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Geometric properties of circular space-times

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Title: Geometric properties of circular space-times

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Abstract: We investigate the geometric properties of circular space-times within the framework of 3+1 and subsequent 2+1+1 decomposition of space-time. The problem that is discussed revolves around the identification of minimal 2-dim. submanifolds, within each space-time included, aimed at constructing an adapted coordinate system to the second decomposition, that leaves the resulting submanifold minimal. We employ a shooting method to solve the Neumann boundary value problem for the second-order non-linear ordinary differential equation $L = 0$. Cases of Minkowski limit, some parameter values of the Levi-Civita metric, and a Weyl metric are solved analytically, and the adapted coordinates system is determined. For other cases, numerical plots and 3D plots of the minimal submanifolds for various geometries are presented.

Keywords: general theory of relativity, circular space-times, 3+1 splitting of space-time

Název práce: Geometrické vlastnosti cirkulárních prostoročasů

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Abstrakt: V rámci formalismu 3+1 a následného 2+1+1 rozštěpení zkoumáme geometrické vlastnosti cirkulárních prostoročasů. Centrálním problémem je nalezení minimálních 2-dim. podvariet v každém vybraných prostoročasů, s cílem napsat přizpůsobené souřadnice druhé dekompozici, která ponechá povrch výsledné podvariety minimální. Využijeme metodu střílení k vyřešení Neumannovy okrajové úlohy pro obyčejnou nelineární diferenciální rovnici druhého řádu $L = 0$. V případě Minkowského limity, některých hodnot parametrů Levi-Civitovy metriky a jednu Weylovu metriku je získáno analytické řešení a jsou explicitně nalezeny přizpůsobené souřadnice. Pro ostatní případy prezentujeme grafy a 3D obrázky minimálních podvariet pro různé geometrie.

Klíčová slova: obecná teorie relativity, cirkulární prostoročasy, 3+1 rozštěpení prostoročasu

Contents

Conventions

Following is the list of conventions used in this thesis. It follows the convention template, as well as conventions themselves from [13], and unambiguously sets the convention-dependent signs in general relativity:

1. $\boldsymbol{g}$ sign:
$$+\boldsymbol{g} = -(\boldsymbol{\omega}^0)^2 + (\boldsymbol{\omega}^1)^2 + (\boldsymbol{\omega}^2)^2 + (\boldsymbol{\omega}^3)^2,$$
2. ***Riemann*** sign:
$$+\mathbb{R}(\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{\nabla}_{\boldsymbol{v}} - \boldsymbol{\nabla}_{\boldsymbol{v}}\boldsymbol{\nabla}_{\boldsymbol{u}} - \boldsymbol{\nabla}_{[\boldsymbol{u},\boldsymbol{v}]},$$
$$+R^{\mu}{}_{\nu\alpha\beta} = \partial_\alpha \Gamma^{\mu}{}_{\nu\beta} - \partial_\beta \Gamma^{\mu}{}_{\nu\alpha} + \Gamma^{\mu}{}_{\sigma\alpha}\Gamma^{\sigma}{}_{\nu\beta} - \Gamma^{\mu}{}_{\sigma\beta}\Gamma^{\sigma}{}_{\nu\alpha},$$
3. Quotient of ***Einstein*** and ***Ricci*** signs:
$$+R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu},$$
4. ***Einstein*** sign:
$$\boldsymbol{Einstein} = +8\pi \boldsymbol{T} - \Lambda \boldsymbol{g},$$
$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = +8\pi T_{\mu\nu} - \Lambda g_{\mu\nu}, \tag{1}$$
5. ”Positive energy density” sign:
$$T_{\hat{0}\hat{0}} \equiv \boldsymbol{T}(\boldsymbol{e}_{\hat{0}}, \boldsymbol{e}_{\hat{0}}) > 0.$$

Rough guide to notation

Symbol	Meaning
$\mathbb{T}_x\mathcal{M}/\mathbb{T}^*_x\mathcal{M}, \mathcal{T}_x\mathcal{M}, \mathcal{T}^*_x\mathcal{M}$	The tangent/cotangent space of a differentiable manifold $\mathcal{M}$ at the point $x \in \mathcal{M}$, resp. the corresponding *Sect*.
$\mathcal{F}\mathcal{M}$	The space of all C^∞ functions on the manifold $\mathcal{M}$.
A, B, C	The uppercase Latin indices take on values $1, 2/1, 3/2, 3$.
a, b, c	The lowercase Latin indices take on values $1, 2, 3$.
α, β, γ	The Greek letter indices take on values $0, 1, 2, 3$.
$\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$	The standard abstract indices.
$\nabla_\nu n^\mu, n^\mu_{;\nu}$	The components of the covariant differential of n^μ .
$A^{\mu\nu\ldots}{}_{[\kappa\lambda]\ldots}$	The antisymmetrization of the indices κ, λ.
$A^{\mu\nu\ldots}{}_{(\kappa\lambda)\ldots}$	The symmetrization of the indices κ, λ.
$\stackrel{*}{=}$	The quality only holds for a vacuum region.

1 Preliminaries

General Relativity (GR) still, after 109 years, remains the most successful gravitational theory ever known to mankind. One of its central ideas is that space-time is not a mere backdrop for celestial events, but a dynamic, curved entity influenced by mass and energy. This curvature of space-time introduces a bunch of non-trivial and fascinating phenomena. Intriguingly, even the simplest space-time configurations can exhibit surprising and complex geometric properties. In this thesis, we examine the geometric properties of a specific class of space-times, whose main representative is the renowned Kerr metric, that serves as the most important example of rotating black holes and their unique characteristics.

In the first chapter, we aim to establish the essential concepts and techniques that are going to be explored in depth in the following sections. Our main goal is to offer a comprehensive and pseudo-rigorous exposition of all foundational elements in one place, ensuring the thesis is largely self-sufficient. This approach not only solidifies the core understanding necessary for subsequent discussions, but also introduces a degree of generality to the problem (more below), enriching the analysis, and providing a broader context for the more-specialized topics addressed later. We begin with a thorough examination of the Frobenius' theorem, establishing the necessary conditions for the existence of space-time decompositions that are going to be analyzed later. This discussion closely follows the framework provided in [11] with additional insights from [4], despite the differing notation and depth of analysis found in the latter.

In the second section, we establish a significant covariant decomposition of a covariant differential of a hypersurface-orthogonal vector field. This decomposition, treated in Subsection 1.2, is going to be articulated in terms of various tensor fields that describe general congruences. Our approach here draws heavily from [1], but we extend the analysis to include an unspecified causal character of the normal field for the purpose of generality, unlike the time-like specification found in [1]. This more general treatment aligns with the brief discussion in the unpublished research [19] and the bachelor thesis [10].

By synthesizing the insights from the first two sections, we are going to be positioned to tackle the broader problem of space-time decomposition in GR. It is assumed that the reader is already familiar with the basics of differential geometry, which serve as the only prerequisites to this foundational chapter. Throughout the text, the usual summation rule is used. The geometrized units are always used as well, unless explicitly noted otherwise.

1.1 Submanifolds and Frobenius' theorem

Definition 1 (Locally injective mapping, immersion, embedding). *Let $\mathcal{M}$, $\tilde{\mathcal{N}}$ be two smooth differentiable manifolds of dimensions dim $\mathcal{M} = m$, and dim $\tilde{\mathcal{N}} = n$, where $n \leq m$ is commonly assumed. We say that a mapping*[1] *$\phi : \tilde{\mathcal{N}} \to \mathcal{M}$ is* ***locally injective*** *at the point $x \in \tilde{\mathcal{N}}$, if the push-forward $\phi_*\big|_{\tilde{x}}$ is injective.*

[1]Note that since the assumption $n \leq m$, ϕ is by definition not a diffeomorphism. We enforce the property of push-forward's injectivity (as it is always well defined) to ensure that directions of vector bases do not degenerate.

From this point on, we shall refer to a locally injective map at every point $\tilde{x} \in \tilde{\mathcal{N}}$ as ***immersion*** *and an injective map in the standard sense as* ***embedding*** *of manifold $\mathcal{N} = \phi\tilde{\mathcal{N}}$ into $\mathcal{M}$.*

Definition 2 (Submanifold)**.** *Given the notions established by the preceding definition,* ***submanifold*** *shall therefore be defined as the image of immersion/embedding, $\phi : \tilde{\mathcal{N}} \to \mathcal{M}$ ($\iff$ $\mathcal{N} = \phi\tilde{\mathcal{N}}$).*

Lemma 1 (Adapted coordinates)**.** *Let $\tilde{\mathcal{N}}$, $\mathcal{N}$, $\mathcal{M}$ and ϕ be defined as above. It is then possible to prove that there exists a subset $V \subset \mathcal{N}$ around a point $x \in \mathcal{N}$, where ϕ is injective. Therefore, there exists a* ***coordinate system*** *$[y^1, ..., y^n]$, covering the neighbourhood $U \subset \mathcal{M}$ of the point ϕx, that can be extended to a coordinate system $[x^1, ..., x^m]$, such for every $w \in U$ the following holds[11]:*

(i) $x^j(\phi w) = y^i(w)$ *for* $i \in \{1,...,n\}$,

(ii) $x^p(\phi w) \equiv const.$ *for* $p \in \{n{+}1,...,m\}$.

(iii) Local injectivity and surjectivity of ϕ is the sufficient condition for ϕ being a diffeomorphism on a neighbourhood of w.

Moreover, in the ideal non-degenerate case, the immersion/embedding of $\mathcal{N}$ into $\mathcal{M}$ as a hypersurface in $\mathcal{M}$ is through the coordinates $y^i, i \in \{1, ..., n\}$ and $x^p, p \in \{n+1, ..., m\}$ fully described by the relations $x^j, j \in \{1, ..., m\} = x^j(x^p, y^i)$. The interpretation is that the choice of $x^p = const.$ "chooses" the particular submanifold, whereas y^i are the intrinsic coordinates of the submanifold.

Now, we wish to define vectors and 1-forms from the (co)tangent bundle of $\mathcal{N}$ for further tensorial treatment of the problem. It is possible to start by identifying the tangent vectors of $\mathcal{N}$ with the image of ϕ_* because the push-forward of vectors $\phi_* \mathbb{T}_x \tilde{\mathcal{N}} \equiv \mathbb{T}_{\phi x} \mathcal{N} \subset \mathbb{T}_{\phi x} \mathcal{M}$ is always a well defined map, thanks to the Definitions 1 and 2. On the other hand, with 1-forms, the analogous mapping from $\tilde{\mathcal{N}}$ to $\mathcal{N}$ is not immediately clear. The previous thoughts are summarized and solved by the following definition:

Definition 3 (Tangent vectors and normal forms of a submanifold)**.** *Let $\mathcal{N}$ be a submanifold of $\mathcal{M}$. Employing the notation introduced earlier, denote the corresponding mapping by $\phi : \tilde{\mathcal{N}} \to \mathcal{M}$. Lets us define*

(i) ***Tangent vectors to $\mathcal{N}$*** *(objects from $\mathbb{T}_x\mathcal{N}$) as the image of $\phi_* \mathbb{T}_x \tilde{\mathcal{N}}$.*

(ii) ***Normal 1-forms*** *as objects from $\mathbb{N}^*_x\mathcal{N} \subset \mathbb{T}^*_x\mathcal{M}$ defined via the relation*

$$\omega \in \mathbb{N}^*_x\mathcal{N} \iff \omega(\phi_* a_1, ..., \phi_* a_n) = 0 \; \forall a_k \in \mathbb{T}_x\mathcal{N}.$$

Note that the previous definition gives the duality between the spaces $\mathbb{T}_x\mathcal{N}$ and $\mathbb{N}^*_x\mathcal{N}$ in the following sense

$$\omega \in \mathbb{N}^*_x\mathcal{N} \iff \forall a \in \mathbb{T}_x\mathcal{N} : \omega \cdot a = 0,$$

$$a \in \mathbb{T}_x\mathcal{N} \iff \forall \omega \in \mathbb{N}^*_x\mathcal{N} : \omega \cdot a = 0.$$

Definition 4 (Distribution Δ of subspaces of tangent vectors of dimension n)**.** *A collection of n-dimensional subspaces* $\Delta_x \subset \mathbb{T}_x\mathcal{M}$ *of a differentiable manifold* $\mathcal{M}$ *of dimension m is called a* ***distribution*** *at x, if it is generated by smooth (in the sense of* $\mathcal{FM}$*) vector fields* $a_j \in \mathcal{TM}$ *for* $j \in \{1, ..., n\}$*. Therefore, the bulk distribution* Δ *is* $\forall x \in \mathcal{M}$ *generated by* $a_j(x)$*, and we say that* $a \in \mathcal{TM}$ *is an element of* $\Delta \iff \forall x : a(x) \in \Delta_x$*.*

Definition 5 (Codistribution Δ^* of subspaces of 1-forms of codimension n)**.** *A collection of (m-n)-dimensional subspaces* $\Delta_x^* \subset \mathbb{T}_x^*\mathcal{M}$ *of a differentiable manifold* $\mathcal{M}$ *of dimension m is called a* ***codistribution*** *at x if it is generated by smooth (in the sense of* $\mathcal{FM}$*) 1-form fields* $\alpha^p \in \mathcal{T}^*\mathcal{M}$ *for* $p \in \{n+1, ..., m\}$*. Therefore, the bulk codistribution* Δ^* *is* $\forall x \in \mathcal{M}$ *generated by* $\alpha^p(x)$*, and we say that* $\alpha \in \mathcal{T}^*\mathcal{M}$ *is an element of* $\Delta^* \iff \forall x : \alpha(x) \in \Delta_x^*$*.*

Definition 6 (Complementary distributions)**.** *(Co)distributions* Δ, Δ^* *are called* ***complementary distributions****, if the following holds:*

(i) *dimension of* Δ *is n and codimension of* Δ^* *is n,*

(ii) $\forall a \in \Delta_x \; \forall \alpha \in \Delta_x^* : \alpha \cdot a = 0$.

The definition of complementary distributions allows us to classify two complementary (co)distributions just by classifying the (co)distribution because, analogously as before, the complementary (co)distribution is given unambiguously by

$$a \in \Delta \iff \forall \alpha \in \Delta^* : \alpha \cdot a = 0,$$
$$\alpha \in \Delta^* \iff \forall a \in \Delta : \alpha \cdot a = 0,$$

that allows us to employ a shorthand notation: $\alpha \in \Delta^* \iff \alpha|_\Delta = 0$. Let us define one more notion regarding distributions. From this point on, we are assuming that the distributions of interest, denoted by Δ, Δ^*, are complementary in the sense defined by the previous definition.

Definition 7 (Involutive distribution)**.** *Let* Δ *be a distribution of dimension n. We say the distribution* Δ *is* ***involutive****, if it is generated by smooth vector fields* a_i*, where* $i \in \{1, ..., n\}$*, and the Lie bracket* $[a_j, a_k]$ *can be written as a "linear" combination:* $\sum_{l=1}^{n} f_{jk}^l a_l$ *for arbitrary* $f_{jk}^l \in \mathcal{FM}$ *,* $\forall j, k \in \{1, ..., n\}$*.*

And the induced complementary notion by the complementarity:

Definition 8 (Differential codistribution)**.** *Let* Δ^* *be a codistribution of codimension n. We say the codistribution is* ***differential****, if it is generated by smooth 1-form fields* α^p*, where* $p \in \{n+1, ..., m\}$*, and the exterior derivative* $\mathbf{d}\alpha^p$ *can be written as a "linear" combination* $\sum_{q=n+1}^{dim\Delta^*+n} \Theta_q^p \wedge \alpha^q$*, for an arbitrary 1-form* Θ_q^p*,* $\forall p \in \{n+1, ..., dim\Delta^* + n\}$[2].

[2] The wedge product is obviously convention-dependent. The convention is here taken from [11], so for K forms of degrees $p_1, ..., p_K$, where $\sum_{i=1}^{K} p_i$ is abbreviated to p, the wedge product reads

$$(\alpha^1 \wedge \alpha^2 \wedge ... \wedge \alpha^K)_{\boldsymbol{a_1 ... a_p}} := \frac{p!}{p_1!\, p_2! \, ... \, p_K!} \alpha^1_{[\boldsymbol{a_1 ... a_{p_1}}} \alpha^2_{\boldsymbol{a_{p_1+1} ... a_{p_2}}} ... \alpha^K_{\boldsymbol{a_{p-p_K} ... a_p}]}.$$

Theorem 2. *Let $\mathcal{M}$ be a differentiable manifold of dimension m, Δ distribution of dimension n and Δ^* the complementary codistribution. Then Δ is involutory if and only if Δ^* is differential[11].*

Proof. Δ^* being differential means that $\forall p \in \{n+1, ...m\} : \mathrm{d}\alpha^p\Big|_\Delta = 0$; therefore: one has $\forall i, j \in \{1, ..., n\}$, $\forall p \in \{n+1, ..., m\} : 0 = a_i \cdot \mathrm{d}\alpha^p \cdot a_j$, where a_i, a_j denote the generators of the complementary distribution. By using the known Cartan's formula (for the formulation and proof, see [11]), one gets the following: $0 = a_i \cdot \mathrm{d}(a_j \cdot \alpha^p) - a_j \cdot \mathrm{d}(a_i \cdot \alpha^p) - [a_i, a_j] \cdot \alpha^p$, but the first two terms are identically zero because of the presumption that the (co)distributions are complementary. Therefore, we get the equivalence with $\forall i, j \in \{1, ..., n\} : [a_i, a_j]$ being an element of Δ, which is equivalent to the definition of involutive distribution. □

With the aid of this theorem, we finally arrive at the question of integrability, which is our main matter of interest. Another two definitions are necessary to formulate the main theorem of this section

Definition 9 (Integral submanifold)**.** *A manifold $\mathcal{N}$ is an **integral submanifold** of the corresponding complementary co(distributions) Δ, Δ^* on $\mathcal{N}$, if:*

$$\forall x \in \mathcal{N} : \begin{cases} a \in \Delta_x \iff a \in \mathbb{T}_x\mathcal{N}, \\ \alpha \in \Delta^*_x \iff \alpha \in \mathbb{N}^*_x\mathcal{N}. \end{cases}$$

If a differentiable manifold $\mathcal{N}$ is an integral submanifold, the distinction between Δ_x and $\mathbb{T}_x\mathcal{N}$, respectively Δ^*_x and $\mathbb{N}^*_x\mathcal{N}$, is no longer needed, and it is possible to identificate them: $\Delta_x \equiv \mathbb{T}_x\mathcal{N}$, respectively $\Delta^*_x \equiv \mathbb{N}^*_x\mathcal{N}$.

And the "inverse" definition:

Definition 10 (Integrable distribution)**.** *The complementary (co)distributions Δ, Δ^* on differentiable manifold $\mathcal{M}$ are called **integrable** on U, if every point $x \in U \subset \mathcal{M}$ is crossed by an integral submanifold.*

Claim 3. *Let Δ, Δ^* be (co)distributions on a differentiable manifold $\mathcal{M}$ $\dim \mathcal{M} = m$ of (co)dimensions n, let $[x^k]$ denote a coordinate chart defined on a neighbourhood of point $x \in \mathcal{M}$ denoted by $U \subset \mathcal{M}$. Then[11]*

$$\Delta, \Delta^* \text{ are integrable} \iff \exists [x^k]_{k=1}^m : \begin{cases} \boldsymbol{\partial}_i, i \in \{1, ..., n\} \\ \mathbf{d}x^p, p \in \{n+1, ..., m\}. \end{cases}$$

And finally, we arrive at the Frobenius' theorem, which we shall not prove in this form, for the proof is quite extensive and can be found in the non-public lecture notes of [11], apart from differential geometry textbooks:

Theorem 4 (Frobenius)**.** *Let Δ, Δ^* be mutually complementary (co)distributions on a differentiable manifold. Then:*

$$\Delta, \Delta^* \textit{ are integrable} \iff \Delta \textit{ is involutive} \iff \Delta^* \textit{ is differential.}$$

Proof. The rightmost equivalence has already been shown in this text, so just two implications remain. The implication " $\Longrightarrow$ " can be quite easily shown from definition, but the "$\Longleftarrow$" implication is much more complicated and one has to recourse to induction. □

Remark. In the ideal case, every point of the manifold $\mathcal{M}$ is then passed by exactly one integral submanifold. Then we say that the manifold is *foliated* by integral submanifolds.

1.2 Integral congruences of vector fields

Now, we shall apply the general knowledge gained from the previous subchapter to the problem of hypersurface-orthogonality in GR, and acquire the sufficient condition when in general a $(m-1)$-dimensional foliation of unspecified causal character submanifolds exists. Let $\mathcal{M}$ denote the (space-time) manifold, of dimension $m \leq 4$, equipped with a non-degenerate[3] tensor field $\boldsymbol{g} \in \mathcal{T}^0_{(2)}\mathcal{M}$ with Lorentzian signature, that defines the isometry between $\mathbb{T}\mathcal{M}$ and $\mathbb{T}^*\mathcal{M}$. Therefore, the normal vectors of a submanifold can be defined through the isometry, and the problem can be formulated in the "dual" form:

Definition 11 (Hypersurface-orthogonal vector field)**.** *Given the above established notions, let* $\boldsymbol{n}$ *denote a smooth vector field defined in* $\mathbb{T}\mathcal{M}$*, that for every* $x \in \mathcal{M}$*, defines a (m-1)-dimensional hyperspace in the corresponding* $\mathbb{T}_x\mathcal{M}$*, that has* $\boldsymbol{n}$ *as its normal vector field. A vector field, for which such (m-1)-dimensional foliation exists, is from now on being referred to as* ***hypersurface-orthogonal vector field.***

Before we dive deeply into the sufficient condition mentioned above, it is necessary to establish in the preceding paragraphs mentioned covariant decomposition of the covariant differential of a hypersurface-orthogonal vector field. For the purpose of which, we shall investigate the kinematics of space-time congruences with general space-time character. Although, excluding the degenerated light-like case, that clearly has to be treated separately.

So, let us begin with the "4→ 3" case. Let a general one-parameter congruence of integral curves in $U \subset \mathcal{M}$ be parameterized, after the choice of a coordinate chart on U $[x^\mu]^4_{\mu=0}$[4] by two real parameters $l, \tau \in \mathbb{R}$ as $x^\mu = x^\mu(l;\tau)$, where the map $(l,\tau) \mapsto x^\mu(l;\tau)$ is assumed to be a diffeomorphism, by virtue of Lemma 1, so that every point of U is passed through by exactly one integral curve. It is assumed that the parameter l continously numbers the members of the congruence, and τ is the standard parameter of the curves for a given l. Let us introduce the components of the tangent vector field of the congruence, given the nature of parameters (l,τ), by:

$$u^\mu := \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \text{ with the assumption } u_\mu u^\mu := \varepsilon \neq 0 \; \forall x \in U. \tag{1.1}$$

Analogously, we introduce the components of the relative position vector of the curves:

$$\delta x^\mu := \frac{\mathrm{d}x^\mu}{\mathrm{d}l} \text{ with the assumption } u_\mu \delta x^\mu = 0 \; \forall x \in U. \tag{1.2}$$

Now, from the assumption that the mapping $(l,\tau) \mapsto x^\mu(l;\tau)$ is a diffeomorphism, it follows, that total derivatives of the mapping by the two parameters commute.

[3]Non-degeneracy in the sense that $\exists \boldsymbol{g}^{-1} \in \mathcal{T}^{(2)}_0\mathcal{M} : g_{\mu\sigma}(g^{-1})^{\sigma\nu} = \delta^\nu_\mu$.
[4]Here assumed to be covering the whole region $U \subset \mathcal{M}$ for brevity

Therefore, one can immediately deduce that the components of the Lie derivative of one vector field with respect to the other one vanishes, as shown below

$$(\mathcal{L}_{\boldsymbol{u}}\delta x)^\mu \equiv [u, \delta x]^\mu := (\delta x^\mu)_{,\nu} u^\nu - (u^\mu)_{,\nu}\delta x^\nu = \frac{\mathrm{d}(\delta x^\mu)}{\mathrm{d}\tau} - \frac{\mathrm{d}u^\mu}{\mathrm{d}l}$$
$$\implies (\mathcal{L}_{\boldsymbol{u}}\delta x)^\mu \equiv (\mathcal{L}_{\boldsymbol{\delta x}})^\mu \equiv 0.$$

Moreoever, if the corresponding affine connection on fiber-bundles of $\mathcal{M}$ is of the Levi-Civita type[5], then:

$$\nabla_{\delta x} u := u^\mu_{;\nu}(\delta x)^\nu \equiv \frac{\mathrm{d}u^\mu}{\mathrm{d}l} + \Gamma^\mu{}_{\kappa\lambda}(\delta x)^\kappa u^\lambda = \frac{\mathrm{d}(\delta x)^\mu}{\mathrm{d}\tau} + \Gamma^\mu{}_{\kappa\lambda} u^\kappa(\delta x)^\lambda =: \nabla_u(\delta x)^\mu.$$

This is the reason why the decomposition of the covariant differential of the vector field u^μ is so crucial, as it specifies the evolution of transversal properties of the congruence.

Now, we lay out the ground work for the decomposition:

Definition 12 (Projector to the orthogonal space of a hypersurface-orthogonal vector field)**.** *Let* $\boldsymbol{n}$ *be a hypersurface-orthogonal vector field in the sense of Definition 11. We then define* ***the projector to the subspace orthogonal to the vector field*** $\boldsymbol{n}$ *as*

$$h^\mu_\nu := \delta^\mu_\nu - \varepsilon\, n^\mu n_\nu,$$

which is indeed a projector. The orthogonality is trivial. The idempotency of the projector can also be explicitly checked: directly from the definition we can show that

$$h^\mu_\sigma h^\sigma_\nu = (\delta^\mu_\sigma - \varepsilon n^\mu n_\sigma)(\delta^\sigma_\nu - \varepsilon n^\sigma n_\nu) \equiv h^\mu_\nu.$$

One naturally deduces that the tensor h^μ_ν *at any point* $x \in \mathcal{M}$ *indeed projects on the three dimensional subspace spanned by the vectors orthogonal to the local* $\boldsymbol{n}$*.*

Using $\boldsymbol{g}$*, which defines the isometry between* $\mathcal{TM}$ *and* $\mathcal{T}^*\mathcal{M}$*, one can further deduce that*

$$h_{\mu\nu} \equiv g_{\mu\nu} - \varepsilon n_\mu n_\nu,$$

where $g_{\mu\nu}$ *denotes the components of the metric* $\boldsymbol{g}$ *of* $\mathcal{M}$ *in the coordinate basis of the above chosen coordinates* $[x^\mu]^4_{\mu=0}$ *as given in abstract notation by* $g_{\mu\nu} := \boldsymbol{g}(\boldsymbol{\partial}_\mu\, \boldsymbol{\partial}_\nu)$*, or, equivalently, the tensor* $\boldsymbol{g}$ *itself.* ε *here stands for the analogous norm of* n^μ *as in (1.1). It is evident that* $h_{\mu\nu}$ *plays the role of the metric tensor components of the orthogonal subspace.*

Let us also define the corresponding acceleration (referred to as acceleration regardless of its causal character) of the vector field $\boldsymbol{n}$ in contravariant form as given in components by $a_\mu := n_{\mu;\nu}n^\nu$, and project it by h^μ_ν in its both indices (exploiting the fact that n^μ is normalized, henceforth $(n_\mu n^\mu)_{;\nu} = 0$ at any given point, see (1.19))

$$n_{\kappa;\lambda} h^\kappa_\mu h^\lambda_\nu = (\delta^\kappa_\mu - \varepsilon\, n^\kappa_\mu)(\delta^\lambda_\nu - \varepsilon\, n^\lambda_\nu) = n_{\mu;\nu} - \varepsilon\, a_\mu n_\nu.$$

[5] i.e. the corresponding covariant derivative annihilates the associated metric $\boldsymbol{g}$, and it is torsion-free ($\mathrm{Tor}[\boldsymbol{\nabla}] = 0$ in pre-fix notation.)

We hereby invoke the decomposition of the covariant differential $n_{\mu;\nu}$ on the right-hand side of the equation as being of the following form:

$$\boxed{n_{\mu;\nu} := \omega_{\mu\nu} + \sigma_{\mu\nu} + \frac{1}{d}\Theta h_{\mu\nu} + \varepsilon a_\mu n_\nu,} \tag{1.3}$$

where d in general denotes the dimension of the orthogonal subspace as the image of the mapping represented by the tensor h^μ_ν. It now follows that the process can be repeated once more, by the choice of another normal vector field, producing 2-dimensional submanifolds, *et cetera*, see Section 1.4.

Definition 13 (Vorticity, expansion and shear tensors)**.** *By invoking this decomposition, we have therefore defined the following tensors:*

(i) ***the vorticity tensor****:* $\omega_{\mu\nu} := n_{[\kappa;\lambda]}h^\kappa_\mu h^\lambda_\nu = n_{[\mu;\nu]} - \varepsilon\, a_{[\mu}n_{\nu]}$,

(ii) ***the expansion tensor****:* $\Theta_{\mu\nu} := n_{(\kappa;\lambda)}h^\kappa_\mu h^\lambda_\nu = n_{(\mu;\nu)} - \varepsilon\, a_{(\mu}n_{\nu)} \equiv \frac{1}{2}\mathcal{L}_{\boldsymbol{n}} h_{\mu\nu}$,

(iii) ***the expansion scalar*** *as the metric trace of* $\Theta_{\mu\nu}$*:* $\Theta := h^{\mu\nu}\Theta_{\mu\nu} \equiv n^\mu{}_{;\mu}$,

(iv) ***the shear tensor*** *as expansion tensor's traceless part:* $\sigma_{\mu\nu} := \Theta_{\mu\nu} - \frac{1}{d}\Theta h_{\mu\nu}$.

Remark. Note that the decomposition (1.3) is covariant. The individual terms are of tensorial character, hence describe the transversal properties of a congruence regardless of the choice of a coordinate chart. Every single tensor, that was defined, is orthogonal to the defining normal vector field $n^\mu\ \forall x \in U$ because of the definition via the projector h^μ_ν, and the normalisation of the normal (1.19). This induces the following identities:

$$\varepsilon\, a_\mu n_\nu n^\mu = n^\mu\omega_{\mu\nu} = n^\mu\sigma_{\mu\nu} = n^\mu h_{\mu\nu} \equiv 0\,(\iff n^\mu A^{\cdots\cdots}_{\cdots\mu\cdots} \equiv 0\,\forall A \in \mathcal{T}^k_l\Sigma_t). \tag{1.4}$$

The interpretation of the result may proceed through the analogy with mechanics of continuum in 3 dimensions. For example the antisymmetric vorticity tensor is known from the theory of continuum as the angular velocity vector $\vec{\omega} = \frac{1}{2}\vec{\nabla} \times \vec{v}$. Because in that case it is given by only 3 independent components, and thus is fully specified by a corresponding pseudovector via the Hodge star operator.

One can subsequently interpret the other tensorial quantities as [1]:

(i) Θ describes the isotropic expansion of a 3-dimensional element of the flow described by the congruence, and is the only tensor that changes volume of the flow element, i.e., for the case $\Theta < 0$ one gets the congruence converging, and vice versa.

(ii) $\omega_{\mu\nu}$ describes how (in the time-like normal case) world-lines of the congruence entwine through each other like individual fibres of a rope.

