Filozofická fakulta Univerzity Karlovy v Praze

Ústav filozofie a religionistiky

Bakalářská práce

THE PHILOSOPHICAL PROBLEM OF IDENTITY AND CATEGORY THEORY

ŠIMON POŠTA

Prague 12.8.2024 Supervisor: Mgr. Vít Punčochář, Ph.D.

Abstract

Logically foundational mathematical theories such as ZFC suggest, by way of their underlying pragmatic drive, a grim disregard for the ontological substrate. This thesis attempts to inspirit the foundation of a comparative metaontology over a uniformly structured scheme of ontology-formation, capable of hosting involved formal systems and providing thereby structural insight into the ontological impositions made by the adoption of a given mathematical or non-mathematical formal framework. Beyond this, it searches for understanding of the place of identity.

The spine of a universal ontological structure is excavated. The essential role of identity is divulged as it permeates each layer of the system itself and over it facilitates the game of ignorance formative of the confines of mathematical investigation. The canonical first order logic presentation of ZFC is gradually formulated for the first test case of an ontological interpretation, accomodating here the multiverse view. A comparative is given by a lightly modified Lawverian categorical foundation CCAF, encompassing Lawvere's original categorical set theory ETCS. Identity criteria shift is followed from Set extensionality of ZFC to a dynamical setting natural to Category Theory-based foundations. Full ontological interpretation is provided for the enriched CCAF along with elaboration on the extent of abstraction of such foundational theories with respect to their models. Final is the comparison of ZFC and CCAF on ontological and epistemological grounds.

Keywords

mathematical foundations, foundational theories, categorical foundations, CCAF, ETCS, ontology, metaontology, mathematical ontology, identity, identity criteria

Abstrakt

Logicky zakládající matematické teorie, jakou je kupříkladu ZFC, vedou v důsledku pragmatických vůdčích principů, jimž se podřizovalo jejich založení, k představě nebezpečného zanedbání vlastního ontologického substrátu. Má práce se pokouší o položení základů komparativní metaontologie budované nad jednotně strukturovaným schématem produkce samotných ontologií, schopných hostit vnitřně komplikované systémy, čímž chce poskytnout strukturní vhled v ontologické závazky přinášené přijetím daného matematického či zcela nematematického formálního rámce. Nad tím se pokouší porozumět roli identity.

Představují se základy univerzální ontologické struktury. Ukazuje se podstatnost kritérií identity sahajících napříč všemi úrovněmi onoho systému. Rozvádí se návaznost hry předstírané nevědomosti na povahu identit uvnitř ontologií. Postupně se předkládá kanonická prvořádová prezentace ZFC jakožto první zkušební případ ontologické interpretace formálního systému, který se ukáže být překloněný mnohem spíše množinovému multiverse view než universe view. Pro porovnání je zpracována také lehce upravená zakládající teorie v Lawverově stylu zvaná CCAF, pojímající přímo jeho původní kategorickou teorii množin ETCS. Je zkoumána proměna pojetí identit, kterou sledujeme v přechodu od množinové extenzionality ZFC k dynamickému rozvržení identit přirozenějšímu zakládajícím teoriím věrně povahou odpovídajícím teorii kategorií. Také obohacená teorie CCAF dostává úplnout ontologickou interpretaci, spolu s podrobnějším výkladem vztahu míry abstrakce zakládajících teorií vzhledem k jejich vlastním modelům. Závěrem jsou obě představené teorie podrobeny porovnání charakteristik ontologických a epistemologických.

Klíčová slova

matematické základy, zakládající teorie, kategorické základy, CCAF, ETCS, ontologie, metaontologie, matematická ontologie, identita, kritéria identity

Acknowledgements

I would like to thank my supervisor Vít Punčochář, Ph.D. for all his help. The consultations, notes and advice have been instrumental.

Prohlašuji, že jsem bakalářskou práci vypracoval samostatně, že jsem řádně citoval všechny použité prameny a literaturu a že práce nebyla využita v rámci jiného vysokoškolského studia či k získání jiného nebo stejného titulu.

. . . .

V Praze dne 12. srpna 2024

Contents

List of Figures

Acronyms

iff if and only if

wrt with respect to

st such that

FOL first order logic

ACU absolute conceptual universe

SO source ontology

CoP collection of preobjects

TO target ontology

ZFC Zermelo Fraenkel with Choice

CH continuum hypothesis

CCAF category of categories and functors

ETCS elementary theory of categories and sets

ETCS+R elementary theory of categories and sets with replacement

Chapter 1

On identity in ontology

1.1 Global introduction

The purpose of this thesis is two-fold: first developing a uniform conceptual system capable of grounding the ontologically-foundational claims of mathematical theories such as Zermelo Fraenkel with Choice (ZFC) and Category of Categories and Functors (CCAF), in an actual ontological framework, posed in such a way as to allow for genuinely ontological structural comparison, on both the formal framing as a symbolic system and the universe produced as therein captured virtual world; second understanding the internal structure of the concept of identity as it lays itself out in the formation of a general ontology – what kinds of identities are relevant for the building of such a system, which of them are necessary, which contingent additions, what decides among those that are not necessary, what constitutes naturality of an identity with respect to a species of ontology as well as the identity of such species, and more specifically the differences in identity native to two mathematically prominent species – set-theoretic and category-theoretic foundational theory, particular and structural respectively. Apart from its intrinsic value as at least an exceptionally robust virtual world, mathematical foundational theories make for a perfect test case, due to already being essentially pre-formatted over, in our choice, a canonical symbolic system of first order logic (FOL). Giving them a uniquely appealing uniformity of presentation, consequently facilitating not only the individual translation process, but more importantly the comparison itself. The contrast of Set theory and Category theory is then justified in consulting [24] by Jean-Pierre Marquis, wherein he argues for the inclusion of other identity criteria than is the set-theoretic extensionality into the class of concepts seen as fundamental and natural to modern mathematics, that is by considering a wider sense of identity criteria for mathematical entities, in particular discussing the equivalence of Categories as its explicit weakening and foundational systems other than ZFC better suited to hosting them even going so far as making them structurally integral. Our vantage point in the second aspect of the program will thus be the set-theoretical extensionality itself, from which we will diversify by adding criteria intensional, taken in the most vulgar terminology under Feferman's [9, p.2] slogan "getting along without extensionality". [3, p.S1195]

1.2 Local introduction

The purpose of the first chapter is to explore the plurality of relevant identities in ontology and organize them according to their applicability conditions establishing thereby incidentally also their foundation with respect to the ontological system, to which end we shall propose a model of ontologies intended to cover the formalization of foundational mathematical theories and host their comparison. In doing so we must stand on Frege's shoulders, for it is his brilliant questioning I suspect mainly responsible for the cultural heritage of extensive logical formalism applicable to questions of ontology in philosophy and mathematics, through his parallel treatment of sign and concept word under the fundamental analysis of Sense/Sinn and Meaning/Bedeutung [10, 11, 12] finding its way through Russell [31], Wittgenstein [33] and ultimately Carnap [6] to the notion of Extension so deeply imbued in modern logic and mathematics. Extension we find pre-eminent in the classical foundational theory of mathematics following Cantor's innovation, with whom we are going to enter into disputations in the course of the second chapter. Classical Set theory of Zermelo and Fraenkel along with the axiom of choice sets the notion of extension at the very core of mathematical formalism via the axiom of extensionality: $\forall u(u \in x \iff u \in y) \rightarrow (x = y)$ which under the standard logical axioms defining behaviour of equality "=" yield the notorious formulation of the extensional identity of objects: $(x = y) \iff \forall u(u \in x \iff u \in y)$ - two objects x and y are identical if and only if so are the collections of their constituents seen as loose collections rather than entities doing the collecting. We are going to develop an open model of a general ontological system with the intention of subsuming ZFC and a rivalling foundation coming from Category Theory as its instances further set within a larger system capable of comparing them on at least quasi-formal grounds. In its development we shall encounter a variety of kinds of identity criteria relevant to the system, of extensional and intensional natures alike, whom we will try to contextualize and problematize.

The first chapter itself gradually builds up all the essential structural features of the general model of ontological systems, which will be used to interpret the two axiomatic theories in chapters 2 and 3. As such, it can serve as a reference point for the notions utilized within the three remaining chapters. We consider the motivating questions that lead to the formalization of the general ontological system, its common structure shared by all particulars, provide it a loose epistemological grounding, (the resolution of which can be modified depending on the degree of reflection admitted into the system), we then capture some essential identity criteria over the layers of the system, and finally delineate a way of comparing two ontologies, which will be the content of the concluding fourth chapter. The features we discuss are the essentials necessary for the very first attempt at interpreting an actual real-world system in a semi-uniform manner through the formal ontological framework.

1.3 The motivating questions, the preobjects and the source ontology

There are two ways to first considerations of basic equality identity check "=" on objects of an ontology, equality being a particular element of a class of identities. Our begins by noticing that in its natural form it is not a positive question asking whether A is the same as B, but instead a derivation of the original negative meaning of when and how can we differentiate an object into two distinct entities A and B. It is establishing the constraints of the shape of our objects in general. And yet in asking ?x = y on one level we are actually stipulating their difference, as for the possibility of asking the question at all there must be such virtually distinct objects x and y. We thus have the ontological foundation and over it the epistemic level of ignorance, together requiring a base collection of "preobjects", which are objects only virtually with respect to the constructed ontology. It is the collection of possibly distinct entities we project as being objective when the ontological foundations are fully laid down. However, such a collection of preobjects cannot be considered in isolation, and so we shall henceforth refer to a Source Ontology (SO) hosting it as the foundation upon which to conceive the constructed ontology. As we explore ZFC and CCAF we will zone in on what preobjects are plausible, for now they might be as different as Frege's Morning star and Evening star, $\{\emptyset\}$ and $\{N\}$ or the discrete category we will come to know as **1** of one object and one enforced identity morphism and a category **1'** of three objects, three identity morphisms, and 3 pairs of isomorphic morphisms making them into a triangle. Of each pair of the proposed preobjects we could ask "are they identical?" and answer yes for each. But as preobjects of whom the question is asked also give three negative answers in turn, precisely because it can be asked. The question is thus incomplete. If it is to be answered properly, it must specify the context and level of resolution. As such it is part of a game of pretend ignorance, with many variations across mathematics in particular, often under an implicit expectation of agreement made more palletable by a shared accepted logical environment.

1.4 First sketch of a target ontology and the absolute conceptual universe

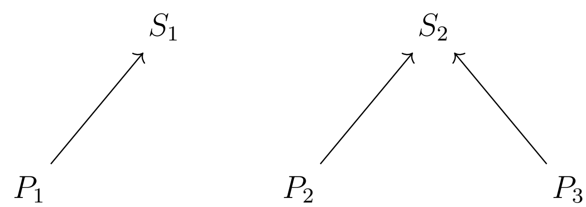
The second part of our system, the Target Ontology (TO) has as one of its motivations fixing the setting of such questions, so that answers can be extracted uniformly. The collections of preobjects entering these questions must very clearly be in some way structured. In the extreme case we won't broach as processed symbols x, y anchored in a surprisingly rich context of intermingling concepts hoped to be ignored in an abstraction they are supposed to act out. Such context, or structuring as we shall call it henceforth, should not be assumed wiped through abstraction without explicit acknowledgement. It ought, further, to be epistemologically grounded, even if the grounding place should be excessively general and in virtue of it so very close to triviality as ours is.

We will speak of the absolute conceptual universe (ACU) as a collection of all presently accessible "concepts" structured by self-application - interconnected by relations of modalities determined by the concepts themselves, meaning the concept of multiplicity doesn't exist on just one level of homogeneously posed entities, but applies to all the concepts together, as does unity to each individually as well as to complexes and so on. Concepts not as entities of language but that to which language directly or indirectly refers, that which language carries on the tongue, which incidentally calls on gustatory, olfactory, tactile and auditory and visual forms of experience because they also refers to, and carry the concepts. We name it a universe for the most naive illustration of a homogeneous plane inhabited by points (the concepts) constituting the core of any frame of reference with a genuine ambition to absoluteness, as that which must necessarily be accessible to anyone who might ask for such a thing in the first place. Not as a static thing to be undermined in retrospect reproof, but as the fluidum of life. As such it says very little except that it must fit the function for which we commission it – grounding whatever preobjects we might conceive of as that over which they come about through a nested, repeating, self-structuring selection over the total homogeneous collection of concepts in accordance with the naive illustration we wish to maintain, and thus the singular seat of all structured collections of preobjects (CoP). The SO will in accordance consist of ACU as well as the selection determining a particular CoP, whatever they may be.

1.5 The joining of SO and TO, brute identity criterion

The TO will in turn consist of an underlying foundation of CoP inherited from its SO, among which it selects once more to produce a “sortal layer” of the target ontology by a particular choice of identity criteria collapsing or differentiating some features captured within the structure of the inherited CoP and producing thereby the sortal layer’s sorts such as might be the traditional sorts Substances S and Properties P – for example S having constituents S_1, S_2, P having constituents P_1, P_2, P_3 characterized respectively under a single sort by a shared pattern of high-level identity criteria, and differentiated as constituents of the same sort dually by a disjoint pattern of low-level identity criteria. Their interrelation, interaction and dependence is then the content of the “behavioural layer” binding them together in erecting the final level of the TO - the “universe produced” taken as a network of all the sortal constituents, the space to host them.

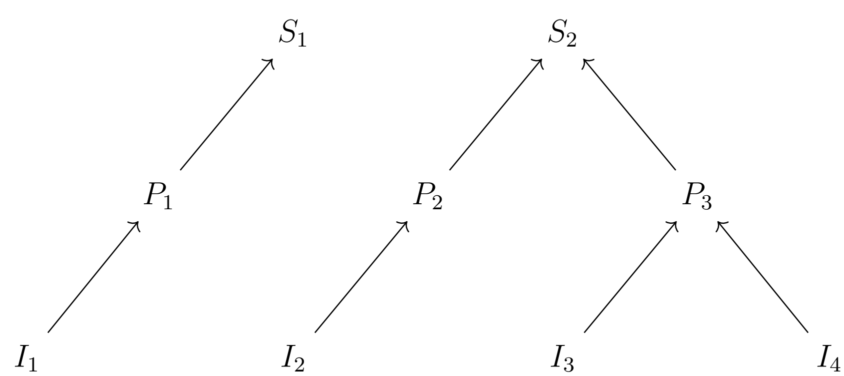
Falling back onto our example of two sorts, Substances and Properties, the behavioural layer embodies the choices of identity criteria for both the sorts as sorts and the sortal constituents of the same one sort between eachother. S_1 may be characterized by instantiating P_1, whereas S_2 instantiates P_2 as well as P_3, these are in turn said to be determined by the respective Substance sort constituents. (determination and instantiation are simply dual)



Imagine eating two dishes, given as a pill each. Pill 1 floods your mind with a singular reference, bitterness. Pill 2, however, comes in a mixture of sour sweetness. Each dish is thus determined (apart from the distracting context of its pill form) by reference to these 3 concepts of a shared form. Because you know nothing about them in this abstraction, they become determined only by instantiating the 3 concepts.

That is an aspect of a relative form of dependence we will encounter throughout the text – that of referential sort over a basic sort. It is a dependence within a sequential perspective of a universe stratified – separated by levels of dependence, each level partition representing a perspectival shift of the basic-referential.

