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Coherent sheaves on singular curves of genus one

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I would like to thank my supervisor for many useful discussions and for his patience in answering all my questions.

Title: Coherent sheaves on singular curves of genus one

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Abstract: We study properties of coherent torsion sheaves on smooth and singular curves and classify such sheaves on a nodal singular curve. We investigate the bounded derived category of coherent sheaves on a singular Weierstrass curve of genus one. As a main tool we will use Siedel-Thomas twist functors. The notion of semi-stability and the numerical invariants degree and rank are essential for understanding of the complexity of such a category. We show that any category of semi-stable coherent sheaves of a given phase is equivalent to the category of torsion coherent sheaves on a singular Weierstrass curve.

Keywords: semi-stability, phase, torsion sheaves, coherent sheaves, Weierstrass curve

Název práce: Koherentné zväzky na singulárnych krivkách rodu jedna

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Abstrakt: Budeme skúmať vlastnosti koherentných torzných zväzkov na hladkých a singulárnych krivkách. Ďalej klasifikujeme torzné koherentné zväzky na nodálnej singulárnej krivke. Budeme skúmať ohraničenú derivovanú kategóriu koherentných zväzkov na singulárnej Weierstrassovej krivke rodu jedna. Ako základný nástroj použijeme Siedelove a Thomasove rotačné funktory. Pojem semi-stability a numerické invarianty rank a stupeň tvoria základ pochopenia štruktúry danej kategórie. Ukážeme, že kategórie semi-stabilných koherentných zväzkov na singulárnej Weierstrassovej krivke danej fázy sú ekvivalentné kategórii torzných koherentných zväzkov.

Klíčová slova: semi-stabilita, fáza, torzné zväzky, koherentné zväzky, Weierstrassová krivka

Contents

Introduction

One of the central objects of study in algebraic geometry are varieties and morphisms between them, and the study of varieties corresponds to the study of sheaves on them, to a large extent. Of particular interest is the study of quasi-coherent and coherent sheaves, respectively. Naturally, this leads to the study of their corresponding Abelian categories $Coh(X)$ and $Qcoh(X)$ and the functors between them. For instance, the functors as pushforward f_* or pullback f^* induced by a morphism f are not exact, which leads to complications. Historically, the notion of derived functors was introduced to correct non-exact functors. Later, the technique was developed by Grothendieck, which led to the new concept: to study derived categories and derived functors between them.

One type of functors between derived categories are of particular interest. The so called Fourier-Mukai transforms. Let X and Y be smooth projective varietis. We have the fiber product $X \times Y$ together with two projections $q : X \times Y \to X$ and $p : X \times Y \to Y$. Then any object $\mathcal{P} \in D^b(X \times Y)$ defines an exact functor

$$\Phi_{\mathcal{P}} : D^b(X) \longrightarrow D^b(Y), \quad \mathcal{E} \longmapsto p_*(q^*\mathcal{E} \otimes \mathcal{P}),$$

where p_*, q^* and $\otimes$ are derived functors. We call such functors Fourier-Mukai transforms. In fact, the famous result of Orlov is, that any equivalence between derived categories of smooth projective varieties is geometric in nature, i.e., of Fourier-Mukai type, see Theorem 5.14 in [1], hovever, not so much is known in singular care as it is pointed out in [2] pg. 2.

It turned out that the study of derived categories is closely related to string theory. Indeed, the study of derived categories of coherent sheaves on smooth projective varieties attracted a lot of interest mainly driven by homological mirror symmetry conjecture, see [2], Introduction. Furthermore, the key notion of a stability condition, an invariant of the derived category D with its natural topology was motivated by work of Douglas on Π- stability for Dirichlet branes. For more details, see [3] part 1.4. and Introduction.

As we can see, the motivation to study and investigate the structure of the derived category of coherent sheaves is clear.

The aim of this thesis is to give a better description of the bounded derived category of coherent sheaves $\mathcal{D}^b(Coh(\mathbf{E}))$ on a singular Weierstrass curve $\mathbf{E}$ of arithmetic genus one. We follow the work of Burban and Kreussler [4].

In the smooth case, such a structure was described by Atiyah's work in [5]. In the singular case, however, some crucial properties fail, e.g., Serre duality, in general, is no longer true and the homological dimension of $Coh(X)$ is infinite among others. These differences are described in [4] Table 1.

We focus on the key result of Chapter 4 in [4]. They prove, that the Fourier-Mukai transform given by Thomas-Siedel twist functors [6] preserves semi-stability of sheaves. The direct consequence of their result yields the better description of the category in the sense of Corollary 4.3 in [4]. It says, that the Abelian subcategory of semi-stable coherent sheaves of given phase is equivalent to the category of torsion coherent sheaves on **E**. We prove this corollary in all its details in Chapter 4. One can refer to these subcategories as 'slices', the meaning is obvious from [4] Chapter 4, Figure 1.

For this, in Chapter 1 we recall basic definitions and constructions necessary for understanding the Chapter 2 and Chapter 4. Our main objects of interest are sheaves, schemes as well as their derived (bounded) categories.

In chapter 2, we focus on torsion sheaves. We will investigate the behaviour of torsion coherent sheaves on a smooth and singular curve. We will see that they do not share all properties. Torsion sheaves play a crucial role in Chapter 4.

In Chapter 3, we will see that torsion sheaves supported at the singularity of the nodal singular curve given by the equation $y^2 - x^2(x+1)$ decompose into indecomposable finitely generated string and band modules. Representation theory and the completion will be used as the main tool. We will classify torsion coherent sheaves supported at the singularity, thanks to the work of Crawley-Boevey [7].

Finally, Chapter 4 is devoted to understanding the structure of the bounded derived category of coherent sheaves on a singular Weierstrass curve **E**. We prove that the category is k- linear and define the notions of phase and stability. As a main result, we provide the proof of Corollary 4.3 in [4].

1 Preliminaries

In this section we provide the necessary definitions of sheaves and schemes as well as some elementary examples. We recall the basic definitions and constructions of the triangulated and derived categories. Finally, we define the notion of a curve and the particular example of a curve, namely, a Weierstrass curve.

1.1 Sheaves

Definition 1 (Presheaf)**.** *Let X be a topological space. A presheaf $\mathcal{F}$ of abelian groups on X consists of the following data:*

1. *for any open subset $U \subseteq X$, $\mathcal{F}(U)$ is an abelian group,*
2. *for any pair of subsets $V \subseteq U$ we have a morphism of abelian groups $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$,*

such that

1. *$\mathcal{F}(\emptyset) = 0$,*
2. *$\rho_{UU} : \mathcal{F}(U) \to \mathcal{F}(U)$ is the identity morphism, and*
3. *for three open subsets $W \subseteq V \subseteq U$ we have the relation $\rho_{VW} \circ \rho_{UV} = \rho_{UW}$.*

Remark 2.
- *Elements $s \in \mathcal{F}(U)$ are called sections of $\mathcal{F}$ over U.*
- *Elements $s \in \mathcal{F}(X)$ are called global sections.*
- *The morphisms ρ_{UV} are called restriction maps. We will denote the image of a section s by the restriction map as $s|_V$ and call it the restriction of s from U to V.*
- *For $\mathcal{F}(U)$ we will alternatively use notations $\Gamma(U, \mathcal{F})$ or $H^0(U, \mathcal{F})$.*

Definition 3 (Sheaf)**.** *A presheaf $\mathcal{F}$ on topological space X is a sheaf if it satisfies the following two conditions for any open U and any open cover $\{V_i\}$ of U:*

1. *if $s, t \in \mathcal{F}(U)$ such that $s|_{V_i} = t|_{V_i}$ for every i then $s = t$;*
2. *for any collection of sections $\{s_i\}$ where $s_i \in \mathcal{F}(V_i)$ such that $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, there exists a section $s \in \mathcal{F}(U)$ such that $s|_{V_i} = s_i$.*

Remark 4. *Condition* 1. *in the above definition is called locality and condition* 2. *glueing.*

Definition 5 (Stalk)**.** *If $\mathcal{F}$ is presheaf on a topological space X, and if P is a point of X, we define stalk $\mathcal{F}_P$ of $\mathcal{F}$ to be the direct limit of abelian groups $\mathcal{F}(U)$ for all open sets containing P. That is $\mathcal{F}_P := \varinjlim \mathcal{F}(U)$.*

Remark 6. *For any point $x \in U \subseteq X$, the image of a section $s \in \mathcal{F}(U)$ under the canonical map $\mathcal{F}(U) \to \mathcal{F}_x$ is denoted as s_x. We call s_x germ of the section s at x.*

Definition 7 (Morphism of presheaves)**.** *Let X be a topological space, $\mathcal{F}$ and $\mathcal{G}$ presheaves on X. A morphism of presheaves $\varphi : \mathcal{F} \to \mathcal{G}$ is a collection of morphisms of abelian groups $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for every open subsets $V \subseteq U \subseteq X$ such that the diagram*

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\ {\scriptstyle \rho^{\mathcal{F}}_{UV}}\big\downarrow & & \big\downarrow{\scriptstyle \rho^{\mathcal{G}}_{UV}} \\ \mathcal{F}(V) & \xrightarrow{\varphi(V)} & \mathcal{G}(V) \end{array}$$

commutes.

Definition 8 (Morphism of sheaves)**.** *Let X be a topological space. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on X. A morphism of sheaves $\mathcal{F}$ and $\mathcal{G}$ is defined as a morphism of the corresponding presheaves.*

Proposition 9. *Given a presheaf $\mathcal{F}$, there is a sheaf $\mathcal{F}^+$ and a morphism $\theta : \mathcal{F} \to \mathcal{F}^+$, with the property that for any sheaf $\mathcal{G}$ and any morphism $\varphi : \mathcal{F} \to \mathcal{G}$, there is a unique morphism $\psi : \mathcal{F}^+ \to \mathcal{G}$ such that $\varphi = \psi \circ \theta$. Furthemore the pair $(\mathcal{F}^+, \theta)$ is unique up to unique isomorphism.*

Proof. See [8] II Proposition 1.2. □

Remark 10. *We will refer to this process as the sheafification of a presheaf.*

Definition 11. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{F}$ be a sheaf on X and $\mathcal{G}$ be a sheaf on Y. Then we define the direct image of $\mathcal{F}$ along f as $f_*\mathcal{F}(U) := \mathcal{F}(f^{-1}(U))$ for every open subset $U \subset X$. Moreover, we define the inverse image of $\mathcal{G}$ along f as the sheafification of the presheaf $f^{-1,pre}\mathcal{G}$ defined as*

$$f^{-1,pre}\mathcal{G}(U) := \varinjlim_{f(U) \subset V} \mathcal{G}(V)$$

.

Definition 12 (Skyscraper sheaf)**.** *Let X be a topological space and let $x \in X$ be a point. Denote $i_x : \{x\} \to X$ the inclusion map. Consider an abelian group A as a sheaf on one point topological space $\{x\}$. Then we define skyscraper sheaf on X as $i_{x,*}A$. Similarly, we call a sheaf of abelian groups $\mathcal{F}$ a skyscraper sheaf if there exists a point $x \in X$ and an abelian group A such that $\mathcal{F} \cong i_{x,*}A$ as sheaves of abelian groups.*

Lemma 13. *Let X be a topological space, $x \in X$ a point. Then for any point $x' \in X$ the stalk of the skyscraper sheaf $i_{x,*}A$ is*

$$i_{x,*}A_{x'} = \begin{cases} A & if x' \in \overline{\{x\}} \\ 0 & if x' \notin \overline{\{x\}} \end{cases}$$

Proof. Omitted. □

Proposition 14 (Glueing of morphism of sheaves)**.** *Let X be a topological space with an open covering $\{U_i\}_{i \in I}$. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on X. Each family of morphisms of sheaves $\phi_i : \mathcal{F}_{U_i} \to \mathcal{G}_{U_i}$ such that $\phi_i|_{U_{ij}} = \phi_j|_{U_{ji}}$ gives rise to a unique morphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ such that $\phi|_{U_i} = \phi_i$.*

Proof. [9], Lemma 6.33.1 □

1.2 Schemes

Definition 15. *Let A be a ring. We equip the spectrum $SpecA$ with a topology by defining closed subsets to be $V(I) := \{\mathfrak{p} \in SpecA \mid I \subset \mathfrak{p}\}$ for each ideal $I \subset A$. We call it Zariski topology. Let $f \in A$. By $D(f)$ we denote the complement of the closed subset $V(f)$, that is, $D(f) = \{\mathfrak{p} \in SpecA \mid f \notin \mathfrak{p}\}$. These subsets are clearly open and we call them distinguished open sets or principal open sets.*

Lemma 16. *Let A be a ring and $\{\mathfrak{a}_i\}_{i \in I}$ a family of ideals in A. Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals in A. Then the following is true:*

1. $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a}\mathfrak{b})$;
2. $V(\sum_i \mathfrak{a}_i) = \bigcap_i V(\mathfrak{a}_i)$;
3. $V(A) = \emptyset$, $V(0) = SpecA$.

Proof. Omitted. □

Proposition 17. *Let A be a ring. If $\mathfrak{p}$ is a prime ideal of A, the closure $\overline{\{\mathfrak{p}\}}$ is equal to $V(\mathfrak{p})$,* *Proof.* [9], Lemma 10.26.1. □

Lemma 18. *Let A be a ring. The open sets $D(f)$ form a basis for the Zariski topology on $SpecA$ when f runs through the elements of A.*

Proof. Indeed, for any open subset U in $SpecA$ we have the following equalities:

$$U = U^c = V(\mathfrak{a})^c = V(\textstyle\sum_i (f_i))^c = (\bigcap_i V(f_i))^c = \bigcup_i D(f_i).$$

Here U^c denote the complement of U and $\{f_i\}$ is a set of generators of $\mathfrak{a}$. □

Now we define the sheaf of rings $\mathcal{O}_{SpecA}$ on the topological space $X = SpecA$ as follows, for each open subset $U \subset X$ we define the set of functions

$$\mathcal{O}_X(U) := \{s : U \to \amalg A_{\mathfrak{p}} \mid s \; satisfies \; (1) \; and \; (2)\},$$

where

1. for each $\mathfrak{p} \in U, s(\mathfrak{p}) \in A_{\mathfrak{p}}$, and
2. for each point $\mathfrak{p} \in U$ there is a neighbourhood V of $\mathfrak{p}$ in U and there are elements $f, a \in A$ such that for every $\mathfrak{q} \in V$, $f \notin \mathfrak{q}$ and $s(\mathfrak{q}) = \frac{a}{f}$ in $A_{\mathfrak{q}}$.

The ring structure and the unit element is clear. Moreover, it is easy to see that this is a sheaf of rings.

Proposition 19. *Let A be a ring.*

1. *For any $\mathfrak{p} \in SpecA$, the stalk of the structure sheaf $\mathcal{O}_{SpecA}$ is isomorphic to the local ring $A_{\mathfrak{p}}$.*
2. *For any element $f \in A$, the ring $\mathcal{O}_{SpecA}(D(f))$ is isomorphic to the localized ring A_f.*
3. *In particular, $\Gamma(SpecA, \mathcal{O}_{SpecA}) \cong A$.*

Proof. [8] II, Proposition 2.2. □

Definition 20 (Ringed Space)**.** *A ringed space is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ on X. A morphism of ringed spaces from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair $(f, f^\#)$ of a continuous map $f : X \to Y$ and a map $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of rings on Y.*

Definition 21 (Locally ringed space)**.** *A locally ringed space is a ringed space $(X, \mathcal{O}_X)$ such that stalk $\mathcal{O}_{X,x}$ is a local ring for every point $x \in X$. A morphism of locally ringed spaces is a morphism of ringed spaces such that $f_x^\# : \mathcal{O}_{Y,f(x)} \to (f_*\mathcal{O}_X)_{f(x)} \to \mathcal{O}_{X,x}$ is local for every point x.*

Remark 22. *If $(A, \mathfrak{m}_A)$ and $(B, \mathfrak{m}_B)$ are local rings, a homomorphism $\varphi : A \to B$ is called a local homomoprhism if $\varphi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$.*

Example 23. *$(SpecA, \mathcal{O}_{SpecA})$ is a locally ringed space for every ring A.*

Proof. [8] II, Proposition 2.3. □

Definition 24 (Scheme)**.** *An affine scheme is a locally ringed space $(X, \mathcal{O}_X)$ which is isomorphic to the locally ringed space $(SpecA, \mathcal{O}_{SpecA})$ for some ring A. A scheme is a locally ringed space $(X, \mathcal{O}_X)$ such that there exists an open cover $\{U_i\}$ such that every $(U_i, \mathcal{O}_{U_i})$ is an affine scheme. A morphism of schemes is a morphism of them as locally ringed spaces.*

Remark 25. *The sheaf of rings $\mathcal{O}_X$ is called the structure sheaf.*

Remark 26. *By abuse of notation we will sometimes write simply X for the scheme $(X, \mathcal{O}_X)$.*

Now we define an important example of projective schemes. This class of schemes are constructed from graded rings. Let S be a graded ring. Denote by S^+ the ideal $\bigoplus_{d>0} S_d$. We define the set $X = Proj(S)$ as the set of all homogenuous prime ideals $\mathfrak{p}$ such that $S^+ \not\subseteq \mathfrak{p}$. For $\mathfrak{a}$ homogenuous ideal of S we define the closed subset $V(\mathfrak{a}) := \{\mathfrak{p} \in ProjS \,|\, \mathfrak{a} \subset \mathfrak{p}\}$, thus we can define the Zariski topology on $Proj(S)$. For more detailed explanation and construction of the structure sheaf, see [8] pg. $76 - 77$.

