \frac{1\pm\sqrt{5}}{2}$ for greater clarity).

$$\begin{aligned}
\alpha &= \frac{X}{1 - X - X^2} \\
&= \frac{1}{\sqrt{5}}\frac{\phi_-}{\phi_- - X} - \frac{1}{\sqrt{5}}\frac{\phi_+}{\phi_+ - X} \\
&= \frac{1}{\sqrt{5}}\frac{1}{1 - \frac{X}{\phi_-}} - \frac{1}{\sqrt{5}}\frac{1}{1 - \frac{X}{\phi_+}} \\
&= \frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}\frac{X^n}{\phi_-^n} - \frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}\frac{X^n}{\phi_+^n} \\
&= \sum_{n=0}^{\infty}\frac{\phi_+^n - \phi_-^n}{\sqrt{5}}X^n \\
&= \sum_{n=0}^{\infty} f_n X^n,
\end{aligned}$$

where in the fourth equality we used the fact that $\phi_- = \frac{1}{\phi_+}$.

3. A proof of the Lagrange-Jacobi theorem

We can now use the machinery of formal series to prove a few more interesting theorems, crowned by a well-known theorem - Lagrange-Jacobi four square theorem.

Theorem 3.1 (Lagrange-Jacobi)**.** *Every integer can be expressed as a sum of four squares. Furthermore, the number of different ways to do this is given by*

$$q(n) = |\{(a,b,c,d) \in \mathbb{Z}^4; n = a^2+b^2+c^2+d^2\}| = 8 \sum_{4 \nmid d | n} d.$$

We will define some more specific notions needed for the proof of the main theorem, which we introduce below.

Definition 3.2 (Gauss product)**.** For $n \in \mathbb{N}_0$ we define

$$X^n! = \prod_{k=1}^{n} (1-X^k) = (1-X)(1-X^2)...(1-X^n).$$

For $0 \leq k \leq n$, we define

$$\left\langle {n \atop k} \right\rangle = \frac{X^n!}{X^k!X^{n-k}!} = \frac{(1-X)(1-X^2)...(1-X^n)}{(1-X)...(1-X^k)(1-X)...(1-X^{n-k})},$$

and for other $k \in \mathbb{Z}$ we define $\left\langle {n \atop k} \right\rangle = 0$.

We will now continue with a practical identity with the Gauss products.

Lemma 3.3. For $n \in \mathbb{N}_0$ and $k \in \mathbb{Z}$ the following equality holds

$$\left\langle {n+1 \atop k} \right\rangle = X^k \left\langle {n \atop k} \right\rangle + \left\langle {n \atop k-1} \right\rangle \tag{3.1}$$

$$= \left\langle {n \atop k} \right\rangle + X^{n+1-k} \left\langle {n \atop k-1} \right\rangle. \tag{3.2}$$

Proof. We begin by realising that for any term with $k > n+1$ or $k < 0$ (0 by definition) both of the formulas hold. About either equality, we have

$$1 = \left\langle {n+1 \atop 0} \right\rangle = X^0 \left\langle {n \atop 0} \right\rangle + \left\langle {n \atop -1} \right\rangle = 1 + 0$$
$$= \left\langle {n \atop 0} \right\rangle + X^{n+1} \left\langle {n \atop -1} \right\rangle = 1 + 0,$$

and the case for $k = n+1$ is completely analogous.
It therefore suffices to prove the lemma for $0 < k < n+1$. We will start from both resulting sums and simplify them into the single term on the left

$$X^k \left\langle {n \atop k} \right\rangle + \left\langle {n \atop k-1} \right\rangle = X^k \frac{X^n!}{X^k!X^{n-k}!} + \frac{X^n!}{X^{k-1}!X^{n-k+1}!}$$

$$
\begin{aligned}
&= X^k \frac{(1-X^n)...(1-X)}{(1-X^k)...(1-X)(1-X^{n-k})...(1-X)} \\
&+ \frac{(1-X^n)...(1-X)}{(1-X^{k-1})...(1-X)(1-X^{n-k+1})...(1-X)} \\
&= \left(X^k \frac{1-X^{n-k+1}}{1-X^k} + 1\right) \frac{(1-X^n)...(1-X)}{(1-X^{k-1})...(1-X)(1-X^{n-k+1})...(1-X)} \\
&= \left(\frac{X^k - X^{n+1} + 1 - X^k}{1-X^k}\right) \frac{X^n!}{X^{k-1}!X^{n-k+1}!} \\
&= \frac{1-X^{n+1}}{1-X^k} \frac{X^n!}{X^{k-1}!X^{n-k+1}!} \\
&= \frac{X^{n+1}!}{X^k!X^{n-k+1}!} \\
&= \left\langle {n+1 \atop k} \right\rangle.
\end{aligned}
$$

For the second equality we will use $\left\langle {n \atop k} \right\rangle = \left\langle {n \atop n-k} \right\rangle$ and the first equality, that we have already proven.

$$
\begin{aligned}
X^{n+1-k} \left\langle {n \atop k-1} \right\rangle + \left\langle {n \atop k} \right\rangle &= X^{n+1-k} \left\langle {n \atop n-k+1} \right\rangle + \left\langle {n \atop n-k} \right\rangle \\
&\overset{(3.1)}{=} \left\langle {n+1 \atop n-k+1} \right\rangle = \left\langle {n+1 \atop k} \right\rangle.
\end{aligned}
$$

□

We will now show an example of use of the preceding lemma.

Example 3.4.

$$
\begin{aligned}
\left\langle {5 \atop 3} \right\rangle &= X^3 \left\langle {4 \atop 3} \right\rangle + \left\langle {4 \atop 2} \right\rangle \\
&= X^6 \left\langle {3 \atop 3} \right\rangle + X^3 \left\langle {3 \atop 2} \right\rangle + X^2 \left\langle {3 \atop 2} \right\rangle + \left\langle {3 \atop 1} \right\rangle \\
&= X^6 + (X^5 + X^4 + X^2) \left\langle {2 \atop 2} \right\rangle + (X^3 + X^2 + 1) \left\langle {2 \atop 1} \right\rangle \\
&= X^6 + X^5 + X^4 + X^2 + (X^4 + X^3 + X) \left\langle {1 \atop 1} \right\rangle + (X^3 + X^2 + 1) \left\langle {1 \atop 0} \right\rangle \\
&= X^6 + X^5 + X^4 + X^2 + X^4 + X^3 + X + X^3 + X^2 + 1 \\
&= X^6 + X^5 + 2X^4 + 2X^3 + 2X^2 + X + 1.
\end{aligned}
$$

Theorem 3.5 (Gauss' binomial theorem)**.** *For any $n \in \mathbb{N}$ and $\alpha \in \mathbb{C}[[X]]$ we have the following identities:*

$$
\begin{aligned}
(i) \prod_{k=0}^{n-1} \left(1 + \alpha X^k\right) &= \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle \alpha^k X^{\binom{k}{2}}, \\
(ii) \prod_{k=1}^{n} \frac{1}{1-\alpha X^k} &= \sum_{k=0}^{\infty} \left\langle {n+k-1 \atop k} \right\rangle \alpha^k X^k.
\end{aligned}
$$

The theorem is proved by technical manipulation of Gauss binomial coefficients.

Proof. We prove both parts of the theorem by induction on n. In the case of $n = 1$ both sides of (i) reduce to $1+\alpha$, while (ii) becomes the geometric series

$$\prod_{k=1}^{1} \frac{1}{1-\alpha X^k} = \sum_{k=0}^{\infty} \left\langle {k \atop k} \right\rangle \alpha^k X^k = \sum_{k=1}^{\infty} \alpha^k X^k.$$

The induction step of (i) first expands the sum from $-\infty$ to ∞. This can be done since the Gauss binomial is by definition 0 on all of the terms we added, and it will allow us to manipulate the series more easily. When we sum over all of $\mathbb{Z}$, we can shift indexes by any fixed value and even sum in the opposite direction $(k \to -k)$ because none of these add or take any terms of the sum. Assume the identity holds for n. Using basic operations and Lemma 3.2, we then get

$$\begin{aligned}\prod_{k=0}^{n} \left(1+\alpha X^k\right) &= (1+\alpha X^n) \sum_{k=-\infty}^{\infty} \left\langle {n \atop k} \right\rangle \alpha^k X^{\binom{k}{2}} \\ &= \sum_{k=-\infty}^{\infty} \left\langle {n \atop k} \right\rangle \alpha^k X^{\binom{k}{2}} + \sum_{k=-\infty}^{\infty} \left\langle {n \atop k} \right\rangle \alpha^{k+1} X^{\binom{k}{2}+n} \\ &= \sum_{k=-\infty}^{\infty} \left\langle {n \atop n-k} \right\rangle \alpha^k X^{\binom{k}{2}} + \sum_{k=-\infty}^{\infty} X^{n-k} \left\langle {n \atop n-k} \right\rangle \alpha^{k+1} X^{\binom{k+1}{2}}.\end{aligned}$$