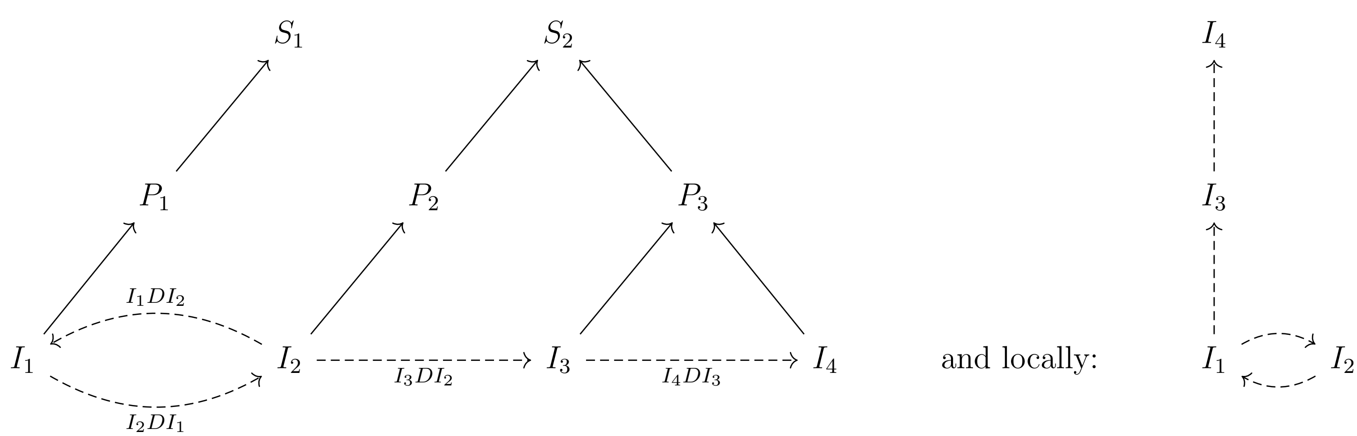
To illustrate this, we introduce a third sort Information (I) of four constituents I_1, I_2, I_3 and I_4, s.t. I_1 is instantiated by P_1, I_2 by P_2, I_3 and I_4 by P_3



We can distinguish local and global dependence of the basic-referential pairs. I is globally basic (meaning I exists in this system only in the basic modality, it is not referential to any sort and if it is at all related to it, it must be basic to its referentiality) but locally basic only to P. P is neither basic nor referential globally, but locally referential to I and basic to S. S is then globally referential, locally referential to P. In our depiction being relatively basic is represented by being the source of some arrow, relatively referential by being the target. The basic must be fixed before we can speak of its referential complement, because its identity is characterized by the basic, which must be made available to this end. This gives a sequence of dependence within the pretend openness externally plastered over the ontology, and shows the dichotomy of perspectives – open and closed. The latter still welcomes the dependence of this type, but unlike the open perspective it doesn't host ignorance as to the identity character of the constituents – the behavioural level is until the last step of the sequence indeterminate w.r.t. the possibility space established by that which has already been fixed below. We might for example ask within this open perspective what properties distinguish the two substances, assuming merely that they indeed happen to be just two in number – that is already a play on a "brute" identity by "multiplicity" which we can see take place for Informations at the base of the sequence. – We say merely there are 4 Informations, but refrain from any external reference in their characterization. In our depiction the level of vertical brute identity is signified by the absence of any arriving arrows - Informations here are not targets of any arrow. We have nothing to refer to in saying one is different from the other, whereas the other two sorts are not so "brute". This brute identity by multiplicity must be at the bottom of any such complex systems in one variation or another, it is utterly unavoidable except if we allow for extra-ontological reference, which can ultimately be sortified, translocating the burden onto one such later addition. Brute identity is at the foundation of extensional as well as intensional identity criteria, it is by itself a coarse setting of the bedrock.

1.6 Sequential and simultaneous perspectives

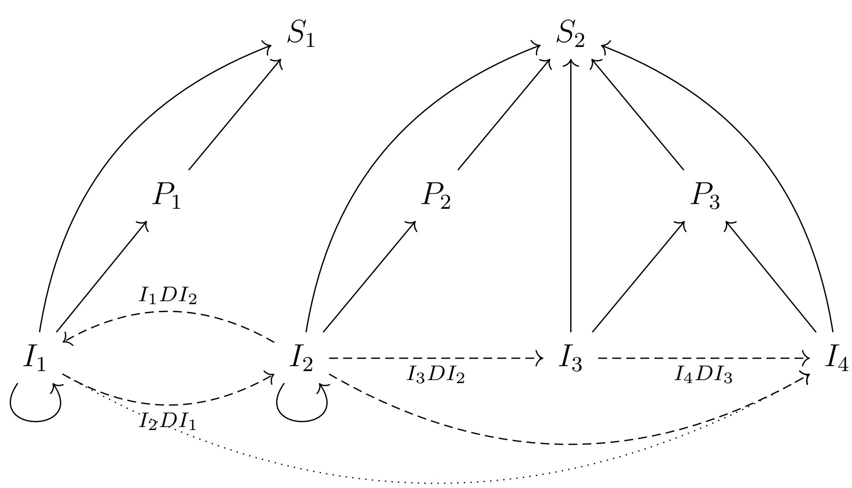
So far we have only seen the sequential perspective, but it too has a complement in the simultaneous perspective. For that, we will consider again the third Information sort of our running example. Instead of mere identity criteria by brute multiplicity, we will now reform the Information sort so that the constituent I_1 itself refers to I_2, which refers back to I_1. I_3 then refers to I_1 and I_4 to I_3. Informations can be thought of as the basic building blocks bootstrapping all experience as entities embodying its self-description, giving thus rise to every concept, hence also Properties and through them Substances.



We call such sort an "autonomous amphibian". Meaning, none of its constituents depend in their characterization on anything outside the sort itself, and they are both basic and referential – this time internally. Not only can the sorts themselves be relatively basic and referential, but so can their constituents. The vertical dependence of the higher on the lower sort, the referential on the basic sort, can also have analogue inside the sorts themselves - I_3 depends on I_1 but the novelty comes by having a two-way dependence, hither and thither - I_1 depends on I_2 and conversely I_2 depends on I_1 so they must be on the same dependence level. In particular I_3 is basic to I_4's referential, I_3 is referential to I_1's basic, but importantly I_1 and I_2 as a pair refer to nothing outside. They constitute the bottom stratum of the Information sequential hierarchy. I_1 is basic to I_2's referential, but conversely so is I_2 also basic to I_1's referential. Sequentiality fails there, they must be established and fixed together, this is the meaning of simultaneity. Each stratum of the sequential hierarchy is characterized by simultaneity – all the constituents must be fixed simultaneously, together. Over the first stratum raises the sequential dependence of I_3 and over it again I_4. Over the whole sort then the Property sort and over it the Substance sort. Informations taken in isolation might, perhaps, be interpreted as functors of a 1-sorted presentation of some categorical ontology, where objects are represented by their respective identity morphisms.

We can then notice that the structure we attributed to the property sort P of applicability, is in an abstracted way internalized for Informations. Applicability captures something akin to arity – the number of instantiators of the constituent. Properties are here seen as unary. In contrast, Informations due to their range of application confinement to their own sort, can be seen as binary, one information describes another. This carries certain directionality of the meaning of this description. I_1 describes I_2 (denoted I_1DI_2) meaning I_1 carries within itself this binary description. And inversely again I_2 describes I_1. Thus in all, the internal structure is of the number of applicants, distinction between homogeneity or heterogeneity and positionality. Only the first of which is really inextricable, the others are extra structure we needn't assume – Properties here lack the other two steps, since being binary is insufficiently rich to host it.

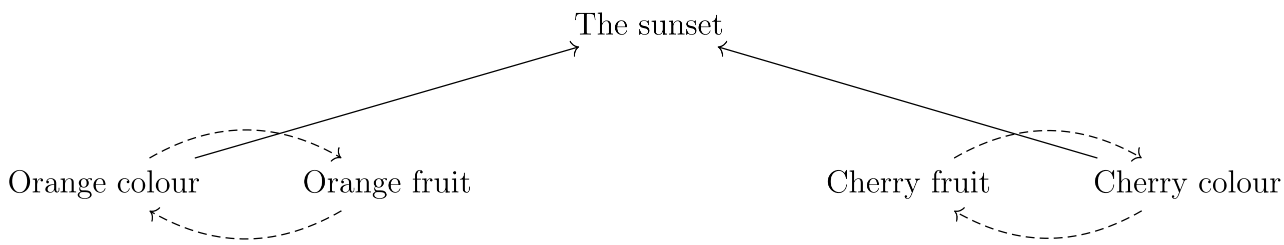
Peter Freyd in his formalization of diagrammatic language [14] uses similar directed graphs as we do, but with added composition conditions simulating categorical compositionality. We could alter our presentation to capture what we later speak of as inheritance by having just such extensions as this:



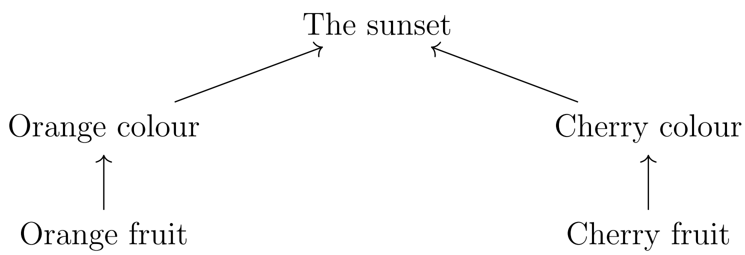
plus a few more on account of the cycles. Such extension being justified at least naively as dependence can really be thus transitive – S depends on P being fixed and

P on I being fixed hence S also depends on I being fixed, and the composition yields a unique overarching basic-referential arrow in accordance. But for the content of the thesis we will restrain ourselves since it hurts the demanded clarity. Regardless, it shows that there must be a corresponding diagrammatic formal language to contain the sortal and behavioural layers of an ontology. We will consider it in detail in another work, as well as its possible Freyd-style FOL translation (although he didn't begin with FOL for another reason we will make clear later). The informality will allow us greater expressivity. We will abuse the informality even in the categorical diagrams of Chapter 3. Suffice it to say it is safe to expect it to facilitate making explicit some identity criteria for ontologies on the level of the sortal and behavioural layers - as an analogous example, consider Aczel's extensionality over bisimulation sketched in the second chapter.

Imagine the late evening summer sky. You are French for convenience. There you see "*Cerise. Orange.*" (Cherry and Orange) Yet these colours do not emerge in isolation. Unlike the bitterness of the previous example, these concepts are grounded. The fruits carry their own colours. The object you discover looking at the horizon exists in referential dependence on both, either in simultaneity of the second sequential stratum over the object itself as seen in the first diagram below, or in three strata total – the sunset is referential to the basic sort Colour's two constituents Cherry and Orange, which are in turn referential to the basic Fruits sort. Like the dishes before, the sunset is now determined by the Colour sort, which is itself determined by Fruits sort – we show the two perspectives it can have: simultaneity or deeper sequentiality.

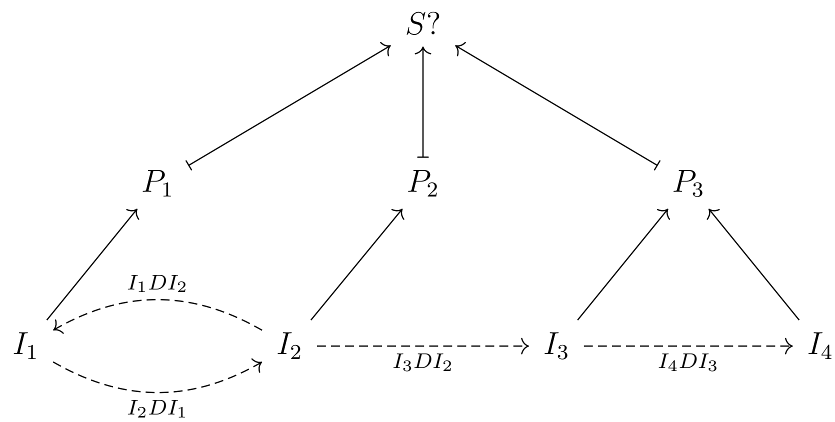


or...

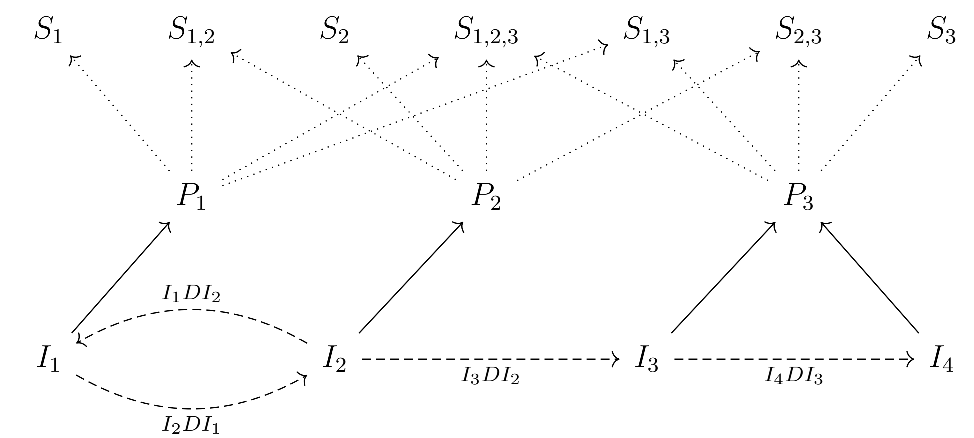


1.7 Identity criteria for the sortal and behavioural layers of a TO

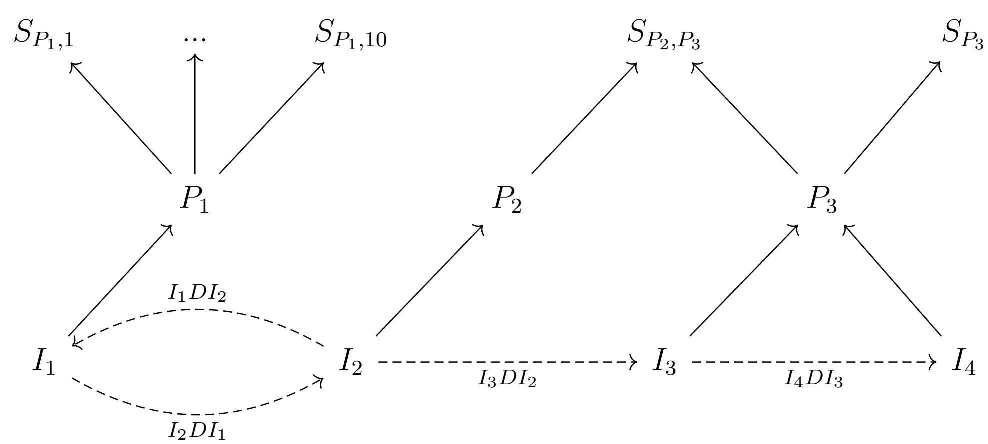
We will show one more variation that turns out to be quite important in the pretend ignorance game. We take the system as is but forget all we said about the Substance sort.



We do not know how many substances there are now. We only know they are referential to Properties. How then could they possibly be characterized by properties? First, there are the obvious combinations of the three properties corresponding to the subsets in a skewed way: each property must be instantiated at least once, all sorts must have their application specified (as all functors must have their source and target category specified, and so must the membership relation $\in$ have its domain and codomain given). The globally referential might then be free of this burden, except listing what sorts they determine retrospectively. We could have 3 substances with just one property each, then 3 substances of 2 properties each and just 1 instantiating all 3 properties together – all of them would be characterized uniquely.



This we can call identity by combination selection. A brute difference enters when we have nothing to refer to. Thus, we may say there are ten substances of the same property-wise characteristic, all instantiating P_1 only.



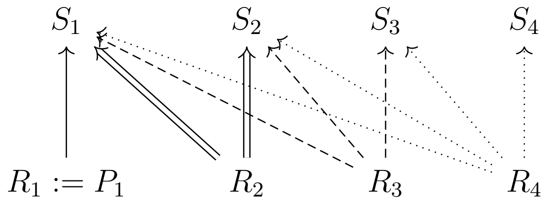
That we might call brute identity by number of instantiators. We have no way of distinguishing them except in relative comparison because there is nothing internal to them to distinguish any one apart within this equivalence class.

To give an example, these $S_{P_1,1}$ ... $S_{P_1,10}$ could be the abstract elements of an unstructured set produced by the axiom of separation satisfying a shared formula, as they are seen w.r.t. the formula from their superset looking inwards. Meaning they are indistinguishable w.r.t. this property and make an equivalence class on it and yet are 1O in number. Except for the substances we have no other properties to distinguish them from each other.

