Evaluation of Master Thesis by Supervisor

Michael Jančík: Coherent sheaves on singular curves of genus one

The goal of the thesis was to understand the structure of the category of coherent sheaves (or its bounded derived category) on projective singular curves of arithmetic genus one. The structure of coherent sheaves on smooth projective curves of genus one (= elliptic curves) was described by Atiyah already in the 1950's. In the presence of a singularity, however, the problem is much harder and existing results are less complete. Nevertheless, there is a natural stability structure on the bounded derived category and at least for nodal singularities, the categories of semistable objects are fairly completely understood. The thesis gives an account on these results, based on Burban, Kreußler, *Derived categories of irreducible projective curves of arithmetic genus one*, Compos. Math., 2006.

The topic turned out to be quite involved, since it uses a lot of algebraic geometry and homological algebra, well beyond what is taught in the Master program at Charles University. The author had the advantage of studying algebraic geometry much deeper during his Erasmus stay in Bonn, yet it was necessary for him to build bridges between the very abstract theory and concrete computations for curves such as (the projective closure of) the nodal singularity $y^2 = x^2(x+1)$. This, together with collecting all the necessary preliminaries, occupies most of the thesis in the end. Some of the main results from Burban's and Kreußler's paper are discussed in Section 4 (10-11 pages out of 62).

The thesis without doubt shows the author's ability to work with involved techniques in algebraic geometry. The main contribution consists of guiding the reader all the way to the results by Burban and Kreußler and completing the discussion with many details. In some places, however, the presentation could be improved:

1. The decision what concepts to define and which definitions to omit without explanation feels sometimes quite arbitrary. Usually, this concerns relatively well known concepts, but for example the notion of finitely controlled module (Theorem 136, p. 42) is not such and should have been put into context in some way.

2. Some explanations include logical inconsistencies or individual steps are not ordered correctly:

   (a) Theorem 86 (p. 25) says, that $H^i(X, \mathcal{F})$ is a finitely generated A-module under the conditions there. However, at the very end of §1.4 (p. 24) the text says that "regardless of the structures of X and $\mathcal{F}$, ... we always take cohomology ... regarding $\mathcal{F}$ simply as a sheaf of abelian groups ...." That is, there is a priori no A-module structure either on an injective resolution of $\mathcal{F}$ used to compute the cohomology or on the cohomology itself. A hint why there is such a structure in the situation in the theorem is only given later in Proposition 98 (p. 27).

(b) The proof of Lemma 101 (p. 28) obfuscates the main reason why we have an isomorphism of the Ext functors in the statement: the equivalence of the corresponding abelian categories $\operatorname{mod} A$ and $\operatorname{coh} X$ given in Corollary 47 (p. 16). An arbitrary resolution by locally free sheaves is in general not suitable for computing values of Ext functors in $\operatorname{coh} X$, unless of course X is affine as in that case locally free sheaves are projective objects of $\operatorname{coh} X$. However, this point is not mentioned in the proof either.

(c) Principal open sets were defined only for affine schemes (Definition 15), but are used in the context of not necessarily affine schemes in the proof of Lemma 105 (p. 29) and in Remark 111 (p. 31). This does not make sense, in both the cases one should have spoken of *affine* open covers or subsets.

(d) The proof of Theorem 113 (p. 32) is not easy to read. In particular, it starts with the sentence "Firstly, we prove the theorem in the affine case," but then it is unclear where the proof of the special case finishes and a proof of the general situation begins.

(e) The meaning of the term $\mathbf{R}\mathrm{Hom}(E, F) \otimes E$ in the displayed triangle on p. 44 is only explained in §4.2 a page later.

3. The explanation of the classification result in Section 3 is suboptimal. As mentioned above, there is quite specific terminology (like finitely controlled modules) which is neither explained nor at least informally put into the context. Moreover, indecomposable band modules could have been described much more concretely: They correspond to indecomposable finite dimensional $k[T, T^{-1}]$-modules, but $k[T, T^{-1}]$ is a Dedekind domain, so Theorem 161 in Appendix A.1.1 would perfectly apply. There is more than that if k is algebraically closed—maximal ideals are precisely those of the form $(x - \lambda)$, where λ runs over $k \setminus \{0\}$, so the description of indecomposable band modules is really hands on in that case.

4. It might be nice to mention what the set $\mathbf{Q}$ on p. 50 looks like, in the sense that all rational slopes do appear.

5. I would prefer to have the reference list ordered and typed at the very end of the file, after the appendices.

Conclusion

I recommend to recognize this work as a Master thesis. The suggested grading will be communicated directly to the examination (sub)committee.

In Murcia, September 4, 2024 doc. RNDr. Jan Šťovíček, Ph.D.