(iii) $\sigma_{\mu\nu}$ describes the shear deformation (for example imagine stepping on a ball, thus turning it into an ellipsoid).

It is evident that the three deformations, that the newly-defined tensors describe, correspond to what one might call *"Euler angles of deformation"*.

The previous considerations guide us to a bunch of much more practical formulations of the hypersurface-orthogonality notion, than the Frobenius theorem. The following holds:

Theorem 5. *When working on a topologically trivial region of a differentiable manifold of dimension 4, the following statements are equivalent:*

(i) An arbitrary smooth vector field $\boldsymbol{n}$ *is hypersurface-orthogonal in the sense of Definition 11.*

(ii) There exist smooth functions of a chosen coordinate chart $\Phi(x^\mu)$, $f(x^\mu)$ *(*f *is just a proportional factor), such that the corresponding components of the 1-form dual to* n^μ*, obtained via the metric tensor of the metric* $\boldsymbol{g}$*, are given by:*

$$n_\mu = \varepsilon f \tfrac{\partial \Phi}{\partial x^\mu}.$$

(iii) Given the previously obtained 1-form n_μ*, the following identity holds:*

$$n_{[\mu;\nu}n_{\kappa]} = 0.$$

(iv) The corresponding vorticity tensor, defined by n^μ*, constructed in the sense of Definition 13 and the preceding decomposition (1.3), is identically zero.*

Proof. $(i) \iff (ii)$: Holds due to the original Frobenius' theorem. The codimension is equal to 1 here, so the exterior derivative of n_μ, which is a 2-form, can in a topologically trivial region always be written in the form mentioned in Definition 8. Due to the sum being actually trivial, it is sufficient to choose $\Theta_\mu \equiv \frac{f_{,\mu}}{f}$, and the equality holds, because $\Phi(x^\mu) \in \mathcal{FM} \implies \Phi_{,\nu\mu} = \Phi_{,\mu\nu}$. Therefore the original Frobenius' theorem gives us the equivalence.

$(ii) \implies (iii)$: Given the first equivalence, it is sufficient to submit the explicit form of the normal from (ii) into the relation, and one can easily check that it indeed vanishes, because of the assumed smoothness and implied commutativity of partial derivatives of Φ, like in the case of (ii):

$$n_{[\mu;\nu}n_{\kappa]} = n_{[\mu,\nu}n_{\kappa]} = \frac{1}{6}(n_{\mu,\nu}n_\kappa + n_{\kappa,\mu}n_\nu + n_{\nu,\kappa}n_\mu - \{\mu \leftrightarrow \nu\}) = 0.$$

$(iii) \implies (iv)$: Considering (1.1), the norm of n^μ was already denoted by ε, and recall that it is assumed not to be equal to zero.[6]On the other hand, multiplying the equation from the previous point by n^κ yields the vorticity tensor:

$$\begin{aligned} 0 =& \frac{1}{3}[(n_{\mu;\nu} - n_{\nu;\mu})n_\kappa n^\kappa + (n_{\kappa;\mu} - n_{\mu;\kappa})n_\nu n^\kappa + (n_{\nu;\kappa} - n_{\kappa;\nu})n_\mu n^\kappa] = \\ =& \varepsilon(n_{\mu;\nu} - n_{\nu;\mu}) + a_\nu n_\mu - a_\nu n_\mu = -2\omega_{\mu\nu}. \end{aligned}$$

[6]As mentioned before, we consider the ε normalisation factor to not be equal to zero: that would represent the null congruence case that needs to be treated separately. Rather, we shall be interested primarily in the case $\varepsilon = -1$, or $\varepsilon = 1$, see below.

$(iv) \implies [(i) \iff (ii)]$: One can get this implication from the well-known theorem $\mathbf{d} = \boldsymbol{\nabla}_\wedge$.[7] Here, without the torsion term though, because we wish to prove our theorem only for applications in GR and coordinate bases. When one writes out what vanishing vorticity tensor means in components, one can immediately convert the covariant derivative to the exterior derivative, yielding:

$$n_{[\mu;\nu]} - \varepsilon a_{[\mu} n_{\nu]} \equiv 0 \implies \mathrm{d}_\mu n_\nu = \varepsilon(a_\nu n_\mu - a_\mu n_\nu) \underset{(8)}{\overset{!}{=}} \Theta_\mu \wedge n_\nu.$$

Thus, the corresponding 1-form Θ_μ, from Definition 8, is here equal to $-\varepsilon a_\mu$ and again, because of the Frobenius' theorem, that concludes the proof. □

The original Frobenius' theorem was posing the question of integrability of submanifolds tangent to distributions, generated by smooth vector fields, as well as the following complementary question. Is it possible to foliate a differentiable manifold by complementary-dimensional integral submanifolds, which are at every point orthogonal, to the complementary distribution's generating vector fields? The Frobenius' theorem gave us an equivalence relation between these two statements. Moreover, the corollary Theorem 5 tells us that both these properties are guaranteed by the condition $\omega_{\mu\nu}$, defined as a tensor field (i.e. everywhere corresponding to the normal n^μ), being identically equal to zero for the whole region of interest. Furthermore, according to the previous theorem, defining the normal covector by $n_\mu := \epsilon f \Phi_{,\mu}$ in the sense of point (ii) guarantees zero-ness of the vorticity tensor and thus implies the rest of the aforementioned properties. In summary, for non-degenerate cases, the choice of the normal, as the previously-mentioned form, guarantees surface-orthogonality as well as integrability.

1.3 General 3+1 decomposition of space-time

1.3.1 Introduction and extrinsic curvature

The most important result of the last subchapter, addressed in the last paragraph, shall be addressed once more, although with the normal's causal character still undetermined.

Let $\boldsymbol{\nabla}$ denote the covariant derivative corresponding to the Levi-Civita connection (in components denoted by a semicolon). Let the congruence from the last subchapter be defined globally, and the respective foliation by smooth hypersurfaces be denoted by $(\Sigma_t)_{t\in\mathbb{R}}$, where t is an unspecified real parameter transitioning in the case of time-like normal (and therefore space-like integral submanifolds) to time[8]. In our case, a globally well-defined function (in the context of Frobenius' theorem and consequent theorems denoted by Φ), which monotonously increases along any time-like world line. In Definition 12, it has been established already that the metric of Σ_t has the following form:

$$h_{\mu\nu} = g_{\mu\nu} - \varepsilon\, n_\mu n_\nu. \tag{1.5}$$

[7] For the proof, see again [11].

[8] Not necessarily the time of any physical observer, although in the case of the FLRW-type space-times it even represents proper time of the cosmic fluid everywhere, as noted in [1].

Because we assume the hypersurfaces, that are everywhere orthogonal to the field n^μ, to be smooth, the latter is always normalize-able. That allows us to rethink the idea of the before-chosen $\varepsilon := g_{\mu\nu}n^\mu n^\nu$ to be only taking on values ± 1 (depending on the causal character of the normal) and to normalize the normal with an arbitrary function N instead (i.e. $\varepsilon \mapsto \mathrm{sign}(\varepsilon)N$).

A subtle observation: in the special case of $\omega_{\mu\nu} \equiv 0$, which we are interested in, there is little sense in the symmetrization in the definition of the expansion tensor $\Theta_{\mu\nu}$, because together with the previously defined "acceleration" a^μ, it fully specifies the decomposition of the covariant differential of the normal (1.3), see Definition 13. In a circumstance like this, the expansion tensor is referred to as the **extrinsic curvature tensor** (see also Section 3.1)[1]. Thus, from Definition 13, we have $K_{\mu\nu} := n_{\mu;\nu} - \varepsilon\, a_\mu n_\nu$. It is also commonly referred to as the *second fundamental form*. Together with the metric tensor, the *first fundamental form*, it fully specifies *(i)* the standard intrinsic curvature of the (sub)manifold, which is obtainable by exploiting (for example) the Ricci identities, *(ii)* the exterior curvature, that specifies the properties of the corresponding immersion (see Definition 1). Meaning that the symmetric (by definition, in the case $\omega_{\mu\nu} \equiv 0$) tensor field $\boldsymbol{K} \in \mathcal{T}^0_{(2)}\Sigma_t$ describes how the submanifolds Σ_t of $\mathcal{M}$ are curved as submanifolds immersed/embedded in $\mathcal{M}$[9]. One can get an intuition behind why this statement is true by computing the projection of decomposition (1.3) on Σ_t. The Lie derivative equality is especially intuitive, and easy to prove, the components really correspond 1:1

$$n_{\mu;\nu}h^\mu_\kappa h^\nu_\lambda \overset{(1.3)}{=} \varepsilon\, a_\mu n_\nu h^\mu_\kappa h^\nu_\lambda + K_{\mu\nu}h^\mu_\kappa h^\nu_\lambda \overset{(1.4)}{=} K_{\mu\nu}h^\mu_\kappa h^\nu_\lambda \overset{(12)}{=} K_{\kappa\lambda} \overset{(13)}{=} \frac{1}{2}\mathcal{L}_{\boldsymbol{n}} h_{\mu\nu}.$$

$K_{\mu\nu}$ quantifies the projection of the covariant differential of the normal on $\mathcal{T}^*\Sigma_t$. Let us summarize the results with the main definition of this section:

Definition 14 (Extrinsic curvature)**.** ***Extrinsic curvature*** *of a submanifold $\mathcal{N}$, immersed or embedded in $\mathcal{M}$, is a symmetric tensor field $\boldsymbol{K} \in \mathcal{T}^0_{(2)}\mathcal{N}$ defined via the corresponding projector in the sense of Definition 12 as:*

$$K_{\mu\nu} := n_{\kappa;\lambda}h^\kappa_\mu h^\lambda_\nu \equiv n_{\mu;\nu} - \varepsilon a_\mu n_\nu.$$

Hence, is by definition symmetric $K_{\mu\nu} \overset{(13)}{=} K_{\nu\mu}$, and specifies the projection of the covariant differential of the defining hypersurface-orthogonal field on $\mathcal{T}^*\mathcal{N}$.

Remark. In Definition 13, it has already been established, that in general the expansion tensor's metric trace, called the expansion scalar, is given by the four-covariant divergence of the normal field. Therefore, in the case $\omega_{\mu\nu} \equiv 0$, it holds:

$$K := g^{\mu\nu}K_{\mu\nu} \overset{(1.4)}{=} h^{\mu\nu}K_{\mu\nu} \overset{(13)}{\equiv} n^\mu_{;\mu}. \tag{1.6}$$

Its geometrical interpretation was already established after the aforementioned definition as $\mathrm{sign}(K) \lessgtr 0 \overset{\Longrightarrow \text{contraction}}{\Longrightarrow \text{expansion}}$ of the normal field (therefore, the covariant-divergence makes sense here).

[9]The most notorious example of this is the cylinder, where parallel lines on the underlying manifold do not expand, yet, one would not call an embedded cylinder in $\mathbb{E}^3$ flat.

Remark. Mixed components of the extrinsic curvature tensor represent what is sometimes being called the *shape operator* [1], [19], from which the usual geometrical curvature-describing scalars can be constructed. Let d denote the dimension of the differentiable manifold of interest. Then, in the non-degenerate case, the eigenvalues of the shape operator are what is called the *principal curvatures* κ_i, $i \in \{1, ..., d\}$. Afterwards, the *mean curvature* H can be obtained as $\frac{K}{d}$ which, because, the trace in general is equivalent to the sum of eigenvalues, is the same as $\frac{1}{d}\sum_{i=1}^{d}\kappa_i$. The well known *Gauss-Kronecker curvature* κ corresponds to the determinant of the shape operator. Using Lemma 1, one can find adapted coordinates of the submanifold(s), so that the whole system of submanifolds has the same adapted coordinates. This diagonalizes the shape operator, yielding $\kappa = \prod_{i=1}^{d}\kappa_i$.

1.3.2 Decomposition of the field equations

Next effort shall go into finding the relations between (tensorial) objects, defined on $\mathcal{M}$, and their respective analogues, defined on Σ_t. The general[10] 3+1 decomposition of the Einstein field equations. One has to begin somewhere, and the logical first step, to define the tensorial quantities, is finding the components of the induced covariant derivative corresponding to the connection of the Levi-Civita type compatible with the metric $\boldsymbol{h}$ of Σ_t. The following Lemma holds:

Lemma 6 (Covariant differential on Σ_t)**.** *The operation defined as:*[11]*.*

$$^{(3)}\nabla_\kappa T^{\mu\dots}_{\nu\dots} \equiv T^{\mu\dots}_{\nu\dots|\kappa} := T^{\alpha\dots}_{\beta\dots;\lambda} h^\mu_\alpha h^\beta_\nu h^\lambda_\kappa \dots,$$

where the semicolon part denotes the components of the standard covariant differential compatible with $\boldsymbol{g}$*, represent the components of the covariant differential corresponding to the connection of the Levi-Civita type compatible with the metric* $\boldsymbol{h}$*. In other words fulfilling the following properties for arbitrary* $r \in \mathbb{R}$*,* $f \in \mathcal{F}\Sigma_t$*, and* $A, B \in \mathcal{T}^k_l\Sigma_t$*:*

(i) $^{(3)}\boldsymbol{\nabla} : \mathcal{T}^k_l\Sigma_t \to \mathcal{T}^k_{l+1}\Sigma_t$,

(ii) $(A^{\mu\dots}_{\nu\dots} + rB^{\mu\dots}_{\nu\dots})_{|\kappa} = A^{\mu\dots}_{\nu\dots|\kappa} + rB^{\mu\dots}_{\nu\dots|\kappa}$,

(iii) $(A^{\mu\dots}_{\nu\dots}B^{\kappa\dots}_{\lambda\dots})_{|\rho} = A^{\mu\dots}_{\nu\dots|\rho}B^{\kappa\dots}_{\lambda\dots} + A^{\mu\dots}_{\nu\dots}B^{\kappa\dots}_{\lambda\dots|\rho}$,

(iv) $A^{\dots\kappa\dots}_{\dots\kappa\dots|\rho} = \delta^\mu_\nu A^{\dots\mu\dots}_{\dots\nu\dots|\rho}$,

(v) $f_{|\kappa} = \mathbf{d}_\kappa f$,

(vi) $h_{\mu\nu|\rho} = 0$,

(vii) $f_{|\mu\nu} - f_{|\nu\mu} \equiv -{}^{(3)}T^\rho_{\mu\nu} f_{|\rho} = 0$,
where $^{(3)}T^\rho_{\mu\nu}$ *stands for the components of the induced torsion tensor* $^{(3)}\boldsymbol{T}$*.*

[10] General meaning still performed with a general ε, therefore with general causal character of the normal and of the hypersurfaces, meaning that "t", in Σ_t, is still general, and does not yet necessarily correspond to any physical quantity, such as the suggestive *time*.

[11] The covariant derivative on Σ_{ts} itself shall be, if needed, denoted by $^{(2)}\boldsymbol{\nabla}$.

Proof. The projection modifies nothing we already had with the standard $\boldsymbol{\nabla}$, concerning the items $(i)-(v)$, so let us prove (vi) first. Exploiting the Levi-Civita property of $\boldsymbol{\nabla}$, we have:

$$h_{\mu\nu|\rho} := h_{\kappa\lambda;\sigma} h^\kappa_\mu h^\lambda_\nu h^\sigma_\rho \overset{(1.5)}{=} (n_{\kappa;\sigma} n_\lambda + n_\kappa n_{\lambda;\sigma}) h^\kappa_\mu h^\lambda_\nu h^\sigma_\rho \overset{(1.4)}{=} 0.$$

Hence, only the torsion-free property remains:

$$\begin{aligned} f_{|\mu\nu} :=& (f_{;\kappa} h^\kappa_\rho)_{;\lambda} h^\rho_\mu h^\lambda_\nu \overset{(1.5)}{=} f_{;\kappa\lambda} h^\kappa_\mu h^\lambda_\nu + f_{;\kappa}(n^\kappa_{;\lambda} n_\rho + n^\kappa n_{\rho;\lambda}) h^\rho_\mu h^\lambda_\nu = \\ =& (f_{,\kappa})_{;\lambda} h^\kappa_\mu h^\lambda_\nu + f_{,\kappa} n^\kappa n_{\rho;\lambda} h^\rho_\mu h^\lambda_\nu \overset{(14)}{=} (f_{,\kappa\lambda} - \Gamma^\sigma_{\kappa\lambda} f_{,\sigma}) h^\kappa_\mu h^\lambda_\nu + f_{,\kappa} n^\kappa K_{\mu\nu} = \\ =& (f_{,\lambda\kappa} - \Gamma^\sigma_{\lambda\kappa} f_{,\sigma}) h^\kappa_\mu h^\lambda_\nu + f_{,\kappa} n^\kappa K_{\nu\mu} \equiv f_{|\nu\mu} \implies {}^{(3)}\boldsymbol{T} \equiv 0, \end{aligned}$$

where just the symmetry property of $\boldsymbol{K}$ and Levi-Civita property of $\boldsymbol{\nabla}$ have been employed in the last row; therefore the proof is concluded. □

The following road plan is simple, yet quite tedious. To obtain the components of the 3-dimensional analogues of tensorial quantities, it is needed to, through the exact same procedure as in 4 dimensions, define the corresponding *Riemann tensor*, via the covariant differential commutator, in standard theory known as the Ricci identities. Although, for this purpose, it is first needed to obtain the relation identities between $\boldsymbol{\nabla}$, ${}^{(3)}\boldsymbol{\nabla}$, on Σ_t, for 1-forms. Let us then calculate the relation for arbitrary covector $v_\mu \in \mathcal{T}^*\Sigma_t$, and vector $v^\mu \in \mathcal{T}\Sigma_t$, respectively. We use (1.5) repetitively keeping in mind, that the covariant differential of $v_\alpha n^\alpha$ is trivially zero, due to the orthogonality identities. therefore, one is, in the parts of an expression, which are being summed over, allowed to "per partes" the semicolon to the second quantity

$$\begin{aligned} v_{\mu|\nu} :=& v_{\kappa;\lambda} h^\kappa_\mu h^\lambda_\nu = v_{\mu;\lambda} h^\lambda_\nu - \varepsilon v_{\kappa;\lambda} n^\kappa n_\mu h^\lambda_\nu \overset{(12)}{=} v_{\mu;\lambda} h^\lambda_\nu + \varepsilon v_\kappa n^\kappa_{;\lambda} n_\mu h^\lambda_\nu \overset{(14)}{\equiv} \\ \overset{(14)}{\equiv}& v_{\mu;\lambda} h^\lambda_\nu + \varepsilon K^\kappa_\nu v_\kappa n_\mu \overset{(12)}{=} v_{\mu;\nu} + \varepsilon(n_\mu K^\kappa_\nu v_\kappa - n_\nu \nabla_{\boldsymbol{n}} v_\mu), \end{aligned} \tag{1.7}$$

$$\begin{aligned} v^{\mu|\nu} :=& v^{\kappa;\lambda} h^\mu_\kappa h^\nu_\lambda = v^{\mu;\lambda} h^\nu_\lambda - \varepsilon v^{\kappa;\lambda} n_\kappa n^\mu h^\nu_\lambda \overset{(12)}{=} v^{\mu;\lambda} h^\nu_\lambda + \varepsilon v^\kappa n_\kappa^{;\lambda} n^\mu h^\nu_\lambda \overset{(14)}{\equiv} \\ \overset{(14)}{\equiv}& v^{\mu;\lambda} h^\nu_\lambda + \varepsilon K^\nu_\kappa v^\kappa n^\mu \overset{(12)}{=} v^{\mu;\nu} + \varepsilon(n^\mu K^\nu_\kappa v^\kappa - n^\nu \nabla_{\boldsymbol{n}} v^\mu). \end{aligned} \tag{1.8}$$

Here, $\nabla_{\boldsymbol{n}}$ denotes the covariant derivative in the direction of $\boldsymbol{n}$. As planned, we continue by calculating the second induced covariant differential of an arbitrary 1-form, $v_\mu \in \mathcal{T}^*\Sigma_t$, by once more differentiating (1.7):

$$\begin{aligned} v_{\nu|\kappa\lambda} =& v_{\rho;\sigma\gamma} h^\rho_\nu h^\sigma_\kappa h^\gamma_\lambda - \varepsilon v_{\rho;\sigma}[(n_{\alpha;\gamma} n^\rho + n_\alpha n^\rho_{;\gamma}) h^\alpha_\nu h^\sigma_\kappa h^\gamma_\lambda + (n_{\beta;\gamma} n^\sigma + n_\beta n^\sigma_{;\gamma}) h^\rho_\nu h^\beta_\kappa h^\gamma_\lambda] \\ \overset{(1.4)}{=}& v_{\rho;\sigma\gamma} h^\rho_\nu h^\sigma_\kappa h^\gamma_\lambda - \varepsilon v_{\rho;\sigma}(n_{\alpha;\gamma} n^\rho h^\alpha_\nu h^\sigma_\kappa h^\gamma_\lambda + n_{\beta;\gamma} n^\sigma h^\rho_\nu h^\beta_\kappa h^\gamma_\lambda) \overset{(14)}{=} \\ \overset{(14)}{=}& v_{\rho;\sigma\gamma} h^\rho_\nu h^\sigma_\kappa h^\gamma_\lambda - \varepsilon v_{\rho;\sigma} n^\rho h^\sigma_\kappa K_{\nu\lambda} - \varepsilon v_{\rho;\sigma} h^\rho_\nu n^\sigma K_{\kappa\lambda} \equiv \\ \equiv& v_{\rho;\sigma\gamma} h^\rho_\nu h^\sigma_\kappa h^\gamma_\lambda + \varepsilon v_\rho n^\rho_{;\sigma} h^\sigma_\kappa K_{\nu\lambda} - \varepsilon v_{\rho;\sigma} h^\rho_\nu n^\sigma K_{\kappa\lambda} \overset{(14)}{=} \\ \overset{(14)}{=}& v_{\rho;\sigma\gamma} h^\rho_\nu h^\sigma_\kappa h^\gamma_\lambda + \varepsilon(v_\rho K^\rho_\kappa K_{\nu\lambda} - v_{\rho;\sigma} h^\rho_\nu n^\sigma K_{\kappa\lambda}). \end{aligned}$$

So, the commutator yields, using the standard Ricci identities in the covariant form:

$$v_{\rho;\sigma\gamma} - v_{\rho;\gamma\sigma} = R^\mu{}_{\rho\gamma\sigma} v_\mu,$$

where $\boldsymbol{R} \in \mathcal{T}^1_3\mathcal{M}$ denotes the standard ("4D") Riemann curvature tensor:

$$\begin{aligned}\implies v_{\nu|\kappa\lambda} - v_{\nu|\lambda\kappa} &= v_{\rho;\sigma\gamma}h^\rho_\nu(h^\sigma_\kappa h^\gamma_\lambda - h^\sigma_\lambda h^\gamma_\kappa) + \varepsilon v_\rho(K^\rho_\kappa K_{\nu\lambda} - K^\rho_\lambda K_{\nu\kappa}) \equiv \\ &\equiv (v_{\rho;\sigma\gamma} - v_{\rho;\gamma\sigma})h^\sigma_\kappa h^\gamma_\lambda h^\rho_\nu + \varepsilon v_\rho(K^\rho_\kappa K_{\nu\lambda} - K^\rho_\lambda K_{\nu\kappa}) = \\ &= [R^\mu{}_{\rho\gamma\sigma}h^\rho_\nu h^\gamma_\kappa h^\sigma_\lambda + \varepsilon(K^\mu_\kappa K_{\nu\lambda} - K^\mu_\lambda K_{\nu\kappa})]v_\mu.\end{aligned}$$

The "operator equality" holds on Σ_t, so using the analogous ("3D") Ricci identities, that we use to define the Riemann tensor of Σ_t denoted by ${}^{(3)}\boldsymbol{R} \in \mathcal{T}^1_3\Sigma_t$:

$$ {}^{(3)}R^\mu{}_{\nu\kappa\lambda}v_\mu := v_{\nu|\kappa\lambda} - v_{\nu|\lambda\kappa}, \tag{1.9}$$

one arrives, by projecting in the remaining index to Σ_t, to the *Gauss equation*:

$$\boxed{R^\alpha{}_{\beta\gamma\delta}h^\mu_\alpha h^\beta_\nu h^\gamma_\kappa h^\delta_\lambda = {}^{(3)}R^\mu{}_{\nu\kappa\lambda} - \varepsilon(K^\mu_\kappa K_{\nu\lambda} - K^\mu_\lambda K_{\nu\kappa}).} \tag{1.10}$$

The Gauss equation is a well known result from differential geometry, which relates the (1.9) Riemann tensor of Σ_t to the projection of Riemann tensor of $\mathcal{M}$ onto Σ_t. By projecting the ordinary Ricci identity with the normal, one may obtain the other two non-trivial projections of the Riemann tensor of $\mathcal{M}$, namely $R^\alpha{}_{\beta\gamma\delta}n_\alpha h^\beta_\nu h^\gamma_\kappa h^\delta_\lambda$ and $R^\alpha{}_{\beta\gamma\delta}n_\alpha h^\beta_\nu n^\gamma h^\delta_\lambda$. This part is somewhat convention-dependent, although due to the known Riemann tensor (anti)symmetries, the rest of the projections are indeed always trivial for any convention choice. Obtaining the remaining two equations becomes easy, if one first evaluates the at first glance random expression $K_{\beta\gamma;\delta} - K_{\beta\delta;\gamma}$:

$$\begin{aligned}K_{\beta\gamma;\delta} - K_{\beta\delta;\gamma} :=&(n_{\beta;\gamma} - \varepsilon a_\beta n_\gamma)_{;\delta} - (n_{\beta;\delta} - \varepsilon a_\beta n_\delta)_{;\gamma} \\ =&n_{\beta;\gamma\delta} - n_{\beta;\delta\gamma} - \varepsilon a_{\beta;\gamma}n_\gamma + \varepsilon a_{\beta;\gamma}n_\delta - \varepsilon(n_{\gamma;\delta} - n_{\delta;\gamma})a_\beta.\end{aligned}$$

This can be further treated upon using the symmetry relation, that $\boldsymbol{K}$ satifies

$$K_{\gamma\delta} = K_{\delta\gamma} \implies n_{\gamma;\delta} - n_{\delta;\gamma} \equiv \varepsilon(a_\gamma n_\delta - a_\delta n_\gamma).$$

By substituting the identity into the previous relation and rearranging the terms, we immediately obtain a different form of the commutator on the right side of the Ricci identities for the normal.

$$n_{\beta;\gamma\delta} - n_{\beta;\delta\gamma} = K_{\beta\gamma;\delta} - K_{\beta\delta;\gamma} + \varepsilon(a_{\beta;\delta} - \varepsilon a_\beta a_\delta)n_\gamma - \varepsilon(a_{\beta;\gamma} - \varepsilon a_\beta a_\gamma)n_\delta.$$

From this, by substituting into the previously mentioned Ricci identities, one obtains

$$R^\alpha{}_{\beta\gamma\delta}n_\alpha = K_{\beta\gamma;\delta} - K_{\beta\delta;\gamma} + \varepsilon(a_{\beta;\delta} - \varepsilon a_\beta a_\delta)n_\gamma - \varepsilon(a_{\beta;\gamma} - \varepsilon a_\beta a_\gamma)n_\delta.$$

By projecting in the rest of the indices on Σ_t and using (1.4), the projection $R^\alpha{}_{\beta\gamma\delta}n_\alpha h^\beta_\nu h^\gamma_\kappa h^\delta_\lambda$ is complete

$$R^\alpha{}_{\beta\gamma\delta}n_\alpha h^\beta_\nu h^\gamma_\kappa h^\delta_\lambda = (K_{\beta\gamma;\delta} - K_{\beta\delta;\gamma})h^\beta_\nu h^\gamma_\kappa h^\delta_\lambda.$$

This equation in its most canonical form is called the *Codazzi equation*. It is derived directly by applying Lemma 6:

$$\boxed{R^\alpha{}_{\beta\gamma\delta}n_\alpha h^\beta_\nu h^\gamma_\kappa h^\delta_\lambda = K_{\nu\kappa|\lambda} - K_{\nu\lambda|\kappa}.} \tag{1.11}$$

While the other aforementioned possible projection $R^{\alpha}{}_{\beta\gamma\delta}n_{\alpha}h^{\beta}_{\nu}n^{\gamma}h^{\delta}_{\lambda}$ can be obtained by, instead of projecting on Σ_t in the rest of the indices, projecting γ with the corresponding vector n^{γ} instead, again, using Lemma 6 and (1.4):

$$\begin{aligned}R^{\alpha}{}_{\beta\gamma\delta}n_{\alpha}h^{\beta}_{\nu}n^{\gamma}h^{\delta}_{\lambda} &= [K_{\beta\gamma;\delta} - K_{\beta\delta;\gamma} + \varepsilon(a_{\beta;\delta} - \varepsilon a_{\beta}a_{\delta})n_{\gamma} - \\ &- \varepsilon(a_{\beta;\gamma} - \varepsilon a_{\beta}a_{\gamma})n_{\delta}]h^{\beta}_{\nu}n^{\gamma}h^{\delta}_{\lambda} = \\ &= \varepsilon^2 a_{\nu|\lambda} - \varepsilon^3 a_{\nu}a_{\lambda} + (K_{\beta\gamma;\delta} - K_{\beta\delta;\gamma})h^{\beta}_{\nu}n^{\gamma}h^{\delta}_{\lambda} = \\ &= a_{\nu|\lambda} - \varepsilon a_{\nu}a_{\lambda} - K_{\beta\gamma}h^{\beta}_{\nu}n^{\gamma}_{;\delta}h^{\delta}_{\lambda} - K_{\beta\delta;\gamma}h^{\beta}_{\nu}h^{\delta}_{\lambda}n^{\gamma} \overset{(14)}{=} \\ &\overset{(14)}{=} a_{\nu|\lambda} - \varepsilon a_{\nu}a_{\lambda} - K_{\nu\gamma}K^{\gamma}_{\lambda} - K_{\beta\delta;\gamma}h^{\beta}_{\nu}h^{\delta}_{\lambda}n^{\gamma}.\end{aligned}$$

By explicitly evaluating the last term, one can get a different expression:

$$\begin{aligned}K_{\beta\delta;\gamma}h^{\beta}_{\nu}h^{\delta}_{\lambda}n^{\gamma} &\overset{(12)}{\equiv} K_{\nu\lambda;\gamma}n^{\gamma} - \varepsilon K_{\nu\delta;\gamma}n^{\gamma}n_{\lambda}n^{\delta} - \varepsilon K_{\beta\lambda;\gamma}n^{\gamma}n_{\nu}n^{\beta} + \\ &+ \varepsilon^2 K_{\beta\delta;\gamma}n^{\gamma}n_{\nu}n_{\lambda}n^{\beta}n^{\delta} \overset{(12)}{\equiv} \\ &\overset{(12)}{\equiv} K_{\nu\lambda;\gamma}n^{\gamma} + \varepsilon K_{\nu\delta}n_{\lambda}a^{\delta} + \varepsilon K_{\beta\delta}n_{\nu}a^{\beta} \overset{(14)}{=} \\ &\overset{(14)}{=} K_{\nu\lambda;\gamma}n^{\gamma} + K_{\nu\delta}(n^{\delta}_{;\lambda} - K^{\delta}_{\lambda}) + K_{\beta\delta}(n^{\beta}_{;\nu} - K^{\beta}_{\nu}) = \\ &= K_{\nu\lambda;\gamma}n^{\gamma} + K_{\nu\delta}n^{\delta}_{;\lambda} + K_{\beta\delta}n^{\beta}_{\nu} - 2K_{\nu\gamma}K^{\gamma}_{\lambda} \equiv \\ &\equiv \mathcal{L}_{\boldsymbol{n}}K_{\nu\lambda} - 2K_{\nu\gamma}K^{\gamma}_{\lambda}.\end{aligned} \tag{1.12}$$

