AN INTRODUCTION TO INFORMATION THEORY

Symbols, Signals & Noise

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Second, Revised Edition

Dover Publications, Inc.
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TO CLAUDE AND BETTY SHANNON

This Dover edition, first published in 1980, is an unabridged and revised version of the work originally published in 1961 by Harper & Brothers under the title *Symbols, Signals and Noise: The Nature and Process of Communication.*

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Preface to the Dover Edition

THE REPUBLICATION OF THIS BOOK gave me an opportunity to correct and bring up to date *Symbols, Signals and Noise,*[1] which I wrote almost twenty years ago. Because the book deals largely with Shannon's work, which remains eternally valid, I found that there were not many changes to be made. In a few places I altered tense in referring to men who have died. I did not try to replace *cycles per second* (*cps*) by the more modern term, *hertz* (*hz*) nor did I change everywhere *communication theory* (Shannon's term) to *information theory*, the term I would use today.

Some things I did alter, rewriting a few paragraphs and about twenty pages without changing the pagination.

In Chapter X, Information Theory and Physics, I replaced a background radiation temperature of space of "2° to 4°K" (Heaven knows where I got that) by the correct value of 3.5°K, as determined by Penzias and Wilson. To the fact that in the absence of noise we can in principle transmit an unlimited number of bits per quantum, I added new material on quantum effects in communication.[2] I also replaced an obsolete thought-up example of space communication by a brief analysis of the microwave transmission of picture signals from the Voyager near Jupiter, and by an exposition of new possibilities.

[1] Harper Modern Science Series, Harper and Brothers, New York, 1961.

[2] See *Introduction to Communication Science and Systems*, John R. Pierce and Edward C. Posner, Plenum Publishing Corporation, New York, 1980.

In Chapter VII, Efficient Encoding, I rewrote a few pages concerning efficient source encoding of TV and changed a few sentences about pulse code modulation and about vocoders. I also changed the material on error correcting codes.

In Chapter XI, Cybernetics, I rewrote four pages on computers and programming, which have advanced incredibly during the last twenty years.

Finally, I made a few changes in the last short Chapter XIV, Back to Communication Theory.

Beyond these revisions, I call to the reader's attention a series of papers on the history of information theory that were published in 1973 in the *IEEE Transactions on Information Theory*[3] and two up-to-date books as telling in more detail the present state of information theory and the mathematical aspects of communication.[2,4,5]

Several chapters in the original book deal with areas relevant only through application or attempted application of information theory.

I think that Chapter XII, Information Theory and Psychology, gives a fair idea of the sort of applications attempted in that area. Today psychologists are less concerned with information theory than with cognitive science, a heady association of truly startling progress in the understanding of the nervous system, with ideas drawn from anthropology, linguistics and a belief that some powerful and simple mathematical order must underly human function. Cognitive science of today reminds me of cybernetics of twenty years ago.

As to Information Theory and Art, today the computer has replaced information theory in casual discussions. But, the ideas explored in Chapter XIII have been pursued further. I will mention some attractive poems produced by Marie Borroff[6,7], and, es-

[3] *IEEE Transactions on Information Theory*, Vol. IT–19, pp. 3–8, 145–148, 257–262, 381–389 (1973).

[4] *The Theory of Information and Coding*, Robert J. McEliece, Addison-Wesley, Reading, MA, 1977.

[5] *Principles of Digital Communication and Coding*, Andrew J. Viterbi and Jim K. Omura, McGraw Hill, New York, 1979.

[6] "Computer as Poet," Marie Borroff, *Yale Alumni Magazine*, Jan. 1971.

[7] *Computer Poems*, gathered by Richard Bailey, Potagannissing Press, 1973.

pecially a grammar of Swedish folksongs by means of which Johan Sundberg produced a number of authentic sounding tunes.[8]

This brings us back to language and Chapter VI, Language and Meaning. The problems raised in that chapter have not been resolved during the last twenty years. We do not have a complete grammar of any natural language. Indeed, formal grammar has proved most powerful in the area of computer languages. It is my reading that attention in linguistics has shifted somewhat to the phonological aspects of spoken language, to understanding what its building blocks are and how they interact—matters of great interest in the computer generation of speech from text. Chomsky and Halle have written a large book on stress,[9] and Liberman and Prince a smaller and very powerful account.[10]

So much for changes from the original *Signals, Symbols and Noise*. Beyond this, I can only reiterate some of the things I said in the preface to that book.

When James R. Newman suggested to me that I write a book about communication I was delighted. All my technical work has been inspired by one aspect or another of communication. Of course I would like to tell others what seems to me to be interesting and challenging in this important field.

It would have been difficult to do this and to give any sense of unity to the account before 1948 when Claude E. Shannon published "A Mathematical Theory of Communication."[11] Shannon's communication theory, which is also called information theory, has brought into a reasonable relation the many problems that have been troubling communication engineers for years. It has created a broad but clearly defined and limited field where before there were many special problems and ideas whose interrelations were not well

[8] "Generative Theories in Language and Musical Descriptions," Johan Sundberg and Bjorn Lindblom, *Cognition*, Vol. 4, pp. 99–122, 1976.

[9] *The Sound Pattern of English*, N. Chomsky and M. Halle, Harper and Row, 1968.

