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Operators of harmonic analysis, related function spaces and applications

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Contents

1 Preface

The present thesis focuses on the investigation of boundedness properties of various operators of harmonic analysis, including Fourier multiplier operators, singular integral operators and their multilinear variants. A special emphasis is put on the role that function spaces play in these results. Part of our work involving function spaces is also motivated by possible applications in partial differential equations.

The thesis consists of nine papers, whose list is given below.

[A] Angela Alberico, Andrea Cianchi, Luboš Pick, and Lenka Slavíková. "Fractional Orlicz-Sobolev embeddings". In: *J. Math. Pures Appl. (9)* 149 (2021), pp. 216–253. ISSN: 0021-7824,1776-3371. DOI: 10.1016/j.matpur.2020.12.007

[B] Angela Alberico, Andrea Cianchi, Luboš Pick, and Lenka Slavíková. "Boundedness of functions in fractional Orlicz-Sobolev spaces". In: *Nonlinear Anal.* 230 (2023), Paper No. 113231, 26. ISSN: 0362-546X,1873-5215. DOI: 10.1016/j.na.2023.113231

[C] Loukas Grafakos and Lenka Slavíková. "A sharp version of the Hörmander multiplier theorem". In: *Int. Math. Res. Not. IMRN* 15 (2019), pp. 4764–4783. ISSN: 1073-7928,1687-0247. DOI: 10.1093/imrn/rnx314

[D] Lenka Slavíková. "On the failure of the Hörmander multiplier theorem in a limiting case". In: *Rev. Mat. Iberoam.* 36.4 (2020), pp. 1013–1020. ISSN: 0213-2230,2235-0616. DOI: 10.4171/rmi/1157

[E] Loukas Grafakos, Mieczysław Mastyło, and Lenka Slavíková. "A sharp variant of the Marcinkiewicz theorem with multipliers in Sobolev spaces of Lorentz type". In: *J. Funct. Anal.* 282.3 (2022), Paper No. 109295, 36. ISSN: 0022-1236,1096-0783. DOI: 10.1016/j.jfa.2021.109295

[F] Loukas Grafakos, Danqing He, and Lenka Slavíková. "$L^2 \times L^2 \to L^1$ boundedness criteria". In: *Math. Ann.* 376.1-2 (2020), pp. 431–455. ISSN: 0025-5831,1432-1807. DOI: 10.1007/s00208-018-1794-5

[G] Georgios Dosidis and Lenka Slavíková. "Multilinear singular integrals with homogeneous kernels near L^1". Math. Ann., to appear. DOI: 10.1007/s00208-023-02691-x

[H] Polona Durcik, Lenka Slavíková, and Christoph Thiele. "Local bounds for singular Brascamp-Lieb forms with cubical structure". In: *Math. Z.* 302.4 (2022), pp. 2375–2405. ISSN: 0025-5874,1432-1823. DOI: 10.1007/s00209-022-03148-8

[I] Polona Durcik, Lenka Slavíková, and Christoph Thiele. "Norm-variation of triple ergodic averages for commuting transformations". Preprint, arXiv:2307.07372

In the next section, we summarize the historical background that motivated our research and we explain how our contribution fits into that context. Section 3 then provides a more detailed summary of the attached papers.

2 Introduction

2.1 Fractional Sobolev spaces and their generalizations

In the modern theory of partial differential equations, a function does not need to satisfy a given equation in a pointwise sense to be called its solution. Instead, a certain weaker equality involving integration against a suitable family of test functions is required. The solutions of these equations are then naturally found in the family of Sobolev-type spaces, and a good understanding of various properties of these function spaces is therefore indispensable in the study of partial differential equations.

Let $n \in \mathbb{N}$. Given a positive integer m and $p \in [1, \infty]$, we define the *Sobolev space* $W^{m,p}(\mathbb{R}^n)$ as the space of all m-times weakly differentiable functions u on $\mathbb{R}^n$ such that u belongs to $L^p(\mathbb{R}^n)$ together with all its weak derivatives up to order m. The space $W^{m,p}(\mathbb{R}^n)$ is equipped with the norm

$$\|u\|_{W^{m,p}(\mathbb{R}^n)} = \sum_{k=0}^{m} \| |\nabla^k u| \|_{L^p(\mathbb{R}^n)},$$

where $\nabla^k u$ stands for the vector of all k-th order weak derivatives of u, with the convention that $\nabla^0 u = u$.

