Appendix A

The overflowing mathematics

A.1 ZFC

A.1.1 Basic Language of Set Theory

Our ZFC presentation is given in FOL with a specifically tailored set theory language of the following symbols:

1. unrestrictedly many variables for sets x, y, z
2. binary predicate symbol $=$ for equality
3. binary predicate symbol $\in$ for membership
4. some basic some redundant logical connectives for ease of expression: $\neg P$, $P \ \& \ Q$, $P \vee Q$, $P \rightarrow Q$, $P \iff Q$
5. quantifiers $\forall$ and $\exists$
6. auxiliary symbols for brackets

Among them $\in$ is a special symbol of the set theory language, whereas the others are logical, native to the language of FOL. In all, this is called the basic language of set theory. [4, p.33]

Well-formed formulas (wff) defined in the usual way.

A.1.2 Axioms

1. Ax.1 Axiom of set existence

$$\exists x(x = x)$$

- The ZFC ontology has a universe such that it contains at least one set entity.

2. For classes we chose ZFC instead of NGB, and the difference must be reflected. Any formula $\phi(x)$ can be seen to filter the universe into two possibly distinct areas, one of sets that satisfy it, the other which do not. If all satisfy it $(x = x)$ then we can speak of the whole universe of ZFC. Any such area characterized by the formula ϕ we can call a Class. It is customary to denote class A of x sets as $A = \{x|\phi(x)\}$ but it hides the obvious formula $\forall x(x \in A \iff \phi(x))$.

3. Ax.2 Axiom of set extensionality

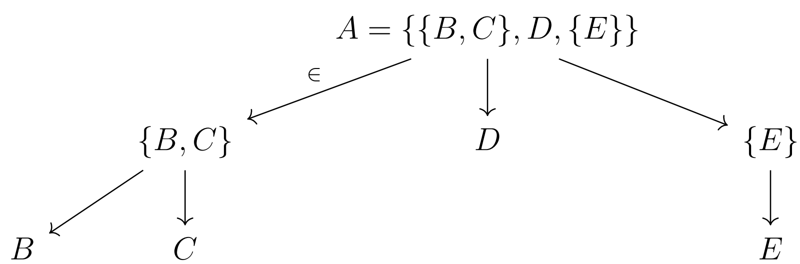
$$\forall u(u \in x \iff u \in y) \to (x = y)$$

where "=" is the classical FOL congruence over $\in$ (reflexivity, symmetry, transitivity and substitutability under $\in$) yields

$$(x = y) \iff \forall u(u \in x \iff u \in y)$$

- It is a definitional axiom establishing the full meaning of "=" for sets, of whom we "knew" already the three properties.

4. Aczel's Non-well-founded set theory A short mention is warranted of an alternative set theory allowing for such loops due to Peter Aczel.[1] There he uses top-down "accessible pointed graphs"(aps) or more classically rooted directed graphs, to depict sets such as:



Arrow representing the target being an element of the source. Such depiction characterizes it adequately w.r.t. membership. Whence over it we can define well-founded set as a set which has no infinite paths or cycles in its graph depiction (non-wellfounded otherwise). His antifoundation axiom (AFA) states that every graph pictures a unique set. AFA is then taken instead of the axiom of foundation in ZFC yielding a non-well-founded "Hyperset Theory" denoted ZFC^-/AFA, which allows for the existence and depiction of non-well-founded sets and consequently contains the ZFC universe V in its own. Aczel then supplies his own "extensionality" via bisimulation: a binary relation between two aps F, G of points f, g is a bisimulation iff

(a) fRg

(b) if nRm then

for every edge $n \to n'$ of F, there exists an edge $m \to m'$ in G s.t. $n'Rm'$ and conversely for every edge $m \to m'$ in G there is an edge $n \to n'$ in F s.t. $n'Rm'$

F and G are "bisimilar" if there is a bisimulation between them. Meaning both apgs picture the same set and a set is completely determined by any graph that pictures it. [2]

5. Ax.3 Axiom schema of separation

$$\forall a \exists z \forall x (x \in z \iff x \in a \wedge \phi(x))$$

-For each $\phi(x)$ formula not containing z as a free variable, this is an axiom.

For any set a there also exists a set z consisting of exactly those elements of a satisfying the formula. Any formula specifies a set relative to some fixed set a of elements of a satisfying it. It is an axiom scheme because it tells us how to construct the axiom when the formula is accessible, and must be given in a scheme since there are infinitely many such formulas.

Separation schema is sometimes called the axiom schema of restricted comprehension in homage to unrestricted comprehension of naive set theory, which has it that

$$\forall a_1...a_n \exists z \forall x(x \in z \iff x \in a \land \phi(x, a_1...a_n))$$

lacking the above variable restriction and leading thereby to Russel's paradox by taking $\phi = (x \notin x)$.

Separation is a way of building sets out of existing sets, namely subsets of existing sets. (set whose elements can only be elements of the superset) Such axioms we call raising.

6. Ax.4 Axiom of Power set

$$\forall x \exists y \forall z(z \in y \iff (\forall w(w \in z \to (w \in x))))$$

- shorthand notation $\forall x \exists y \forall z(z \in y \iff z \subseteq x)$ - For every set x there exists a set y ($P(A)$) whose elements are only the subsets of x: sets z whose elements w are necessarily also elements of x.

7. The problem of the powerset axiom comes around at the limit steps such as instead of continuing with succesor ascension of individual natural numbers, the move to ω_0. (A.1.2 9.) Here the operator becomes hazy, the result of which is undecidability of CH, which says: there are no sizes between ω_0 and ω_1 or also $\omega_1 = P(\omega_0)$ the size of which is 2^ω

   Against CH, there is also a plausibility argument: that CH"... cannot be decided in ZFC because its axioms do not say exactly what makes up a subset of ω (that is the limit step mentioned); hence we cannot relate the size of

P(ω) to an infinite cardinal numbers..." [7, p.1] This limitation of ZFC gives a prompt for Gödel's extension by constructibility axiom into constructible set theory, which attempts to remedy what is seen as a defect of ZFC grounding itself in definability.

Separation doesn't tell us that all the subsets of A exist, but gives a formal condition for them to exist - being definable. Then and only then can the Powerset axiom properly take off. Separation considers the already fixed set a in which it characterizes subsets by the satisfied formulas of its elements. Thus, Ax.4 is raising over Ax.3. The latter can be seen to provide a limit bridge w.r.t. the P rule in the sequential perspective. With the aid of the axiom of infinity which guarantees some infinite I set materially, it is possible to construct the $P(I)$, but separation specifies what this means - yet still it is insufficient to solve CH, because the powerset axiom carries an important constructivity blemish.

It calls back to the role of the chosen formulaic language, as Ax.4 rests on Ax.3 and he in turn on wffs true and false. It is also possible to experiment with sortification of formulas in the TO through some Gödel-style encoding leading to its arithmetization, and grounding the universe in the TO rather than admitting to externality of one kind or another. FOL is here accepted for a background, but FOL has a set-theoretic formalization (semantics), which itself rests on some deeper still logic, speaking of a spiralling hierarchy of background reference. A formal FOL is admitted over a naive "classes theory". But soberly, rank 1 displays the advertised self-reference capabilities of ZFC

– defining FOL, grounding it within itself, which then also gives form to how to go about formalizing formulas inside the ontology - they might be sets of particular kind, encoding the basic language of ZFC we began with. Devlin also includes the addition of Axiom of choice (AC) as it enriches the I set with certain choice sets and well-orderings.

