REVIEW BY THE OPPONENT OF THE BACHELOR THESIS

Thesis title: Proving combinatorial identities via formal power series
Thesis author: Thomáš Sklenář

SUMMARY OF THE THESIS CONTENT

In this thesis, the author aims to provide a comprehensive understanding of formal power series, detailing their properties and applications in well-known combinatorial identities, such as the Roger-Ramanujan identities, Jacobi's triple product, Jacobi's cubic formula, and the Lagrange-Jacobi four-square theorem. The work is structured into three chapters, with the first chapter establishing formal power series as an extension of polynomials while incorporating a canonical metric structure with ultrametric properties that merge algebraic structures with the analytical notion of convergence. Following this, the author establishes various properties of formal power series over a commutative field (in particular, when $K = \mathbb{C}$), demonstrating that they form a vector space and an integral domain, along with important theorems regarding convergence and composition. The second chapter focuses on definitions and properties of the formal exponential, logarithm, and derivative, providing detailed proofs and examples to enhance clarity. In the final chapter, the author applies the developed theory to prove the above-mentioned combinatorial identities, including the Lagrange-Jacobi four-square theorem, which states that every positive integer n can be represented as a sum of four squares, with the number of such representations given by the formula $8 \sum_{4\nmid d|n} d$.

OVERALL EVALUATION OF THE THESIS

Thesis topic. This topic is appropriately challenging for a bachelor's thesis and meets the assignment's requirements by combining theoretical depth with applications. The author successfully presents the material in a clear and detailed manner, making it accessible to readers with minimal prerequisite knowledge.

Author's contribution. While the thesis primarily serves as a detailed exposition of the existing source material ("An invitation to formal power series" by Benjamin Sambale), it also includes the author's original contributions. These contributions consist of additional examples and intuitive explanations that enhance the clarity of the exposition. For example, the author does a good job of explaining the reason behind certain conditions needed to obtain a well-defined expression of the composition of two formal series on page 15, and further derives the identity $\sin^2 X + \cos^2 X = 1$ (and other trigonometrical identities) using the formal power series expression of $\sin X$ and $\cos X$ in Example 2.14, which are absent in the source material. The author effectively specifies these contributions throughout the thesis, particularly in the expanded proofs (for example, the proofs of Jacobi's cubic formula in Theorem 3.14 and the lemma (Lemma 3.13) leading up to it) and corrections of inaccuracies found in the source material. This approach further helps in the reader's comprehension.

Mathematical level. The mathematical level of the thesis is appropriate for a bachelor's degree, containing rigorous and correctly formulated mathematical text. The author demonstrates a solid grasp of the subject matter by deriving results that are not present in the main source material.

Work with sources. The source material and other references are correctly cited in the thesis. This thesis does not contain verbatim copied or word-to-word translated passages to the best of my knowledge, and it further includes original materials derived and explained by the author.

Formal editing. The thesis is well-structured. It successfully achieves its goal and provides a coherent and well-explained exposition of the proposed study.

COMMENTS AND QUESTIONS

1. However, the thesis contains a few minor typos, in particular:

   In example 3.10 (page 35), the α is taken to be $-X^3$, while the referenced theorem (Theorem 3.8) is applicable only when $\alpha \in \mathbb{C}[[X]] \setminus (X^2)$. I believe the theorem remains valid even when $\alpha \in \mathbb{C}((X)) \setminus \{0\}$, which would justify the choice of α in this example.

CONCLUSION

The thesis is well-organized and effectively conveys its key insights. The author's intuitive explanations and additional examples enhance the reader's understanding of the material. In conclusion, I consider the thesis to be excellent and recommend that it be accepted as a bachelor's thesis.

Proposed Grade (1-2-3-4): 1

The classification will be communicated by the supervisor/opponent to the chair of the examination committee.

Name of opponent, signature: Subham Roy Subham Roy

Department: Department of Algebra (Faculty of Mathematics and Physics)

Date: 24/08/2024