Evaluation of Master's thesis

Title: Coherent sheaves on singular curves of genus one
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Summary of the thesis

The thesis is mainly focused on the exposition of two results: the first result by Burban-Kreussler establishes an equivalence between the category of torsion sheaves on a Weierstrass nodal cubic with support at the singular point and categories of semi-stable sheaves of different slope. The second is the classification of indecomposable (and hence all) coherent torsion sheaves via the classification results for modules over string algebras due to Crawley-Boevey which is applied to the algebra $k[[x,y]]/(xy)$. The thesis recalls some of the background material such as schemes, the foundations of sheaf theory and derived categories as well as spherical twist functors.

Evaluation of the thesis

Topic of the thesis. The topic and its mathematical difficulty seem appropriate for a Master's thesis. The thesis exposites several research papers from the last twenty years.

Contribution of the student. The thesis contains an exposition of some of the results in the research paper by Burban-Kreussler (2006). It also carries out part of the classification of torsion sheaves at the singular point over the Weierstrass nodal cubic by using classification results of indecomposable modules over string algebras due to Crawley-Boevey which are applied to the algebra $k[[x,y]]/(xy)$. While the classification of such torsion sheaves also follows from work of Burban-Drozd (2004, Duke Mathematical Journal, Appendix A), the approach via Crawley-Boevey's classification is seemingly new.

Mathematical quality. To my best knowledge, the mathematical content of the thesis is correct and showcases the author's ability to understand and present a new topic in mathematics. Overall, the thesis leaves room for improvement with regards to its mathematical rigor and presentation. A selection of instances of this are the following:

1. **Missing conditions or assumptions in definitions:**
   - Across Section 1.2., which introduces schemes, the ground ring is never mentioned to be commutative which is vital for the theory. Only from page 19 onward, rings are correctly assumed to be commutative.
   - In Definition 1, p.8., the subsets $U \subset V$ are not assumed to be open.
   - Lemma 55, p.17. The map $\mathcal{F}_2 \to \mathcal{F}_1$ needs to be injective.
   - Definition 58. This is usually called a "preadditive" category. In order to obtain the definition of an additive category one needs to add the assumption that finite (bi)products exist.
2. **Undefined notation or use of notation beyond defined cases:**
   - The notation $\mathcal{F}_U$ for the restriction of a sheaf $\mathcal{F}$ on a topological space X to a sheaf on an open subset $U \subset X$ is not introduced despite the goal of the section to introduce the foundations of sheaf theory which are necessary for the thesis.

- Definition 29, p.12, uses notation and mentions the support of a sheaf. This notion is not introduced or previously mentioned despite introducing the foundations of sheaf theory. In any case, an explanation of the notation is due. The same goes for the quotient of a sheaf which involves a sheafification process. In general, neither the existence nor the constructions of kernels and cokernels for sheaves are discussed despite their appearance in the thesis.
- Proposition 4.4. 4), p.15: I was not able to locate the meaning of the notation N_A for a module N in the thesis.
- Definition 4.5., p.15. This variant of the definition of "coherent" is only correct if the scheme is assumed to be noetherian.
- Proposition 52, p.17. The notions of "quasi-compact" and "separated" morphisms were neither discussed nor defined.
- Proposition 68. The notation $\mathcal{D}^+(\mathcal{A})$ is undefined.
- Definition 83. The notation $\mathbb{P}^n_Y$ is never introduced nor is its meaning (the relative projective space).
- Definition 91. The notion of a "generic point" is neither explained nor referenced.
- Throughout Section 4.1. The structure sheaf of the curve E, previously denoted by $\mathcal{O}_E$ is now referenced as $\mathcal{O}$ without further explanation. The author of the thesis seems to follow the notation of the paper by Burban-Kreussler (2004) but without stating it.
- Lemma 4.3. The normalisation of a nodal cubic is used but no reference is given that the normalisation (which is not introduced in the thesis) is the projective line.
- Section 4.3, before Definition 144. The extension of the degree and rank to complexes should be explicitly given as they do not require much space or technicalities.

In summary, the thesis could be more consistent with the level of knowledge of the reader that it assumes: on one hand it introduces many basic concepts (which I see positively) but then assumes a beyond-beginner familiarity of the reader when it comes to notations which are frequently used without comment or introduction.

A selection of other notable comments are the following.

1. The classification of indecomposable torsion sheaves via strings and bands stops at citing Crawley-Boevey's work but does not actually write out the final result.
2. Remark 79. It is not clear if the field is generally assumed to be algebraically closed or not. In this case, identifying the notions of smoothness and regularity makes them rather ambiguous in subsequent parts of the thesis.
3. The author name "Seidel" of an for the thesis important paper by Seidel-Thomas is consistently misspelled as "Siedel" throughout the thesis.
4. Proof of Lemma 101 (after the second sequence). The way in which right derived functors are introduced uses injective resolutions whereas here locally-free resolutions are used. From this point of view, it should be clarified why this correctly computes the derived functors $\mathrm{Ext}^\bullet(-, \tilde{N})$.
5. Proof of Lemma 105. It should be explained in greater detail how Proposition 163, part 2, is used to deduce the statement.
6. Proof of Proposition 109. How is the fact that the curve has dimension 1 used exactly to deduce the statement?

7. Proof of Proposition 114, first sentence. Why can one assume this condition without loss of generality? It seems that the following sentence does not provide sufficient arguments for this as the kernel $\mathcal{F}/t\mathcal{F}$ does not split in the same way as the torsion subsheaf $t\mathcal{F}$.
8. Proof of Lemma 146. The proof appears to be incomplete. The proof shows that every non-zero morphism is an isomorphism. However, one requires an additional argument to conclude that the morphism space is exactly one-dimensional.
9. Page 41. It should be specified why the map φ_* is a quasi-inverse when restricted to torsion sheaves. In particular, where is the assumption on torsion used?
10. Page 41. The meaning of the symbol $\times$ in $(\mathfrak{m})^n \times x$ (and similar in subsequent parts of the page) is unclear to me.
11. Page 42. The classification is not carried out explicitly. In particular, it appears that the notation $N_{\lambda,\infty}$ has not been introduced.
12. Section 4.2. There seems to be a misunderstanding of the meaning of k-linearity. It is not the k-linearity that needs to be established for the derived category (as it always holds in these circumstances) but the fact that morphisms spaces are finite-dimensional vector spaces.

Usage of the literature. The thesis should include a reference to the work of Burban-Drozd which classifies the respective objects in the derived category of the algebra $k[[x, y]]/(xy)$ in its Appendix. This leads to a classification of torsion sheaves over the singular point of the Weierstrass nodal cubic. Overall the thesis does correctly attribute its results to the respective authors but a more transparent way of doing so would be desirable. In several places and without prior knowledge it would have been unclear to me if results were obtained by the author of this thesis or by others. This is partially due to the use of phrases such as "we will prove (result X)" without immediate attribution of the proof to its original authors. It was not entirely apparent to me if the classification of torsion sheaves via Crawley-Boevey's results (carried out partially in this thesis) is new and due to the author, or reproduced from the literature. Either case should be clearly marked as such.

Formal aspects. The thesis meets the usual formal standards.

Conclusion

I recommend to recognize this work as a Master's thesis. I suggest an overal grading of **B** (**2** in the Czech grading system).

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