We shift the indexing with $k \to k-1$ in the latter sum:

$$\begin{aligned} &= \sum_{k=-\infty}^{\infty} \left\langle {n \atop n-k} \right\rangle \alpha^k X^{\binom{k}{2}} + \sum_{k=-\infty}^{\infty} X^{n-k+1} \left\langle {n \atop n-k+1} \right\rangle \alpha^k X^{\binom{k}{2}} \\ &\overset{(3.2)}{=} \sum_{k=-\infty}^{\infty} \left\langle {n+1 \atop n-k+1} \right\rangle \alpha^k X^{\binom{k}{2}} \\ &= \sum_{k=-\infty}^{\infty} \left\langle {n+1 \atop k} \right\rangle \alpha^k X^{\binom{k}{2}}.\end{aligned}$$

Now we follow with the induction step for (ii). We will show the following

$$(1-\alpha X^{n+1}) \sum_{k=0}^{\infty} \left\langle {n+k \atop k} \right\rangle \alpha^k X^k = \sum_{k=0}^{\infty} \left\langle {n+k-1 \atop k} \right\rangle \alpha^k X^k,$$

where using the induction step and dividing both sides by $(1-\alpha X^{n+1})$ finishes the proof. We multiply

$$\begin{aligned}&(1-\alpha X^{n+1}) \sum_{k=0}^{\infty} \left\langle {n+k \atop k} \right\rangle \alpha^k X^k \\ &\qquad = \sum_{k=0}^{\infty} \left\langle {n+k \atop k} \right\rangle \alpha^k X^k - X^n \sum_{k=0}^{\infty} \left\langle {n+k \atop k} \right\rangle \alpha^{k+1} X^{k+1},\end{aligned}$$

and will now again shift the indexing in the last sum on the right-hand side with $k \to k-1$. This will only add a new first term and we have to show it is zero -

however the Gauss coefficient is zero by definition, so the reindexation was correct.

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} \left\langle {n+k \atop k} \right\rangle \alpha^k X^k - X^n \sum_{k=0}^{\infty} \left\langle {n+k-1 \atop k-1} \right\rangle \alpha^k X^k \\
&= \sum_{k=0}^{\infty} \left(\left\langle {n+k \atop k} \right\rangle - X^n \left\langle {n+k-1 \atop k-1} \right\rangle \right) \alpha^k X^k \\
&\overset{(3.2)}{=} \sum_{k=0}^{\infty} \left(\left\langle {n+k-1 \atop k} \right\rangle + X^{(n+k)-k} \left\langle {n+k-1 \atop k-1} \right\rangle - X^n \left\langle {n+k-1 \atop k-1} \right\rangle \right) \alpha^k X^k \\
&= \sum_{k=0}^{\infty} \left\langle {n+k-1 \atop k} \right\rangle \alpha^k X^k \\
&\overset{IS}{=} \prod_{k=1}^{n} \frac{1}{1-\alpha X^k}.
\end{aligned}
$$

We have received the formula for $n+1$, so the proof is finished:

$$
\begin{aligned}
(1-\alpha X^{n+1}) \sum_{k=0}^{\infty} \left\langle {n+k \atop k} \right\rangle &= \prod_{k=1}^{n} \frac{1}{1-\alpha X^k} \\
\sum_{k=0}^{\infty} \left\langle {n+k \atop k} \right\rangle &= \prod_{k=1}^{n+1} \frac{1}{1-\alpha X^k}.
\end{aligned}
$$

□

Remark 3.6. The importance of the preceding theorem is best highlighted by noticing that α is independent of X. We can therefore substitute any $\beta \in (X)$ for X without the need of to use $\alpha(\beta)$.
If we compute the limit of the Gauss' coefficients, we can also use the infinite version of the theorem, contributed to Euler.

$$\lim_{n\to\infty} \left\langle {n \atop k} \right\rangle = \frac{1}{X^k!} \lim_{n\to\infty} (1-X^{n-k+1})...(1-X^n) = \frac{1}{X^k!}. \tag{3.3}$$

Since all the non-one terms in the product have arbitrarily large power of X, they also have arbitrarily small norms (recall $|\alpha| = 2^{-\inf\{\alpha\}}$) and so tend to 0.

Theorem 3.7 (Euler)**.** *For all $\alpha \in \mathbb{C}[[X]]$ the following identities hold:*

$$\prod_{k=0}^{\infty} \left(1+\alpha X^k\right) = \sum_{k=0}^{\infty} \frac{\alpha^k X^{\binom{k}{2}}}{X^k!}, \tag{3.4}$$

$$\prod_{k=1}^{\infty} \frac{1}{1-\alpha X^k} = \sum_{k=0}^{\infty} \frac{\alpha^k X^k}{X^k!}. \tag{3.5}$$

If αX^{-1} is a well-defined power series, that is, $\alpha \in (X)$, we can substitute it for X in (3.5) and get

$$\prod_{k=0}^{\infty} \frac{1}{1-\alpha X^k} = \sum_{k=0}^{\infty} \frac{\alpha^k}{X^k!}. \tag{3.6}$$

We will now state and prove Jacobi's triple product formula, one of the most important tools in the whole theory of formal power series.

Theorem 3.8 (Jacobi's triple product formula)**.** *For every $\alpha \in \mathbb{C}[[X]] \setminus (X^2)$ we have the formula*

$$\prod_{k=1}^{\infty}(1-X^{2k})(1+\alpha X^{2k-1})(1+\alpha^{-1}X^{2k-1}) = \sum_{k=-\infty}^{\infty} \alpha^k X^{k^2}. \tag{3.7}$$

The main tool for the proof will be the Euler corollary (Theorem 3.7). Most of the proof is simply using the identities at the right time. It is important to note, that α and X are independent in the identities, so we can freely use substitutions for both X and α separately, as given in Remark 3.6.

Proof. We will be using the equality (3.4):

$$\prod_{k=0}^{\infty}\left(1+\alpha X^k\right) = \sum_{k=0}^{\infty} \frac{\alpha^k X^{\binom{k}{2}}}{X^k!},$$

and (3.6):

$$\prod_{k=0}^{\infty} \frac{1}{1-\alpha X^k} = \sum_{k=0}^{\infty} \frac{\alpha^k}{X^k!}.$$

We will prove three secondary formulas by choosing the right substitutions into the Euler corollary. Next, we will prove the triple-product formula, using these three secondary formulas. For the first equation, we will substitute $X \to X^2$ and $\alpha \to \alpha^{-1}X$. We compute

$$\prod_{k=1}^{\infty}\left(1+\alpha^{-1}X^{2k+1}\right) = \sum_{k=0}^{\infty} \alpha^{-k} X^{2\binom{k}{2}+k} \prod_{i<k} \frac{1}{1-X^{2i}} = \sum_{k=0}^{\infty} \frac{\alpha^{-k}X^{k^2}}{(1-X^2)...(1-X^{2k})}. \tag{3.8}$$

Next we will substitute $X \to X^2$ and $\alpha \to -X^{2k+2}$ for $k \in \mathbb{N}_0$ to the same formula, (3.4). We get

$$\prod_{\ell=0}^{\infty}(1-X^{2k+2}X^{2\ell}) = \sum_{\ell=0}^{\infty} \frac{(-X)^{(2k+2)\ell}X^{2\binom{\ell}{2}}}{(1-X^2)...(1-X^{2k})} = \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}X^{2k\ell+\ell+\ell^2}}{(1-X^2)...(1-X^{2k})}. \tag{3.9}$$

The final preparatory computation substitutes $X \to X^2$ and $\alpha \to -\alpha X$ into the second equation at the beginning of the proof, (3.6).

$$\prod_{k=0}^{\infty} \frac{1}{1+\alpha X^{2k+1}} = \sum_{k=0}^{\infty} \frac{(-1)^k \alpha^k X^k}{(1-X^2)...(1-X^{2k})}. \tag{3.10}$$