8. Ax.5 Axiom of the Sum

$$\forall a \exists z \forall x (x \in z \iff \exists y ((x \in y) \wedge (y \in a)))$$

- For any set a there exists a set z of all elements of its elements.

This defines the standard operation of union:

first the union set $\bigcup a = \{x | \exists (x \in y \wedge y \in a)\}$

then specifically for a = {b,c} $\bigcup a = \{x | (x \in b \vee x \in c)\}$

hence finally $b \cup c = \{x | (x \in b \vee x \in c)\}$

9. The universe of set theory is denoted V for von Neumann, as well as the visual cue suggestion. We must define ordinal numbers and the proper class of them On from which the union draws in V = $\bigcup_{\alpha \in On} V_\alpha$ within the construction (or inconstructively V=$\{x | x = x\}$ - which carries some baggage: Just as we had problem with $\emptyset$'s definition, now V comes up as $\{x | x = x\}$ and by russel's paradox it's a proper class - by contradiction from separation axiom there exists $z = \{x | x \in V \ \ \& \ \ x \notin x\} \rightarrow z \in z \iff z \notin z$ - contradiction)

- Tarski's definition of finite set - set x is finite Fin(x) if each nonempty subset y $\subseteq$ P(x) has a maximal element w.r.t. inclusion. Over this Fin = $\{x|Fin(x)\}$
- (Theorem 6.8) Principle of induction for finite sets: If x is a class for which the 2 following conditions hold

  1. $\emptyset \in X$

  2. $x \in X \to \forall y(x \cup \{y\} \in X)$

  then Fin $\subseteq$ X
- (Lemma 6.9) Fin(x) $\to$ Fin(P(x))

  -by powerset iteration we cannot get out of the sequence of natural numbers - we need a limit step collecting all particular naturals as we have mentioned w.r.t. CH
- von Neumann's natural numbers (ω_0 or also just ω): the basic idea is a that natural number is a set of all smaller natural numbers:

  0 - {}, 0 = 0, 0 is empty

  1 - {{}} = {0}, 1 = 0 $\cup$ {}, 1 has one element (0)

  2 - {{},{{}}} = $0, 1$, 2 = 1 $\cup${1}, 2 has two elements (0,1)
- Cartesian product $A \times B = \{< a, b > |a \in A \wedge b \in B\}$
- (Lemma 4.11) for any two sets x,y $x \times y$ is also a set
- Inductive set: w is an inductive set if it holds that

  $\emptyset \in w \wedge \forall v \in w(v \cup \{v\} \in w)$

As we will see the axiom of infinity postulates the existence of at least one inductive set, thus the intersection of the class of inductive sets is nonempty and it is a further fact that its intersection must be a set, hence the definition to follow makes sense.

- The set of all natural numbers $\omega = \bigcap\{w|w \text{ is inductive set}\}$
- (Lemma 6.16) ω is the least inductive set
- Successor function: s(n) = $n \cup \{n\}$
- (Theorem 6.18) Principle of induction for natural numbers: If X is a set of natural numbers, for which the two conditions hold:

  1. $\emptyset \in X$
  2. $x \in X \rightarrow s(X) \in X$

  then ω = X
- (Theorem 6.20)

  1. every natural number is finite
  2. ω and any inductive set is infinite

  Ordinal numbers extend metaphorically natural numbers beyond finite sets. They are types of all well-ordered sets, whereas natural numbers are just types of all well-orders of finite sets.
- Transitive sets and classes: class x is transitive iff $x \in X \rightarrow x \subseteq X$

  called transitive for X because it also means $y \in x \in X \rightarrow y \in X$
- well-ordering is a total ordering (reflexive, symmetric, transitive, linearly connected) s.t. every nonempty subset has a least element.

as have been some of the definitions already, it is quantifying over subsets, and hence cannot be first-order

- Set x is an ordinal number iff x is transitive and $\in$ is a strict well-ordering on X (strict meaning irreflexive rather than reflexive, asymmetric rather than symmetric) and On $= \{x|x$ is an ordinal number$\}$
- (chapter 2 Lemma 1.6) On is a transitive class
- (Lemma 1.8) $\in$ is well-ordering on On
- (Lemma 1.9) On is not a set
- (Corollary 1.13) Ordinal ω is the supremum of the set of all natural numbers in the class On. (meaning ω is the least ordinal number and finite ordinals are just natural numbers)
- (Lemma 1.14) If α is ordinal, then $\alpha \cup \{\alpha\}$ is the least ordinal $\in$-higher than α ($\alpha \cup \{\alpha\}$ is called the successor of α and α is the predecessor of $\alpha \cup \{\alpha\}$ $\in$-wise
- Ordinal number α is isolated if it has no predecessor

  Ordinal α is limit ordinal iff it is nonzero and doesn't have a predecessor

  -each natural number is isolated, but ω is the first limit ordinal

  Now we can finally sketch the cumulative hierarchy construction
- The Powerset construction of the universe:

  $V_0 = \emptyset$

  $V_{\alpha+1} = P(V_\alpha)$

$V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$ where λ is a limit ordinal

And finally: WF $= \bigcup\{V_\alpha | \alpha \in On\}$

Now

- (Lemma 6.16) for each ordinal α it holds that

  1. V_α is a transitive set
  2. $V_\beta \subseteq V_\alpha$ for each $\beta < \alpha$

  The consequence of which is that the V sets make up the cumulative hieararchy and WF is a transitive class.

  With this we can invoke the set universe V.

- (Theorem 6.32) The following are equivalent over ZFC without foundation axiom:

  1. Foundation axiom
  2. V = WF
  3. $\forall x \exists \alpha (x \in V_\alpha)$

  Meaning we have to involve the Foundation axiom in order for WF to make up the entire universe of set theory and we can only regard the unrestricted P operation as capable of generating the ZFC universe under this axiom directly. It provides a global characterization of the universe produced and leads all sets to a proper foundation in the empty set $\emptyset$.

10. Ax.6 Axiom of Foundation

$$\forall a((a \neq \emptyset) \to \exists x(x \in a \wedge x \cap a = \emptyset) \text{ where } x \cap a = \{y | y \in x \wedge y \in a\}$$

- For every set a, if a is not empty, then it has some element x and no element of this x is an element of a

The nonempty sets a have to have as their constituents at least something that does not exist directly on the level of elements of a, that is non-self-referential directly within this one step of elementhood w.r.t. a. This is a powerful global characterization preventing the existence of certain unsavoury sets - namely sets that loop in any way w.r.t. $\in$, so self-founding $x \in x$ or co-self-founding $x \in y \in x$, but even better - it prevents all infinitely descending chains on $\in$, wherefore also the name "foundation".

This has a neat proof. [5] Assume contradiction both the axiom and that we can have an infinitely descending chain - A $= \{a_n | n \in \omega\}$ s.t. $a_{n+1} \in a_n$ so naturals are indexing the chain with 0 for the top going down $...a_2 \in a_1 \in a_0$ Now if x $\in A$, then for some n $\in \omega$, $a_n = x \in A$ and thus there must also be $a_{n+1} \in a_n \wedge a_{n+1} \in A$. A is nonempty so it must satisfy the consequent of the axiom's implication, but any x being a_n we get $A \cap a_n = a_{n+1} \neq \emptyset$ - contradiction. And we can encode any loop this way if we can make a set object over it using the union axiom (for $x \in y \in z \in x$ we just take $z \cup y \cup x$ labelled arbitrarily, and construct A naturally by their iteration and successor indexing).