Therefore, we obtain two different but equivalent expressions of the *Ricci equation*

$$\boxed{R^{\alpha}{}_{\beta\gamma\delta}n_{\alpha}h^{\beta}_{\nu}n^{\gamma}h^{\delta}_{\lambda} = a_{\nu|\lambda} - \varepsilon a_{\nu}a_{\lambda} - K_{\nu\gamma}K^{\gamma}_{\lambda} - K_{\beta\delta;\gamma}h^{\beta}_{\nu}h^{\delta}_{\lambda}n^{\gamma},} \tag{1.13}$$

$$\boxed{R^{\alpha}{}_{\beta\gamma\delta}n_{\alpha}h^{\beta}_{\nu}n^{\gamma}h^{\delta}_{\lambda} = a_{\nu|\lambda} - \varepsilon a_{\nu}a_{\lambda} + K_{\nu\gamma}K^{\gamma}_{\lambda} - \mathcal{L}_{\boldsymbol{n}}K_{\nu\lambda}.} \tag{1.14}$$

With the knowledge of the relations for all non-trivial projections of the Riemann tensor, let us proceed to examine the Ricci tensor and curvature scalar, enabling us to decompose the field equations (1). The number of all non-trivial projections of the Ricci tensor is the same as in the Riemann tensor case, also the procedure is analogous. The Ricci tensor on $\mathcal{M}$ reads, expressed via relevant quantities:

$$R_{\beta\delta} := g^{\alpha\gamma}R_{\alpha\beta\gamma\delta} \overset{(12)}{=} h^{\alpha\gamma}R_{\alpha\beta\gamma\delta} + \varepsilon R_{\alpha\beta\gamma\delta}n^{\alpha}n^{\gamma}.$$

Firstly, we tackle the projection to Σ_t using Gauss equation and Ricci equation

$$\begin{aligned}R_{\beta\delta}h^{\beta}_{\nu}h^{\delta}_{\lambda} &\equiv h^{\mu\kappa}R_{\alpha\beta\gamma\delta}h^{\alpha}_{\mu}h^{\beta}_{\nu}h^{\gamma}_{\kappa}h^{\delta}_{\lambda} + \varepsilon R_{\alpha\beta\gamma\delta}n^{\alpha}h^{\beta}_{\nu}n^{\gamma}h^{\delta}_{\lambda} \overset{(1.10)}{=} \\ &\overset{(1.10)}{=} {}^{(3)}R_{\nu\lambda} - \varepsilon KK_{\nu\lambda} + \varepsilon K^{\kappa}_{\lambda}K_{\nu\kappa} + \varepsilon R_{\alpha\beta\gamma\delta}n^{\alpha}h^{\beta}_{\nu}n^{\gamma}h^{\delta}_{\lambda} \overset{(1.13)}{=} \\ &\overset{(1.13)}{=} {}^{(3)}R_{\nu\lambda} - \varepsilon KK_{\nu\lambda} + \varepsilon K^{\kappa}_{\lambda}K_{\nu\kappa} + \varepsilon a_{\nu|\lambda} - \varepsilon^2 a_{\nu}a_{\lambda} - \varepsilon K^{\kappa}_{\lambda}K_{\nu\kappa} - \\ &- \varepsilon K_{\beta\delta;\gamma}h^{\beta}_{\nu}h^{\delta}_{\lambda}n^{\gamma} \equiv \\ &\equiv {}^{(3)}R_{\nu\lambda} - a_{\nu}a_{\lambda} + \varepsilon(a_{\nu|\lambda} - KK_{\nu\lambda} - K_{\beta\delta;\gamma}h^{\beta}_{\nu}h^{\delta}_{\lambda}n^{\gamma}).\end{aligned} \tag{1.15}$$

Secondly, the "mixed" projection utilizes the Codazzi equation, along with the (anti)symmetries of the Riemann tensor, and incorporates the pre-established expression for Ricci on $\mathcal{M}$

$$\begin{aligned} R_{\beta\delta}n^{\beta}h^{\delta}_{\lambda} = h^{\mu\kappa}R_{\alpha\beta\gamma\delta}h^{\alpha}_{\mu}n^{\beta}h^{\gamma}_{\kappa}h^{\delta}_{\lambda} \equiv -h^{\nu\kappa}R_{\alpha\beta\gamma\delta}n^{\alpha}h^{\beta}_{\nu}h^{\gamma}_{\kappa}h^{\delta}_{\lambda} \overset{(1.11)}{=} \\ \overset{(1.11)}{=} -h^{\nu\kappa}(K_{\nu\kappa|\lambda} - K_{\nu\lambda|\kappa}) \equiv K^{\kappa}_{\lambda|\kappa} - K_{|\lambda}. \end{aligned} \tag{1.16}$$

The last projection can be obtained directly by contraction of the Ricci equation

$$R_{\alpha\gamma}n^{\alpha}n^{\gamma} = a^{\lambda}_{|\lambda} - \varepsilon a^{\lambda}a_{\lambda} - K^{\lambda}_{\kappa}K^{\kappa}_{\lambda} - h^{\beta\delta}K_{\beta\delta;\gamma}n^{\gamma}.$$

so, by contracting (1.7), with the metric $\boldsymbol{h}$, which is annihilated by ${}^{(3)}\nabla$, and further simplifying the covariant-3-divergence to the following

$$v^{\nu}_{|\nu} \equiv h^{\mu\nu}v_{\mu|\nu} \overset{(1.7)}{=} h^{\mu\nu}(v_{\mu;\lambda}h^{\lambda}_{\nu} + \varepsilon K^{\kappa}_{\nu}v_{\kappa}n_{\mu}) \overset{(1.4)}{=} v_{\mu;\lambda}h^{\mu\lambda} \overset{(12)}{=} v^{\lambda}_{;\lambda} - \varepsilon v_{\mu;\lambda}n^{\mu}n^{\lambda}. \tag{1.17}$$

Then explicitly evaluating the last expression to a useful identity

$$\begin{aligned} h^{\beta\delta}K_{\beta\delta;\gamma}n^{\gamma} \overset{(12)}{=} (g^{\beta\delta}K_{\beta\delta})_{;\gamma}n^{\gamma} - \varepsilon K_{\beta\delta;\gamma}n^{\beta}n^{\gamma}n^{\delta} \overset{(14)}{=} \\ \overset{(14)}{=} K_{;\gamma}n^{\gamma} - \varepsilon(n_{\beta;\delta} - \varepsilon a_{\beta}n_{\delta})_{;\gamma}n^{\beta}n^{\gamma}n^{\delta} \equiv K_{,\gamma}n^{\gamma}. \end{aligned} \tag{1.18}$$

We repetitively used the following well known identity, arising from the normalisazion of the normal

$$(n_{\mu}n^{\mu})_{;\nu} = 0 \implies n_{\mu;\nu}n^{\mu} \equiv 0. \tag{1.19}$$

Meaning the last non-trivial projection of the Ricci tensor takes the final form:

$$R_{\alpha\gamma}n^{\alpha}n^{\gamma} = a^{\lambda}_{;\lambda} - K^{\lambda}_{\kappa}K^{\kappa}_{\lambda} - K_{,\gamma}n^{\gamma}. \tag{1.20}$$

The curvature scalar of $\mathcal{M}$ can be expressed in two equivalent ways, that yield different decompositions:

$$\begin{aligned} R := g^{\beta\delta}R_{\beta\delta} \overset{(12)}{=} h^{\beta\delta}R_{\beta\delta} + \varepsilon R_{\beta\delta}n^{\beta}n^{\delta} \equiv \\ \equiv h^{\nu\lambda}R_{\beta\delta}h^{\beta}_{\nu}h^{\delta}_{\lambda} + \varepsilon R_{\beta\delta}n^{\beta}n^{\delta} \equiv h^{\mu\kappa}h^{\nu\lambda}R^{\alpha}{}_{\beta\gamma\delta}h^{\mu}_{\alpha}h^{\beta}_{\nu}h^{\gamma}_{\kappa}h^{\delta}_{\lambda} + 2\varepsilon R_{\beta\delta}n^{\beta}n^{\delta}. \end{aligned}$$

The first expression on the second row can be further handled by using the two already obtained projections of the Ricci tensor:

$$\begin{aligned} R \overset{(1.15)}{=} h^{\nu\lambda}[{}^{(3)}R_{\nu\lambda} - a_{\nu}a_{\lambda} + \varepsilon(a_{\nu|\lambda} - KK_{\nu\lambda} - K_{\beta\delta}h^{\beta}_{\nu}h^{\delta}_{\lambda}n^{\gamma})] + \varepsilon R_{\beta\delta}n^{\beta}n^{\delta} \underset{(1.17)}{\overset{(1.20)}{=}} \\ \underset{(1.17)}{\overset{(1.20)}{=}} {}^{(3)}R + \varepsilon(a^{\lambda}_{;\lambda} - K^{2} - K_{\beta\delta;\gamma}h^{\beta\delta}n^{\gamma}) + \varepsilon(a^{\lambda}_{;\lambda} - K^{\kappa}_{\lambda}K^{\lambda}_{\kappa} - K_{,\gamma}n^{\gamma}) \overset{(1.18)}{=} \\ \overset{(1.18)}{=} {}^{(3)}R + \varepsilon(2a^{\lambda}_{;\lambda} - K^{2} - 2K_{,\gamma}n^{\gamma} - K^{\kappa}_{\lambda}K^{\lambda}_{\kappa}). \end{aligned} \tag{1.21}$$

Whereas, the second expression on the second row yields directly, after using the Gauss equation, the following decomposition:

$$\begin{aligned} R \overset{(1.10)}{=} h^{\mu\kappa}h^{\nu\lambda}({}^{(3)}R_{\mu\nu\kappa\lambda} - \varepsilon K_{\mu\kappa}K_{\nu\lambda} + \varepsilon K_{\mu\lambda}K_{\nu\kappa}) + 2\varepsilon R_{\beta\delta}n^{\beta}n^{\delta} = \\ = {}^{(3)}R + \varepsilon(2R_{\beta\delta}n^{\beta}n^{\delta} + K^{\kappa}_{\lambda}K^{\lambda}_{\kappa} - K^{2}). \end{aligned} \tag{1.22}$$

The Einstein tensor of $\mathcal{M}$, defined by (1), is given by just a "linear" combination of already decomposed terms; therefore not many manipulations are neccessary. Analogously, like in the case of the Ricci tensor, three distinguished non-trivial projections exist

$$\begin{aligned} G_{\beta\delta}h^{\beta}_{\nu}h^{\delta}_{\lambda} &:= R_{\beta\delta}h^{\beta}_{\nu}h^{\delta}_{\lambda} - \frac{1}{2}Rh_{\nu\lambda} \overset{(1.15)}{\underset{(1.21)}{=}} \\ &\overset{(1.15)}{\underset{(1.21)}{=}} {}^{(3)}G_{\nu\lambda} - a_{\nu}a_{\lambda} + \varepsilon[a_{\nu|\lambda} - KK_{\nu\lambda} - K_{\beta\delta;\gamma}h^{\beta}_{\nu}h^{\delta}_{\lambda}n^{\gamma} + \\ &+ h_{\nu\lambda}(K_{,\gamma}n^{\gamma} + \frac{1}{2}K^{\kappa}_{\lambda}K^{\lambda}_{\kappa} + \frac{K^2}{2} - a^{\lambda}_{;\lambda})]. \end{aligned} \tag{1.23}$$

The "mixed" projection remains unchanged. Thanks to the orthogonality relations (1.4):

$$G_{\beta\delta}n^{\beta}h^{\delta}_{\lambda} \overset{(1.16)}{\equiv} K^{\kappa}_{\lambda|\kappa} - K_{|\lambda}. \tag{1.24}$$

Whereas, the twice to-normal-projected Einstein tensor is obtained immediately by plugging (1.22):

$$G_{\beta\delta}n^{\beta}n^{\delta} \overset{(1.4)}{=} R_{\beta\delta}n^{\beta}n^{\delta} - \frac{\varepsilon^3}{2}R \overset{(1.22)}{=} \frac{1}{2}(K^2 - K^{\kappa}_{\lambda}K^{\lambda}_{\kappa} - \varepsilon\,{}^{(3)}R). \tag{1.25}$$

Finally, submitting the last two results into the respectively-projected field equations (1) yields two equations, whose terms on the left side all $\in \Sigma_t$. This explains the noun *constraint*: those are the equations that must be obeyed for every parameter value. Therefore, for example, for space-like Σ_t, $\forall t \in \mathbb{R}$, i.e. at any *given moment*, meaning

$$\forall\,\Sigma_t : \begin{cases} K^{\kappa}_{\lambda|\kappa} - K_{|\lambda} = 8\pi T_{\beta\delta}n^{\beta}h^{\delta}_{\lambda} \ \dots \ \textit{momentum constraint}, \\ K^2 - K^{\kappa}_{\lambda}K^{\lambda}_{\kappa} - \varepsilon\,{}^{(3)}R = 16\pi T_{\beta\delta}n^{\beta}n^{\delta} + 2\Lambda \ \dots \ \textit{Hamiltonian constraint}. \end{cases} \tag{1.26}$$

The remaining equation (1.23) can be substituted into the field equations (1) in two ways. One can substitute directly (1.23) into the field equations or alternatively, submit it into the second known form of the equations. This one can get by tracing (1) and substituting the result back into the equation, as follows:

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu} &\xrightarrow{\textit{Trace}} R = 4\Lambda - 8\pi T \\ &\implies R_{\mu\nu} = 8\pi(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}) + \Lambda g_{\mu\nu}. \end{aligned} \tag{1.27}$$

Using the first option mentioned, one obtains the following identity

$$\begin{aligned} &{}^{(3)}G_{\nu\lambda} - a_{\nu}a_{\lambda} + \varepsilon[a_{\nu|\lambda} - KK_{\nu\lambda} - K_{\beta\delta;\gamma}h^{\beta}_{\nu}h^{\delta}_{\lambda}n^{\gamma} \\ &+ h_{\nu\lambda}(K_{,\gamma}n^{\gamma} + \frac{1}{2}K^{\kappa}_{\lambda}K^{\lambda}_{\kappa} + \frac{K^2}{2} - a^{\lambda}_{;\lambda})] = \\ &= 8\pi T_{\beta\delta}h^{\beta}_{\nu}h^{\delta}_{\lambda} - \Lambda h_{\nu\lambda}. \end{aligned}$$

Whereas, the second mentioned option yields, after utilizing (1.15), the known *evolution equation* for $\boldsymbol{K}$ in the direction of n^{γ}:

$$\begin{aligned} {}^{(3)}R_{\nu\lambda} - a_{\nu}a_{\lambda} + \varepsilon(a_{\nu|\lambda} - KK_{\nu\lambda} - K_{\beta\delta;\gamma}h^{\beta}_{\nu}h^{\delta}_{\lambda}n^{\gamma}) &\overset{(12)}{=} 8\pi(T_{\beta\delta}h^{\beta}_{\nu}h^{\delta}_{\lambda} - \frac{1}{2}Th_{\nu\lambda}) + \\ &+ \Lambda h_{\nu\lambda} \equiv 8\pi T_{\beta\delta}h^{\beta}_{\nu}h^{\delta}_{\lambda} + (\Lambda - 4\pi T)h_{\nu\lambda}. \end{aligned} \tag{1.28}$$

Subtracting the two options leads immediately to:

$$-\frac{1}{2}{}^{(3)}Rh_{\nu\lambda} + \varepsilon h_{\nu\lambda}(a_{\nu|\lambda} - KK_{\nu\lambda} - K_{\beta\delta;\gamma}h^{\beta}_{\nu}h^{\delta}_{\lambda}n^{\gamma}) = -2(\Lambda - 2\pi T)h_{\nu\lambda}.$$

After tracing and some elementary algebra, this becomes:

$${}^{(3)}R + \varepsilon(2a^{\kappa}_{;\kappa} - K^2 - K^{\kappa}_{\lambda}K^{\lambda}_{\kappa} - 2K_{,\gamma}n^{\gamma}) = 4(\Lambda - 2\pi T) \equiv R. \tag{1.29}$$

On the other hand, substituting the second form of the field equations (1.27) into (1.20), to ensure the desired property ${}^{(3)}R_{\nu\lambda} = {}^{(3)}R$, yields the following equation

$$\begin{aligned} a^{\lambda}_{;\lambda} - K^{\kappa}_{\lambda}K^{\lambda}_{\kappa} - K_{,\lambda}n^{\lambda} \equiv a^{\lambda}_{|\lambda} - \varepsilon a^{\lambda}a_{\lambda} - K^{\kappa}_{\lambda}K^{\lambda}_{\kappa} - K_{,\lambda}n^{\lambda} = \\ = 8\pi(T_{\kappa\lambda} - \frac{1}{2}Tg_{\kappa\lambda})n^{\kappa}n^{\lambda} + \Lambda g_{\kappa\lambda}n^{\kappa}n^{\lambda} \overset{(12)}{=} \\ \overset{(12)}{=} 8\pi T_{\kappa\lambda}n^{\kappa}n^{\lambda} + \varepsilon(\Lambda - 4\pi T). \end{aligned} \tag{1.30}$$

1.3.3 Adapted coordinates and the vacuum cases

In the previous section we have been, guided by Theorem 5, employing the 3+1 decomposition of $\mathcal{M}$ successfully, although, yet without employing any specific coordinates whatsoever. Let us then choose the parameter t as the zeroth coordinate of codimension 1 and use the Frobenius theorem to construct hypersurfaces of the constant time coordinate. Remember that, as we already know, the whole point is valid if and only if the vorticity tensor $\omega_{\mu\nu}$ of the normal vanishes. Simply apply the equivalence from Theorem 5 and define the normal as

$$n_{\alpha} := \varepsilon N\frac{\partial t}{\partial x^{\alpha}} \iff n^{\alpha} := \frac{1}{N}\frac{\partial x^{\alpha}}{\partial t}, \tag{1.31}$$

where N is an arbitrary normalization factor (it's actually not so arbitrary, see Chapter 2). A perceptive reader will surely notice, that in the case of the normal being time-like, the tangent vector to the direction of a chosen time coordinate, $t^{\alpha} := \frac{\partial x^{\alpha}}{\partial t}$, is going to have the "inverse" structure, so let us compute:

$$\varepsilon t_{\alpha}n^{\alpha} = \varepsilon^2 N\frac{\partial x^{\alpha}}{\partial t}\frac{\partial t}{\partial x^{\alpha}} \equiv N.$$

Which tells us, that N in the case, when $\varepsilon = -1 \iff \Sigma_t$ space-like, respresents the proportionality factor between the ordinary four-velocity u^{α}.

Now, thanks to the above definition of the normal, by directly evaluating the expression $2n_{[\mu;\nu]}$, one can get hands on the explicit expression of the acceleration of the n_{μ} field in terms of the normalization factor N:

$$2n_{[\mu;\nu]} \equiv 2n_{[\mu,\nu]} \overset{(1.31)}{=} 2\frac{N_{[,\nu}n_{\mu]}}{N} \implies a_{\mu} \overset{(1.19)}{=} 2n_{[\mu;\nu]}n^{\nu} \overset{(1.31)}{=} -\varepsilon\frac{N_{,\nu}h^{\nu}_{\mu}}{N} \overset{(6)}{=} -\varepsilon\frac{N_{|\mu}}{N}. \tag{1.32}$$

So the induced covariant derivative, defined by the Lemma 6 of a_{μ}, then becomes

$$a_{\mu|\nu} \overset{(1.32)}{=} \varepsilon\frac{N_{|\mu}N_{|\nu}}{N^2} - \varepsilon\frac{N_{|\mu\nu}}{N} \overset{(1.32)}{\implies} \varepsilon a_{\mu}a_{\nu} - a_{\mu|\nu} = \varepsilon\frac{N_{|\mu\nu}}{N}. \tag{1.33}$$

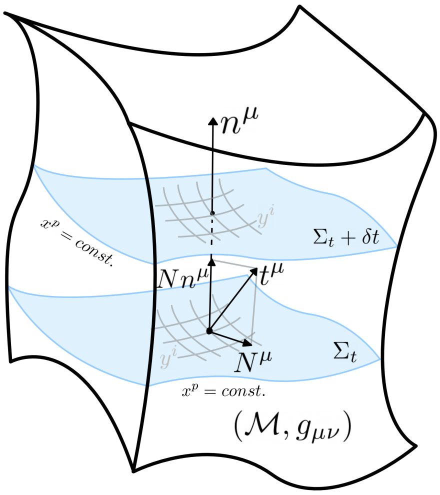


Figure 1.1 An illustration of the foliation for a time-like normal with the previously defined quantities. Here $x^p = const.$, equivalent to $t = const.$ chooses the hypersurface, as mentioned before.

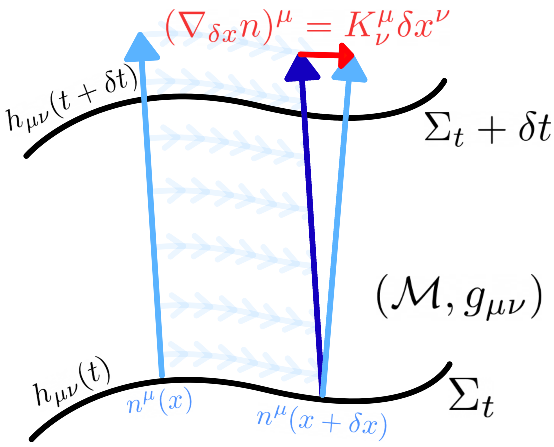


Figure 1.2 A vertical cut through the Figure 1.1, demonstrating the physical meaning of the shape operator (mixed components of $\boldsymbol{K}$), as being responsible for the difference between the orientation of the normal for two infinitesimally close points within the hypersurface Σ_t (see Definition 14).

Thus, again, using Lemma 6, the expression for $a_{\mu|\nu}$ is symmetric, thanks to the vanishing induced torsion tensor. This fact will be used in the next section. Next, let the projection of t^α onto Σ_t be denoted by N^α. Therefore let N^α be defined as follows (this vector is commonly called the *shift*, see Figure 1.1 for the intuition)

$$N^\alpha := t^\beta h^\alpha_\beta \overset{(12)}{=} (\delta^\alpha_\beta - \varepsilon n^\alpha n_\beta) t^\beta. \tag{1.34}$$

Thanks to the known mathematical identity of adding zero and the previous definition, one is able to decompose t^α into the normal, respectively tangent part, with respect to Σ_t

$$t^\alpha \equiv \delta^\alpha_\beta t^\beta \equiv (\delta^\alpha_\beta \pm \varepsilon n^\alpha n_\beta) t^\beta \overset{(1.34)}{=} N n^\alpha + N^\alpha. \tag{1.35}$$

From this decomposition, one can deduce that t^α is not, even in the time-like case discussed frequently in this text, generally proportional to the normal n^α, as demonstrated in Figure 1.1. In Chapter 2, we are going to see that the proportionality holds for time-like n^α only if the underlying (circular) space-time is static.

From Lemma 1, one knows that if the immersion/embedding of the submanifolds Σ_t has reasonable properties, there exists a coordinate system adapted to the immersion/embedding in the sense of Lemma 1. If the coresponding mapping ϕ, in the sense of Definition 1, is assumed to be an embedding, so it has no cusps, edges or self-intersections (so the normal is always normalize-able), those are the sufficient reasonable properties. Thus, denote by $[y^i]^3_{i=1}$, the intrinsic coordinates of the submanifolds Σ_t, and $[x^\mu]^3_{\mu=0}$ the corresponding extended coordinate system, such that $x^0 = const.$ correspond to $t = const.$ and $x^j \equiv \delta^j_i y^i$. Moreover, thanks to the Frobenius' theorem and Theorem 5, via the Definition of the normal, the submanifolds are integral, and the corresponding distributions are integrable. Thus, the aforementioned adapted coordinates are defined globally. As it was formulated in the subsequent paragraph of Lemma 1, the embedding is fully described by the relations $x^\mu = x^\mu(t, y^i)$, which by differentiation yields in components

$$dx^\mu = \frac{\partial x^\mu}{\partial t}dt + \frac{\partial x^\mu}{\partial y^i}dy^i \overset{(1.35)}{=} Nn^\mu dt + N^\mu dt + \frac{\partial x^\mu}{\partial y^i}dy^i.$$

Notice that N^μ is, by definition, tangent to Σ_t, so one can abbreviate its decomposition in the coordinate basis y^i of Σ_t, like $N^i := N^\mu \frac{\partial y^i}{\partial x^\mu}$, and get the decomposition of the metric line element, $ds^2 := g_{\mu\nu}dx^\mu dx^\nu$, by simply multiplying out the definition

$$ds^2 = g_{\mu\nu}\left[Nn^\mu dt + (N^i dt + dy^i)\frac{\partial x^\mu}{\partial y^i}\right]\left[Nn^\nu dt + (N^j dt + dy^j)\frac{\partial x^\nu}{\partial y^j}\right].$$

One can futher manipulate the previous expression for the line element, via the components of the covariant version of $\boldsymbol{g}$. For example, the term $n_\nu \frac{\partial x^\nu}{\partial y^j}$ is identically equal to zero, because the partial derivatives represent just the coordinate components of a vector tangent to Σ_t. Moreover, the last term involves the expression $g_{\mu\nu}\frac{\partial x^\mu}{\partial y^i}\frac{\partial x^\nu}{\partial y^j}$, in which one recognizes the tensor of type (0,2) transformation formula; therefore, the last term yields, after utilizing (1.5), just

$$g_{\mu\nu}\frac{\partial x^\mu}{\partial y^i}\frac{\partial x^\nu}{\partial y^j} \overset{(1.5)}{=} h_{\mu\nu}\frac{\partial x^\mu}{\partial y^i}\frac{\partial x^\nu}{\partial y^j} \equiv h_{ij}.$$

Here the orthogonality of n^μ and Σ_t was used again, as in the case of $n_\nu \frac{\partial x^\nu}{\partial y^j}$. The line element decomposition therefore becomes the known ADM formula[5], there eq. 4.48

$$\begin{aligned} \mathrm{d}s^2 &= \varepsilon N^2 dt^2 + h_{ij}(N^i dt + dy^i)(N^j dt + dy^j) \equiv \\ &\equiv (\varepsilon N^2 + N_k N^k)dt^2 + 2N_i dy^i dt + h_{ij}dy^i dy^j. \end{aligned} \tag{1.36}$$

The components of the defined-above quantities in the adapted coordinates can also be determined. If one begins by identifying

$$\text{directly from the definitions} \implies \begin{cases} n_\alpha = \varepsilon N \delta^0_\alpha, \\ t^\alpha = \delta^\alpha_0, \\ N^\alpha \equiv N^i. \end{cases} \tag{1.37}$$

The dual forms of n_α and t^α can also be obtained. From (1.35), one can immediately obtain the dual components of n_α

$$n^\alpha \overset{(1.35)}{=} \frac{1}{N}(t^\alpha - N^\alpha) \overset{(1.37)}{=} \frac{1}{N}(\delta^\alpha_0 - N^\alpha).$$

Considering the covariant form of (1.35), the dual components of t^α, are as follows

$$t_\alpha \overset{(1.35)}{=} Nn_\alpha + N_\alpha = -N^2\delta^0_\alpha + N_\alpha.$$

Knowing the components of the dual forms of n_α and t^α, one can reconstruct the components of the contravariant metric, via the relation $n^\mu = g^{\mu\nu}n_\nu \overset{(1.37)}{=}$

$-Ng^{\mu 0} \implies g^{\mu 0} = \frac{1}{N^2}(\varepsilon\delta^\mu_0 - N^\mu)$, by substituting the dual form of n_α into the expression $g^{ij} = h^{ij} + \varepsilon n^i n^j$, one can find, that

$$\mathrm{d}s^2 = \frac{\varepsilon}{N^2}\frac{\partial^2}{\partial t^2} - 2\varepsilon\frac{N^i}{N^2}\frac{\partial}{\partial y^i}\frac{\partial}{\partial t} + \left(h^{ij} + \varepsilon\frac{N^i N^j}{N^2}\right)\frac{\partial}{\partial y^i}\frac{\partial}{\partial y^j}. \tag{1.38}$$

Summarizing the above results, the following holds:

$$h_{\mu j} = g_{\mu j},\ h^\mu_j = \delta^\mu_j,\ h^0_\mu = 0,\ h^i_0 = N^i,\ h^{0\mu} = 0. \tag{1.39}$$

The contravariant form of the metric also implies that $N^2 = \frac{\varepsilon}{g^{tt}}$. From there, by the Laplace row (column) expansion of the respective matrices, one gets a very important relation between the determinants of the previously-defined metrics $\boldsymbol{g}$ and $\boldsymbol{h}$, abbreviating $\det(\boldsymbol{g}) := g$ and $\det(\boldsymbol{h}) := h$ [5], [19]

$$\varepsilon g = \frac{h}{\varepsilon g^{tt}} = N^2 h. \tag{1.40}$$

That is the last relation we shall mention without referencing to any (more concrete) adapted coordinates.

When working in a vacuum region of spacetime, $(\mathcal{M}, \boldsymbol{g})$, one has from (1) that $R_{\mu\nu} \stackrel{*}{\equiv} 0$[12]. This in turn implies that $R \stackrel{*}{\equiv} 0$, as well. Submitting the result into the main results of the last chapter, i.e., equations (1.15), (1.26), (1.28) and (1.29), assuming $\Lambda = 0$, yields, respectively

$$\begin{aligned} {}^{(3)}R_{\nu\lambda} &\stackrel{*}{=} a_\nu a_\lambda + \varepsilon(KK_{\nu\lambda} + K_{\beta\delta;\gamma}h^\beta_\nu h^\delta_\lambda n^\gamma - a_{\nu|\lambda}) \stackrel{(1.13)}{=} \\ &\stackrel{(1.13)}{=} \varepsilon(KK_{\nu\lambda} - R_{\alpha\nu\gamma\lambda}n^\alpha n^\gamma - K_{\nu\gamma}K^\gamma_\lambda), \end{aligned} \tag{1.41}$$

$$K^\kappa_{\lambda|\kappa} \stackrel{*}{=} K_{|\lambda}, \tag{1.42}$$

$${}^{(3)}R \stackrel{*}{=} \varepsilon(K^2 - K^\kappa_\lambda K^\lambda_\kappa), \tag{1.43}$$

$${}^{(3)}R \stackrel{*}{=} \varepsilon(2K_{,\gamma}n^\gamma + K^2 + K^\kappa_\lambda K^\lambda_\kappa - 2a^\kappa_{;\kappa}). \tag{1.44}$$

On the other hand, from (1.30) one gets

$$a^\kappa_{;\kappa} \stackrel{*}{=} K^\kappa_\lambda K^\lambda_\kappa + K_{,\gamma}n^\gamma. \tag{1.45}$$

Which is clearly not independent of the preceding two results above. This is the last result which we shall, without really employing specific coordinates for spatial dimensions, discuss.