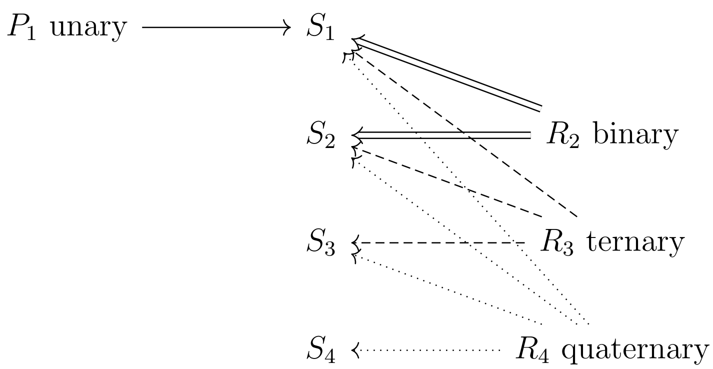
1.8 Choosing our identity criteria

Such identity criteria are then within a limited path endless in possibilities, and we should have some system to capture the process of choice rather than its manifestation over a concrete CoP. This function is met by various value-hierarchies meeting into a metavalue hierarchy ordering them in their interrelation. We explicitly recognize Naturality-arbitrariness value hierarchy, Simplicity value hierarchy, and Expressivity value hierarchy. The first of them captures the distance of a concept to the conceptual base of the ACU, which might be thought of as a class of categorical concepts (not in the sense of category theory, more so Aristotle and Kant), whose self-structuring gives rise to the verticality of the system.

To illustrate, consider a sortal layer containing the sort Substances and, instead of just Properties, also Relations. The Property sort could be considered as a subclass of the Relation sort characterized by being unary, or established as completely separate and independent:



for the first case, and...



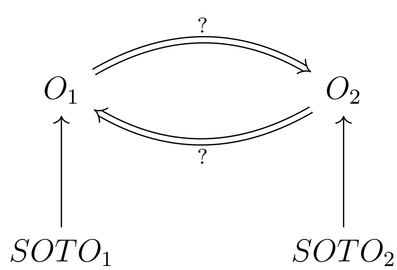
for the second.

The collapse here occurs over the separation of unary Properties and m-ary Relations (Arity of ≥ 2). We could distinguish sorts with justifiable recourse to naturality by the distinction in arity of Relations in general, not the deeper one-many distinction, but the natural numbers or in mathematically more involved systems even sizes of infinity, making a new sort for each distinct arity of a Relation. Here is then an example of a proper collapse that occurs based on the shared feature of reference to multiplicity. The collapse occurs over a distinctly strong naturality, which is overcome by stronger yet naturality of unity against multiplicity. So it is in a sense more costly as it loses the natural distinction that is far stronger than being applied to different objects, instead it is being applied to differently many objects. Through some such value hierarchy we are able to obliterate the inherent natural distinction over Relations. That is what collapse means - obliteration of a lower valued feature capable of separation in favour of the other's unification feature. (Russell attributes relations very high value, e.g. [32])

In terms of the other two value hierarchies, we haven't helped ourselves at all, the systems are intertranslatable – no change in simplicity nor in expressivity in the presentation we offer (we could of course find some feature the relation sort would obliterate of the property sort in a slightly different presentation).

Simplicity and expressivity come about in the example one before where we discussed brute identity by number of instantiators. By allowing this kind of identity criterion in the system, we loose unique characterization of the referential constituents, which is a particular strain of expressivity. But at the same time gain the power of describing much richer systems. Another strain. Collapsing all the property-wise indistinguishable substances into a single substance is on the other hand distinctly gaining in simplicity of the system. There is simplicity in terms of number of constituents, number of sorts, and in the complexity of their interrelation; in particular the relevant senses of dependence. This is what the structure of the respective hierarchies must reflect, ordering one facet over another, so that any input of SO would necessarily produce a uniquely determined TO. The two value hierarchies clearly interact, as the maximally simple system of a single sort and a single constituent would also be minimally expressive. But neither is a clear concept and may even involve something akin to logical expressivity, so that over Gödel's incompleteness theorems systems of certain well-defined complexity/power are limited in their expressivity by being unable to prove their own consistency – meaning if a system is simple enough it gains on this kind of expressivity. Whence it is not its direct complement competing for resources, as in the case of naturality arbitrariness value hierarchy.

The value hierarchy subsystem (VHS) takes such value hierarchies interwoven as one, so that the system can be judged on all they offer together, and they may influence each other, whereas otherwise they might be independent. They have no claim to complete enumeration, rather they are just the obvious ones entering into the choice that determines the shape of the sorts. Not only does the VHS capture the choice of identity criteria to be applied w.r.t. the sorts, but what is equally important, it selects identity criteria for ontologies themselves – given an ontology O_1 over SOTO_1 overlooking an ontology O_2 over SOTO_2 it is the highest question we may ask in the pretend ignorance game – whether these are identical or different. Here again we might want to think of them as preobjects stipulated in difference whether brute or substantial.



This is part of the VHS each TO carries – what ontologies would it identify, or differentiate. Such determination is thus TO-relevant rather than global and absolute. Even as we consider the two test cases, we might inconspicuously look at reconstructing the fantastically more complex operational framework from which the comparison is conducted. Hilariously, then, the VHS as a formal part of the system may pose itself as a feature on which two ontologies are to be differentiated.

Recapitulation

We distinguish two parts of a formal ontological system the source ontology SO and the target ontology TO.

1. SO consists necessarily of the only unique absolute conceptual universe ACU, which hosts any conceivable concept at a given point - either as conceptual category, or built up as their complex or as a virtual entity of language without any proper conceptual reference.

2. And of the structured collection of preobjects CoP gathered by nested selection over the categorical concepts of the ACU.

And

1. TO consists of the inherited structured CoP in the layer called underlying foundation, connecting the TO to a particular input SO.

2. Then of the value hierarchy subsystem VHS which is the meta-value hierarchy subsuming naturality-arbitrariness, simplicity, expressivity and any other value hierarchies the system permits, and ordering them in such a way as to produce unambiguous identity criteria for the sorts and the resultant ontology itself.

3. Third is the sortal layer made of sorts like Substances and Properties, each having some particular sortal constituents like S_1 and P_1, P_2.

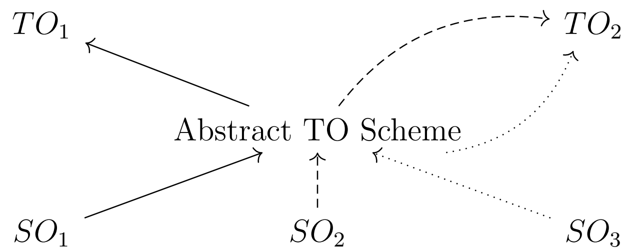
4. Their behaviour is encoded in the behavioural layer with assumed features of multiplicity, and optional features of distinguishing heterogeneous from homogeneous and over heterogeneous then the feature of positionality.

5. Last is then the universe produced as that which the sorts depict.

The ontology finally comes about from the union of SOTO - either internally as it sees itself in the universe produced, or externally as it is seen by others.

1.9 Abstract TO

For our purposes, it will also be necessary to mention what an abstract target ontology might look like. Because TO is tightly-knit with the SO's structured CoP as inherited in the underlying foundation, any particular ontology is over a particular SO-TO pair. That there should be an abstract TO preserving the essential features is therefore not possible because the sorts are specified by the selection over the structured CoP necessitating access to it – the VHS is in some sense just capturing the particular differentiation instances over a fixed structured CoP, although our presentation gives it a principiant aspect. It might address some shared high-level features of the CoP inputs, in such a way as to have the differing low-level features determine the ontology uniquely. What we consider is then an abstract scheme capable of fixing particular TOs over varying inputs of SOs – namely, over varying structured CoPs. We will call it an abstract TO scheme throughout the following chapters.



Its effect is then to capture just the features that can from some perspective (we will use that of FOL in both ZFC and CCAF considerations) be seen as uniformly shared over each TO produced through the interpretation of the uniquely different structured CoP input. Of course the point stands that axiom of Powerset has a different in-system meaning from one TO to another, and our naming is then over the assumed uniform perspective – as is after all calling all the global referential sorts we have considered throughout our exampled "Substances".

1.10 Identifying ontologies

Finally, regarding the joining of identity criteria for an identification of ontologies. Each partial identity criterion can be seen to exist within a class of competing possibilities – within a particular level of the formal system, as a single possibility the choosing of which for actual means the disqualification of all the others. There are then several such levels in rough strokes corresponding to the strata of the SO-TO pair. But not necessarily going bottom up by, for example, differentiation on the sortal constituents, but even taking the higher level features which ignore the innards to a lesser or greater extent. Such features may then be composed, so that a requirement for identification of two ontologies might depend on a concord on 1 or 2 or 3 levels on some one shared identity criterion. There is for example what can be in a loose metaphor called soft global existential quantifier – satisfying at least one identity criteria identification on each level, the universe produced, the behavioural layer, the sortal layer, the VHS and the underlying foundation. This incorporates all the aspects into the final evaluation.

With it comes the question of symmetry. As we have mentioned already, implicitly we overlook the field of ontologies from an undisclosed location. But we may consider the identification that occurs within said field from the perspective of the ontologies. $SOTO_1$ might for example have such identity criteria as to identify itself with $SOTO_2$ on meeting the aforementioned soft global existence quantifier identity criterium for ontologies, regardless of which (up to a point respecting the demands of the identity criterium choice) the latter may still differentiate itself on a utterly specific and even generally uninteresting identity criteria such as the banal number of sorts in the sortal layer. Hence, creating an asymmetric situation where we cannot speak of identification from top-down, but must instead look from either perspective individually. The overlooking view then requires at least this much, that both mutually comparing SOTO systems consider the ontology they produce identical, by whatever identity criteria it may be, with the other.

Last note then before we examine the set theoretic test case will concern the translation necessary for a comparison. An ontology cannot address another except as it models it within itself. The global comparison we consent to by force of practicality, is seriously lacking in the formal grounding that would otherwise be required of a comparison and even a single particular description of a system. Locally, then, we need to provide a translation from how the systems appear globally to a single fixed system as our vantage point – a way to internally represent the other ontology in comparison to itself. There needs to be an implication of depth in the representation, else it is merely a subsystem of the universe, which cannot conceivably be greater than the system itself. (It is possible we fall in the size argument at the concluding chapter for a confusion of this exact character.) This means the running example system SOTO_1 must for the comparison internally represent itself as an ontology O_1 as well as the O_2 over SOTO_2 against which the comparison is staged. The question then comes to the fidelity of the representation on either end. If we say the former representing both identifies them, either the comparison is vacuous as the identification occurs actually in the representation itself – there are no two stipulated different represented systems, rather the representation already is a manifestation of the preferred identity criteria of the home (vantage point) ontology; or it captures some differences among the systems as they are represented and only through the representation of its own VHS choosing process implies an identification on the internal level – that which is represented cannot be the ontology itself as the identification would by assumption collapse both, but rather certain structural feature of the SOTO system on which the home system SOTO_1 can draw any difference whatsoever. Thus to conclude, the local comparison requires some unspecified means of translating one system into the other allowing us to speak of at least the asymmetric situation, which could conceivably imply the dual direction of the comparison also if its identity criteria were of a particular strictness preventing the onset of a deep-enough distortion as could undermine the accurate capturing of its comparand's own identity criteria choice as embodied by the VHS. It is my naive and completely unjustified assumption that this happens over our final comparison of the two test cases. We rely on the throughput of the conduit of FOL connecting the two axiomatic presentations to follow immediately.

Chapter 2

Chapter 2 ZFC

2.1 Introduction

The rest of the text will be devoted to: first - the exposition of real living systems, one that is canonical and almost implicit throughout the mathematician's world, another on the very fringe, as axiomatic theories with a claim to logical and ontological mathematical foundations [22]; second - their interpretation within the framework sketched in the first chapter taking their ontological ambitions seriously; third - investigating the system's expressivity by exploring its ability to handle such intentionally simple and formally concise test cases one by one as well as within the concluding chapter by an involved comparison; fourth - observing the emergence of in-relevance competing notions of identity criteria as they affect the formal and the ontological in parallel. For Zermelo Fraenkel set theory with choice, we will begin without much ambition in simple commented presentation of its axioms together with the first attempts at SOTO integration. Because of how simple the system is formally, its TO might not give very much insight at all, but rather provide a base case for the follow-up chapter on categories. It will, at least, help us shift perspective to sets as ontological entities, and its universe V as something cogent. Only with the introduction of the competing ontology of CCAF will we begin noticing what was amiss in the ZFC case, a structural insight arising from an unsuspecting juxtaposition of positive and negative. The third chapter will thus be significantly richer on all 4 points. Due to the interreferential nature of axiomatic exposition, a full meaning can be extracted over an acquired familiarity, hence the nonlinearity invites to second reading of the second and third chapters if anything seemed out of order, because it might be and might be by necessity. Be forewarned that throughout the following chapters, we will heavily refer to the appendix.

Light historical introduction

In the year of our Lord 1902, Bertrand Russell alarmed the mathematical community reliant on a naive theory of sets as collections of mutually distinct objects of thought termed elements, with origins in 18th century and originator Bernard Bolzano for the purpose of investigating mathematical infinity, back then in a potential modality only. In his works throughout the years 1873-1897 Georg Cantor brought the inherited sets to infinity actual by allowing the sets themselves to become infinite, most notoriously the natural numbers. In doing so, he founded the mathematical discipline of set theory. His we call the naive set theory. Russell's paradox abuses the naive all-embracing universality of sets - it assumes the existence of a collection of all elements which aren't their own elements, this would, under the naive set theory, yield a particular set. In modern notation $x = \{y | y \notin y\}$. By case analysis if x is its own element it must satisfy the defining conditions and hence not be its own element, else if it is not its own element, it satisfies the defining condition and becomes its own element - contradiction. The remedy came from the elden method of axiomatic foundation, as glorified by Euclid's Elements. Erns Zermelo was responsible for the first segment of axioms delimiting the behaviour of the naive sets with a later addition of Fraenkel's replacement scheme, his own foundation axiom and later still the canonization of the axiom of choice. [4]

2.2 The Axioms

Our presentation is taken directly from [4] and properly begins with language (see Appendix A - A.1.1).

Language can then be followed by the first axiom of ZFC:

2.2.1 Existence

Ax.1 (see Appendix A A.1.2 1.)

What is a set? We take it to mean that there is set sort S and a constituent S_1 at least. Note the condition $x = x$. We will have to deal with "=" in detail over the rest of the axioms, but now it seems like a part of the language game of ignorance in asking for such condition as though $x \neq x$ could properly mean anything - any identity criterion among whom = must belong, cannot but maintain $x = x$. So, asking the question ?x=x is misunderstanding the function of "=" for consideration of real comparable preobjects. It takes on different meaning in the realm of formal language games of logic, but we do not care for it, it has no absolute canonicity we would have to respect.

We must sincerely question ?x=x as a formal condition within the ontology. For the same reasons does the empty set as a class $\{x|x \neq x\}$ seem to involve a bizarre ontological definition. On this strangeness we must introduce the dichotomy of ontological and formal, which is going to reappear in mentioning the axiom schemes of separation and replacement and again in the third chapter over categorical separation. The theory of ZFC is within the presentation captured by the formal apparatus constituted by the basic language of set theory over the standard shared language of FOL. Its descriptions of sets are indirect insofar as they utilize initially abstract and flexible set variables x, y, z in order to represent the modelled entities. Consider the $\emptyset = \{x|x \neq x\}$ or even more illustratively $\emptyset = \{x|x \in x\}$ over foundation axiom. The characterizing formulas are not satisfied by any set x, hence the set determined really is empty of elements taken by separation from an arbitrary set as guaranteed by existence axiom. The class is established internally through its elements x. The anonymity is what interests us here. The ontology captures on the level of sorts only the particulars, as a formal system it has no way of speaking of abstract entities, because actually there are none such. Recall the three sorts S, P, I captured in the last diagram of subsection 1.7. and their individual constituents. Importantly, there is something of a trickery even there in naming the individuals mutually externally indistinguishable as when employing the brute identity criterion by number of instantiators - suggesting the S sort to contain $S_{P_1,1}$, $S_{P_1,2}$ $\ldots$ $S_{P_1,10}$. All 10 are Property-wise indistinguishable, as they only instantiate P_1 each. Naming them violates their essential indistinguishability. There is an overview which represents the multiplicity accurately, but inauthentically their property-wise indistinguishability. In thus far, it is a formal system analogous to that describing the ZFC theory and its models. In thus far, the TO is also separated from the universe produced, described by the sorts. There is also a separation from the ontology produced over the SOTO, but that is a matter of perspective of the SOTO as it sees itself.