[10] "On Stress and Linguistic Rhythm," Mark Liberman and Alan Prince, *Linguistic Inquiry*, Vol. 8, No. 2, pp. 249–336, Spring, 1977.

[11] The papers, originally published in the *Bell System Technical Journal*, are reprinted in *The Mathematical Theory of Communication*, Shannon and Weaver, University of Illinois Press, first printing 1949. Shannon presented a somewhat different approach (used in Chapter IX of this book) in "Communication in the Presence of Noise," *Proceedings of the Institute of Radio Engineers*, Vol. 37, pp. 10–21, 1949.

understood. No one can accuse me of being a Shannon worshiper and get away unrewarded.

Thus, I felt that my account of communication must be an account of information theory as Shannon formulated it. The account would have to be broader than Shannon's in that it would discuss the relation, or lack of relation, of information theory to the many fields to which people have applied it. The account would have to be broader than Shannon's in that it would have to be less mathematical.

Here came the rub. My account could be *less* mathematical than Shannon's, but it could not be *nonmathematical.* Information theory is a mathematical theory. It starts from certain premises that define the aspects of communication with which it will deal, and it proceeds from these premises to various logical conclusions. The glory of information theory lies in certain mathematical theorems which are both surprising and important. To talk about information theory without communicating its real mathematical content would be like endlessly telling a man about a wonderful composer yet never letting him hear an example of the composer's music.

How was I to proceed? It seemed to me that I had to make the book self-contained, so that any mathematics in it could be understood without referring to other books or without calling for the particular content of early mathematical training, such as high school algebra. Did this mean that I had to avoid mathematical notation? Not necessarily, but any mathematical notation would have to be explained in the most elementary terms. I have done this both in the text and in an appendix; by going back and forth between the two, the mathematically untutored reader should be able to resolve any difficulties.

But just how difficult should the most difficult mathematical arguments be? Although it meant sliding over some very important points, I resolved to keep things easy compared with, say, the more difficult parts of Newman's *The World of Mathematics.* When the going is very difficult, I have merely indicated the general nature of the sort of mathematics used rather than trying to describe its content clearly.

Nonetheless, this book has sections which will be hard for the nonmathematical reader. I advise him merely to skim through these, gathering what he can. When he has gone through the book in this manner, he will see why the difficult sections are there. Then he can turn back and restudy them if he wishes. But, had I not put these difficult sections in, and had the reader wanted the sort of understanding that takes real thought, he would have been stuck. As far as I know, other available literature on information theory is either too simple or too difficult to help the diligent but inexpert reader beyond the easier parts of this book. I might note also that some of the literature is confused and some of it is just plain wrong.

By this sort of talk I may have raised wonder in the reader's mind as to whether or not information theory is really worth so much trouble, either on his part, for that matter, or on mine. I can only say that to the degree that the whole world of science and technology around us is important, information theory is important, for it is an important part of that world. To the degree to which an intelligent reader wants to know something both about that world and about information theory, it is worth his while to try to get a clear picture. Such a picture must show information theory neither as something utterly alien and unintelligible nor as something that can be epitomized in a few easy words and appreciated without effort.

The process of writing this book was not easy. Of course it could never have been written at all but for the work of Claude Shannon, who, besides inspiring the book through his work, read the original manuscript and suggested several valuable changes. David Slepian jolted me out of the rut of error and confusion in an even more vigorous way. E. N. Gilbert deflected me from error in several instances. Milton Babbitt reassured me concerning the major contents of the chapter on information theory and art and suggested a few changes. P. D. Bricker, H. M. Jenkins, and R. N. Shepard advised me in the field of psychology, but the views I finally expressed should not be attributed to them. The help of M. V. Mathews was invaluable. Benoit Mandelbrot helped me with Chapter XII. J. P. Runyon read the manuscript with care, and Eric Wolman uncovered an appalling number of textual errors, and made valuable suggestions as well. I am also indebted to Prof. Martin Harwit, who persuaded me and Dover that the book was

worth reissuing. The reader is indebted to James R. Newman for the fact that I have provided a glossary, summaries at the ends of some chapters, and for my final attempts to make some difficult points a little clearer. To all of these I am indebted and not less to Miss F. M. Costello, who triumphed over the chaos of preparing and correcting the manuscript and figures. In preparing this new edition, I owe much to my secretary, Mrs. Patricia J. Neill.

September, 1979 J. R. PIERCE

CHAPTER I *The World and Theories*

IN 1948, CLAUDE E. SHANNON published a paper called "A Mathematical Theory of Communication"; it appeared in book form in 1949. Before that time, a few isolated workers had from time to time taken steps toward a general theory of communication. Now, thirty years later, communication theory, or information theory as it is sometimes called, is an accepted field of research. Many books on communication theory have been published, and many international symposia and conferences have been held. The Institute of Electrical and Electronic Engineers has a professional group on information theory, whose *Transactions* appear six times a year. Many other journals publish papers on information theory.