1.4 General 2+1+1 decomposition of space-time

It has already been mentioned in this text, that the expression (1.33) is symmetric. Dor example, due to the above proven fact, that the induced covariant derivative, given by Lemma 6, is of the Levi-Civita type. Therefore, due to $a_\mu \in \mathcal{T}^*\Sigma_t$ (see (1.4)) analogously as in Definition 13, the corresponding vorticity

[12] The asterisk above the equal sign denotes that the equality only holds in a vacuum region of space-time.

tensor of the acceleration a_μ of the normal field n_μ is going to be, independently of any afterwards projection of $a_{[\mu|\nu]}$, identically equal to zero. *Familiar?*

The above mentioned argument gives one a reason to believe, that, defined possibly by the intersections with another 3D foliation by different hypersurfaces $(\Sigma_s)_{s\in\mathbb{R}}$, which in general necessarily need not to be orthogonal to the Σ_t foliation, a foliation of Σ_t by 2-dimensional submanifolds exists, denoted symbolically by Σ_{ts}, analogously to the "3+1" case. The aforementioned vorticity tensor (and the rest of the tensors defined in Definition 13) corresponding to the acceleration a_μ would thus be defined as the projection of the expression $a_{[\mu|\nu]}$ via the projector to Σ_{ts}.

Given the (again assumed to be smooth) Σ_s foliation, denote by r_μ a unit[13] projection of the defining-normal of Σ_s onto Σ_t. Moreover, analogously to ε, denote by $\delta := \operatorname{sign}(h_{\mu\nu}r^\mu r^\nu)$ the constant, that says, whether the defining-normal of Σ_s is time-like ($\delta = -1$) or space-like ($\delta = +1$). Let us, at this moment, stress that, given that we only work in 4 dimensions now, we automatically have the implication $\varepsilon = -1 \implies \delta = +1$, because the foliations Σ_t and Σ_s are assumed to be *different*. Next, note that r^μ is introduced as the unit projection onto Σ_t, so one, by definition, has $n_\mu r^\mu \equiv 0$. Now, analogously as in Definition 12, the metric of the 2-dimensional subfoliation defined by the intersections of Σ_t and Σ_s, denoted by Σ_{ts}, reads [19]:

$$f_{\mu\nu} := g_{\mu\nu} - \varepsilon n_\mu n_\nu - \delta r_\mu r_\nu \overset{(12)}{=} h_{\mu\nu} - \delta r_\mu r_\nu. \tag{1.46}$$

That is is completely analogous to Definition 12, as well as the corresponding (analogous as well) definition of the projector onto Σ_{ts}

$$f^\mu_\nu := \delta^\mu_\nu - \varepsilon n^\mu n_\nu - \delta r^\mu r_\nu \overset{(12)}{=} h^\mu_\nu - \delta r^\mu r_\nu. \tag{1.47}$$

Again, from above, analogously to (1.4), orthogonality indentities have been induced

$$r_\mu n^\mu = f^\mu_\nu n^\nu \equiv h^\mu_\nu n^\nu \equiv 0 \text{ (and again } r^\mu A^{\dots\dots\dots}_{\dots\mu\dots} \equiv 0 \,\forall A \in \mathcal{T}^k_l \Sigma_{ts}). \tag{1.48}$$

In the case of $r^\mu \in \mathcal{T}\Sigma_t$, the projector onto Σ_t obviously acts on it as an identity

$$h^\mu_\nu r^\nu = r^\mu. \tag{1.49}$$

This is the case even for the projector onto Σ_{ts} itself, as one can explicitly confirm with the aid of definition (1.47)

$$f^\mu_\sigma h^\sigma_\nu \overset{(1.47)}{=} (h^\mu_\sigma - \delta r^\mu r_\sigma) h^\sigma_\nu \equiv h^\mu_\nu - \delta r^\mu r_\nu \overset{(1.49)}{=} f^\mu_\nu. \tag{1.50}$$

Analogously to (1.19), one can prove the corollary of r^μ being normalized to the constant δ

$$(r_\mu r^\mu)_{|\nu} = 0 \implies r_{\mu|\nu} r^\mu \equiv 0. \tag{1.51}$$

Given the previously established notions and the clear analogy of the proposed decomposition to the Section 1.3, one is guided what to do next. One needs to

[13] Note that r^μ is always normalize-able again due to the assumed smoothness of the foliation Σ_s.

construct the "acceleration" of the r_μ field to be able to define the corresponding exterior curvature tensor $\boldsymbol{L}$ of Σ_{ts}. Let us then denote its components by b_μ and define it through the induced covariant derivative ${}^{(3)}\boldsymbol{\nabla}$ associated with $\boldsymbol{h}$, naturally, because $r_\mu \in \mathcal{T}^*\Sigma_t$. Moreover, this time, utilizing the "per-partes" trick and Definition 14, one can further rewrite the result into the following

$$\begin{aligned} b_\mu &:= r_{\mu|\nu} r^\nu \overset{(6)}{=} r_{\kappa;\lambda} h^\kappa_\mu r^\lambda \overset{(12)}{=} r_{\mu;\lambda} r^\lambda - \varepsilon r_{\kappa;\lambda} n^\kappa r^\lambda n_\mu \equiv r_{\mu;\lambda} r^\lambda + \varepsilon r^\kappa n_{\kappa;\lambda} r^\lambda n_\mu \overset{(14)}{=} \\ &\overset{(14)}{=} r_{\mu;\lambda} r^\lambda + \varepsilon (K_{\kappa\lambda} + \varepsilon a_\kappa n_\lambda) r^\kappa r^\lambda n_\mu \overset{(1.48)}{=} r_{\mu;\lambda} r^\lambda + \varepsilon K_{\kappa\lambda} r^\kappa r^\lambda n_\mu . \end{aligned} \tag{1.52}$$

Knowing this, one is able to construct the corresponding exterior curvature tensor of Σ_{ts}, denoted by $\boldsymbol{L}$. Similarly to Definition 14, the corresponding metric trace L can one further simplify using the previous result, while the "per-partes" trick is again repetitively enforced

$$\begin{aligned} L_{\mu\nu} &:= r_{\kappa|\lambda} f^\kappa_\mu f^\lambda_\nu \overset{(1.47)}{=} r_{\kappa|\lambda} (h^\kappa_\mu - \delta r^\kappa r_\mu) f^\lambda_\nu \overset{(1.51)}{=} r_{\mu|\lambda} f^\lambda_\nu \overset{(1.47)}{=} r_{\mu|\nu} - \delta b_\mu r_\nu \\ &= r_{\alpha;\beta} h^\alpha_\kappa h^\beta_\lambda f^\kappa_\mu f^\lambda_\nu \overset{(1.50)}{=} r_{\alpha;\beta} f^\alpha_\mu f^\beta_\nu \overset{(1.47)}{=} r_{\alpha;\beta} (h^\alpha_\mu - \delta r_\mu r^\alpha) f^\beta_\nu \overset{(1.51)}{=} r_{\alpha;\beta} h^\alpha_\mu f^\beta_\nu \overset{(12)}{=} \\ &\overset{(12)}{=} r_{\alpha;\beta} (\delta^\alpha_\mu - \varepsilon n_\mu n^\alpha) f^\beta_\nu \overset{(14)}{=} r_{\mu;\beta} f^\beta_\nu + \varepsilon r^\alpha (K_{\alpha\beta} + \varepsilon a_\alpha n_\beta) f^\beta_\nu n_\mu \overset{(1.48)}{=} \\ &\overset{(1.48)}{=} r_{\mu;\beta} f^\beta_\nu + \varepsilon K_{\alpha\beta} r^\alpha f^\beta_\nu n_\mu \overset{(1.47)}{=} \\ &\overset{(1.47)}{=} r_{\mu;\beta} h^\beta_\nu - \delta b_\mu r_\nu + \varepsilon K_{\alpha\nu} r^\alpha n_\mu . \end{aligned} \tag{1.53}$$

With aid of the expression (1.52), one can actually show that

$$b^\mu r_\mu \overset{(1.52)}{=} (r^\mu_{;\lambda} r^\lambda + \varepsilon K_{\kappa\lambda} r^\kappa r^\lambda n^\mu) r_\mu \underset{(1.48)}{\overset{(1.51)}{=}} 0.$$

As a verification of (1.48). The expression for L then reads

$$L := f^{\mu\nu} L_{\mu\nu} \equiv L^\mu_\mu \overset{(1.53)}{=} r^\mu_{|\mu} - \delta b^\mu r_\mu - \delta\varepsilon K_{\kappa\nu} r^\kappa r^\nu n^\mu r_\mu \overset{(1.48)}{=} r^\mu_{|\mu} \overset{(1.17)}{=} r^\mu_{;\mu} + \varepsilon a^\mu r_\mu . \tag{1.54}$$

1.4.1 Further general decomposition of the field equations

The procedure is analogous to Section 1.3. The first step to finding the general 2+1+1 decomposition of the Einstein field equations is finding the components of the induced covariant derivative corresponding to the connection of the Levi-Civita type compatible with the metric $\boldsymbol{f}$ of Σ_{ts}. One could construct a lemma analogous to the Lemma 6, although the points $(i) - (v)$ are again trivial. That is due to the fact that the projection by (1.47) changes nothing regarding the properties of the derivative (excluding the tensor spaces). The only semi-non-trivial point is the point (v). However, due to the theorem $\mathbf{d} = \boldsymbol{\nabla}_\wedge$, mentioned (although not proven) in Theorem 5, together with the further down-proven torsion-less the proof of (v), is trivial, because the wedge product consists of nothing but one term.

Let us then prove only the remaining non-trivial points (vi) and (vii) from Lemma 6, provided the generalized form as follows, denoting the operation by two vertical strokes.

Lemma 7 (Covariant differential on Σ_{ts}). *The operation defined as* [14].

$$ {}^{(2)}\nabla_\kappa T^{\mu\dots}_{\nu\dots} \equiv T^{\mu\dots}_{\nu\dots||\kappa} := T^{\alpha\dots}_{\beta\dots|\lambda} f^\mu_\alpha f^\beta_\nu f^\lambda_\kappa \cdots , $$

where the vertical stroke part denotes the components of the induced covariant differential compatible with $\boldsymbol{h}$*, represent the components of the covariant differential corresponding to the connection of the Levi-Civita type compatible with the metric* $\boldsymbol{f}$*. Therefore, according to the above-agreed simplification, satisfying the following for arbitrary* $k \in \mathcal{F}\Sigma_{ts}$*, and* $A, B \in \mathcal{T}^k_l\Sigma_{ts}$

(i) $h_{\mu\nu||\rho} = 0$,

(ii) $k_{||\mu\nu} - k_{||\nu\mu} \equiv -{}^{(2)}T^\rho_{\mu\nu} k_{||\rho} = 0$,
where ${}^{(2)}T^\rho_{\mu\nu}$ *stands for the components of the induced torsion tensor* ${}^{(2)}\boldsymbol{T}$.

Proof. Annihilation of the metric $\boldsymbol{h}$ follows immediately from the Levi-Civita property of $\boldsymbol{g}$

$$ f_{\mu\nu||\rho} := f_{\kappa\lambda|\sigma} f^\kappa_\mu f^\lambda_\nu f^\sigma_\rho \overset{(12)}{=} f_{\alpha\beta;\gamma} h^\alpha_\kappa h^\beta_\lambda h^\gamma_\sigma f^\kappa_\mu f^\lambda_\nu f^\sigma_\rho \underset{(1.47)}{\overset{(1.50)}{=}} $$
$$ \underset{(1.47)}{\overset{(1.50)}{=}} (g_{\alpha\beta;\gamma} - \varepsilon(n_\alpha n_\beta)_{;\gamma} - \delta(r_\alpha r_\beta)_{;\gamma}) f^\kappa_\mu f^\lambda_\nu f^\sigma_\rho \overset{(1.48)}{\equiv} 0. $$

Whereas, the torsion-free property is easily obtained by, among other properties, the Levi-Civita property of $\boldsymbol{h}$, this time of course with respect to ${}^{(3)}\boldsymbol{\nabla}$

$$ k_{||\mu\nu} := (k_{|\kappa} f^\kappa_\rho)_{|\lambda} f^\rho_\mu f^\lambda_\nu = k_{|\kappa\lambda} f^\kappa_\mu f^\lambda_\nu + g_{|\kappa}(h^\kappa_\rho - \delta r_\rho r^\kappa)_{|\lambda} f^\rho_\mu f^\lambda_\nu \overset{(1.47)}{=} $$
$$ \overset{(1.47)}{=} k_{|\kappa\lambda} f^\kappa_\mu f^\lambda_\nu - \delta k_{|\kappa} \left[r_\rho r^\kappa (h^\rho_\mu h^\lambda_\nu - \delta h^\rho_\mu r_\nu r^\lambda - \delta h^\lambda_\nu r_\mu r^\rho + \delta^2 r_\mu r_\nu r^\rho r^\lambda) \right] = $$
$$ = k_{|\kappa\lambda} f^\kappa_\mu f^\lambda_\nu - \delta k_{|\kappa} \left(r_\mu r^\kappa h^\lambda_\nu - \delta r_\mu r^\kappa r_\nu r^\lambda - h^\lambda_\nu r^\kappa r_\mu + \delta r^\kappa r_\mu r_\nu r^\lambda \right) \equiv k_{|\kappa\lambda} f^\kappa_\mu f^\lambda_\nu. $$

Furthermore, writing out the ${}^{(3)}\boldsymbol{\nabla}$, by definition, yields (using the connection component expression of $\boldsymbol{\nabla}$)

$$ k_{|\kappa\lambda} f^\kappa_\mu f^\lambda_\nu \overset{(6)}{=} k_{;\alpha\beta} h^\alpha_\kappa h^\beta_\lambda f^\kappa_\mu f^\lambda_\nu \overset{(1.50)}{=} (k_{,\alpha\beta} - \Gamma^\sigma{}_{\alpha\beta} f_{,\sigma}) f^\alpha_\mu f^\beta_\nu \equiv k_{;\beta\alpha} f^\alpha_\mu f^\beta_\nu =: k_{||\nu\mu} $$
$$ \implies {}^{(2)}\boldsymbol{T} \equiv 0. $$

□

From the last line of the proof, one can immediately deduce that the definition's correctness is independent of the respective covariant differential, i.e.,

$$ {}^{(2)}\nabla_\kappa T^{\mu\dots}_{\nu\dots} \equiv T^{\mu\dots}_{\nu\dots||\kappa} := T^{\alpha\dots}_{\beta\dots|\lambda} f^\mu_\alpha f^\beta_\nu f^\lambda_\kappa \dots \equiv T^{\alpha\dots}_{\beta\dots;\lambda} f^\mu_\alpha f^\beta_\nu f^\lambda_\kappa \cdots . \tag{1.55} $$

Again, to define the corresponding Riemann tensor, one has to have some relation between the covariant differentials $\boldsymbol{\nabla}$ and ${}^{(2)}\boldsymbol{\nabla}$ for 1-forms in order to

[14]The covariant derivative on Σ_{ts} itself shall be, if needed, denoted by ${}^{(2)}\boldsymbol{\nabla}$

simplify the commutator from Ricci identities. Let us then, as before, calculate the relation for arbitrary covector $v_\mu \in \mathcal{T}^*\Sigma_{ts}$, respectively vector $v^\mu \in \mathcal{T}\Sigma_{ts}$

$$\begin{aligned} v_{\mu||\nu} &:= v_{\kappa;\lambda} f^\kappa_\mu f^\lambda_\nu \overset{(1.47)}{=} v_{\kappa;\lambda}(\delta^\kappa_\mu - \varepsilon n_\mu n^\kappa - \delta r_\mu r^\kappa) f^\lambda_\nu \underset{(1.53)}{\overset{(14)}{=}} \\ &\underset{(1.53)}{\overset{(14)}{=}} v_{\mu;\lambda} f^\lambda_\nu + \varepsilon n_\mu v^\kappa (K_{\kappa\lambda} + \varepsilon a_\kappa n_\lambda) f^\lambda_\nu + \delta r_\mu v^\kappa (L_{\kappa\lambda} + \delta b_\kappa r_\lambda - \varepsilon K_{\alpha\lambda} r^\alpha n_\kappa) f^\lambda_\nu \\ &\overset{(1.48)}{=} v_{\mu;\lambda} f^\lambda_\nu + \varepsilon K^\kappa_\lambda v_\kappa f^\lambda_\nu n_\mu + \delta L^\kappa_\nu v_\kappa r_\mu \\ &\overset{(1.47)}{=} v_{\mu;\nu} + \varepsilon (K^\kappa_\nu v_\kappa n_\mu - n_\nu \nabla_{\boldsymbol{n}} v_\mu) + \delta (L^\kappa_\nu v_\kappa r_\mu - r_\nu \nabla_{\boldsymbol{r}} v_\mu), \end{aligned} \tag{1.56}$$

$$\begin{aligned} v^{\mu||\nu} &:= v^{\kappa;\lambda} f^\mu_\kappa f^\nu_\lambda \overset{(1.47)}{=} v^{\kappa;\lambda}(\delta^\mu_\kappa - \varepsilon n^\mu n_\kappa - \delta r^\mu r_\kappa) f^\nu_\lambda \underset{(1.53)}{\overset{(14)}{=}} \\ &\underset{(1.53)}{\overset{(14)}{=}} v^{\mu;\lambda} f^\nu_\lambda + \varepsilon n^\mu v_\kappa (K^{\kappa\lambda} + \varepsilon a^\kappa n^\lambda) f^\nu_\lambda + \delta r^\mu v_\kappa (L^{\kappa\lambda} + \delta b^\kappa r^\lambda - \varepsilon K^{\alpha\lambda} r_\alpha n^\kappa) f^\nu_\lambda \\ &\overset{(1.48)}{=} v^{\mu;\lambda} f^\nu_\lambda + \varepsilon K^\lambda_\kappa v^\kappa f^\nu_\lambda n^\mu + \delta L^\nu_\kappa v^\kappa r^\mu \\ &\overset{(1.47)}{=} v^{\mu;\nu} + \varepsilon (K^\nu_\kappa v^\kappa n^\mu - n^\nu \nabla_{\boldsymbol{n}} v^\mu) + \delta (L^\nu_\kappa v^\kappa r^\mu - r^\nu \nabla_{\boldsymbol{r}} v^\mu). \end{aligned} \tag{1.57}$$

As before, one continues by evaluating the second induced covariant differential of an arbitrary 1-form, $v_\mu \in \mathcal{T}^*\Sigma_{ts}$, by the same procedure as in Section 1.3. However, a little remark on dimensionality is perhaps on point here.

The Σ_{ts} submanifolds are 2-dimensional, which implies that the Riemann tensor of Σ_{ts} has but one independent component obviously proportional to the *Kulkarni-Nomizu product* of the metric $\boldsymbol{f}$ with itself [11], [19]

$$\begin{aligned} {}^{(2)}R_{\mu\nu\kappa\lambda} &= \frac{{}^{(2)}R}{4} f_{\mu\nu} \owedge f_{\kappa\lambda} \equiv \\ &\equiv \frac{{}^{(2)}R}{4} (f_{\mu\kappa} f_{\lambda\nu} + f_{\nu\lambda} f_{\kappa\mu} - f_{\nu\kappa} f_{\lambda\mu} - f_{\mu\lambda} f_{\kappa\nu}) \overset{(1.47)}{=} {}^{(2)}R f_{\kappa[\mu} f_{\nu]\lambda}. \end{aligned}$$

Hence, substituting to the ("2D") Ricci identities yields

$$v_{\nu||\kappa\lambda} - v_{\nu||\lambda\kappa} = \frac{{}^{(2)}R}{2} (v_\kappa f_{\nu\lambda} - v_\lambda f_{\nu\kappa}).$$

This fact may be proven immediately from the definition of the *Levi-Civita tensor* of Σ_{ts}, because the tensor product of the Levi-Civita tensor with its inverse has to be proportional to the projector onto the tensor subspace of antisymmetric tensors [11], i.e.,

$$\operatorname{sign}(f) \varepsilon_{\mu\nu} \varepsilon^{\kappa\lambda} = {}^{[2]}\delta^{\kappa\lambda}_{\mu\nu} \equiv 2\delta^{[\kappa}_\mu \delta^{\lambda]}_\nu = \delta^\kappa_\mu \delta^\lambda_\nu - \delta^\lambda_\mu \delta^\kappa_\nu \implies \varepsilon_{\mu\nu} \varepsilon_{\kappa\lambda} = f_{\mu\kappa} f_{\nu\lambda} - f_{\mu\lambda} f_{\nu\kappa}.$$

Therefore, by lowering the indices, one is immediately left with a tensor that shares the same (anti)symmetries as the Riemann tensor. This means that, by setting it proportional to the 2-dimensional Riemann tensor and finding the proportionality factor via tracing, one arrives at the identity discussed above.

Further generalisation of the formula into 3 dimensions can be obtained, via considering the well-known decomposition of the Riemann tensor of general dimension d into irreducible trace-free parts [11], where ${}^{(d)}\boldsymbol{C}$ stands for the d-dimensional Weyl tensor and ${}^{(d)}\boldsymbol{g}$ for the corresponding metric tensor

$$ {}^{(d)}R_{\mu\nu\kappa\lambda} = {}^{(d)}C_{\mu\nu\kappa\lambda} + \frac{1}{d-2}\,{}^{(d)}R_{\mu\nu} \owedge {}^{(d)}g_{\kappa\lambda} - \frac{{}^{(d)}R}{2(d-1)(d-2)}\,{}^{(d)}g_{\mu\nu} \owedge {}^{(d)}g_{\kappa\lambda}. $$

By setting $d = 3$, one then arrives directly at the generalisation of the above formula in 3 dimensions[15]

$$ {}^{(3)}R_{\mu\nu\kappa\lambda} = {}^{(3)}R_{[\mu|[\kappa}h_{\lambda]|\nu]} - \frac{{}^{(3)}R}{4}h_{[\mu|[\kappa}h_{\lambda]|\nu]}. $$

Summarizing the previous results, in 3 dimensions, the Riemann tensor is fully determined by the Ricci tensor and the scalar curvature. In 2 dimensions, this further simplifies to the Riemann tensor being fully determined by the scalar curvature. Because the 2-dimensional analogues of the projections of the Riemann tensor relate the projections of the 3-dimensional Riemann tensor and 2-dimensional Riemann tensor, using the previously obtained identities, those are actually relations between 3-dimensional Ricci tensor together with 3-dimensional scalar curvature and 2-dimensional scalar curvature.

To continue with the decomposition, by obtaining the ("2D") analogue of (1.9), one is able to write the aforementioned projections, in this text referred to as the Gauss, Codazzi and Ricci equations, this time for the Riemann tensor ${}^{(3)}R^{\alpha}{}_{\beta\gamma\delta}$. The procedure of obtaining and the form of which are completely analogous to (1.10), (1.11), (1.13) and (1.14) [19]

$$ \boxed{{}^{(3)}R^{\alpha}{}_{\beta\gamma\delta}f^{\mu}_{\alpha}f^{\beta}_{\nu}f^{\gamma}_{\kappa}f^{\delta}_{\lambda} = {}^{(2)}R^{\mu}{}_{\nu\kappa\lambda} - \delta(L^{\mu}_{\kappa}L_{\nu\lambda} - L^{\mu}_{\lambda}L_{\nu\kappa}),} \tag{1.58} $$

$$ \boxed{{}^{(3)}R^{\alpha}{}_{\beta\gamma\delta}r_{\alpha}f^{\beta}_{\nu}f^{\gamma}_{\kappa}f^{\delta}_{\lambda} = L_{\nu\kappa||\lambda} - L_{\nu\lambda||\kappa},} \tag{1.59} $$

$$ \boxed{{}^{(3)}R^{\alpha}{}_{\beta\gamma\delta}r_{\alpha}f^{\beta}_{\nu}r^{\gamma}f^{\delta}_{\lambda} \overset{(1.47)}{=} {}^{(3)}R_{\alpha\nu\gamma\lambda}r^{\alpha}r^{\gamma} = b_{\nu||\lambda} - \delta b_{\nu}b_{\lambda} - L_{\nu\gamma}L^{\gamma}_{\lambda} - L_{\beta\delta|\gamma}f^{\beta}_{\nu}f^{\delta}_{\lambda}r^{\gamma}.} \tag{1.60} $$

Let us stress here the forms of the expressions when the above established dimensionality-induced simplifications are introduced to the equation. Namely, the ("2D") Gauss equation in covariant form (over)simplifies to

$$ \begin{aligned} &{}^{(3)}G_{\kappa[\mu}f_{\nu]\lambda} - {}^{(3)}R_{\lambda[\mu}f_{\nu]\kappa} - \delta r_{\kappa}r^{\alpha\,(3)}R_{\alpha[\mu}f_{\nu]\lambda} + \delta r_{\lambda}r^{\alpha\,(3)}R_{\alpha[\mu}f_{\nu]\kappa} + \\ &+ r_{[\mu}f_{\nu]\lambda}\left[r_{\kappa}\left(r^{\alpha\,(3)}R_{\alpha\beta}r^{\beta} + \frac{\delta}{2}\,{}^{(3)}R\right) - \delta r^{\alpha\,(3)}R_{\alpha\kappa}\right] + \\ &+ r_{[\mu}f_{\nu]\kappa}\left[\delta r^{\alpha\,(3)}R_{\alpha\lambda} - r_{\lambda}r^{\alpha\,(3)}R_{\alpha\beta}r^{\beta}\right] = \frac{{}^{(2)}R}{2}f_{\kappa[\mu}f_{\nu]\lambda} - \frac{\delta}{2}(L^{\mu}_{\kappa}L_{\nu\lambda} - L^{\mu}_{\lambda}L_{\nu\kappa}). \end{aligned} \tag{1.61} $$

The ("2D") Codazzi equation simplifies to

$$ R_{\alpha\beta}r^{\alpha}f^{\beta}_{[\kappa}f_{\lambda]\nu} = \frac{1}{2}\left(L_{\nu\kappa|\lambda} - L_{\nu\lambda|\kappa}\right). \tag{1.62} $$

[15] Of course, the vertical stroke does not denote any induced derivative here, it simply emphasizes that the two antisymmetrizations are independent, due to the nature of the relation, we believe that no confusion may arise.

Finally, the ("2D") Ricci equation simplifies to

$$\begin{aligned}\delta\,^{(3)}G_{\nu\lambda} + {}^{(3)}R_{\alpha\beta}r^\alpha r^\beta h_{\nu\lambda} + \frac{^{(3)}R}{2}r_\nu r_\lambda - 2r^\alpha\,^{(3)}R_{\alpha(\nu}r_{\lambda)} = \\ = b_{\nu||\lambda} - \delta b_\nu b_\lambda - L_{\nu\gamma}L^\gamma_\lambda - L_{\beta\delta|\gamma}f^\beta_\nu f^\delta_\lambda r^\gamma.\end{aligned} \tag{1.63}$$

Now, again, analogously to Section 1.3, one can write the corresponding decompositions of the Ricci tensor and curvature scalar, utilizing the same procedures. Those of the Ricci tensor read, respectively [19]

$$\begin{aligned}{}^{(3)}R_{\beta\delta}f^\beta_\nu f^\delta_\lambda = {}^{(3)}R_{\nu\lambda} + \delta\left(L^\kappa_\lambda L_{\nu\kappa} - LL_{\nu\lambda} + {}^{(3)}R_{\alpha\nu\gamma\lambda}r^\alpha r^\gamma\right) = \\ = {}^{(2)}R_{\nu\lambda} - a_\nu a_\lambda + \delta\left(b_{\nu||\lambda} - LL_{\nu\lambda} - L_{\beta\delta|\gamma}f^\beta_\nu f^\delta_\lambda r^\gamma\right),\end{aligned} \tag{1.64}$$

$$R_{\beta\delta}r^\beta f^\delta_\lambda = L^\kappa_{\lambda||\kappa} - L_{||\lambda}. \tag{1.65}$$

Again, the simplification of the induced covariant divergence of an arbitrary vector, $v^\nu \in \mathcal{T}\Sigma_{ts}$, calculated using (1.17)

$$\begin{aligned}v^\nu_{||\nu} \equiv f^{\mu\nu}v_{\mu||\nu} \overset{(1.56)}{=} f^{\mu\nu}(v_{\mu;\lambda}f^\lambda_\nu + \varepsilon K^\kappa_\lambda v_\kappa f^\lambda_\nu n_\mu + \delta L^\kappa_\nu v_\kappa r_\mu) \overset{(1.47)}{=} \\ \overset{(1.47)}{=} v^\lambda_{;\lambda} - \varepsilon v_{\mu;\lambda}n^\mu n^\lambda - \delta v_{\mu;\lambda}r^\mu r^\lambda.\end{aligned} \tag{1.66}$$

This implies, for the "acceleration" of the r^μ field

$$b^\lambda_{||\lambda} = b^\lambda_{|\lambda} + \delta b^\lambda b_\lambda$$

and the equation analogous to (1.18)

$$L_{\beta\delta|\gamma}f^{\beta\delta}r^\gamma = L_{,\gamma}r^\gamma, \tag{1.67}$$

allows one to write the last projection as

$${}^{(3)}R_{\alpha\gamma}r^\alpha r^\gamma = b^\lambda_{|\lambda} - L^\lambda_\kappa L^\kappa_\lambda - L_{,\gamma}r^\gamma. \tag{1.68}$$

The projection of the scalar curvature takes the (again analogous) form, like (1.21) and (1.22)

$$\begin{aligned}{}^{(3)}R = {}^{(2)}R + \delta(2b^\lambda_{|\lambda} - L^2 - 2L_{,\gamma}r^\gamma - L^\kappa_\lambda L^\lambda_\kappa) = \\ {}^{(2)}R + \delta(2\,^{(3)}R_{\beta\delta}r^\beta r^\delta + L^\kappa_\lambda L^\lambda_\kappa - L^2).\end{aligned} \tag{1.69}$$

2 Space-times considered

In this chapter, we are going to investigate some solutions of the ten independent field equations (1), expressed through the metric tensor in a four-member coordinate chart, referred to as space-time. Our primary focus is on vacuum solutions without the cosmological constant Λ. More specifically, *stationary (or static)*, *asymptotically flat* and *axisymmetric* space-times. This chapter offers a concise overview of the fundamental concepts and key characteristics of these specific solutions.

2.1 Basic notions

Let us start with defining the notions one needs to rigorously define the four properties of interest mentioned in the preceding paragraph.