Definition 27. *Let A be a ring and let $n > 0$. We define n - projective space to be the scheme $\mathbb{P}^n_A = ProjA[x_0, ..., x_n]$ where $A[x_0, ..., x_n]$ is the polynomial ring with standard graded structure.*

Definition 28. *Let $(X, \mathcal{O}_X)$ be a scheme. An ideal sheaf on $(X, \mathcal{O}_X)$ is a subsheaf $\mathcal{I} \subset \mathcal{O}_X$ such that $\mathcal{I}(U)$ is an ideal for every open subset $U \subset X$.*

Definition 29 (closed subscheme)**.** *Let $(X, \mathcal{O}_X)$ be a scheme. A closed subscheme of $(X, \mathcal{O}_X)$ is a subscheme of the form $(Supp(\mathcal{O}_X/\mathcal{I}), \mathcal{O}_X/\mathcal{I})$ for some sheaf of ideals $\mathcal{I}$.*

1.2.1 Properties of schemes

Definition 30. *A scheme is irreducible if its topological space is irreducible.*

Definition 31. *A scheme X is reduced if for every open set U, the ring $\mathcal{O}_X(U)$ has no nilpotent elements.*

Definition 32. *A scheme X is integral if for every open set $U \subset X$, the ring $\mathcal{O}_X(U)$ is an integral domain.*

Proposition 33. *A scheme is integral if and only if it is irreducible and reduced.*

Proof. [8], Proposition 3.1. □

Definition 34. *A scheme X is locally Noetherian if it can be covered by open affine subsets $SpecA_i$, where each A_i is a Noetherian ring. A scheme X is Noetherian if it is locally Noetherian and quasi-compact.*

Proposition 35. *A scheme X is locally Noetherian if and only if for every open affine subset $U = SpecA$, A is a Noetherian ring.*

Proof. [8], Proposition 3.2. □

Definition 36. *A morphism $f : X \to Y$ of schemes is locally of finite type if there exists a covering of Y by open affine subsets $V_i = SpecB_i$, such that for each i, $f^{-1}(V_i)$ can be covered by open affine subsets $U_{ij} = SpecA_{ij}$, where each A_{ij} is a finitely generated B_i- algebra. The morphism f is of finite type if in addition each $f^{-1}(V_i)$ can be covered by finite numbers of the U_{ij}.*

1.3 Sheaves of $\mathcal{O}_X$ - modules

Definition 37. *Let $(X, \mathcal{O}_X)$ be a ringed space.*

1. *Sheaf of $\mathcal{O}_X$ - modules (or simply an $\mathcal{O}_X$ - module) is a sheaf $\mathcal{F}$ on X such that for every open subset $U \subseteq X$, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$- module and for each inclusion of open sets $V \subseteq U$, the restriction homomorphism $\mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with the module structures via the ring homomorphism $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$.*

2. *A morphism $\mathcal{F} \to \mathcal{G}$ of sheaves of $\mathcal{O}_X-$ modules is a morphism of sheaves, such that for each open set $U \subseteq X$, the map $\mathcal{F}(U) \to \mathcal{G}(U)$ is a homomorphism of $\mathcal{O}_X(U)$- modules.*

3. *An $\mathcal{O}_X$- module $\mathcal{F}$ is free if it is isomorphic to a direct sum of copies of the structure sheaf $\mathcal{O}_X$ It is called locally free if X can be covered by open sets U such that $\mathcal{F}|_U$ is a free $\mathcal{O}_X|_U$- module. The rank of $\mathcal{F}$ on such open set is the number of copies of the structure sheaf needed. Note that, if X is connected, the rank of a locally free sheaf is the same everywhere. A locally free sheaf of rank* 1 *is called an invertible sheaf.*

Definition 38. *We define the tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ of two $\mathcal{O}_X$- modules as the sheafification of the presheaf $U \to \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$ for each open $U \subseteq X$. We will write just $\mathcal{F} \otimes \mathcal{G}$ for brevity.*

Definition 39. *Let $(f, f^{\#}) : (X.\mathcal{O}_X) \to (Y.\mathcal{O}_Y)$ be a morphism of ringed spaces.*

1. *We define the direct image functor*

$$f_* : Mod(X, \mathcal{O}_X) \to Mod(Y, \mathcal{O}_Y)$$

as the composition of the pushforward f_* and the restriction of scalars via the map $f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X$.

2. *We define the inverse image functor*

$$f^* : Mod(Y, \mathcal{O}_Y) \to Mod(X, \mathcal{O}_X)$$

as the composition of the inverse image f^{-1} and the extension of scalars via the map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. Namely, to be tensor product $f^{-1}(-) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$.

Remark 40.

1. *For an $\mathcal{O}_X$- module $\mathcal{F}$ and an $\mathcal{O}_Y$- module $\mathcal{G}$, $f_*\mathcal{F}$ is an $f_*\mathcal{O}_X$- module and $f^*\mathcal{G}$ a sheaf of $f^*\mathcal{O}_Y$- modules in natural way.*
2. *The map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ comes from the adjunction $f^{-1} \dashv f_*$.*

Definition 41. *Let A be a ring and M an A- module. We define sheaf of $\mathcal{O}_X$- modules associated to module M as follows. We define the group*

$$\widetilde{M}(U) := \{s : U \to \amalg M_{\mathfrak{p}} \,|\, s \; satisfies \; (1) \; and \; (2)\},$$

where

1. *for each $\mathfrak{p} \in U, s(\mathfrak{p}) \in M_{\mathfrak{p}}$ and*
2. *for each point $\mathfrak{p} \in U$ there is a neighbourhood V of $\mathfrak{p}$ in U and there are elements $m \in M$ and $a \in A$ such that for every $\mathfrak{q} \in V$, $a \notin \mathfrak{q}$ and $s(\mathfrak{q}) = \frac{m}{a}$ in $M_{\mathfrak{q}}$.*

Remark 42. *The above set of function really forms a group by pointwise addition and restriction maps are defined $\rho_{UV} s := s|_V$. That is, $\widetilde{M}$ is indeed a sheaf, moreover sheaf of $\mathcal{O}_X$- modules.*

Proposition 43. *Let A be a ring. M an A- module and $\widetilde{M}$ associated sheaf of $\mathcal{O}_X$- modules to M on $X = SpecA$. Then:*

1. *for each* $\mathfrak{p} \in X$, *the stalk* $(\widetilde{M})_{\mathfrak{p}}$ *is isomorphic to the localized module* $M_{\mathfrak{p}}$;

2. *for each* $a \in A$, *the* A_a*- module* $\widetilde{M}(D(a))$ *is isomorphic to the localized module* M_a;

3. *in particular,* $\Gamma(X, \widetilde{M}) = M$.

Proof. [8] II, Proposition 5.1. □

Proposition 44. *Let* A *be a ring and let* $X = SpecA$ *be the corresponding affine scheme. Also let* $\varphi : A \to B$ *be a ring homomorphism and* $f : SpecB \to SpecA$ *the corresponding morphism of affine schemes. Then:*

1. *the map* $M \to \widetilde{M}$ *gives an exact, fully faithful functor from the category of* A*- modules to the category of* $\mathcal{O}_X$*- modules;*

2. *if* M *and* N *are two* A*- modules, then* $(\widetilde{M \otimes_A N}) \cong \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$;

3. *if* $\{M_i\}$ *is any family of* A*- modules, then* $(\widetilde{\bigoplus M_i}) \cong \bigoplus \widetilde{M_i}$;

4. *for any* B*- module* N *the direct image* $f_*(\widetilde{N}) \cong \widetilde{N_A}$;

5. *for any* A*- module* M *the inverse image* $f^*(\widetilde{M}) \cong \widetilde{M \otimes_A B}$.

Proof. [8] II, Proposition 5.2. □

1.3.1 Quasi-coherent and coherent sheaves

Definition 45. *Let* $(X, \mathcal{O}_X)$ *be a scheme.*

1. *A sheaf of* $\mathcal{O}_X$*- modules* $\mathcal{F}$ *is quasi-coherent if* X *can be covered by open affines subsets* $U_i = SpecA_i$ *such that* $\mathcal{F}|_{U_i} \cong \widetilde{M}_i$ *for some* A_i*- module* M_i *for each* i.

2. *We say* $\mathcal{F}$ *is coherent if furthermore each* M_i *is finitely generated* A_i*- module and each* A_i *is a Noetherian ring.*

Proposition 46. *Let* X *be a scheme. Then an* $\mathcal{O}_X$*- module* $\mathcal{F}$ *is quasi-coherent if and only if for every open affine subset* $U = SpecA$ *of* X, *there is an* A*- module* M *such that* $\mathcal{F}|_U \cong \widetilde{M}$. *Moreover, if* X *is Noetherian then* $\mathcal{F}$ *is coherent if and only if the same is true with* M *finitely generated* A*- module.*

Proof. [8] II, Proposition 5.4. □

Corollary 47. *Let A be a ring and $X = SpecA$ affine scheme. The tilde functor $\widetilde{(-)}$ gives an equivalence of categories between the category of A- modules and the category of quasi-coherent $\mathcal{O}_X$- modules. The same holds between categories of coherent $\mathcal{O}_X$- modules and finitely generated A- modules provided that A is Noetherian.*

Proof. [8] II, Corollary 5.5. □

Proposition 48. *Let X be a scheme. The kernel, cokernel and image of any morphism of quasi-coherent sheaves are quasi-coherent. Any extension of quasi-coherent sheaf is quasi-coherent. The same holds for coherent sheaves if X is a Noetherian scheme.*

Proof. [8] II, Proposition 5.7. □

It is also convenient to have an easy way to check quasi-coherence. In fact, a quasi-coherent sheaf $\mathcal{F}$ is equivalent to the data of one module for each affine open subset, such that the module over a principal open subset $D(a)$ is given by localizing the module over $SpecA$. We form this to the following lemma.

Lemma 49. *Let X be a scheme and let $\mathcal{F}$ be an $\mathcal{O}_X$- module. Then $\mathcal{F}$ is a quasi-coherent sheaf if and only if the map $\mathcal{F}(SpecA)_f \to \mathcal{F}(D(f))$ induced by the universal property of the localization and restriction map is an isomorphism for every open affine $SpecA$ and $f \in A$.*

Proof. One implication is given in [8], Lemma 5.3. Conversely, define $M := \mathcal{F}(SpecA)$. Then sheaf axioms and Proposition 43 give the proof. □

Lemma 50. *Let X be a Noetherian scheme and $\mathcal{F}$ a coherent $\mathcal{O}_X$- module. Then every ascending chain of quasi-coherent subsheaves of $\mathcal{F}$*

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset ... \subset \mathcal{F}$$

stabilizes.

Proof. [9], Lemma 30.10.1. □

Lemma 51. *The support of a coherent sheaf $\mathcal{F}$ on a Noetherian scheme X is closed.*

Proof. If M is a finitely generated module then $SuppM = V(Ann(M))$ (see [10] pg. 25-26), which is a closed subset in $SpecA$. Now, if the scheme X is Noetherian then we can choose a finite cover, and the restriction of the coherent sheaf $\mathcal{F}$ on every such open affine corresponds to finitely generated modules M_i. Then $Supp(\mathcal{F}_{|U_i}) = V(Ann(M_i)$. We observe that $Supp(\mathcal{F}) = \bigcup Supp(F_{|U_i})$ and this finishes the proof. □

Proposition 52. *Let $f : X \to Y$ be a morphism of schemes.*

1. *If $\mathcal{G}$ is a quasi-coherent sheaf of $\mathcal{O}_Y$- modules, then $f^*\mathcal{G}$ is a quasi-coherent sheaf of $\mathcal{O}_X$- modules.*

2. *If X and Y are Noetherian, and $\mathcal{G}$ is coherent, then $f^*\mathcal{G}$ is coherent.*

3. *If X Noetherian or f quasi-compact and separated then if $\mathcal{F}$ is a quasi-coherent sheaf of $\mathcal{O}_X$- modules, $f_*\mathcal{F}$ is a quasi-coherent sheaf of $\mathcal{O}_Y$- modules.*

Proof. [8] II, Proposition 5.8. □

1.3.2 Sheaf $\mathcal{H}om$

Let $(X, \mathcal{O}_X)$ be a ringed space. If $\mathcal{F}$ and $\mathcal{G}$ are two $\mathcal{O}_X$- modules, the presheaf

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) : U \to Hom_{\mathcal{O}_{X|U}}(\mathcal{F}_{|U}, \mathcal{G}_{|U})$$

is a sheaf, which we call the *sheaf $\mathcal{H}om$*. Moreover, it is also an $\mathcal{O}_X$- module. This short subsection is devoted to show that this sheaf is coherent under some assumptions.

Definition 53. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a $\mathcal{O}_X$- module. We say $\mathcal{F}$ has a presentation if X can be covered by open subsets U such that we have an exact sequence*

$$\bigoplus_{i \in I} \mathcal{O}_U \to \bigoplus_{j \in J} \mathcal{O}_U \to \mathcal{F}_U \to 0.$$

When the sets I and J are finite, we say $\mathcal{F}$ is of finite presentation.

Proposition 54. *Every coherent sheaf $\mathcal{F}$ on a noetherian scheme is of finite presentation.*

Proof. Let X be a noetherian scheme. By definition of coherence, we can cover X with open affine sets U_i such that $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$ where M_i is finitely generated A-module over a noetherian ring. Thus, we can find an exact sequence of the form $A^{\oplus n_2} \to A^{\oplus n_1} \to M_i \to 0$. The tilde functor $\widetilde{(-)}$ is exact and preserves direct sums, hence this sequence correspond to the exact sequence $\mathcal{O}_{U_i}^{\oplus n_1} \to \mathcal{O}_{U_i}^{\oplus n_2} \to \mathcal{F}_{U_i} \to 0$. □

Lemma 55. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}, \mathcal{G}$ be $\mathcal{O}_X$- modules.*

1. *If $\mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F}$ is an exact sequence of $\mathcal{O}_X$- modules, then*

$$0 \to \mathcal{H}om(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om(\mathcal{F}_1, \mathcal{G}) \to \mathcal{H}om(\mathcal{F}_2, \mathcal{G})$$

is exact.

Proof. [9], Lemma 17.22.2. □

Lemma 56. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of coherent $\mathcal{O}_X$- modules. Then, $Ker(\varphi)$ and $Coker(\varphi)$ are coherent.*

Proof. [9], Lemma 17.12.4. □

Proposition 57. *Let $(X, \mathcal{O})_{\mathcal{X}}$ be a ringed space. Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}_X$- modules such that $\mathcal{F}$ is of finite presentation and $\mathcal{G}$ is coherent. Then $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is a coherent sheaf.*

Proof. $\mathcal{F}$ is of finite presentation. Thus we have

$$\bigoplus_{i \in I} \mathcal{O}_U \to \bigoplus_{j \in J} \mathcal{O}_U \to \mathcal{F}_U \to 0$$

with I, J finite. By the Lemma above, we get an exact sequence

$$0 \to \mathcal{H}om(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om(\mathcal{O}_X^{\oplus n_1}, \mathcal{G}) \to \mathcal{H}om(\mathcal{O}_X^{\oplus n_2}, \mathcal{G}).$$

Moreover, this is isomorphic to the sequence

$$0 \to \mathcal{H}om(\mathcal{F}, \mathcal{G}) \to \mathcal{G}^{\oplus n_1} \to \mathcal{G}^{\oplus n_2}.$$

That is, the *sheaf* $\mathcal{H}om$ is isomorphic to the kernel of the map between coherent sheaves. Thus, it is coherent. □

In particular, the *sheaf* $\mathcal{H}om$ of the coherent sheaves is coherent.

1.4 Towards derived categories

In Chapter 4 we investigate the bounded derived category of coherent sheaves. Hence, we now introduce basic notions of the triangulated category as well as we recall the brief idea of the construction of the derived categories. We define the notion of right derived functors. In the end, we define the sheaf cohomology.

1.4.1 Additive categories

Definition 58. *A category $\mathcal{A}$ is an additive category if for any two objects $A, B \in \mathcal{A}$ the set $Hom(A, B)$ is endowed with abelian group structure such that the following three conditions are satisfied:*

1. *The composition map $\circ$ is bilinear.*

2. *There exists a zero object* $0 \in \mathcal{A}$ *that is, the object for which* $Hom(0,0)$ *is the trivial abelian group.*

3. *For any two objects* $A_1, A_2 \in \mathcal{A}$ *there exist an object* $B \in \mathcal{A}$ *with morphisms* $j_i : A_i \to B$ *and* $p_i : B \to A_i$ *which make* B *the direct sum and the direct product of* A_1 *and* A_2.

We assume the usual compatibilities $p_i \circ j_i = 1$, $p_2 \circ j_1 = p_1 \circ j_2 = 0$ *and* $j_1 \circ p_1 + j_2 \circ p_2 = 1$ *which hold automatically up to automorphism of* B.

Definition 59. *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be additive categories. A functor* $F : \mathcal{C} \to \mathcal{D}$ *is called additive if* $F : Hom_{\mathcal{C}}(X,Y) \to Hom_{\mathcal{D}}(FX, FY)$ *is a homomorphism of abelian groups for all* $X, Y \in \mathcal{C}$.

Definition 60. *Let* k *be a field. A* k*- linear category* $\mathcal{A}$ *is an additive category such that the groups* $Hom(A,B)$ *are* k*- vector spaces and such that all compositions are* k*- bilinear. Moreover, additive functors between two* k*- linear additive categories over a common base field* k *is defined as* k*- linear functor. That is, the map* $F : Hom(A,B) \to Hom(F(A), F(B))$ *is* k*- linear for every* $A, B \in \mathcal{A}$.

Definition 61. *An additive category* $\mathcal{C}$ *is called abelian if:*

1. *any morphism admits a kernel and a cokernel;*

2. *the natural map* $Coimf \to Imf$ *is an isomorphism for every* $f \in Hom(A,B)$.

Recall that the image Imf *is a kernel for a cokernel* $B \to Cokerf$ *and the coimage* $Coimf$ *is a cokernel for a kernel* $Kerf \to A$.

Let us provide some examples of abelian categories.

Example 62. *Let* X *be any scheme. Let* R *be a commutative ring.*

1. *The category of modules* $Mod(R)$ *is abelian.*

2. *The category of* $\mathcal{O}_X$*- modules* $Mod(X, \mathcal{O}_X)$ *is abelian ([9], Lemma 17.3.1.).*

3. *The category of quasi-coherent sheaves* $Qcoh(X)$ *is abelian ([11] pg. 381-382).*

4. *The category of coherent sheaves* $Coh(X)$ *is abelian ([9], Lemma 17.12.4.).*

1.4.2 Triangulated categories

Definition 63 (triangle)**.** *Let $\mathcal{D}$ be an additive category. Let $[1] : \mathcal{D} \to \mathcal{D}$ be a additive auto-equivalence functor such that $E \mapsto E[1]$. The functor $[1]$ is called the shift functor.*

1. *A triangle is a sextuple (X, Y, Z, f, g, h) where $X, Y, Z \in \mathcal{D}$ where $X, Y, Z \in \mathcal{D}$ and $f : X \to Y$, $g : Y \to Z$ and $h : Z \to X[1]$ are morphism in $\mathcal{D}$.*

2. *A morhpism of triangles $(X, Y, Z, f, g, h) \to (X', Y', Z', f', g', h')$ is given by morphisms $a : X \to X'$, $b : Y \to Y'$ and $c : Z \to Z'$ of $\mathcal{D}$ such that $b \circ f = f' \circ a$, $c \circ g = g' \circ b$ and $a[1] \circ h = h' \circ c$.*

Definition 64. *A triangulated category is a triple $(\mathcal{D}, \{[n]\}_{n\in\mathbb{Z}}, \mathcal{T})$ where*

1. *$\mathcal{D}$ is an additive category,*

2. *$\{[n]\}_{n\in\mathbb{Z}}$ is a collection of additve auto-equivalences such that $[n]\circ[m] = [n+m]$ for $n, m \in \mathbb{Z}$. We denote $[0] = id$, and $[n]$ is the n-fold composition of $[1]$ and $[-n]$ is equal to n-the fold composition of $[-1]$ for $n > 0$, and*

3. *set of triangles $\mathcal{T}$ called the distinguished triangles.*

subjected to the following axioms

TR1 Any triangle of the form $(X, X, 0, 1, 0, 0)$ is a distinguished triangle. Any triangle isomorphic to a distinguished triangle is distinguished. Every morphism $f : X \to Y$ can be complete into a distinguished triangle of the form (X, Y, Z, f, g, h).