We start by observing that $\alpha^{-k}X^{2k}$ and $1+\alpha^{-1}X^{2k-1}$ are actually power series for any k. We have assumed $\alpha \in \mathbb{C}[[X]] \setminus (X^2)$ for this step. The power of X in α is lower than 2 — otherwise the triple-product identity (3.7) could consist of terms X^{-k}, which we haven't defined. This way, both sides of (3.7), the formula we are proving, are well-defined. Let us start with one of the factors on the left-hand side and calculate

$$\prod_{k=1}^{\infty}(1+\alpha^{-1}X^{2k-1}) = \prod_{k=0}^{\infty}(1+\alpha^{-1}X^{2k+1})$$

$$\overset{(3.8)}{=} \sum_{k=0}^{\infty} \frac{\alpha^{-k} X^{k^2}}{(1-X^2)...(1-X^{2k})}$$
$$= \sum_{k=0}^{\infty} \frac{\alpha^{-k} X^{k^2}}{(1-X^2)...(1-X^{2k})} \cdot \frac{(1-X^{2k+2})(1-X^{2k+4})...}{(1-X^{2k+2})(1-X^{2k+4})...}$$
$$= \left(\prod_{k=1}^{\infty} \frac{1}{1-X^{2k}}\right) \sum_{k=0}^{\infty} \alpha^{-k} X^{k^2} \prod_{\ell=0}^{\infty} \left(1 - X^{2+2k+2\ell}\right).$$

where we force in another term to expand the product to infinity and factor out the denominator. We will multiply both sides of the equation with $\prod_{k=1}^{\infty} 1 - X^{2k}$. It is possible to change the lower bound of the sum from 0 to $-\infty$ since for those k one of the factors of the inner product is going to be zero, meaning we did not add any new terms. We use the second secondary formula and get

$$\prod_{k=1}^{\infty}(1+\alpha^{-1}X^{2k-1})(1-X^{2k}) = \sum_{k=-\infty}^{\infty} \alpha^{-k} X^{k^2} \prod_{\ell=0}^{\infty} \left(1 - X^{2+2k+2\ell}\right)$$
$$\overset{(3.9)}{=} \sum_{k=-\infty}^{\infty} \alpha^{-k} X^{k^2} \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell} X^{\ell^2+\ell+2k\ell}}{(1-X^2)...(1-X^{2l})}$$
$$= \sum_{\ell=1}^{\infty} \frac{(-\alpha X)^{\ell}}{(1-X^2)...(1-X^{2l})} \sum_{k=-\infty}^{\infty} \alpha^{-k-\ell} X^{(k+\ell)^2}$$
$$= \sum_{\ell=1}^{\infty} \frac{(-\alpha X)^{\ell}}{(1-X^2)...(1-X^{2l})} \sum_{k=-\infty}^{\infty} \alpha^{k} X^{k^2}.$$

We shift the indexing of the inner sum with $k \to -k - \ell$ and the sum becomes independent of ℓ. We finally use the last substitution

$$\prod_{k=1}^{\infty}(1+\alpha^{-1}X^{2k+1})(1-X^2k) = \sum_{\ell=1}^{\infty} \frac{(-\alpha X)^{\ell}}{(1-X^2)...(1-X^{2l})} \sum_{k=-\infty}^{\infty} \alpha^{k} X^{k^2}$$
$$\overset{(3.10)}{=} \prod_{k=0}^{\infty} \frac{1}{1+\alpha X^{2k+1}} \sum_{k=-\infty}^{\infty} \alpha^{k} X^{k^2}$$
$$= \prod_{k=1}^{\infty} \frac{1}{1+\alpha X^{2k-1}} \sum_{k=-\infty}^{\infty} \alpha^{k} X^{k^2},$$

and the proof is complete. □

Example 3.9. Using different series for α in combination with substitutions can give rise to many nice identities. Some of them are derived further, mainly for future use.
We can substitute $\alpha \in \{\pm 1, X\}$ and derive useful identities, which we will use in the proof of the four-square theorem:

$$\prod_{k=1}^{\infty}(1-X^{2k})(1+X^{2k-1})^2 = \sum_{k=-\infty}^{\infty} X^{k^2}, \tag{3.11}$$
$$\prod_{k=1}^{\infty} \frac{(1-X^k)^2}{1-X^{2k}} = \prod_{k=1}^{\infty}(1-X^{2k})(1-X^{2k-1})^2 = \sum_{k=-\infty}^{\infty} (-1)^k X^{k^2}, \tag{3.12}$$
$$\prod_{k=1}^{\infty}(1+X^{2k-2})(1-X^{2k})(1+X^{2k}) = 2\prod_{k=1}^{\infty}(1-X^{2k})(1+X^{2k})^2$$

$$= \sum_{k=-\infty}^{\infty} X^{k^2+k} = 2\sum_{k=0}^{\infty} X^{k^2+k}. \tag{3.13}$$

The first one comes from a simple use of the theorem. The equality in the second equation comes from us noticing the factors of the middle product are just odd and even parts of $(1 - X^k)$, so we rewrite the product in terms of this simpler term since it often gets canceled out and the formula gets more practical. In the third equation we realise that the first and third factors are the same, just shifted, so we shift the exponent of the first factor by 2 (this is just a reindexation) and multiply the product by 2 since that was the first term for $k = 1$, which we removed. We can then rewrite the sum from 0 since

$$\sum_{k=-\infty}^{-1} X^{k^2+k} = \sum_{k=0}^{\infty} X^{(-k-1)^2+(-k-1)} = \sum_{k=0}^{\infty} X^{k^2+k}.$$

We also notice that in the last formula, all powers of X are even, so we can substitute X for X^2 and get the following formula

$$\prod_{k=1}^{\infty}(1 - X^k)(1 + X^k)^2 = \sum_{k=0}^{\infty} X^{\frac{k^2+k}{2}}. \tag{3.14}$$

Example 3.10. We substitute X^5 for X with $\alpha = -X$ and $\alpha = -X^3$ respectively to get identities we will be using later. We get

$$\begin{aligned}
\textstyle\prod_{k=1}^{\infty}(1-(X^5)^{2k})(1+(-X)(X^5)^{2k-1})(1+(-X)^{-1}(X^5)^{2k-1}) &= \sum_{k=-\infty}^{\infty} (-X)^k (X^5)^{k^2} \\
\prod_{k=1}^{\infty}(1 - X^{10k})(1 - X^{10k-4})(1 - X^{10k-6}) &= \sum_{k=-\infty}^{\infty} (-1)^k X^{5k^2+k} \\
\prod_{k=1}^{\infty}(1 - X^{5k})(1 - X^{5k-2})(1 - X^{5k-3}) &= \sum_{k=-\infty}^{\infty} (-1)^k X^{\frac{5k^2+k}{2}}.
\end{aligned} \tag{3.15}$$

$$\begin{aligned}
\textstyle\prod_{k=1}^{\infty}(1-(X^5)^{2k})(1+(-X^3)(X^5)^{2k-1})(1+(-X^3)^{-1}(X^5)^{2k-1}) &= \sum_{k=-\infty}^{\infty} (-X^3)^k (X^5)^{k^2} \\
\prod_{k=1}^{\infty}(1 - X^{10k})(1 - X^{10k-2})(1 - X^{10k-8}) &= \sum_{k=-\infty}^{\infty} (-1)^k X^{5k^2+3k} \\
\prod_{k=1}^{\infty}(1 - X^{5k})(1 - X^{5k-1})(1 - X^{5k-4}) &= \sum_{k=-\infty}^{\infty} (-1)^k X^{\frac{5k^2+3k}{2}}.
\end{aligned} \tag{3.16}$$

We will now follow with an extremely technical lemma we need for the same theorem — Roger-Ramanujan identities.

Lemma 3.11. For $n \in \mathbb{N}_0$, we have

$$\sum_{k=0}^{\infty} \left\langle {n \atop k} \right\rangle X^{k^2} = \sum_{k=-\infty}^{\infty} (-1)^k X^{\frac{5k^2+k}{2}} \left\langle {2n \atop n+2k} \right\rangle, \tag{3.17}$$

$$\sum_{k=0}^{\infty} \left\langle {n \atop k} \right\rangle X^{k^2+k} = \sum_{k=-\infty}^{\infty} (-1)^k X^{\frac{5k^2-3k}{2}} \left\langle {2n+1 \atop n+2k} \right\rangle. \tag{3.18}$$

We prove the identities using a tricky calculation. We will prove 2 inductive formulas holding between the sums on the left-hand sides, which will allow us to calculate the sum for any n just from the constant ones for $n = 0$. We will then prove the same formulas for the right-hand sides. When we show the identities are equal for $n = 0$, they will be proven.