Definition 15 (Group action on space-time manifold, orbit)**.** *Let $\mathcal{G}$ be a group, and let $\mathcal{M}$ denote the space-time manifold of interest. By* ***action of $\mathcal{G}$ on $\mathcal{M}$****, we understand a mapping $\Phi : \mathcal{G} \times \mathcal{M} \longrightarrow \mathcal{M}$, therefore if $a \in \mathcal{G}$ and $x \in \mathcal{M}$, the pair (a, x) is mapped onto $\Phi(a, x) \equiv a(x)$, such that $\forall x \in \mathcal{M} : \mathrm{id}(x) \equiv x$, and $\forall (a, b) \in \mathcal{G} \times \mathcal{G} : a(b(x)) = ab(x)$, where id denotes the identity element of $\mathcal{G}$. The full set $\{a(x)\, \forall a \in \mathcal{G}\} \subset \mathcal{M}$ is called an* ***orbit****. If $\mathcal{G}$ is a one-dimensional Lie group, the orbit of a point $x \in \mathcal{M}$ is either the point itself, in the case when x is an invariant point of the group action, or a one-dimensional curve in $\mathcal{M}$ parameterized by some parameter $t \in \mathbb{R}$. The tangent vector corresponding to t is then called the* generator of $\mathcal{G}$ associated with the parameterization by the parameter t. *The generator associated with the parameter t will be denoted by the greek letter $\boldsymbol{\eta}$[6].*

In the case of $\mathcal{G}$ being a one-dimensional group of differomorphisms, it is sometimes being referred to as *flow on the manifold $\mathcal{M}$* and has a close relationship to the Lie derivative in the direction of the generator $\boldsymbol{\eta}$ (actually, it is its differential version, as stated in [11])

Definition 16 (Stationary space-time)**.** *A spacetime $(\mathcal{M}, \boldsymbol{g})$ is called* ***stationary*** *if there exist a group $\mathcal{G}_1$ such that the group's action on $\mathcal{M}$ fulfills the following properties[6]:*

(i) $\mathcal{G}_1$ is isomorphic to $(\mathbb{R}, +)$, where $+$ denotes the standard addition operation,

(ii) $\forall x \in \mathcal{M}$ are the orbits timelike curves in $\mathcal{M}$, i.e. their tangent vector's space-time norm defined via $\boldsymbol{g}$ is always negative,

(iii) $\boldsymbol{g}$ is invariant under the $\mathcal{G}_1$ group action, in differential form meaning that for $\mathcal{G}_1$ generator $\boldsymbol{\eta}$ in the sense of Definition 15 one has $\mathcal{L}_{\boldsymbol{\eta}} \boldsymbol{g} = 0$ [1].

[1]This, of course, using the components of the Lie derivative and the Levi-Civita property of $\boldsymbol{g}$, trivially corresponds to $\boldsymbol{\eta}$ being a *Killing vector*, i.e. satisfying $\nabla_{(\mu}\eta_{\nu)} = 0$.

Definition 17 (Asymptotically flat space-time)**.** *A vacuum spacetime* $(\mathcal{M}, \boldsymbol{g})$ *is called asymptotically flat at null and spatial infinity if there exists a spacetime* $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ *with* $\tilde{\boldsymbol{g}}$ *being* C^∞ *everywhere except possibly at a point* i^0 *(spatial infinity), and a conformal isometry* $\psi : \mathcal{M} \to \psi(\mathcal{M}) \subset \tilde{\mathcal{M}}$ *with the conformal factor* Ω *satisfying:*

(i) $\tilde{\mathcal{M}} = \bar{J}^+(i^0) \cup \bar{J}^-(i^0)$ *and* $\mathcal{M} = \tilde{\mathcal{M}} - \partial\tilde{\mathcal{M}}$*, where the boundary* $\partial\tilde{\mathcal{M}}$ *consists of* i^0, $\mathscr{I}^+$ *(future null infinity), and* $\mathscr{I}^-$ *(past null infinity). Here,* $\bar{J}^+(i^0)$ *and* $\bar{J}^-(i^0)$ *are the causal future and past of* i^0*, respectively.*

(ii) There exists an open neighborhood V *of* $\partial\tilde{\mathcal{M}}$ *such that* $(V, \boldsymbol{g})$ *is strongly causal, meaning that no causal curve intersects itself more than once.*

(iii) Ω *(the conformal factor) can be extended to a function on all of* $\tilde{\mathcal{M}}$ *which is* C^2 *at* i^0 *and* C^∞ *elsewhere.*

(iv) On $\mathscr{I}^+$ *and* $\mathscr{I}^-$*:*

  (a) $\Omega = 0$ *and* $\tilde{\nabla}_\mu \Omega \neq 0$*, where* $\tilde{\nabla}$ *denotes the covariant differential associated with* $\boldsymbol{g}$*.*

  (b) $\Omega(i^0) = 0$, $\lim_{i^0} \tilde{\nabla}_\mu \Omega = 0$*, and* $\tilde{\nabla}_\mu \tilde{\nabla}_\nu \Omega = 2\tilde{g}_{\mu\nu}(i^0)$*.*

(v) The map of null directions $n^\mu = \tilde{g}^{\mu\nu}\tilde{\nabla}_\nu \Omega$ *on* $\mathscr{I}^+$ *and* $\mathscr{I}^-$ *is a diffeomorphism.*

(vi) For a smooth function ω *on* $\mathcal{M} \cup \mathscr{I}^+ \cup \mathscr{I}^-$ *with* $\omega > 0$ *on* $\mathcal{M} \cup \mathscr{I}^+ \cup \mathscr{I}^-$, $\tilde{\nabla}_\mu(\omega^{-1} n^\mu)$ *is complete (can be extended to any value of the parameter) on* $\mathscr{I}^+$ *and* $\mathscr{I}^-$*.*

From now on we shall reserve ourselves to the case of asymptotically flat space-times.

It follows from the last definition that in order to reflect the Killing symmetries to the adapted coordinates in the sense of (1.31), one has to at last choose the causal orientation of the normal to $\varepsilon = -1$, because then the normal will be proportional to the Killing vector η^μ arising from the stationarity of the space-time, thus, t will be representing the proper time of the asymptotically inertial observer at rest with respect to the source of curvature [19].

Definition 18 (Axisymmetric space-time)**.** *A space-time* $(\mathcal{M}, \boldsymbol{g})$ *is called* ***axisymmetric*** *if there exists a group* $\mathcal{G}_2$ *such that the group's action on* $\mathcal{M}$ *fulfills the following properties[6]:*

(i) $\mathcal{G}_2$ *is isomorphic to SO(2) (special orthogonal group)*

(ii) $\boldsymbol{g}$ *is invariant under the* $\mathcal{G}_2$ *group action, again, in differential form meaning that* $\mathcal{L}_{\boldsymbol{\xi}}\boldsymbol{g}$*, where* $\boldsymbol{\xi}$ *denotes the generator of* $\mathcal{G}_2$ *with parameter* ϕ*, which is taken to be periodic with* $\phi \in [0, 2\pi)$*, and* 2π *is identified with* 0*.*

In Carter's paper named "The commutation property of a stationary, axisymmetric system"[2], a theorem can be found, which states that under the condition of asymptotical flatness of the space-time and the points $(i)-(ii)$ from the previous definition, the set of points invariant under the group $\mathcal{G}_2$ action (therefore the set where $\boldsymbol{\xi}$ vanishes) is a time-like 2-dimensional subspace of $\mathcal{M}$. This set will be from now on referred to as the *axis of rotation* W_2. It follows that the tensor field

$$\mathcal{H}_{\mu\nu} := (\nabla_\mu \xi^\kappa)(\nabla_\nu \xi_\kappa)$$

is at any point $q \in W_2$ the projection tensor to the space orthogonal to W_2. To ensure Lorentzian geometry in the vicinity of W_2, the length of a $\boldsymbol{\xi}$ orbit passing through the point p in this vicinity should be at first relevant order equal to 2π times the distance between q, which is equivalent to [21]

$$X := \xi^\mu \xi_\mu, \text{ with the property } \lim_{x \to q} \left(\frac{X_{,\mu} X^{,\mu}}{4X} \right) = 1. \tag{2.1}$$

This regularity condition known as *elementary flatness* is assured if there exists an expansion of the $\boldsymbol{\xi}$ vector field's components around the point $q \in W_2$

$$\begin{aligned} &\xi^\mu(x^\nu) = (x^\nu - q^\nu)\nabla_\nu \xi^\mu|_q + \mathcal{O}(x-q)^2 \\ &\implies X := \xi^\mu \xi_\mu = (x^\mu - a^\mu)(x^\nu - q^\nu)\mathcal{H}_{\mu\nu} + \mathcal{O}(x-a)^3. \end{aligned}$$

If the elementary flatness condition is violated, there are conical singularities present on W_2. Therefore, the elementary flatness condition corresponds to a boundary condition on metric functions in the field equations.

To summarize, the stationarity and axisymmetry of a space-time is a sufficient condition for the existence of two Killing vector fields. The stationarity together with asymptotical flatness imply the existence of $\boldsymbol{\eta}$ in the above sense, therefore(at least for $\varepsilon = -1$) a well-behaved and in the standard sense time-like (at least asymptotically) vector field. As was briefly mentioned in Preliminaries, in 4-dimensional spacetimes $\varepsilon = -1 \implies \delta = +1$ in the sense of (1.47). Therefore, the second Killing field $\boldsymbol{\xi}$ is of space-like causal character and generates closed orbits around the axis of rotation, i.e. symmetry axis. In [2], it is further shown that no generality is lost by assuming that besides $(\mathcal{M}, \boldsymbol{g})$ being invariant under the group action of groups isomorphic to $(\mathbb{R}, +)$ and SO(2) separately, it is invariant under the action of $\mathbb{R} \times \mathrm{SO}(2)$ as well. Therefore, in the language of the group action generators, implying that $[\boldsymbol{\eta}, \boldsymbol{\xi}] \equiv 0$. This is equivalent to the existence of a holonomic basis (see [11]), where t, ϕ coincide with the generators of group $\mathcal{G}_1$ and $\mathcal{G}_2$ action, respectively

$$\boldsymbol{\partial}_t \equiv \boldsymbol{\eta}, \ \boldsymbol{\partial}_\phi \equiv \boldsymbol{\xi}. \tag{2.2}$$

This statement, when put into components, translates into partial derivatives of the $\boldsymbol{g}$ metric components only, henceforth yielding the equivalent $\frac{\partial g_{\mu\nu}}{\partial t} = \frac{\partial g_{\mu\nu}}{\partial \phi} \equiv 0$. In [7], it is stated that the general line element of a space-time possessing above mentioned properties can be, again without loss of generality, written in the following form, covering the complementary two directions with arbitrary coordinates[2] x^1, x^2

$$\boxed{ds^2 = e^{-2U} \left(\gamma_{MN} dx^M dx^N + W^2 d\phi^2 \right) - e^{2U} \left(dt + A d\phi \right)^2 ,} \tag{2.3}$$

[2]Let us note that these coordinates are not necessarily uniquely given. In particular, transforming the t and ϕ coordinates via adding an arbitrary function of ρ and z changes nothing regarding the Killing-ness. Even more so, one can equivalently choose (r, θ), meaning spherical-type coordinates.

where the metric functions U, γ_{MN}, W, and A depend only on the coordinates $x^M = (x^1, x^2)$ orthogonal to the orbits and t, ϕ denote the coordinates adapted to the space-time symmetries in the sense above [21]. In [9], a different form is assumed, although it can quickly be found to be equivalent. This line element's vacuum field equations (1) yield a system of nonlinear PDEs, although they are known to be integrable[7].

The above mentioned basis together with x^1, x^2 is fully holonomic in the case when the complementary two directions are locally orthogonal to both the Killing vectors $\boldsymbol{\eta}, \boldsymbol{\xi}$. Let us further precise this statement:

Definition 19 (Orthogonal transitivity, circular space-time)**.** *The* $\mathbb{R} \times \mathrm{SO}(2)$ *group action is called* ***orthogonally transitive*** *if, besides the given collection of 2-dimensional surfaces labeled unambiguously by* $x^1 = const.$ *and* $x^2 = const.$ *(distributions generated by the two Killing vector fields), there exists another collection of 2-dimensional surfaces such that the two collections are always orthogonal. A space-time region possessing the properties defined by Definition 16, Definition 17, Definition 18, in which the action of the* $\mathbb{R} \times \mathrm{SO}(2)$ *group is orthogonally transitive is referred to as a region of a* ***circular space-time****.*

Because one, from the Frobenius' theorem, knows that the property will be integrable if and only if the corresponding distribution generated by the two remaining coordinates' tangent vector fields is involutive, one is able to formulate the following theorem:

Theorem 8 (Sufficient conditions for circularity of a space-time)**.** *Let a space-time* $(\mathcal{M}, \boldsymbol{g})$ *be endowed with a lorentzian metric and the corresponding Levi-Civita type connection, which expresses the components of the corresponding covariant derivative* $\boldsymbol{\nabla}$*. Let* η^μ *and* ξ^μ *be two Killing vectors fields defined as above. The 2-dimensional surfaces everywhere orthogonal to* η^μ *and* ξ^μ *are integrable if*

$$\eta_{[\mu}\xi_\nu\nabla_\kappa\eta_{\lambda]} = \eta_{[\mu}\xi_\nu\nabla_\kappa\xi_{\lambda]} \equiv 0 \implies \eta^\alpha R_\alpha^{[\beta}\eta^\gamma\xi^{\delta]} = \xi^\alpha R_\alpha^{[\beta}\eta^\gamma\xi^{\delta]} \equiv 0.$$

Proof. The proof is here divided into a few steps, for it is quite lengthy.

(i) **Integrability and left side of the implication.** For the integrability statement, it is sufficient to only apply the Frobenius' theorem. According to the equivalence in the theorem, one has to check whether the exterior derivative of the generating Killing vector fields (its corresponding Killing 1-forms via the isometry defined by $\boldsymbol{g}$) can be written in the form from Definition 8 That is of course because the distributions are complementary in the sense of Definition 6. By employing the known theorem $\mathbf{d} = \boldsymbol{\nabla}_\wedge$, used frequently in this text, the integrability conditions follow in this form

$$2\nabla_{[\mu}\eta_{\nu]} = \Theta^1_{[\mu}\eta_{\nu]} + \Theta^2_{[\mu}\xi_{\nu]},$$

$$2\nabla_{[\mu}\xi_{\nu]} = \Theta^3_{[\mu}\eta_{\nu]} + \Theta^4_{[\mu}\xi_{\nu]},$$

which is of course equivalent to the left side of the implication, due to the properties of antisymmetrization.

(ii) **A needed property of a general Killing vector field.** Let k^μ denote a general Killing vector field. Let us once more use the Ricci identities, this time combined with the Killing equation

$$R^\sigma{}_{\nu\kappa\lambda}k_\sigma := k_{\nu;\kappa\lambda} - k_{\nu;\lambda\kappa} = k_{\nu;\kappa\lambda} + k_{\lambda;\nu\kappa}.$$

Writing this equation in all its cyclic permutations yields

$$R^\sigma{}_{\nu\kappa\lambda}k_\sigma = k_{\nu;\kappa\lambda} + k_{\lambda;\nu\kappa},$$

$$R^\sigma{}_{\lambda\nu\kappa}k_\sigma = k_{\lambda;\nu\kappa} + k_{\kappa;\lambda\nu},$$
$$R^\sigma{}_{\kappa\lambda\nu}k_\sigma = k_{\kappa;\lambda\nu} + k_{\nu;\kappa\lambda}.$$

Therefore adding the first and the third equation and subtracting the second one immediately gets

$$k_{\nu;\kappa\lambda} = -R^\sigma{}_{\lambda\nu\kappa}k_\sigma \implies k_{\nu;}{}^\kappa{}_\kappa = -R^\sigma_\nu k_\sigma.$$

(iii) **General formula for Killing vector field vorticity's covariant differential.** Left side of the implication can also be understood as being the inner product of one of the Killing vector fields and the other one's vorticity vector defined via Hodge star operator of the corresponding vorticity tensor (see Definition 13) as mentioned in Preliminaries. Therefore, for example, in the case of $\boldsymbol{\eta}$, one has $\omega^{(\boldsymbol{\eta})}_\mu = \frac{1}{2}\varepsilon_{\mu\nu\kappa\lambda}\eta^\nu\eta^{\kappa;\lambda}$, from which, by differentiating and again applying the Hodge dual, one gets

$$\star(\mathrm{d}_\beta\omega^{(\boldsymbol{\eta})}_\mu) = \frac{1}{4}\varepsilon^{\mu\beta\gamma\delta}\varepsilon_{\mu\nu\kappa\lambda}\nabla_{\beta\wedge}(\eta^\nu\eta^{\kappa;\lambda}).$$

Combining the fact that the tensor product of Levi-Civita tensors is proportional to the projector on the subspace of antisymmetric tensors[3] and the partial summation formula for the projector[4], it follows that

$$(\star\mathrm{d}\omega^{(\boldsymbol{\eta})})^{\gamma\delta} = -(\eta^\nu\eta^{\kappa;\lambda})_{;\beta}\,{}^{[3]}\delta^{\beta\gamma\delta}_{\nu\kappa\lambda} \overset{(ii)}{=} \eta^\gamma R^\delta_\sigma\eta^\sigma - \eta^\delta R^\gamma_\sigma\eta^\sigma.$$

Applying the Hodge star operator once more and using that in 4 dimensions $\star^2 = (-1)^{p+1}$, where p denotes the degree of the form being "Hodged", one finally arrives at (the form in the parentheses can be found in [8]):

$$\omega^{(\boldsymbol{\eta})}_{[\mu;\beta]} = \frac{1}{2}\varepsilon_{\beta\mu\gamma\delta}\eta^\delta R^\gamma_\sigma\eta^\sigma \;\left(\iff \mathbf{d}\boldsymbol{\omega}^{(\boldsymbol{\eta})} = \star[\boldsymbol{\eta}\wedge(\boldsymbol{Ric}\cdot\boldsymbol{\eta})]\right).$$

(iv) **The proof itself.** One has $[\boldsymbol{\eta},\boldsymbol{\xi}] = 0$, as has already been discussed above. This, by definition, implies that $\mathcal{L}_{\boldsymbol{\eta}}\boldsymbol{\xi} = \mathcal{L}_{\boldsymbol{\xi}}\boldsymbol{\eta} \equiv 0$. Therefore, by computing the exterior derivative (for example) of the expression $\xi^\nu\omega^{(\boldsymbol{\eta})}_\nu$, first using the well-known *Cartan identity* (for proof, see [11]), and then (again) the

[3] $\varepsilon^{a_1\ldots a_d}\varepsilon_{b_1\ldots b_d} = d!\ \mathrm{sign}(g)\,{}^{[d]}\delta^{a_1\ldots a_d}_{b_1\ldots b_d}$, for the proof, see [11] (note that $\varepsilon = -1$ is again needed here).

[4] ${}^{[k]}\delta^{a_1\ldots a_l r_1\ldots r_{k-l}}_{b_1\ldots b_k r_1\ldots r_{k-l}} = \frac{(k-l)!l!}{(d-k)!k!}\,{}^{[l]}\delta^{a_1\ldots a_l}_{b_1\ldots b_l}$, for the proof, see again [11] (d denotes the dimension of the vector space in question, denoted for example by V).

relationship between the exterior derivative and covariant differential, one can demonstrate, that indeed $\boldsymbol{\omega}^{(\boldsymbol{\eta})} = 0 \implies \eta^{\alpha} R^{[\beta}_{\alpha} \eta^{\gamma} \xi^{\delta]} = 0$:

$$\mathrm{d}_{\mu} \xi^{\nu} \omega^{(\boldsymbol{\eta})}_{\nu} = -\xi^{\nu} \mathrm{d}_{\mu} \omega^{(\boldsymbol{\eta})}_{\nu} = -2\xi^{\nu} \omega^{(\boldsymbol{\eta})}_{[\nu;\mu]} \overset{(iii)}{=} -\varepsilon_{\mu\nu\gamma\delta} \eta^{\sigma} R^{\gamma}_{\sigma} \xi^{\nu} \eta^{\delta}.$$

And because the second case (i.e. $\boldsymbol{\omega}^{(\boldsymbol{\xi})} = 0 \implies \xi^{\alpha} R^{[\beta}_{\alpha} \eta^{\gamma} \xi^{\delta]} = 0$) is completely analogous, the proof is therefore concluded.

□

Remark. In fact, some authors [8], [1] refer to the implied condition from the last theorem as *Ricci-circularity*, whereas here and for example in Wald's book [22], no special name is put forward since it is a corollary of circularity, as was proven above. Also note that by using (1) with $\Lambda = 0$, one can formulate an equivalent criterion via $T_{\mu\nu}$. It is also worth noting that the conditions of orthogonal transitivity are thanks to the right side of the implication always satisfied in vacuum regions of any asymptotically flat, stationary, and axisymmetric space-time.

2.2 Stationary axisymmetric space-times

This text is primarily interested in the vacuum solutions. We shall due to Theorem 8 refer in this chapter with mediocre loss of generality to circular space-times instead. It is self-explanatory that circular space-times are a subset of asymptotically flat, stationary, and axisymmetric space-times. In this case, when the two Killing vector fields satisfy the assumptions of Theorem 8, one can choose the remaining two coordinates x^1, x^2, such that the line element without loss of generality takes the ansatz [1], [10], [19], [22]:

$$\boxed{ds^2 = -N^2 \mathrm{d}t^2 + g_{\phi\phi}(\mathrm{d}\phi - \omega \mathrm{d}t)^2 + g_{AB} \mathrm{d}x^A \mathrm{d}x^B.} \tag{2.4}$$

We are denoting the coordinate chart again by $\{x^{\mu}\}^{3}_{\mu=0}$, like in Preliminaries, thus $\{x^{A}\}^{2}_{A=1}$ represent the rest of the coordinate chart chosen to cover the (integrable) 2-surfaces orthogonal to the two Killing vector fields $\boldsymbol{\eta}, \boldsymbol{\xi}$, whereas coordinates t, ϕ are adapted to the Killing symmetries, in the sense that, for example in components, one has:

$$\eta^{\mu} := \frac{\partial x^{\mu}}{\partial t}, \; \xi^{\mu} := \frac{\partial x^{\mu}}{\partial \phi}. \tag{2.5}$$

The metric functions thus read [1], [19]

$$g_{tt} = g_{\mu\nu} \eta^{\mu} \eta^{\nu}, \;\; g_{\phi\phi} = g_{\mu\nu} \xi^{\mu} \xi^{\nu}, \;\; g_{t\phi} = g_{\mu\nu} \eta^{\mu} \xi^{\nu}. \tag{2.6}$$

Above, in the line element (2.4), we have abbreviated

$$\omega := -\frac{g_{t\phi}}{g_{\phi\phi}} = -\frac{\eta_{\mu} \xi^{\mu}}{\xi_{\mu} \xi^{\mu}}, \;\; N^2 := -(g_{tt} + g_{t\phi}\omega) \equiv -(\eta_{\mu} + \omega \xi_{\mu})(\eta^{\mu} + \omega \xi^{\mu}). \tag{2.7}$$

2.2.1 Notes on the Kerr metric

The vacuum Kerr metric is the most astrophysically significant representative of the class of circular space-times. The metric includes two functions (M, a) which, under coordinate transformations, behave like scalars, i.e. those are the parameters of the metric and its interpretations are well known. The Kerr metric is obviously a special case of (2.4), more generally of (2.3). The remaining two non-Killing coordinates can most notably be chosen as either of spheroidal type (r, θ) known as the *Boyer-Lindquist coordinates*, or cylindrical type (ρ, z) known as the *Kerr-Schild cylindrical coordinates.* The line element in the Boyer-Lindquist coordinates (t, r, θ, ϕ) reads

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + g_{\phi\phi}(\mathrm{d}\phi - \omega\mathrm{d}t)^2 + \frac{\Sigma}{\Delta}\mathrm{d}r^2 + \Sigma\mathrm{d}\theta^2, \tag{2.8}$$

with the following common abbreviations that are frequently used

$$\Sigma := r^2 + a^2\cos^2\theta,$$

$$\Delta := r^2 - 2Mr + a^2,$$

$$g_{\phi\phi} = \frac{\mathcal{A}}{\Sigma}\sin^2\theta,$$

$$\mathcal{A} := \Sigma\Delta + 2Mr(r^2 + a^2).$$

The Killing-constructed functions ω and N are in these coordinates specifically of the form

$$\omega := -\frac{\eta_\mu\xi^\mu}{\xi_\mu\xi^\mu} = \frac{2Mar}{\mathcal{A}}, \quad N^2 := -(\eta_\mu + \omega\xi_\mu)(\eta^\mu + \omega\xi^\mu) = \frac{\Sigma\Delta}{\mathcal{A}}.$$

The fact that the letter N is used again in (2.8), in conflict with the normalisation constant of the (for $\varepsilon = -1$) time-like normal from Preliminaries, is no coincidence. Let us recall that in Preliminaries it played the role of the norm of the time-like Killing vector $\boldsymbol{\eta}$. One could further verify that the function is actually the same in both cases by evaluating first the four-velocity of observers circularly orbiting the surfaces given by $r = const.$, $\theta = const.$ with time-independent angular velocity defined as $\Omega = \frac{\mathrm{d}\phi}{\mathrm{d}t}$ specifically in the Boyer-Lindquist coordinates

$$u^\mu = \frac{\eta^\mu + \Omega\xi^\mu}{|\eta^\mu + \Omega\xi^\mu|} = \frac{1}{\sqrt{N^2 - g_{\phi\phi}(\Omega - \omega)^2}}(\delta^\mu_0 + \Omega\delta^\mu_3),$$

and second, the corresponding four-acceleration, because for the case $\Omega = \omega$, which corresponds to the known *ZAMO* congruence, it comes out in the covariant form as

$$a_\mu = \frac{N_{,\mu}}{N}.$$

This exactly corresponds to (1.32), because above we have already chosen ε to be -1, therefore the norm of the Killing vector $\boldsymbol{\eta}$ is proportional to the Lapse function. The above mentioned fact is very well known so let us not derive it completely and focus on the consequences instead. Because the ZAMO congruence is at rest with respect to the rotating space-time geometry characterized by the dragging angular velocity ω, the covariant four-velocity of the congruence is given only by the temporal part, thus, the four-velocity is orthogonal to any vector that possesses only spatial components. That is because, by construction in Preliminaries, any vector that has only spatial components is at the same time tangent to the hypersurfaces given by $t = const.$, i.e. the world-lines of the ZAMO congruence are orthogonal to Σ_t. Nevertheless, various other properties of the Kerr metric that were omitted here can be found for example in [1].

Let us, for completeness, at least state what form the line element takes in the aforementioned Kerr-Schild cylindrical coordinates (T, ρ, z, ψ) [1]

$$\mathrm{d}s^2 = -\mathrm{d}T^2 + \mathrm{d}\rho^2 + \rho^2\mathrm{d}\psi^2 + \mathrm{d}z^2 + \frac{2Mr^3}{r^4 + a^2z^2}\left(\mathrm{d}T + \frac{r\rho\mathrm{d}\rho + a\rho^2\mathrm{d}\psi}{r^2 + a^2} + \frac{z\mathrm{d}z}{r}\right)^2, \tag{2.9}$$

where r is an oblate radius given as a solution to the fourth degree algebraic equation (therefore the definition depends on the relationship between ρ, a and z)

$$r^4 + r^2(\rho^2 + a^2 + z^2) - a^2z^2 = 0.$$

One can, from this form of the line element, transform back to the Boyer-Lindquist coordinates with the aid of relations

$$\mathrm{d}T = \mathrm{d}t - \frac{2Mr}{\Delta}\mathrm{d}r,\ \mathrm{d}\psi = \mathrm{d}\phi - \frac{2Mar}{\Delta(r^2 + a^2)}\mathrm{d}r,\ \rho = \sqrt{r^2 + a^2\sin^2\theta},\ z = r\cos\theta.$$

2.2.2 3+1 decomposition of circular space-times

Because of the Killing-ness of the coordinates in circular space-times, one is able to say quite a lot about the 3+1 splitting of circular space-times invariantly, using the Killing vector adapted coordinates. This problem is treated in [19] and is only adopted here because of the proof that for a circular space-time, the slicing with respect to the Killing time coordinate t leaves the resulting submanifold Σ_t extremal, in the sense described in detail in Chapter 3.

Let us define the normal vector n^μ to the hypersurfaces of the constant Killing time t by a unit component of η^μ orthogonal to ξ^μ. The norm can be immediately obtained from (1.31) and the afterward discussion, considering that from there it follows that $n_\mu\eta^\mu = -N$ [10], [19]

$$n^\mu := \frac{1}{N}(\eta^\mu + \omega\xi^\mu). \tag{2.10}$$

The corresponding acceleration of the n^μ field immediately follows and yields the same expression as in the previous chapter or in Preliminaries. One also has the orthogonality relations somewhat analogous to (1.4)

$$n_\mu\xi^\mu = a_\mu\eta^\mu = a_\mu\xi^\mu \equiv 0, \tag{2.11}$$

and the projections of the Killing vectors onto Σ_t may be explicitly evaluated to

$$\eta^\nu h^\mu_\nu \overset{(12)}{=} \eta^\nu\left[\delta^\mu_\nu + \frac{1}{N^2}(\eta^\mu + \omega\xi^\mu)(\eta_\nu + \omega\xi_\nu)\right] \overset{(2.11)}{=} -\omega\xi^\mu \tag{2.12}$$

in the case of $\boldsymbol{\eta}$ and to

$$\xi^\nu h^\mu_\nu \overset{(12)}{=} \xi^\nu\left[\delta^\mu_\nu + \frac{1}{N^2}(\eta^\mu + \omega\xi^\mu)(\eta_\nu + \omega\xi_\nu)\right] \overset{(2.11)}{=} \xi^\mu \tag{2.13}$$

in the case of $\boldsymbol{\xi}$. One can now, using the relations above, evaluate the the components of the extrinsic curvature tensor of Σ_t to be [10], [19]

$$K_{\mu\nu} := n_{(\alpha;\beta)} h^{\alpha}_{\mu} h^{\beta}_{\nu} = \frac{1}{N}\left[\eta_{(\alpha;\beta)} - N_{;(\beta} n_{\alpha)} + \omega \xi_{(\alpha;\beta)} + \omega_{;(\beta}\xi_{\alpha)}\right] h^{\alpha}_{\mu} h^{\beta}_{\nu} = \frac{1}{N}\xi_{(\mu}\omega_{,\nu)}. \tag{2.14}$$

Therefore, the trace of the extrinsic curvature tensor of Σ_t comes out in a general asymptotically flat stationary and axisymmetric space-time as

$$K := K_{\mu\nu}h^{\mu\nu} - \frac{1}{N}\omega_{,\nu}\xi^{\nu} \equiv 0, \tag{2.15}$$

which, according to the discussion following (2.2), is identically equal to zero, because of the assumption that the space-time in question is axisymmetric, thus, $\boldsymbol{g}$ (and thus the function ω) is assumed not to depend on the coordinate ϕ. Moreover, for example for the Kerr metric, this means that the ZAMO congruence's tangent vector covariant differential decomposition has only the part given by the shear tensor (see (1.3)).

The fact that the trace of $\boldsymbol{K}$ is indeed in general identically equal to zero will be very important in the forecoming chapters, because from this result, one concludes that Σ_t is an extremal submanifold of $\mathcal{M}$. We now refer to [10], [19], where one can find expressions for all the main terms, which played a role in the general 3+1 decomposition discussed in the preceding chapters, only using the two Killing vector fields $\boldsymbol{\eta}, \boldsymbol{\xi}$ of a general stationary and axisymmetric space-time. This shall be put to a further use in the upcoming research.

2.3 Static axisymmetric space-times

2.3.1 Weyl metrics

The family of Weyl metrics is an important example of static axisymmetric space-times. Before introducing it, let us precise the staticity notion

Definition 20 (Static space-time)**.** *A space-time $(\mathcal{M}, \boldsymbol{g})$ is called static if the following holds[6]:*

(i) the space-time is stationary in the sense of Definition 16,

(ii) the time-like Killing vector $\boldsymbol{\eta}$ associated with stationarity generates a distribution, whose complementary distribution in the sense of Definition 6 and the isometry defined by $\boldsymbol{g}$ is involutive[5].