11. For most set theorists, this is a familiar idea, so we only discuss it briefly here.

There are two views of a model, external and internal - internal is how the models sees itself, and external is how the model appears from some other perspective - from another model or abstract models, or an absolute view. (this we will specify as we explore the question) The most prominent difference of the two views is in well-foundedness. Every model of ZFC is internally well-founded - it sees every set constituting it as well-ordered by $\in$, the classes, and even On. It is due to the axiom of foundation. It is integral to ZFC that it see itself only as $\in$-wellfounded. But there could be models (M,E) $\models$ ZFC where E is not the expected $\in$ relation. There can be ω-models that are nonstandard retaining the expected natural numbers or models having nonstandard natural numbers be responsible for the infinite descending chains of E. (meaning there must be a nonstandard number n which is not a numeral 0 1, 2... but can be subtracted from, yielding an infinite descending chain) Now by Mostowski's collapse lemma we know that every well-founded model is isomorphic to a standard model of set theory. [8]

To introduce it however, we will again have to reach for a few ingredients taken directly from [16]

- We generalize the notion of well-founded relations to relations on proper classes. With it we extend the method of induction to well-founded relations.
- Extension: Let E be a binary relation on a class P. For each x$\in$ P, we let $ext_E(x) = \{z \in P | zEx\}$ be the extension of x.
- Well-founded relation generalized: A relation E on P is well-founded, if:

  1. every nonempty set $x \subset P$ has an E-minimal element;

2. $ext_E(x)$ is a set, for every $x \in P$

(Condition 2. is vacuous if P is a set.) Note that the relation $\in$ is well-founded on any class ,by the Axiom of Regularity.

- (Lemma 6.9): If E is a well-founded relation on P, then every nonempty class $C \subset P$ has an E-minimal element.

- (Theorem 6.10) Well-Founded Induction: Let E be a well-founded relation on P. Let ϕ be a property. Assume that:

  1. every E-minimal element x has property ϕ

  2. if $x \in P$ and if $\phi(z)$ holds for every z s.t. zEx, then phi (x).

  Then every $x \in P$ has property ϕ

- Extensional well-founded relation: A well-founded relation E on a class P is extensional if $ext_E(X) \neq ext_E(Y)$ whenever X and Y are distinct elements of P.

- A class M is extensional if the relation $\in$ on M is extensional, i.e., if for any distinct X and Y$\in M$, $X \cap M \neq Y \cap M$.

  Mostowski's theorem shows that the transitive collapse of an extensional well-founded relation is one-to-one, and that every extensional class is $\in$-isomorphic to a transitive class.

- (Theorem 6.15, p.69) Mostovski's collapsing Theorem:

  1. If E is a well-founded and extensional relation on a class P, then there is a transitive class M and an isomorphism π between (P,E) and (M,$\in$)

  The transitive class M and the isomorphism π are unique.

2. In particular, every extensional class P is isomorphic to a transitive class M.

The transitive class M and the isomorphism π are unique.

3. In case 2. if T⊂P is transitive, then $\forall x \in T(\pi x = x)$

It provides an easy routine for restating any ZFC sentence as an isomorphism-invariant statement about well-founded extensional relational systems, which is in ZFC provably equivalent to the original.

- Corollary: every well-founded model is isomorphic to a standard model. [8]

12. How Mirage principle influences the notion of standard model.

Internally every model must by necessity see itself as a standard model of ZFC insofar as its binary E is behaving as expected in well-foundedness. Thus differentiating standard from nonstandard happens from a fixed TO, which might but might not coincide - if it is said to be nonstandard, then it certainly cannot be the universe we are addressing reflexively speaking of itself precisely because internally it sees itself as ∈-conforming and so must see itself as a well-founded model even externally as the two levels collapse. By the Mirage principle, any TO we are considering is seen by some TO to be nonstandard when the distinction is first made. By Mostowski all the standard models are mutually isomorphic and as we will see the right notion of identity for sets being isomorphism it is not much of a stretch to consider isomorphic models identical either. But there still remain the ill-founded counterpart model subclass and the necessary choice of a privileged model, forming the dichotomy. And seen from the multiverse view, there is no reasonable stopping point to fend off the well-foundedness mirage.

13. Ax.7 Pairing axiom

$$\forall a \forall b \exists z \forall x (x \in z \iff (x = a \lor x = b))$$

- For any two sets a, b there exists a set z whose every element is either a or b itself.

It has exactly those two elements - two except if b=a, yielding only $\{a\}$. By extensionality, it is clearly unique, but it is unique also insofar as all we can say internally of the set $\{a, b\}$ is said by it, because both a and b are themselves already fixed - and yet there is an addendum to even this: if the fixed strata include the axiom of P in the application as an operation, openness might be introduced as happens in the CH case - it is there w.r.t. the multiverse and abstraction of features of its models. Just by the P iteration alone, we cannot therefore determine which models we speak of, meaning which of them is the TO-produced ontology.

We must then be overlooking it from a high-enough level to ignore the determining structured CoP, which fixes the openness in question completely. The games we play we play over the multiverse of models and models properly construed - this must come first. It is played by zoning in on an area while moving between the levels fluidly and without concern for safety as well as looking for the high-level consequences of the low-level brute identity criteria over the CoP, even admitting the conceivability of only partially fixing the CoP in asking the CH.

14. Ax.8 Axiom of infinity

$$\exists z(\emptyset \in z \wedge \forall x(x \in z \rightarrow (x \cup \{x\} \in z)))$$

-There exists an inductive set z, whose element is $\emptyset$ and for any other of its elements x so is $x \cup \{x\}$ (the successor)

If $x \notin x$ then $x \cup \{x\}$ is different from x, making it a stepup and a progression, justifying the successor naming. For each x it has to bring in another element reached by a precise construction method, but apart from the $\emptyset$ linear progression we don't know any other set therein present (such as can be the nonstandard number not reachable by any finite iteration of successor on $\emptyset$.) It materially establishes this new set by its lone existential quantification, but also raises the sets within itself over $\emptyset$ whom it shares or establishes by itself, as well as any other nonstandard element that finds its way into it through the structured CoP. It does not specify the operation by which it is sequentially constructed as a set, apart from having to involve at some state a union of its constituents, as ω hyperlink17. rdoes.

15. Ax.9 Axiom scheme of replacement

First we must recall two elementary definitions:

- An n-ary relation R is a class $R \subseteq V \times V \times .... \times V$ V taken n-times.
- Relation F is a function iff $\forall u \forall v \forall w((u, v) \in F \wedge (u, w) \in F) \rightarrow (v = w)$

Now finally Ax.9:

$$\forall u \forall v \forall w((\phi(u, v) \wedge \phi(u, w)) \rightarrow v = w) \rightarrow \forall a \exists z \forall v(v \in z \iff \exists u(u \in a \wedge \phi(u, v))$$

- For any definable function ϕ (which can be a proper class), the image of any set a is also a set

For once we will lift a motivation also: class being a set depends only on the cardinality of the class, not on the rank of its elements

- A cardinal number κ is an ordinal s.t. $\forall \beta < \alpha$ where β is an ordinal, α cannot be injected into β via any definable function.
- Rank of a set a is the least α s.t. $a \subseteq V_\alpha$ - the least stratum of V containing a - its index

  Jech even speaks of the purpose of the foundation axiom as enabling the definition of rank. [16]

So if any class is small enough cardinality-wise to be a set (we care about cardinals rather than ordinals in determining it) and there exists a surjection from this set (this is guaranteed by the consequent of the implication) then the ϕ-images of the set form a set also.