The fact that the weak derivatives of a given function belong to $L^p(\mathbb{R}^n)$ does not imply that the function itself belongs to $L^p(\mathbb{R}^n)$ as well. In order to understand how integrability properties of the weak derivatives affect integrability properties of the function, it is therefore useful to consider the homogeneous variant of the space $W^{m,p}(\mathbb{R}^n)$, denoted $V^{m,p}(\mathbb{R}^n)$ and consisting of all m-times weakly differentiable functions whose m-th order weak derivatives belong to $L^p(\mathbb{R}^n)$, with a seminorm given by

$$|u|_{m,p,\mathbb{R}^n} = \| |\nabla^m u| \|_{L^p(\mathbb{R}^n)}.$$

The functional $|\cdot|_{m,p,\mathbb{R}^n}$ vanishes at all polynomials of order at most $m-1$, and it is thus not a norm. It becomes, however, a norm when restricted to the subset $V_d^{m,p}(\mathbb{R}^n)$ of $V^{m,p}(\mathbb{R}^n)$ containing those $u \in V^{m,p}(\mathbb{R}^n)$ which vanish near infinity, in the sense that

$$|\{x \in \mathbb{R}^n : |\nabla^k u(x)| > \lambda\}| < \infty \quad \text{for } k \in \{0, \dots, m-1\} \text{ and } \lambda > 0. \tag{2.1}$$

We illustrate the relationship between $W^{m,p}(\mathbb{R}^n)$ and $V_d^{m,p}(\mathbb{R}^n)$ on the particular case when $1 \le p < n/m$. By the classical Sobolev embedding theorem, any $u \in V_d^{m,p}(\mathbb{R}^n)$ then belongs to $L^{np/(n-mp)}(\mathbb{R}^n)$, and thus also to $L^p_{\text{loc}}(\mathbb{R}^n)$. Nevertheless, u does not need to belong to $L^p(\mathbb{R}^n)$ globally. This is in contrast with the space $W^{m,p}(\mathbb{R}^n)$, which is contained in $L^p(\mathbb{R}^n)$ by its very definition.

In various applications, an extension of the above-mentioned function spaces to the setting involving derivatives of non-integer order comes into play. Notably, these applications include nonlocal problems in partial differential equations, see, e.g., the introduction of the expository paper [23] for references. Nevertheless, the focus of this thesis will be on a different type of applications, namely those involving boundedness properties of Fourier multiplier operators. These applications will be discussed in the next subsection.

Somewhat surprisingly, there are several natural ways how fractional Sobolev spaces can be defined, and these different notions are not equivalent to each other except in the particular case of an L^2-based Sobolev space. Let $s \in (0,1)$ and $1 \le p < \infty$. One possible way of introducing an L^p-based *fractional Sobolev space* of order s involves the seminorm

$$|u|_{s,p,\mathbb{R}^n} = \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left(\frac{|u(x) - u(y)|}{|x-y|^s} \right)^p \frac{dxdy}{|x-y|^n} \right)^{\frac{1}{p}}. \tag{2.2}$$

This seminorm gives rise to the homogeneous fractional Sobolev space $V^{s,p}(\mathbb{R}^n)$, which consists of all measurable functions u for which the functional (2.2) is finite. Subsequently, the higher-order fractional Sobolev space $V^{s,p}(\mathbb{R}^n)$ associated with the smoothness parameter $s \in (1,\infty) \setminus \mathbb{N}$ is defined to be the set of all functions u in $W^{\lfloor s \rfloor,1}_{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$|u|_{s,p,\mathbb{R}^n} = |\nabla^{\lfloor s \rfloor} u|_{\{s\},p,\mathbb{R}^n} < \infty.$$

Here and in what follows, $\lfloor s \rfloor$ denotes the integer part of s and $\{s\} = s - \lfloor s \rfloor$ is the fractional part of s. In both situations, the inhomogeneous counterpart $W^{s,p}(\mathbb{R}^n)$ of $V^{s,p}(\mathbb{R}^n)$ equals the intersection of $V^{s,p}(\mathbb{R}^n)$ with $W^{\lfloor s \rfloor,p}(\mathbb{R}^n)$, with the convention that $W^{0,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$.