Proof. We define α, $\tilde{\alpha}$ the left and right-hand sides of the first equation, and do the same for the second equation with β and $\tilde{\beta}$. Note that all of these sums are actually finite since on the left-hand side of both identities, for k greater than n every term is zero, while on the right the terms can only be non-zero for $-\frac{n+1}{2} \le k \le \frac{n+1}{2}$. We will show that for $n = 0$, they are all reduced to 1. On the left-hand side just by realising that the Gauss products will simplify to

$$\left\langle {0 \atop k} \right\rangle$$

for $n = 0$. The only non-zero term will therefore be the one for $k = 0$, which is 1 for both identities. On the right-hand side, we have

$$\left\langle {0 \atop 2k} \right\rangle, \left\langle {1 \atop 2k} \right\rangle$$

respectively. Again, the only non-zero term will come from the index $k = 0$, and it is again 1.
We intend to show

$$\alpha_n = \alpha_{n-1} + X^n \beta_{n-1}, \tag{3.19}$$

$$\tilde{\alpha}_n = \tilde{\alpha}_{n-1} + X^n \tilde{\beta}_{n-1}, \tag{3.20}$$

$$\beta_n - X^n \alpha_n = (1 - X^n)\beta_{n-1}, \tag{3.21}$$

$$\tilde{\beta}_n - X^n \tilde{\alpha}_n = (1 - X^n)\tilde{\beta}_{n-1}. \tag{3.22}$$

Since they all share the same value for $n = 0$, (3.19) forces a unique value to the next α_n ((3.20) forces the same value to $\tilde{\alpha}_n$) and (3.21) uses this to give a unique value to β_n ((3.22) again gives the same to $\tilde{\beta}_n$). Therefore, after proving all of the formulas, induction will yield the lemma for every $n \in \mathbb{N}$.
We will prove the relations in the same order, starting with (3.19)

$$\begin{aligned}
\alpha_n = \sum_{k=0}^{\infty} \left\langle {n \atop k} \right\rangle X^{k^2} &\overset{(3.2)}{=} \sum_{k=0}^{\infty} \left(\left\langle {n-1 \atop k} \right\rangle + X^{n-k} \left\langle {n-1 \atop k-1} \right\rangle \right) X^{k^2} \\
&= \sum_{k=0}^{\infty} \left\langle {n-1 \atop k} \right\rangle X^{k^2} + \sum_{k=0}^{\infty} X^{n-k} \left\langle {n-1 \atop k-1} \right\rangle X^{k^2} \\
&= \alpha_{n-1} + X^n \sum_{k=0}^{\infty} \left\langle {n-1 \atop k-1} \right\rangle X^{k^2-k} = \alpha_{n-1} + X^n \beta_{n-1}.
\end{aligned}$$

We continue with (3.20)

$$\begin{aligned}
\beta_n - X^n \alpha_n &= \sum_{k=0}^{\infty} \left\langle {n \atop k} \right\rangle X^{k^2+k} - X^n \sum_{k=0}^{\infty} \left\langle {n \atop k} \right\rangle X^{k^2} \\
&= \sum_{k=0}^{\infty} \left\langle {n \atop k} \right\rangle X^{k^2+k} - \sum_{k=0}^{\infty} X^{n-k} \left\langle {n \atop k} \right\rangle X^{k^2+k}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty}(1-X^{n-k})\left\langle{n \atop k}\right\rangle X^{k^2+k} \\
&= \sum_{k=0}^{\infty}\frac{X^n!}{X^k!X^{n-k}!}(1-X^{n-k})X^{k^2+k} \\
&= \sum_{k=0}^{\infty}\frac{1-X^n}{1-X^{n-k}}\frac{X^{n-1}!}{X^k!X^{n-k-1}!}(1-X^{n-k})X^{k^2+k} \\
&= (1-X^n)\sum_{k=0}^{\infty}\left\langle{n-1 \atop k}\right\rangle X^{k^2+k} = (1-X^n)\beta_{n-1}.
\end{aligned}
$$

To prove the same relations for the other 2 series, we will need another formula:

$$
\sum_{k=-\infty}^{\infty}(-1)^k\left\langle{2n-2 \atop n+2k}\right\rangle X^{\frac{5k^2+5k}{2}} = 0. \tag{3.23}
$$

We will again be using the transformation $k \to -k-1$ onto this formula, we will show the k-th term and the $(-k-1)$-st term cancel out.

$$
\begin{aligned}
\sum_{k=-\infty}^{-1}(-1)^k\left\langle{2n-2 \atop n+2k}\right\rangle X^{\frac{5k^2+5k}{2}} &= \sum_{k=0}^{\infty}(-1)^{-k-1}\left\langle{2n-2 \atop n+2(-k-1)}\right\rangle X^{\frac{5(-k-1)^2+5(-k-1)}{2}} \\
&= \sum_{k=0}^{\infty}(-1)^{k+1}\left\langle{2n-2 \atop n-2k-2}\right\rangle X^{\frac{5k^2+10k+5-5k-5}{2}} \\
&= -\sum_{k=0}^{\infty}(-1)^{k}\left\langle{2n-2 \atop 2n-2-(n+2k)}\right\rangle X^{\frac{5k^2+5k}{2}} \\
&= -\sum_{k=0}^{\infty}(-1)^{k}\left\langle{2n-2 \atop n+2k}\right\rangle X^{\frac{5k^2+5k}{2}},
\end{aligned}
$$

since by definition

$$
\left\langle{n \atop k}\right\rangle = \left\langle{n \atop n-k}\right\rangle.
$$

Armed with this, we can finish the proof and prove (3.21).

$$
\begin{aligned}
\tilde{\alpha}_n - \tilde{\alpha}_{n-1} &= \sum_{k=-\infty}^{\infty}\left((-1)^k X^{\frac{5k^2+k}{2}}\left\langle{2n \atop n+2k}\right\rangle\right) \\
&\quad - \sum_{k=-\infty}^{\infty}\left((-1)^k X^{\frac{5k^2+k}{2}}\left\langle{2n-2 \atop n-1+2k}\right\rangle\right) \\
&= \sum_{k=-\infty}^{\infty}\left(\left\langle{2n \atop n+2k}\right\rangle - \left\langle{2n-2 \atop n-1+2k}\right\rangle\right)(-1)^k X^{\frac{5k^2+k}{2}} \\
&\overset{(3.2)}{=} \sum_{k=-\infty}^{\infty}\Bigg(\left\langle{2n-1 \atop n+2k}\right\rangle + X^{n-2k}\left\langle{2n-1 \atop n-1+2k}\right\rangle \\
&\quad - \left\langle{2n-2 \atop n-1+2k}\right\rangle\Bigg)(-1)^k X^{\frac{5k^2+k}{2}} \\
&\overset{(3.1)}{=} \sum_{k=-\infty}^{\infty}\Bigg(X^{n+2k}\left\langle{2n-2 \atop n+2k}\right\rangle + \left\langle{2n-2 \atop n-1+2k}\right\rangle
\end{aligned}
$$

$$+ X^{n-2k}\left\langle {2n-1 \atop n-1+2k} \right\rangle - \left\langle {2n-2 \atop n-1+2k} \right\rangle\Bigg)(-1)^k X^{\frac{5k^2+k}{2}}$$

$$= \sum_{k=-\infty}^{\infty} X^{n+2k}\left\langle {2n-2 \atop n+2k} \right\rangle (-1)^k X^{\frac{5k^2+k}{2}}$$

$$+ \sum_{k=-\infty}^{\infty} X^{n-2k}\left\langle {2n-1 \atop n-1+2k} \right\rangle (-1)^k X^{\frac{5k^2+k}{2}}$$

$$= X^n \sum_{k=-\infty}^{\infty} \left\langle {2n-2 \atop n+2k} \right\rangle (-1)^k X^{\frac{5k^2+5k}{2}}$$

$$+ X^n \sum_{k=-\infty}^{\infty} \left\langle {2n-1 \atop n-1+2k} \right\rangle (-1)^k X^{\frac{5k^2-3k}{2}}$$

$$\overset{(3.23)}{=} 0 + X^n \tilde{\beta}_{n-1}.$$

Now we continue with (3.22).

$$\tilde{\beta}_n - X^n\tilde{\alpha}_n = \left(\sum_{k=-\infty}^{\infty} (-1)^k X^{\frac{5k^2-3k}{2}} \left\langle {2n+1 \atop n+2k} \right\rangle \right)$$

$$- X^n \left(\sum_{k=-\infty}^{\infty} \left\langle {2n \atop n+2k} \right\rangle (-1)^k X^{\frac{5k^2+k}{2}} \right)$$

$$\overset{(3.1)}{=} \sum_{k=-\infty}^{\infty} \left(X^{n+2k}\left\langle {2n \atop n+2k} \right\rangle + \left\langle {2n \atop n+2k-1} \right\rangle \right) (-1)^k X^{\frac{5k^2-3k}{2}}$$

$$- \left(\sum_{k=-\infty}^{\infty} X^{n+2k}\left\langle {2n \atop n+2k} \right\rangle (-1)^k X^{\frac{5k^2-3k}{2}} \right)$$

$$= \sum_{k=-\infty}^{\infty} \left\langle {2n \atop n+2k-1} \right\rangle (-1)^k X^{\frac{5k^2-3k}{2}}$$

$$\overset{(3.2)}{=} \sum_{k=-\infty}^{\infty} \Bigg(\left\langle {2n-1 \atop n+2k-1} \right\rangle$$

$$+ X^{n-2k+1}\left\langle {2n-1 \atop n+2k-2} \right\rangle \Bigg)(-1)^k X^{\frac{5k^2-3k}{2}}$$

$$= \sum_{k=-\infty}^{\infty} \left\langle {2n-1 \atop n+2k-1} \right\rangle (-1)^k X^{\frac{5k^2-3k}{2}}$$

$$+ \sum_{k=-\infty}^{\infty} X^{n-2k+1}\left\langle {2n-1 \atop n+2k-2} \right\rangle (-1)^k X^{\frac{5k^2-3k}{2}}$$

$$= \tilde{\beta}_{n-1} + X^n \sum_{k=-\infty}^{\infty} \left\langle {2n-1 \atop n+2k-2} \right\rangle (-1)^k X^{\frac{5k^2-7k+2}{2}}.$$