All that is to say there must be a way to capture the entities and structure of the universe produced formally within the TO, and it might be that the means of capturing skew their projections onto the universe produced. Exactly the same happens when the formal language of set theory is interpreted ontologically. The semantics-equipped formulas do not project directly and there is a nebulous stage prior to uniquely determining the model of the theory by fixing the semantic interpretation with a particular structure. These layers must be distinguished at least here because we will be moving across them with often inobvious indices only. There is the layer of wffs, then the wffs exulted in satisfaction of the general theory of ZFC, then such as capture the particular models (whom we will consider not confined to FOL but moving beyond it up to the hoped for unique characterization, as ought to correspond to the structure of the CoP in at least one interpretation we suggest). These are then translated in the interpretation to the SOTO formal system and projected from the sorts to the universe produced therein. We speak of them as the outer perspective layers.

Differently do we treat a particular TO and an abstract TO-producing scheme w.r.t. the usage of the layers of outer perspectives. The latter even adds another layer by itself which must be through the plugging in of a particular input solidified through a similar projection as that which we have been describing. The language too of the formal ontological system is abstracted to accommodate the height of the high-level features presided over. The restriction to particulars of the sorts is lifter as to allow for the integration of the language of the interpreted theory. What remains is to speak of structural features, patterns standing across the produced TOs. We can still maintain the presence of two sorts, one as meets some features characteristic of sets, the other of membership. By turning to the axioms, we look backwards at the 3 original language layers. But whenever we discuss the universal features structurally recognizable, we step down to the abstract TO scheme instead. The 3 layers interact with the ontology either as inherited within the encoding of the structured CoP forming the TO, or externally as extraontological motivation discussed once more in the third chapter. Both cases involve structural enforcement of a different kind, the latter the proper. The abstract TO scheme does not run over an arbitrary input, instead over a class of structured CoP sharing some high-level feature which the abstract TO could process and as has in the final analysis some one, not necessarily unique, model correlate of the original theory. The abstract TO scheme is a template windowing over the system, harnessing its commonalities to handle multiple variations simultaneously. The system might then be retrofitted so that the particular SOTOs assume the roles of the structured preobjects wherefore instead of considering the ontologies locally, the abstract TO scheme fits the role of a global ontology of the multiverse itself.

For class (see Appendix A A.1.2 2.)

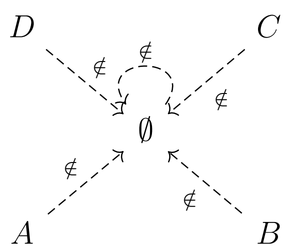
Second we present...

2.2.2 Extensionality

Ax.2 (see Appendix A A.1.2 3.)

Explicitly, it establishes the identity criterion for the set sort constituents. Note that there is behind it reference to $\in$ as something primary and basic. So what is this really in the TO itself? We will conform to the presentation in adopting a binary relation-like sort "membership" over a substance-like Set sort.

We claim to adhere to the cited preformatting of the original presentation in our choice of the second sort membership, yet recognize there not only the positive belonging $\in$, but equally well its negative dual $\notin$. Why so if one can be defined in terms of the other? Because such definition as employs negation falls under the realm of the outer perspective layers we've distinguished earlier and functions as a means of expression. No doubt the same could in some lesser modality be said of the formal system of an ontology, yet the latter carries a promise of fidelity of directness. Essential to it is the universe produced in its representation. In establishing the sortal layer, the absence of $\in$ is itself an implicit encoding on the structure of the universe produced understood from the operational framework, s.t. there is still in the background context a virtuality engaged regarding anything given by negation or absence; it is a dual parity of the positively given playing an insubstantiable role. We say a belongs to b, which carries with it what not belonging means in complement, unspoken, implicit but necessary. Within the ontology itself, however, both must have an equally positive presence, regardless of our manner of formal presentation. For one reason added, without the presence of the negation on the universe produced, if it were possible there could emerge areas of $\in$ networks wholly disconnected and hence incapable of satisfying the demand of each set having a global characterization determining its role in the universe in relating all others - $\notin$ supplants this requirement, so that all sets are mutually interconnected explicitly on the universe produced disregarding thus the limitation of the interpreted meaning of the directionality of $\in$. The behavioural layer can thusly be adequately manifested. (We might then ask for the same for functors on CCAF, but the system of CCAF is cluttered enough that it might be of injury on visibility). Imagine in example describing the empty set in positive formulation, without the presence of $\notin$ allowing the categorical analogy of a $\notin$ terminal object.



$\notin$ being definable as negated $\in$ on the formulaic layer of analysis thus doesn't translate, wherefore $\notin$ prevents some expressivity costs on the ontological system. Even further, such definability must be of different level of the ignorance game because the negative parity $\notin$ needs be encoded on the third layer to structural authenticity by way of negation, simplifying conceptually, not structurally.

By the reference to elements, we need them to have been fixed already in this expressedly sequential way of thinking. We can compare set entities, but the elements are themselves set entities, and there looms circularity. We need some bottom to be fixed before all this. Good bet is $\emptyset$ because it avoids the question entirely. We establish the set as one that has no elements, which eliminates the question of its internal identity, whereto extensionality refers.

The sequential approach cannot be taken to $\emptyset$'s self-reference, and simultaneous must be adopted instead. Each layer of the hierarchy is horizontally simultaneous, involving both parities of membership. And yet there is another sequential perspective capturing just $\in$ upwards to the current level - the construction perspective. But considering the negative dual breaks it immediately establishing the global alternative. We can consider cut-off branches leading to the empty set root, which are locally isolated by $\in$ but globally interconnected via $\notin$. Thus, over a step-back we see that the proper complete characterization of a set needs its connections by both parities, for the ontological set takes its meaning in context.

Returning to extensionality, why don't we sortify set equality? It would be a principle of structural projection we will encounter dealing with categories. Not merely abiding the VHS choice of identity criteria, as is inescapable, but capturing the choosing itself as a sort. Such self-referentiality is offered to ZFC TO, but we direct our decision by the preceding classification on the ZFC symbols.

The axiom seems to give us incidentally also the identity criterion for the membership sort. $\in_1 = \in_2$ if and only if (iff) their domains and codomains agree because such "membership" must by its nature be unique. Which invokes circularity, solved easily by sequential stratification. But we immediately notice the difference of this form of a relation-like sort to the universal relations we played around in chapter 1 (CH1) - R_2 had multiple instantiators as it was a basic sort with referential counterpart of various Substance sort constituents. Still just one such relation constituent covered in it all the substance instantiators. We did not drop deeper into differentiating subsorts. That was determined by the choice of perspective and its apparent uniqueness. If the related sort could be itself basic with relations referential, the same could be reformulated there. For ZFC this latter perspective is not only conceivable but as we shall see much better fits a categorically authentic (meaning structural) reconstruction of sufficient logical adequacy. What sets these apart is actually the vantage point. The membership is seen thus from the sets themselves as they look inside on their elements, they see such projections of the membership sort. And again when they look up. This is how they see $\in$ and $\notin$. And it is also how extensionality determines the membership sort, because the projection is genuinely reflective. It can work only in the stratification of the universe as it presents itself. In it the strata are given in simultaneity, and vertically step in the sequential perspective which sustains the projections only as the sets of the strata below and their relations are fixed already.

How then looks the founding stratum? Is it a single set ($\emptyset$?), is it a loop, do we have just deeper and deeper layers endlessly ungrounded? With our two axioms, we can take our guaranteed set x and ask $?x = \emptyset$ - is it empty? If not we could run into the three scenarios, loop, finite descending chain or infinite one, only the middle leading to $\emptyset$.

For alternative (see Appendix A.1.2 4.)

Which opens up...

2.2.3 Classification of axioms

The existence axiom we will call material axiom as it by itself provides the raw material for the edifice built by other construction axioms. The axioms can be considered alone, sequentially, or of completely free relativity within or without the confines of the theory. Extensionality is an identity axiom for the set sort constituents as they appear impurely referential but still in the confines of the sequential stratification. For constituents of membership sort, their duality is carried over extensionality, although the axiom itself speaks of it only implicitly. It is so in a less direct manner even than the exposed set-seen projections, which correspond to it, albeit have no formal TO presence.

To shortly relate set variables, they might be too poor to constitute the preobjects of our initial motivation, despite standing before the actual universe - they are somewhere in the middle, part of the language game. And yet they stand as the quantification domain of the formulas, which in turn actually inform the universe through, for example, the separation axiom to be exposed shortly. This raises a fundamental question regarding the proper place for wffs. Recall that membership is not merely basic to the referential set sort, but through the stratification assumes both perspectives depending on the stage considered, with the bottom $\emptyset$ set fixed as referential by saying "it has no elements". Set is thus a dynamic sort, which explains the reach of its identity axiom as well as the system's tightness.

We turn to discuss the formulaic equality for set variables rather than set entities. Is the meaning of $=$ provided by the logic entirely or partially, or entirely by ZFC? Are the formulas grounded at least in the SO or do they make up a wholly extraontological input to shape the TO by structural enforcement. Since the extensionality axiom caters to the contextual rather than the isolated mode of inquisition, does it bind the formulaic set variables to actual sets? This reaches the issue of satisfaction as satisfiability must reflect the axiom-defined identity and the separation axiom we are thinking of considers as relevant only ever the formulas satisfied. And satisfaction really is embodied in the semantics and so bound to the universe produced.

Now, if the variables were part of the preobjects, their structuring would have to capture the wffs. And over this satisfaction would take place. But VHS does select among the SO input offered CoPs. So for this to work, all wffs would be in the same CoP from which the satisfied wffs form the actual sorts. One branch of the mathematical game is then precisely determining the overlap between wff and true ones in howevermany approaches this admits. Yet the satisfied formulas go beyond first order and its structuring of the CoP selected by VHS; they are richer structure of the CoP, whose part we choose to see only by the setting. This structuring must also reflect the $\in$ and $\notin$ exclusivity translated to $\neg$ presentation.

The formulas consulted by the separation axiom are not just the satisfied wffs. Its complement collapses in determining a single entity of the universe - the empty set. This comes about trivially as $\in$ is dual to $\notin$ and by Ax.2 $\forall x(x = x)$ for all sets whence its dual $\forall x(x \neq x)$ must not be satisfied by any set - hence Ax.3 yields a set of no elements if this is its definition.

2.2.4 Separation

Ax.3 (see Appendix A A.1.2 5.)

We can then use a particular sense of the word constructive insofar as it sequentially fully internally characterizes the set to be construed.

With this we may consider the Powerset constructor and question whether it indeed produces all subsets. That it does, at least within the bounds of natural numbers:

$\exists b \in x \forall c \in x(b = c)$ - separation over this formula produces the subset of all singletons

$\exists b \in x \exists c \in x((b \neq c) \wedge \forall d \in x(d = b \vee d = c))$ - which produces all "pair" subsets

and so on...

But as it turns out it fails on the transfinite, whence the trouble with continuum hypothesis(CH). [4]

The axiom captures the notion of set definability, and suggests that the complete characterization of any set might be approached through formulable properties of raising order. But what about the expected powerset axiom promising all subsets? Doesn't the production of all subsets of a set A simply supersede the separation occurring therein?

2.2.5 Power set

Ax.4 (see Appendix A A.1.2 6.)

The axiom actually does not access the possibility space of subsets, it works with only those provided in adherence to the sequential perspective. Meaning, it does not guarantee that all the conceivable subsets of a fixed A exist as elements of its unique powerset $P(A)$, rather just that any that do so exist among those conceivable constitute the $P(A)$. But without the axiom of separation, only the $\subseteq$-wise bottom and top ($\emptyset$ and the set itself) are guaranteed to exist and would be all the unrestricted powerset operator has to work with.

There are some technical issues of Powerset and Separation (see Appendix A A.1.2 7.)

Still, we cannot understand the Powerset axiom fully without reference to the cumulative hierarchy of sets as stratified by the P operation and union, over the lightning rod of an axiom - Foundation. We will discuss these informally for motivation purposes until we have built up enough to formalize them naturally. We begin with an easy one.

2.2.6 Sum

Ax.5 (see Appendix A A.1.2 8.)

These will be necessary for the standard presentation of the cumulative hierarchy construction. In addition, we will gloss over few other complex definitions and results regarding them, whom the comprehension of the final definition presupposes. see Appendix A A.1.2 9.

2.2.7 Foundation

Ax.6 (see Appendix A A.1.2 10.)

In $\emptyset$ the axiom grounds the foundation of the entire set-theoretic universe, because sets must lessen as we look inside, and we must reach the bottom of this lessening in finitely many steps. It is the first occurrence of a purely restrictive axiom, which rather than guaranteeing the existence of sets, prevents some sets from entering the universe. It is effectively tidying the universe of ZFC. In a sense, the extensionality axiom was also restrictive, as without it the set sort could be produced in whatever way independent on $\in$.

2.2.8 Models and Mostowski

Something we have yet to consider are models w.r.t. the axiom of extensionality. If it was a full identity criterion on TO, the model would have to be determined uniquely, or uniquely up to its own identity criterion established within the VHS - such as isomorphism of models. Each model must correspond to a particular choice of an ontology, whence to speak of the ontology of ZFC we must fall back onto the abstract TO scheme briefly shadowed in CH1.

There is also a technical discussion of models and results of Mostowski: see Appendix A A.1.2 11

It shows an interesting collapse of the externally well-behaved models of ZFC. Over what we have suggested, it is then possible to consider the ZFC axioms uniquely determining a standard-isomorphic model. But as we will discover next in discussion of [15], the very notion of standard model might be ill-conceived.

2.2.9 Universe and Multiverse views

The universe view sees a unique absolute background manifested in the corresponding absolute set-theoretic universe(the cumulative hierarchy), in which every set-theoretic assertion has a definite value. [15, p.416] It is the home perspective from which the models we considered can be externally well-founded or ill-founded. It privileges a model that sees the world as it really is, and has a penetrating eye. Provability takes on the ignorance-game perspective, whereas truth is reserved for that which is irrespective of our knowledge. It is clear then at least that the ZFC axioms are insufficient descriptors of this universe, as they recognizedly carry the openness of independence.

The multiverse view has it that "there are diverse distinct concepts of set, each instantiated in a corresponding set-theoretic universe, which exhibit diverse set-theoretic truths. Each such universe exists independently..."[15, p.416]

The multiverse view leads to the dissolution of the standard model imagery, seen most poignantly by the

- Well-foundedness Mirage principle: every universe V is ill-founded from the perspective of another, better universe. [15]

This completely relativizes the identity criteria and is an example of what we have blindly stumbled upon in CH1 regarding TO-relative judgements.

For formal integration of the principle (see Appendix A A.1.2 12.)

In the abstract TO scheme, each model is a proper fixation of ZFC ontology. VHS accepts the axioms of ZFC, but because there can be differently many differently behaving sets, which all in their respective separate systems satisfy the axioms and give them different meaning VHS can fix identity criteria for the sortal and behavioural layers together in accordance and still produce widely ranging difference in models. The problem with the universe view is its immediate annihilation of the epistemological utility of the abstractness for the mathematical game.

2.2.10 Concrete and Abstract Target Ontologies

Abstract TO is a scheme for constructing TOs over varying SO inputs in some unified manner. They can then be compared on effect to the same class of SO inputs or on the way of acting on them. For that the models must be in their individuality determined by the input structured CoPs and differ from one another essentially on those.

Different VHSs can coincide on a single sortal setting between different inputs, which means models are not in difference by the CoP input alone. What then are the high-level features safeguarded by the abstract TO scheme? - at least there is a set sort and an E relation sort, satisfying the ZFC axioms for themselves. They carry openness for the structured CoPs to play with and solidify into individual models.

The differentiation into well and ill-founded occurs from the perspectives of the ZFC TOs, yet the abstract TO scheme is seen from our operational framework-associated ontology, where all that matters is the definition of the system that is internal, because it requires a particular canonical fixed ZFC-TO and our TO when viewing the scheme is not a ZFC TO so there is no framing into internal-external, and we instead see them as constituting the multiverse.

2.2.11 Pairing

Ax.7 (see Appendix A A.1.2 13.)