Wherefore it really says: If a class a is small enough to be a set and there exists (is definable) a surjective function from it to another class b (so that we can cover the entire class b with just elements of a), then b is itself also a set.

16. Ax.10 Axiom of choice (AC)

First we need to define Selector:

- Selector on a set X is a function f defined on the set X of nonempty sets s.t.

  $\forall A((A \in X) \to f(A) \in A)$

  - Selector on X picks one element out of each nonempty set X (that's the image of f)).

Ax. 1O

$$\forall X(\emptyset \notin X \to \exists f : X \to \bigcup_{A \in X} (\forall A \in X f(A) \in A)$$

- On any set X of nonempty sets A, there exists a selector.

- For any set X of nonempty sets A, there exists some parallel choice of a representative element for each A.

Due to its clarity, we will use on the side of sets an equivalent statement using cartesian product.

- (Lemma 7.6) AC' - the cartesian product $X(a_i|i \in X)$ - nonempty family of nonempty sets, is itself nonempty.

AC' is equivalent to AC

So we can speak of them interchangeably. But why choose this over AC? We prefer cartesian product to functions, because functions speak of the choice made by AC only indirectly within its constitutive pairs - as the images. Here, however, we can see the choice immediately in the guaranteed nonempty cartesian product.

Additionally

- (Theorem 7.23) AC is equivalent to maximality principle (whom we will happily ignore) and Well-ordering principle.

The WO principle states that every set can be well-ordered. SO AC effectively provides also for each set A a relation R that is a well-order on A. This again invokes an earlier point that the feature of a given presentation of an axiom is only essential so far as it is presentation invariant. Whence the classification suffers certain isolated usefulness.

A.2 CCAF

A.2.1 EML axioms

Usually this is done adopting the Eilenberg-MacLane category axioms over a particular 2-sorted FOL language displayed in Fig 1 below. (All 5 figures are taken directly from McLarty's [27])

Types: $A, B, C \ldots$ for objects, and $f, g, h \ldots$ for arrows.

Operators:

Dom takes arrows to objects, read "domain of."
Cod takes arrows to objects, read "codomain of."
$1\_$ takes objects to arrows, read "identity arrow of."

Relation: $C(x, y; z)$ applies to arrows, read "z is the composite of x and y."

Axioms:

Domain and codomain: $\forall\ f, g, h,$ if $C(f, g; h)$ then

$$\mathrm{Dom}\, f = \mathrm{Dom}\, h \quad \text{and} \quad \mathrm{Cod}\, f = \mathrm{Dom}\, g \quad \text{and} \quad \mathrm{Cod}\, g = \mathrm{Cod}\, h.$$

Existence and uniqueness of composites:

$$\forall\ f, g, \quad \text{if } \mathrm{Cod}\, f = \mathrm{Dom}\, g \text{ then } \exists!\ h \text{ such that } C(f, g; h).$$

Identity arrows: $\forall\ A,\ \mathrm{Dom}\, 1_A = \mathrm{Cod}\, 1_A = A$. And

$$\forall\ f,\ C(1_{(\mathrm{Dom}\, f)}, f; f) \text{ and } C(f, 1_{(\mathrm{Cod}\, f)}; f).$$

Associativity of composition:

$$\forall\ f, g, h, i, j, k,\ \text{if } C(f, g; i) \text{ and } C(g, h; j) \text{ and } C(f, j; k) \text{ then } C(i, h; k).$$

Figure A.1: the abstract category axioms

There are two types of variables in the language, originally [20] for objects and arrows, within CCAF for categories and functors. Thus instead of structurally enforcing the makeup of a category w.r.t. its constituents and their properties being primitives of the language, instead of such bottom-up setting where categories are seen as constructs of these formally primitive notions, CCAF levels the ground. We begin with categories and end up with categories. Objects and morphisms have been transposed to the 1st level immediately but will be retrieved through finite categorical forms **1, 2, 3**.

The prefaced description of a category serves as a guidance for intuition, but the names category and functor must be subjected to only the requirements of the official theory. Thus, simply any entities and operations satisfying these axioms form a Category. Notice the circularity. We say our variables are of two species "categories" and "functors" and all variables comporting to the category axioms form a "category" proper. Yet we call the variables of one sort categories as we had done for sets before and so must make sure that they do in fact agree. Thus, we ought to have Category made of categories and functors and Functors then made of Categories in turn. But where do we begin? If on categories of Categories, then even each variable of the category type must be so behaved as to formally meet these axiomatic criteria internally. Every basic entity of the system which is a category is structurally bound to have an internal structure on its own level - of Categories and Functors. Functors are typically [18] [20] defined once we are finished with Categories, which cannot be the case here. It is both before and after Categories.

In the same spirit and despite the inclination to think of natural transformations and objects and morphisms separately, they must be special cases of Categories and Functors as they are the elementary concepts. We shouldn't speak of functors prior to integrating the 4 axioms for **1, 2, 3**, but it is necessary in order to do just that.

A.2.2 The diagrammatic behaviour of 1,2,3

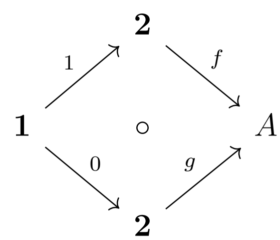
The definition of object o:**1**→ A as a functor is based on the unique translatability of objects of A with functors from any 1-object category targetting them - such functor selects precisely one object of its target category and disregards, due to its complete unity, the source category.

The correspondence is found analogous for morphisms, only requiring the **2** category to itself host two distinct objects met by an arrow. This captures the second axiom stating **2** has exactly 3 automorphisms - identity $1_{\mathbf{2}}$ carrying its image. The other two compose the guaranteed functor from **2** to **1** with one of the two functors from **1** to **2** comforting us in our expectation - 0 and 1 must be there to signify **2** has precisely 2 objects and the composition 1_0 together with 1_1 captures the order of **2** w.r.t. source (the former) and target (the latter).

Thirdly we have the category **3** being the form of commutative triangles of morphisms as well as the basic complex building block of interesting categorical properties. If f, g have the same source and target, then commutativity means they must equal, it is denoted by $\circ$ within our diagrams. Triangles, squares and so on just take f and g to be composite morphisms.

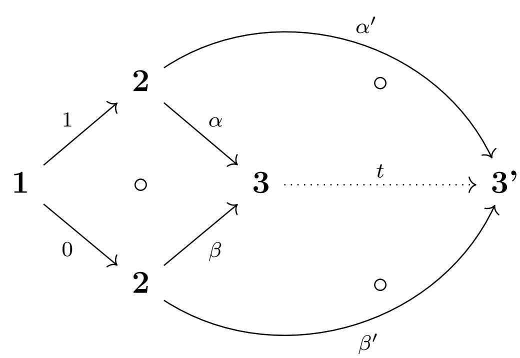
The first two axioms equip us with just enough expressivity to address composition of arrows in any category A: two arrows f, g composing into a single arrow

"$g \circ f$" is captured by the diagram

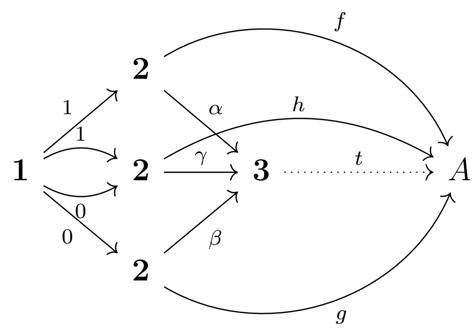


which can be read - the target of f is the source of g in A.