We will shift the indexing in the remaining sum by $k \to 1-k$, which will yield

$$\tilde{\beta}_n - X^n\tilde{\alpha}_n = \tilde{\beta}_{n-1} + X^n \sum_{k=-\infty}^{\infty} \left\langle {2n-1 \atop n+2(1-k)-2} \right\rangle (-1)^{(1-k)} X^{\frac{5(1-k)^2-7(1-k)+2}{2}}$$

$$= \tilde{\beta}_{n-1} - X^n \sum_{k=-\infty}^{\infty} \left\langle {2n-1 \atop n-2k} \right\rangle (-1)^k X^{\frac{5k^2-10k+5+7k-7+2}{2}}$$

$$= \tilde{\beta}_{n-1} - X^n \sum_{k=-\infty}^{\infty} \left\langle {2n-1 \atop n-2k} \right\rangle (-1)^k X^{\frac{5k^2-3k}{2}}$$
$$= \tilde{\beta}_{n-1} - X^n \tilde{\beta}_{n-1} = (1-X^n)\tilde{\beta}_{n-1}. \qquad \square$$

Finally, using the last Lemma 3.11 and formulas from the previous example, Example 3.10, we can prove a famous theorem:

Theorem 3.12 (Roger-Ramanujan identities)**.** *We have*

$$\prod_{k=1}^{\infty} \frac{1}{(1-X^{5k-1})(1-X^{5k-4})} = \sum_{k=0}^{\infty} \frac{X^{k^2}}{X^k!}, \tag{3.24}$$
$$\prod_{k=1}^{\infty} \frac{1}{(1-X^{5k-2})(1-X^{5k-3})} = \sum_{k=0}^{\infty} \frac{X^{k^2+k}}{X^k!}. \tag{3.25}$$

The proof uses the knowledge of (3.3):

$$\lim_{n\to\infty} \left\langle {n \atop k} \right\rangle = \frac{1}{X^k!} = \prod_{i=1}^{k} \frac{1}{1-X^i},$$

which we know from Remark 3.6. We replace the factorial term from the denominator with the limit and use the formulas we have prepared.

Proof. We start with the right-hand side and use formulas from this chapter:

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{X^{k^2}}{X^k!} &\overset{(3.3)}{=} \sum_{k=0}^{\infty} \lim_{n\to\infty} \left\langle {n \atop k} \right\rangle X^{k^2} \\
&= \lim_{n\to\infty} \sum_{k=0}^{\infty} \left\langle {n \atop k} \right\rangle X^{k^2} \\
&\overset{(3.17)}{=} \lim_{n\to\infty} \sum_{k=-\infty}^{\infty} (-1)^k X^{\frac{5k^2+k}{2}} \left\langle {2n \atop n+2k} \right\rangle \\
&= \sum_{k=-\infty}^{\infty} (-1)^k X^{\frac{5k^2+k}{2}} \lim_{n\to\infty} \left\langle {2n \atop n+2k} \right\rangle \\
&\overset{(3.3)}{=} \sum_{k=-\infty}^{\infty} (-1)^k X^{\frac{5k^2+k}{2}} \prod_{i=1}^{\infty} \frac{1}{1-X^i} \\
&\overset{(3.7)}{=} \frac{\prod_{k=1}^{\infty}(1-X^{5k})(1-X^{5k-2})(1-X^{5k-3})}{\prod_{k=1}^{\infty} 1-X^k} \\
&= \prod_{k=1}^{\infty} \frac{1}{(1-X^{5k-1})(1-X^{5k-4})}.
\end{aligned}$$

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{X^{k^2+k}}{X^k!} &\overset{(3.3)}{=} \sum_{k=0}^{\infty} \lim_{n\to\infty} \left\langle {n \atop k} \right\rangle X^{k^2+k} \\
&= \lim_{n\to\infty} \sum_{k=0}^{\infty} \left\langle {n \atop k} \right\rangle X^{k^2+k} \\
&\overset{(3.18)}{=} \lim_{n\to\infty} \sum_{k=-\infty}^{\infty} (-1)^k X^{\frac{5k^2-3k}{2}} \left\langle {2n+1 \atop n+2k} \right\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=-\infty}^{\infty} (-1)^k X^{\frac{5k^2-3k}{2}} \lim_{n\to\infty} \left\langle {2n+1 \atop n+2k} \right\rangle \\
&\overset{(3.3)}{=} \sum_{k=-\infty}^{\infty} (-1)^k X^{\frac{5k^2-3k}{2}} \prod_{k=1}^{\infty} \frac{1}{1-X^k} \\
&\overset{(3.7)}{=} \frac{\prod_{k=1}^{\infty}(1-X^{5k})(1-X^{5k-1})(1-X^{5k-4})}{\prod_{k=1}^{\infty} 1-X^k} \\
&= \prod_{n=1}^{\infty} \frac{1}{(1-X^{5k-2})(1-X^{5k-3})}.
\end{aligned}
$$

□

We will continue with the last technical lemma and theorem needed for the grand finale, the four-square theorem.

Lemma 3.13. For every $n \in \mathbb{N}$ it holds

$$
\prod_{k=1}^{n}(1-X^k)^2 = \sum_{k=0}^{n}(-1)^k(2k+1)X^{\frac{k^2+k}{2}} \left\langle {2n+1 \atop n-k} \right\rangle. \tag{3.26}
$$

Proof. Just like most other finite theorems, we will prove this one by induction. For $n = 1$, we calculate

$$
\prod_{k=1}^{1}(1-X^k)^2 = (1-X)^2 = 1-2X+X^2
$$

$$
\begin{aligned}
\sum_{k=0}^{1}(-1)^k(2k+1)X^{\frac{k^2+k}{2}} \left\langle {3 \atop 1-k} \right\rangle &= \left\langle {3 \atop 1} \right\rangle - 3X \left\langle {3 \atop 0} \right\rangle \\
&\overset{(3.2)}{=} \left\langle {2 \atop 1} \right\rangle + X^2 \left\langle {2 \atop 0} \right\rangle - 3X \\
&\overset{(3.2)}{=} \left\langle {1 \atop 0} \right\rangle + X \left\langle {1 \atop 1} \right\rangle + X^2 - 3X \\
&= 1 + X + X^2 - 3X = 1 - 2X + X^2.
\end{aligned}
$$

We want to be able to index the right-hand side over the whole $\mathbb{Z}$, to do that we realise

$$
\begin{aligned}
&\sum_{k=0}^{n}(-1)^k(2k+1)X^{\frac{k^2+k}{2}} \left\langle {2n+1 \atop n-k} \right\rangle \\
&\qquad = \sum_{k=-n-1}^{-1} (-1)^{-k-1}(2(-k-1)+1)X^{\frac{(-k-1)^2+(-k-1)}{2}} \left\langle {2n+1 \atop n-(-k-1)} \right\rangle \\
&\qquad = -\sum_{k=-n-1}^{-1} (-1)^k(-2k-1)X^{\frac{(k^2+2k+1-k-1)}{2}} \left\langle {2n+1 \atop n+k+1} \right\rangle \\
&\qquad = \sum_{k=-n-1}^{-1} (-1)^k(2k+1)X^{\frac{(k^2+k)}{2}} \left\langle {2n+1 \atop n-k} \right\rangle \\
&\qquad = \frac{1}{2} \sum_{k=-n-1}^{n} (-1)^k(2k+1)X^{\frac{k^2+k}{2}} \left\langle {2n+1 \atop n-k} \right\rangle.
\end{aligned}
$$