All of us use the words communication and information, and we are unlikely to underestimate their importance. A modern philosopher, A. J. Ayer, has commented on the wide meaning and importance of communication in our lives. We communicate, he observes, not only information, but also knowledge, error, opinions, ideas, experiences, wishes, orders, emotions, feelings, moods. Heat and motion can be communicated. So can strength and weakness and disease. He cites other examples and comments on the manifold manifestations and puzzling features of communication in man's world.

Surely, communication being so various and so important, a theory of communication, a theory of generally accepted soundness and usefulness, must be of incomparable importance to all of us. When we add to *theory* the word *mathematical,* with all its implications of rigor and magic, the attraction becomes almost irresistible. Perhaps if we learn a few formulae our problems of communication will be solved, and we shall become the masters of information rather than the slaves of misinformation.

Unhappily, this is not the course of science. Some 2,300 years ago, another philosopher, Aristotle, discussed in his *Physics* a notion as universal as that of communication, that is, motion.

Aristotle defined motion as the fulfillment, insofar as it exists potentially, of that which exists potentially. He included in the concept of motion the increase and decrease of that which can be increased or decreased, coming to and passing away, and also being built. He spoke of three categories of motion, with respect to magnitude, affection, and place. He found, indeed, as he said, as many types of motion as there are meanings of the word *is.*

Here we see motion in all its manifest complexity. The complexity is perhaps a little bewildering to us, for the associations of words differ in different languages, and we would not necessarily associate motion with all the changes of which Aristotle speaks.

How puzzling this universal matter of motion must have been to the followers of Aristotle. It remained puzzling for over two millennia, until Newton enunciated the laws which engineers still use in designing machines and astronomers in studying the motions of stars, planets, and satellites. While later physicists have found that Newton's laws are only the special forms which more general laws assume when velocities are small compared with that of light and when the scale of the phenomena is large compared with the atom, they are a living part of our physics rather than a historical monument. Surely, when motion is so important a part of our world, we should study Newton's laws of motion. They say:

1. A body continues at rest or in motion with a constant velocity in a straight line unless acted upon by a force.

2. The change in velocity of a body is in the direction of the force acting on it, and the magnitude of the change is proportional to the force acting on the body times the time during which the force acts, and is inversely proportional to the mass of the body.

3. Whenever a first body exerts a force on a second body, the second body exerts an equal and oppositely directed force on the first body.

To these laws Newton added the universal law of gravitation:

4. Two particles of matter attract one another with a force acting along the line connecting them, a force which is proportional to the product of the masses of the particles and inversely proportional to the square of the distance separating them.

Newton's laws brought about a scientific and a philosophical revolution. Using them, Laplace reduced the solar system to an explicable machine. They have formed the basis of aviation and rocketry, as well as of astronomy. Yet, they do little to answer many of the questions about motion which Aristotle considered. Newton's laws solved the problem of motion as Newton defined it, not of motion in all the senses in which the word could be used in the Greek of the fourth century before our Lord or in the English of the twentieth century after.

Our speech is adapted to our daily needs or, perhaps, to the needs of our ancestors. We cannot have a separate word for every distinct object and for every distinct event; if we did we should be forever coining words, and communication would be impossible. In order to have language at all, many things or many events must be referred to by one word. It is natural to say that both men and horses run (though we may prefer to say that horses gallop) and convenient to say that a motor runs and to speak of a run in a stocking or a run on a bank.

The unity among these concepts lies far more in our human language than in any physical similarity with which we can expect science to deal easily and exactly. It would be foolish to seek some elegant, simple, and useful scientific theory of running which would embrace runs of salmon and runs in hose. It would be equally foolish to try to embrace in one theory all the motions discussed by Aristotle or all the sorts of communication and information which later philosophers have discovered.

In our everyday language, we use words in a way which is convenient in our everyday business. Except in the study of language itself, science does not seek understanding by studying words and their relations. Rather, science looks for things in nature, including our human nature and activities, which can be grouped together and understood. Such understanding is an ability to see what complicated or diverse events really do have in common (the planets in the heavens and the motions of a whirling skater on ice, for instance) and to describe the behavior accurately and simply.

The words used in such scientific descriptions are often drawn from our everyday vocabulary. Newton used force, mass, velocity, and attraction. When used in science, however, a particular meaning is given to such words, a meaning narrow and often new. We cannot discuss in Newton's terms force of circumstance, mass media, or the attraction of Brigitte Bardot. Neither should we expect that communication theory will have something sensible to say about every question we can phrase using the words communication or information.

A valid scientific theory seldom if ever offers the solution to the pressing problems which we repeatedly state. It seldom supplies a sensible answer to our multitudinous questions. Rather than rationalizing our ideas, it discards them entirely, or, rather, it leaves them as they were. It tells us in a fresh and new way what aspects of our experience can profitably be related and simply understood. In this book, it will be our endeavor to seek out the ideas concerning communication which can be so related and understood.

When the portions of our experience which can be related have been singled out, and when they have been related and understood, we have a *theory* concerning these matters. Newton's laws of motion form an important part of *theoretical physics,* a field called *mechanics.* The laws themselves are not the whole of the theory; they are merely the basis of it, as the axioms or postulates of geometry are the basis of geometry. The theory embraces both the assumptions themselves and the mathematical working out of the logical consequences which must necessarily follow from the assumptions. Of course, these consequences must be in accord with the complex phenomena of the world about us if the theory is to be a valid theory, and an invalid theory is useless.