Thus there should be exactly one commutative triangle in A with f after g and this is exactly what **3** is used for - Ax.3 here states **3** is a pushout, that means if there was any **3'** s.t. A mirrors the structural position of **3**, then there must also exist a unique morphism t:**3** $\rightarrow$ **3'** producing the two new commutative triangles saying: if in any category **3'** there are two composable morphisms α and β then there must be a commutative triangle in **3'** given by the functor t, which is formed by composing the two aforementioned morphisms. Giving us something like this:

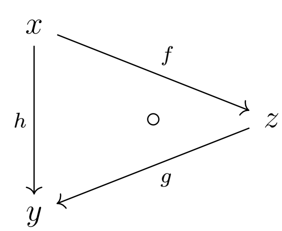


3 will also have to satisfy the same if **3'** is **3** itself, imprinting unto itself its own form externally. The third axiom however hides a somewhat more complex diagram (the pictures of category theory have a formal backing - diagram of shape S in a category A is a functor D from the indexing category S to A, whose source patterns the focused area of A - it is a tool of illustration allowing us to fix the indexing set S and vary the diagram functor D in its targets)



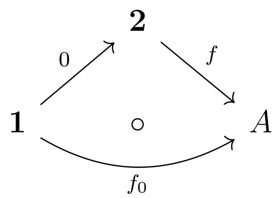
where we avoid cluttering notation of commutativity ∘.

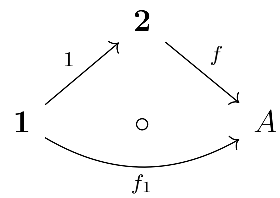
with



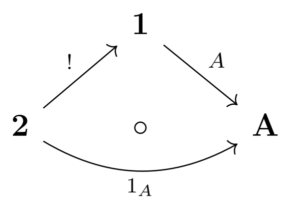
inside A, where $h = g \circ f :=$ gf lets us define the composite of morphisms on the level of categories. Within the diagram we capture γ to be of the same source as α and target as β. The target of f is the source of g and t fills the missing composite g ∘ f. That is then the condition of composability taken by category axioms to actual composites, all captured within this beautiful diagram as a functor from **3**. Allowing us finally to speak of functors in satisfaction of our motivation by way of commuting diagrams:

For morphisms taken as functors from **2**, we can diagrammatically encode the source, target and identity as:

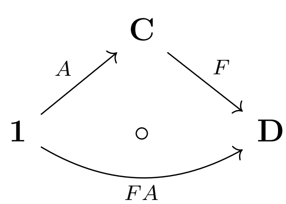


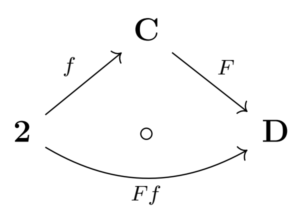


f_0 is the source object of f in A and f_1 its target - so if f is taken to be a functor F, F_0 is its source category. Similarly, given an object A of **A** we can speak of its identity morphism using the uniquely defined functor ! to **1** like this:

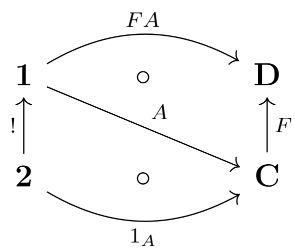


And now for functors specially: each functor F: **C**→**D** takes objects and arrows of **C** to the same in **D** just by functor composition





And by associativity of functor composition, F preserves domains codomains and identity arrows. So for the $F(1_A) = 1_{F(A)}$ and F$(g \circ f) = F(g) \circ F(f)$ conditions, we can just go through associativity diagrams like:

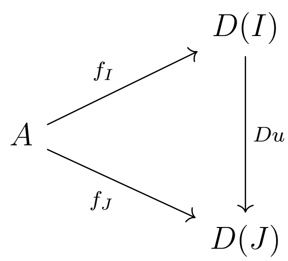


and those analogous to it.

A.2.3 The construction axioms

The product coproduct axiom illustrates the universal property of product $A \times B$ with projections π_1 and π_2 and coproduct A + B with injections i_1 and i_2 - they are mutually dual definitions, meaning one is reached from the other by inverting the arrows (alternative axiomatizations [25] formally introduce operator ^{op} which takes a category A and yields its dual A^{op}, so that whenever A has a product A^{op} has a coproduct) Coproduct can be read as a disjoint union and is the same kind of conic phenomenon we have explained of **3** Ax. guaranteeing it to be a pushout - it is covered by a general notion of limit and colimit.

- A cone on diagram D is an object A of **A** (the vertex of the cone) together with a family $(f_I : A \to D(I))_{Iin\mathbf{I}}$ of arrows in **A** s.t. for all arrows $u : I \to J$ in **I** the following triangle commutes.



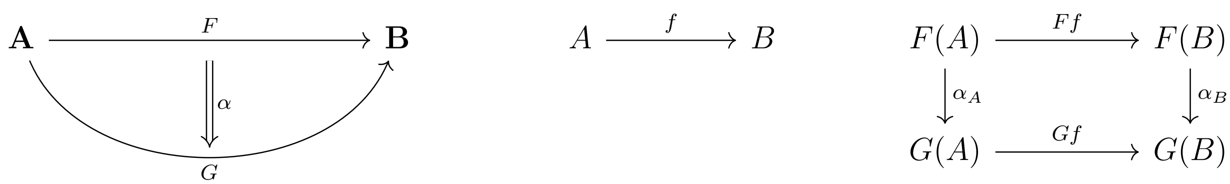
- A limit of D is a cone $(p_I : L \to D(I))_{Iin\mathbf{I}}$ with the property that for any cone on D, there exists a unique map f':$A \to L$ s.t. $p_I \circ f' = f_I$ for all I in **I**. The maps p_I are called projections of the limit [18, p.118]

The universal property is then analogous - consider A+B and a category T structurally mirroring it by having injection-like functors F and G for A and B respectively, then there must exist a unique functor (F,G) from A+B $\to$ T meaning any such category T factors through A+B. Dually for the product.

For equalizer and coequalizer again as particular instances of limit and colimit, the universal property is shown on T where u is the unique induced functor respectively object because these structures are often seen inside a category. Coequalizer which interest us primarily can be seen as a quotient by an equivalence relation, such as occurs for example in the famous Lindeubaum-Tarski algebra used for algebraization of classic FOL.

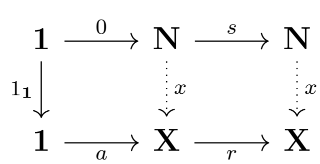
Third axiom here has been used to justify the earlier definition using functor categories - the scaling up of the categorical universe allows for the formation of categories whose objects are functors and morphisms natural transformations - from a top-downperspective it is a morphism connecting two functors, which manifests itself in their common target category B. Bottomup perspective is illustrated here:

$\alpha : F \to G$ $\quad$ $Inside\mathbf{A}:$ $\quad$ $Inside\mathbf{B}:$ $\quad$ commuting



for any morphism f:A→B of **A** we get a commuting square in **B** sequentially connected in an expanding series of composed commuting diagrams. Functors and natural transformations again satisfy the axioms of EML forming a category - here we guarantee that any two categories **A, B** induce a, possibly empty, category of functors connecting them - $\mathbf{B}^{\mathbf{A}}$. Just as was the case for ZFC the produced entities tend to collapse under other axioms if not alone as happens when two categories are perfectly disconnected. Perfect disconnectedness is incidentally the defining feature of discrete categories whose only morphisms are identities. These will play the role of sets soon enough.