The axioms of ZFC give the universe a certain outer shape, yet leave empty space to be filled depending on the input's structured CoP. By the abstract TO scheme an inductive set, empty set over it, pairs, unions, power sets are all guaranteed in every TO thus produced (except for the empty extreme). We have touched of perfect constructivity with pairing axiom, open constructivity with Powerset, but still there is a variation on the classification that comes with the axiom of infinity.

2.2.12 Infinity

Ax. 8 (see Appendix A A.1.2 14.)

Under the classification it is therefore an open constructive material-raising axiom. For the abstract TO scheme, it guarantees the commonality of some inductive set and launches the transfinite strata as a higher order vantage point for a P series and in turn the classical stratification of V. Infinity also leads us to expand the classification based on the axiom's form of effect on the behavioural layer - general principles such as extensionality, structural principles such as pairing, which builds from inside, and particular effects such as the material aspect of infinity. The extensionality axiom, as after all do the others although in a weaker way, speaks of a partial identity criterion directly, rather than by "extension" of the bruteness.

Antefinal is the replacement scheme:

2.2.13 Replacement

Ax.9 (see Appendix A A.1.2 15.)

It enriches the ZFC ontology by guaranteeing the existence of some infinite sets like $\omega \cdot 2$ which is the first ordinal that needs it. ω_1 must then also use replacement in its construction. It is called replacement because taking any a we can replace all its elements by prefixed sets and still get a set. Separation is redundant through derivation from the axiom scheme over infinity's provision of $\emptyset$. It too is raising. It too depends in its formation and guarantee of sets on definable formulas of FOL. And assures us that certain large classes really do make up sets. We spoke of $\omega \cdot 2$ but any limit ordinals above ω are subject to the same. Thus, whereas separation works on the insides only, replacement moves beyond.

2.2.14 Choice

Ax.10 (see Appendix A A.1.2 16.)

The point of the axiom is then to say there needn't be any differentiation features recognized for the nonempty elements A of the fixed set X for there to be a "choice" picking simultaneously one representative, used as index, from each. This invokes a manner of establishing identity criteria in a brute fashion. In effect, it is an axiom of just such bruteness regarding the justification of the choice - there is none, and none is needed, says AC. Hence, we know that there is guaranteed in the universe such raised set and whence hail its elements. That is clearly a positive degree of constructivity, but we cannot know how such a choice could be epistemically performed and inthusfar it really is inconstructive, because the choice among the possibility space for AC or even infinity is independent on the axioms beyond the acknowledged enclosure in which it takes place. It is the effect of the structured CoP, but the bruteness doesn't disappear through this realization, it is transposed instead to the SO, yet remains the same in its responsibility.

The essence of an axiom is what it guarantees - for AC both bottom-up raising functionality and top-down organizational ordering by WO and many others, whose additions cumulate to give more and more accurate description of the axiom's effect.

2.2.15 Backwards glance

With all axioms of ZFC out of the way we will review the abstract TO scheme and the ontologies produced before moving onto CCAF. The extensionality axiom as central to ZFC establishes the playground for its models. It does indeed function as the identity criterion for sets - collapsing those which share the same elements into a single ontological entity and doing so in a way as to ground also the set-seen projections of memberships. In that it is universal, but still much remains which an identity criterion could affix and does not. Much of the responsibility is externalized. The restrictive axioms (extensionality and foundation) form the outer bounds of the identity criteria. The material axioms lend it the fabric. The raising axioms weave the pattern. Openness is of that which slips between the hardwood floor. The bruteness we know to be necessary at the very bottom of any ontology. The axioms are admixed references to varying levels of identity criteria and our discussion shows that something that can be in one presentation given in a brutish fashion can also be given structure-schematically, which can in turn take the form of a principle. Once gripped as a frame, they open themselves up to comparison with other axiomatics. We will compare the multiverse of ZFC and of CCAF along matching lines as revealed through a side-by-side observance to follow, allowing for variance and differentiation over a shared abstract form. The ZFC ontology is compact and amazingly expressive for such simplicity of formalized containment.

Let us now involve categories.

Chapter 3

Chapter 3 CCAF

3.1 Introduction

As a categorical comparative, we will uncover the axiomatics of an esoteric category of categories and functors (CCAF). The lowest fruit to pluck would be ontologizing a categorical set theory. This is not it. At least not exactly. But what even is a category? Samuel Eilenberg and Saunders Mac Lane's prodigious child seen as a collection, not a class nor set, of objects and morphisms/arrows, s.t. each morphism has a source (domain) object and target (codomain) object. It is structured by identity morphisms for each object, allowing for a lossless translation between one and two-sorted settings [20], a ternary relation of composition binding two well-posed morphisms together by joining them to a third one sharing the origin source and end target of the pair. It further satisfies associativity of morphism composition ((hg)f=h(gf)), their enforced existence and uniqueness together with the basic identity laws.

Categories have the amusing capacity to scale up through self-application into a stratified hierarchy that retains the basic categorical structure while adjusting gradually to the raising complexity. Objects and morphisms of a category are usually taken as purely dependent and lacking ontological substantiality w.r.t. categories.

They are seen as abstract structureless parts, although they are liable to any fitting interpretation, which so often ends up equipping them with structure implicitly and externally. Such objects could be members of a family preordered by age, they could be natural numbers with morphisms matrices as transformations, they could be plain unstructured sets of discrete independence, graphs, or perhaps formulas and proofs for morphisms. 0th level being morphisms and objects, 1st categories themselves and "morphisms" raised to repeat the pattern under the name "functors" - connecting categories and homomorphically-like translating between their structures by preservation of identities and sending composable morphisms of one category to those of the other under a global coherence. Categories and functors can form categories of a higher rank, where one takes on the role of objects, the other of morphisms. But so can functors and natural transformations between them step up and natural transformations and their bizarre morphisms. [18] At each stratum the basic constituents have a different natural notion of identity, which will be formulated explicitly as we give our special presentation of categories which takes on Bill Lawvere's inspired approach in Colin McLarty's artful adaptation. [27] Much of his exposition will be taken word-for-word, arrow-for-arrow.

3.2 CCAF Axioms

3.2.1 Category axioms

First we need the hand-wavy notion of a category formalized in a typical way: (see Appendix A A.2.1 EML)

3.3 Axioms on 1,2,3

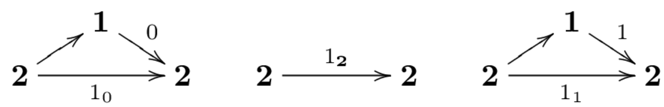
3.3.1 Introducing 1,2,3

These forms were Lawvere's development of a fundamental categorical principle of externality-internality, as captured formally most briskly by Yoneda Lemma [17] and its corollaries;m aintaining that any category may be characterized equally well in a uniquely mapping fashion internally by its objects and morphisms among them, and externally by other categories and its functors thither. Thus, external characterization has, exactly as was the case with sets, the obvious globally inclusive character, demanding the express knowledge of all relations from a fixed entity, regardless of direction, to every other entity of the appropriate class and level (although this level distinction fades out for ZFC). Category theory however splits even this into the consideration of Hom Categories and functors, differentiating precisely based on direction of the functors - what our fixture sees - the functor category of all those having it for its source and dually those which see it.

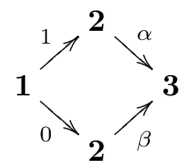
It thus splits the universe (which just like V in ZFC doesn't make a proper entity of the system, but an imagined spatial constraint as is the paper to our dots and arrows) into 3 areas, functors and categories disconnected immediately, those seeing

Every category **C** has a unique functor $\mathbf{C}\to\mathbf{1}$.

The category **2** has exactly two functors from **1** and three to itself:



The category **3** is a pushout:



and there is a functor $\gamma\colon\mathbf{2}\to\mathbf{3}$ with $\gamma_0 = \alpha_0$ and $\gamma_1 = \beta_1$.

Arrow extensionality: $\forall\, \mathbf{F},\mathbf{G}\colon\mathbf{A}\to\mathbf{B}$ if $\mathbf{F}\neq\mathbf{G}$ then $\exists f\colon\mathbf{2}\to\mathbf{A}$ with $\mathbf{F}f\neq\mathbf{G}f$

Figure 3.1: The CCAF axioms on **1, 2, 3**

and those seen. Of the first it could further distinguish immediate and complete separation, but these are the matters of interpretation relating to ignorance. And yet the external characterization Lawvere picks up on needs only these latter two, seeing and seen, and in a mildly weaker sense just those seeing and if we focus on particular essential features we will only need 3 specific categorical forms, which see always (empty category disregarding).

Lawvere devised a foundationally externalist project in which he built all but 3 categories bottom-up. Their nature is codified by the axioms of figure 3.2

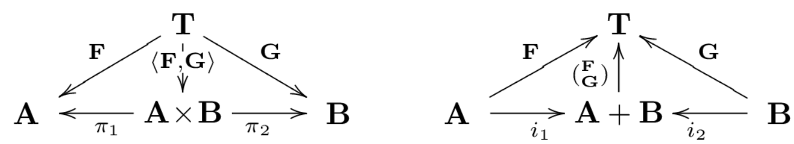
3.3.2 Domain and codomain, existence and uniqueness of composites, identity arrows for 1,2,3

The axiomatic sequence properly begins with **1**. Its role is to externally resurrect the notion of an object. The first axiom then identifies the terminal category of the universe and names it **1**. Meaning a category I to which every category has exactly one functor - it is unique because stipulating I' terminal, the composition functor is forced to equal the identity functor on either side, making it isomorphic, making I and I' collapse under equivalence of categories, which is the identity criterion of categories weaker and therefore subsuming of isomorphisms - meaning I and I' can differ only formally (as preobjects) but as their form is embodied in the universe they collapse. **1** thus connects all categories as that which they see. It is said to have precisely one object internal to it, as its terminality necessitates precisely one automorphic functor (automorphic functor F is s.t. its source = its target) and object of a category A is being freshly defined as a functor o:**1**→ A.

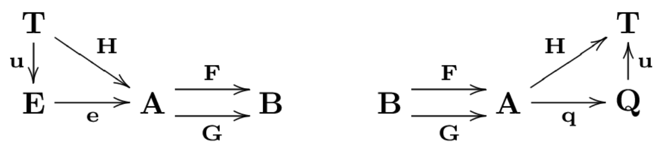
For diagrammatic description of the behaviour of **1, 2, 3** (see Appendix A A.2.2)

Before we get to treat the axioms by TO-interpretation and address the categorical extensionality axiom, we must finalize the project of building categories up from our three forms by adequate external characterization. The four axioms given thus far lead to arrows of arbitrary quasicategory **A** constructed by **1, 2** and **3** to satisfy the Eilenberg-MacLane (EML) category axioms with the exception of associativity. For that we will need

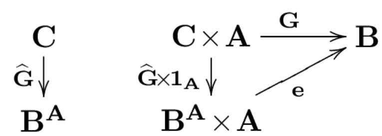
Every pair of categories $\mathbf{A}, \mathbf{B}$ has a product and a coproduct:



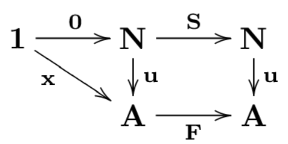
Every parallel pair of functors $\mathbf{F}, \mathbf{G} : \mathbf{A} \to \mathbf{B}$ has an equalizer and a coequalizer:



There is a functor category from each category $\mathbf{A}$ to each category $\mathbf{B}$:



There is a natural number category $\mathbf{N}, \mathbf{0}, \mathbf{S}$:



Choice: For every onto functor between discrete categories $\mathbf{F} : \mathbf{D}' \to \mathbf{D}$, there is a functor $\mathbf{H} : \mathbf{D} \to \mathbf{D}'$ with $\mathbf{FH} = 1_{\mathbf{D}}$.

Figure 3.2: The CCAF construction axioms

3.4 The construction axioms

3.4.1 Associativity through 1,2,3

Once we guarantee **A** is associative in composition we reapply the 3 terms to themselves and from informal quasicategories reach categories proper. For that, we need the axioms of Figure3.3 on the preceding page.

To follow McLarty's explanation we must then (see Appendix A A.2.3)

3.4.2 Laying the foundation for an ontological interpretation

With this, we can ask again: what is the basic layout of the CCAF ontology class? Ordinarily, once we can see any positive category it must satisfy the EML axioms, but conversely whatever entities and operations satisfy these points do form a category sequentially beginning with objects and morphisms in the classical setting. In CCAF however as far as we have introduced it and by guarantee of my word everforeafter we expect the variables to stand in one sort for categories and in the other for functors. Every instantiated variable then, if it is to meet our demands, must structurally enforce its own unraveling (explicatio) in accordance with the axioms - there must be such entities as can stand in the role of objects and morphisms satisfying meanwhile the prescribed etiquette of associativity, identity and composition in good order.

For that we can clearly have a category of categories and functors as abide, but what of objects and morphisms if both are just functors? It would be at once lower and higher to it, for functors as objects and natural transformations as morphisms are of a higher order and yet represent uniquely the lowest. Is there anything besides functors and categories in the ontology? Another question natural in light of our set theoretic escapade would be where is the sequential bottom in all this? - is there anything analogous to ZFC's $\emptyset$ over whom the whole universe might be concocted? The triplet terms **1, 2** and **3** seem fair candidates. What then do they make up in the ontology?

We did discuss the formal 2-sorted language of 3 unary function symbols dom, cod, id (the other common name "id" will be used instead of the original) and a ternary relation or else binary partial function of composition over two variable kinds, one intended for categories, the other for functors. The straightforward adaptation would be to sortify all of them, but domain and codomain can in light of what we have gone through be integrated to the behavioural layer, asserting simply each morphism to be instantiated by one or two entities.

For id arrow of objects, the categorical axioms enforce the derivability of objects from arrows based on having already provided these special identity morphisms. The problem we could encounter if we integrated id haphazardly however is that there are multiple automorphisms, for example on **2**, among whom the identities ought to be somehow well distinguished. Similarly, can there be multiple different morphisms for the source of f to the target of g where f and g share the middle link, and we need the composite $g \circ f$ to stand apart. When we speak of objects and morphisms from hereonout, we are doing so from within CCAF as aforementioned w.r.t. self-instantiation.

We affirm the presentation choice of two sorts, although we could rid ourselves of objects entirely and speak of morphism with domain given by one identity morphism and codomain by another, on the steep cost of visibility, which dictates the original differentiation to begin with. We therefore begin with one sort for categories C, another F for functors. Now, not only must categories be behaviourally linked to functors by satisfying the 4 category axioms, but they must be satisfied in mediation through the 3 category forms **1,2,3** as they externally condition the satisfaction of these axioms by their interrelation. We best begin with them as they condition themselves:

3.4.3 1,2,3 in the target ontology

Consider then **1** as a category with the single terminality-mandated (and behaviourally by multitude brutely established) automorphic identity functor $1_{\mathbf{1}}$. This represents and externally characterizes it having just one "object". But "object" is an informal notion, just as class was for ZFC (albeit not for NGB). There we see the character of the littoral floor. How is **1** to satisfy the EML axioms? Well, how about over its only autofunctor? If we speak of its identity functor aren't we laying ground for a supercategory above it? Or can **1** fund itself?

All $1_{\mathbf{1}}$ can compose with is itself and it really is composable. It thus satisfies the 4 axioms trivially, although not vacuously, it substantially satisfies them still. It must therefore form some category A, that has at least one object constituting it in our modelled environment, so a functor from **1** thereto. But **1** is terminal so there is the unique functor $!{:}A \to \mathbf{1}$ and their composite $! \circ X$ where X is the object of A and this composite is an automorphism on **1** which can only sustain 1 such, wherefore it must equal the identity on **1** (the X functor is called the section). This is a unidirectional isomorphism and since as we will see isomorphism suffices to collapse categories, from what **1** sees there can only be 1 category in the universe, except if there is an empty set, whom it cannot see by its design either. But for the other direction, because the category given by the automorphism $1_{\mathbf{1}}$ was established by assuming there were no other "morphisms"(functors) for whom the EML conditions would have to be checked w.r.t. to $1_{\mathbf{1}}$ and itself as it stood alone and a category of just one object couldn't by the extensionality axiom (which requires **2**, so sequentially this is untenable and requires simultaneity) be but isomorphic fully and hence collapsing to **1** from the other side also, because it would have to act on some one morphism differently but if the only object of A was **1** itself, it would have just the identity morphism on it, so there is no separator, and hence the functor $X \circ ! = 1_A$ is the final sign of isomorphism proper. Meaning just that **1** really is self-founding.