TR2 The triangle (X, Y, Z, f, g, h) is distinguished if and only if

$$(Y, Z, X[1], g, h, -f[1])$$

is a distinguished triangle.

TR3 Given a solid diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle a} \\
X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & X'[1]
\end{array}$$

whose rows are distinguished triangles and leftmost square commutes, there exists a morphism $c : Z \to Z'$ such that (a, b, c) is a morhpism of triangles.

TR4 *Given objects* X, Y, Z *of* $\mathcal{D}$ *and morphisms* $f : X \to Y$, $g : Y \to Z$ *and distinguished triangles* (X, Y, Q_1, f, p_1, d_1), $(X, Z, Q_2, g \circ f, p_2, d_2)$, *and* (Y, Z, Q_3, g, p_3, d_3), *there exist morphisms* $a : Q_1 \to Q_2$ *and* $b : Q_2 \to Q_3$ *such that*

- *a* $(Q_1, Q_2, Q_3, a, b, p_1[1] \circ d_3)$ *is a distingiushed triangle,*
- *b The triple* (id_X, g, a) *is a morphism of triangles* $(X, Y, Q_1, f, p_1, d_1) \to (X, Z, Q_2, g \circ f, p_2, d_2)$, *and*
- *c The triple* (f, id_Z, b) *is a morphism of triangles* $(X, Z, Q_2, g \circ f, p_2, d_2) \to (Y, Z, Q_3, g, p_3, d_3)$.

Definition 65. *An additive functor* $F : \mathcal{D} \to \mathcal{D}'$ *between triangulated categories is called exact if the following conditions are satisfied:*

i) There exists a functor isomorphism

$$F \circ [1] \to [1] \circ F.$$

ii) F *preserves distinguished triangles, in other words, any distinguished triangle*

$$A \to B \to C \to A[1]$$

in $\mathcal{D}$ *is mapped to a distinguished triangle*

$$F(A) \to F(B) \to F(C) \to F(A)[1]$$

in $\mathcal{D}'$, *where* $F((A)[1])$ *is identified with* $F(A)[1]$ *via the functor isomorphism in i).*

Definition 66. *Let* $\mathcal{D}$ *and* $\mathcal{D}'$ *be two triangulated categories. We say the categories are equivalent if there exists an exact equivalence* $F : \mathcal{D} \to \mathcal{D}'$.

1.4.3 Derived categories

Here we just recall the brief idea behind the construction of the derived categories. The aim behind the construction is the following: quasi-isomorphic objects should become isomorphic in the derived category. The construction proceeds in several steps which can be reflected by the following sequence of functors

$$\mathcal{A} \to \mathcal{C}(\mathcal{A}) \to \mathcal{K}(\mathcal{A}) \to \mathcal{D}(\mathcal{A}).$$

Here $\mathcal{A}$ is an Abelian category, $\mathcal{C}(\mathcal{A})$ is the category of cochain complexes, $\mathcal{K}(\mathcal{A})$ its homotopy category and finally $\mathcal{D}(\mathcal{A})$ the derived category. Recall the definition of the homotopy category. The homotopy category $\mathcal{K}(\mathcal{A})$ is the category of complexes of $\mathcal{A}$ with morphisms given up to homotopy.

Now, the objects of the derived category are pretty straightforward. We define $Ob(\mathcal{D}(\mathcal{A})) := Ob(\mathcal{K}(\mathcal{A})) = Ob(\mathcal{C}(\mathcal{A}))$. Finally, the morphisms are obtained by formally inverting the class of the quasi-isomorphisms, but we will not provide the technical details. Note that the derived category is not Abelian in general. However, it is always triangulated.

Finally, we provide one useful lemma, which we will use in Chapter 4. It says that a complex of vector spaces over a field k is isomorphic to its cohomology complex in the derived category of vector spaces over a field k.

Lemma 67. *Let $\mathcal{A} := Vec(k)$ be an Abelian category of k - vector spaces. Then any complex $A^\bullet \in \mathcal{D}(\mathcal{A})$ is isomorphic to its cohomology complex $\bigoplus H^i(A^\bullet)[-i]$.*

Proof. Let us have a chain complex of vector spaces $V^\bullet = (... \to V^i \xrightarrow{d^i} V^{i+1} \xrightarrow{d^{i+1}} V^{i+2} \to ...)$. We form a complex $H^\bullet(V^\bullet) = (... \to H^i(V^\bullet) \xrightarrow{0} H^{i+1}(V^\bullet) \xrightarrow{0} H^{i+2}(V^\bullet) \to ...)$ with obvious zero differentials. Now, as we are in the category of vector spaces the short exact sequence $0 \to B^i(V^\bullet) \to Z^i(V^\bullet) \to H^i(V^\bullet) \to 0$ splits for every i, that is we have a section $s^i : H^i(V^\bullet) \to Z^i(V^\bullet)$. The composition with the inclusion $Z^i(V^\bullet) \to V^i$ gives a well-defined chain map $H^\bullet(V^\bullet) \to V^\bullet$. It is easy to see that this map is a quasi-isomorphism, hence it is an isomorphism in $\mathcal{D}(\mathcal{A})$. Then the isomorphism $H^\bullet(V^\bullet) \cong \bigoplus H^i(V^\bullet)[-i]$ holds by definition of direct sum of complexes. □

1.4.4 Derived functors

We recall the construction of right derived functors provided that an Abelian category $\mathcal{A}$ has enough injectives.

Proposition 68. *Let $\mathcal{A}$ be an Abelian category. Suppose $\mathcal{A}$ contains enough injectives, i.e any object in $\mathcal{A}$ can be embedded into injective one. Then the natural functor*

$$\iota : \mathcal{K}^+(\mathcal{I}) \to \mathcal{D}^+(\mathcal{A})$$

is an equivalence. Here $\mathcal{K}^+(\mathcal{I})$ is the bounded below homotopy category of the full additive subcategory $\mathcal{I} \subset \mathcal{A}$ of all injectives.

Proof. [1] Proposition 2.40. □

Definition 69. *Let $\mathcal{A}$ be an Abelian category with enough injectives. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor between two Abelian categories. The right derived functor of F is the functor*

$$RF := Q_{\mathcal{B}} \circ K(F) \circ \iota^{-1} : \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{B}).$$

Here $K(F) : \mathcal{K}^+(\mathcal{A}) \to \mathcal{K}^+(\mathcal{B})$ is well-defined functor that maps $(... \to A^{i-1} \to A^i \to A^1 \to ...)$ to $(... \to F(A^{i-1}) \to F(A^i) \to F(A^1) \to ...)$ and $Q_{\mathcal{B}} : \mathcal{K}^+(\mathcal{B}) \to \mathcal{D}^+(\mathcal{B})$ is the natural functor.

Proposition 70. *The right derived functor $RF : \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{B})$ is an exact functor of triangulated categories.*

Proof. [1], Proposition 2.47. □

Definition 71. *Let $RF : \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{B})$ be the right derived functor of a left exact functor $F : \mathcal{A} \to \mathcal{B}$. Then for any complex $A^\bullet \in \mathcal{D}^+(\mathcal{A})$ we define:*

$$R^iF(A^\bullet) := H^i(RF(A^\bullet)) \in \mathcal{B}.$$

We call the induced additive functors

$$R^iF : \mathcal{A} \longrightarrow \mathcal{B}$$

the higher derived functors of F.

Proposition 72. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories such that $\mathcal{A}$ has enough injectives. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor and R^iF its higher derived functors. Then any short exact sequence*

$$0 \to A \to B \to C \to 0$$

in $\mathcal{A}$ gives a rise to a long exact sequence

$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow R^1F(A) \longrightarrow ...$$

$$... \longrightarrow R^iF(B) \longrightarrow R^iF(C) \longrightarrow R^{i+1}F(A) \longrightarrow ...$$

Proof. [1], Corollary 2.50. □

Before we define the sheaf cohomology, we need the following proposition and corollary.

Proposition 73. *Let $(X, \mathcal{O}_X)$ be a ringed space. Then the category $Mod(X, \mathcal{O}_X)$ has enough injectives.*

Proof. [8] chapter III, Proposition 2.2. □

Corollary 74. *If X is a topological space, then the category $Sh(X)$ of sheaves of abelian groups on X has enough injectives.*

Proof. [8] chapter III, Corollary 2.3. □

Now we are ready to define the sheaf cohomology. Let X be a topological space. Let $\Gamma(X, -) : Sh(X) \to Ab$ be the left exact global section functor, where $Sh(X)$ is the category of sheaves of abelian groups. We define the cohomology functors $H^i(X, -)$ to be the right derived functors of $\Gamma(X, -)$. For any sheaf $\mathcal{F}$ the groups $H^i(X, \mathcal{F})$ are the *cohomology groups* of $\mathcal{F}$. Note that regardless of the structures of X and $\mathcal{F}$, e.g., X scheme and $\mathcal{F}$ quasi-coherent sheaf, we always take cohomology in this sense, regarding $\mathcal{F}$ simply as a sheaf of abelian groups on the underlying topological space X.

Similarly, let $(X, \mathcal{O}_X)$ be a ringed space and let $\mathcal{F}$ be an $\mathcal{O}_X$- module. We define the functors $Ext^i(\mathcal{F}, -)$ as the right derived functors of $Hom(\mathcal{F}, -)$ and $\mathcal{E}xt^i$ as the right derived functors of $\mathcal{H}om(\mathcal{F}, -)$. Here, if $\mathcal{F}$, and $\mathcal{G}$ are $\mathcal{O}_X$- modules, then by $Hom(\mathcal{F}, \mathcal{G})$ we denote the group of $\mathcal{O}_X$- modules.

1.5 Curves

Note that when we talk about a scheme with some implicit structure, for example a scheme over a field k, we think of the scheme via the structure morphism $X \to Speck$.

Definition 75 (Variety)**.** *Let k be a field. A variety is an integral scheme X over k such that the structure morphism $X \to Speck$ is separated*[1] *and of finite type.*

Definition 76 (Curve)**.** *Let k be a field. A curve is variety of dimension 1 over k.*

Remark 77. *A curve is Noetherian scheme.*

Definition 78 (Regular Scheme)**.** *Let X be a locally Noetherian scheme. It is called regular at a point $x \in X$ if the stalk $\mathcal{O}_{X,x}$ is regular. The scheme X is called regular if it is regular at every point $x \in X$. It is called non-regular or singular if the stalk $\mathcal{O}_{X,x}$ is non-regular for some point $x \in X$.*

Remark 79. *We will refer to a regular point or a regular scheme as a smooth point or a smooth scheme, respectively. Strictly speaking, these notions are different, but they coincide whenever we talk about curves over an algebraically closed field. [11] 12.2.10.*

Definition 80 (Dedekind Scheme)**.** *We call a Noetherian integral scheme X a Dedekind scheme if $\mathcal{O}_X(U)$ is a Dedekind ring for every open affine $U \subseteq X$. In other words, a Dedekind scheme is Noetherian integral regular scheme of dimension* ≤ 1.

[1] for the definition, see [8] on page 96.

Example 81. *Every regular curve over a field k is a Dedekind scheme.*

In the following lemma we use the notion of an effective Cartier divisor, see the Appendix A.3.

Lemma 82. *Let X be a curve over a field k and $x \in X$ closed point. We think of x as a reduced closed subscheme of X with sheaf of ideals $\mathcal{I}$. Then following are equivalent:*

1. *$\mathcal{O}_{X,x}$ is regular,*
2. *$\mathcal{O}_{X,x}$ is normal,*
3. *$\mathcal{O}_{X,x}$ is discrete valuation ring,*
4. *$\mathcal{I}$ is an invertible $\mathcal{O}_x$ - module,*
5. *x is an effective Cartier divisor on X.*

Proof. [9], Lemma 33.43.8. □

Definition 83 (Projective morphism). *Let $f : X \to Y$ be a morphism of schemes. We call f projective morphism if it factors into a closed immersion $\iota : X \to \mathbb{P}^n_Y$ for some integer n, followed by the projection $\mathbb{P}^n_Y \to Y$.*

Remark 84. *Thus, when we talk about a projective curve over an algebraically closed field k we think of it with the embedding in the ambient space $\mathbb{P}^n_k$.*

Now we introduce two important theorems which play a crucial role in upcoming definitions. With aid of these theorems we define the Euler characteristic of a coherent sheaf, subsequently we define the arithmetic genus, an important invariant of a curve. We will see the significant meaning of these theorems in Chapter 4.

Theorem 85 (A Vanishing Theorem of Grothendieck). *Let X be a Noetherian topological space of dimension n. Then for all $i > n$ and all sheaves of Abelian groups $\mathcal{F}$ on X the sheaf cohomology $H^i(X, \mathcal{F})$ vanishes.*

Proof. [8] chapter III, Theorem 2.7. □

Theorem 86. *Let X be a projective scheme over a noetherian ring A. Let $\mathcal{F}$ be a coherent sheaf on X. Then:*

1. *for each $i \geq 0$, $H^i(X, \mathcal{F})$ is a finitely generated A-module.*

Proof. [11], Theorem 18.1.4. □

Definition 87 (Euler characteristic)**.** *Let X be a projective scheme over a field k and let $\mathcal{F}$ be a coherent sheaf on X. We define the Euler characteristic of $\mathcal{F}$ as*

$$\chi(\mathcal{F}) = \sum dim_k (-1)^i H^i(X, \mathcal{F}).$$

Remark 88. *Due to the above theorem, all of the cohomologies are finitely generated k- vector spaces.*

Lemma 89. *Let X be a projective scheme over a field k. The Euler characteristic is additive on short exact sequences. That is, for any short exact sequence $0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0$ of coherent sheaves on X, we have $\chi(\mathcal{E}) = \chi(\mathcal{F}) + \chi(\mathcal{G})$.*

Proof. Let us have a short exact sequence of coherent sheaves $0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0$. We have the long exact sequence of cohomologies

$$0 \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{E}) \to \Gamma(X, \mathcal{G})$$

$$\to H^1(X, \mathcal{F}) \to H^1(X, \mathcal{E}) \to H^1(X, \mathcal{G}) \to ...$$

All of these are given the k vector space structure. Then the rank-nullity theorem for vector spaces completes the proof. □

Definition 90 (Arithmetic genus)**.** *Let k be a field. Let X be a projective scheme of dimension r over the field k. We define the arithmetic genus p_a of X by*

$$p_a(X) = (-1)^r(\chi(X, \mathcal{O}_X) - 1).$$

In our thesis we work with a projective scheme (with a projective curve to be precise) over an algebraically closed field k. For such scheme we can nicely describe the space of global sections of the structure sheaf. In fact, we get $\Gamma(X, \mathcal{O}_X) = H^0(X, \mathcal{O}_X) = k$. This is an important and non-trivial fact. It can be found in [11] pg. 295. And for such schemes the arithmetic genus

$$p_a = \sum_{i=0}^{r-1} (-1)^i dim_k H^{r-i}(X, \mathcal{O}_X).$$

Indeed, it is a direct consequence of the definition and the Grothendieck Vanishing Theorem. In particular, the arithmetic genus of a projective curve X is $p_a(X) = dim_k H^1(X, \mathcal{O}_X)$. In other words, the arithmetic genus and the genus of the projective curve coincide.

Now we define numerical invariants of a coherent sheaf on a curve X. Their importance and meaning will be obvious in Chapter 4.

Definition 91 (Rank)**.** *We define the rank of a coherent sheaf $\mathcal{F}$ on a curve X as $rk(\mathcal{F}) := dim_{\mathcal{O}_{X,\eta}} \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,\eta}$. Here by η we denote the generic point of the curve X.*

Remark 92. *Note that definition of the rank of a coherent sheaf does not depend on the choice of the point $x \in X$.*

Definition 93 (Degree)**.** *Let X be a scheme over a field k. The degree of a coherent sheaf $\mathcal{F}$ on the scheme X is defined as $deg(\mathcal{F}) = \chi(\mathcal{F}) - \chi(\mathcal{O}_X)rk(F)$.*

Remark 94. *If X is a projective curve over an algebraically closed field k of arithmetic genus one, then the definition of the Euler characteristic and the degree coincide as $\chi(\mathcal{O}_X) = 0$ (see the discussion above for the structure sheaf $\mathcal{O}_X$).*

Finally, we define a special type of a projective curve, namely a Weierstrass curve.

Definition 95. *Let k be an algebraically closed field of characteristic $\neq 2, 3$. A Weierstrass curve X is a projective curve such that it is isomorphic to a cubic curve in $\mathbb{P}^2_k$ given by an equation*

$$y^2 z = 4x^3 - g_2 x z^2 - g_3 z^3,$$

where $[x : y : z]$ are homogeneous coordinates in $\mathbb{P}^2_k$ and g_2, g_3 constants in k.

Remark 96. *Every such curve has at most one singular point. If $g_2 = g_3 = 0$ then the singular point is a cusp. Otherwise, it is a node. Moreover, Weierstrass curves are curves of arithmetic genus one.*

For the rest of this section we provide some useful propositions.