The ideas and assumptions of a theory determine the *generality* of the theory, that is, to how wide a range of phenomena the theory applies. Thus, Newton's laws of motion and of gravitation are very general; they explain the motion of the planets, the timekeeping properties of a pendulum, and the behavior of all sorts of machines and mechanisms. They do not, however, explain radio waves.

Maxwell's equations[1] explain all (non-quantum) electrical phenomena; they are very general. A branch of electrical theory called *network theory* deals with the electrical properties of electrical circuits, or networks, made by interconnecting three sorts of idealized electrical structures: resistors (devices such as coils of thin, poorly conducting wire or films of metal or carbon, which impede the flow of current), inductors (coils of copper wire, sometimes wound on magnetic cores), and capacitors (thin sheets of metal separated by an insulator or dielectric such as mica or plastic; the Leyden jar was an early form of capacitor). Because network theory deals only with the electrical behavior of certain specialized and idealized physical structures, while Maxwell's equations describe the electrical behavior of any physical structure, a physicist would say that network theory is *less* general than are Maxwell's equations, for Maxwell's equations cover the behavior not only of idealized electrical networks but of all physical structures and include the behavior of radio waves, which lies outside of the scope of network theory.

Certainly, the most general theory, which explains the greatest range of phenomena, is the most powerful and the best; it can always be specialized to deal with simple cases. That is why physicists have sought a unified field theory to embrace mechanical laws and gravitation and all electrical phenomena. It might, indeed, seem that all theories could be ranked in order of generality, and, if this is possible, we should certainly like to know the place of communication theory in such a hierarchy.

Unfortunately, life isn't as simple as this. In one sense, network theory is less general than Maxwell's equations. In another sense,

[1] In 1873, in his treatise *Electrictity and Magnetism,* James Clerk Maxwell presented and fully explained for the first time the natural laws relating electric and magnetic fields and electric currents. He showed that there should be *electromagnetic waves* (radio waves) which travel with the speed of light. Hertz later demonstrated these experimentally, and we now know that light is electromagnetic waves. Maxwell's equations are the mathematical statement of Maxwell's theory of electricity and magnetism. They are the foundation of all electric art.

however, it is more general, for all the mathematical results of network theory hold for vibrating mechanical systems made up of idealized mechanical components as well as for the behavior of interconnections of idealized electrical components. In mechanical applications, a spring corresponds to a capacitor, a mass to an inductor, and a dashpot or damper, such as that used in a door closer to keep the door from slamming, corresponds to a resistor. In fact, network theory might have been developed to explain the behavior of mechanical systems, and it is so used in the field of acoustics. The fact that network theory evolved from the study of idealized electrical systems rather than from the study of idealized mechanical systems is a matter of history, not of necessity.

Because all of the mathematical results of network theory apply to certain specialized and idealized mechanical systems, as well as to certain specialized and idealized electrical systems, we can say that in a sense network theory is *more* general than Maxwell's equations, which do not apply to mechanical systems at all. In another sense, of course, Maxwell's equations are more general than network theory, for Maxwell's equations apply to all electrical systems, not merely to a specialized and idealized class of electrical circuits.

To some degree we must simply admit that this is so, without being able to explain the fact fully. Yet, we can say this much. Some theories are very strongly *physical* theories. Newton's laws and Maxwell's equations are such theories. Newton's laws deal with mechanical phenomena; Maxwell's equations deal with electrical phenomena. Network theory is essentially a *mathematical* theory. The terms used in it can be given various physical meanings. The theory has interesting things to say about different physical phenomena, about mechanical as well as electrical vibrations.

Often a mathematical theory is the offshoot of a physical theory or of physical theories. It can be an elegant mathematical formulation and treatment of certain aspects of a general physical theory. Network theory is such a treatment of certain physical behavior common to electrical and mechanical devices. A branch of mathematics called *potential theory* treats problems common to electric, magnetic, and gravitational fields and, indeed, in a degree to aerodynamics. Some theories seem, however, to be more mathematical than physical in their very inception.

We use many such mathematical theories in dealing with the physical world. Arithmetic is one of these. If we label one of a group of apples, dogs, or men 1, another 2, and so on, and if we have used up just the first 16 numbers when we have labeled all members of the group, we feel confident that the group of objects can be divided into two equal groups each containing 8 objects ($16 \div 2 = 8$) or that the objects can be arranged in a square array of four parallel rows of four objects each (because 16 is a perfect square; $16 = 4 \times 4$). Further, if we line the apples, dogs, or men up in a row, there are 2,092,278,988,800 possible sequences in which they can be arranged, corresponding to the 2,092,278,-988,800 different sequences of the integers 1 through 16. If we used up 13 rather than 16 numbers in labeling the complete collection of objects, we feel equally certain that the collection could not be divided into any number of equal heaps, because 13 is a prime number and cannot be expressed as a product of factors.

This seems not to depend at all on the nature of the objects. Insofar as we can assign numbers to the members of any collection of objects, the results we get by adding, subtracting, multiplying, and dividing numbers or by arranging the numbers in sequence hold true. The connection between numbers and collections of objects seems so natural to us that we may overlook the fact that arithmetic is itself a mathematical theory which can be applied to nature only to the degree that the properties of numbers correspond to properties of the physical world.