Alternatively, fractional Sobolev spaces can be introduced by using the notion of the *fractional Laplace operator*. Given $s > 0$, this operator is denoted by $(I - \Delta)^{\frac{s}{2}}$ and it is defined via multiplication by $(1 + 4\pi^2|\xi|^2)^{\frac{s}{2}}$ on the frequency side, namely

$$\widehat{(I - \Delta)^{\frac{s}{2}} u}(\xi) = (1 + 4\pi^2|\xi|^2)^{\frac{s}{2}} \widehat{u}(\xi), \tag{2.3}$$

where $\widehat{u}$ stands for the Fourier transform of u. Given also $1 \le p \le \infty$, the associated fractional Sobolev space $L^p_s(\mathbb{R}^n)$ consists of all tempered distributions u satisfying

$$\|u\|_{L^p_s(\mathbb{R}^n)} = \|(I - \Delta)^{\frac{s}{2}} u\|_{L^p(\mathbb{R}^n)} < \infty.$$

We note that the homogeneous variant of the space $L^p_s(\mathbb{R}^n)$ can be defined similarly by employing the homogeneous fractional Laplace operator

$$\widehat{(-\Delta)^{\frac{s}{2}} u}(\xi) = (2\pi|\xi|)^s \widehat{u}(\xi)$$

in place of (2.3).

It is of interest to understand the different relations between the Sobolev spaces defined above. To start with, if s is an integer then the space $L^p_s(\mathbb{R}^n)$ coincides with $W^{s,p}(\mathbb{R}^n)$ as long as $1 < p < \infty$. If s is not an integer then $L^p_s(\mathbb{R}^n) = W^{s,p}(\mathbb{R}^n)$ is only true when $p = 2$. For $p \in (1,2)$, we have the strict inclusion $W^{s,p}(\mathbb{R}^n) \subset L^p_s(\mathbb{R}^n)$, while for $p \in (2,\infty)$ the reverse inclusion $L^p_s(\mathbb{R}^n) \subset W^{s,p}(\mathbb{R}^n)$ holds. For more information on the different types of Sobolev spaces described above, see, e.g., [65, Chapter V].

In the recent years, the study of nonlocal problems featuring functionals of non-standard growth has attracted a lot of attention [6, 10, 11, 29, 31, 58]. In this connection, a variant of the spaces $V^{s,p}(\mathbb{R}^n)$ in which the p-th power in (2.2) is replaced by a more general convex function came into play. These spaces are associated with functionals of the form

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} A\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dxdy}{|x-y|^n},$$

where $s \in (0,1)$ and A is a Young function, namely a nonnegative convex function on $[0,\infty)$ vanishing at 0. The corresponding fractional Orlicz-Sobolev seminorm is then defined in analogy with the definition of the Luxemburg norm in Orlicz spaces by

$$|u|_{s,A,\mathbb{R}^n} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} A\left(\frac{|u(x)-u(y)|}{\lambda|x-y|^s}\right)\frac{dxdy}{|x-y|^n} \leq 1\right\}, \tag{2.4}$$

and the space of all measurable functions u for which the functional (2.4) is finite is the *fractional Orlicz-Sobolev space* $V^{s,A}(\mathbb{R}^n)$. The space $V^{s,A}(\mathbb{R}^n)$ also has its higher-order variant, defined via the seminorm

$$|u|_{s,A,\mathbb{R}^n} = |\nabla^{\lfloor s\rfloor} u|_{\{s\},A,\mathbb{R}^n},$$

where $s \in (1,\infty) \setminus \mathbb{N}$. Basic properties of fractional Orlicz-Sobolev spaces were established in [21, 32] under certain technical assumptions on the Young function A.

Similarly to the constructions from the previous paragraph, the spaces $L^p_s(\mathbb{R}^n)$ can be extended to a more general setting as well. Let $X(\mathbb{R}^n)$ be a *rearrangement-invariant space*, that is, roughly speaking, a Banach space of functions whose norm depends only on the measure of level sets of the absolute value of a given function; for a precise definition of rearrangement-invariant spaces, see [2, Chapter 2]. Given $s > 0$, one can then consider the Sobolev-type space consisting of all tempered distributions u satisfying

$$\|(I-\Delta)^{\frac{s}{2}} u\|_{X(\mathbb{R}^n)} < \infty. \tag{2.5}$$

A case of particular interest is the one when $X(\mathbb{R}^n)$ is the *Lorentz space* $L^{p,q}(\mathbb{R}^n)$, defined as

$$\|u\|_{L^{p,q}(\mathbb{R}^n)} = \|t^{\frac{1}{p}-\frac{1}{q}} u^*(t)\|_{L^q(0,\infty)}.$$