Theorem 97. *Let X be a projective scheme over Noetherian ring A. Then any coherent sheaf $\mathcal{F}$ can be written as quotient of a sheaf $\mathcal{E}$, where $\mathcal{E}$ is a finite direct sum of invertible sheaves.*

Proof. [8] chapter II, Corollary 5.18. □

Proposition 98. *Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_X$- modules. Then we have:*

1. *$Ext^i(\mathcal{O}_X, \mathcal{G}) \cong H^i(X, \mathcal{G})$, for all $i \geq 0$.*

Proof. [8] chapter III, Proposition 6.3. □

Proposition 99. *Let $\mathcal{L}$ be a loally free sheaf of finite rank, and let $\mathcal{L}^\vee = \mathcal{H}om(\mathcal{L}, \mathcal{O}_X)$ be its dual. Then for any $\mathcal{F}, \mathcal{G} \in Mod(X, \mathcal{O}_X)$ and any $i \geq 0$ we have*

$$Ext^i(\mathcal{F} \otimes \mathcal{L}, \mathcal{G}) \cong Ext^i(\mathcal{F}, \mathcal{G} \otimes \mathcal{L}^\vee).$$

Proof. [8] chapter III, Proposition 6.7. □

Proposition 100. *If* $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ *is a short exact sequence of* $\mathcal{O}_X$*-modules, then for any* $\mathcal{O}_X$*- module* $\mathcal{G}$ *we have a long exact sequence*

$$0 \to Hom(\mathcal{F}'', \mathcal{G}) \to Hom(\mathcal{F}, \mathcal{G}) \to Hom(\mathcal{F}', \mathcal{G})$$

$$\to Ext^1(\mathcal{F}'', \mathcal{G}) \to Ext^1(\mathcal{F}, \mathcal{G}) \to Ext^1(\mathcal{F}', \mathcal{G}) \to \dots,$$

and similarly for $\mathcal{E}xt$ *sheaves.*

Proof. [8] chapter III, Proposition 6.4. □

Lemma 101. *Let* $X = SpecA$ *be an affine locally noetherian scheme. Let* M *and* N *be* A*- modules such that* M *is finitely generated* A*- module. Then*

$$Ext^i_X(\widetilde{M}, \widetilde{N}) \cong Ext^i_A(M, N).$$

Proof. The module M is finitely generated over the noetherian scheme X. Thus we can find a free resolution

$$... \to A^{\oplus n_2} \to A^{\oplus n_1} \to M \to 0.$$

Applying the exact tilde functor $\widetilde{(-)}$ we get the resolution

$$... \to \mathcal{O}_X^{\oplus n_2} \to \mathcal{O}_X^{\oplus n_1} \to \widetilde{M} \to 0.$$

This resolution consist of locally free sheaves and hence computes the right derived functor $Ext(-, \widetilde{N})$. Note that $Hom(A^{\oplus n_k}, N) \cong N^{\oplus n_k}$ and $Hom(\mathcal{O}_X^{\oplus n_k}, \widetilde{N}) \cong \Gamma(X, \widetilde{N})^{\oplus n_k} \cong N^{\oplus n_k}$. Thus we have two isomorphic complexes, in particular, quasi-isomorphic. And this finish the proof. □

Proposition 102. *Let* X *be noetherian scheme, let* $\mathcal{F}$ *be a coherent sheaf on* X*. Let* $\mathcal{G}$ *be an* $\mathcal{O}_X$*- module and let* $x \in X$ *be a point. Then we have*

$$\mathcal{E}xt^i(\mathcal{F}, \mathcal{G})_x \cong Ext^i_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$$

for any $i \geq 0$*, where the right-hand side is Ext over the local ring* $\mathcal{O}_{X,x}$*.*

Proof. [8], Proposition 6.8. □

2 Torsion Sheaves

In this chapter, we introduce the notion of torsion and torsion free sheaves. We prove some elementary properties as well as some more important and difficult properties of such sheaves. Our main interest is to explore the behaviour of such sheaves on smooth and singular curves. At the end of this section, we will see the common properties they share as well as their differences.

Definition 103. *We call a quasi-coherent sheaf $\mathcal{F}$ on an integral scheme X torsion, if its stalk at the generic point is zero.*

Definition 104. *If X is a scheme, we call an $\mathcal{O}_X$- module $\mathcal{F}$ torsion free if it is torsion free as an $\mathcal{O}_{X,x}$- module for every point $x \in X$.*

Lemma 105. *Let X be an integral scheme with the generic point η. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$- module. Let $U \subset X$ be a non-empty open subset and $s \in \mathcal{F}(U)$ a section of $\mathcal{F}$ over U. The following are equivalent*

1. *for some $x \in U$ the image of s in $\mathcal{F}_x$ is torsion,*
2. *for all $x \in U$ the image of s in $\mathcal{F}_x$ is torsion,*
3. *the image of s is zero in $\mathcal{F}_\eta$.*

Proof. 3) $\implies$ 1) is trivial.
1) $\implies$ 2). Firstly, we show this for the affine case $U = SpecA$. So let us have a section $s \in \mathcal{F}(X)$ and some point x such that s_x is torsion. $\mathcal{F}$ is quasi-coherent so the section corresponds to an element $m \in M$ for module M such that $\widetilde{M} \cong \mathcal{F}$. As s_x is torsion in $\mathcal{F}_x$, then $m_\mathfrak{p}$ is torsion in $M_\mathfrak{p}$ under the canonical map $M \to M_\mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal corresponding to the point $x \in X$. This means we have a non-zero element $a/s \in A_\mathfrak{p}$ such that $a/s \cdot m/1 = 0$. From the definition of localization we have an element $t \in S^{-1} := A \backslash \mathfrak{p}$ such that $tam = 0$. Now, to show that m is torsion in any other localization it is enough to take the section $ta \in A$. Indeed, we have that $ta/1 \in S'^{-1}A$ for any localization set S'. Then $ta/1 \cdot m/1 = 0$ as $1 \in S'$. Thus, $m/1$ is torsion as $A_\mathfrak{q}$ - element for localization at any primes. It is left to show that $ta \neq 0$ in $A_\mathfrak{q}$ for any $\mathfrak{q}$ but this follows from the assumption that X is an integral scheme, in particular, A is an integral domain.
To show the general case, choose some open subset $U \subset X$. We can cover U by principal opens $\{D(f)\}$. In particular $x \in D(f)$ for some $f \in A$ and for some A. By above, as s is torsion at the point x it is torsion at every point in $D(f)$. As X is integral, any intersection $D(f) \cap D(g)$ is non-empty (just take the generic point corresponding to the zero ideal). This finishes the proof.

2) $\implies$ 3) We show the affine case $U = SpecA$. The general case follows from the fact that the generic point is contained in any open subset. Quasi-coherence implies that $\mathcal{F}$ is given by some module over this open affine subset. Then Proposition 163 part 2 gives the proof. □

Remark 106. *The section from the above lemma is called a torsion section.*

Lemma 107. *Let X be an integral scheme and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$- module. The following are equivalent*

1. *$\mathcal{F}$ is torsion-free,*
2. *for each open affine $U \subseteq X$, $\mathcal{F}(U)$ is a torsion free $\mathcal{O}_X(U)$- module.*

Proof. This is the direct consequence of Proposition 162. and quasi-coherence of the sheaf $\mathcal{F}$. □

Proposition 108. *Let X be an integral scheme and $\mathcal{F}$ be a quasi-coherent sheaf. The following are equivalent*

1. *$\mathcal{F}$ is a torsion quasi-coherent sheaf,*
2. *$\mathcal{F}_x$ is a $\mathcal{O}_{X,x}$- torsion module for every point x of the scheme X,*
3. *for each open affine $U \subseteq X$ $\mathcal{F}(U)$ is a torsion $\mathcal{O}_X(U)$- module.*

Proof. We will show the implication 1) $\implies$ 3). The converse holds by the same arguments. Assume that $U = SpecA$. Quasi-coherence gives $\mathcal{F} \cong \widetilde{M}$ for some A-module M. We show that $\mathcal{F}(X)$ is a torsion $\mathcal{O}_X(X)$ - module. This is equivalent with the fact that $\mathcal{F}(X) \otimes_{\mathcal{O}_X(X)} Quot(\mathcal{O}_X(X)) = M \otimes_A Quot(A) = 0$ (see Proposition 163). Again, from the Proposition 157 we know that $\mathcal{F}_\eta \cong M_{(0)} \cong M \otimes_A Quot(A)$ and $\mathcal{F}_\eta = 0$ by the assumption.

1) $\iff$ 2)

The implication from right to left follows directly from $\mathcal{F}_\eta \cong \mathcal{F}_\eta \otimes_{\mathcal{O}_{X,\eta}} \mathcal{O}_{X,\eta} = 0$ as the stalk at the generic point is torsion (again, 163).

Conversely, from the local global property of modules we know that a module is zero if and only if the localization at every prime ideal is zero. That is, $M \otimes_A Quot(A) \cong M_{(0)} \cong \mathcal{F}_\eta = 0$ if and only if $S^{-1}(M \otimes_A Quot(A)) = 0$ for every multiplicative set which comes from the complement of some prime ideal. Basic commutative algebra gives $S^{-1}(M \otimes_A Quot(A)) = S^{-1}M \otimes_{S^{-1}A} Quot(A) = 0$. In particular, the latter is equal to $\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} Quot(\mathcal{O}_{X,x}) = M_\mathfrak{p} \otimes_{A_\mathfrak{p}} Quot(A)$ for any prime ideal. This finishes the proof.

□

Proposition 109. *Let X be a curve over an algebraically closed field. Every coherent torsion sheaf $\mathcal{F}$ on X is supported at finitely many points.*

Proof. We show that the support of a coherent torsion sheaf $\mathcal{F}$ is a proper closed subset. The fact that the support is closed is due to 51. The sheaf $\mathcal{F}$ is torsion, hence, $\mathcal{F}_\eta = 0$ and the support does not contain all points of X. Since the curve X is of dimension one this finishes the proof. □

Proposition 110. *Let X be an integral, Noetherian scheme. For a coherent sheaf $\mathcal{F}$ on X there exists a canonical short exact sequence of quasi-coherent $\mathcal{O}_X$- modules*

$$0 \to t\mathcal{F} \to \mathcal{F} \to \mathcal{F}/t\mathcal{F} \to 0,$$

such that $t\mathcal{F}$ is a maximal torsion subsheaf of $\mathcal{F}$ and $\mathcal{F}/t\mathcal{F}$ is torsion free.

Proof. Denote $\mathcal{S} = \{S \subseteq \mathcal{F} \,|\, S \; is \; torsion\}$. That is the set of all torsion subsheaves. Obviously the zero subsheaf is torsion, hence $\mathcal{S} \neq \emptyset$. If $\mathcal{F}$ does not contain a maximal torsion subsheaf then there exists S_1 such that $0 \subseteq S_1$. If S_1 is not a maximal then we can find a torsion subsheaf S_2 such that $S_1 \subset S_2$. By repeating this argument we can find an ascending chain of quasi-coherent torsion sheaves

$$0 \subset S_1 \subset S_2 \subset S_3 \cdots .$$

Due to Lemma 50, this ascending chain of quasi-coherent sheaves has to stabilize. Hence, there exists a maximal torsion subsheaf of $\mathcal{F}$.
Now we show that $\mathcal{F}/t\mathcal{F}$ is torsion free. We prove that being quasi-coherent torsion free sheaf is equivalent with the fact that every torsion subsheaf is zero. Indeed, if $\mathcal{G}$ is quasi-coherent torsion free and contains some non-zero torsion subsheaf $\mathcal{T} \subseteq \mathcal{G}$ then as taking a stalk is an exact functor we get $\mathcal{T}_x \subseteq \mathcal{G}_x$. Take $x \in Supp\mathcal{T}$, the stalk $\mathcal{T}_x$ is torsion by definition, hence $\mathcal{G}$ is not torsion free, again, by definition. Conversely, for the sake of contradiction, assume $\mathcal{G}$ is not torsion free. That is, there exists a point $x \in X = SpecA$ such that $\mathcal{G}_x$ is not a torsion free $\mathcal{O}_{X,x}$- module. Let us define the subsheaf $\mathcal{T} \subseteq \mathcal{G}$ as $\mathcal{T}(U) := \{s \in \mathcal{G}(U) \,|\, s_x \; torsion \; for \; every \; x \in U\}$. This sheaf is non-zero due to 105 and the assumption. It has clearly the structure of $\mathcal{O}_X$- modules and it is quasi-coherent (see Remark 111). Moreover, it is torsion by the construction, hence the contradiction.
So, to show $\mathcal{F}/t\mathcal{F}$ is torsion free is enough to show that it does not contain a non-zero torsion subsheaf. Let us denote $\mathcal{G} = \mathcal{F}/t\mathcal{F}$. Assume $\mathcal{G}$ is not torsion free. By the above, there exist non-zero torsion subsheaf $\mathcal{T} \subseteq \mathcal{G}$. The fact that the just constructed sheaf is of the form $\mathcal{E}/t\mathcal{F}$ contradicts the fact that $t\mathcal{F}$ is a maximal torsion subsheaf. Hence, this finishes the proof.

□

Remark 111. *The constructed sheaf $\mathcal{T}$ is indeed quasi-coherent. This can be checked on the principal opens 49. Let $U = SpecA$ be an affine subset of X and $f \in A$. From quasi-coherence of $\mathcal{F}$ we have an isomorphism $\psi : \mathcal{F}(SpecA)_f \to \mathcal{F}(D(f))$. We show that the restriction $\psi|_{\mathcal{T}(SpecA)_f} : \mathcal{T}(SpecA)_f \to \mathcal{T}(D(f))$ is an isomorphism. Note it is given by $s/f^k \to s|_{D(f)}$. Take a section $t \in \mathcal{T}(D(f))$ and consider it as a section in $\mathcal{F}(D(f))$. Then, via isomorphism ψ, this corresponds to some element r/a^l for $r \in \mathcal{F}(SpecA)$. But $r|_{D(f)} = t$, so it is torsion on some subset of $SpecA$ and thus torsion on the whole space $SpecA$. That is, $r \in \mathcal{T}(SpecA)$ and this show surjectivity. To show injectivity take s/f^k such that $s|_{D(f)} = 0$. Again, $\mathcal{F}$ quasi-coherent gives some power of f such that it annihilates the section s which finish the proof. In particular, the torsion sheaf $\mathcal{T}$ is quasi-coherent.*

Remark 112. *Every non-generic point of a curve is closed.*

Proof. Proposition 17. □

Theorem 113. *Let $\mathcal{F}$ be a coherent torsion sheaf on a regular curve X. Then it decomposes into direct sums of skyscraper sheaves.*

Proof. Firstly, we prove the theorem in the affine case. Let us assume $X = SpecA$ is affine. Then $\mathcal{F} = \widetilde{M}$ for some finitely generated A - module M. As $\mathcal{F}$ is torsion, $Supp\mathcal{F}$ is finite. Let us assume that $Supp\mathcal{F} = \{p\}$. M_p is a finitely generated A_p - module over a Dedekind domain. Thus, $\widetilde{M_p} \in Coh(Spec\mathcal{O}_{X,p})$. We have a morphism of schemes $\iota : Spec\mathcal{O}_{X,p} \to SpecA$ induced by the natural map $\mathcal{O}_X(X) \to \mathcal{O}_{X,p}$. Pushforward of $\widetilde{M_p}$ along this map is isomorphic to ${}_A\widetilde{M_p}$. The structure theorem for finitely generated torsion module over a Dedekind domain 161 gives decomposition $M_p = \bigoplus A_p/pA_p^{n_i}$. From commutative algebra we know that quotients commute with taking localization. In particular, we localize at the maximal ideal p. Hence, we get an isomorphism of A - modules $\bigoplus A_p/pA_p^{n_i} \cong \bigoplus A/p^{n_i}$. The latest is finitely generated as an A - module so M_p is. Thus, by the above arguments the pushforward of the coherent sheaf $\widetilde{M_p}$ along the map ι is coherent, in other words, $\iota_*\widetilde{M_p} \in Coh(SpecA)$. It is easy to see (remind $\mathcal{F}_\eta = 0$ and the uniqueness of the direct limit yields $\mathcal{F}(\{\eta\}) = 0$) that

$$\iota_*\tilde{M}_p(U) = \begin{cases} 0 & \text{if } p \notin U \\ M_p & \text{if } p \in U \end{cases} \quad and \quad (\iota_*\tilde{M}_p)_x = \begin{cases} 0 & \text{if } x \neq p \\ M_p & \text{if } x = p \end{cases}.$$

That is, we have constructed a coherent skyscraper sheaf on $SpecA$. Then extending the just constructed coherent sheaf by zeros we get a coherent skyscraper sheaf on the curve $X = \bigcup_{i=1}^n SpecA_i$. Now we define the morphism of coherent $\mathcal{O}_X$- modules $\phi^p : \mathcal{F} \to \iota_*\widetilde{M_p}$ as follows, $\phi^p(U) = \begin{cases} 0 & \text{if } p \notin U \\ \mathcal{F}(U) \to \mathcal{F}_p & \text{if } p \in U \end{cases}$.

It is easy to check that this map induces isomorphism on stalks thus ϕ^p is an isomorphism.
The case when the support consists of n points we construct the skyscraper sheaves for each point and define $\begin{pmatrix} \phi^{p_1} \\ \vdots \\ \phi^{p_n} \end{pmatrix} : \mathcal{F} \to \bigoplus_{i=1}^n \mathcal{F}_{p_i}$. Using the fact that stalk commutes with taking direct sums, this finishes the proof.

□

Proposition 114. *Let $\mathcal{F}$ be a coherent sheaf on a regular curve X. Then the canonical short exact sequence*

$$0 \to t\mathcal{F} \to \mathcal{F} \to \mathcal{F}/t\mathcal{F} \to 0$$

splits, where $t\mathcal{F}$ is a torsion module and $\mathcal{F}/t\mathcal{F}$ is torsion free.

Proof. Without lose of generality, we can assume $Supp(t\mathcal{F}) = \{p\}$ where p is a closed point of X. If $Supp(t\mathcal{F}) = \{x_1, \dots, x_n\}$ then from the decomposition of torsion sheaves on regular curve 113 and homological algebra we get

$$Ext^1(\mathcal{F}/t\mathcal{F}, t\mathcal{F}) \cong Ext^1(\mathcal{F}/t\mathcal{F}, \bigoplus_{x\in Z} \mathcal{F}_x) \cong \bigoplus_{x\in Z} Ext^1(\mathcal{F}/t\mathcal{F}, \mathcal{F}_x),$$

where Z denote the support of $t\mathcal{F}$.

We want to find a morphism $s : \mathcal{F}/t\mathcal{F} \to \mathcal{F}$, such that the composition with the morphism $\mathcal{F} \to \mathcal{F}/t\mathcal{F}$ is identity. Now cover the curve X with open affines such that $p \in U_1$ and $p \neq U_i$ for $i \neq 1$. This is possible since p is closed point. Indeed, let us have finite affine cover $\{U_i\}$ and WLOG assume $p \in U_1$. Then $X\backslash\{p\}$ is open and cover it by open affines $\{V_k\}$ then take a new cover $U_1 \bigcup \{V_k\}$. Taking open affine sections with coherent sheaves is exact functor, hence the sequence

$$0 \to t\mathcal{F}(U_i) \to \mathcal{F}(U_i) \to \mathcal{F}/t\mathcal{F}(U_i) \to 0$$

is exact. If $i \neq 1$ then $t\mathcal{F}(U_i) = 0$ and we have well-defined isomorphisms $s_i\ \mathcal{F}(U_i) \cong \mathcal{F}/t\mathcal{F}(U_i)$ for such index i. if $i = 1$ then $\mathcal{F}/t\mathcal{F}(U_1)$ is torsion free 107, and over the Dedekind domain $\mathcal{O}_X(U_1)$ this is equivalent with projectivity. Thus, the above sequence splits and we get a well-defined split monomorphism $s_1 : \mathcal{F}/t\mathcal{F}(U_1) \to \mathcal{F}(U_1)$. The collection of morphisms (s_i) is compatible on intersections (note that we chose a cover such that $t\mathcal{F}$ has support just on U_1, thus on intersections it is zero) and by $\widetilde{(-)}$ functor it corresponds to morphisms of sheaves $(\mathcal{F}/t\mathcal{F})_{U_i} \to \mathcal{F}_{U_i}$. Hence, by glueing 14 we get the proof. □

As we have seen earlier, the canonical short exact sequence $0 \to t\mathcal{F} \to \mathcal{F} \to \mathcal{F}/t\mathcal{F} \to 0$ always splits when a curve is regular. We now provide an example which shows that this is not always true if the curve is singular.