Physicists tell us that we can talk sense about the total number of a group of elementary particles, such as electrons, but we can't assign particular numbers to particular particles because the particles are in a very real sense indistinguishable. Thus, we can't talk about arranging such particles in different orders, as numbers can be arranged in different sequences. This has important consequences in a part of physics called *statistical mechanics.* We may also note that while Euclidean geometry is a mathematical theory which serves surveyors and navigators admirably in their practical concerns, there is reason to believe that Euclidean geometry is not quite accurate in describing astronomical phenomena.

How can we describe or classify theories? We can say that a theory is very narrow or very general in its scope. We can also distinguish theories as to whether they are strongly physical or strongly mathematical. Theories are strongly physical when they describe very completely some range of physical phenomena, which in practice is always limited. Theories become more mathematical or abstract when they deal with an idealized class of phenomena or with only certain aspects of phenomena. Newton's laws are strongly physical in that they afford a complete description of mechanical phenomena such as the motions of the planets or the behavior of a pendulum. Network theory is more toward the mathematical or abstract side in that it is useful in dealing with a variety of idealized physical phenomena. Arithmetic is very mathematical and abstract; it is equally at home with one particular property of many sorts of physical entities, with numbers of dogs, numbers of men, and (if we remember that electrons are indistinguishable) with numbers of electrons. It is even useful in reckoning numbers of days.

In these terms, communication theory is both very strongly mathematical and quite general. Although communication theory grew out of the study of electrical communication, it attacks problems in a very abstract and general way. It provides, in the *bit*, a universal measure of amount of information in terms of choice or uncertainty. Specifying or learning the choice between two equally probable alternatives, which might be messages or numbers to be transmitted, involves one bit of information. Communication theory tells us how many bits of information can be sent per second over perfect and imperfect communication channels in terms of rather abstract descriptions of the properties of these channels. Communication theory tells us how to measure the rate at which a message source, such as a speaker or a writer, generates information. Communication theory tells us how to represent, or *encode*, messages from a particular message source efficiently for transmission over a particular sort of channel, such as an electrical circuit, and it tells us when we can avoid errors in transmission.

Because communication theory discusses such matters in very general and abstract terms, it is sometimes difficult to use the understanding it gives us in connection with particular, practical problems. However, because communication theory has such an abstract and general mathematical form, it has a very broad field of application. Communication theory is useful in connection with written and spoken language, the electrical and mechanical transmission of messages, the behavior of machines, and, perhaps, the behavior of people. Some feel that it has great relevance and importance to physics in a way that we shall discuss much later in this book.

Primarily, however, communication theory is, as Shannon described it, a *mathematical* theory of communication. The concepts are formulated in mathematical terms, of which widely different physical examples can be given. Engineers, psychologists, and physicists may use communication theory, but it remains a mathematical theory rather than a physical or psychological theory or an engineering art.

It is not easy to present a mathematical theory to a general audience, yet communication theory is a mathematical theory, and to pretend that one can discuss it while avoiding mathematics entirely would be ridiculous. Indeed, the reader may be startled to find equations and formulae in these pages; these state accurately ideas which are also described in words, and I have included an appendix on mathematical notation to help the nonmathematical reader who wants to read the equations aright.

I am aware, however, that mathematics calls up chiefly unpleasant pictures of multiplication, division, and perhaps square roots, as well as the possibly traumatic experiences of high-school classrooms. This view of mathematics is very misleading, for it places emphasis on special notation and on tricks of manipulation, rather than on the aspect of mathematics that is most important to mathematicians. Perhaps the reader has encountered theorems and proofs in geometry; perhaps he has not encountered them at all, yet theorems and proofs are of primary importance in all mathematics, pure and applied. The important results of information theory are stated in the form of mathematical theorems, and these *are* theorems only because it is possible to prove that they are true statements.

Mathematicians start out with certain assumptions and definitions, and then by means of mathematical arguments or proofs they are able to show that certain statements or theorems are true. This is what Shannon accomplished in his "Mathematical Theory of Communication." The truth of a theorem depends on the validity of the assumptions made and on the validity of the argument or proof which is used to establish it.

All of this is pretty abstract. The best way to give some idea of the meaning of *theorem* and *proof* is certainly by means of examples. I cannot do this by asking the general reader to grapple, one by one and in all their gory detail, with the difficult theorems of communication theory. Really to understand thoroughly the proofs of such theorems takes time and concentration even for one with some mathematical background. At best, we can try to get at the content, meaning, and importance of the theorems.

The expedient I propose to resort to is to give some examples of simpler mathematical theorems and their proof. The first example concerns a game called *hex,* or *Nash.* The theorem which will be proved is that the player with first move can win.

Hex is played on a board which is an array of forty-nine hexagonal cells or spaces, as shown in Figure I-1, into which markers may be put. One player uses black markers and tries to place them so as to form a continuous, if wandering, path between the black area at the left and the black area at the right. The other player uses white markers and tries to place them so as to form a continuous, if wandering, path between the white area at the top and the white area at the bottom. The players play alternately, each placing one marker per play. Of course, one player has to start first.