Given an axisymmetric space-time that satisfies the conditions above, the function A in the line element (2.3) can be put to zero [21][6] and the line element takes the following form (x^1 and x^2 again denote the coordinates chosen to cover the complementary two directions to $\boldsymbol{\eta}, \boldsymbol{\xi}$)

$$\mathrm{d}s^2 = -\mathrm{e}^{2\nu}\mathrm{d}t^2 + \mathrm{e}^{-2\nu}\left[\mathrm{e}^{2\lambda}\left((\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2\right) + \rho^2\mathrm{d}\phi^2\right], \tag{2.16}$$

[5]Equivalently, because of the Frobenius' theorem and the corollary Theorem 5, the space-time is called static if the Killing vector field $\boldsymbol{\eta}$ associated with the stationarity property is orthogonal to a family of hypersurfaces, here Σ_t.

[6]This can be better seen from the form (2.4) and the definition of the normal (2.10), because if $\boldsymbol{n}$ and $\boldsymbol{\eta}$ are to be colinear, $\omega \equiv 0$ has to hold.

where ρ, ν and λ are now functions of only x^1, x^2. The resulting metric describes a static axially symmetric space-time for any choices of the functions ν, λ [7]. Analogously as in the Kerr metric, one could cover the rest of the space-time manifold with spherical-type coordinates, or cylindrical-type coordinates. The cylindrical-type coordinates are commonly being called *Weyl's canonical coordinates* ρ, z, because with $\Lambda = 0$ the vacuum field equations (1) imply that [7]

$$\begin{aligned} &\nu_{,\rho\rho} + \frac{1}{\rho}\nu_{,\rho} + \nu_{,zz} = 0, \\ &\lambda_{,\rho} = \rho\left[(\nu_{,\rho})^2 + (\nu_{,z})^2\right], \\ &\lambda_{,z} = 2\rho\nu_{,\rho}\nu_{,z}. \end{aligned} \tag{2.17}$$

Keeping in mind the classical Newtonian limit of GR as in [1] $g_{tt} \sim -1 - 2\Phi$, where Φ denotes the standard Newtonian gravitational potential and the fact that the function ν obeys the *Laplace's equation* for an axially symmetric function (2.17), one can regard the function ν as the analogue of a classical Newtonian gravitational potential. This statement is further reinforced by the fact that the choice $\nu \equiv \lambda \equiv 0$ compatible with (2.17) yields the line element of the Minkowski metric in cylindrical coordinates[7]. The line element in the Weyl canonical coordinates takes the form [7], [21]

$$\mathrm{d}s^2 = -\mathrm{e}^{2\nu}\mathrm{d}t^2 + \mathrm{e}^{-2\nu}\left[\mathrm{e}^{2\lambda}(\mathrm{d}\rho^2 + \mathrm{d}z^2) + \rho^2\mathrm{d}\phi^2\right]. \tag{2.18}$$

Whereas the other version of the line element that arises when one decides to cover the complementary directions to the ones given by $\boldsymbol{\eta}, \boldsymbol{\xi}$ by spherical-type coordinates of course takes with aid of the transformation $\{\rho \to r\sin\theta, z \to r\cos\theta\}$ the form [7]

$$ds^2 = -\mathrm{e}^{2\nu}\mathrm{d}t^2 + \mathrm{e}^{-2\nu}\left[\mathrm{e}^{2\lambda}(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2) + r^2\sin\theta^2\mathrm{d}\phi^2\right]. \tag{2.19}$$

Therefore the Λ-less vacuum Laplace field equation (2.17) is in the spherical coordinates of the following form

$$\nu_{,rr} + \frac{2}{r}\nu_{,r} + \frac{1}{r^2}\nu_{,\theta\theta} + \frac{\cot\theta}{r^2} = 0. \tag{2.20}$$

An important realisation so far is that the transformation between the cylindrical and spherical version of the coordinate system leaves the Killing coordinates t, ϕ intact, therefore both versions are, in this sense, in fact equally canonical and can easily be switched around back and forth. Because of this, we shall from now on refer to these two coordinate systems in short as *cylindrical Weyl coordinates*, respectively *spherical Weyl coordinates*.

The analogy of the function ν and the Newtonian gravitational potential prompts a very interesting solution-generating procedure based on taking ν to be an exact Newtonian gravitational potential and using (2.17) to evaluate λ, because one can then interpret the resulting solution's properties within the analogy of the chosen Newtonian gravitational potential.

[7] [7] mentions that the choice $(\nu = \log(\rho),\ \lambda = \log(\rho))$ and even another one of the form $\left(\nu = \frac{1}{2}\log\left(\sqrt{\rho^2+z^2}+z\right),\ \lambda = \frac{1}{2}\log\left(\frac{\sqrt{\rho^2+z^2}+z}{2\sqrt{\rho^2+z^2}}\right)\right)$ provide a flat solution as well. For the sake of simplicity, this fact has not been put into a serious consideration, since these two line elements are related to a uniformly accelerated metric.

2.3.2 Majumdar-Papapetrou solution and its disc

An important example of the solution-generating procedure is the Majumdar-Papapetrou solution, which can be obtained by considering that the factor multiplying $\mathrm{d}t^2$ in (2.18) is actually the lapse function N, as one knows from Notes on the Kerr metric (thus $\nu = \ln N$). It was even previously established that the Lapse function follows the same trend as the Lorentz factor and should therefore, far apart from a source of curvature, tend to unity. Inspired by the above thoughts, one may choose the following potential

$$\frac{1}{N} = 1 + \sum_{j=1}^{n} \frac{M_j}{|\vec{r} - \vec{r}_j|} \implies \nu(r) = -\ln\left(1 + \sum_{j=1}^{n} \frac{M_j}{|\vec{r} - \vec{r}_j|}\right), \tag{2.21}$$

therefore a potential that in the first order corresponds to the exact classical newtonian potential of point particles (in fact extremal black holes) of masses M_j situated at positions $\vec{r}_j$. In order for the solution to be stable and achieve equilibrium, one has to introduce a non-zero stress energy tensor generating an electromagnetic field such that the electromagnetic repulsion cancels out the gravitational attraction completely, namely $|Q_i| = M_i \; \forall i \in \{1, ..., n\}$[8]. Recall that for a Reissner-Nordström black hole, the locations of the two horizons are in Schwarzschild-type coordinates $r_\pm = M + \sqrt{M^2 - Q^2}$, therefore the extremality condition is that $|Q| = M$. This implies that the corresponding electromagnetic field is given by the potential that has the temporal component $A_\mu = N\delta^0_\mu = \mathrm{e}^\nu \delta^0_\mu$ [17], thus, the solution is not vacuum.

By virtue of the space-time's axisymmetry, it's natural to consider a special constellation of the extreme Reissner-Nordström black holes compatible with Weyl cylindrical coordinates - ring of radius a (like in the Kerr case), lying in the $z = 0$ plane and centered at the origin. Consider a point in space with coordinates (ρ, ϕ, z). A point on the ring can then be parameterized by the angle ϕ_0 and the coordinates of any point on the ring are $(a, \phi_0, 0)$. The denominator of the sum term in (2.21) then takes the form

$$\begin{aligned}|\vec{r} - \vec{r}_0| &= \sqrt{(\rho\cos\phi - a\cos\phi_0)^2 + (\rho\sin\phi - a\sin\phi_0)^2 + z^2} = \\ &= \sqrt{\rho^2 + a^2 - 2a\rho(\cos\phi\cos\phi_0 + \sin\phi\sin\phi_0) + z^2} = \\ &= \sqrt{\rho^2 + a^2 + 2a\rho\cos(\phi - \phi_0) + z^2}.\end{aligned}$$

Assuming that M denotes the total mass of the ring, by limit transition one gets the following form of the choice of the potential in (2.18)

$$\nu(\rho, z) \equiv \ln\left(\frac{1}{N}\right) = -\ln\left(1 + \frac{M}{2\pi}\int_0^{2\pi} \frac{1}{\sqrt{\rho^2 + a^2 + 2a\rho\cos(\phi - \phi_0) + z^2}}\mathrm{d}\phi_0\right).$$

The integral term can be manipulated further if one recalls that the *complete elliptic integral of the first kind* is defined as

$$K(k) := \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - k^2\sin^2\alpha}}\mathrm{d}\alpha, \tag{2.22}$$

[8]This would read $|Q| = M\sqrt{4\pi\epsilon_0 G}$ in non-geometrized units.

therefore, with abbreviations

$$\ell_\pm := \sqrt{(\rho \pm a)^2 + z^2}, \tag{2.23}$$

the potential function ν takes the following form

$$\nu(\rho, z) = -\ln\left(1 + \frac{2MK(k)}{\pi \ell_+}\right), \tag{2.24}$$

where the argument of the complete elliptic integral of the first kind (modulus) is equal to

$$k^2 := 1 - \left(\frac{\ell_-}{\ell_+}\right)^2 \overset{(2.23)}{=} \frac{4a\rho}{(\rho + a)^2 + z^2}. \tag{2.25}$$

Given that the complete elliptic integral of the first kind is a special function, one cannot, in general, simplify this result further. The only analytic simplification happens on the axis $\rho = 0$, where $\nu = -\ln\left(1 + \frac{M}{\sqrt{z^2+a^2}}\right)$. From this point, it is straightforward to verify that the metric (2.18) with ν given by (2.24) and the choice of the metric function $\lambda \equiv 0$ is in fact a solution of the Einstein-Maxwell equations, c.f. [7], [12], [15], [17], [18].

2.3.3 Levi-Civita metric

Another simple exact Newtonian potential that can be considered to generate a solution could be that of an infinite uniform line source, with mass per length density denoted by σ, which is known to solve the Laplace equation. Therefore, the ν metric function, in this case obviously independent of z, would read

$$\nu(\rho) = 2\sigma \log(\rho), \tag{2.26}$$

by inserting the exact potential into the rest of the vacuum field equations (2.17), which are in this case of the form

$$4\sigma^2 = \rho\lambda_{,\rho}, \; 4\sigma^2 + \rho^2(\lambda_{,\rho\rho} + \lambda_{,zz}) = 0,$$

one finds a valid choice for the metric function λ also independent of z

$$\lambda(\rho) = 4\sigma^2 \log(\rho) + \log(k), \tag{2.27}$$

where k is a constant, which puts the line element in the form

$$\mathrm{d}s^2 = -\rho^{4\sigma}\mathrm{d}t^2 + k^2\rho^{4\sigma(2\sigma-1)}(\mathrm{d}\rho^2 + \mathrm{d}z^2) + \rho^{2(1-2\sigma)}\mathrm{d}\phi^2. \tag{2.28}$$

The corresponding space-time appears to be, in addition to axisymmetric, even cylindrically symmetric (there exists an additional Killing vector arising from the translational symmetry). Considering the spherical version of the metric as well as for the rest of the space-times mentioned here does not make much sense in this context. For further information about the space-time and its interpretations, see [20].

2.3.4 The Weyl solutions

Another solution-generating procedure possibility is: consider the line element (2.19) in spherical coordinates and solve the axisymmetric Laplace equation explicitly (2.20) via separation of variables to obtain the general form of the solution for the ν metric function, assuming asymptotic flatness. This approach in the angular part yields, after a substitution, provided that the separation constant κ obeys $\kappa = n(n+1)$, the *Legendre differential equation* of the following general form for a function $y(x)$

$$(1-x^2)y_n''(x) - 2xy_n'(x) + n(n+1)y_n(x) = 0,$$

the solutions of which are the well-known *Legendre polynomials* in the argument $\cos(\theta)$, given for example by the Rodrigues formula in the argument x as

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left((x^2-1)^n\right).$$

The radial part yields a standard ordinary Euler differential equation of the second order. After enforcing the asymptotical flatness property to eliminate one of the integration constants, the general solution becomes

$$\nu(r,\theta) = -\sum_{n=0}^{\infty} a_n r^{-(n+1)} P_n(\cos\theta), \tag{2.29}$$

where the coefficients a_n can be found to have, in classical Newtonian theory, a correspondence with the sequence of multipole moments [7]. The general expression for λ obeying the field equations is then, using the properties of the Legendre polynomials to simplify the field equations in the process, the following [7]

$$\lambda = -\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} a_l a_m \frac{(l+1)(m+1)}{(l+m+2)} \frac{(P_l(\cos\theta)P_m(\cos\theta) - P_{l+1}(\cos\theta)P_{m+1}(\cos\theta))}{r^{l+m+2}}. \tag{2.30}$$

The simplest member of this family of solutions resembles a one-particle case of the Majumdar-Papapetrou solution.

2.3.5 Curzon-Chazy solution

As has already been mentioned, the Curzon-Chazy solution is the simplest member of the family of the Weyl solutions. With the choices of coefficients $a_0 = M$, $\forall i \geq 1 : a_i \equiv 0$ (a_0 corresponds to monopole moment) the metric functions read

$$\nu = -\frac{M}{r}, \ \lambda = -\frac{M^2\sin^2(\theta)}{2r^2}, \tag{2.31}$$

therefore, this solution formally arises when, in the solution-generating procedure, one takes ν to be a spherically symetric Newtonian potential of a point-particle located at $r = 0$, although, the resulting space-time is evidently not spherically symmetric and constains a naked singularity [7]. The line element in the spherical coordinates reads

$$\mathrm{d}s^2 = -e^{-\frac{2M}{r}}\mathrm{d}t^2 + e^{2(\frac{M^2\sin^2\theta}{2r^2} - \frac{M}{r})}(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2) + e^{\frac{2M}{r}} r^2 \sin^2\theta \mathrm{d}\phi^2. \tag{2.32}$$

It is straight-forward to transform the solution to the cylindrical coordinates. The naked singularity is still located at $\rho = 0,\ z = 0$ and the line element obviously takes the form

$$\mathrm{d}s^2 = -e^{-\frac{2M}{\sqrt{\rho^2+z^2}}}\mathrm{d}t^2 + e^{2\left(\frac{M^2\rho^2}{2(\rho^2+z^2)^2} - \frac{M}{\sqrt{\rho^2+z^2}}\right)}\left(\mathrm{d}\rho^2 + \mathrm{d}z^2\right) + e^{\frac{2M}{\sqrt{\rho^2+z^2}}}\rho^2\mathrm{d}\phi^2. \quad (2.33)$$

For additional information about the Curzon-Chazy solution, reach out to [21], or [7].

3 Formulation of the problem

In this chapter, we will formally state the main problem addressed in this thesis, which revolves around finding minimal 2-dimensional submanifolds within a given (circular) space-time context. Consider a stationary, asymptotically flat space-time where the 3+1 decomposition described in Preliminaries with respect to the Killing time vector $\boldsymbol{\eta}$ exists. Additionally, as outlined in the chapter about further decomposition, assume the existence of an alternative, different foliation of space-time denoted by Σ_s. This foliation necessarily intersects with the Σ_t foliation, resulting in a family of 2-dimensional surfaces Σ_{ts} within Σ_t. It was proven in the previous chapter using coordinates adapted to the first foliation that the extrinsic curvature tensor of Σ_t yields trace equal to zero, i.e. the hypersurface Σ_t yields *zero mean curvature* H. This implies that Σ_t is an extremal submanifold of the space-time manifold $\mathcal{M}$, as we shall see for the 2-dimensional case in this chapter. That means, loosely speaking, that it locally cannot be perturbed without increasing/decreasing its area (for minimal/maximal hypersurface), therefore the signature-non-dependent unifying condition is that Σ_t is the stationary point of an area functional. As a consequence of extremality of the hypersurface the expressions involving K in the 3+1 decomposition simplify considerably. This, provided that one is then able to find a closed-form prescription for the adapted coordinates, could have some implications: for example further simplification of the expression for the 2+1+1 decomposition of the Kretschmann scalar obtained in [19] and partly verified by [10]. The argument about the implications of this can be found in both the above-mentioned cited works.

3.1 Geometry of surfaces in 3D

Because the first decomposition is trivial in a circular space-time regarding the issue of extremality, let us concentrate on the case where $\mathcal{M}$ is 3-dimensional. It follows that if one was to construct some adapted coordinates $\{x^a\}_{a=1}^3$ such that setting a specific coordinate x^1 to a constant would yield an extremal 2-dimensional surface, the 2+1+1 decomposition of Σ_t hypersurfaces would simplify similarly as in the 3+1 case. The unit normal vector field would then be defined as

$$\boldsymbol{r} = \frac{1}{\sqrt{|\boldsymbol{e}_2 \times \boldsymbol{e}_3|}} \left(\boldsymbol{e}_2 \times \boldsymbol{e}_3 \right). \tag{3.1}$$

Here $\boldsymbol{e}_2, \boldsymbol{e}_3$ denote the coordinate holonomic basis corresponding to the rest of the adapted coordinates chart (x^2, x^3) and the cross denotes the ordinary vector product known from $\mathbb{R}^3$, because Σ_t is a 3-dimensional Riemannian manifold. We shall investigate the further meaning of the extrinsic curvature tensor of a surface embedded in 3D to obtain the *Weingarten formula*. For clarity, abstract indices are omitted in this section. Denote now by $\{y^A\}_{A=2}^3$ the intrinsic coordinates of the 2-dimensional Σ_{ts} surface in the sense of Lemma 1. The immersion/embedding of Σ_{ts} into Σ_t is thus fully described by the relations $x^a(y^A)$. As above, denote the basis vectors of the holonomic basis by

$$\boldsymbol{e}_2 := \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} y^2}, \boldsymbol{e}_3 := \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} y^3}.$$

Furthermore, to derive the Weingarten formula, consider a parameterized curve $z(\alpha) \in \Sigma_{ts}$ parameterized in such a way that $z(0) = x \in \Sigma_{ts}$ by some parameter $\alpha \in I \subset \mathbb{R}$. The coordinate expression of the curve is from this point being shortened as $y^A(z(\alpha)) \equiv y^A(\alpha)$. The components of the tangent vector $\boldsymbol{t}$ to the curve $z(\alpha)$ at point $x \in \Sigma_{ts}$ read

$$t^A = \frac{\mathrm{d}}{\mathrm{d}\alpha}\left[y^A(\alpha)\right]\Big|_{\alpha=0}, \tag{3.2}$$

whereas the components of the principal normal vector, respectively its normalized version denoted by $\boldsymbol{p}$ read

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}t^A = \frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}\left[y^A(\alpha)\right]\Big|_{\alpha=0},\ p^A = \frac{1}{\zeta}\frac{\mathrm{d}}{\mathrm{d}\alpha}t^A. \tag{3.3}$$

If one denotes by ϑ the angle between the two unit vectors $\boldsymbol{r}, \boldsymbol{p}$, the following equation follows

$$\boldsymbol{p}\cdot\boldsymbol{r} = \cos\vartheta \overset{(3.3)}{\Longrightarrow} \zeta\cos\vartheta = \frac{\mathrm{d}}{\mathrm{d}\alpha}\Big|_{\alpha=0}\boldsymbol{t}\cdot\boldsymbol{r}, \tag{3.4}$$

from which, by explicitly expressing the components of the principal normal vector, one gets

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\Big|_{\alpha=0}\boldsymbol{t} = \frac{\mathrm{d}^2y^A}{\mathrm{d}\alpha^2}\Big|_{\alpha=0}\boldsymbol{e}_A + \boldsymbol{e}_{A,B}\frac{\mathrm{d}y^A}{\mathrm{d}\alpha}\Big|_{\alpha=0}\frac{\mathrm{d}y^B}{\mathrm{d}\alpha}\Big|_{\alpha=0}. \tag{3.5}$$

Since $\boldsymbol{e}_A\cdot\boldsymbol{r} = 0$ by definition, one has:

$$\zeta\cos\vartheta \overset{(3.4)}{=} (\boldsymbol{e}_{A,B}\cdot\boldsymbol{r})\frac{\mathrm{d}y^A}{\mathrm{d}\alpha}\Big|_{\alpha=0}\frac{\mathrm{d}y^B}{\mathrm{d}\alpha}\Big|_{\alpha=0},$$

where the expression in brackets is independent of the curve $z(\alpha)$ and a property of Σ_{ts} alone. It is actually equal to the components of the extrinsic curvature tensor of Σ_{ts} denoted analogously to Preliminaries by L_{AB} [3]:

$$\boldsymbol{L} = \boldsymbol{e}_{A,B}\cdot\boldsymbol{r}\ \mathbf{d}y^A\mathbf{d}y^B. \tag{3.6}$$

Because by virtue of (3.1) we assume $\boldsymbol{r}$ to be normalized, one can thanks to the normalization (see (1.51)) express the extrinsic curvature tensor equivalently as

$$\boldsymbol{L} = -\boldsymbol{e}_A\cdot\boldsymbol{r}_{,B}\mathbf{d}y^A\mathbf{d}y^B. \tag{3.7}$$

The Weingarten formula can now be obtained by differentiating the normalization relation

$$(\boldsymbol{r}\cdot\boldsymbol{r} = 1)_{,A} = 0 \implies \boldsymbol{r}_{,A} = M^B_A\boldsymbol{e}_B \overset{(3.7)}{\Longrightarrow} \boldsymbol{r}_{,A} = -L^B_A\boldsymbol{e}_B. \tag{3.8}$$

That is, knowing the meaning of covariant differentiation ${}^{(2)}\boldsymbol{\nabla}$ associated with the Levi-Civita type connection on tangent and cotangent bundles of Σ_{ts} with respect to which the metric $\boldsymbol{f}$ is covariantly constant, it is not difficult to derive that in general (see Figure 1.2 for the correspondence)

$${}^{(2)}\boldsymbol{\nabla}_A\boldsymbol{r} = -L^B_A\boldsymbol{e}_B. \tag{3.9}$$

Let us continue further, to preparations of the proof that extremal 2-surfaces yield zero mean curvature H. First, we consider x^a to be describing an equilibrium shape. Second, we consider a strictly normal variation scaled by a sufficiently smooth (including its derivatives) function $\psi(y^A)$:

$$\boldsymbol{x} \to \boldsymbol{x} + \delta\boldsymbol{x} = \boldsymbol{x} + \psi\boldsymbol{r},$$

now, assuming that the operation of variation commutes with partial differentiation, one can express the variation of a tangent holonomic basis vector directly from the definitions [3], [24]

$$\begin{aligned} \delta\boldsymbol{e}_A := \delta\left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} y^A}\right) = \delta\left(\frac{\partial x^a}{\partial y^A}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^a}\right) = \frac{\partial}{\partial y^A}\delta(x^a)\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^a} = \frac{\partial(\psi\, r^a)}{\partial y^A}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^a} \overset{(3.8)}{=} \\ \overset{(3.8)}{=} \psi_{,A}\boldsymbol{r} - \psi L^B_A \boldsymbol{e}_B. \end{aligned} \tag{3.10}$$

Knowing this, the induced metric variation follows directly from the corresponding definition in the form of $\delta f_{AB}(\boldsymbol{x}) := f_{AB}(\boldsymbol{x}+\delta\boldsymbol{x}) - f_{AB}(\boldsymbol{x})$

$$\begin{aligned} \implies \delta f_{AB} &= \frac{\partial\boldsymbol{x}+\delta\boldsymbol{x}}{\partial y^A}\cdot\frac{\partial\boldsymbol{x}+\delta\boldsymbol{x}}{\partial y^B} + -\frac{\partial\boldsymbol{x}}{\partial y^A}\cdot\frac{\partial\boldsymbol{x}}{\partial y^B} = \\ &= \frac{\partial\delta\boldsymbol{x}}{\partial y^A}\cdot\frac{\partial\boldsymbol{x}}{\partial y^B} + \frac{\partial\boldsymbol{x}}{\partial y^A}\cdot\frac{\partial\delta\boldsymbol{x}}{\partial y^B} + \frac{\partial\delta\boldsymbol{x}}{\partial y^A}\cdot\frac{\partial\delta\boldsymbol{x}}{\partial y^B} = \\ &= \delta\boldsymbol{e}_A\cdot\boldsymbol{e}_B + \boldsymbol{e}_A\cdot\delta\boldsymbol{e}_B + \delta\boldsymbol{e}_A\cdot\delta\boldsymbol{e}_B \underset{(3.1)}{\overset{(3.10)}{=}} \\ &\underset{(3.1)}{\overset{(3.10)}{=}} -2\psi L_{AB} + \psi_{,A}\psi_{,B} + \psi^2 L^C_A L_{CB}, \end{aligned} \tag{3.11}$$

we see that the variation of the induced metric terminates after the second order. Note that the metric trace of the expression in the sense $f^{AB}\delta f_{AB}$ up to the first order in ψ yields $-4\psi H$, since $2H = L^A_A$. The variation of the metric determinant f follows from the identity [24]

$$f = \det\boldsymbol{f} := \frac{1}{2}\varepsilon^{AC}\varepsilon^{BD} f_{AB} f_{CD},$$

which yields, in a similar fashion as above

$$\begin{aligned} \delta f &= f f^{AB}\delta f_{AB} - 2\psi^2\varepsilon^{AC}\varepsilon^{BD} L_{AB} L_{CD} = \\ &= f(-4\psi H + f^{AB}\psi_{,A}\psi_{,B} + \psi^2(4H^2 + 2\kappa)) + O(\psi^3), \end{aligned} \tag{3.12}$$

where $\kappa = \frac{L}{f}$ denotes the determinant of the shape operator, known as the Gauss-Kronecker curvature. We are now ready to formulate the theorem, although we shall not be interested in questions such as whether the below mentioned coordinate chart exists, because these questions have already been addressed in Preliminaries. The character of the extremum of the area functional shall not be investigated further either, it is well treated in [3]. We prove this version of the theorem for simplicity and because this version is of much more importance to the issue, the more general theorem can be found in differential geometry textbooks.

Theorem 9. *Let Σ_{ts} denote a set of 2-dimensional surfaces, which are of the property that each one is covered by a single coordinate chart $(\{y^A\}^3_{A=2}, U_{ts})$ on a neighbourhood U_{ts} of a given surface Σ_{ts}. Let A_{ts} denote the area functional of a given surface from the set Σ_{ts}. The necessary condition for extremality of the area functional is $H = 0$.*

Proof. The area functional of Σ_{ts} reads

$$A_{ts} = \int_{y^A[\Sigma_{ts}]} \sqrt{f}\, \mathrm{d}y^2 \mathrm{d}y^3.$$

Extremality of a functional means that the variation up to the first order vanishes, therefore one only needs to evaluate the variation of $\sqrt{f}$ up to the first order

$$\delta\sqrt{f} = \frac{\partial\sqrt{f}}{\partial f}\delta f + O(\delta f)^2 = -2\psi H\sqrt{f} + O(\psi^2).$$

Thus, up to the first order one has, using the fundamental lemma of calculus of variations

$$\delta A_{ts} = -2\int_{y^A[\Sigma_{ts}]} \sqrt{f} H\psi\, \mathrm{d}y^2 \mathrm{d}y^3 = 0\; \forall\psi \iff H \equiv 0.$$

□

3.2 In general circular space-times

As has already been mentioned, the first decomposition of a general stationary asymptotically flat space-time with respect to the normalized time-like Killing vector is trivial in the sense that it yields zero trace of the extrinsic curvature tensor $\boldsymbol{K}$ of the hypersurface of constant Killing time coordinate Σ_t in the form (2.15) and consequently Σ_t is always an extremal submanifold of the space-time manifold $\mathcal{M}$. Authors of the paper [9] realised this and investigated the 2-surfaces of constant mean curvature H as if it had been only in a 3-dimensional space-time. This is correct because the associated connection of the Levi-Civita type is unique for a given metric (for the proof, see [11]).

The procedure for a vacuum case (thus, by virtue of Theorem 8 a circular space-time) is as follows: one starts from the general line element of a circular space-time (2.4), from which, obviously, the line element of the $t = const.$ slice can be obtained via setting $t = const.$. The line element of Σ_t in a general circular space-time then reads

$$^{(3)}\mathrm{d}s^2 = g_{AB}\mathrm{d}x^A\mathrm{d}x^B + g_{\phi\phi}\mathrm{d}\phi^2. \tag{3.13}$$

Now, the choice of the coordinate chart has to be made. Because of the above mentioned fact that the rest of the coordinate chart $\{x^A\}_{A=1}^2$ can be chosen to be either of spherical-type or cylindrical-type, combined with the fact that the transformation between these two coordinate systems leaves the Killing coordinates t, ϕ untouched, the two cases can be treated at once. The radial-type coordinates read r in spherical-type and ρ in cylindrical-type, therefore we introduce an arbitrary function $x^1 = x^1(x^2)$ that defines the particular surface Σ_{ts} in question. By substituting, the metric (3.13) of the hypersurface $t = const.$ then transitions into a metric of the 2-dimensional surface Σ_{ts} defined by the relation $x^1 = x^1(x^2)$ which reads, upon denoting $x^{1'} := \frac{\mathrm{d}x^1}{\mathrm{d}x^2}$

$$^{(2)}\mathrm{d}s^2 = \left[g_{11}(x^{1'})^2 + g_{22}\right](\mathrm{d}x^2)^2 + g_{\phi\phi}\mathrm{d}\phi^2, \tag{3.14}$$

from which the contravariant components in the coordinates adapted to the Σ_t slicing of the outward pointing unit normal $\boldsymbol{r}$ of the 2-surface Σ_{ts} can be deduced and then easily verified to be [9]

$$r^i = \frac{1}{\sqrt{g_{11}(x^{1'})^2 + g_{22}}} \left(\sqrt{\frac{g_{22}}{g_{11}}} \delta_1^i - \sqrt{\frac{g_{11}}{g_{22}}} x^{1'} \delta_2^i \right). \tag{3.15}$$

Now, by using (1.6), the trace of the extrinsic curvature tensor of the 2-surface Σ_{ts} can be written as

$$L = -r^i_{;i}, \tag{3.16}$$

where the semicolon denotes the components of the covariant differential associated with the metric corresponding to the line element (3.13). [9] further notes that the explicit form of the equation (3.16) reads

$$\begin{aligned} H = \frac{L}{2} = &\frac{1}{4\sqrt{g_{11}g_{22}}g_{\phi\phi}(g_{11}(x^{1'})^2 + g_{22})^{3/2}} (2g_{11}g_{22}g_{\phi\phi}x^{1''} + (x^{1'})^3 g_{11}(g_{11}g_{\phi\phi})_{,\theta} + \\ &+ (x^{1'})^2 (g_{11,x^1}g_{22}g_{\phi\phi} - 2g_{11}g_{22,x^1}g_{\phi\phi} - g_{11}g_{22}g_{\phi\phi,x^1}) + \\ &+ x^{1'}(2g_{11,x^2}g_{22}g_{\phi\phi} - g_{11}g_{22,x^2}g_{\phi\phi} + g_{11}g_{22}g_{\phi\phi,x^2}) - g_{22}(g_{22}g_{33})_{,x^1}), \end{aligned} \tag{3.17}$$

where, after setting this identically equal to zero, inserting the metric components, and evaluating the partial derivatives, the function x^1 has to be replaced with $x^1(x^2)$. This procedure yields a second order non-linear ordinary differential equation for the function $x^1(x^2)$, which defines the surface in question. Furthertmore, to find a regular axisymmetric surface with zero mean curvature given by the function $x^1(x^2)$ a boundary problem with the Neumann boundary conditions has to be solved, i.e. the solution of the resulting differential equation (3.17) with a particular choice of coordinates has to satisfy in spherical-type for the function $r(\theta)$, respectively in cylindrical-type coordinates for the function $\rho(z)$ the following (for a given choice of z_0):

$$r'(0) = r'(\pi) = 0, \quad \text{resp. } \rho'(z_0) = \rho'(-z_0) = 0. \tag{3.18}$$

3.3 xAct implementation

In this thesis, namely for the purpose of Chapter 4, the xAct package was extensively utilized to facilitate the implementation of various calculations. xAct is a suite of free packages for tensor computer algebra in Mathematica, designed to perform symbolic (xTensor module) and component (xCoba module) tensor computations and much more with a high degree of automatization and efficiency. The primary reason for employing xAct is the inherent complexity and lengthiness of the calculations involved in the analysis. Manual computation of these tensor operations is not only time-consuming but also prone to errors. By leveraging xAct, these calculations can be performed quite swiftly and error-freely, provided that the code is appropriately tested. For example, in [10] the xTensor module was used to confirm the results of [19] with success. The xCoba module provides tools for manipulating coordinate bases, which are essential for the calculations that this thesis relies on. It allows for the automatic handling of coordinate transformations and the simplification of expressions.