Forth is the natural number category advertised. Perhaps more clarificatory diagram is this slight alternative:



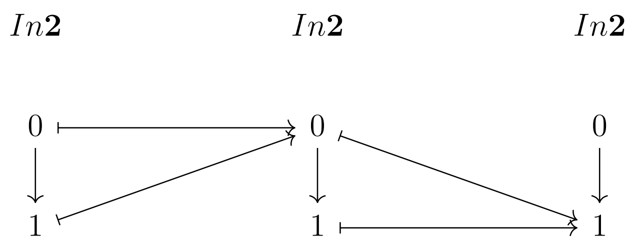
with each square commuting.

It is an abstraction of the ordinary notion of a sequence defined recursively - given a set X, $a \in X$ r:X$\rightarrow$ X $\exists!(x_n)_{n=0}^{\infty}$ sequence in X s.t. $\forall n \in N$ $x_0 = a$, $x_{n+1} = r(x_1)$ So sequence in x is just a function $\mathbf{N} \rightarrow X$, hence we can define sequences in N and N has the bottom element 0 together with the succesor function s:$N \rightarrow N$ given by s(n)=n+1 (or in ZFC $s(A) = A \cup \{A\}$) [19]. The diagram can be read in light of this categorically as: given category X, a an object of X, r an automorphic functor X$\rightarrow$X, there exists a unique sequence x:$N \rightarrow X$ s.t. its initial point is a (x(0) = a) and x(s(n)) = r(x(n)) (the topdown direction = the downbottom direction in the right square). Natural number category is also abreviated NNO for natural number object.

The choice axiom is a paraphrase of the standard categorical equivalent of AC which says every surjective functor F has a section s:$D \rightarrow D'$ meaning $F \circ S = 1_D$

A.2.4 2,3 in the target ontology

Considering the same for the constitution of **2** with instead the 3 automorphic functors $1_{\mathbf{2}}$ 1_0 and 1_1, we must ask if on one hand these satisfy our 4 category axioms, on the other if 1_0 can encode one of the objects of **2**, 1_1 the second, and $1_{\mathbf{2}}$ the morphism beginning with either, ending with another? The autos compose by default, and we see Ax1 is done away with trivially, 3 also, but 2 and 4 need a little tinkering. For 2. consider either composition $1_0 \circ 1_1$ or $1_1 \circ 1_0$. One way to encode 1_0 as the **2**-morphism's source is to collapse all constituents of **2** into the 0 object - where we take already there to be two functors 0 and 1 from **1** as its only objects. Dually because $0^{op} = 1$ and $1^{op} = 0$ [25] 1_1 collapses 0 and 1 to 1 and the morphism $0 \rightarrow 1$ to 1_1 identity morphism on 1 - that is the $1_{\mathbf{2}}$ identity on **2**. Hence we really have 0 given by 1_0, 1 by 1_1 and $0 \to 1$ as $1_{\mathbf{2}}$. In picture:



$$1_1 \circ 1_0 = 1_1$$

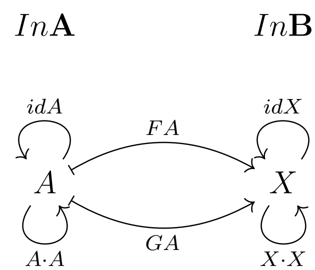
and similarly $1_0 \circ 1_1 = 1_0$ according to this interpretation and the cases of $1_{\mathbf{2}}$ are covered in the Ax.3. Thus, Ax.2 holds. And considering the few combinations, a little calculation shows Ax.4 holds also. Still this works with objects, so we might have to use the virtual world it is modelling as our defining matrix for the behaviour of these functors - according to it will the functors compose and satisfy the axioms, making **2** the desired category of 3 automorphisms. **3** would then take just the same approach.

A.2.5 Arrow extensionality

It simply says **2** is a separator - If two formal parallel functors F, G from A to B are brute different in the ontology, then there must exist a morphism f in A given by the morphism functor $\mathbf{2} \to A$ on whom them differ($F \circ f \neq G \circ f$) - the axiom simply guardblocks these two preobjects from becoming the same produced functor. The presence of composition is readily apparent here, whence its relevance to our questioning.

Functors are not differentiated on how they treat the objects, but the arrows.

Consider the following example



Two categories, **A, B** of one object each (for illustration purposes only) of two automorphisms each. Two parallel functors F,G:**A** → **B** F which sends id to id and $A \cdot A$ to $X \cdot X$ whereas G sends id to id but collapses twice A to id also. Such functors are brute different, and still act on objects the same. To separate them, the arrow extensionality kicks in - F and G must act differently on some morphism of **A**, here on the functor $A \cdot A$:**2** → **A** - the functor composites capture this inequality.

A.2.6 Technical reconsideration of the encountered axioms

Taken in over the NNO Ax.1 among the **1,2,3** axioms is raising in the size by one functor for each category, making **1** terminal as motivation dictates because **1** having just one constituent object and one morphism, there really is a single possibility of how any category might be mapped onto it - by complete forgetting collapse. This establishes the triangle providing representation for identity morphisms of objects that we saw earlier. Any non-empty category would then be bound in this unique way for each of its object in going from and to **1**.

The **2** axiom is material and limiting. It provides **2** as it is externally, but only onto itself, not as it acts on others in the global externality - that is instead given by

Ax.4, which establishes its behaviour as a separator. Note that in ZFC this external aspect was missing from our presentation, yet ETCS corrects for it by maintaining global elements as $\mathbf{1} \to A$ function from its terminal object.

The second and third axioms then over them fund the behaviour of the composition sort on its projection as well as the projection of its participant morphisms and objects. Each of the three has a special universal property. **1** is a terminal category. **2** is a separator. **3** a pushout. It is again a limiting axiom which has much more of an impact in constraining the existents of the particular form, whilst guaranteeing that the composition conditions enforce an archetypal instantiation of the composition form.

The construction axioms

Both product and equalizer pair axioms are plain raising of a structure, but through the universal property also immediately limiting the categories that can thusly be produced to one of each species per each pair, because just as before any structural mirroring brings about isomorphism and then by identity criterion on categories, collapse. Meaning, we are guaranteeing a collapse in a way less explicit formally, more ontologically, than was the case for composition's own uniqueness and existence axiom. To disambiguate, composition itself permits structural mirroring by having a morphism parallel to the composite, which needn't collapse, but simply shares the source and target, while acting differently on some morphism as dictated by arrow extensionality. In the cases of the universal properties, the acting distinction would not do as a local differentiation feature - it is by these axioms ignored in favour of the unification feature of structural mirroring. The third axiom too is a raising one, but without any constraint to go along. There could be different functor categories of the inbetween of **A, B** categories, since there could just be different natural transformations among them which are standing for morphisms in the category.