For other $k < -n-1$ or $k > n$, the Gauss coefficient is zero, so it is fine to index by all of $\mathbb{Z}$. We will use Lemma 3.3 multiple times:

$$\begin{aligned}
2\sum_{k=0}^{n}(-1)^k(2k+1)X^{\frac{k^2+k}{2}}\left\langle{2n+1 \atop n-k}\right\rangle &= \sum_{k=-\infty}^{\infty}(-1)^k(2k+1)X^{\frac{k^2+k}{2}}\left\langle{2n+1 \atop n-k}\right\rangle \\
&\overset{(3.1)}{=} \sum_{k=-\infty}^{\infty}(-1)^k(2k+1)X^{\frac{k^2+k}{2}}X^{n-k}\left\langle{2n \atop n-k}\right\rangle \\
&\quad + \sum_{k=-\infty}^{\infty}(-1)^k(2k+1)X^{\frac{k^2+k}{2}}\left\langle{2n \atop n-k-1}\right\rangle \\
&\overset{(3.2)}{=} \sum_{k=-\infty}^{\infty}(-1)^k(2k+1)X^{\frac{k^2+k}{2}}X^{n-k}\left\langle{2n-1 \atop n-k}\right\rangle \\
&\quad + \sum_{k=-\infty}^{\infty}(-1)^k(2k+1)X^{\frac{k^2+k}{2}}X^{n-k+n-k}\left\langle{2n-1 \atop n-k-1}\right\rangle \\
&\quad + \sum_{k=-\infty}^{\infty}(-1)^k(2k+1)X^{\frac{k^2+k}{2}}\left\langle{2n-1 \atop n-k-1}\right\rangle \\
&\quad + \sum_{k=-\infty}^{\infty}(-1)^k(2k+1)X^{\frac{k^2+k}{2}}X^{n+k}\left\langle{2n-1 \atop n-k-2}\right\rangle \\
&= X^n\sum_{k=-\infty}^{\infty}(-1)^k(2k+1)X^{\frac{k^2-k}{2}}\left\langle{2n-1 \atop n-k}\right\rangle \\
&\quad + X^{2n}\sum_{k=-\infty}^{\infty}(-1)^k(2k+1)X^{\frac{k^2+k}{2}}\left\langle{2n-1 \atop n-k-1}\right\rangle \\
&\quad + \sum_{k=-\infty}^{\infty}(-1)^k(2k+1)X^{\frac{k^2+k}{2}}\left\langle{2n-1 \atop n-k-1}\right\rangle \\
&\quad + X^n\sum_{k=-\infty}^{\infty}(-1)^k(2k+1)X^{\frac{k^2+3k}{2}}\left\langle{2n-1 \atop n-k-2}\right\rangle .
\end{aligned}$$

We will now rearrange and adjust the sums into the right-hand side term for $n-1$, which will prepare them for the induction. We will use reindexations $k \to k-1$ and $k \to k+1$ to match the binomial coefficients.

$$\begin{aligned}
&2\sum_{k=0}^{n}(-1)^k(2k+1)X^{\frac{k^2+k}{2}}\left\langle{2n+1 \atop n-k}\right\rangle \\
&\quad = X^n\sum_{k=-\infty}^{\infty}(-1)^{k+1}(2(k+1)+1)X^{\frac{(k+1)^2-(k+1)}{2}}\left\langle{2n-1 \atop n-(k+1)}\right\rangle \\
&\qquad + X^n\sum_{k=-\infty}^{\infty}(-1)^k(2(k-1)+1)X^{\frac{(k-1)^2+3(k-1)}{2}}\left\langle{2n-1 \atop n-(k-1)-2}\right\rangle \\
&\qquad + (1+X^{2n})\sum_{k=-\infty}^{\infty}(-1)^{k-1}(2k+1)X^{\frac{k^2+k}{2}}\left\langle{2n-1 \atop n-k-1}\right\rangle \\
&\quad \overset{IS}{=} -X^n\sum_{k=-\infty}^{\infty}(-1)^k(2k+3)X^{\frac{k^2+k}{2}}\left\langle{2n-1 \atop n-k-1}\right\rangle \\
&\qquad - X^n\sum_{k=-\infty}^{\infty}(-1)^k(2k-1)X^{\frac{k^2+k}{2}}\left\langle{2n-1 \atop n-k-1}\right\rangle
\end{aligned}$$

$$
\begin{aligned}
&\quad + (1+X^{2n})2\prod_{k=0}^{n-1}(1-X^k)^2 \\
&= -2X^n \sum_{k=-\infty}^{\infty}(-1)^k(2k+1)X^{\frac{k^2+k}{2}} \left\langle \begin{matrix} 2n-1 \\ n-k-1 \end{matrix} \right\rangle \\
&\quad + (1+X^{2n})2\prod_{k=0}^{n-1}(1-X^k)^2 \\
&\overset{IS}{=} -2X^n 2\prod_{k=0}^{n-1}(1-X^k)^2 + (1+X^{2n})2\prod_{k=0}^{n-1}(1-X^k)^2 \\
&= (1-X^n)^2 2\prod_{k=0}^{n-1}(1-X^k)^2 = 2\prod_{k=0}^{n}(1-X^k)^2. \qquad \square
\end{aligned}
$$

The infinite version of the preceding lemma is contributed to Jacobi.

Theorem 3.14 (Jacobi cubic formula)**.** *It holds*

$$
\prod_{k=1}^{\infty}(1-X^k)^3 = \sum_{k=0}^{\infty}(-1)^k(2k+1)X^{\frac{k^2+k}{2}}. \tag{3.27}
$$

Proof. We use the previous lemma

$$
\begin{aligned}
\prod_{k=0}^{\infty}(1-X^k)^3 &= \lim_{n\to\infty}\prod_{k=0}^{n}(1-X^k)^3 \\
&\overset{(3.26)}{=} \lim_{n\to\infty}\sum_{k=0}^{n}(-1)^k(2k+1)X^{\frac{k^2+k}{2}} \left\langle \begin{matrix} 2n+1 \\ n-k \end{matrix} \right\rangle \prod_{k=0}^{n}(1-X^k) \\
&= \lim_{n\to\infty}\sum_{k=0}^{n}(-1)^k(2k+1)X^{\frac{k^2+k}{2}} \\
&\qquad \cdot \frac{(1-X)...(1-X^{2n+1})\cdot(1-X)...(1-X^n)}{(1-X)...(1-X^{n+k+1})\cdot(1-X)...(1-X^{n-k})} \\
&= \lim_{n\to\infty}\sum_{k=0}^{n}(-1)^k(2k+1)X^{\frac{k^2+k}{2}} \\
&\qquad \cdot(1-X^{n+k+2})...(1-X^{2n+1})\cdot(1-X^{n-k+1})...(1-X^n) \\
&= \sum_{k=0}^{\infty}(-1)^k(2k+1)X^{\frac{k^2+k}{2}} \\
&\qquad \cdot \lim_{n\to\infty}(1-X^{n+k+2})...(1-X^{2n+1})\cdot(1-X^{n-k+1})...(1-X^n) \\
&= \sum_{k=0}^{\infty}(-1)^k(2k+1)X^{\frac{k^2+k}{2}}.
\end{aligned}
$$

where we use again the fact, that all the non-1 terms in the limit have only arbitrarily large powers of X, which have arbitrarily small norms. $\square$

Now we finally reach the four-square theorem. Recall

Theorem 3.1(Lagrange-Jacobi) Every integer can be expressed as a sum of four squares. Furthermore, the number of different ways to do this is given by

$$
q(n) = |\{(a,b,c,d)\in\mathbb{Z}^4; n = a^2+b^2+c^2+d^2\}| = 8\sum_{4\nmid d|n} d.
$$

We will, however, start with a lemma which is the most important part of the proof. The most important idea is the fact we find the formal power series with coefficients equal to $q(n)$. We realise that when we use the series

$$\left(\sum_{n=-\infty}^{\infty} X^{n^2}\right)^4,$$

every term in the expansion has the form $X^{a^2+b^2+c^2+d^2}$, thus the coefficient of X^n counts the number of ways n can be written as a sum of four squares, namely $q(n)$. Therefore we (for now not completely formally) realise the most important part of the proof:

$$\left(\sum_{n=-\infty}^{\infty} X^{n^2}\right)^4 = \sum_{n=1}^{\infty} q(n)X^n.$$

The next lemma manipulates this sum in a technical way and prepares it for the four-square theorem.

Lemma 3.15 (main lemma for proving the four square theorem)**.** We have

$$\left(\sum_{k=-\infty}^{\infty} X^{k^2}\right)^4 = 1 + 8\sum_{k=1}^{\infty} \frac{2kX^{2k}}{1+X^{2k}} + \frac{(2k-1)X^{2k-1}}{1-X^{2k-1}}.$$

Proof. We start with the latter Jacobi theorem, the cubic formula (Theorem 3.14), yielding

$$\begin{aligned}
&\prod_{k=0}^{\infty}(1-X^k)^3 = \frac{1}{2}\left(2\sum_{k=0}^{\infty}(-1)^k(2k+1)X^{\frac{k^2+k}{2}}\right)\\
&= \frac{1}{2}\left(\sum_{k=1}^{\infty}(-1)^k(2k+1)X^{\frac{k^2+k}{2}} + \sum_{k=-\infty}^{-1}(-1)^{(-k-1)}(2(-k-1)+1)X^{\frac{(-k-1)^2+(-k-1)}{2}}\right)\\
&= \frac{1}{2}\left(\sum_{k=1}^{\infty}(-1)^k(2k+1)X^{\frac{k^2+k}{2}} + \sum_{k=-\infty}^{-1}(-1)^k(2k+1)X^{\frac{k^2+k}{2}}\right)\\
&= \frac{1}{2}\sum_{k=-\infty}^{\infty}(-1)^k(2k+1)X^{\frac{k^2+k}{2}},
\end{aligned}$$

with the second equality being just a reindexation $k \to -k-1$. We square both sides and denote the result by α

$$\alpha = \prod_{k=1}^{\infty}(1-X^k)^6 = \frac{1}{2}\sum_{k,\ell=-\infty}^{\infty}(-1)^{k+\ell}(2k+1)(2\ell+1)X^{\frac{k^2+k+\ell^2+\ell}{2}}.$$