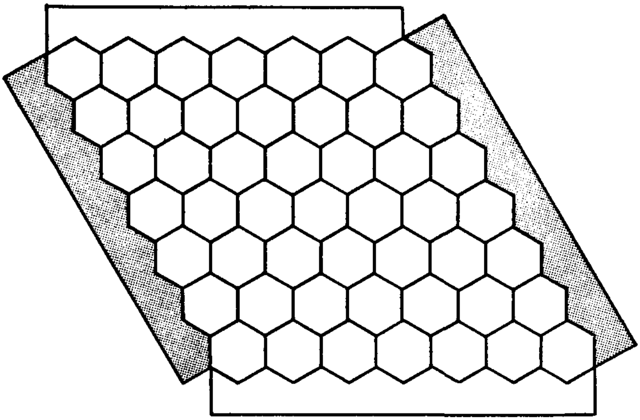

Fig. I-1

In order to prove that the first player can win, it is necessary first to prove that when the game is played out, so that there is either a black or a white marker in each cell, one of the players must have won.

Theorem I: *Either one player or the other wins.*

Discussion: In playing some games, such as chess and ticktack-toe, it may be that neither player will win, that is, that the game will end in a draw. In matching heads or tails, one or the other necessarily wins. What one must show to prove this theorem is that, when each cell of the hex board is covered by either a black or a white marker, either there must be a black path between the black areas which will interrupt any possible white path between the white areas or there must be a white path between the white areas which will interrupt any possible black path between the black areas, so that either white or black must have won.

Proof: Assume that each hexagon has been filled in with either a black or a white marker. Let us start from the left-hand corner of the upper white border, point I of Figure I-2, and trace out the boundary between white and black hexagons or borders. We will proceed always along a side with black on our right and white on our left. The boundary so traced out will turn at the successive corners, or vertices, at which the sides of hexagons meet. At a corner, or vertex, we can have only two essentially different con-

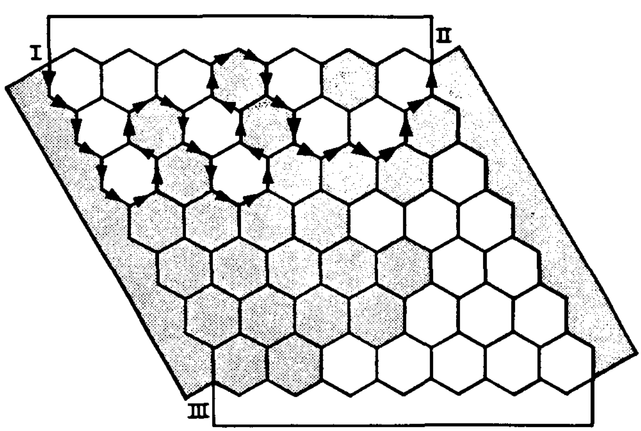

Fig. I-2

ditions. Either there will be two touching black hexagons on the right and one white hexagon on the left, as in *a* of Figure I-3, or two touching white hexagons on the left and one black hexagon on the right, as shown in *b* of Figure I-3. We note that in either case there will be a continuous black path to the right of the boundary and a continuous white path to the left of the boundary. We also note that in neither *a* nor *b* of Figure I-3 can the boundary cross or join itself, because only one path through the vertex has black on the right and white on the left. We can see that these two facts are true for boundaries between the black and white borders and hexagons as well as for boundaries between black and white hexagons. Thus, along the left side of the boundary there must be a continuous path of white hexagons to the upper white border, and along the right side of the boundary there must be a continuous path of black hexagons to the left black border. As the boundary cannot cross itself, it cannot circle indefinitely, but must eventually reach a black border or a white border. If the boundary reaches a black border or white border with black on its right and white on its left, as we have prescribed, at any place except corner II or corner III, we can extend the boundary further with black on its right and white on its left. Hence, the boundary will reach either point II or point III. If it reaches point II, as shown in Figure I-2, the black hexagons on the right, which are connected to the left black border, will also be connected to the right black border, while the white hexagons to the left will be connected to the upper white border only, and black will have won. It is clearly impossible for white to have won also, for the continuous band of adjacent

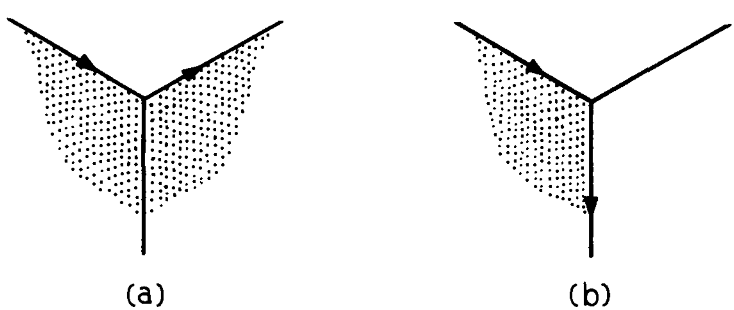

Fig. I-3

black cells from the left border to the right precludes a continuous band of white cells to the bottom border. We see by similar argument that, if the boundary reaches point III, white will have won.