Example 115. *Consider $X = Spec(k[x,y]/(y^2 - x^2(x+1))$ for an algebraically closed field k of $char(k) \neq 2$. $M := (\overline{x}, \overline{y}) \leq R$ where $R = k[x,y]/(y^2 - x^2(x+1))$. The module M is a torsion free. We have a short exact sequence*

$$0 \to K \to R \oplus R \xrightarrow{(\overline{x},\overline{y})} M \to 0$$

with non-zero kernel K. Localizing the short exact sequence at the maximal ideal $M = (\overline{x}, \overline{y})$ gives rise to a short exact sequence of the form

$$0 \to K_M \to R_M \oplus R_M \xrightarrow{(\overline{x},\overline{y})} M_M \to 0.$$

Lemma 156 and Proposition 155 implies that the module M_M is not a free module and due to 159 we get the non-vanishing $Ext^1_{(R_M)}(M_M, (R/M)_M)$. Moreover, from the proof of 159 we get an isomorphism

$$Ext^1_{R_M}(M_M, (R/M)_M) \cong Hom_{R_M}(K_M, (R/M)_M).$$

To sum up, we have the following isomorphisms $0 \neq Ext^1_{R_M}(M_M, (R/M)_M) \cong Hom_{R_M}(K_M, (R/M)_M) \cong Hom_R(K, R/M)_M$, where the latter isomorphism comes from the elementary properties of localization. Non-zero localization of the hom module gives us $0 \neq f \in Hom_R(K, R/M)$, such that the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & R \oplus R & \xrightarrow{g} & M & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow \varphi & & \downarrow 1 & & \\ 0 & \longrightarrow & R/M & \xrightarrow{f'} & E & \xrightarrow{g'} & M & \longrightarrow & 0 \end{array} \qquad (\epsilon)$$

is commutative.

Moreover, the non-vanishing extension group for M_M implies the non-splitting short exact sequence

$$0 \to (R/M)_M \to E_M \to M_M \to 0, \qquad (\epsilon_M)$$

for the R_M- module E_M. From the general fact that $Ext^1_{R_M}(M_M, (R/M)_M) = Ext^1_R(M, (R/M))_M$ and the fact that the ϵ_M does not split we get that ϵ does split neither. Moreover $tE = R/M$ and $M = E/tE$. By applying the tilde functor and 101 we get the result.

3 Torsion sheaves and representation theory

Our aim is to classify torsion coherent sheaves supported at the node (x, y) of the nodal singular curve $\mathbf{E} = Spec(k[x, y]/y^2 - x^2(x + 1))$, where k is a field of $char(k) \neq 2$. We have an induced functor between the category of modules over a local ring and the category of modules over the completion of the ring. It turns out to be an equivalence on a level of so-called finitely generated I- **torsion** subcategories for some particular ideals I. This allows us to view the stalk of a coherent torsion sheaf as finitely generated module over the infinite dimensional string algebra $k[x, y]/(xy)$, subsequently, we decompose it into indecomposable string and band modules thanks to the results of Crawley-Boevey's [7].

3.1 Completion

We start out this chapter with recalling the definition of a completion of a ring and its basic properties. The completion was introduced to catch the more local behaviour in comparing with the localization. We provide a crucial example of the completion of the ring $A = k[x, y]/(y^2 - x^2(x + 1))$ which will be a necessary in our classification.

Definition 116. *Let A be a ring, $I \leq A$ an ideal and M an A- module. Then we define the completion of A with respect to I to be the inverse limit*

$$\widehat{R} := \varprojlim A/I^n.$$

An element of $\widehat{A}$ is given by a sequence of elements $f_n \in A/I^n$ such that $f_n \equiv f_{n+1}$ mod I^n for all n. Similarly, we define the completion of M with respect to I to be the inverse limit

$$\widehat{M} := \varprojlim M/I^n M.$$

An element of $\widehat{M}$ is given by a sequence of elements $m_n \in M/I^n M$ such that $m_n \equiv m_{n+1}$ mod $I^n M$ for all n. We will view $\widehat{M}$ as $\widehat{A}$- module. Note that we always have canonical maps

$$M \to \widehat{M} \text{ and } M \otimes_A \widehat{A} \to \widehat{M}.$$

Moreover, for every map $\varphi : M \to N$ there is an induced map $\widehat{\varphi} : \widehat{M} \to \widehat{N}$ that is natural, namely we have a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N \\ \downarrow & & \downarrow \\ \widehat{M} & \xrightarrow{\widehat{\varphi}} & \widehat{N} \end{array}.$$

In general, the completion is not an exact functor, but the following lemmas say under what condition a short exact sequence is preserved.

Lemma 117. *Let A be a ring and $I \leq A$ its ideal.*

1. *If $0 \to K \to M \to N \to 0$ is a short exact sequence of A- modules and N is a flat A- module, then $0 \to \widehat{K} \to \widehat{M} \to \widehat{N} \to 0$ is a short exact sequence.*

Proof. [9], Lemma 10.96.1. □

Lemma 118. *Let A be a Noetherian ring and I its ideal. By $\widehat{(-)}$ we denote the completion with respect to this ideal. If $0 \to K \to N \to M \to 0$ is a short exact sequence of finite A- modules, then $0 \to \widehat{K} \to \widehat{N} \to \widehat{M} \to 0$ is a short exact sequence.*

Proof. [9], Lemma 10.97.1. □

Definition 119. *Let A be a ring, $I \leq A$ an ideal of A and M an A- module. We say M is I- adically complete if the map*

$$M \to \widehat{M} := \varprojlim M/I^n M$$

is an isomoprhism. We say A is I- adically complete if it is I- adically complete as A- module.

Example 120. *Let $R = S[x_1, ..., x_n]$ be a polynomial ring over a ring S and $\mathfrak{m} = (x_1, ..., x_n)$. Then the completion of the ring R with respect to the ideal $\mathfrak{m}$ is the ring of formal power series $\widehat{R} = S[[x_1, ...x_n]]$.*

Proof. See an example in [12], pg. 181. □

Rings we will work with are always local rings. Hence, if $(A, \mathfrak{m})$ is a local ring with the maximal ideal $\mathfrak{m}$ then by the completion of the local ring we always mean the completion with respect to the maximal ideal $\mathfrak{m}$ and usually write just as $\widehat{A}$.

Now we recall some useful properties of the completion of a local ring.

Theorem 121. *Denote $(A, \mathfrak{m})$ a local Noetherian integral domain. Then we have:*

1. $\bigcap_{n \geq 0} \mathfrak{m}^n = (0)$.

2. *For a finite A -module M and its submodule N, $\bigcap_{n \geq 0}(N + \mathfrak{m}^n M) = N$.*

3. *The completion $\widehat{A}$ is faithfully flat over A, hence $A \subseteq \widehat{A}$ and $I\widehat{A} \bigcap A = I$ for every ideal $I \subseteq A$.*

4. *The completion $\widehat{A}$ is again Noetherian local with maximal ideal $\widehat{\mathfrak{m}} = \mathfrak{m}\widehat{A}$, moreover $\widehat{A}/\mathfrak{m}^n\widehat{A} = A/\mathfrak{m}^n$ for very $n \geq 0$.*

5. *If A is a complete local ring, then for any ideal $I \neq A$, A/I is again a complete local ring.*

Proof. [10] pg. 62-63 and [13] Proposition 10.16. □

Theorem 122. *Let A be a Noetherian ring I an ideal of A and M a finite A - module. Writing $\widehat{A}, \widehat{M}$ for the I- adic completions of M and A we have*

$$M \otimes_A \widehat{A} \cong \widehat{M}.$$

Proof. [10], Theorem 8.7. □

Theorem 123. *Let A be a Noetherian ring, I an ideal and $\widehat{A}$ the I- adic completion of A. Then $\widehat{A}$ is flat over A.*

Proof. [10], Theorem 8.8. □

Theorem 124. *Let A be a Noetherian ring with a maximal ideal $\mathfrak{m}$. Then $\widehat{A_{\mathfrak{m}}} \cong \widehat{A}$. Here for $\widehat{A}$ we think the completion with respect to the maximal ideal $\mathfrak{m}$.*

Proof. By definition, $\widehat{A} = \varprojlim A/\mathfrak{m}^n$ and $\widehat{A}_{\mathfrak{m}} = \varprojlim A_{\mathfrak{m}}/(\mathfrak{m}A_{\mathfrak{m}})^n$. Since localization is an exact functor it commutes with taking quotient, that is $(A/\mathfrak{m}^n)_{\mathfrak{m}} = A_{\mathfrak{m}}/(\mathfrak{m}A_{\mathfrak{m}})^n$. But $(A/\mathfrak{m}^n)$ is local, so everything outside of the maximal ideal is already invertible. Thus, $(A/\mathfrak{m}^n)_{\mathfrak{m}} = (A/\mathfrak{m}^n)$. □

We provide two particular example of a completion.

Example 125. *Let k be a field. Let $k[x, y]$ be a ring of polynomials in two variables and $\mathfrak{m} = (x, y)$ an ideal of the ring $k[x, y]$. We compute the completion of the ring $B := k[x, y]/(xy)$ with respect to the maximal ideal $\mathfrak{m}$. We have the following short exact sequence of finitely generated $k[x, y]$- modules*

$$0 \to (xy) \to k[x, y] \to k[x, y]/(xy) \to 0.$$

Thus by Lemma 118 we get a short exact sequence of $k[[x, y]]$- modules

$$0 \to \widehat{(xy)} \to k[[x, y]] \to \widehat{B} \to 0.$$

Now, by Theorem 122 and Theorem 123 we get that $\widehat{(xy)} = (xy)k[[x, y]]$ is an ideal in $k[[x, y]]$. Thus we get $\widehat{B} = k[[x, y]]/(xy)$.

Example 126. *Let k be a field. Let us have a look at the completion of the ring $A = k[x,y]/(y^2 - x^2(x+1))$ at the maximal ideal $\mathfrak{m} = (x,y)$. We have a short exact sequence*

$$0 \to (y^2 - x^2(x+1)) \to k[x,y] \to k[x,y]/(y^2 - x^2(x+1)) \to 0.$$

Again, by Lemma 118 the completion of this short exact sequence is again exact, hence we get

$$0 \to (y^2 - \widehat{x^2(x}+1)) \to k[[x,y]] \to \widehat{A} \to 0$$

The completion $(y^2 - \widehat{x^2(x}+1)) = (y^2 - x^2(x+1))$ is an ideal in $k[[x,y]]$ by the same arguments as above. Thus $\widehat{A} = k[[x,y]]/(y^2 - x^2(x+1))$. If $char(k) \neq 2$ then due to the fact that there exists $u \in k[[x]]$ ([12] pg 185-186) such that $u^2 = 1 + x$ and $1 + x$ is a unit in $k[[x]]$ (the inverse is given by a geometric series), we have an isomorphism of rings $\widehat{A} \cong k[[u', v']]/(u'v')$.

3.2 String and Band modules

In this section, we recall the basic definitions from the Crawley-Boewey's paper [7]. We firstly define a string algebra and we will proceed to introduce the notion of the so-called *word*. In the end of this section, we define string and band modules. With aid of these objects, we classify the coherent torsion sheaves on **E**.

Definition 127 (string algebra)**.** *By a string algebra we mean an algebra of the form $\Lambda = kQ/(\rho)$ where k is a field, Q is a quiver, not necessarily finite, kQ is a path algebra and ρ is a set of zero relations in kQ, that is, paths of length at least 2. As usual (ρ) denotes an ideal generated by ρ. Moreover, we suppose that*

1. *Any vertex of Q is the head of at most two arrows and the tail of at most two arrows, and*
2. *Given any arrow $y \in Q$, there is at most one path xy of length 2 such that $xy \notin \rho$ and at most one path zy of length 2 with $zy \notin \rho$.*

Example 128. *$k[x,y]/(xy)$ is a string algebra which arises from a quiver with one vertex and loops x and y with $\rho = \{xy, yx\}$.*

We will classify the finitely generated modules over the string algebra with aid of the so-called string and band modules. To do so, we firstly introduce a notion of words and letters.

Let Q be a quiver. When we talk about a letter l we mean an arrow $x \in Q$ or its formal inverse x^{-1} with obvious head and tail. Now, let I be a one of the following sets: $\{0,1,2,...n\}$ for some $n \geq 0$, $\mathbb{N} = \{0,1,2...\}$, $-\mathbb{N} = \{0,-1,-2,...\}$ or $\mathbb{Z}$. Then we define an I- word C as follows. If $I \neq \{0\}$, then C consists of a sequence of letters C_i for all $i \in I$ such that $i-1 \in I$. That is,

$$C = \begin{cases} C_1C_2\dots C_n & \text{if} \quad I = \{0,1,2,\dots n\} \\ C_1C_2\dots & \text{if} \quad I = \mathbb{N} \\ \dots C_{-2}C_{-1}C_0 & \text{if} \quad I = -\mathbb{N} \\ \dots C_{-1}C_0|C_1C_2 & \text{if} \quad I = \mathbb{Z} \end{cases}$$

(a bar shows the position of C_0 and C_1 in the latter case) satisfying:

1. if C_i and C_{i+1} are consecutive letters, then the tail of C_i is equal to the head of C_{i+1};
2. if C_i and C_{i+1} are consecutive letters, then $C_i^{-1} \neq C_{i+1}$; and
3. no zero relations $x_1 \dots x_m \in \rho$, nor its inverse $x_m^{-1} \dots x_1^{-1}$ occurs as a sequence of consecutive letters in C.

Moreover, if $I = \{0\}$ there are trivial I- words $1_{v,\epsilon}$ for each vertex $v \in Q$ and $\epsilon = \pm 1$. By a word, we mean an I- word for some I. We define the inverse letter C^{-1} of a word C by inverting the letters of C and reversing their order. If C is a $\mathbb{Z}$-word and $n \in \mathbb{Z}$ then we define the shift $C[n]$ by moving the bar to the following position $\dots C_n|C_{n+1}\dots$. We say that a word C is periodic if it is $\mathbb{Z}$- word and $C = C[n]$ for some $n > 0$. The minimal such n is called the period. We extend the shift to any I- word C by defining $C[n] := C$ for $n \in \mathbb{Z}$ and $I \neq \mathbb{Z}$. We define an equivalence relation on the set of all words by $D \sim C$ if and only if $D = C[n]$ or $D = C^{-1}[n]$ for some $n \in \mathbb{Z}$.

Definition 129 (Modules given by words)**.** *Let Λ be a string algebra. Given any I- word C, we define a Λ- module $M(C)$ with basis b_i, for $i \in I$, as a vector space, and the action of Λ is given by*

$$e_v b_i = \begin{cases} b_i & if \quad (v_i(C) = v) \\ 0 & otherwise \end{cases}$$

for a trivial path $e_v \in \Lambda$, and vertex $v \in Q$ and

$$x b_i = \begin{cases} b_{i-1} & if \quad i-1 \in I \quad and \quad C_i = x \\ b_{i+1} & if \quad i+1 \in I \quad and \quad C_{i+1} = x^{-1} \\ 0 & otherwise \end{cases}$$

for any arrow $x \in Q$.

Remark 130. *We have a natural isomorphism $M(C) \cong M(C^{-1})$ for any word C, and for a $\mathbb{Z}$- word C and $n \in \mathbb{Z}$ we have $t_{C,n} : M(C) \cong M(C[n])$ given by $b_i \mapsto b_{i-n}$. Moreover, if C is a periodic word of period n, then $M(C)$ becomes a $\Lambda - k[T, T^{-1}]$- bimodule with T acting as $t_{C,n}$ and we define*

$$M(C, V) := M(C) \otimes_{k[T,T^{-1}]} V$$

for a $k[T, T^{-1}]$- module V. Note that $M(C, V)$ is finite dimensional if and only if V is a finite dimensional k - vector space.

Now we are ready to define the string and band modules.

Definition 131 (String and band modules)**.** *Let $\Lambda = kQ/(\rho)$ be a string algebra. We say that a module M is a string module if it is equal to $M = M(C)$ for some non-periodic word C, and a module M is called a band module if it is of form $M(C, V)$ for some periodic word C and some indecomposable $k[T, T^{-1}]$- module V.*

3.3 Classification

To view the coherent torsion sheaves as modules over the string algebra $k[x, y]/(xy)$ we need one particular functor. The canonical map between a local ring and its completion induce the restriction functor between the categories of modules over the rings. We will show that in fact this is an equivalence between the subcategory of finitely generated $\mathfrak{m}$ - **torsion** modules and the subcategory of finitely generated $\widehat{\mathfrak{m}}$ - **torsion** modules. Finally, we use the results of Crawley-Boevey's and classify the sheaves.

Definition 132. *Let A be a ring. Let M be an A- module.*

1. *Let $I \subset A$ be an ideal of A. We say M is a I- power torsion module if for every $m \in M$ there exists an $n > 0$ such that $I^n m = 0$.*

2. *Let $f \in A$. We say M is an f- power torsion module if for every $m \in M$ there exists an $n > 0$ such that $f^n m = 0$.*

Lemma 133. *Let I be a finitely generated ideal of a ring A. The I- power torsion modules form a Serre subcategory of the Abelian category $Mod(A)$.*

Proof. [9], Lemma 15.88.5. □

Notation Let us denote this subcategory of modules by I - **torsion**.

Lemma 134. *Let $(A, \mathfrak{m})$ be a Noetherian local integral domain of dimension one. Then for any non-zero ideal $I \subset A$ such that $I \subset \mathfrak{m}$ we have $\mathfrak{m}^n \subset I$ for some $n >> 0$.*

Proof. The ring A/I has Krull dimension zero and its nilradical is equal to $\overline{\mathfrak{m}}$, and as the ring A is Noetherian, $\mathfrak{m}$ is finitely generated so $\overline{\mathfrak{m}}^n = 0$ for some $n >> 0$ in A/I. This means that $\mathfrak{m}^n \subset I$. □

As a direct consequence of the above lemma, we get that $\mathcal{F}_\mathfrak{m}$ is $\mathfrak{m}A_\mathfrak{m}$- power torsion $A_\mathfrak{m}$- module, where $\mathcal{F}$ is a torsion coherent sheaf on $\mathbf{E}$ and $A = k[x,y]/(y^2 - x^2(x+1)$.