The objects of a category cannot be naively asked to satisfy the EML conditions for objects, as they are a level above categories. Formally what constitutes a category can only be the functor and category variables abiding EML, but there is a parallel side of representation accurate to the modelled world where satisfaction of the EML conditions in particular actually occurs, rather than accurate ontologically in the means by which the representation occurs. That being said, functors can still take the role of objects in their respective functor categories with natural transformations over them as morphisms, and in them then satisfying the conditions, but it would have to overcome that formally both objects and morphisms were just functors, despite **1** searing itself into **2** so that working with the latter might perhaps be sufficient to cover the participation of both.

This might be possible even without considering **2** as **1**'s vessel, through an alternative characterization of any morphism f in an arbitrary category A, moving away from the functor f:$\mathbf{2} \rightarrow A$ to two functors a,b:$\mathbf{1} \rightarrow A$ and a natural transformation α (which is itself sequentially suspect) from the intended source to the intended target, as well as the natural identities automorphic on their respective functors - so that the two level stratification of morphisms over objects is retained. Taking then the category of all global elements (the object functors) of A with all the natural transformations connecting them as morphisms represents A uniquely due to natural transformations sequentially expanding in the appropriate manner and preserving the important structural features. [18], [27, p.45]

Because it is the functor category $A^{\mathbf{1}}$ it is also guaranteed to exist by the third CCAF construction axiom. There we can take for the **C** category, our **1**, whence the Ĝ functor becomes the object of the functor category. But then we also take **1** for **A** and we have G an object of **B**, $\mathbf{B}^{\mathbf{1}}$ the category of all **B** objects, and since $\mathbf{1}\times\mathbf{1}$ is isomorphic to **1** alone, the objects of $\mathbf{B}^{\mathbf{1}}$ correspond to functors from $\mathbf{1} \rightarrow \mathbf{B}$ which are the defined objects of B. Meaning the functors from **1** to an arbitrary **B** defining objects really correspond to objects of the functor-natural transformation category. Which provides us with an extra level of security binding these notions in their representation grounded in the category-functor-only toolbox of our CCAF ontology. Yet even then there is some way to go in making this fully explicit.

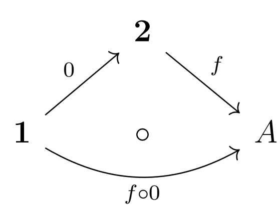
So if there is any leakage whatsoever, the metaphor bootstraps itself into substantiality, as we acknowledge a modelled virtual world which the objects and morphisms formal in CCAF reconstruct, that is itself external to the TO but must accord with it. We have discussed the role of motivation on VHS's choice of identity criteria, which here comes into fruition. The virtual world must, says the determining motivation, in the materially guaranteed as well as by raising added categories, find the described correlate. Delectably, we may thus reconstruct the modelled world from the "model", in parts at least. There must therefore be a primal category **1** of one proper object and a single identity morphism constituting it. **2** of 2 objects and 3 morphisms, **3** of 3 and 6.

To watch **2** treated (see Appendix A A.2.4)

With this complaint taken care of, we are to address whether "id" and C(, ;) should be separate sorts.

3.4.4 Identity arrow and composition in CCAF TO

We begin with composition - what does it matter to the ontology that the morphism $g \circ f$ is sequentially formed from f and g composed? It seems important to have some such connection between morphisms to capture their binding. Does the ontology loose anything if it does not have sorts for composition, and if, what? We may speak of $\beta \circ \alpha$ formally, but not for the $\mathbf{2} \to \mathbf{3}$ morphisms. Can we at least reach the source of A category's morphism f in the source-encoding triangle

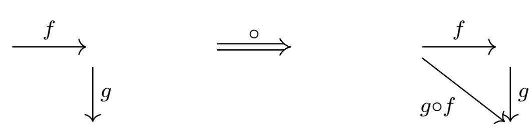


?

The issue is that the message of the triangle can only be understood over a coherent concept of composition, which we stipulate missing. Whereas for ZFC the formulas were not of a coherent virtual world as they were mutually incongruent, CCAF bows to a proper virtual world over the ACU.

Composition may thus fall back on this extra layer of data in some way if nothing else can save it. Since internally to the ontology, it might not be possible to connect 3 morphisms/functors as such, we will look at a functor M connected behaviourally to two instantiating category sort constituents C_1, C_2, which is in the nature of Functors just as there was for the binary relation $\in$ the basic behavioural feature of arity and directionality. Functor may apply to other functors if it is taken as an amphibian sort - applied to particular designated identity functors. The need for these identity functors is supplied by our decision to instead have in the presentation the category sort functioning as object of a category. It again does the unique determination of the virtual modelled correlate. Else, we might need to supplement it with the identity morphism sort to designate the first stratum of the sequential perspective that is in the other presentation. Here its sortification becomes wholly redundant over the stratification's expulsion. Yet the need for composition in some form remains. Couldn't then functors encode the composition relation alone?

What we need of it is to say something like this



So it would have to apply this connecting representation of composition relation to two functors f, g and itself be this produced $g \circ f$ composite. But $g \circ f$ exists on the same stratum. And so the produced functor would by the condition of its production have to exist on the higher stratum, which shouldn't be permitted.

And yet it might be possible if we encode the two functors f, g into just one, wherein they would have to be positioned (as it might matter which goes first as in the $1_1 \circ 1_0$ example even over the same source) and so have a composition functor over two functors i, j, one of whom captures two composable functors, the other their composite.

For all functors, the arrow extensionality of the fourth axiom for **1,2,3** ought to weight in appropriately: (see Appendix A A.2.5)

The identity criterion for functors is thus dependent on the lower levels of the sequential stratification - on the morphisms and objects constituting the category in both the virtual modelled world and its **1,2,3** representations. **2** is the global separator, but we will also discover that within the categorical set theory ETCS, sets have extensionality w.r.t. the separator **1**; that distinct functions between sets S and S' differ on some element of S given by $\mathbf{1} \to S$.

In ETCS functions are fundamental, so this translates to set theoretic extensionality through a characteristic function which uniquely captures the $\in$-contents of a set. **1** thus acts as the set separator, suggesting separators encode abstract extensionality. We can furthermore clearly distinguish the places where extensionality occurs for ZFC and CCAF. In terms of ontologically relevant structure, sets have nothing to order but membership. Categories have two levels, an expressivity increase, where the identity of the most fundamental entity - Functor - can be separated by variation on the level of objects which structurally enforces the morphisms' going along, but the objects can remain shared, which set theory is incapable of accepting and providing its equivalent representation for. The CCAF entities simply are richer, which need not translate to the CCAF itself being wealthier apart from certain structural core, which would have to be compensated for. Notice again the external-internal symphony of categories on even the most base questions - morphisms are inside but the distinction occurs by composition that is soon outside.

With this we return to composition considerations. Without composition the axiom of extensionality could not be internally formulated on the behavioural layer, because the composite would be missing as composite - instead there would be an arbitrary conveniently named yet essentially disconnected morphism. And so the notion of image-taking analogous to the behaviour of set-theoretic functions would too be inaccessible and with it questioning ?F(f)=G(f). By the level-mismatch argument above we have shown that encoding it should not be possible by the remaining 2 sorts alone. And even if something slipped by, the clarity of ordinary categorical exposition is sure to increase greatly in its primitivity, justifying our pre-emptive acceptance.

3.4.5 Sequential dependence

For Composition we must then separate the sequential constructive approach of the composite morphism as the result of the components. It induces an image of two levels of the sequential view, the upper dependent on the fixedness of the lower,

$$\xrightarrow{\quad g\circ f \quad}$$

$$\xrightarrow{\ f\ }\ \xrightarrow{\ g\ }$$

but ontologically the dependence is not of this kind, ontologically we consider the applicability - the dependence of the morphism on the objects constituting their source and target. Morphism f:$A \rightarrow B$ must be of higher dependency than just A and B which can exist alone without the morphism, functor on category, category on its object and morphisms, natural transformation on functors. This again invokes an appearance of circularity, which comes through the necessary employment of the simultaneous perspective over the sequential.

Whereas the ZFC universe V was properly raised, here we have a disruption to the monotonicity of V - a nondiscrete category of categories (nondiscrete because then there are functors which are of a higher level than the categories which stand for their domains and codomains). The CCAF universe O consists of areas, even enclosed areas, which perturb the above suggested order of dependence. Sets like categories are nested, but they are shallow in their nestedness. Categories trickle down in their biotopes through both applicability considerations and the structurally enforced capturing of the virtual world. Categories are to be made on one side of categories and functors over them, on the other of just functors. These two perspectives must oblige, they must find coherence, even if structurally enforced.

Applicability is the dependence we have described above, but even there we don't have internal formalization of definitions above functors. For objects, morphisms or natural transformation, all we do is provide a structural significance and recognizable pattern within the universe. That too is the meaning of externalization with which category theory and CCAF in particular is loaded, that certain high-level concepts are thus structurally represented despite not having a direct sortal explication to match them. And further, they don't exist just passively, but their intended definition is authentically (in terms of uniqueness) expressed within the universe. Similar is the notion of inner model that captures this elsewhere[25] and even has its set-theoretic flashes. The choices of simplicity cost us this strain of expressivity in preventing the virtual world from being embodied immediately. Thus, it still plays an essential role for the lowest strata, where a type match between the actual and the expected cannot yet be managed.

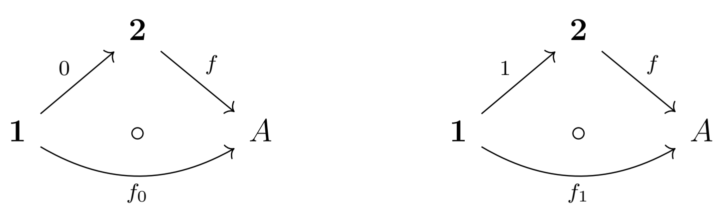
3.4.6 The problems of the composition sort

Composition in O clearly doesn't open itself up to the same kind of nestedness as categories and functors and is itself in service of them by commuting triangle diagrams establishing their behaviour externally - gf composite is after all defined for morphisms through the pushout **3** and maintains for functor morphisms in the same spirit. It provides a connection to morphisms, which internally distinguishes three roles, two of participants that are source-target compatible, the third of a unique compatible representation of the two within the positional arity form.

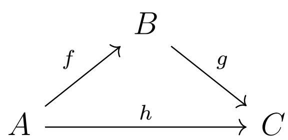
Something akin to arity is universal to the behavioural layer of a multitude-admitting sort, as we have come across in CH1. The positionality is the distinction between an edge and an arrow - it captures first the simplest distinction between the forms of multitude-admitting sorts that are internally heterogeneous or homogeneous, then if heterogeneous it is an "ordering". We have taken this to be in the basic competence of sort's behavioural layer due to its sheer potency and in virtue of its expressivity still simplicity-inbound. It is thus a two-fold step-up from the bare necessity, and we keep it for utility alone, although it warrants a proper delineation as to what complexity entry each step constitutes.

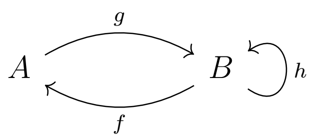
Now just as we had done with the membership sort, order is included, but how can ternary composition say the third of its coinstantiators is the composite of the first two? How can it internalize the two triangles?

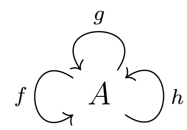
Sort might use something akin to typechecking of its instantiators relative to the positions - e.g. position 1 is sort category, 2 category, 3 functor. Hence, if the composite could be seen as having different type in any system-legitimate sense, that would suffice to designate it as the composite of the others. But if there was certain levelness about the positioning - homogeneity of that which the order orders - and we had three functors indexed by 1 2 3, how could the system distinguish the composite among them? Recall the triangle diagrams encoding source and target



We can distinguish through these examples three structural forms a composition can take - triangle, line segment and a point:



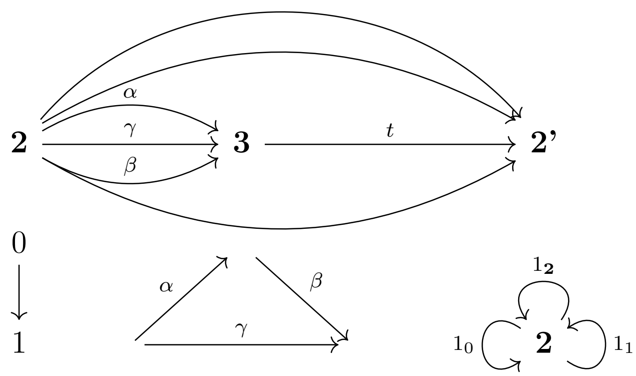




For triangle it is not sufficient even to say the composite has the source of other and target of another in common, because h and g both fit the description, so the positionality must come into play. There are three distinct functors, hence the shape and name s.t. $g \circ f = h$ in accordance to arrow extensionality, their behaviour on the Composition C(f,g,h) sort with source and target conditions is uniquely given so that only one can be the composite accounting for positionality. In a different way but to the same result does the line segment uniquely determine its composite morphism as the only one among them that is automorphic. The problem hides in the point form on A. There we distinguish the three cases

1. All three functors are the same
2. Two of three are the same
3. Each is different

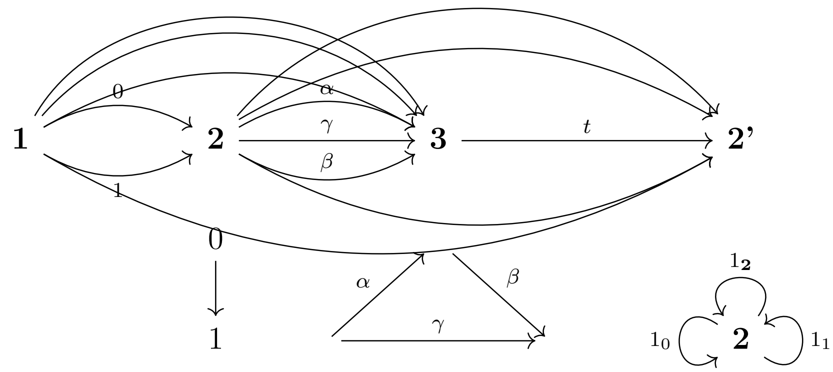
For 1. The functor need not be identity as in $1_0 \circ 1_0 = 1_0$ on $\mathbf{2}$. But once more is there just one possibility of a composite, trivially. But in the next step $F \circ F = G$ / $F \circ G =$ F / $G \circ F = F$ we encounter a pathological situation. Take for example ^{op2} which sends 0 to 1 and 1 to 0 in $\mathbf{2}$ and $^{op2}\circ^{op2}$ as the composite, so then two such inversions after one-another mean $^{op2}\circ^{op2} = 1_{\mathbf{2}}$. Or even the $1_0 \circ 1_1 = 1_0$ which shares functor as composite and participant. Here because the composite can occur on either side of the equator, it is structurally no longer unique even accounting for positionality. For resolution, we will invoke the structural enforcement reliant on the virtual world correspondence.



Read the diagram with vertical alignment saying what are the relevant contents of **2** then **3** then **2'**

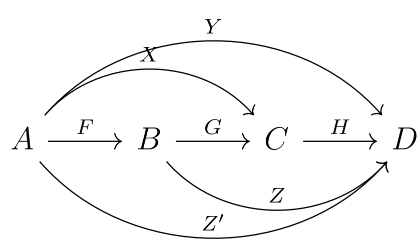
Due to the existence of concrete vestiges of the virtual world inside the universe O, we can move its burden to the level of externality, wherein the positionality extends to encode the sources and targets in the triangle form of **3** imprinting itself onto whichever category the considered point lives in, so that the composite corresponds to the t-image of γ. Meaning we can establish the behaviour of composition sort on the point case by utilizing the as yet incomplete composition on triangle form alone.

When we also extend the diagram by **1** on the left and all the object functors, it gets little harder to navigate



but we get proper access to source and target encoding as triangle case compositions from **1** to **2** to **3**. Which is also where associativity axiom needs to enter, as it carries precisely the imprinting of the commuting triangle structure with positionality over 4 total categories. With this, we may acknowledge that the point might have several compositions on it, each of whom must structurally enforce an invitation of a different functor from **3**.