Here, $1 \leq p, q \leq \infty$ and u^* stands for the *non-increasing rearrangament* of the function u, namely, the unique non-increasing right-continuous function on $(0,\infty)$ equimeasurable with u, in the sense that

$$|\{x \in \mathbb{R}^n : \ |u(x)| > \lambda\}| = |\{t \in (0,\infty) : \ u^*(t) > \lambda\}| \quad \text{for } \lambda > 0.$$

The s-th order *Lorentz-Sobolev space* associated with the Lorentz space $L^{p,q}(\mathbb{R}^n)$ via equation (2.5) is then denoted by $L^{p,q}_s(\mathbb{R}^n)$. Lorentz-Sobolev spaces arise naturally as optimal domain spaces for the embedding into the space $L^\infty(\mathbb{R}^n)$ of essentially bounded functions, see [17, 64]. Various other embeddings between Lorentz-Sobolev spaces, as well as their generalizations into the context of Besov and Triebel-Lizorkin spaces, were studied in [63].

The purpose of the first two articles of this thesis [A, B] is to provide a complete description of embeddings of the spaces $V^{s,A}(\mathbb{R}^n)$ into Orlicz spaces and, more generally, into rearrangement-invariant spaces. On the other hand, the papers [C], [D] and [E] are centered around the spaces $L^{p,q}_s(\mathbb{R}^n)$, and they establish sharp variants of classical theorems in Fourier analysis by making use of these function spaces. In the next subsection, we briefly summarize the Fourier-analytical background for the papers [C–E].

2.2 Fourier multiplier operators

With any bounded function m in $\mathbb{R}^n$ we associate the *Fourier multiplier operator* T_m, which alters the Fourier transform of a given Schwartz function f by multiplying it by m, namely

$$\widehat{T_m f} = m\widehat{f}.$$

As a consequence of the Plancherel theorem, each operator T_m admits a bounded extension from $L^2(\mathbb{R}^n)$ into itself. In fact, Fourier multiplier operators are exactly those translation-invariant operators that are bounded on $L^2(\mathbb{R}^n)$, and they include many standard operators of harmonic analysis, such as the Hilbert transform and the Riesz transforms, as special cases.

While the L^2-boundedness of T_m is straightforward, the operator T_m may in general not be bounded on $L^p(\mathbb{R}^n)$ if $p \neq 2$. A well known example of this phenomenon is the Fourier multiplier operator associated with the characteristic function of a ball in dimension 2 or higher, which is L^p-bounded only for $p = 2$, see [30]. The problem of characterizing those bounded functions m for which the operator T_m extends to a bounded operator from $L^p(\mathbb{R}^n)$ into itself is difficult and still far from being resolved, and therefore various sufficient conditions for the L^p-boundedness have been established as a substitute. These conditions are often called *multiplier theorems*.

The classical *Mikhlin multiplier theorem* [55] implies that if the symbol m satisfies

$$|\partial^\alpha m(\xi)| \leq C_\alpha |\xi|^{-|\alpha|}, \quad \xi \neq 0 \tag{2.6}$$

for all multiindices α up to order $\lfloor n/2 \rfloor + 1$, then the operator T_m is bounded from $L^p(\mathbb{R}^n)$ into itself for any $p \in (1, \infty)$. We next discuss improvements of this result that make use of fractional Sobolev spaces. We start with the classical *Hörmander multiplier theorem* [44], which says, roughly speaking, that one does not need the bound (2.6) to be true in a pointwise sense, but only in the L^2-mean uniformly with respect to all dyadic annuli. To state the theorem precisely, we introduce an auxiliary function ϕ that is smooth, supported in the unit annulus on $\mathbb{R}^n$ and satisfies

$$\sum_{k\in\mathbb{Z}} \phi(2^k \xi) = 1 \quad \text{for } \xi \neq 0.$$

A fractional variant of Hörmander's theorem asserts that if the symbol m fulfills the condition

$$\sup_{k\in\mathbb{Z}} \|\phi(\xi) m(2^k \xi)\|_{L^2_s(\mathbb{R}^n)} < \infty \tag{2.7}$$

for some $s > n/2$ then T_m is L^p-bounded for all $p \in (1, \infty)$. In addition, it turns out that relaxing the lower bound on the smoothness parameter s still yields the