We will not distribute the terms on the right-hand side since we intend to first introduce the following substitution. The sum can be split into two, where $k+l$ is even in one and odd in the other. Then we set $k = s+t$, $\ell = s-t$ in the sum with even terms and $k = s+t$, $\ell = t-s-1$ in the other one.
We have to show the substitution is correctly used, namely that all s, t are indeed integers with each pair corresponding to exactly one pair of k, ℓ. However, we have

$$s = \begin{cases} \frac{k+\ell}{2} & k+\ell \text{ even.} \\ \frac{k-\ell+1}{2} & k+\ell \text{ odd.} \end{cases} \qquad t = \begin{cases} \frac{k-\ell}{2} & k+\ell \text{ even.} \\ \frac{k+\ell+1}{2} & k+\ell \text{ odd.} \end{cases}$$

So for every pair of (k, ℓ), we have at most one pair of (s, t), and we do have it by definition. It is also clear that s and t are always integers. The next steps are therefore valid.

$$\begin{aligned}
\alpha &= \frac{1}{2} \sum_{k,\ell=-\infty}^{\infty} (-1)^{k+\ell}(2k+1)(2\ell+1)X^{\frac{k^2+k+\ell^2+\ell}{2}} \\
&= \frac{1}{4} \sum_{s,t=-\infty}^{\infty} (2s+2t+1)(2s-2t+1)X^{\frac{(s+t)^2+s+t+(s-t)^2+s-t}{2}} \\
&\qquad - \frac{1}{4} \sum_{s,t=-\infty}^{\infty} (2s+2t+1)(2t-2s-1)X^{\frac{(s+t)^2+s+t+(t-s-1)^2+t-s-1}{2}} \\
&= \frac{1}{4} \sum_{s,t=-\infty}^{\infty} \left((2s+1)^2 - (2t)^2\right) X^{s^2+s+t^2} - \frac{1}{4} \sum_{s,t=-\infty}^{\infty} \left((2t)^2 - (2s+1)^2\right) X^{s^2+s+t^2} \\
&= \frac{1}{2} \sum_{s,t=-\infty}^{\infty} \left((2s+1)^2 - (2t)^2\right) X^{s^2+s+t^2} \\
&= \frac{1}{2} \sum_{t=-\infty}^{\infty} X^{t^2} \sum_{s=-\infty}^{\infty} \left((2s+1)^2\right) X^{s^2+s} - \frac{1}{2} \sum_{s=-\infty}^{\infty} X^{s^2+s} \sum_{t=-\infty}^{\infty} \left((2t)^2\right) X^{t^2}.
\end{aligned}$$

We can now introduce series β and γ defined as follows:

$$\begin{aligned}
\beta &= \sum_{t=-\infty}^{\infty} X^{t^2}, & \beta' &= \sum_{t=-\infty}^{\infty} t^2 X^{t^2-1}, \\
\gamma &= \frac{1}{2} \sum_{s=\infty}^{\infty} X^{s^2+s}, & \gamma' &= \frac{1}{2} \sum_{s=\infty}^{\infty} (s^2+s) X^{s^2+s-1}.
\end{aligned}$$

This will allow us to greatly simplify the expression for α using these two series and their derivatives. We rewrite the first summand

$$\begin{aligned}
\frac{1}{2} \sum_{t=-\infty}^{\infty} X^{t^2} \sum_{s=-\infty}^{\infty} \left((2s+1)^2\right) X^{s^2+s} &= \frac{1}{2}\beta \sum_{s=-\infty}^{\infty} (4s^2+4s+1) X^{s^2+s} \\
&= \frac{1}{2}\beta X \sum_{s=-\infty}^{\infty} (4s^2+4s) X^{s^2+s-1} + \frac{1}{2}\beta \sum_{s=-\infty}^{\infty} X^{s^2+s} \\
&= 4\beta\gamma' X + \beta\gamma,
\end{aligned}$$

and the second summand

$$\begin{aligned}
\frac{1}{2} \sum_{s=-\infty}^{\infty} X^{s^2+s} \sum_{t=-\infty}^{\infty} \left((2t)^2\right) X^{t^2} &= \gamma \cdot 4X \sum_{t=-\infty^{\infty}} t^2 X^{t^2-1} \\
&= 4\gamma\beta' X.
\end{aligned}$$

Therefore together we have

$$\alpha = 4\beta\gamma' X + \beta\gamma + 4\gamma\beta' X = \beta\gamma + 4X(\beta\gamma' - \beta'\gamma).$$

Now we will use the examples derived from Theorem 3.8(Jacobi's triple product formula) to turn our sum into a product. We will then repeatedly use Lemma 2.9 about the properties of the derivative. The result will then be substituted into the new formula for α.

$$\beta' = \left(\sum_{t=-\infty}^{\infty} X^{t^2} \right)'$$

$$
\begin{aligned}
&\overset{(3.11)}{=} \left(\prod_{k=1}^{\infty}(1-X^{2k})(1+X^{2k-1})^2\right)' \\
&\overset{2.9(II)}{=} \left(\prod_{k=1}^{\infty}(1-X^{2k})\right)' \prod_{k=1}^{\infty}(1+X^{2k-1})^2 + \prod_{k=1}^{\infty}(1-X^{2k}) \left(\prod_{k=1}^{\infty}(1+X^{2k-1})^2\right)' \\
&\overset{2.9(V)}{=} \left(\prod_{k=1}^{\infty}(1-X^{2k})\right)' \prod_{k=1}^{\infty}(1+X^{2k-1})^2 \\
&\qquad + \prod_{k=1}^{\infty}(1-X^{2k}) 2 \prod_{k=1}^{\infty}(1+X^{2k-1}) \left(\prod_{k=1}^{\infty}(1+X^{2k-1})\right)' \\
&\overset{2.9(III)}{=} \left(\prod_{k=1}^{\infty}(1-X^{2k})(1+X^{2k-1})^2\right) \sum_{k=1}^{\infty} \frac{(-X^{2k})'}{1-X^{2k}} \\
&\qquad + \left(\prod_{k=1}^{\infty}(1-X^{2k})(1+X^{2k-1})^2\right) \sum_{k=1}^{\infty} 2\frac{(X^{2k-1})'}{(1+X^{2k-1})} \\
&= \left(\prod_{k=1}^{\infty}(1-X^{2k})(1+X^{2k-1})^2\right) \sum_{k=1}^{\infty} \frac{-2kX^{2k-1}}{1-X^{2k}} \\
&\qquad + \left(\prod_{k=1}^{\infty}(1-X^{2k})(1+X^{2k-1})^2\right) \sum_{k=1}^{\infty} 2\frac{(2k-1)X^{2k-2}}{1+X^{2k-1}} \\
&= \beta \sum_{k=1}^{\infty} \left(\frac{2(2k-1)X^{2k-2}}{1+X^{2k-1}} - \frac{2kX^{2k-1}}{1-X^{2k}}\right).
\end{aligned}
$$

Analogous calculation for γ yields

$$
\begin{aligned}
\gamma' &= \left(\frac{1}{2}\sum_{s=\infty}^{\infty} X^{s^2+s}\right)' \\
&\overset{(3.13)}{=} \left(\prod_{k=1}^{\infty}(1-X^{2k})(1+X^{2k})^2\right)' \\
&\overset{2.9(II)}{=} \left(\prod_{k=1}^{\infty}(1-X^{2k})\right)' \prod_{k=1}^{\infty}(1+X^{2k})^2 + \prod_{k=1}^{\infty}(1-X^{2k}) \left(\prod_{k=1}^{\infty}(1+X^{2k})^2\right)' \\
&\overset{2.9(V)}{=} \left(\prod_{k=1}^{\infty}(1-X^{2k})\right)' \prod_{k=1}^{\infty}(1+X^{2k})^2 \\
&\qquad + \prod_{k=1}^{\infty}(1-X^{2k}) 2 \prod_{k=1}^{\infty}(1+X^{2k}) \left(\prod_{k=1}^{\infty} 1+X^{2k}\right)' \\
&\overset{2.9(III)}{=} \left(\prod_{k=1}^{\infty}(1-X^{2k})(1+X^{2k})^2\right) \sum_{k=1}^{\infty} \frac{(-X^{2k})'}{1-X^{2k}} \\
&\qquad + \left(\prod_{k=1}^{\infty}(1-X^{2k})(1+X^{2k})^2\right) \sum_{k=1}^{\infty} 2\frac{(X^{2k})'}{(1+X^{2k})} \\
&= \left(\prod_{k=1}^{\infty}(1-X^{2k})(1+X^{2k})^2\right) \sum_{k=1}^{\infty} \frac{-2kX^{2k-1}}{1-X^{2k}} \\
&\qquad + \left(\prod_{k=1}^{\infty}(1-X^{2k})(1+X^{2k})^2\right) \sum_{k=1}^{\infty} 2\frac{2kX^{2k-1}}{(1+X^{2k})} \\
&= \gamma \sum_{k=1}^{\infty} \left(2\frac{2kX^{2k-1}}{1+X^{2k}} - \frac{2kX^{2k-1}}{1-X^{2k}}\right).
\end{aligned}
$$