This can be strengthened even more by utilizing sequentiality of a particular kind in the external encoding via associativity:



There we consider two parallel progressions C(F,G,H) → C(X,H,Y) and C(H,G,Z)→ C(F,Z,Z'), and use them as a naive typechecking differentiation feature - associativity has a sequence of composition building, participation on which is structurally unique in a particular sense our second approach tries to exploit. This added to the structural enforcement above, we have remedied the pathology of the point case and established the composition sort of the abstract CCAF TO scheme in fullness.

3.5 The remaining axioms

Let us now speak of the universe and account for the rest of the axioms of CCAF.

What does the axiom of functorial associativity mean to the universe? The existence and uniqueness of the namesake axiom sequentially raise the composition participants, raised further by associativity in the interaction of composites - the former structures composition beginning on the lowest level, the latter structures complexes of composition. The uniqueness of composition is the more interesting because it is thus extended by associativity to ternary and through it all m-ary composition. It means that the scaling of the composition leads always to a unique composite resulting functor or allows us to take two as a composite single unique functor and reapply the axiom to yield the result a level higher.

The functor associativity is also responsible for their preservation of domain, codomain and identity arrows. Through arrow extensionality, the top-down direction of the sequence can also be taken up as a perspective of its own (we begin with the outmost composite and step down inside it by the morphisms it differentiates).

3.5.1 Technical background to the axioms thus far

The axioms of figures 1 and 2 can be given a more technical context: (see Appendix A A.2.6)

We are in a strange position, since the axioms we are going to introduce effectively simulate ZFC within CCAF, and ZFC has different identity criterion for sets than is that for categories in CCAF. Thus, for the first time we can speak of local identity criteria particular to an area of the universe by some justification separated from its complement. And yet the universal arrow extensionality must hold there no less, it just happens to point instead of the connecting morphisms to the identity morphisms and through them the objects they uniquely characterize. The set extensionality is a particular local brand of the arrow extensionality, where the role of the building-block forms shifts from the dependence on **2** directly to **1**. **1** becomes a local separator. This brings a drop in complexity of the subsystem seeing itself as the entirety of being, because the central identity criterion becomes dependant in its judgement on a significantly simpler structure of a single constituent.

Categories as forms

Another question concerns the archetypality of categories suggested by **1,2,3**: as far as the build-up of the category sort is concerned, we already know that the sequential raising as ordered by $\in$ in ZFC cannot be straight here in reaching up to pushbacks, equalizers, and so on. Relating CCAF TO to CH1's substance/relation systems, do categories behave like substances or as relations so that we can say Category C_1 is instantiated 5 times by listed constituents; do we speak of categorical and functorial forms? A limit is instantiated by any product which is instantiated through every pair **A,B** uniquely, but is it a particular or a universal? The functors in a sense function as the instantiators of the particular categorical forms **1,2,3** - they capture the category as it is forming its target category and the functors must preserve this form.

What can happen, however, is the loss of resolution, of fidelity as various aspects of it collapse. If **2** is taken by the unique functor ! to **1**, nothing of it can be seen in **1**. Amazingly, certain categories can be seen as representing notions of identity in their roles as either sources or targets of functors - namely, **1** as target captures the notion of unification and the varying source determines what of. What structure is being obliterated. **1** has a special role in the universe viewed as a form, because it is never compressed, never loses its own structure (even if there was an empty category that was initial, **1** couldn't functor to it without there arising isomorphism and hence collapse) and uniformly serves as a target means of structure collapse. **2** already can lose itself by targeting **1**, it still projects its morphism structure, but it is no longer binary nor accordingly distinguishes source and target, which is the entirety of the structure of a morphism really. These form instantiating functors must therefore be all injective on morphisms (which by identity morphisms means also on objects). These can be used to track the instantiation of a categorical form into another category that is resolution lossless, elsewise it is really difficult to speak of instantiation convincingly.

It is not the case that the target category is instantiating the form by its own, rather that internally it carries in itself the form of the source (also called embedding of the functor). So it is a loose complex of constituent objects and morphisms that instantiate it in particular. But as tiger instantiates beauty and complexity, they cannot each make him up entirely, nor needn't they make any one organ either.

These specific functors could then be seen as conveying the proper instantiation, and corresponding to the number of on-ontology formalized instantiators of any given category. Which takes on a different meaning for the presence of the category as a form on its own. That is unique as any two categories of the same makeup would have an obvious instantiation of themselves as a form through identity functor, but then also going to the other and since the whole category corresponds to the shape of the source, it must be isomorphic, thus identical and collapsing.

The complexity of forms

Only certain aspects of the basic forms must be elementary: **1** brings objects, **2** connecting morphisms, **3** composed morphisms. The rest can be made of them - say a commuting square: it can be reduced to a triangle by partial composition injected from **3** along one angle, so composition is stacking over these **3** unproblematically and as we saw through associativity uniquely. But **1** is the only form not instantiating any other. **2** takes only **1** but adds the connection as its complete novelty, along with the directionality it carries. **3** is then made of 3 varied instantiations of the **2** form with an added ordering on the directional data. Seeing then as forms can be complex, any category poses a form and the distinction of particular and form comes from being instantiated nontrivially by own identity functor.

There is a technical method of going beyond this: (see Appendix A A.2.7)

3.5.2 Separation

Before we move onto the Set axioms of CCAF, there is the categorical separation scheme to take care of (see Appendix A A.2.9)

Again are we pulled to contemplation of formal language as it enters the raising process, but unlike for ZFC the language is not wholly external, but grounded in familiar externality. The notion of subcategory clearly mirrors that of a subset insofar as it poses the existence of an organ as a thing in itself, rather than just a constituent.

3.5.3 The Set axioms

Last pair to cover contains all the set axioms. We will begin with ETCS and fine-tune it only afterwards.

1. There is a category Set satisfying the ETCS+R axioms
2. Fullness: For all sets I,J:$\mathbf{1} \to Set$, every functor ($\mathbf{1}$↓I)→($\mathbf{1}$↓J) equals ($\mathbf{1}$↓f) for some functor f:I→ J in Set (where f:I→ J is reconstructed as ($\mathbf{1}$↓I)→($\mathbf{1}$↓J))

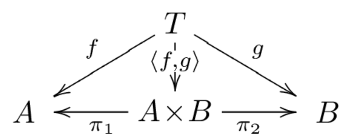
Now then, we may rejoice, for much of ETCS axioms is covered already in the anteceding axioms:

- 1) is our **1** already given
- 2) products we already have
- 3) equalizers we also have

There is a singleton 1: $\forall S\ \exists!\ S \to 1$

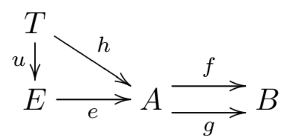
Every pair of sets A, B has a product:

$$\forall T, f, g \text{ with } f\colon T \to A,\ g\colon T \to B,\ \exists!\ \langle f, g\rangle\colon T \to A \times B$$



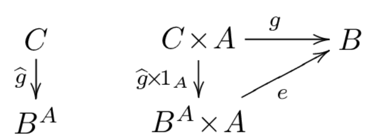
Every parallel pair of functions $f, g\colon A \to B$ has an equalizer:

$$\forall T, h \text{ with } fh = gh\ \exists!\ u\colon T \to E$$



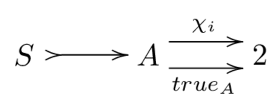
There is a function set from each set A to each set B:

$$\forall\ C \text{ and } g\colon C \times A \to B,\ \exists!\ \widehat{g}\colon C \to B^A$$



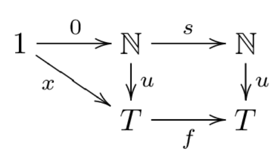
There is a truth value $true\colon 1 \to 2$:

$$\forall\ A \text{ and monic } S \rightarrowtail A,\ \exists!\ \chi_i \text{ making } S \text{ an equalizer}$$



There is a natural number triple $\mathbb{N}, 0, s$:

$$\forall\ T \text{ and } x\colon 1 \to T \text{ and } f\colon T \to T,\ \exists!\ u\colon \mathbb{N} \to T$$



Extensionality: $\forall\ f \neq g\colon A \to B,\ \exists x\colon 1 \to A$ with $f(x) \neq g(x)$.

Non-triviality: $\exists\ false\colon 1 \to 2$ such that $false \neq true$.

Choice: $\forall$ onto function $f\colon A \to B,\ \exists\ h\colon B \to A$ such that $fh = 1_A$.

Figure 3.3: the ETCS axioms

- 6) NNO object we have
- 9) Choice once more
- 10) implicit associativity of composition (gf)(x)=g(f(x)) comes from the EML category axioms

The axioms 5) and 8) are to be read together. They guarantee the existence of a particular subobject classifier. (see Appendix A A.2.8)

4) is just a specific instance of the functor category we have guaranteed already. Note with McLarty that the powerset of a set **A** is just $\mathbf{A^2}$ the function set from **2** to **A**.

Finally 7) brings back the structural version of the set extensionality axiom.

-For two different functions f,g on sets $A \to B$ there is an element on A whom they treat differently.

Membership-based set theories say a set is determined by its elements. Categoric set theories say a function is determined by its effect on elements [19, p.27]

$\in$ in ETCS is "Homogeneous" as the sets and elements are formally of the same type, no element has internal structure nor is any set an internal structue of other.

Membership relation $\in$ is seen as a special function from $\mathbf{1} \to S$ just as we have seen objects in CCAF to be a particular functor instantiating the singleton archetype. For ETCS, sets and functions are the objects and morphisms satisfying the EML axioms of Set. Functions are not within sets as morphism, but traverse between sets seen as particular discrete categories. Just as the category of all categories CAT making up the CCAF universe is not really a category, Set is not a set. It is however a category. Set is a local embodied universe.

ETCS is however significantly weaker than ZFC. For an adequate simulation, we require the axiom scheme of replacement [26]: (see Appendix A A.2.10)

The very last axiom to discuss is the fullness condition. **1**↓I is the slice category of all elements of the set I. All its arrows are identities, so it is discrete. Each functor F: I→ J is reconstructed as a functor from (**1**↓I)→(**1**↓J) It says that reconstructing sets as discrete categories this way makes no difference to the pattern of functions between sets. As an axiom it guarantees the adequate integration of the set-theoretic universe made of ETCS+R into CCAF's overarching O. Meaning it not only makes sense to itself as ETCS+R seeing just its own insides, but also seen from the categorical context of CCAF's structure are the essential constituents - sets and functions making sense categorically as particular discrete categories and functors inbetween them. McLarty explains that fulness says discretes are "basically" sets, because each set has a corresponding discrete, but not all discretes have a corresponding set, some are as large as Set itself, or larger so the universe of all discrete categories will be a large extension of the universe of discrete categories coming as correlates of Set constituents. [27, p. 51] On the set representation of categories much is said in [25]]expanding on internal categories.

The axioms for Set guarantee that it behaves like ZFC in terms of provability. And still we must realize that the other axioms of CCAF don't cease in influence even for this isolated system inside the categorical universe O. That the axioms of ZFC are satisfied under the translation we discuss in CH4 is guaranteed, but it holds even for just ETCS. The CCAF background then enforces through its own extra strength that the models of their subuniverse, in the class of models of this abstract TO scheme, are abiding of these CCAF axioms on top of those for ZFC. But these CCAF axioms do not apply to it in isolation, nor guarantee that the Set category be closed on the constructive axioms of CCAF .

Clearly the Set category makes O significantly richer, and it is conceivable to consider the ETCS+R axioms as a standalone system, which however provides a set-theoretic foundation presented categorically rather than CCAF's purely categorical foundation indulging in the set-theoretic simulation, but fully now grounded. Due to the artificial and admittedly naive addition of ETCS+R where McLarty takes after Lawvere natural to CCAF just ETCS without replacement, there could be internal conflicts emerging from the interaction of replacement over the rest of the CCAF axioms within the Set category, ETCS+R alone wouldn't have to deal with, none of these are covered, making it a very unsafe ground for your mathematics. The comparison of ontologies we have been heading for atleast offers thanks to its enrichment an obvious path forth to be discussed in the concluding chapter.

Chapter 4

Conclusion

In ZFC we used total emptiness as the vantage point in $\emptyset$. Although it was not used as the backbone of its identity criterion, it served the role for the sequential foundation for P and union. The similarity of arrow and set extensionality role is in establishing the conditions of unification and differentiation of its essential entity - morphism/Functor, and yet not essential in the ZFC sense where just one comes into consideration. For ZFC there is $\in$ and $\notin$ of membership, but its categorical analogue is Composition over functors and functors over categories. It is an essential relation of a higher order. As is the arrow extensionality of higher order than set extensionality in the sequential dependence. Its role is to establish a reference point for the differentiation/unification - the feature VHS chooses as relevant and responsible for the distinction. In ZFC presentation, for sets it was their elements, in ETCS it's the functions differing on how they treat elements. In CCAF then finally it's functors differing on how they treat morphisms - and as we mentioned not just the objects - there hides the sinuous shadow of extra expressivity over ZFC.

Arrow extensionality externalizes the identity criterion as part of the universe -

the interrelation of the sorts, responsible for identity criteria, that was implicit for ZFC, becomes in CCAF a substructure. It is thus present on both levels. That which is so drawing on categorical ontologies is precisely this inside-outside fluctuation, the metatheory leaves a recognizable vestige on the world so that the immersed perspective can see over its own shoulder. Lawvere's genius just illustrates this aspect of category theory in the most fundamental, yet simple still, manner.

Once more is the model in the abstract TO scheme given by nailing the identity criteria down, which could only be the case if extensionality was an identity criterion form leaving great openness in its complement within the formal choice. In CCAF we have to distinguish the identity within and without - not only are we dealing with functors, categories and composition in terms of their own identity criteria, but with the entire scale starting with objects. The virtual correspondence must maintain the identity criteria form for objects being isomorphism, meaning a joint employment of all 3 sorts, as was the case for set extensionality. Object A of category **A** is seen as identical to B of **A** if there is an isomorphism $A \to B$ $B \to A$ the composition of which to both sides yields identity. Yet it is a higher level identity, because ontology distinguishes the objects through global elementhood functors from **1**, but doesn't, in the naive manner, distinguish isomorphic objects at the modelled world - the objects and their binding isomorphism-constituting morphisms are still formal entities; these identity criteria are leaps in perspective, there is a stratification of identities, of the proper "entities".

Although it is formal, it is so only partially, as is the equivalence of categories the level just above, not directly. It is the metaphor of objects and morphisms, but the step-up is recognized and accounted for because the appropriate notion of identity for categories is the equivalence of categories: $\mathbf{A} = \mathbf{B}$ iff there exist two converse functors F:$\mathbf{A} \to \mathbf{B}$ and G:$\mathbf{B} \to \mathbf{A}$ s.t. $G \circ F \cong 1_{\mathbf{A}} \wedge F \circ G \cong 1_B$ as natural transformations with each α component isomorphic (called natural isomorphisms). The identity criterion is a weakening of isomorphism and is weakened still more by each upward step. Just like for ZFC, the pretend bottom must be brute - morphisms and objects brute differentiated. ZFC is made plane as extensionality suffices throughout. Categorical ontologies internalize the epistemic aspect of perspective, of sequentiality and simultaneity, they integrate it as the formal ontology projects itself onto the universe, so that its pattern can be recognized and at least partially reconstructed from within.

There are again two levels at play: the formulaic and ontological. In ZFC we were able to use formulas to describe sets that were different as structured preobjects, but ontologically could be one. Just the same for CCAF, we might describe two categories **A, B** which are not isomorphic and differ in constituents, yet the identity criterion for the sort can be equivalence of categories, which rests on composition and functors as does the implied identity for morphisms and objects - but it can be structurally enforced(embodied) identity, meaning because there is a representation of objects and morphisms and their composition that is formal by **1,2,3** inside the universe explicitly, the universe can be made to meet the identity criteria for the represented entities. These functors whom we call objects and morphisms will be, by the identity criterion we have no form container for, made to satisfy these conditions - there will only be such functors as would make identical two isomorphic objects.