Let $(A_\mathfrak{m}, \mathfrak{m}A_\mathfrak{m})$ be a localization of a Noetherian local ring $(A, \mathfrak{m})$. We have the canonical map $\varphi : A_\mathfrak{m} \to \widehat{A_\mathfrak{m}}$. This map induces functor of restriction of scalars $\varphi_* : Mod(\widehat{A_\mathfrak{m}}) \to Mod(A_\mathfrak{m})$. We show, that this functor is an equivalence of subcategories of finitely generated $\mathfrak{m}A_\mathfrak{m}$- **torsion** modules and subcategory of $\widehat{\mathfrak{m}A_\mathfrak{m}}$- **torsion** modules. Indeed, the quasi-inverse is given as follows:

$$\begin{aligned} \varphi_* : \quad Mod(\widehat{A_\mathfrak{m}}) &\longrightarrow Mod(A_\mathfrak{m}) \\ M &\longmapsto M_A \\ \widehat{N} &\longleftarrow N \end{aligned}$$

for any finitely generated $M \in \widehat{\mathfrak{m}A_\mathfrak{m}}$- **torsion** and any finitely generated $N \in \mathfrak{m}A_\mathfrak{m}$- **torsion**

Let us prove this in more details. For clarity, let us do everything in the case of the local rings $(A, \mathfrak{m})$ and $(\widehat{A}, \widehat{\mathfrak{m}})$. Let M be an $\widehat{\mathfrak{m}}$ - power torsion $\widehat{A}$ - module. For every $x \in M$ there is some n such that $(\widehat{\mathfrak{m}})^n \times x = 0$. Theorem 121 part 4 gives us $(\widehat{\mathfrak{m}})^n = \mathfrak{m}^n\widehat{A}$. Thus $\mathfrak{m}^n\widehat{A} \times x = 0$, in particular, $\mathfrak{m}^n A \times x = 0$. Conversely, note that a finitely generated $\mathfrak{m}$- torsion A - module M is $\mathfrak{m}$ - adically complete, in other words, we have $\widehat{M} = M$. Thus if we have a finitely generated $\mathfrak{m}$- power torsion module N then we can consider it as $\widehat{A}$- module. By definition we have $\mathfrak{m}^k n = 0$ for every $n \in N$ and some $k > 0$ and the fact that it is $\widehat{\mathfrak{m}}$- torsion module, holds by Theorem 121 part 4.

Before we proceed further, we have to state the main results of the Crawley-Boevey's paper. For the given string algebra $k[x,y]/(xy)$ we will classify the finitely generated modules. Note that Crawley-Boevey relaxes the condition of finiteness and classifies also the so-called finitely controlled modules. The notion of finitely controlled module is a generalization of finitely generated modules when one talk about a quiver with infinitely many vertices. We provide the theorem in this general form, though we focus just on the finitely generated modules.

Theorem 135. *String modules and finite-dimensional band modules are indecomposable. Moreover, there only exist isomorphisms between such modules when the corresponding words are equivalent: there are no isomorphisms between string modules and modules of the form $M(C,V)$; string modules $M(C)$ and $M(D)$ are* *isomorphic if and only if* $C \sim D$*; and* $M(C,V) \cong M(D,W)$ *if and only if* $D = C[m]$ *and* $W \cong V$ *or* $D = C^{-1}[m]$ *and* $W \cong res_\iota V$ *for some* m*.*

Here ι is the automorphism map of $k[T,T^{-1}]$ exchanging T and T^{-1} and res_ι denotes the restriction map via ι.

Proof. [7], Theorem 1.1, section 12. □

Theorem 136. *Every finitely controlled* Λ*- module is isomorphic to a direct sum of copies of string modules and finite-dimensional band modules.*

Proof. [7], Theorem 1.2, section 12. □

Theorem 137 (Krull-Remak-Schmidt property)**.** *If a finitely controlled module is written as a direct sum of indecomposables in two different ways, then there exists bijection between the summands in such a way that corresponding summands are isomorphic.*

Proof. [7], Theorem 1.4, section 12. □

Now we are ready to classify torsion coherent sheaf over the singular curve $\mathbf{E} = Spec(A = k[x,y]/(y^2 - x^2(x+1))$ at the singular point $\mathfrak{m}_A = (x,y) \subset k[x,y]$. Let $\mathcal{F}$ be a coherent torsion module over $\mathbf{E}$ supported at the singular point. Let $B = k[x',y']/(x'y')$ be the ring and $\widehat{B} = k[[x',y']]/(x'y')$ the completion of B with respect to the ideal $\mathfrak{m}_B = (x',y') \subset k[x',y']$, see Example 125. As we have seen earlier the module $\mathcal{F}_\mathfrak{m}$ is also $\mathfrak{m}A_\mathfrak{m}$- power torsion $\mathcal{O}_{X,\mathfrak{m}}$- module. For clarity, we will denote the local ring $(A_{\mathfrak{m}_A}, \mathfrak{m}_A A_{\mathfrak{m}_A})$ just as the local ring $(A, \mathfrak{m}_A)$. Now we have the following sequence of equivalences, that is, $\mathfrak{m}_A$- **torsion** $\cong \widehat{\mathfrak{m}}_{\widehat{A}}$ - **torsion** $\cong \widehat{\mathfrak{m}}_{\widehat{B}}$ - **torsion** $\cong \mathfrak{m}_B$ - **torsion**. The first equivalence comes from the equivalence of the categories of the finitely generated $\mathfrak{m}_A$- **torsion** and $\widehat{\mathfrak{m}}_{\widehat{A}}$ - **torsion** modules and the fact that $\widehat{A_\mathfrak{m}} = \widehat{A}$, 124. The second equivalence holds due to the fact that $\widehat{A} \cong \widehat{B}$, see Examples 125 and 126. Finally, the last equivalence is again the equivalence of the subcategories of finitely generated $\mathfrak{m}_B$ and $\mathfrak{m}_{\widehat{B}}$ - **torsion** modules. Thus, $\mathcal{F}_\mathfrak{m} \cong \bigoplus_{\lambda \in \Phi} N_{\lambda,\infty}$. Note that finitely generated torsion modules are of finite dimension over the field k. Thus, they are exactly the finite direct sums of finite dimensional string and band modules with support at $\mathfrak{m}$. For more details of this decomposition, see [7] pg. 24.

4 The bounded derived category of coherent sheaves

The aim of this section is to give a better description of the bounded derived category of coherent sheaves on a Weierstrass curve. We describe the category with aid of Abelian subcategories of semi-stable sheaves of a given phase. To do so, we define numerical invariants rank and degree of coherent sheaves and then extend the definitions for any bounded complex. The central role here is played by the Abelian subcategory of torsion semi-stable sheaves. In fact, we see that any subcategory of semi-stable coherent sheaves and their shifts of a given phase is equivalent to the subcategory of torsion sheaves. To prove such equivalence, we use a sequence of exact equivalences defined by Thomas and Siedel.

Notation Throughout this chapter, $\mathbf{E}$ denotes a Weierstrass singular curve with the singular point $s \in \mathbf{E}$ over an algebraically closed field k. We denote $\mathcal{D}^b(\mathbf{E}) := \mathcal{D}^b(Coh(\mathbf{E}))$ the bounded derived category of coherent sheaves on $\mathbf{E}$.

4.1 Twist functors

Siedel and Thomas defined so-called twist functors

$$T_E : \mathcal{D}^b(\mathbf{E}) \longrightarrow \mathcal{D}^b(\mathbf{E}),$$

depending on an object $E \in \mathcal{D}^b(\mathbf{E})$, see Definition 2.5. and Proposition 2.4 in [6]. They proved that if the object E is spherical, then these functors are equivalences ([6], Definition 2.14 and the text below). The spherical objects in the context of the derived category of coherent sheaves on a Weierstrass curve are given in [2] Definition 2.3 and Corollary 2.6. Thus, we define the following:

Definition 138 (Spherical sheaf)**.** *A coherent sheaf $\mathcal{F}$ on $\mathbf{E}$ is called spherical if*

1. *$\mathcal{F}$ has a finite resolution by locally free sheaves and,*
2. *$\mathcal{F}$ is a simple sheaf.*

Remark 139. *We call a coherent sheaf $\mathcal{F}$ simple if $End(\mathcal{F}) \cong k$.*

Example 140. 1. *the structure sheaf $k(x)$ is spherical for a regular point $x \in \mathbf{E}$.*

2. *The structure sheaf $\mathcal{O}_{\mathbf{E}}$ is spherical, moreover any simple locally free sheaf is spherical.*

Remark 141. *Note, that the structure sheaf* $\mathbf{k}(s)$ *for the singular point* $s \in \mathbf{E}$ *is not spherical because it does not have any finite locally free resolution (Example 3.2 in [4]).*

Moreover, any of these functors gives a rise to the distinguished triangle of the form

$$\mathbf{R}Hom(E, F) \otimes E \to F \to T_E(F) \xrightarrow{+}$$

for any $F \in \mathcal{D}^b(\mathbf{E})$. We will work with two particular functors and their composition namely, $T_{k(p)}$, $T_{\mathcal{O}}$ and the composition $\mathbb{F} := T_{k(p)} T_{\mathcal{O}} T_{k(p)}$ for some regular point $p \in \mathbf{E}$. Now we give an explicit description of some particular twists acting on some particular sheaves.

Example 142. *For the structure sheaf* $k(x)$ *of a regular point* x*, the twist functor* $T_{k(x)}(-)$ *corresponds to tensoring with the sheaf associated to the Cartier divisor of the point* x*. To be more precise* $T_{k(x)}(-) \cong - \otimes \mathcal{O}(x)$*. (See [6], pg. 28, (3.11)).*

Lemma 143. *Let* x, y *be closed points in* $\mathbf{E}$ *and suppose* x *is regular. Then we have the following isomorphisms:*

$$T_{\mathcal{O}}(k(y)) \cong \mathcal{I}_y[1], \quad T_{\mathcal{O}}(\mathcal{O}(x)) \cong k(x) \quad \text{and } T_{\mathcal{O}}(\mathcal{O}) \cong \mathcal{O}.$$

Here $\mathcal{I}_y$ *is the ideal sheaf corresponding to the closed subscheme* y*. Recall that if it is regular, then it is an effective Cartier divisor 82. Moreover, if* $s \in \mathbf{E}$ *is the singular point and* $n : \mathbb{P}^1 \to \mathbf{E}$ *a normalisation then*

$$T_{\mathcal{O}}(\widetilde{\mathcal{O}}) \cong k(s),$$

for $\widetilde{\mathcal{O}} := n_* \mathcal{O}$*.*

Proof. [2], Lemma 2.13. □

4.2 k - linearity of the category

We are going to show that the bounded derived category of coherent sheaves on a Weierstrass curve $\mathcal{D}^b(\mathbf{E})$ is k - linear. Indeed. let $\mathcal{F}, \mathcal{G}$ be coherent sheaves on $\mathbf{E}$. We show $Ext^i(\mathcal{F}, \mathcal{G})$ is a finite dimensional k- vector space. We proceed by induction on i. The case $i = 0$ holds due to the fact that the sheaf $\mathcal{H}om(\mathcal{F}, \mathcal{G})$is coherent, see 1.3.2 and Theorem 86 for index $j = 0$. Now assume $\mathcal{F}$ is a finite direct sum of invertible sheaves. Then $Ext^i(\mathcal{F}, \mathcal{G}) \cong \bigoplus Ext^i(\mathcal{L}, \mathcal{G}) \cong \bigoplus Ext^i(\mathcal{O}_X, \mathcal{L}^* \otimes \mathcal{G})$ where the last isomorphism holds due to 99, and the latter is isomorphic to $H^i(X, \mathcal{L}^* \otimes \mathcal{G})$

98. Then Theorem 86 yields the result. Now consider the general case, where $\mathcal{F}$ is a coherent sheaf. Theorem 97 gives us an exact sequence

$$0 \to \mathcal{K} \to \mathcal{E} \to \mathcal{F} \to 0$$

such that $\mathcal{K}$ is a coherent and $\mathcal{E}$ is a finite direct sum of invertible sheaves. Proposition 100 induces a long exact sequence

$$0 \longrightarrow Hom(\mathcal{F}, \mathcal{G}) \longrightarrow Hom(\mathcal{E}, \mathcal{G}) \longrightarrow Hom(\mathcal{K}, \mathcal{G}) \longrightarrow Ext^1(\mathcal{F}, \mathcal{G}) \longrightarrow \dots$$

$$\dots \longrightarrow Ext^i(\mathcal{K}, \mathcal{G}) \longrightarrow Ext^{i+1}(\mathcal{F}, \mathcal{G}) \longrightarrow Ext^{i+1}(\mathcal{E}, \mathcal{G}) \longrightarrow \dots .$$

By inductive hypothesis $Ext^i(\mathcal{K}, \mathcal{G})$ is finite dimensional, and $Ext^{i+1}(\mathcal{E}, \mathcal{G})$ is finite dimensional by the arguments above. Thus, $Ext^i(\mathcal{F}, \mathcal{G})$ is finite dimensional. One can show that $Ext^i(\mathcal{F}^\bullet, \mathcal{E}^\bullet)$ is also finite dimensional k - vector space, where $\mathcal{F}^\bullet$ and $\mathcal{E}^\bullet \in \mathcal{D}^b(\mathbf{E})$. However, it would be needed a more sophisticated technique, namely, spectral sequences. This is described in [1] Remark 3.7 (ii). In particular, we get that $Hom(\mathcal{F}^\bullet, \mathcal{E}^\bullet)$ is a finite dimensional k - vector space for any $\mathcal{F}^\bullet, \mathcal{E}^\bullet$ in $\mathcal{D}^b(\mathbf{E})$. Thus, from now on, we consider the category $\mathcal{D}^b(\mathbf{E})$ as k - linear and finite dimensional.

In the sequel, we will use the following action of finite dimensional vector spaces on coherent sheaves. Let us have a scheme X over a field k so we have a structure morphism $X \to Speck$. Take some coherent sheaf $\mathcal{E} \in Coh(X)$. Note that the category of coherent sheaves over $Speck$ is equivalent to the category of finite dimensional K - vector spaces. We define an action of $Vec_f(k)$ on $\mathcal{E}$ as $\mathcal{E} \otimes V := \mathcal{E}^{\oplus n}$ for $V \in Coh(Speck)$ considered as a vector space of finite dimension n. This makes sense via pullback of the structure morphism. In other words, we have $f^*(V) = f^{-1}(V) \otimes_{f^{-1}(\mathcal{O}_Y)} \mathcal{O}_X$. In particular, when $Y = Speck$, the structure sheaf $\mathcal{O}_Y = k$ as $\mathcal{O}_Y$ - module. Note that, $f^{-1}(V) = V$ and $f^{-1}(\mathcal{O}_Y) = k$. Thus we get $f^*(V) \otimes_{\mathcal{O}_X} \mathcal{E} = f^{-1}(V) \otimes_{f^{-1}(\mathcal{O}_Y)} \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{E} = V \otimes_k \mathcal{E}$ and the latter is isomorphic to $\mathcal{E}^{\oplus n}$.

4.3 Rank and Deg

We have already defined the numerical invariants for coherent sheaves, namely the degree and rank. It helps us to analyse the structure of the derived category of bounded coherent sheaves on a Weierstrass curve. We extend the definitions to every complex $F \in \mathcal{D}^b(\mathbf{E})$. Moreover, with aid of these invariants we define phase and slope of a coherent sheaf.

Definition 144 (Phase of Sheaf)**.** *The phase φ of a coherent sheaf $\mathcal{F}$ is defined as a unique number from the interval $(0, 1]$ such that the following equality holds:*

$m(\mathcal{F})exp(\pi i\varphi(\mathcal{F})) = irk(\mathcal{F}) - deg(\mathcal{F})$, where $m(\mathcal{F})$ is a positive real number called the mass, $rk(\mathcal{F})$ and $deg(\mathcal{F})$ are the rank of the sheaf and the degree respectively.

Note that, due to characterization of torsion sheaves by its rank, all non-zero torsion coherent sheaves have phase equal to one. The phase of the structure sheaf is $\varphi(\mathcal{O}) = 1/2$, as we assume the arithmetic genus of the curve equal to one.

Definition 145 (slope of sheaf)**.** *Let $\mathcal{F}$ be a coherent sheaf. We define slope as $\mu(\mathcal{F}) := deg(\mathcal{F})/rk(\mathcal{F})$.*

These two definitions are equivalent, but the phase is better adapted when one talks about shifts. We now define the notion of semi-stability. We say a coherent sheaf $\mathcal{F}$ is semi-stable if for any non-trivial short exact sequence of coherent sheaves

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

the inequality $\varphi(\mathcal{E}) \leq \varphi(\mathcal{F})$, or equivalently $\varphi(\mathcal{F}) \leq \varphi(\mathcal{G})$, holds where $\mathcal{E}, \mathcal{G} \neq 0$. We talk about stable sheaves if the inequality is strict. Definition using the slope μ instead is equivalent. Note that in general for any such short exact sequence exactly one of the following possibilities can occur: the phase is strictly increasing/decreasing or it is equal at each term in the sequence. Recall that every semi-stable coherent sheaf of positive rank is torsion free by definition. Similarly, every non-zero coherent torsion sheaf is semi-stable.

We follow Bridgeland's extension of the definition to any shift object from the category $\mathcal{D}^b(\mathbf{E})$. Namely, we define

$$\varphi(\mathcal{F}[n]) := \varphi(\mathcal{F}) + n,$$

where $\mathcal{F} \neq 0$ is a coherent sheaf on $\mathbf{E}$ and $n \in \mathbb{Z}$.

Let us set up some notation. By $\mathbf{P}(\varphi)$ we denote the Abelian subcategory of shifted semi-stable coherent sheaves with phase $\varphi \in \mathbb{R}$. In particular, if $\varphi \notin (0, 1]$ then we define $\mathbf{P}(\phi + n) = \mathbf{P}(\phi)[n]$ such that $\varphi = \phi + n$ and $\phi \in (0, 1]$, $n \in \mathbb{Z}$. Then we call any non-zero object $\mathcal{F}$ in $\mathcal{D}^b(\mathbf{E})$ semi-stable if it is an object from $\mathbf{P}(\varphi)$ for some $\varphi \in \mathbb{R}$. By $\mathbf{P}^s(\varphi)$ we denote the full sub-category of stable shifted sheaves with phase φ. Note that $\mathbf{P}^s(1)$ consists of sheaves $k(x)$ where $x \in \mathbf{E}$ is a closed point. Indeed, any skyscraper sheaf of closed point is torsion of phase 1. Note that here we consider $k(x)$ as skyscraper sheaf via the map $\iota : Spec(\mathcal{O}_x) \to SpecA$ followed by obvious extension by zeros on $\mathbf{E}$. Moreover, take any subsheaf $\mathcal{F} \subset k(x)$. Inclusion is preserved at stalks and this means that $supp(\mathcal{F}) = \{x\}$. Note that $k(x) = \mathcal{O}_x/\mathfrak{m}_x$ is simple as a module so $\mathcal{F}_x$ is either zero or $k(x)$. Thus $k(x)$ is stable as the condition in definition is empty.
Before we show the converse, let us prove the following important and useful lemma.