Therefore, to ensure the reliability of the obtained results, the implemented code was subjected to a series of tests. These tests verified the correctness of the code in various circumstances: the main example being the comparison of the expression obtained by directly substituting into (3.17) with the ”from the ground up built” expression. We include a few snippets from the full code here:

Snippet 1: The Weyl metric in WSC

```
DefChart[wsc,M3,{1,2,3},{r[],\[Theta][],\[Phi][]},ChartColor->Green];

weylWSC=CTensor[{
{Exp[-2*\[Nu]]*Exp[2*\[Lambda]],0,0},
{0,r[]^2*Exp[-2*\[Nu]]*Exp[2*\[Lambda]],0},
{0,0,r[]^2*Sin[\[Theta][]]^2*Exp[-2*\[Nu]]}},
{-wsc,-wsc}]
/.{\[Nu]->\[Nu][r[],\[Theta][]],\[Lambda]->\[Lambda][r[],\[Theta][]]};

weylWSC[-\[Mu],-\[Nu]]

SetCMetric[weylWSC,wsc,SignatureOfMetric->{3,0,0}]

CD3=LC[weylWSC];
```

The first input defines the chart of Weyl spherical coordinates on the 3-dimensional manifold named `M3` defined by the command (the first argument is the name, the second argument is the dimension and the third argument represents the abstract indices)

```
    DefManifold[M3,3,{\[Alpha],\[Beta],\[Gamma],\[Delta]}];
```

under the name `wsc`. The second input sets up the components of the fully covariant Weyl metric in the `wsc` chart, substituting λ for a function of the desired coordinates (formally scalar functions on the `M3` manifold, thus the brackets). The third input displays the metric components in a tabular format to visually check the form. The subsequent two commands establish the metric as a coordinate metric in the specified coordinates with the defined signature and set `CD3` as the Levi-Civita covariant derivative compatible with the `weylWSC` metric in virtue of Lemma 7.

Snippet 2: Obtaining the differential equation $L = 0$ in WSC

```
metricCoefficientsWSC={
g11[r[],\[Theta][]]->weylWSC[[1,1,1]],
g22[r[],\[Theta][]]->weylWSC[[1,2,2]],
g33[r[],\[Theta][]]->weylWSC[[1,3,3]]};

nWSC:=CTensor[{Sqrt[g22[r[],\[Theta][]]/(g11[r[],\[Theta][]]
(g11[r[],\[Theta][]] (D[rs[\[Theta][]],\[Theta][]])^2
```

```
+g22[r[],\[Theta][]]))],-Sqrt[g11[r[],\[Theta][]]/(g22[r[],\[Theta][]]
(g11[r[],\[Theta][]](D[rs[\[Theta][]],\[Theta][]])^2
+g22[r[],\[Theta][]]))]D[rs[\[Theta][]],\[Theta][]],0},{wsc}]
normalWSC[\[Alpha]]=nWSC[\[Alpha]]/.metricCoefficientsWSC//FullSimplify

eqnWSC=Simplify[-CD3[-\[Alpha]]@normalWSC[\[Alpha]]==0,
r[]\[Element]PositiveReals]/.r[]->rs[\[Theta][]]
```

The first input defines a replacement rule for the metric components that is used to convert the equation (3.17) adopted from [9] and the prescription for the normal (3.15) into the particular expressions in WSC. This is in the case of the equation (3.17) used to compare it with the obtained result and in the case of the equation (3.15) for the definition of the normal, which is done by the second and third input. The equation $L = 0$ is then saved under the name `eqnWSC` and is calculated by virtue of (3.16). The output of this snippet is equivalent to (4.9).

The following snippet is an example of the aforementioned tests. This one simply compares the result of the calculation via the second snippet and the explicit formula (3.17) from [9], which is saved under the name `eqnKHWSC`:

Snippet 3: A test

```
eqnKHWSC=1/2(2g11[r[],\[Theta][]] g22[r[],\[Theta][]]
g33[r[],\[Theta][]] D[rs[\[Theta][]],\[Theta][],\[Theta][]]
+(D[rs[\[Theta][]],\[Theta][]])^3 g11[r[],\[Theta][]]
D[g11[r[],\[Theta][]] g33[r[],\[Theta][]],\[Theta][]]
+(D[rs[\[Theta][]],\[Theta][]])^2(D[g11[r[],\[Theta][]],r[]]
g22[r[],\[Theta][]] g33[r[],\[Theta][]]-2g11[r[],\[Theta][]]
D[g22[r[],\[Theta][]],r[]]g33[r[],\[Theta][]] -g11[r[],\[Theta][]]
g22[r[],\[Theta][]] D[g33[r[],\[Theta][]],r[]])+
D[rs[\[Theta]],\[Theta]](2D[g11[r[],\[Theta][]],\[Theta][]]
g22[r[],\[Theta][]] g33[r[],\[Theta][]] -g11[r[],\[Theta][]]
D[g22[r[],\[Theta][]],\[Theta][]]g33[r[],\[Theta][]]
+g11[r[],\[Theta][]]g22[r[],\[Theta][]] D[g33[r[],\[Theta][]],
\[Theta][]])-g22[r[],\[Theta][]]D[g22[r[],\[Theta][]]
g33[r[],\[Theta][]],r[]])/(Sqrt[g11[r[],\[Theta][]]
g22[r[],\[Theta][]]]g33[r[],\[Theta][]](g11[r[],\[Theta][]]
(D[rs[\[Theta][]],\[Theta][]])^2 +g22[r[],\[Theta][]])^(3/2))==0
/.metricCoefficientsWSCfunction/.r[]->rs[\[Theta][]]//Simplify

eqnKHWSC-eqnWSC//Simplify
```

This snippet results in an equation, rather than the expected `True`. That is because more assumptions would have to be added, in order for Mathematica to for example cross non-zero terms with the zero on the right side. Nevertheless, it is straightforward to verify by hand that the resulting expression is in fact identically equal to zero. In the case of the analogous snippet in WCC this problem was not encountered and the result of the snippet is the expected `True`.

Overall, the use of xAct has proven to be an invaluable asset in handling the extensive calculations required in this thesis and it will be put to further use in the future. The full code, including the plots, can be found in the attachment.

4 Results

In the present chapter, everything from the preceding chapters comes together for the investigation of the minimal 2-dimensional submanifolds of Σ_t, and construction of the adapted coordinate systems for various space-times. Firstly, we tackle the Kerr space-time, since it was already partly dealt with in [9], and we use the results presented in this paper for the validation of the xAct code (see xAct implementation). The defining second order non-linear ordinary differential equation $L = 0$, obtained via the procedure explained in the Third chapter, is obtained for the Kerr geometry. Then the resulting boundary problem is solved via the *shooting method*. A plot obtained via this procedure exactly agrees with the corresponding plot presented in [9], validating the implementation and proving that the difference between the $r = const.$ and constant mean curvature surfaces is in the per-cent range in the case of the Kerr space-time endowed with the Boyer-Lindquist coordinates.

4.1 Kerr space-time case

The problem from the preceding chapter is firstly studied in the Kerr space-time, since it can provide a validation of the code, as was already mentioned before. The motivation is therefore not to obtain some groundbreaking results in the form of a closed-form solution (which most likely does not exist), but rather to validate the code, before trying to perhaps obtain an analytical solution for a different space-time in the future. From Notes on the Kerr metric, one immediately deduces that the line element, of the Σ_t slice of the Kerr space-time in the Boyer-Lindquist coordinates, reads

$$^{(3)}\mathrm{d}s^2 = \frac{\Sigma}{\Delta}\mathrm{d}r^2 + \Sigma \mathrm{d}\theta^2 + g_{\phi\phi}\mathrm{d}\phi^2 \implies {}^{(2)}\mathrm{d}s^2 = \Sigma(1 + \frac{r'(\theta)^2}{\Delta})\mathrm{d}\theta^2 + g_{\phi\phi}\mathrm{d}\phi^2. \quad (4.1)$$

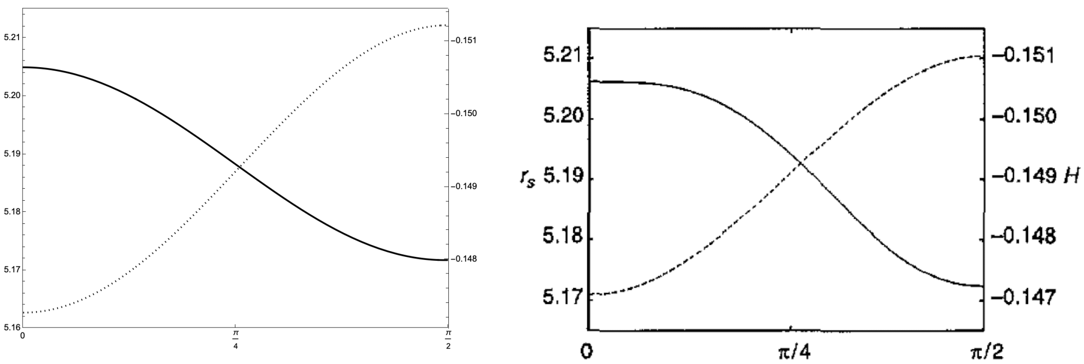
From which one can, by comparing the coefficients with (3.14) and using the general definition of the normal (3.15), obtain the components of the outward-pointing unit normal, to the 2-surface Σ_{ts}, to be

$$r^i = \frac{1}{\sqrt{\Sigma(\Delta + r'(\theta)^2)}}(\Delta\delta^i_1 - r'(\theta)\delta^i_2).$$

Now, the explicit form of the differential equation for $r(\theta)$ can be easily obtained as (for at least some brevity, the argument of r is ommited here)

$$\begin{aligned}2r'^2(a^2\cos(2\theta)(M-r)\Delta + r(-2a^4 + 5a^2(r + \frac{2M}{5})(M-r) + r^3(5M-3r)))+\\ +2r'\cot(\theta)(\mathcal{A} - \Delta a^2\sin^2(\theta))(r'^2 - \Delta) + 2r''\Delta(2a^4\cos^2(\theta) + 4a^2Mr\sin^2(\theta)+\\ +a^2(\cos(2\theta)+3)r^2 + 2r^4) - \Delta^2(2a^2M\sin^2(\theta) + a^2(\cos(2\theta)+3)r + 4r^3) = 0.\end{aligned} \quad (4.2)$$

Few different forms of this equation were obtained. Although, in this case, no further simplification was found, and this form was empirically found to be the most numerically stable. It is self-explanatory that the Neumann boundary value problem, formulated in the Formulation of the problem, has to be solved. For example, the shooting method has to be applied here, in agreement with [9], since not even the built-in Mathematica differential equation solvers give any results. The obtained result was tested via duplicating a plot found in [9] with a relative success. The plot's slight differences may be explicable simply by a worse numerical setting on either side.



(a) Replica of **Figure 2.** from [9] **(b)** Original **Figure 2.** from [9]

Figure 4.1 Side-by-side comparison of the Kerr geometry plots.

Another test, one naturally performs, is based on setting $a = 0$. For such a choice, transforms (2.8) to the known *Schwarzschild metric* in the Schwarzschild coordinates[1], where one suspects that the surfaces in question are of spherical shape, since the r coordinate is defined as the area radius [1]. This should clearly be the case for every choice of a provided that the initial condition is set on the horizon. To verify this, we plot the same as in the Schwarzschild case for $a = 1$, $M = 1$ and the initial condition now being $r(0) = 1$, since the coordinate singularities (horizons) are in the Kerr metric on the Boyer-Lindquist coordinates located at $r_{\pm} = M \pm \sqrt{M^2 - a^2}$ [1].

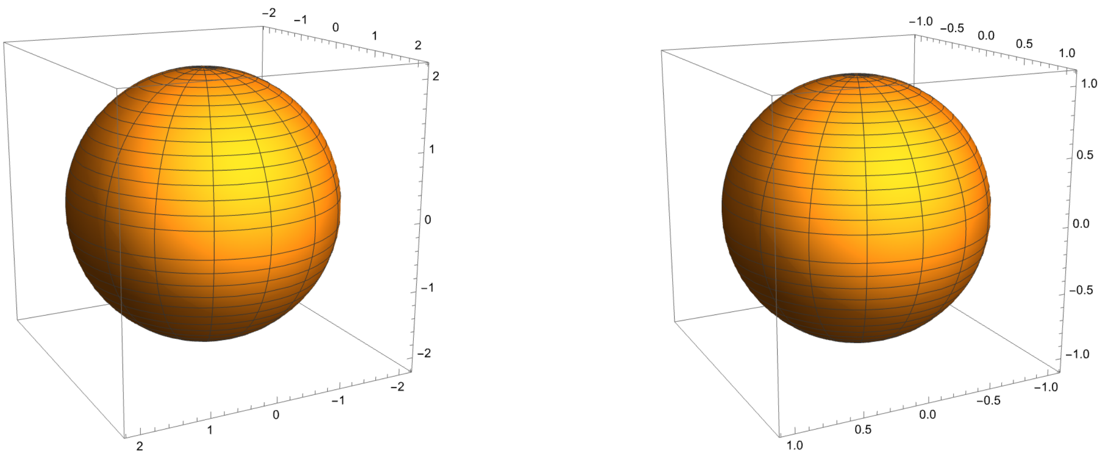
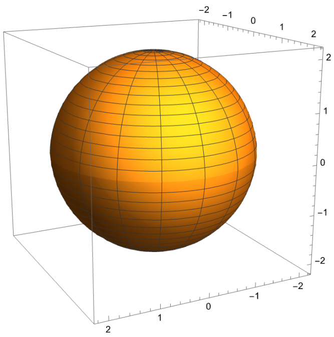
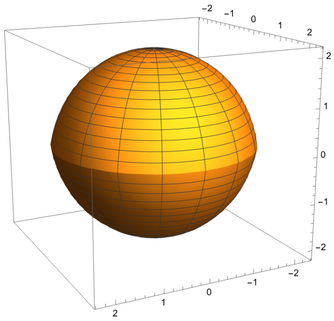


Figure 4.2 3D plot of the solution of (4.2) with $a = 0$, $M = 1$ (left) and $a = 1$, $M = 1$ (right) with the boundary condition $r'(0) = r'(\frac{\pi}{2}) = 0$ and the initial condition $r(0) = 2$ (left) and $r(0) = 1$ (right) obtained via the shooting method.

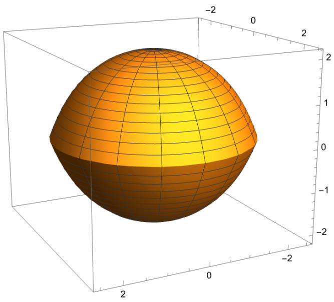
If one allows the a parameter to take on different values, the numerical solution becomes unstable in the asymptotic $\theta \to \pi$ region. This issue was resolved by solving the differential equation (4.2) only on the interval $\theta \in (0, \frac{\pi}{2})$, and mirroring the solution about the $x - y$ plane, since the space-time has to be symmetric about this plane[9]. The following are the 3D plots of the obtained numerical solution with the same boundary condition, M value, and unifying initial condition $r(0) = 2$ obtained via the mirroring technique explained above. This, of course, leaves the $r(\theta)$ function in the plots not exactly smoothly differentiable in the $x - y$ plane, which is an issue, that could be resolved by a different choice of the numerical method used to solve (4.2). In [9], the authors chose a custom shooting method, written in C, and for reference they cite the book [23], which was used as a guidance for the numerical methods presented here as well.



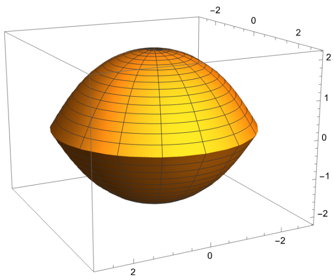
(a) $a = 0.2$



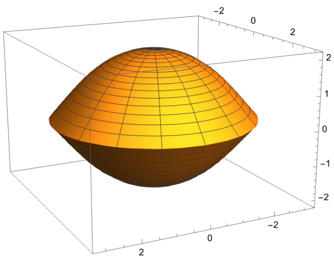
(b) $a = 0.4$



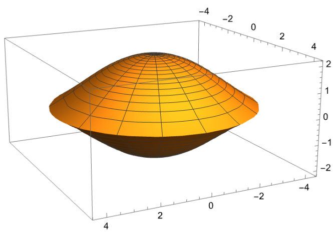
(a) $a = 0.6$



(b) $a = 0.8$



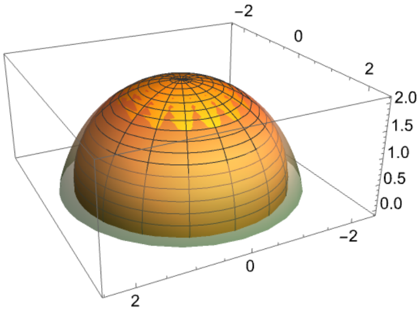
(a) $a = 1$ (Extremal Kerr black hole)



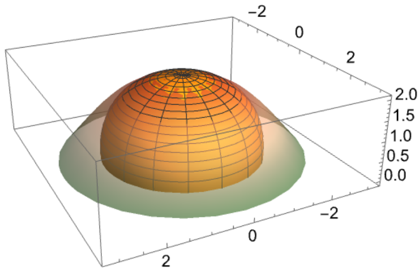
(b) $a = 1.5$ (Over-extremal Kerr black hole)

Perhaps, it is better to directly compare the surfaces plotted for the respective a values in the case of Kerr geometry with the surfaces obtained via setting

$r = const.$. This was 3D plotted again for the same data below for two distinct a values. The weird artifact on top of the surface is due to the finite uncertainty of the shooting method.



(a) $a = 0.5$



(b) $a = 1$

Although, for the Kerr case the analytical solution of the equation (4.2) was confirmed to most probably not exist in a closed form. The results presented here, within the range of the shooting method's uncertainty, closely follow what is already widely known about the Kerr metric. It is self-explanatory that due to the numerical nature of the solution, that was obtained, the construction of the adapted coordinate system, in the sense of Lemma 1, is impossible.

We therefore continue with the investigation of Σ_t minimal submanifolds with different underlying space-times, which either have a greater chance of the equation $L = 0$ being analytically solvable, or exhibit some interesting properties, like in the case of the Kerr geometry.

4.2 Weyl metrics case

The problem presented in the preceding chapter is in this thesis mainly studied within the framework of the Weyl family of metrics, because thanks to the wide range of functions ν, λ, that solve the field equations, it is possible that the problem can be solved analytically for some special choices of the functions ν, λ. As has been mentioned in the section about the Weyl family of metrics, apart from others, the choice $\nu = \lambda \equiv 0$, compatible with the constraints given by the vacuum field equations (2.17), actually results in the Minkowski metric line element in the corresponding coordinates that were chosen to cover the two complementary directions to the two Killing vectors $\boldsymbol{\eta}, \boldsymbol{\xi}$. Because, one naturally has the suspicion that the equation (3.17) will be of the most well-behaved form for the Minkowski metric, we treat this case of a Weyl metric first.

4.2.1 Minkowski case

The general line element of the Σ_t hypersurface within the Weyl metrics in Weyl cylindrical coordinates (2.18) with the above mentioned choice $\nu = \lambda \equiv 0$ takes the known form

$$^{(3)}\mathrm{d}s^2 = \mathrm{d}\rho^2 + \mathrm{d}z^2 + \rho^2 \mathrm{d}\phi^2. \tag{4.3}$$

By explicitly inserting the metric components $g_{11} = g_{22} \equiv 1$ and $g_{\phi\phi} = \rho^2$ into the general prescription of the outward pointing unit normal of the 2-surface Σ_{ts}, given by (3.15), we get the contravariant components of the latter

$$r^i = \frac{1}{\sqrt{\rho'(z)^2+1}}(\delta^i_1 - \rho'(z)\delta^i_2),$$

and the explicit form of the equation (3.17) then reads

$$1 + \rho'(z)^2 = \rho(z)\rho''(z). \tag{4.4}$$

This equation, even though it is of second order and non-linear, is autonomous and luckily straightforward to solve. The general solution can be easily obtained via guessing, because the left-hand side resembles the known hyperbolic functions identity. One can easily verify that the general solution reads

$$\rho(z) = C_1 \cosh\left(\frac{z + C_2}{C_1}\right). \tag{4.5}$$

Though, after imposing the initial condition $\rho'(0) = 0$ (which eliminates C_2), the resulting line element corresponding to the metric of Σ_{ts}

$${}^{(2)}\mathrm{d}s^2 = \cosh^2\left(\frac{z}{C}\right)\left[\mathrm{d}z^2 + C^2\mathrm{d}\phi^2\right], \tag{4.6}$$

is the line element of a surface widely known as *the catenoid*, and the curve defined by (4.5) with the above-chosen initial condition, given by

$$\rho(z) = C\cosh\left(\frac{z}{C}\right), \tag{4.7}$$

is known as *the catenary*. The catenoid can be generated via rotating the catenary about its directrix. Catenoid was first formally described by Leonhard Euler in 1744. The parametric equations for the catenoid in cartesian coordinates are:

$$\begin{aligned} x &= C\cosh\left(\frac{v}{C}\right)\cos u, \\ y &= C\cosh\left(\frac{v}{C}\right)\sin u, \\ z &= v, \end{aligned}$$

where $u \in [-\pi, \pi)$, and $v \in \mathbb{R}$, with C being a non-zero real constant. Historically, the catenoid was the first non-trivial minimal surface discovered apart from the plane. It satisfies the minimal surface partial differential equation in 3 dimensions, derived by Euler, of the form

$$(1 + z_{,x}^2)z_{,yy} - 2z_{,x}z_{,y}z_{,xy} + (1 + z_{,y}^2)z_{,xx} = 0.$$

Since the main motivation for the identification of the minimal submanifolds of Σ_t is to construct an adapted coordinate system in the sense of Lemma 1 to simplify expressions involved in the 2+1+1 decomposition of space-time, let us

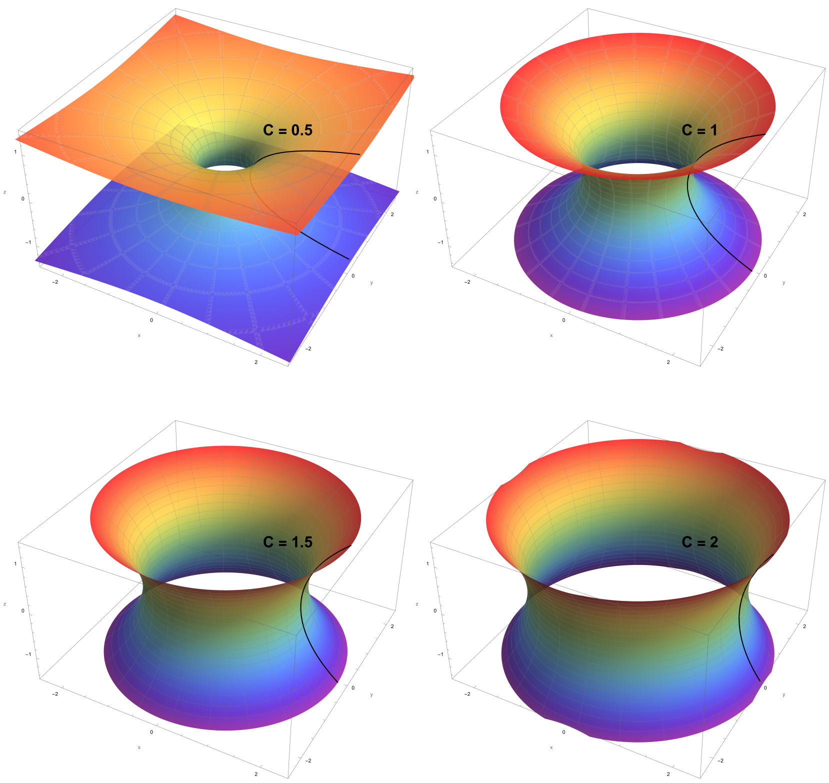


Figure 4.7 3D plots of the catenoid for 4 distinct C values with the black line representing the particular catenary.

do so for the case of the Minkowski space-time. The transformation from the generalised cartesian coordinates into the new *catenoid coordinates* reads

$$
\begin{aligned}
t &= t, \\
R &= \frac{\sqrt{x^2+y^2}}{\cosh\left(\frac{z}{\sqrt{x^2+y^2}}\right)}, \\
\phi &= \arctan\left(\frac{y}{x}\right), \\
z &= z,
\end{aligned}
$$

where the R coordinate has been defined such that the choice of ($t = const.$, $R = const.$) results in a catenoid for every choice of the constant C except zero. Therefore, in the case of the space-time decompositions being performed first with respect to constant Killing time coordinate and the second with respect to constant R coordinate, rendering the Σ_{ts} submanifold of $\mathcal{M}$ minimal. The newly-defined catenoid coordinate system is well-behaved except the origin, which is an expected property of a general radial coordinate. The inverse transformation reads

$$
\begin{aligned}
t &= t, \\
x &= R\cosh\left(\frac{z}{R}\right)\cos(\phi), \\
y &= R\cosh\left(\frac{z}{R}\right)\sin(\phi), \\
z &= z.
\end{aligned}
$$

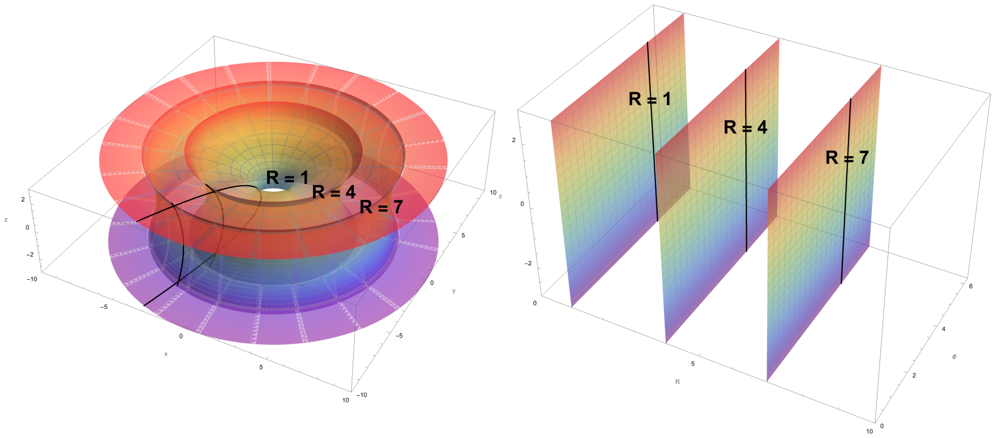


Figure 4.8 3D plots of the $R = const.$ surfaces for three distinct R values highlighted in the image for the ϕ coordinate value $\frac{7\pi}{4}$ in the cartesian coordinates, respectively the corresponding image in the catenary coordinates.

From this, coordinate transformations between different coordinate systems are easily-obtainable via standard techniques.

If one were to construct the analogous differential equation to (4.4), covering the meridional planes with $x^1 = r, x^2 = \theta$, i.e., the above-discussed spherical Weyl coordinates, the equation analogous to (4.4) comes out as

$$
2r(\theta)^3 + 3r(\theta)r'(\theta)^2 = \cot\theta\, r'(\theta)^3 + r(\theta)^2(\cot\theta\, r'(\theta) + r''(\theta)). \tag{4.8}
$$

This equation (4.8) can be further manipulated. Multiplying by $\frac{1}{r(\theta)^2 r'(\theta)}$, and the substitution $\Xi(\theta) := \frac{r'(\theta)}{r(\theta)}$ together yield the *Abel differential equation of the first kind* of the particular form

$$
\Xi'(\theta) = -\cot\theta\, \Xi^3 + 2\Xi^2 - \cot\theta\, \Xi + 2 := f_3(\theta)\Xi^3 + f_2(\theta)\Xi^2 + f_1(\theta)\Xi + f_0(\theta).
$$

In [16], one can find the substitution

$$
\Xi(\zeta) = E(\zeta)F(\zeta) - \frac{f_2(\zeta)}{3f_3(\zeta)}, \quad \zeta = \int f_3(\theta)E(\theta)\mathrm{d}\theta,
$$

where

$$
E(\theta) = \exp\left\{\int (f_1(\theta) - \frac{f_2^2(\theta)}{3f_3(\theta)})\right\},
$$

which takes the general form of the equation into the so-called normal form

$$F'(\zeta) = F(\zeta)^3 + \Phi(\zeta),$$

where the function $F(\zeta)$ is defined parametrically via

$$F(\zeta) := \frac{1}{f_3(\zeta)E(\zeta)^3}(f_0(\zeta) - \frac{f_1(\zeta)f_2(\zeta)}{3f_3(\zeta)} + \frac{2f_2(\zeta)^3}{27f_3(\zeta)^2} + \frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}\zeta}\frac{f_2(\zeta)}{f_3(\zeta)}).$$

Exact parametric or closed form solutions were treated in [14]. However, in this case, the ζ-defining integral likely does not exist in closed form, apart from an expression involving a non-trivial multiple of the hypergeometric function ${}_2F_1$ of goniometric arguments, Therefore, even if the solution existed, it would be really unsightful and impractical for the construction of an adapted coordinate system. This claim can be further supported by the fact that when one tries to trivially transform the above-obtained general solution (4.5) into the spherical Weyl coordinates, the expression is not invertible using elementary methods. We therefore leave this matter to a numerical study.

4.2.2 General space-time described by the Weyl metric

Given the previously established and verified method, it is straightforward to analogously, as in the previous cases, calculate the components of the normal for a completely general Weyl metric. It comes out in cylindrical coordinates as

$$r^i = \frac{1}{\sqrt{\rho'(z)^2 + 1}}(\delta^i_1 e^{2(\nu-\lambda)} - \delta^i_2 \rho'(z) e^{\nu-\lambda}).$$

Thus, the form of the differential equation in this case, for a space-time described by the general Weyl family metric with the line element (2.18) in Weyl cylindrical coordinates, is

$$\boxed{\rho'' + \rho'\left[(1+\rho'^2)(\lambda_{,z} - 2\nu_{,z}) - \rho'(\frac{1}{\rho} + \lambda_{,\rho} - 2\nu_{,\rho})\right] + 2\nu_{,\rho} - \lambda_{,\rho} - \frac{1}{\rho} = 0.} \tag{4.9}$$

Whereas, in spherical coordinates the equation takes the form

$$\boxed{r'' - 3\frac{r'^2}{r} - 2r - (\lambda_{,r} - 2\nu_{,r})(r'^2 + r^2) + r'(1 + \frac{r'^2}{r^2})(\cot(\theta) + \lambda_{,\theta} - 2\nu_{,\theta}) = 0.} \tag{4.10}$$

There is, of course (again), little hope that a closed form solution of these equations exists. Although, from this form it follows that indeed, for some particular choices of the metric functions, the closed form solution may be explicitly found.