Theorem II: *The player with the first move can win.*

Discussion: By *can* is meant that there exists a way, if only the player were wise enough to know it. The method for winning would consist of a particular first move (more than one might be allowable but are not necessary) and a chart, formula, or other specification or recipe giving a correct move following any possible move made by his opponent at any subsequent stage of the game, such that if, each time he plays, the first player makes the prescribed move, he will win regardless of what moves his opponent may make.

Proof: Either there must be some way of play which, if followed by the first player, will insure that he wins or else, no matter how the first player plays, the second player must be able to choose moves which will preclude the first player from winning, so that he, the second player, will win. Let us assume that the player with the second move does have a sure recipe for winning. Let the player with the first move make his first move in any way, and then, after his opponent has made one move, let the player with the first move apply the hypothetical recipe which is supposed to allow the player with the second move to win. If at any time a move calls for putting a piece on a hexagon occupied by a piece he has already played, let him place his piece instead on any unoccupied space. The designated space will thus be occupied. The fact that by starting first he has an extra piece on the board may keep his opponent from occupying a particular hexagon but not the player with the extra piece. Hence, the first player can occupy the hexagons designated by the recipe and must win. This is contrary to the original assumption that the player with the second move can win, and so this assumption must be false. Instead, it must be possible for the player with the first move to win.

A mathematical purist would scarcely regard these proofs as rigorous in the form given. The proof of theorem II has another curious feature; it is not a *constructive* proof. That is, it does not show the player with the first move, who can win in principle, how to go about winning. We will come to an example of a constructive proof in a moment. First, however, it may be appropriate to philosophize a little concerning the nature of theorems and the need for proving them.

Mathematical theorems are inherent in the rigorous statement of the general problem or field. That the player with the first move can win at hex is necessarily so once the game and its rules of play have been specified. The theorems of Euclidean geometry are necessarily so because of the stated postulates.

With sufficient intelligence and insight, we could presumably see the truth of theorems immediately. The young Newton is said to have found Euclid's theorems obvious and to have been impatient with their proofs.

Ordinarily, while mathematicians may suspect or conjecture the truth of certain statements, they have to prove theorems in order to be certain. Newton himself came to see the importance of proof, and he proved many new theorems by using the methods of Euclid.

By and large, mathematicians have to proceed step by step in attaining sure knowledge of a problem. They laboriously prove one theorem after another, rather than seeing through everything in a flash. Too, they need to prove the theorems in order to convince others.

Sometimes a mathematician needs to prove a theorem to convince himself, for the theorem may seem contrary to common sense. Let us take the following problem as an example: Consider the square, 1 inch on a side, at the left of Figure I-4. We can specify any point in the square by giving two numbers, y, the height of the point above the base of the square, and x, the distance of the point from the left-hand side of the square. Each of these numbers will be less than one. For instance, the point shown will be represented by

$x = 0.547000 \ldots$ (ending in an endless sequence of zeros)
$y = 0.312000 \ldots$ (ending in an endless sequence of zeros)

Suppose we pair up points on the square with points on the line, so that every point on the line is paired with just one point on the square and every point on the square with just one point on the line. If we do this, we are said to have *mapped* the square onto the line in a *one-to-one* way, or to have achieved a *one-to-one mapping* of the square onto the line.

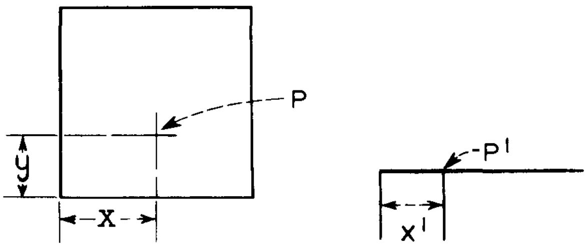

Fig. 1-4

Theorem: *It is possible to map a square of unit area onto a line of unit length in a one-to-one way.*[2]

Proof: Take the successive digits of the height of the point in the square and let them form the first, third, fifth, and so on digits of a number x'. Take the digits of the distance of the point P from the left side of the square, and let these be the second, fourth, sixth, etc., of the digits of the number x'. Let x' be the distance of the point P' from the left-hand end of the line. Then the point P' maps the point P of the square onto the line uniquely, in a one-to-one way. We see that changing either x or y will change x' to a new and appropriate number, and changing x' will change x and y. To each point x,y in the square corresponds just one point x' on the line, and to each point x' on the line corresponds just one point x,y in the square, the requirement for one-to-one mapping.[3]

In the case of the example given before

$$x = 0.547000\ldots$$
$$y = 0.312000\ldots$$
$$x' = 0.351427000\ldots$$

In the case of most points, including those specified by irrational numbers, the endless string of digits representing the point will not become a sequence of zeros nor will it ever repeat.

Here we have an example of a constructive proof. We show that we can map each point of a square into a point on a line segment in a one-to-one way by giving an explicit recipe for doing this. Many mathematicians prefer constructive proofs to proofs which

[2] This has been restricted for convenience; the size doesn't matter.

[3] This proof runs into resolvable difficulties in the case of some numbers such as ½, which can be represented decimally .5 followed by an infinite sequence of zeros or .4 followed by an infinite sequence of nines.

are not constructive, and mathematicians of the intuitionist school reject nonconstructive proofs in dealing with infinite sets, in which it is impossible to examine all the members individually for the property in question.