Lemma 146. *Let X be a projective scheme over a field k. Let $\mathcal{F}$ and $\mathcal{E}$ be coherent stable sheaves of the same phase. Then $Hom(\mathcal{F}, \mathcal{E})$ is either zero or it is equal to the field k.*

Proof. Take some non-zero morphism $w : \mathcal{F} \to \mathcal{E}$. This factorizes through the composition $\mathcal{F} \to Im(w) \to \mathcal{E}$. If $Im(w)$ is a proper subsheaf of $\mathcal{E}$ then it yields a non-trivial short exact sequences such that one gets $\mu(\mathcal{F}) < \mu(Im(w)) < \mu(\mathcal{E})$ as both sheaves are stable. Thus $Im(w) = \mathcal{E}$. Now, we claim that the kernel $Ker(w)$ is zero. If not, then we have the non-trivial short exact sequence $0 \to Ker(w) \to \mathcal{F} \to Im(w) \to 0$. But $\mathcal{F}$ is stable and hence $\mu(Ker(w)) < \mu(\mathcal{F})$. But $\mu(\mathcal{F}) = \mu(\mathcal{E}) = \mu(Im(w))$ contradicts the assumption. □

We will also need the notion of support and the following lemma.

Definition 147 (support of a complex)**.** *Let X be a scheme. The support of a complex $\mathcal{F} \in \mathcal{D}^b(X)$ is the union of the support of all its cohomology sheaves. That is,*

$$supp(\mathcal{F}) := \bigcup supp(H^i(\mathcal{F})).$$

Lemma 148. *Let $\mathcal{F} \in \mathcal{D}^b(X)$ be a complex. Suppose $supp(\mathcal{F}) = Z_1 \sqcup Z_2$, where $Z_i \subset X$ are disjoint closed subsets. Then $\mathcal{F} \cong \mathcal{F}_1 \oplus \mathcal{F}_2$ with $supp(\mathcal{F}_i) \subset Z_i$.*

Proof. [1], Lemma 3.9. □

Now we have all instances to finish our proof. Let $0 \neq \mathcal{F} \in \mathbf{P}^s(1)$. Lemma 146 shows that any stable sheaf is in fact simple. And simple torsion sheaves are precisely skyscraper sheaves of the form $k(x)$ for some closed point $x \in \mathbf{E}$. Indeed, as we have seen in the first chapter, a torsion sheaf on a curve has finite support. We will show that $\mathcal{F}$ is supported exactly at one point. If not, then by the Lemma above it would decompose into a direct sum, that is, we would have a non-invertbile morphism $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F}_1 \oplus \mathcal{F}_2 \cong \mathcal{F}$, which contradicts $\mathcal{F}$ being simple. From commutative algebra we know that if a module M is finite over a local Noetherian ring $(A, \mathfrak{m})$ such that $supp(M) = \{\mathfrak{m}\}$, then we have well-defined projection and inclusion $M \twoheadrightarrow A/\mathfrak{m} \hookrightarrow M$. Thus, we have well-defined non-zero composition $\mathcal{F} \to k(x) \to \mathcal{F}$, and the fact that $\mathcal{F}$ is simple implies the morphism is invertible, hence $\mathcal{F} \cong k(x)$.

To define the notion of rank and degree for a complex $F \in \mathcal{D}^b(\mathbf{E})$ we need to find the right derived functor $\mathbf{R}Hom(\mathcal{O}_\mathbf{E}, -) : \mathcal{D}^b(\mathbf{E}) \to \mathcal{D}^b(Vec_f(k))$. Here $\mathcal{D}^b(Vec_f(k))$ denotes the bounded derived category of finite dimensional vector spaces over k. Before we proceed further we need the following proposition.

Proposition 149. *Let X be a Noetherian scheme. Then the natural functor*

$$\mathcal{D}^b(X) \to \mathcal{D}^b(Qcoh(X))$$

defines an equivalence between the category of $\mathcal{D}^b(X)$ and the full triangulated subcategory of bounded quasi-coherent sheaves with coherent cohomology $\mathcal{D}^b_{Coh(X)}(Qcoh(X))$.

Proof. [1], Proposition 3.5. □

Thus, the construction is the following. The category $Coh(\mathbf{E})$ does not contain enough injectives thus we have to pass to a bigger category, namely $Qcoh(\mathbf{E})$, see [9] Proposition 28.23.4. The functor $Hom(\mathcal{O}_{\mathbf{E}}, -) : Qcoh(\mathbf{E}) \to Vec(k)$ is left exact, which yields well-defined $\mathbf{R}Hom(\mathcal{O}_{\mathbf{E}}, -) : \mathcal{D}^+(Qcoh(\mathbf{E})) \to \mathcal{D}^+(Vec(k))$. Moreover, it restricts to $\mathbf{R}Hom(\mathcal{O}_{\mathbf{E}}, -) : \mathcal{D}^b(Qcoh(\mathbf{E})) \to \mathcal{D}^b(Vec(k))$ due to Theorem of Grothendieck 85 and Corollary 2.68 (ii) in [1]. For $\mathcal{F}^\bullet \in \mathcal{D}^b(\mathbf{E})$ the complex $\mathbf{R}Hom(\mathcal{O}_{\mathbf{E}}, \mathcal{F}^\bullet) \cong \bigoplus_{i\geq 0} Ext^i(\mathcal{O}, \mathcal{F}^\bullet)[-i]$ see Lemma 67, and in section 4.2 we have proved that these extensions are finite dimensional k- vector spaces. Finally, the composition of this functor with Proposition 149 yields the desired functor.

Now we are able to define the rank and degree for a complex $F \in \mathcal{D}^b(\mathbf{E})$. Let $\mathcal{O}_{E,\eta} = K$ be the function field of the Weierstrass curve E with the generic point η. The base change $\eta : Spec(K) \to E$ is flat from the definition and by Lemma 177 its pullback is an exact functor. Thus, we consider its derived functor simply as pulling back its representatives in $\mathcal{D}^b(\mathbf{E})$. Then we define $rk(F) := \chi(\eta^*(F))$ where χ is the alternating sum of the dimensions of the cohomology spaces of the complex $\eta^*(F)$. To define the degree, we use the right derived functor from the above and define $deg(F) := \chi(\mathbf{R}Hom(\mathcal{O}_{\mathbf{E}}, F))$. It is easy to see that the rank and degree are additive on distinguished triangles. Finally, we provide the action of twist functors $T_{\mathcal{O}}$ and $T_{k(p)}$ on the degree and rank, see[4] pg. 1243. Thus, we have $rk(T_{\mathcal{O}}(F)) = rk(F) - deg(F)$ and $deg(T_{\mathcal{O}}(F)) = deg(F)$. And the effect of $T_{k(p)}$ is $rk(T_{k(p)}(F)) = rk(F)$ and $deg(T_{k(p)}(F)) = deg(F) + rk(F)$.

4.4 The categories $\mathbf{P}(\varphi)$

Before we prove the main result, let us recall some facts about the group $SL(2, \mathbb{Z})$. By $SL(2, \mathbb{Z})$ we denote the non-abelian group of invertible matrices with determinant equal to 1 and integer entries. The standard generators of the group are $g_1 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ and $g_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. In our case it will be convenient to express the group with different generators, namely we have two particular elements:

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Note that the matrix S is of order 4 and T is of infinite order, i.e. $T^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$.

Theorem 150. *The matrices S and T generate $SL(2, \mathbb{Z})$.*

Proof. [14], Theorem 1.1. □

By $\widetilde{SL(2, \mathbb{Z})}$ we denote the group given by the following presentation

$$\langle A, B, T \mid ABA = BAB, (AB)^6 = T^2, [A, T] = [B, T] = 1 \rangle.$$

The projection to $SL(2, \mathbb{Z})$ sends generators A, B, T to

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \text{ and } \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{respectively.}$$

This group is in fact a central extension of $SL(2, \mathbb{Z})$ by $\mathbb{Z}$ and the normal subgroup is generated by T. That is, we have a short exact sequence

$$0 \to \mathbb{Z} \to \widetilde{SL(2, \mathbb{Z})} \to SL(2, \mathbb{Z}) \to 0.$$

Then the action of $\widetilde{SL(2, \mathbb{Z})}$ on $\mathcal{D}^b(\mathbf{E})$ is obtained by sending A, B, T to $T_{\mathcal{O}}, T_{k(p_0)}$ and translation functor [1], see [2] Remark 2.19. Note that under this action, the matrix S correspond to the quasi-inverse $\mathbb{F}^{-1}$ of Fourier-Mukai transform $\mathbb{F} = T_{k(p_0)} T_{\mathcal{O}} T_{k(p_0)}$.

We will also need the following important theorems.

Theorem 151. *Let $\mathcal{F}$ be a semi-stable coherent sheaf on $\mathbf{E}$. If the degree of the sheaf $\mathcal{F}$ is negative, then $\mathbb{F}(\mathcal{F})$ is a semi-stable coherent sheaf. If the semi-stable sheaf $\mathcal{F}$ is of a positive degree, the functor $\mathbb{F}$ sends it to a semi-stable sheaf shifted by [1]. Moreover, $\varphi(\mathbb{F}(\mathcal{F})) = \varphi(\mathcal{F}) + 1/2$ for any shifted coherent sheaf $\mathcal{F}$.*

Proof. [4], Theorem 4.1. □

Note that, the proof of Theorem 4.1 in [4] does not coincide with the statement, thus here we provide the theorem which is actually proven.

Theorem 152. *The Abelian category of coherent torsion semi-stable sheaves $\mathbf{P}(1)$ is equivalent to the Abelian category $\mathbf{P}(1/2)$ of coherent semi-stable torsion free sheaves of degree zero*

Proof. [2], Theorem 2.21. □

Theorem 153. *It holds $\mathbb{F} \circ \mathbb{F} \cong \iota^*[1]$. Consequently $\mathbb{F}^4 \cong [2]$.*

Proof. [2], Theorem 2.18. □

Let us describe the functors $\mathbb{F}$, $T_{k(p)}$ and $T_{\mathcal{O}}$. As one can see in the proof of Theorem 151, the functor $\mathbb{F}$ sends a semi-stable sheaf of the non-negative degree to a semi-stable sheaf. If the degree is positive, then the Fourier-Mukai transform sends a semi-stable sheaf to a semi-stable shift in the degree minus 1.

Recall that the functor $T_{k(p)}$ is given by $- \otimes \mathcal{O}(p)$ and the action on degree and rank is given by $(d, r) \mapsto (d + r, r)$. Tensoring by locally free sheaf preserves sheaves and to prove the semi-stability we proceed as follows. Let $\mathcal{F}$ be a semi-stable non-zero coherent sheaf. Let $\mathcal{F}(p) \to \mathcal{G}$ be a non-zero epimorphism such that $\mu(\mathcal{F}(p)) > \mu(\mathcal{G})$. This corresponds to the epimorphism $\mathcal{F} \to \mathcal{G}(-p)$ and semi-stability of $\mathcal{F}$ gives us $\mu(\mathcal{F}) \leq \mu(\mathcal{G}(-p))$. An easy calculation shows that the quasi-inverse $T_{k(p)}^{-1}$ of $T_{k(p)}$ is given by $- \otimes \mathcal{O}(-p)$. It preserves sheaves and sends (d, r) to $(d - r, r)$, and using the latter we come to the contradiction by showing $\mu(\mathcal{F}) + 1 \leq \mu(\mathcal{G}) < \mu(\mathcal{F}) + 1$.

Now we have a closer look at the functor $T_{\mathcal{O}} = T_{k(p)}^{-1} \mathbb{F} T_{k(p)}^{-1}$. The functor is given by composition of $\mathbb{F}$ and $T_{k(p)}^{-1}$. To see that the latter functor preserves semi-stability, one can proceed similarly as in the previous proof. Thus, $T_{\mathcal{O}}$ preserves semi-stability. Moreover, $T_{\mathcal{O}}$ sends a semi-stable sheaf to a semi-stable sheaf exactly when the degree is less than or equal to the rank. Otherwise, it sends the semi-stable sheaf to the shift in the degree -1. Indeed, just $\mathbb{F}$ can produce a shift, and this happens for shifts of positive degree.

We denote

$$\mathbf{Q} := \{\varphi \in \mathbb{R} \mid \mathbf{P}(\varphi) \; contains \; a \; non-zero \; object\}.$$

Now we are ready to prove the following corollary.

Corollary 154. *The category $\mathbf{P}(\varphi)$ of semi-stable objects of phase $\varphi \in \mathbf{Q}$ is equivalent to the category of torsion sheaves $\mathbf{P}(1)$. Any such equivalence restricts to an equivalence between $\mathbf{P}^s(\varphi)$ and $\mathbf{P}^s(1)$. Under such an equivalence, stable vector bundles correspond to structure sheaves of smooth points. Moreover, if $\varphi \in (0, 1) \cap \mathbf{Q}$, $\mathbf{P}^s(\varphi)$ contains a unique torsion free sheaf, which is not locally free. It corresponds to the structure sheaf $\mathbf{k}(s) \in \mathbf{P}^s(1)$ of the singular point.*

Proof. The equivalence of $\mathbf{P}(1)$ and $\mathbf{P}(1/2)$ is Theorem 152.
We are going to find a composition of functors $T_{\mathcal{O}}$ and $\mathbb{F}^{-1}$ which gives us the equivalence of categories $\mathbf{P}(1)$ and $\mathbf{P}(\varphi)$ for any φ. Firstly, let $\varphi \in (0, 1) \cap \mathbf{Q} \setminus \{1/2\}$. This phase corresponds to some degree and rank. Denote them by d and r for brevity. WLOG $GCD(r, d) = 1$, if $GCD(r, d) = m$ then we proceed with $r' := r/m$ and $d' := d/m$ (note that these rank and degree define the same phase). The well-known Bezout Theorem gives us the existence of $a, b \in \mathbb{Z}$ such that $ar + bd = 1$. The following matrix $\begin{pmatrix} d & -a \\ r & b \end{pmatrix}$ is the matrix which sends the phase 1 given by

$(d', 0)$ to the desired phase given by (d, r). This matrix belongs to $SL(2, \mathbb{Z})$. Due to Theorem 150, we can express this matrix as a composition of generators S, T, in particular, we get $T^{m_1}ST^{m_2}S \dots T^{m_{k-1}}ST^{m_k}$. By lifting this composition to $\widetilde{SL(2,\mathbb{Z})}$ and using the action on $\mathcal{D}^b(\mathbf{E})$ we get a composition of functors $T_{\mathcal{O}}$, $T_{\mathcal{O}}^{-1}$ and $\mathbb{F}^{-1}$. By Theorem 153 $\mathbb{F}^{-1} = [-1]\mathbb{F}\iota^*$, which shows that the quasi-inverse can produce a shift in degree $+1$ together with $T_{\mathcal{O}}^{-1} = T_{k(p)}\mathbb{F}^{-1}T_{k(p)}$. Note that all of these functors commute with shift functors. This means that any shift produced within the sequence can be controlled by shift functors $[+1]$ and $[-1]$. Note, that this holds for every sheaf from $\mathbf{P}(1)$ because the question if the above functors produce shifts depends solely on the given phase. Hence, we get the desired composition

$$T_{\mathcal{O}}^{m_1}\mathbb{F}^{-1}T_{\mathcal{O}}^{m_2}\mathbb{F}^{-1} \dots T_{\mathcal{O}}^{m_{k-1}}\mathbb{F}^{-1}T_{\mathcal{O}}^{m_k} : \mathbf{P}(1) \longrightarrow \mathbf{P}(\varphi)[n],$$

controlled by the shift $[-n]$ for some $n \in \mathbb{Z}$. The functors $\mathbb{F}^{-1}$ and $T_{\mathcal{O}}$ preserves semi-stability (note that ι^* as well) thus the above functor sends semi-stable coherent torsion sheaf to semi-stable coherent sheaf of given phase φ.
Now, assume $\varphi \in \mathbf{Q}\backslash(0, 1)$ then $\mathbf{P}(\varphi) = \mathbf{P}(\phi + m) = \mathbf{P}(\phi)[m]$ such that $\phi \in (0, 1)$. So we find the functor $\mathbf{P}(1) \to \mathbf{P}(\phi)$ and then compose it with the shift functor $[m]$.

Now we show that any such equivalence restricts to stable objects. Notice that the equivalence is in fact a k - linear equivalence between Abelian categories, thus it preserves simple objects in the sense of the following definition. An object X is called simple if it is not isomorphic to 0 and any subobject of X is either X or 0. And stable objects of $\mathbf{P}(\varphi)$ are precisely simple objects in the sense of this definition, see Lemma 146. Thus, the equivalence restricts to stable objects.

To show that stable vector bundles correspond to structure sheaves of smooth points, it is enough to show that $T_{\mathcal{O}}$ send vector bundles to vector bundles (unless it is torsion sheaf of the form $k(x)$ for x regular e.g, see Lemma 143). Indeed, a pullback of locally free sheaves[1] is again locally free and tensoring locally free sheaf with $-\otimes\mathcal{O}(p)$ or $-\otimes\mathcal{O}(-p)$ is locally free as well. Moreover, direct calculation with the aid of Lemma 143 shows that the functors $T_{\mathcal{O}}, T_{k(p)}, \mathbb{F}$ and their quasi-inverses applied to $k(x)$ produce a locally free sheaf. Thus, it is left to show that $T_{\mathcal{O}}$ preserves locally free sheaves.