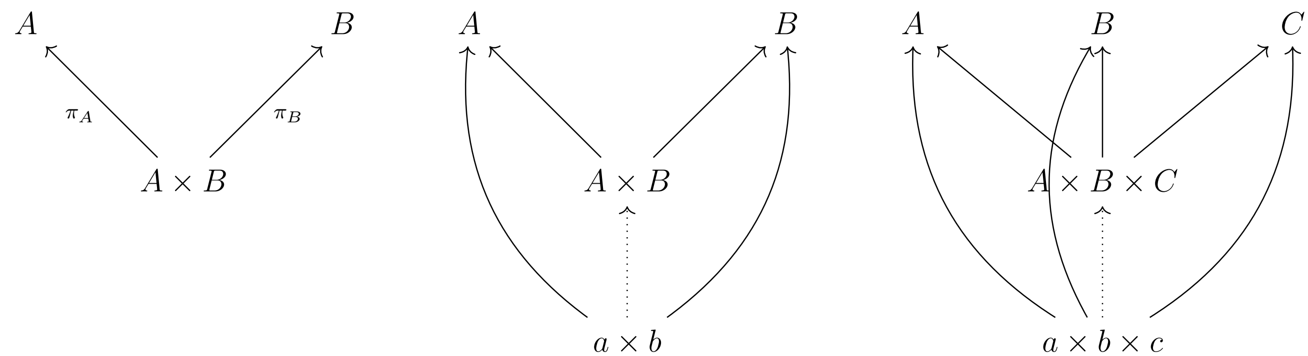
The natural numbers object is another of Lawveree's inventions. It is built within the simulated ZFC by ETCS+R axioms and serves the role of axiom of infinity, so well in fact that in the category of sets Set whose existence we will axiomatically guarantee, the NNO is the set of natural numbers with 0 as the base point and successor morphism s.

It serves also the same role in kickstarting the transfinite. The NNO as defined is bound and confined within the Set category we will characterize. Once more has it a universal property guaranteeing uniqueness of the N,O,s triple. Any two NNOs are isomorphic and hence identical.

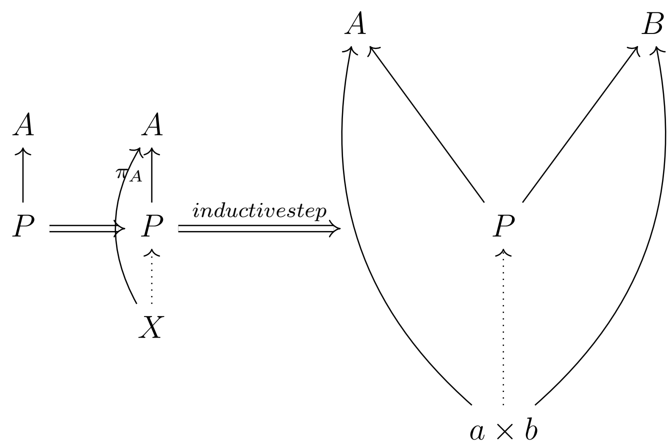
Axiom of choice is also taken straight from ETCS. It is explicitly confined to discretes rather than, as was the case for ZFC, being applied universally. It must hence be a raising axiom guaranteeing the existence of a choice functor.

A.2.7 Compex forms

There is the notion of inductive patterns: Consider product of A, B, objects in **C** category. Binary product alone is just the category together with the two projections, which must then satisfy the defining characteristic property manifest in the presence of a structurally mirroring object. Now generalize to A, B, C like this



It is still a product but no longer binary, as it satisfies the abstract form of the universal property - anything structurally mirroring it has a unique morphism which composes into commuting triangles. A limit step of this progression would move us to the definition of a general limit, suffering the same lack of unique correlate externalizing form. Categories of those shapes are forms if instantiated elsewhere, but their mutual connection is ontologically lacking, despite being of the same pattern. This might be helped by taking a base case and employing some inductive principle as a means of extending the category whilst being outside the pattern's closure (in the end we might get something like transfinite closure on the induction). Base case for product would be:

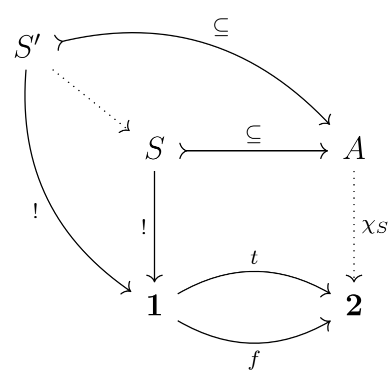


But the problem is with the formulation in terms of a universal property. The X is not actually present, it is a structural enforcement condition. Limit is P with its projection whether or not X is there, so the inductive step has to extend that way also. Thus, if we were able to formulate the inductive principle adequately in terms of the applied combined basic forms as well as the sequentially simpler ones, we could collect a category of these categories and functors - the base case, the constituent basic form, and the induction principle externalized (sounds a little like NNO already). The induction can culminate in an s-like functor, with the function of adding these structural steps, extending the diagram. Which here is not 1-dimensional but 2, so we might have to have variants on the induction s_1, s_2 all part of this highest category constituted by them - if the axioms permit it. If not we might want to adjust them to make this a proper form, s.t. any instance of Limit will be accounted for in this category, but clearly it would be far too complex to permit arrow injective functor into category containing just A, B and their product, so we would have to expand the notion of instantiation accordingly. Needful to say ZFC is empty of any of this, ETCS+R can simulate via functions between sets, however.

Peter Freyd [14] formalizes the language of diagrams, and in doing so actually develops something principally analogous to the idea of inductive patterns - he takes graphs with commutativity conditions, ordered by extension to make a tree with a root of the simplest form extending upwards. In the book [13] he moves to categorical representation proper. So it really is not such a wild idea, and we will consider it within a follow-up paper.

A.2.8 Subset classifier

The subset classifier is captured in the following pullback diagram:



Here, **2** contains two elements, true t and false f, composing into a characteristic function χ which says for every "subset" S of A seen as such by the injection (monomorphic arrows annotated $\subseteq$), which elements of A belong to S by marking them with t and which don't, marked by f.

It is a presentation borrowed from [19] varying from the equalizer-based one here, used because it expresses the same thing more canonically.

Categorical separation scheme: Let $\Psi(x)$ be any formula in the language of CCAF with sole free variable x of functor type. Then the universally quantified conditional

$$\forall \mathbf{A}\ [(\alpha \wedge \beta) \Rightarrow (\gamma \wedge \delta)]$$

is an axiom, for these clauses:

α**:** $(\forall f\!:\!\mathbf{2}\!\rightarrow\!\mathbf{A})\ [\Psi(f) \Rightarrow (\Psi(f_0) \wedge \Psi(f_1))]$

β**:** $(\forall f, g\!:\!\mathbf{2}\!\rightarrow\!\mathbf{A})\ [(f_1 = g_0 \wedge \Psi(f) \wedge \Psi(g)) \Rightarrow \Psi(gf)]$

γ**:** $(\exists \mathbf{B},\ \exists i\!:\!\mathbf{B}\!\rightarrowtail\!\mathbf{A},\ \forall f\!:\!\mathbf{2}\!\rightarrow\!\mathbf{A})\ [\Psi(f) \Leftrightarrow \exists h\!:\!\mathbf{2}\!\rightarrow\!\mathbf{B}\ f = i(h)]$

δ**:** $(\forall i\!:\!\mathbf{B}\!\rightarrowtail\!\mathbf{A},\ i'\!:\!\mathbf{B}'\!\rightarrowtail\!\mathbf{A})$ If i and i' both meet the condition in clause γ then $\exists k\!:\!\mathbf{B}\!\rightarrow\!\mathbf{B}'\ (i = i'k)$

By symmetry of i and i' in δ, it follows that k is an isomorphism of $\mathbf{B}$ and $\mathbf{B}'$.