Let us now consider another matter concerning the mapping of the points of a square on a line segment. Imagine that we move a pointer along the line, and imagine a pointer simultaneously moving over the face of the square so as to point out the points in the square corresponding to the points that the first pointer indicates on the line. We might imagine (contrary to what we shall prove) the following: If we moved the first pointer slowly and smoothly along the line, the second pointer would move slowly and smoothly over the face of the square. All the points lying in a small cluster on the line would be represented by points lying in a small cluster on the face of the square. If we moved the pointer a short distance along the line, the other pointer would move a short distance over the face of the square, and if we moved the pointer a shorter distance along the line, the other pointer would move a shorter distance across the face of the square, and so on. If this were true we could say that the one-to-one mapping of the points of the square into points on the line was *continuous.*

However, it turns out that a one-to-one mapping of the points in a square into the points on a line cannot be continuous. As we move smoothly along a curve through the square, the points on the line which represent the successive points on the square *necessarily* jump around erratically, not only for the mapping described above but for any one-to-one mapping whatever. Any one-to-one mapping of the square onto the line is *discontinuous.*

Theorem: *Any one-to-one mapping of a square onto a line must be discontinuous.*

Proof: Assume that the one-to-one mapping is continuous. If this is to be so then all the points along some arbitrary curve AB of Figure I-5 on the square must map into the points lying between the corresponding points A' and B'. If they did not, in moving along the curve in the square we would either jump from one end of the line to the other (discontinuous mapping) or pass through one point on the line twice (not one-to-one mapping). Let us now choose a point C' to the left of line segment $A'B'$ and D' to the right of $A'B'$ and locate the corresponding points C and D in the

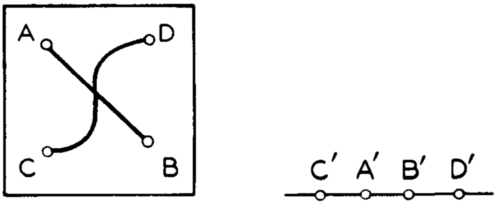

Fig. I-5

square. Draw a curve connecting C and D and crossing the curve from A to B. Where the curve crosses the curve AB it will have a point in common with AB; hence, this one point of CD must map into a point lying between A' and B', and all other points which are not on AB must map to points lying outside of $A'B'$, either to the left or the right of $A'B'$. This is contrary to our assumption that the mapping was continuous, and so the mapping cannot be continuous.

We shall find that these theorems, that the points of a square can be mapped onto a line and that the mapping is necessarily discontinuous, are both important in communication theory, so we have proved one theorem which, unlike those concerning hex, will be of some use to us.

Mathematics is a way of finding out, step by step, facts which are inherent in the statement of the problem but which are not immediately obvious. Usually, in applying mathematics one must first hit on the facts and then verify them by proof. Here we come upon a knotty problem, for the proofs which satisfied mathematicians of an earlier day do not satisfy modern mathematicians.

In our own day, an irascible minor mathematician who reviewed Shannon's original paper on communication theory expressed doubts as to whether or not the author's mathematical intentions were honorable. Shannon's theorems are true, however, and proofs have been given which satisfy even rigor-crazed mathematicians. The simple proofs which I have given above as illustrations of mathematics are open to criticism by purists.

What I have tried to do is to indicate the nature of mathematical reasoning, to give some idea of what a theorem is and of how it may be proved. With this in mind, we will go on to the mathematical theory of communication, its theorems, which we shall not really prove, and to some implications and associations which extend beyond anything that we can establish with mathematical certainty.

As I have indicated earlier in this chapter, communication theory as Shannon has given it to us deals in a very broad and abstract way with certain important problems of communication and information, but it cannot be applied to all problems which we can phrase using the words *communication* and *information* in their many popular senses. Communication theory deals with certain aspects of communication which *can* be associated and organized in a useful and fruitful way, just as Newton's laws of motion deal with mechanical motion only, rather than with all the named and indeed different phenomena which Aristotle had in mind when he used the word *motion.*

To succeed, science must attempt the possible. We have no reason to believe that we can unify all the things and concepts for which we use a common word. Rather we must seek that part of experience which can be related. When we have succeeded in relating certain aspects of experience we have a theory. Newton's laws of motion are a theory which we can use in dealing with mechanical phenomena. Maxwell's equations are a theory which we can use in connection with electrical phenomena. Network theory we can use in connection with certain simple sorts of electrical *or* mechanical devices. We can use arithmetic very generally in connection with numbers of men, stones, or stars, and geometry in measuring land, sea, or galaxies.

Unlike Newton's laws of motion and Maxwell's equations, which are strongly physical in that they deal with certain classes of physical phenomena, communication theory is abstract in that it applies to many sorts of communication, written, acoustical, or electrical. Communication theory deals with certain important but abstract aspects of communication. Communication theory proceeds from clear and definite assumptions to theorems concerning information sources and communication channels. In this it is essentially mathematical, and in order to understand it we must understand the idea of a theorem as a statement which must be proved, that is, which must be shown to be the necessary consequence of a set of initial assumptions. This is an idea which is the very heart of mathematics as mathematicians understand it.