So let F be a semi-stable locally free sheaf. Assume $T_{\mathcal{O}}(F)$ is a sheaf, the case when $T_{\mathcal{O}}(F)$ is a shift is discussed later. Note that we always have the distinguished triangle of the form $\mathbf{R}Hom(\mathcal{O}, F) \otimes \mathcal{O} \to F \to T_{\mathcal{O}}(F) \xrightarrow{+}$. Note that $\mathbf{R}Hom(\mathcal{O}, F) \cong Ext^0(\mathcal{O}, F) \oplus Ext^1(\mathcal{O}, F)[-1]$. Using Serre duality A.4 and the fact that $Hom(\mathcal{F}, \mathcal{E})$ vanishes for semi-stable sheaves such that $\varphi(\mathcal{F}) > \varphi(\mathcal{E})$, one

[1]Vector bundles corresponds to locally free sheaf, see [15] pg. 287 or [11] Chapter 13.1.

of the above *Ext* vanishes depending on the phase F. Thus, we get the following triangles

$$\mathcal{O}^{\oplus n_1} \to F \to T_{\mathcal{O}}(F) \xrightarrow{+}$$

or

$$\mathcal{O}^{\oplus n_1}[-1] \to F \to T_{\mathcal{O}}(F) \xrightarrow{+} .$$

We apply TR2 axiom for the latter and get

$$F \to T_{\mathcal{O}}(F) \to \mathcal{O}^{\oplus n_1} \xrightarrow{+} .$$

Every such distinguished triangle gives a rise to a short exact sequence, in particular we have

$$0 \to F \to T_{\mathcal{O}}(F) \to \mathcal{O}^{\oplus n_1} \to 0$$

or

$$0 \to \mathcal{O}^{\oplus n_1} \to F \to T_{\mathcal{O}}(F) \to 0.$$

In the first case, to show $T_{\mathcal{O}}(F)$ is locally free we can proceed locally. Indeed, being locally free sheaf is a stalk local property for coherent sheaf over Noetherian scheme. The sequence $0 \to \mathcal{O}_{X,x} \to T_{\mathcal{O}}(F)_x \to \mathcal{O}_{X,x}^{\oplus n} \to 0$ of $\mathcal{O}_{X,x}$- modules splits at every point $x \in X$. Hence, $T_{\mathcal{O}}(F)$ is locally free.

In the second case we have to realise that if a coherent sheaf $\mathcal{F}$ on a singular Weierstrass curve $\mathbf{E}$ is torsion free then $depth_{\mathcal{O}_{X,x}}\mathcal{F}_x = dim\mathcal{O}_{X,x} = 1$. Thus, it is Cohen-Macaulay, see definition 182. Since $\mathbf{E}$ is Gorenstein, Theorem 168 yields $Ext^i_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{O}_{X,x}) = 0$ for $i > 0$, thus, we get the vanishing stalks $\mathcal{E}xt^i(\mathcal{F}, \mathcal{O}_X)_x$ for each point $x \in X$ and each index $i > 0$ by Proposition 102. We conclude $\mathcal{E}xt^i(\mathcal{F}, \mathcal{O}) = 0$ for $i > 0$. The sheaf $T_{\mathcal{O}}(F)$ is torsion free, hence, by previous arguments and Proposition 100 we have a short exact sequence

$$0 \to \mathcal{H}om(T_{\mathcal{O}}(F), \mathcal{O}) \to \mathcal{H}om(F, \mathcal{O}) \to \mathcal{H}om(\mathcal{O}^{\oplus n_1}, \mathcal{O}) \to 0.$$

Note that the dual of locally free sheaf is locally free and the kernel of surjective morphism of locally free sheaves is also locally free. In particular, $\mathcal{H}om(T_{\mathcal{O}}(F), \mathcal{O})$ is locally free. Thus, the proof boils down to show that the sheaf $T_{\mathcal{O}}(F)$ is reflexive. We can proceed locally thanks to Lemma 175. This holds for every regular point by Lemma 171 and Lemma 82. It is left to prove that the stalk at the singular point s is reflexive. But this is true by Theorem 168 ($T_{\mathcal{O}}(F)$ is CM sheaf on Gorenstein curve $\mathbf{E}$). Hence, the sheaf $T_{\mathcal{O}}(F)$ is reflexive and this finishes the proof.

Finally, it might happen that $T_{\mathcal{O}}(F)$ is not a sheaf, in that case, it is shifted exactly by the translation functor [1] and this might happen only in the case when the $Ext^1(\mathcal{O}, \mathcal{F})$ vanishes. Thus, we use $TR2$ and we get the distinguished triangle of the form

$$T_{\mathcal{O}}(F)[-1] \to]\mathcal{O}^{\oplus n} \to F \xrightarrow{+}$$

which yields the following short exact sequence

$$0 \to T_{\mathcal{O}}(F)[-1] \to \mathcal{O}^{\oplus n} \to F \to 0.$$

Again, the kernel of a surjective map of locally free sheaf is locally free.

To show the last part of the corollary, we proceed by contradiction. Denote the equivalence from the category of torsion stable sheaves to stable sheaves of given phase φ by $\mathbb{G}$. The sheaf $\mathbb{G}(k(s))$ is torsion free because every semi-stable sheaf of positive rank is torsion free. So let us assume $\mathbb{G}(k(s))$ is locally free. We have already proved that stable vector bundles correspond to structure sheaves of smooth points. Hence, the contradiction. □

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A Appendix

A.1 Commutative Algebra

Proposition 155 (Characterization of DVR). *Let R be domain which is not a field. Then the following is equivalent:*

1. *The ring R is Noetherian, local and the maximal ideal is principal.*
2. *There is an irreducible element $t \in R$ such that every nonzero $z \in R$ may be written uniquely in the form $z = ut^n$, u a unit in R, n a non-negative integer.*

A ring satisfying the above condition is called a discrete valuation ring, written DVR.

Proof. [16], Chapter 2, Proposition 4. □

Lemma 156. *Let $(R, \mathfrak{m})$ be a Noetherian, integral, local ring. If the maximal ideal $\mathfrak{m}$ is not principal, then $\mathfrak{m}$ is not free as R module.*

Proof. Assume $\mathfrak{m}$ is a free R module and not principal. That is, $\mathfrak{m} \cong \oplus x_i R$ for some basis $\{x_1, x_2, ...\}$ of cardinality ≥ 2 (if it would be principal then it is free of rank 1 as R - module). Pick indices $i = 1, 2$, then $x_1 x_2 \in x_1 R \bigcap x_2 R = \{0\}$. But R is an integral domain. □

Theorem 157 (Nakayama Lemma 1). *Let A be a commutative ring. Let M be a finite A-module and $I \leq A$ an ideal of the ring A. If $M = IM$ then there exists $a \in A$ such that $aM = 0$ and $a \equiv 1 \bmod I$. If in addition $I \subseteq rad(A)$ then $M = 0$.*

Proof. [10], Theorem 2.2. □

Theorem 158 (Nakayama Lemma 2). *Let $(A, \mathfrak{m}, k)$ be a local ring and M a finite A- module. Set $\overline{M} = M/\mathfrak{m}M$. Now $\overline{M}$ is a finite dimensional k- vector space, and we write n for its dimension. Then*

1. *If we take a basis $\{\overline{u_1}, ..., \overline{u_n}\}$ for $\overline{M}$ over k, and choose an inverse image $u_i \in M$ of each $\overline{u_i}$, then $\{u_1, ...u_n\}$ is a minimal basis of M;*
2. *conversely, every minimal basis of M is obtained in this way, and so has n elements.*

Proof. [10], Theorem 2.3. □

Proposition 159. *Let* $(R, \mathfrak{m}, k)$ *be a local Noetherian ring and* M *a finitely generated module. Then the following are equivalent:*

1. $Ext^1_R(M, k) = 0$
2. *The module* M *is a free module*

Proof. The implication (2) $\implies$ (1) is basic homological algebra, namely the free module M is projective, hence extension vanishes for every R - module in the second variable, in particular, it vanishes for the R - module k.
Conversely, for the sake of contradiction, assume that the finitely generated module M is not a free. We choose n minimal such that we have a surjection $R^n \to M$. As M is not a free module, then we have a short exact sequence with the kernel $K \neq 0$

$$0 \to K \to R^n \to M \to 0.$$

The right derived functor $\mathbf{R}Hom_R(-, k)$ induces a long exact sequence. R^n is a projective module, which implies vanishing $Ext^i_R(R^n, k) = 0$ in all degrees $n > 0$. That is, $\partial : Hom(K, k) \to Ext^1(M, k)$ is surjective. Moreover, tensoring the map $R^n \to M$ with $- \otimes_R k$ gives an isomorphism

$$(R/\mathfrak{m})^n \cong R^n \otimes_R k \cong M \otimes_R k \cong M/\mathfrak{m}M$$

due to Theorem 158. Now apply $Hom_k(-, k)$ which yields $Hom_k(R^n \otimes_R k, k) \cong Hom_k(M \otimes_R k, k)$. And by $hom, \otimes$ adjunction we get

$$Hom_R(R^n, k) \cong Hom_R(M, k).$$

This implies ∂ is injective. Hence, we get the following important isomorphism, namely $Hom_R(K, k) \cong Ext^1_R(M, k)$. Now, if M is not free then $K \neq 0$ and by Theorem 157, $K/\mathfrak{m}K \neq 0$. But $Hom_R(K, k) \cong Hom_k(K \otimes_R k, k)$ so $Hom_R(K, k) \neq 0$. We conclude, $Ext^1_R(M, k) \neq 0$ and this contradicts the assumption. □

A.1.1 Torsion and Torsion free modules

Definition 160 (Dedekind domain, Wedhorn)**.** *A Dedekind ring is a Noetherian integral domain* A *such that for each maximal ideal* $\mathfrak{m}$ *the localization* $A_\mathfrak{m}$ *is PID. Equivalently, it is Noetherian regular domain of dimension* ≤ 1.

Theorem 161 (Jordan Decomposition)**.** *Let* R *be a Dedekind domain.*

1. *A finitely generated torsion module is indecomposable if and only if it is cyclic and its annihilator is* $\mathfrak{p}^m$ *for some prime ideal* $\mathfrak{p}$ *and natural number* $m \geq 1$. *(Thus* $M \cong R/\mathfrak{p}^m$*).*

2. *Every finitely generated torsion R- module M is a direct sum of indecomoposable modules. That is, we have*

$$M \cong \bigoplus_{i=1}^{s} Rx_i$$

where for each i there is a prime ideal $\mathfrak{p_i}$ and a natural number $m_i \geq 1$ such that the map $R \to Rx_i$ induces an isomorphism $R/\mathfrak{p}^{m_i} \cong Rx_i$. In other words we have

$$M \cong \bigoplus_{i=1}^{s} R/\mathfrak{p}^{m_i}$$

3. *Such a decomposition is a unique in the following sense: If we have two decompositions*

$$M \cong \bigoplus_{i=1}^{s} R/\mathfrak{p}^{m_i} \cong \bigoplus_{j=1}^{k} R/\mathfrak{q}^{n_j}$$

,

then $s = t$ and up to permutation we have $\mathfrak{p_i} = \mathfrak{q_i}$ and $m_i = n_j$.

Proof. [17], Theorem 10.3.9. □

Proposition 162. *Let R be a integral domain. Let M be a module. Then the following are equivalent:*

1. *M is a torsion free module.*

2. *$M_\mathfrak{p}$ is a torsion free $R_\mathfrak{p}$- module for each prime ideal $\mathfrak{p}$.*

3. *$M_\mathfrak{m}$ is a torsion free $R_\mathfrak{m}$- module for each maximal ideal $\mathfrak{m}$.*

Proof. 1) $\implies$ 2) follows from the definition of localization. In fact, $S^{-1}M$ is a torsion free for each multiplicative subset $S \subset R$ which does not contain zero. Implication 2) $\implies$ 3) is obvious. For implication 3) $\implies$ 1) it is enough to noticed that the canonical map

$$M \to \prod_{\mathfrak{m}} M_\mathfrak{m}$$

is injective ([9], Lemma 10.23.1). Thus, if localization at every maximal ideal is torsion free then M is torsion free too. □

Proposition 163. *Let R be an integral domain with the fraction field K. Then the following holds:*

1. *If M is a torsion R - module then $M \otimes_R K = 0$ and if M is not torsion then $M \otimes_R K \neq 0$*

2. $M_{tors} = ker(M \to M \otimes_R K)$ *where* $m \to m \otimes 1$

3. $M \otimes_R K \cong M_{(0)}$

Proof. 3 is Proposition 3.5 in [13]. 2 follows by definition of localization. Finally, 1. is a consequence of 2. □

A.1.2 Cohen-Macaulay rings and Gorenstein rings

Let $(R, \mathfrak{m}, k)$ be a local ring.

Definition 164 (CM module)**.** *Let M be a finitely generated R- module. We say M is maximal Cohen-Macaulay (CM) if $M = 0$ or $depth M = dim R$.*

Definition 165 (CM ring)**.** *The ring R is Cohen-Macaulay (CM) if it is CM as R- module, i.e., $depth R = dim R$.*

Definition 166 (Gorenstein ring)**.** *The ring R is called Gorenstein if $Ext^i(k, R) \cong k$ for $i = dim R$ and zero otherwise.*

Proposition 167. *A local Gorenstein ring is Cohen-Macaulay.*

Proof. [10], Theorem 18.1. □

Theorem 168 (CM modules over Gorenstein local ring)**.** *Let R be a local Gorenstein ring. Let M be a finitely generated Cohen-Macaulay R- module. Then:*

1. *M is reflexive, and*

2. $Ext^i_R(M, R) = 0$ *for all* $i > 0$.

Proof. [18], Theorem 4.8. □

A.2 Reflexivity and Flatness

Definition 169 (reflexive module)**.** *Let R be a domain. We say an R- module M is reflexive if the natural map*

$$j : M \to Hom(Hom(M, R), R)$$

which sends $m \in M$ to the map sending $\varphi \in Hom(M, R)$ to $\varphi(m) \in R$ is an isomorphism.

Remark 170. *We denote the dual of a module M by $^\vee$, in other words $M^\vee = Hom(M, R)$.*

Lemma 171. *Let R be a discrete valuation ring and let M be a finite R - module. Then the map $j : M \to M^{\vee\vee}$ is surjective.*

Proof. [9], Lemma 15.23.3. □

Remark 172. *We call a module M torsion less if the natural map j is injective.*

Similarly, we define the notion of reflexive sheaves.

Definition 173. *Let X be an integral locally noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$ - module. We say $\mathcal{F}$ is reflexive if the natural map*

$$j : \mathcal{F} \to \mathcal{H}om(\mathcal{H}om(\mathcal{F}, \mathcal{O}_X), \mathcal{O}_X)$$

is an isomorphism.

Lemma 174. *Let X be an integral locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$ - module.*

1. *If $\mathcal{F}$ is reflexive then it is torsion free.*
2. *The natural map $j : \mathcal{F} \to \mathcal{F}^{\vee\vee}$ is injective if and only if $\mathcal{F}$ is torsion free.*

Here we denote the dual of the sheaf of $\mathcal{O}_X$ - modules $\mathcal{F}$ by $\mathcal{F}^\vee$.

Proof. [9], Lemma 31.12.4. □

Lemma 175. *Let X be a integral locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$ - module. Then the following are equivalent*

1. *$\mathcal{F}$ is reflexive,*
2. *$\mathcal{F}_x$ is reflexive for every point $x \in X$,*

3. $\mathcal{F}_x$ is reflexive for every closed point $x \in X$.

Proof. [9], Lemma 31.12.5. □

Definition 176. *Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ be a $\mathcal{O}_X$-module.*

1. *We say that $\mathcal{F}$ is flat over Y at point $x \in X$ if $\mathcal{F}_x$ is flat as $\mathcal{O}_{Y,f(x)}$- module via the map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$*
2. *We say that $\mathcal{F}$ is flat over Y if it is flat over Y at every point $x \in X$*
3. *The morphism f is called flat at the point $x \in X$ if $\mathcal{O}_X$ is flat over Y at the point x.*
4. *The morphism f is called flat if it is flat at every point of X.*

Lemma 177. *Let $f : X \to Y$ be a flat morphism of ringed spaces. Then the pullback functor $f^* : Mod(Y, \mathcal{O}_Y) \to Mod(X, \mathcal{O}_X)$ is an exact functor.*

Proof. [9], Lemma 17.20.2. □

A.3 Effective Cartier Divisors

Definition 178. *Let X be a scheme. An effective Cartier divisor on X is a closed subscheme D such that the corresponding ideal sheaf $\mathcal{I}$ is invertible.*

Definition 179. *Let X be a scheme. Let $D \subset X$ be an effective Cartier divisor with ideal sheaf $\mathcal{I}_D$.*

1. *The invertible sheaf $\mathcal{O}_X(D)$ associated to D is defined by*
$$\mathcal{O}_X(D) := \mathcal{H}om(\mathcal{I}_D, \mathcal{O}_X) = \mathcal{I}_D^{-1}$$
2. *We write $\mathcal{O}_X(-D) = \mathcal{I}_D$.*
3. *Given a second effective Cartier divisor $D' \subset X$ we define*
$$\mathcal{O}_X(D - D') = \mathcal{O}_X(D) \otimes \mathcal{O}_X(-D').$$

Lemma 180. *Let X be a scheme. Let D, D' be effective Cartier divisors on X. Then there is a unique isomoprhism*

$$\mathcal{O}_X(D) \otimes \mathcal{O}_X(D') \to \mathcal{O}_X(D + D')$$

which maps $1 \otimes 1$ to 1.

Proof. Omitted. □

A.4 Serre Duality

Definition 181. *A scheme X is called Cohen-Macaulay if all of its local rings are Cohen-Macaulay.*

Definition 182. *Let X be a Noetherian scheme. We say a coherent sheaf $\mathcal{F}$ is maximal Cohen-Macaualay if $depth_{\mathcal{O}_{X,x}}\mathcal{F}_x = dim\mathcal{O}_{X,x}$ for every $x \in X$.*

Definition 183. *Let X be a scheme. We say that X is Gorenstein if X is locally Noetherian and $\mathcal{O}_{X,x}$ is Gorenstein ring for every point $x \in X$.*

Theorem 184 (Duality for a Projective Scheme)**.** *Let X be a projective scheme over an algebraically closed field k. Let ω_X° be a dualizing sheaf on X, and let $\mathcal{O}(1)$ be a very ample sheaf on X. Then:*

1. *for all $i \geq 0$ and $\mathcal{F}$ coherent on X, there are natural functorial maps*

$$\theta^i : Ext^i(\mathcal{F}, \omega_X^\circ) \to H^{n-i}(X, \mathcal{F})^\vee,$$

such that the following conditions are equivalent:

a) X is Cohen-Macaulay and equidimensional (i.e., all irreducible components have the same dimesion);

b) for any locally free sheaf $\mathcal{F}$ on X, we have $H^i(X, \mathcal{F}(-q)) = 0$ for $i < n$ and $q >> 0$;

c) the maps θ^i are isomorphisms for all $i \geq 0$ and all coherent $\mathcal{F}$ on X.

Proof. [8], III, Theorem 7.6. □

We are in the setting of a projective Weierstrass curve $\mathbf{E}$ and such a curve is Gorenstein with the dualizing sheaf $\omega_{\mathbf{E}}^\circ \cong \mathcal{O}_{\mathbf{E}}$, see [2] pg. 5. Thus, any such curve is necessary Cohen-Macaulay and by the above Theorem part a) we have the isomorphisms θ^i. In particular, we have $Ext^1(\mathcal{O}, \mathcal{F}) \cong H^1(X, \mathcal{F})$ and $H^{1-0}(X, \mathcal{F})^\vee \cong Ext^0(\mathcal{F}, \mathcal{O}) \cong Hom(\mathcal{F}, \mathcal{O})$ for $\mathcal{F}$ coherent sheaf on $\mathbf{E}$.