Having at hand equation (4.9), for line elements written using the cylindrical coordinates and (4.10) for line elements written in the spherical coordinates, one can start inserting the explicit formulas for the metric functions, obtained in the second chapter, for individual solutions of the field equations.

4.2.3 Majumdar-Papapetrou case

Since the original case of the Majumdar-Papapetrou solution, given by the choice of the metric function ν and (2.21), is due to the character of the problem investigated in this thesis of not much physical interest, because it in the first order resembles the Curzon-Chazy solution. Let us directly move on to the disc configuration. Insert the choice of the metric function in the form (2.24) into (4.9). Even though the λ metric function is chosen as an identical zero, and one would hope that the equation (4.9) in this case simplifies, the reality is just the opposite, for the final form of the equation reads

$$\begin{aligned}&2M(1+\rho'^2)\{K(k)[2\rho(a+\rho)(z^2+(a-\rho)^2)-(a^2-\rho^2+z^2)\times\\&\times(z^2+(a+\rho)^2)(k-1)-\rho'(z^2+(a+\rho)^2)(k-1)(\rho'(a^2-\rho^2+z^2)-2z\rho)-\\&-2\rho'\rho(z^2+(a-\rho)^2)(z+\rho'(a+\rho))]+E(k)[\rho'(z^2+(a+\rho)^2\times\\&\times(\rho'(a^2-\rho^2+z^2)-2z\rho))-(a^2-\rho^2+z^2)(z^2+(a+\rho)^2)]\}/[a^4+\\&+2a^2(z^2-\rho^2)+(z^2+\rho^2)^2)(\pi\sqrt{(a+\rho)^2+z^2}+2MK(k)]+\rho''\rho=1+\rho'^2.\end{aligned} \tag{4.11}$$

Where $E(k) := \int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\alpha}\,d\alpha$ stands for the *complete elliptic integral of the second kind*. The equation was simplified into this form to see that it resembles the equation obtained for the Minkowski space-time, if it was not for the huge fraction. Recall that $k^2 = \frac{4a\rho}{z^2+(\rho+a)^2}$.

Given the nature of the disc solution, one expects the minimal submanifold to locally exhibit an "inflammation" around the disc's edge, given that the initial condition for (4.11) is set sufficiently close to the edge of the disc. In the Majumdar-Papapetrou disc solution, contrary to the Kerr one, the a parameter represents only the disc radius. Therefore, no generality is lost by setting $a = 1$ apart from the usual $M = 1$ for every discussion of this solution.

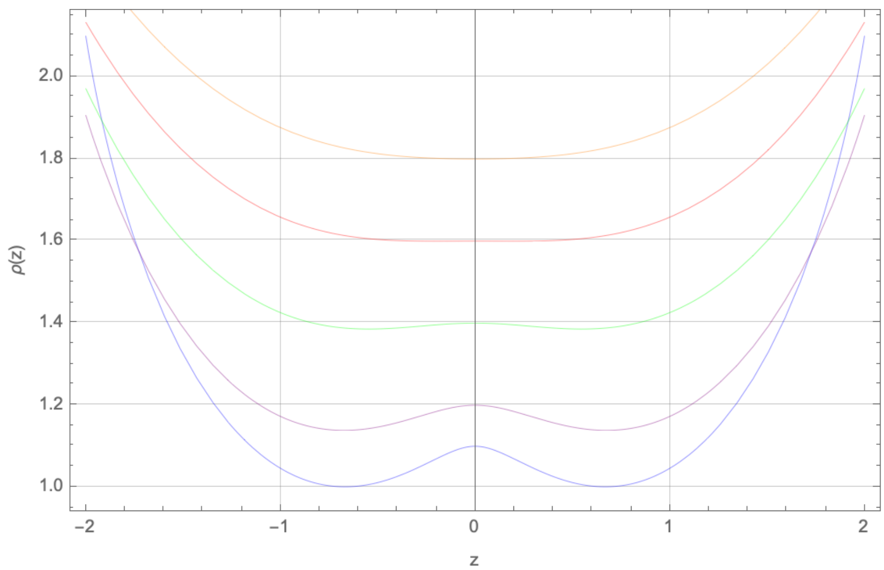


Figure 4.9 Plot of the comparison of $\rho(z)$ functions, for the Majumdar-Papapetrou disc, with the initial condition for $\rho(0)$ equal to respectively 1.1 (Blue), 1.2 (Purple), 1.4 (Green), 1.6 (Red) and 1.8 (Orange), with $a = 1$ everywhere.

From this plot, it is apparent that the $\rho(z)$ solution, obtained again by the shooting method for initial condition value roughly $\rho(0) < \frac{8}{5}a$ and $M = 1$, exhibits two local minima, as was expected. This can be seen in the case of the 3D plot as well. We first include a side view of the plot to appreciate the detail. The disc in all the plots has, again, radius $a = 1$ and is represented finitely thin to be visible.

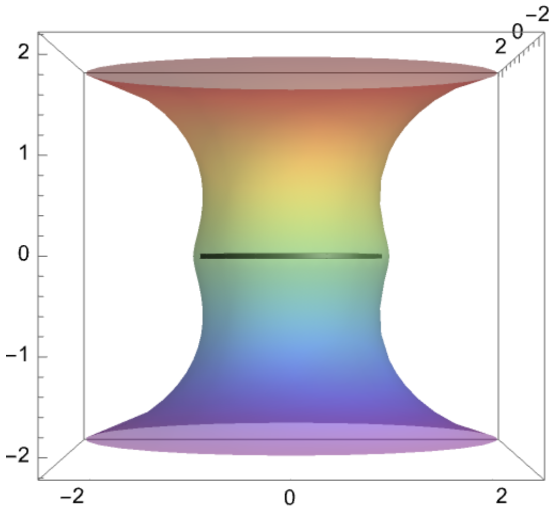
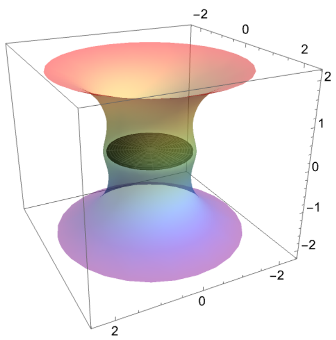
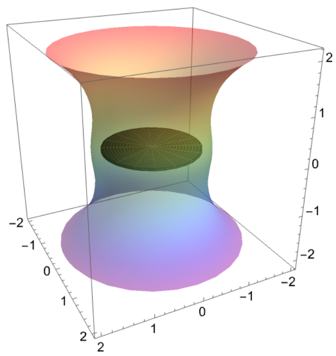


Figure 4.10 3D side-viewed plot of the minimal surface in the Majumdar-Papapetrou geometry with the lowest-found non-divergent initial condition $\rho(0) = 1.08$.

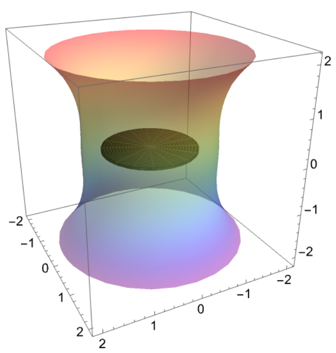
Finally, we include four 3D plots for four initial conditions close to the disc's edge, so one can see the "inflammation", correspondingly to Figure 4.9.



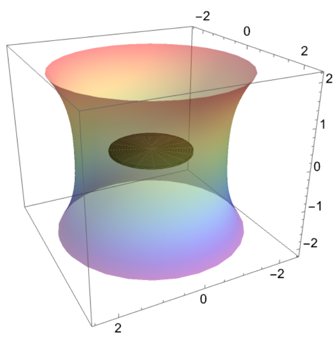
(a) $\rho(0) = 1.08$



(b) $\rho(0) = 1.2$



(a) $\rho(0) = 1.4$



(b) $\rho(0) = 1.8$

4.2.4 Levi-Civita case

Inserting the metric functions in this case renders the general equation (4.9) into the form not much different from the one obtained in the case of the Minkowski space-time

$$(1-2\sigma)^2(1-\rho'(z)^2) = \rho(z)\rho''(z), \tag{4.12}$$

to which it is converted generally for any choice of σ, satisfying that $\sigma^2 = \sigma$ (specially therefore for $\sigma = 0$), as can be easily seen from the left hand side of the equation. This equation has been dealt with in Subsection 4.2.1. In fact, explicit coordinate transformations have been found to transform into the adapted coordinates for Σ_{ts} slicing, that leave the resulting submanifold minimal.

In [7], the Kretschmann scalar for general σ is given as[1]

$$\mathcal{K} := R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda} = \frac{64\sigma^2(1-2\sigma)^2(1-2\sigma+4\sigma^2)}{\rho^{4(1-2\sigma+4\sigma^2)}}.$$

Indicating that the space-time is flat for $\sigma \in \{0, \frac{1}{2}\}$, and in the limit $\sigma \to \infty$. Therefore, the parameter choice $\sigma = 1$ has a curvature singularity at $\rho = 0$, but the form of (4.12) is apparently the same as in the case of the Minkowski space-time (4.4). In [7], it is further, by rescaling coordinates and introducing the parameter C, showed that for $\sigma = 0$, the metric can be taken to the form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}\rho^2 + C^2\rho^2\mathrm{d}\phi^2 + \mathrm{d}z^2.$$

That describes a space-time with a cosmic string along the axis for which the deficit angle of the ϕ coordinate domain is given as $2\pi(1-C)$. Another analytically solvable choice of the σ parameter is $\sigma = \frac{1}{2}$. In which case the space-time is flat, as has already been established, although the equation (4.12) has the general solution $\rho(z) = Az + B$, which has not been observed in the case of Minkowski space-time, and only satisfies the boundary condition as an infinitely-tall cylinder, rendering the solution semi-trivial.

Nevertheless, our main aim with the Levi-Civita metric is to compare the obtained analytical solution, given by either the Minkowski solution or the before-mentioned σ choice, with a numerical solution, obtained again via the shooting method for a general σ value. Firstly, we ensure the validity of the numerical solution by comparing the 3D plot and the $\rho(z)$ function's behaviour between the obtained numerical solution for $\sigma = 0$ and $\sigma = 1$ and the corresponding analytical catenary (see the Minkowski case). This is illustrated by Figure 4.13 with success. The following plots always display on the left the 3D plot of the obtained numerical solution with the initial condition $\rho(0) = 1$ in orange, respectively the analytical solution of the Minkowski-form equation with the same initial condition in transparent grey for comparison. The analytical solution's dependence of ρ on z is on the right displayed as black dashed line. The numerical solution's with the full red line.

[1] Which was used to test the attached xAct code successfully.

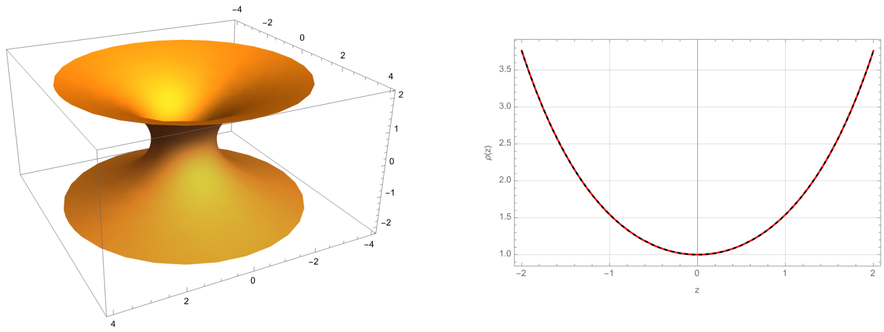


Figure 4.13 The Levi-Civita case $\sigma = 0$ and $\sigma = 1$ render the same result.

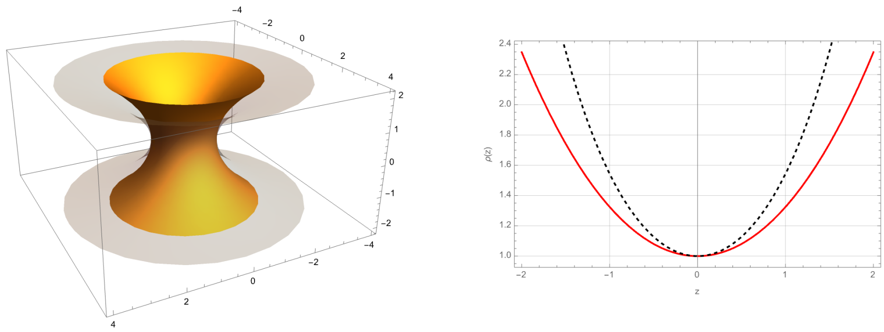


Figure 4.14 The Levi-Civita case $\sigma = 0.1$.

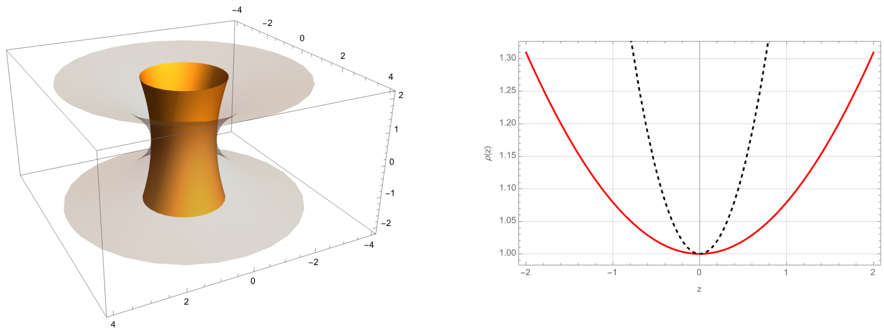


Figure 4.15 The Levi-Civita case $\sigma = 0.3$.

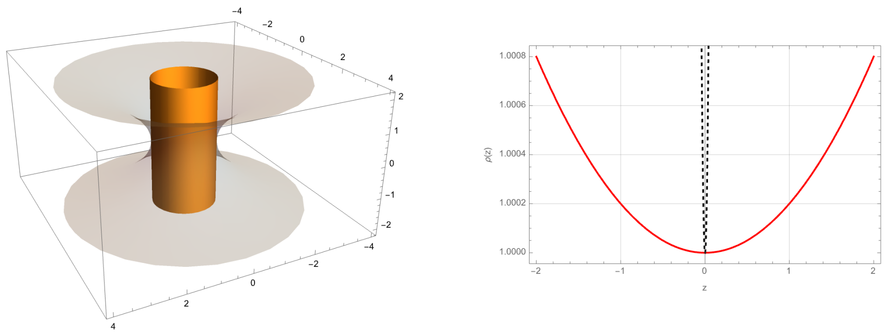


Figure 4.16 The Levi-Civita case $\sigma = 0.49$.

As mentioned in [21] and [7], the solution's behaviour, in terms of the parameter σ, is symmetric about the value $\frac{1}{2}$, which is a trend that was observed in this thesis as well. Recall, that for this choice of the parameter σ the underlying space-time is flat, although the boundary value problem we are interested in has only solution in the form of an infinitely-high cylinder. Figure 4.16 illustrates the closest non-diverging solution that has been found, to stretch the shooting method. This point actually corresponds to the solution $\rho(z)$ switching the left and right branches in the right plot that illustrates the dependence of ρ on z.

Space-times with $\sigma > 1$ typically correspond to exotic matter sources. Nevertheless, we plot the solution for the highest non-diverging value of σ, that has been found (negative values of σ were considered as well, but no lower-valued σ non-diverging solution has been found). The plot shows that, for $\sigma > 1$, the roles of the Minkowski solution and this particular solution interchange in the sense that in the interval $\sigma \in [0,1] \setminus \{\frac{1}{2}\}$ the Levi-Civita space-times's solution's minimal submanifold Σ_{ts} has always been enclosed by the analytical catenoid, for $\sigma > 1$ it is no longer the case. Furthermore, it expands rapidly, suggesting that the catenoid is in this sense somehow unique.

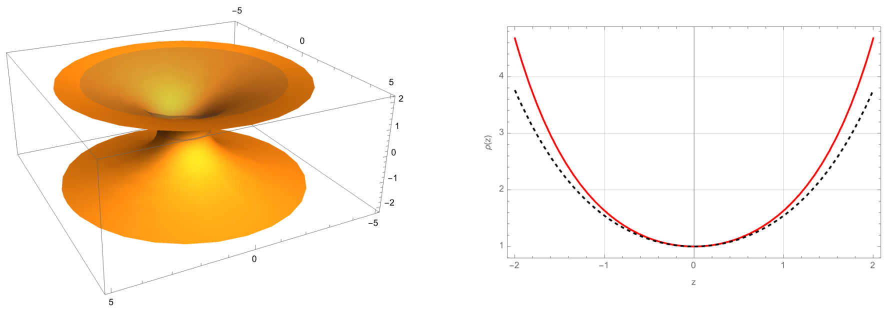


Figure 4.17 The Levi-Civita case $\sigma = 1.03$.

4.2.5 Curzon-Chazy case

Inserting the metric functions in the cylindrical representation case renders the general equation (4.9) into

$$\begin{aligned} &\rho'' \frac{(\rho^2+z^2)^2}{\rho'^2+1} + \rho' \left[\rho^2 - 2(\rho^2+z^2)^{\frac{3}{2}}\right] 2M^2 z + \\ &+ \rho\left[M^2(z^2-\rho^2) + (z^2+\rho^2)(2M\sqrt{\rho^2+z^2} - 3z^2)\right] - \frac{\rho^6+z^6}{\rho} = 0. \end{aligned} \tag{4.13}$$

Whereas, the spherical equation is of the form

$$\begin{aligned} &r'' + 2(M-r) - \frac{M^2}{r}\sin^2\left(1+\frac{r'^2}{r}\right) + \\ &+ r'\left[\cot(\theta)\left(1-\frac{M^2}{r^2}\sin^2(\theta)\right)\left(1+\frac{r'^2}{r^2}\right) + \frac{r'}{r}(3-2M)\right]. \end{aligned} \tag{4.14}$$

Given the complexity of these two differential equations, again, nothing more than a numerical discussion is possible. No significantly different results for this metric, than in the case of other spherical solutions, have been found with much less regular outcomes. Therefore, we do not include the discussion of the equation (4.14) here.

The obtained numerical solution of (4.13) exhibits much more interesting behaviour. Probably suggesting that the cylindrical coordinates are better tailored for the complex directional naked singularity (about which one can obtain further information in [21], or [7]). Apart from the analytical solution of the Minkowski case and for some parameters the Levi-Civita case, its behaviour in the vicinity of the center is the most regular, allowing one to set an initial condition as close to the axis as $\rho(0) = 0.1$. Although ,the solution does not look pleasant in this particular case. We instead plot the 3D plot of the solution, respectively the $\rho(z)$ function for $\rho(0) = 0.4$, and $\rho(0) = 1$. The $\rho(z)$ plot, portrayed by a red line, is plotted in one graph, with the corresponding catenary, for comparison.

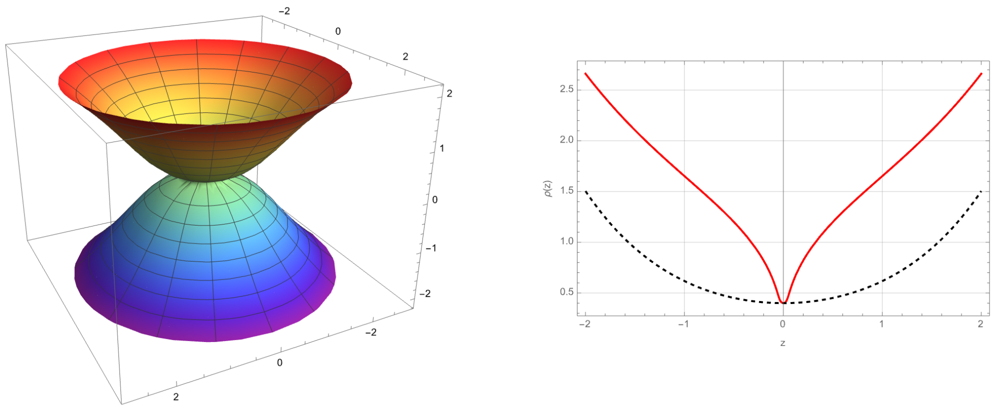


Figure 4.18 The Curzon-Chazy case in cylindrical coordinates for $\rho(0) = 0.4$.

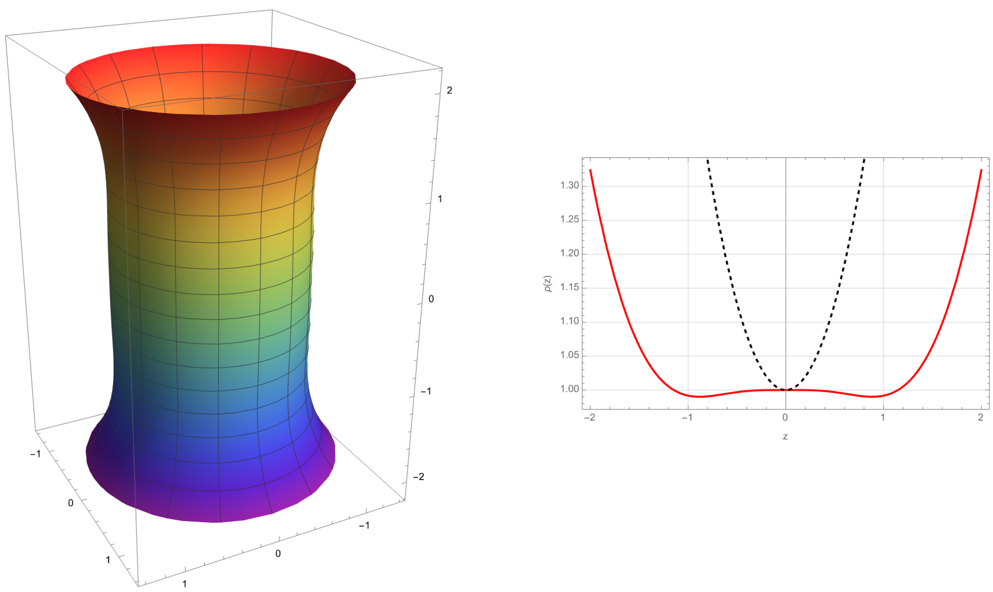


Figure 4.19 The Curzon-Chazy case in cylindrical coordinates for $\rho(0) = 1$.

4.3 Analytical approach

As has already been suggested, equations (4.9) and (4.10) give one some wiggle room for the choice of the metric functions. Therefore, some special choices made in the Weyl metric, for either of the two equations, may render an analytic solution. If not, the analytical approach may at the very least provide some information about the solution's existence in special cases. For this reason, we choose to investigate the system of equations, that represents the sufficient condition for the Weyl metric, to be a solution of the field equations, coupled with the $L = 0$ equation respectively, for both equations (4.9), and (4.10).

Let us handle the cylindrical equation first, and try a naive method. Because this equation explicitly includes the partial derivatives of the λ function, we first introduce the restriction on those derivatives generated by the field equations into the equation $L = 0$, which transforms the system into

$$\begin{aligned} &\nu_{,\rho\rho} - \frac{1}{\rho}\nu_{,\rho} + \nu_{,zz} = 0, \\ &\rho'' + \rho'\left[2\nu_{,z}(1+\rho'^2)(\rho\nu_{,\rho} - 1) - \rho'\left(\frac{1}{\rho} + \nu_{,\rho}(\rho\nu_{,\rho} - 2) + \rho(\nu_{,z})^2\right)\right] + \\ &+ \nu_{\rho}(2 - \rho\nu_{,\rho}) - \rho(\nu_{,z})^2 - \frac{1}{\rho} = 0. \end{aligned} \tag{4.15}$$

From here, several straight-forward simplifying choices are possible. Let us first try the ansatz that annihilates the $(\rho\nu_{,\rho} - 1)$ term. It is evident that it resembles the known flat solution $\nu = \log(\rho) \implies \lambda = \log(\rho)$ mentioned in [7]. Therefore, let us make the ansatz less trivial in the sense that we include a general function of the coordinate z

$$\nu(\rho, z) = \log(\rho) + f(z),$$

which takes the $L = 0$ equation into the form

$$\rho'' = \rho f'(z)^2(1+\rho'^2),$$

and the Laplace equation into the form $f''(z) = 0 \iff f(z) = Az + B$. From this, the final form of the system (4.15) reads simply

$$\rho'' = \rho A^2(1+\rho'^2).$$

As has been mentioned, one already knows, that the choice $A = 0$ is a solution of the field equations. Furthermore, the afterwards equation is solvable easily, but the solution is trivial. Other choice of the constant A does not apper to provide an analytical solution[16].

Another option is to annihilate the second problematic bracket $(\rho\nu_{,\rho} - 2)$. This is very similar, because the ansatz then takes the form

$$\nu(\rho, z) = \log\left(\rho^2\right) + f(z).$$

This ansatz obviously fulfills the Laplace equation, provided that the $f(z)$ function is of the same form as before, and the $L = 0$ equation takes the form

$$\rho'' + f'(z)(1+\rho'^2)(2\rho' - \rho f'(z)) - \frac{1+\rho'^2}{\rho} = 0.$$

This equation is, again, not elementary to solve analytically for any other value of the constant A apart from zero, which just transforms the equation into the Minkowski form (4.3). Therefore the minimal submanifold is, again, of catenoidal shape. The corresponding λ function is then given as

$$\lambda(\rho) = \log\left(\rho^4\right).$$

Another possibility is to consider the system (4.15) only partially coupled, because setting $2\nu_{,\rho} - \lambda_{,\rho} = \frac{1}{\rho}$, and substituting from the field equations only for $\lambda_{,\rho}$ into this simplifying constraint, yields

$$\nu_{,\rho}(2 - \rho\nu_{,\rho}) - \rho(\nu_{,z})^2 - \frac{1}{\rho} = 0.$$

This constraint was chosen, because it renders the equation $L = 0$ in the form

$$\rho'' + 2\rho'\nu_{,z}\left[(1 + \rho'^2)(\rho\nu_{,\rho} - 1)\right] = 0.$$

Provided that there exists such a choice of constants in the Laplace equation's general solution, so that the simplifying constraint holds. Because then one has the last equation for the determination of the λ function, and the problem is well-posed. Two ways of separating variables in the Laplace equation are possible, apparently. One from the ansatz, $\nu(\rho, z) = f(\rho) + g(z)$, gets the general solution in the form

$$\nu(\rho, z) = C_1 \log(\rho) + C_2(z^2 - \frac{\rho^2}{2}) + C_3 z + C_4.$$

Whereas, from the multiplicative ansatz, $\nu(\rho, z) = f(\rho)g(z)$, one gets

$$\nu(\rho, z) = J_0(k\rho)(C1e^{kz} + C_2e^{-kz}) + Y_0(k\rho)(C_3e^{kz} + C_4e^{-kz}).$$

Where J_0 and Y_0 denote the *Bessel functions of the first and second kind*, respectively. Unfortunately, the simplifying condition does not allow any different form of the solution, because it comes out as

$$\frac{2C_1}{\rho} = \frac{1}{\rho} + 2C_2\rho + \rho\left((C_3 + 2C_2z)^2 + (\frac{C_1}{\rho} - C_2\rho)^2\right),$$

in the case of the additive ansatz. In the case of the multiplicative ansatz, the resulting simplifying condition can in fact not be obeyed for any choice of the constants.

In the case of the spherical equation (4.10), one has at hand the general prescription for both metric functions (2.29) and (2.30). From this, maybe after recurrent cutting of the sum some wonder recurrent relation between the coefficients may sum approach follow. However, non-trivially. Another option may be transforming to the catenary or some other better-suited coordinates (since for example in the cylindrical case, the minimal submanifolds follow this trend; therefore, one would expect the $R(z)$ function to not be too wild), but the relation for the implicit derivatives does not appear to simplify anything. We therefore leave further analytical investigation for the upcoming research.

Conclusion

In this thesis, we first formulated Frobenius' theorem, which was crucial for our later analysis, and proved corollary Theorem 5, which generalizes a theorem from [1] and assures the existence of the decompositions. Using this theorem, we performed a general 2+1+1 decomposition of the field equations, omitting the null normal case. It was observed that the decomposition simplifies, if the trace, of the extrinsic curvature tensor of the resulting 2-dimensional submanifolds Σ_{ts}, vanishes. Consequently, we explored several solutions of the field equations in various coordinate systems to find an analytically solvable Neumann boundary value problem for a second-order non-linear ordinary differential equation.

After introducing basic concepts, and proving the sufficient conditions for the circularity of a space-time, we examined individual space-times, focusing on Weyl metrics. We derived the line element forms for the Weyl space-times via analogy of the metric function ν with a Newtonian potential.

We then formulated the main problem addressed in Chapter 3. We proved that zero mean curvature of a surface embedded in 3 dimensions is a sufficient condition for the stationarity of its area functional. After stating the problem and explaining our methods, we began solving it for the specified space-times.

Thanks to the Kerr metric, we validated the attached xAct code with high certainty. The solution matched the expected behavior for the Kerr space-time. We compared our solution plots with those in [9], and found reasonable agreement. For the Minkowski space-time limit, we obtained an analytical solution for $\rho(z)$ of the defining equation K=0, then constructed the adapted coordinates, and plotted isosurfaces of the new radial coordinate to validate our construction. For the Majumdar-Papapetrou disc, we observed the expected behavior of the $\rho(z)$ solution in the vicinity of the disc through plots of the $\rho(z)$ function or the rotational body. Regarding the Levi-Civita metric, we obtained $\rho(z)$ solutions for σ parameter values roughly between -0.1 and 1.1, but no solutions for other values. At $\sigma = 0$, $\sigma = 1$, the expected simplification to the Minkowski equation was observed. At $\sigma = \frac{1}{2}$, the solution is an infinitely-long cylinder, with no other solutions possible under the given Neumann boundary value problem.

We then adopted a more analytical approach, simplifying the equation L=0 to be solvable with the ansatz, $\nu = \log(\rho^2)$, leading to the other metric function, $\lambda = \log(\rho^4)$, and yet another space-time whose minimal Σ_{ts} submanifolds are catenoids. Now, the future research can expand on the findings of this thesis in several directions. One key area mentioned already is the simplification of 2+1+1 decomposition of the main quadratic invariant characteristic of vacuum curvature. The Kretschmann scalar $\mathcal{K}$ and other similar invariant; the Chern-Pontryagin scalar $\mathcal{C}$. In these hypothetical, analytically solvable space-times, simplifying the derived 2+1+1 decomposition of the Kretschmann scalar done in [19] and [10], would be possible, provided that one finds the adapted coordinates. This would be then useful since not even in simple space-times is it effective to calculate the invariants in components. Many of the components may individually diverge for example on a horizon and yet, one knows that the result is regular.

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