Figure A.2: The CCAF separation axiom scheme

A.2.9 Separation

The clauses say:

- α: For each arrow of A, if the ψ property holds of it (it being a functor-arrow), then ψ holds of the source and target objects as well (it is a closure of property on the entire scope of the morphism).

- β: For any two composable arrows of A, f, g, ψ holds on both, it also holds of the composite $g \circ f$(closure of the property on composition)

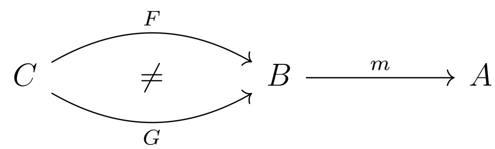
  If both clauses are in effect, then

- γ: There exists a subcategory B and its monomorphism functor is $B \rightarrow A$ s.t. all the morphisms of A satisfying psi are contained in B (taken to B by i)

  and

- δ: Any two subcategories B, B' of A satisfying γ enforce the existence of a commuting functor k from one to the other (through symmetry from the other to the one, making it an isomorphism).

Subcategory is typically determined precisely by a monomorphic (monic) functor m:$B \to A$ s.t.



two parallel functors to B which are brute different, must induce a basic-different composition with m - $m \circ F \neq m \circ G$, which in Set amounts to injective function.

-With this, the whole axiom scheme says: for each unary ψ relation in the language according to the virtual modelled world, for any category A if it maintains a closure of ψ on the vertices of a morphism and composition, then there exists a unique subcategory of A, whose every constituent satisfies the formula ψ. Hence, the name.

As McLarty explains [27, p. 55] the "...scheme says that a predicate $\psi(x)$ on objects and arrows that intuitively ought to define a subcategory i:$B \to A$ does. In ETCS a subcategory of A can be defined as any functor i:$B \to A$ that is one-to-one on arrows..." "It implies that any relation $\psi(x, y)$ of arrows in a category A to those of a category B, which intuitively ought to define a functor A$\to$ B does. And so coequalizers of categories have the properties they intuitively ought to." - "Using the natural number category, every finite sequence of arrows in A which should patch together into a path of composable arrows in the coequalizer Q does, and the composites of these arrows form a subcategory of Q which, by separation, is all of

Q." For ETCS+R "...this implies if 'set' is understood to mean any discrete category (which turns out to be an oversimplification) every description of a category in terms of a *set* A_0 of objects and a set A_1 of morphisms uniquely describes a category A (Lawvere 1966)." (this notion is developed extensively for internal categories [25., p.1250])

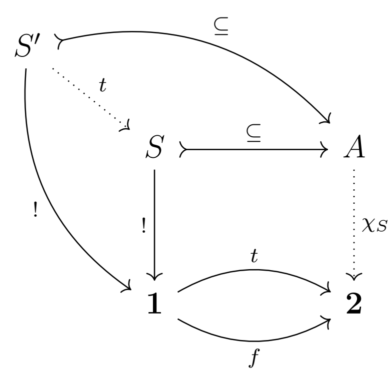
A.2.10 Replacement

For it, we define a shorthand $\exists!_i SP(S)$ meaning there exists a set S unique up to isomorphism $(\exists S \forall X(P(X) \iff X \cong S)$ As Mclarty explains, the reasoning is that each ETCS set provably has the same properties as any set isomorphic to it. So an ETCS formula can only specify a set up to isomorphism, it is too blunt to go underneath it.

R) For each relation R(x,Y) of arrows x to sets Y in ETCS:

For every set A, if $\forall x \in A \exists!_i S_x(R(x, S_x))$, then there is some f:$S \to A$ s.t. for each x$\in A$ the set S(x) is the inverse of x along f.

- think of the diagram:



Suppose each x$\in A$ is assigned a set S_x, unique up to isomorphism by a relation

R(x,Y) expressible in ETCS. Then there is a set S and arrow f:$S \to A$ with this property: for every x$\in A$ and any S_x with R(x,S_x), there is some $t : S_x \to S$ making this a pullback.

So S is a disjoint union of the S_x and f gives the structure of a set of sets $\{S_x | x \in A\}$ is the preimage of its x. With this, we receive ETCS+R. (originally [29])

A.3 Comparison

A.3.1 Translation

- Proofs in ZFC often involve $\in$ in a finite and transfinite membership chains of sets $S_0 \in S_1 ... S_\omega \in S_{\omega+1} ... S_\alpha$ in On

- Those methods gain their full power by using transitive closure to localize iterated membership. - that is, regard the members of any set S, the members of members, their members in turn and so on - as not merely existing "out there" in the whole universe of sets but as all being members of a single set TC(S) called the transitive closure of S.

- There are no membership chains of sets in ETCS since the members of ETCS sets are not sets. And ETCS+R cannot have transitive closures of sets, because all properties of ETCS are isomorphism invariant (ZFC singleton sets $\{\emptyset\}$ and $\{N\}$ are isomorphic, but their transitive closures are not $\emptyset$ is its own transitive closure, but transitive closure of N is infinite)

- But ZFC proves the aforementioned Mostowski's collapsing lemma [16, p.69] - it says every well-founded extensional relation R on any set M is isomorphic to the internal membership relation on a unique ZFC set S. - every well-founded extensional relational system (M,R) is isomorphic to the relational system (TC(S), $\in$) for a unique set. So even if ETCS+R cannot have TR as a set, if it can have an extensional relation system, it is all swell because the sets are determined up to isomorphism any way, and it would have some translation for all such TR sets already.

- Mostowski lets us restate any ZFC sentence as an isomorphism-invariant statement about well-founded extensional relational system, which is in ZFC provably equivalent to the original
- Now, all isomorphism-invariant statements of ZFC translate verbatim into ETCS+R. Mitchell 1972 and Osius 1974 prove the translation preserves and reflects provability of statements on well-founded extensional relational systems.
- So via this two-step translation, ZFC and ETCS+R can formalize all the same concepts and prove all the same theorems.

[27, p.39]

A.3.2 Categorical properties considered

Freyd shows in [14] a counterwitness of equalizers. These are preserver alright, but clearly not reflected: for $F : A \to B$ if there is an equalizer f of g and z in B, it can just happen that A splits the parallel morphisms g,h, so they no longer share source and target in A, preventing the reflective establishment of an equalizer.

In dependent type theory, where not only terms depend on types but inversely types on terms, there is a notion of equivalence type - for two types A, B the equivalence type $A \cong B$ has for terms equivalences between A and B and the relevant mathematical properties are instead determining this equivalence type of categories. These properties he calls diagrammatic over the expectable diagrammatic language he develops and perfects [13], because any FOL fails in its initial expression due to free formulas as well as none of the negations of atomic predicates being preserved by equivalence functors. And yet it is a connection re-established so that the proper properties of category theory can afterall be expressed under translation in FOL. He proves the theorem in chapter 2 that an elementary property on categories is invariant within equivalence types of categories iff it is a diagrammatic property or as Marquis formulates it "If P(C) and C is equivalent to D, then P(D) iff P is a genuine categorical property(diagrammatic)" Over it Freyd shows that any elementary sentence S in the diagrammatic language is invariant within equivalence types iff there is a Freyd-diagrammatic sentence S' s.t. the axioms of category theory imply S $\iff$ S'. [23, p.173]