UNIVERSITY OF BERGEN
Faculty of Mathematics and Natural Sciences

Kundan Kumar, Professor
Department of Mathematics
University of Bergen, Norway,
Allegaten 41, RFG 4A16, 5007 Bergen
+47 55 58 48 64, +47 930 49 146
kundan.kumar@uib.no

Prof. Petr Knobloch
Department of Numerical Mathematics
Charles University, Faculty of Mathematics and Physics,
Sokolovska 83, 186 75 Praha 8, Czech Republic

Dear Prof. Knobloch:

I write below the evaluation of the PhD thesis submitted by the candidate Sunčica Sakić entitled „Numerical solution of degenerate parabolic problems“ to the Faculty of Mathematics and Physics, Charles University, Praha.

The thesis deals with the Richards equation, a canonical example of degenerate nonlinear parabolic partial differential equation with applications to ground water flow. Physically, this describes a water-air system describing the saturated-unsaturated zone in a porous medium. The degeneracy takes place at the interface between the saturated and unsaturated zones. With the evident importance of this equation by the hydrologists and its interesting mathematical character, it has received sustained interest from both the mathematical as well as the porous media communities. The degeneracy introduces computational and mathematical challenges (for example, the finite speed of propagation in porous media equation as in Barenblatt solution) and remains a topic of current research. With the focus of this thesis in developing robust and accurate numerical methods for solving the Richards equation, this thesis makes important contributions to this area of research. The findings of this thesis have been published in a top numerical analysis journal (Congreve et al. Error analysis for local discontinuous Galerkin semidiscretization of the Richards equation, IMA Journal of Numerical Analysis, 2024) and a conference proceeding. I expect at least one more article to follow from this thesis.

The thesis is structured in 6 chapters. The first chapter is on the introduction of the mathematical model followed by a Discontinuous Galerkin (DG) based spatial discretization approach. The preparation of the numerical analysis to come in the third chapter is already made in the second chapter with the introduction of the relevant functional spaces. The main substantive contribution is in the Chapter 3 providing a rigorous numerical analysis of the scheme introduced in the previous chapter. Here, the stability analysis and the error estimates are derived using tools from the numerical analysis of DG method, numerical analysis of degenerate parabolic equations, and a mathematical induction type of argument. The rigorous numerical analysis work is immediately followed by the numerical experiments detailed in chapter 4. This supports the theoretical analysis in the chapter 3. The next two chapters are on the algorithmic development of solving this equation using space time Discontinuous Galerkin approach and then complementing this with an adaptive hp method. This brings out the interplay of discretization, numerical linear algebra, solvers, and a posteriori error estimators. Overall, the thesis combines and introduces new ideas from a variety of numerical world: efficient solvers, rigorous numerical analysis on a problem of high importance and having interesting mathematical properties.

The first two chapters are basic and introduce the problem. In the first chapter, the Richards equation is introduced and the particularities of this equation - its degeneracies in both the fast diffusion and the slow diffusion cases are mentioned. An interesting part is the non-standard boundary condition - the seepage type which introduces a nonlinearity in the boundary condition. In the second chapter, the thesis discusses a local discontinuous Galerkin discretization in space. In time, a continuous formulation is retained. Both pressure based and the hydraulic head based formulations are mentioned which are referred to as ψ and Ψ formulations, respectively. The thesis considers the so-called expanded mixed method where one introduces an additional flux term to take care of the fact that the permeability cannot be inverted.

The third chapter is where the thesis contributes to the numerical analysis of discontinuous Galerkin approach for the Richards equation. The two main results are the stability estimate and the error estimates in the DG framework. Here, one needs to combine several tools to get the result. The DG interpolation estimates here have to be combined with the nonlinear analysis of the pde. The typical approach of standard test functions only provide partial results and a modified test function that smoothens in time and introduces a weight has to be used. This is used in combination with the continuous version of mathematical induction to arrive at the final error estimates. The next chapter is a numerical implementation of the ldg scheme and compares the theoretical estimates - the convergence rates with respect to the mesh size for several benchmark problems.

The next two chapters, chapter 5 and 6 are more in the spirit of scientific computing. They develop space time DG algorithm for solving this model equation and develop a posteriori error estimators. The solution strategy is a combination of mesh adaptivity in hp based framework, regularization, linearization, a damped Newton type method, and a combination of fixed point and Newton type approach, and a posteriori error estimators that resolves the algebraic error, space-algebraic error, and temporal discretization error. Challenging test cases are considered and the performance of the numerical scheme is undertaken. They do not provide any proof of the convergence of the schemes.

The thesis is well written, the proofs are well structured, and the algorithms are well described. However, there are several places where the exposition could have been improved. Scientifically, the thesis makes original contributions and presents material that is worthy of defense. At the same time, this work also raises questions that should be considered by the candidate. Below I describe some of these questions that can be discussed during the defense.

1. The thesis discusses the seepage boundary condition in the introduction. However, it is not clear where it is being used in the thesis. In the numerical computations in chapter 6, it is being mentioned but it is not clear in which way it has been implemented.
2. In the chapter 3, the time continuous semi-discrete scheme has been considered and the time discretization is being considered in the chapter 5. However, no rigorous analysis is being performed for the time discrete problem. What are the difficulties that one encounters if one considers the time discretization, say Backward Euler in the chapter 3 as well?
3. Both in chapter 2 and chapter 3, the seepage boundary conditions are not taken into consideration. Can the proof be extended to include this case as well?
4. In chapter 6, the thesis considers numerical computations for a problem which consists of layered material. These layers have different parametrization of relative permeability. In the analysis considered in chapter 3, the permeability is a function of saturation but there is no explicit dependence of this function on the spatial location. Does the proof in chapter 3 extend to the case considered in the chapter 6?
5. There are two formulations considered in this thesis: the pressure based and the hydraulic head based. In the final chapter, there is some comparison of the two formulations being mentioned. However, we do not get to see any conclusions being drawn - in terms of analysis and in terms of

the computational stability and performance. Which of these formulations should be used and where?

6. The proof in chapter 3 considers the case of fast diffusion. What challenges are encountered if one considers the case of slow diffusion as well?

7. For the lemma 1 on page 15, you may need some consideration of the time derivative as well.

8. In terms of the writing, one could have improved at some places. For example, the Problem 2.2 is not mentioned and one has to make the connection with the enumeration of the equations. Similarly, the ψ model is introduced as a pressure formulation but in the third chapter the analysis is performed with u denoting the pressure. Assumption 2.4 is mentioned but again perhaps it refers to the equation.

To conclude, this thesis presents material of sufficient novelty to proceed for the defense.

Sincerely,

Kundan Kumar

•Allegaten 41, Room RFG 4A16,
5007 Bergen
Postboks 7803, 5020 Bergen
+47 55 58 48 64, +47 930 49 146
kundan.kumar@uib.no