The unification feature chosen for objects is the isomorphism binding them, so it is referential to the basic morphism here. And if objects collapse, there may arise their isomorphism-become identities. In particular this would mean that in the universe either there would be no isomorphism represented by the morphism functors apart from identities (turning all categories into skeletal categories), or it would abide the perspectival shift and the identities would enforce the restructuring of the universe, its perspectival stratification, each stratum corresponding to different level of identity criteria. Morphisms also abide arrow extensionality if their domain is structured by treating arrows differently, if unstructured then by difference on composition (formed on the universal uniqueness) which is basic to its referentiality.

Categories then abide the equivalence criteria, as their identity form relies on functors and composition in just the same manner. Functors rely on arrow extensionality as morphisms do but know their carriers are structured, which the morphisms don't. The equivalence of categories depends on the existence of the functors under natural isomorphisms $\alpha : 1_{\mathbf{A}} \to G \circ F$ and $F \circ G \to 1_{\mathbf{B}}$, which are in turn the identity criteria for functors - it is an isomorphism in the functor category [F,G] or equivalently as a natural transformation whose every component is an isomorphism. Thus, we see the collapse ascend: the functors bound by functors of an internally isomorphic structure become unified $G \circ F = 1_{\mathbf{A}}$ and $F \circ G = 1_{\mathbf{B}}$ so the isomorphism must be taken at the simultaneous unification step, if we wish to remain true to the form. But practically it is impertinent, because the identity criteria are weaker and therefore entail the stricter ones - so if the collapse occurs on functors constituting the categorical equivalence already, then more so are the categories meeting the unification conditions.

[18]

This opens up the question of comparison and strength, to be finalized here. Our CCAF was built in such a way as to explicitly contain ZFC as a subuniverse to simulate it appropriately, which the translation by way of Mostowski manages to achieve.

"ZFC and ETCS+R are intertranslatable preserving provability in both directions." "So far as ordinal induction is concerned, the only difference is that ETCS+R cannot distinguish a selected representative of each well-ordering the way ZFC does with the Von Neumann ordinals..." Which doesn't matter for ZFC and ETCS+R. ETCS+R derives well-orders exactly the same way as ZFC. "They both prove the existence of all the same well-ordered order types and support all the same transfinite orginal inductions, and admit all the same consistent extensions by stronger axioms." [27, p.38]

For the translation of ETCS+R to ZFC: (see Appendix A A.3.1)

We are then comparing two abstract TOs, ZFC and CCAF, and as observed in CH1 we will be in a need of a translation. This translation involves a supersystem. We may then discuss the systems from the perspectives of one another. We can also take the formulaic turn, as these share the same level of language over FOL, with some differences that could be simulated without too much distortion. CCAF by simulating ZFC provides an internal means of comparison on the other hand. So that every TO produced by the scheme will contain in it this set theoretic zone living inside and formulable in its sorts.

We need to set the compared areas evenly so that there is an agreement on some higher structure in whose openness the differences can be drawn. This structure we set out to be the general ontological system of SOTO. Here, however, inside the TO_{CCAF} we are reconstructing the TO of its subsystem captured purely externally. Within the universe O there hides ZFC's universe and so the obvious comparison stems from the universe-produced-dependent identity criteria for the two ontologies, TO_{CCAF} and the reconstructed TO_{ZFC}. Specifically for the TO taken abstractly as they vary over input CoPs and produce different models based thereupon. CCAF obviously sees itself different on this feature as well as on directly expanding the universe produced by ETCS+R.

The aspect of size presents itself there as ETCS+R is not a proper reconstructed sort instantiation even in its own right, but a proper class, a notion ungrounded. In ZFC, the universe cannot be integrated. And yet Set does form a Category. It is a constituent, a particular. CCAF cannot provide the sortal containment of its own universe that is natural and unforced, the same way ZFC cannot address V in terms of its own 2 sorts. The containment of ZFC in CCAF however becomes natural once ZFC is its organic part, not necessitating additional formal structure added to the TO. It is not just a loose collection, but a category, yet the universe O itself remains uncontained; it shows two levels of a perspective at play, the all-embracing world, which stands in as a scattered structure, and its constituents which, like Set, open to incredible depth. Yet CCAF in this form mirrors the structure of ZFC in its outer openness and structurally expressed universe, in comparison once more to NGB, which poses a mismatch. Here the inner is a microcosm of the outer, adding to the naturality of its containment.

If CCAF is then to place ETCS+R on its place as a universe produced, the ontologies it sees of itself and ZFC would be seen different on the universe produced-based criteria. It allows us the categorical reconstruction of the behaviour and sorts of ZFC, where on the outer comparison we must go by the same level of language in FOL and compare two enclosed isolated systems. In saying ZFC and CCAF are the same we hinge on the means of the language; they are the same in terms of the concepts formulable and formulae provable. It would require a treatment of language to adjudicate how relevant such identification is ontologically, but we use FOL conventionally and without reflecting it explicitly as part of the defaulted operational framework. We swim in Land. Adequate comparison needs the involvement of reflection, but that cannot be done in such partial ontologies which do not claim the totality even in suggested direction. They are formal constructions of virtual worlds hosted therein, which cannot but stay virtual until they provide an embodied projection that would justify them.

Within the system, the comparison can instead be made on the ontology, on its impact, rather than just on formal grounds like provability in FOL.

We have discovered a new partial identity criteria - of archetypes - in particular speaking of products, coproducts... (the co-duals missing in ETCS+R in their axiomatic guarantee) there is the in-universe identity criterion on which these differ, structural closure on certain archetypes - that gives the identity criteria for the universe produced.

Moving down the structure, we can differentiate over the sortal and behavioural layer together. There we must compare w.r.t. ETCS+R and ZFC separately, with suggestion of ETCS+R and ZFC comparison alone. For the assumption that ZFC really is in an ontologically-relevant sense ETCS+R, we go along the criteria mathematicians justifiably find most convincing. Even though ontologically it is anything but clear.

For ZFC to reconstruct CCAF to an analogous structural role, it would have to consider CCAF a set, but seeing as it contains in turn ETCS+R which would be the proper class V in its universe, this cannot happen. The fact that CCAF carries further axioms over those of its ZFC reconstruction means simply that there is enforced universe of discretes that is directly above that of ZFC and since the axioms over the ZFC's world only provide additional restraint to its universe rather than its enlargement, it shouldn't based on this alone be possible to reconstruct it. Perhaps the axioms of CCAF might be encoded within ZFC externally, but that would not be a projection of the TO structure either. It could not reconstruct thereby the universe of CCAF in its universe produced, constituting a glaring separation feature. In line with this, CCAF cannot see itself universe-produced-wise identical in TO to ZFC as it reconstructs it. Let us look at other separation features of importance as we delve deeper.

ETCS+R has sets, functions and their composition as reconstructible sorts meeting the same category axioms as categories and functors do above it. Categories clearly are internal structure of categories of categories whereas ETCS+R neither has internal structured elements nor are sets internal structure of other set. Behaviourally, they will differ by the 11 axioms concerning Set with CCAF's rest. And even if there is an overlap on the form of the 3 sorts, a significant difference still stands in virtue of the respective roles of identity criteria.

Sets are distinguished up to isomorphism. Categories up to equivalence of categories. One idea is to speculate that this identity criterion for categories is the proper level at which no genuinely categorical property is lost upon translation - that the significant categorical forms like equalizer and coequalizer are both preserved and reflected by equivalence functors. However, the problem is not so easy (see Appendix A A.3.2)

The special adjunction of equivalence of categories, unlike a general adjunction, cannot collapse nonisomorphic objects to isomorphic - it reflects isomorphism, meaning that over the established identity criteria for objects, two distinct objects cannot be brought to a single object within the ontology. It is a weakening of the isomorphic identity criterion again, as is natural isomorphism for functors, and the tendency obtains. When we move along natural equivalence hither and thither, the object we start with need not remain, we only ask its isomorphic relata [23, p.2148] - but ontologically, we see this is precisely the right expectation as objects under isomorphisms collapse. Thus, the extensionality of sets, as captured by isomorphism in ETCS+R, must by the comparison stand on the lowest level of the categorical hierarchy (relatively if we consider nestedness), but above it the collapse range is wider; unification becomes hungrier as we scale up. Set still maintains it for its objects as a category, meaning the whole Set category is precisely positioned so that its sets are of the lowest caste, and the inherited arrow "extensionality" reflects this.

As noted sets are fundamentally unstructured in their elements. Seeing as the isomorphic singletons $\{\emptyset\}$ and $\{N\}$ collapse. $\{\emptyset\}$ cannot see inside itself anything other than an abstract particular inhabiting it. Categories are then structured least by the morphisms, although the functors that capture objects and those that capture morphisms are themselves yielding abstract entities impenetrable from the category's viewpoint considered in isolation. A sees each object it has by way of the unique Functor ! to **1** composed with it, whence each such triangle yielding identity must be unenlightening.

There are surprising new views encountered within CCAF. For one, the introspection pure and external as the reconstruction of the projecting categorical forms. In it, there is the level of external characterization of a category, but also the global way of interacting with anything whatsoever. There is the umwelt of a category, the transitive closure of information gathered through how A sees B directly but B captures C and so A sees C. Else there is the disembodied perspective of the categories internested in their constituents as they appear to eachother and see eachother and form complex structures while being simultaneously oblivious and privy to it all happening.

Thus, we see in comparison to ZFC a great disorganization. ZFC has a grand central sequential perspective over $\emptyset$ as P iteration with union stacks up to the entire cosmos over the foundation axiom, which unified construction principle and conical origin is entirely missing from CCAF's behavioural layer. On the other hand, the structure is naturally more involved. ETCS+R shows this to be the categorical underpinning at work in the primitives of object morphism and composition, which naturally reach onto universal properties, these proposed inductively built-up forms. That is the categorical analogue, although its sequentiality is very different and often sequential only on the surface, as simultaneity is required in addressing the nested concepts both of category at the top and objects at the bottom at **1**'s feet. This sequentiality is only of an epistemic advance in expressivity, until it becomes embodied, as occurs with the hard dependence of the archetypes in their projections, founding the categorical universe.

Just as simplicity can be two-fold, formal on the formulation of the ontology as a symbolic system and living as the universe produced opens up from the conceptual principles of virtuality, there is also the consideration of expressivity that is formulaic, as we wander around the experimental framework in quasi FOL and ask what all can be spoken of? What properties expressed? What of the underlying motivation for founding the ontological system in descriptive containment is in the power of this combination of concepts-representing symbols? What features can be captured directly, what can be encoded, and what of that which isn't spoken of is our incompetence in finding the encoding?

Final words

The world sees also. Through the aspect of functorial observation we have given heed to, there can be under proper accompaniment an interpretation of Informations which aims at full embodied reflection on even the self-founding of the ontology and of experience at large. CCAF already, through the **1,2,3** categorical forms and behavioural projection, steps onto the field of genuine self-mapping expressed in structural invariants. ZFC ontologies, all else overlooked, are not capable of this feat and must, by virtue of that alone, be seen as philosophically less interesting.

Categories offer much hope to violate the isolation of the mathematical. And as for the discipline of mathematical foundations, it can now, through the system we have developed and demonstrated in action, be investigated from an entirely alien perspective - as an applied ontology.

P. S.

References

- [1] Aczel, Peter. “Non-well-founded Sets.” Palo Alto: Csli Lecture Notes, 1988.
- [2] Akman, Varol, and Müjdat Pakkan. “Nonstandard set theories and information management.” Journal of Intelligent Information Systems 6 (1996), p. 5-31.
- [3] Angere, Staffan. “Identity and intensionality in Univalent Foundations and philosophy.” Synthese 198, no. Suppl 5 (2021), p. 1177-1217.
- [4] Balcar, Bohuslav, and Štěpánek, Petr. “Teorie množin.” Prague: Academia, 1986.
- [5] Brian M. Scott, Axiom of Regularity and infinite sequences, URL (version: 2013-07-08): https:/math.stackexchange.com/q/438594
- [6] Carnap, Rudolf. “Meaning and necessity: A study in semantics and modal logic.” Chicago: University of Chicago Press, 1988.
- [7] Devlin, Keith J.“Constructibility” Cambridge: Cambridge University Press, 2017.

- [8] Dworetzky, Samuel. “The Classification Problem for Models of ZFC,” Mathematical logic quarterly 66 (2020), p. 182-189.
- [9] Feferman, Solomon. “Intensionality in Mathematics.” Journal of Philosophical Logic 14, no. 1 (1985), p. 41–55.
- [10] Frege, Gottlob. “Comments on Sinn and Bedeutung.” In: Gottlob Frege & Michael Beaney (eds.), The Frege reader, p. 172-180. Cambridge: Blackwell, 1997.
- [10] Frege, Gottlob. “Comments on Sinn and Bedeutung.” In: Gottlob Frege & Michael Beaney (eds.), The Frege reader, p. 181-193. Cambridge: Blackwell, 1997.
- [12] Frege, Gottlob. “Comments on Sinn and Bedeutung.” In: Gottlob Frege & Michael Beaney (eds.), The Frege reader, p. 151-172. Cambridge: Blackwell, 1997.
- [13] Freyd, Peter J., and Andre Scedrov. “Categories, allegories.” Amsterdam: Elsevier, 1990.
- [14] Freyd, Peter. “Properties Invariant Within Equivalence Types of Categories.” In: Alex Heller & Myles Tierney (eds.), Algebra, Topology, and Category Theory: A Collection of Papers in Honor of Samuel Eilenberg, p. 55-61. Amsterdam: Elsevier, 1976.
- [15] Hamkins, Joel David. “THE SET-THEORETIC MULTIVERSE.” The Review of Symbolic Logic 5, no. 3 (2012), p. 416–49.

- [16] Jech, Thomas. "Set Theory: The Third Millennium Edition." Berlin, Heidelberg: Springer, 2003.
- [17] Lawvere, F. William. "The category of categories as a foundation for mathematics." In Proceedings of the Conference on Categorical Algebra: La Jolla 1965, p. 1-20. Berlin, Heidelberg: Springer, 1966.
- [18] Leinster, Tom. "Basic category theory." Cambridge: Cambridge University Press, 2014.
- [19] Leinster, Tom. "Rethinking set theory." The American Mathematical Monthly 121, no. 5 (2014), p. 403-415.
- [20] Mac Lane, Saunders. "Categories for the Working Mathematician." New York: Springer, 1978.
- [21] Marquis, Jean-Pierre. "Categorical foundations of mathematics or how to provide foundations for abstract mathematics." The Review of Symbolic Logic 6, no. 1 (2013), p. 51-75.
- [22] Marquis, Jean-Pierre. "Category theory and the foundations of mathematics: Philosophical excavations." Synthese 103 (1995), p. 421-447.
- [23] Marquis, Jean-Pierre. "From a geometrical point of view: A study of the history and philosophy of category theory." Dordrecht: Springer, 2008.
- [24] Marquis, Jean-Pierre. "Mathematical forms and forms of mathematics: leaving the shores of extensional mathematics." Synthese 190 (2013), p. 2141-2164.

- [25] McLarty, Colin. "Axiomatizing a Category of Categories." Journal of Symbolic Logic 56, no. 4 (1991), p. 1243–60.
- [26] McLarty, Colin. "Exploring categorical structuralism." Philosophia Mathematica 12, no. 1 (2004), p. 37-53.
- [27] McLarty, Colin. "Introduction to Categorical Foundations of Mathematics." [Unpublished manuscript] Department of Philosophy, Case Western Reserve University (version of August 14, 2008). https://artscimedia.case.edu/wp-content/uploads/2013/07/14182624/Roskilde-school-814.pdf
- [28] Ochs, Eduardo. "On the the missing diagrams in Category Theory (first-person version)." arXiv preprint arXiv:2204.10630, 2022.
- [29] Osius, Gerhard. "Categorical Set Theory: A Characterization of the Category of Sets." Journal of Pure and Applied Algebra 4, no. 1 (1974), p. 79–119.
- [30] Rodin, Andrei. "Axiomatic Method and Category Theory." Cham: Springer, 2014.
- [31] Bertrand Russell, "II.—ON DENOTING." Mind, Volume XIV, Issue 4 (1905), p. 479–493
- [32] Russell, Bertrand, and Michael Potter. "Introduction to mathematical philosophy." London: Routledge, 2022.
- [33] Wittgenstein, Ludwig. "Tractatus logico-philosophicus." Přeložil Petr Glombíček. Praha: OIKOYMENH, 2017.