And we return to the formula for α

$$\begin{aligned}
\alpha &= \beta\gamma + 4X(\beta\gamma' - \beta'\gamma) \\
&= \beta\gamma + 4X\left(\beta\gamma\sum_{k=1}^{\infty}\left(2\frac{2kX^{2k-1}}{1+X^{2k}} - \frac{2kX^{2k-1}}{1-X^{2k}}\right)\right. \\
&\qquad \left. - \gamma\beta\sum_{k=1}^{\infty}\left(\frac{2(2k-1)X^{2k-2}}{1+X^{2k-1}} - \frac{2kX^{2k-1}}{1-X^{2k}}\right)\right) \\
&= \beta\gamma\left(1 + 4X\left(2\sum_{k=1}^{\infty}\left(\frac{2kX^{2k-1}}{1+X^{2k}} - \frac{kX^{2k-1}}{1-X^{2k}}\right)\right.\right. \\
&\qquad \left.\left. - \left(\frac{(2k-1)X^{2k-2}}{1+X^{2k-1}} - \frac{kX^{2k-1}}{1-X^{2k}}\right)\right)\right) \\
&= \beta\gamma\left(1 + 8\left(\sum_{k=1}^{\infty}\left(\frac{2kX^{2k}}{1+X^{2k}} - \frac{kX^{2k}}{1-X^{2k}}\right) - \left(\frac{(2k-1)X^{2k-1}}{1+X^{2k-1}} - \frac{kX^{2k}}{1-X^{2k}}\right)\right)\right) \\
&= \beta\gamma\left(1 + 8\sum_{k=1}^{\infty}\left(\frac{2kX^{2k}}{1+X^{2k}} - \frac{(2k-1)X^{2k-1}}{1+X^{2k-1}}\right)\right).
\end{aligned}$$

We have shown that

$$\frac{\alpha}{\beta\gamma} = \left(1 + 8\sum_{k=1}^{\infty}\left(\frac{2kX^{2k}}{1+X^{2k}} - \frac{(2k-1)X^{2k-1}}{1+X^{2k-1}}\right)\right).$$

Now we rewrite $\beta\gamma$ back into its product form. Since α is defined as a product, we will be able to cancel out certain terms in $\frac{\alpha}{\beta\gamma}$, which will appear in the proof's final calculation.

$$\begin{aligned}
\beta\gamma &= \prod_{k=1}^{\infty}(1-X^{2k})^2(1+X^{2k-1})^2(1+X^{2k})^2 \\
&= \prod_{k=1}^{\infty}(1-X^{2k})^2(1+X^k)^2.
\end{aligned}$$

We notice the second (third) term in the product is just the odd (even) part of $(1+X^k)$, so we merge them together. We will now rewrite our product in such a way that as many terms as possible cancel out with α when we calculate $\frac{\alpha}{\beta\gamma}$.

$$\begin{aligned}
\beta\gamma &= \prod_{k=1}^{\infty}(1-X^{2k})^2(1+X^k)^2\frac{(1-X^{2k})^2}{(1+X^k)^2(1-X^k)^2} \\
&= \prod_{k=1}^{\infty}(1-X^{2k})^4(1-X^k)^{-2}.
\end{aligned}$$

Now comes the last part of the proof. We remind the reader of the definition of α:

$$\alpha = \prod_{k=1}^{\infty}(1-X^k)^6.$$

We will now use the last example we have prepared after proving Jacobi's triple product formula (Theorem 3.8), as it gives the first equality.

$$\begin{aligned}\left(\sum_{k=-\infty}^{\infty}(-1)^k X^{k^2}\right)^4 &\overset{(3.12)}{=} \left(\prod_{k=1}^{\infty}\frac{(1-X^k)^2}{(1-X^{2k})}\right)^4\\ &= \prod_{k=1}^{\infty}(1-X^k)^6\frac{1}{(1-X^{2k})^4(1-X^k)^{-2}}\\ &= \frac{\alpha}{\beta\gamma}\\ &= 1+8\sum_{k=1}^{\infty}\frac{2kX^{2k}}{1+X^{2k}}-\frac{(2k-1)X^{2k-1}}{1+X^{2k-1}}.\end{aligned}$$

Composing $-X$ with the series gives us the theorem:

$$\begin{aligned}\left(\sum_{k=-\infty}^{\infty}(-1)^k(-X)^{k^2}\right)^4 &= 1+8\sum_{k=1}^{\infty}\frac{2k(-X)^{2k}}{1+(-X)^{2k}}-\frac{(2k-1)(-X)^{2k-1}}{1+(-X)^{2k-1}}\\ \left(\sum_{k=-\infty}^{\infty}X^{k^2}\right)^4 &= 1+8\sum_{k=1}^{\infty}\frac{2kX^{2k}}{1+X^{2k}}+\frac{(2k-1)X^{2k-1}}{1-X^{2k-1}}.\end{aligned}$$ □

Everything is now ready for the proof of the main theorem. We formalize the intuitive understanding of the equality of the sum in the theorem and the sum in the last lemma, and continue with a small trick and some more substitutions.

Proof of Theorem 3.1. We begin by showing that

$$\begin{aligned}\left(\sum_{n=-\infty}^{\infty}X^{n^2}\right)^4 &= \sum_{k=-\infty}^{\infty}X^{k^2}\sum_{\ell=-\infty}^{\infty}X^{\ell^2}\sum_{m=-\infty}^{\infty}X^{m^2}\sum_{n=-\infty}^{\infty}X^{n^2}\\ &= \sum_{k,\ell,m,n\in\mathbb{Z}}X^{k^2+\ell^2+m^2+n^2}\overset{def}{=}\sum_{n=1}^{\infty}q(n)X^n.\end{aligned}$$

We will follow by using the lemma and manipulating the resulting series. Concretely, we force in a new term by adding and subtracting it.

$$\begin{aligned}\sum_{n=1}^{\infty}q(n)X^n &= (\sum_{n=-\infty}^{\infty}X^{n^2})^4\\ &= 1+8\sum_{k=1}^{\infty}\frac{2kX^{2k}}{1+X^{2k}}+\frac{(2k-1)X^{2k-1}}{1-X^{2k-1}}\\ &= 1+8\sum_{k=1}^{\infty}\frac{2kX^{2k}}{1+X^{2k}}-\frac{2kX^{2k}}{1-X^{2k}}+\frac{2kX^{2k}}{1-X^{2k}}+\frac{(2k-1)X^{2k-1}}{1-X^{2k-1}}\\ &= 1+8\sum_{k=1}^{\infty}\frac{-4kX^{4k}}{1-X^{4k}}+\frac{2kX^{2k}}{1-X^{2k}}+\frac{(2k-1)X^{2k-1}}{1-X^{2k-1}}.\end{aligned}$$

We now observe that the latter fractions give two parts of the series $\sum_{k=1}^{\infty}\frac{kX^k}{1-X^k}$ since one gives the terms for k odd and the other for k even. We can therefore substitute

$$\sum_{n=1}^{\infty}q(n)X^n = 1+8\sum_{k=1}^{\infty}\frac{-4kX^{4k}}{1-X^{4k}}+\frac{kX^k}{1-X^k} = 1+8\sum_{4\nmid k}\frac{kX^k}{1-X^k}.$$

We finish the proof by using the geometric sum formula and reindexing the sums

$$\begin{aligned}\sum_{n=1}^{\infty} q(n)X^n &= 1 + 8\sum_{4\nmid k} \frac{kX^k}{1-X^k} \\ &= 1 + 8\sum_{4\nmid k}\sum_{\ell=1}^{\infty} kX^{kl} \\ &= 1 + 8\sum_{n=1}^{\infty}\sum_{4\nmid d|n} dX^n. \qquad \square\end{aligned}$$

Conclusion

In this thesis, we have defined and proved many important notions about formal power series. In our next study, we might continue expanding our theoretical knowledge of the formal power series. We might for example define negative powers of X and define series similar to Laurent series in analysis and explore their value. We can then even consider fractional powers for X. Another option is to go in the direction of linear algebra and consider more different indeterminants $X_1, ..., X_n$ and look for theoretical results there.
The second option is to look to make use of the defined machinery to use it in more practical computations. We can analyze naturally occurring sequences using generating functions (which we outlined in Example 2.17) and use the theoretical formulas in the analysis of the resulting formal power series. This leads for example to the theory of partitions.

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