L^p-boundedness of T_m, though in a limited range of parameters p. This result was proved by Calderón and Torchinski [8] and it asserts that if

$$\frac{s}{n} > \left|\frac{1}{p} - \frac{1}{2}\right| = \frac{1}{r} \tag{2.8}$$

and condition

$$\sup_{k\in\mathbb{Z}} \|\phi(\xi)m(2^k\xi)\|_{L^r_s(\mathbb{R}^n)} < \infty \tag{2.9}$$

is satisfied, then T_m admits a bounded extension from $L^p(\mathbb{R}^n)$ into itself. A further slight improvement of the Calderón-Torchinski result was established in [36], and a certain limiting variant involving a Besov space appeared in [62].

We note that, unlike in the Hörmander original formulation, it is no longer possible to use an L^2-based Sobolev space in assumption (2.9) if $s \le n/2$. This is due to the fact that the symbol of the Fourier multiplier operator necessarily needs to be bounded, but the membership of a function into the Sobolev space $L^2_s(\mathbb{R}^n)$ with $s \le n/2$ does not guarantee its boundedness.

The discussion above suggests that the topic of Fourier multiplier theorems has close connections to the theory of Sobolev embeddings (in particular when Sobolev embeddings into the space $L^\infty(\mathbb{R}^n)$ are concerned). The papers [C], [D] and [E] of the present thesis further strengthen this connection by investigating sharp variants of classical multiplier theorems, including the above-mentioned multiplier theorem by Hörmander and also its multiparameter variant due to Marcinkiewicz [53]. The form of these theorems is inspired by optimal Sobolev embeddings involving Lorentz spaces [17, 64].

2.3 Singular integral operators: from linear to multilinear theory

Another important class of operators in harmonic analysis, closely related to the class of Fourier multiplier operators, is that of *singular integral operators*. We will first focus on singular integral operators of convolution type, having the form

$$Tf(x) = \text{p. v.} \int_{\mathbb{R}^n} f(x-y)K(y)\,dy, \tag{2.10}$$

where f is a Schwartz function on $\mathbb{R}^n$ and K is a function on $\mathbb{R}^n$ which is homogeneous of degree $-n$. The values of K away from the origin are uniquely determined by its restriction to the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$. We denote this restriction by Ω and we will assume throughout that Ω has vanishing integral over $\mathbb{S}^{n-1}$. We observe that the choice $\Omega(y) = y_j$ for $j = 1, \dots, n$ recovers the well-known family of Riesz transforms (and, in particular, the Hilbert transform if $n = 1$). In addition, it is worth pointing out that any Fourier multiplier operator associated with a symbol that is homogeneous of degree 0 and smooth on the unit sphere can be realized as a singular integral operator of the type discussed above, up to a multiple of the identity operator. Such an operator is L^p-bounded for any $p \in (1,\infty)$ by the Hörmander multiplier theorem.

In general, however, the operator T is not well-behaved, and it may even fail to be bounded on $L^2(\mathbb{R}^n)$. On the other hand, if Ω either has good cancellation properties, in the sense that it is odd, or if it enjoys good integrability properties, in the sense that it belongs to the Orlicz space $L\log L(\mathbb{S}^{n-1})$, then the operator T is in fact L^p-bounded for any $p \in (1,\infty)$. We recall that $L\log L(\mathbb{S}^{n-1})$ is the Orlicz space consisting of all functions u satisfying

$$\int_{\mathbb{S}^{n-1}} |u(\theta)| \log_+ |u(\theta)| d\sigma(\theta) < \infty,$$

where $d\sigma$ denotes the surface measure on the unit sphere. In particular, by a simple embedding, the L^p-boundedness holds if $\Omega \in L^q(\mathbb{S}^{n-1})$ for some $q > 1$.

In the 1960s, Calderón introduced objects that are nowadays called *Calderón's commutators* as a part of his programme in the study of partial differential equations. Calderón's commutators are singular integral operators which are more complicated than those discussed above as they are no longer of convolution type. Classical methods turned to be ineffective in dealing with these objects and a radically new approach was necessary for establishing their boundedness. Such an approach was developed by Coifman and Meyer [18, 19], who were the first ones to realize the benefits of interpreting Calderón's commutators as multilinear operators.

Following the pioneering work of Coifman and Meyer, multilinear aspects of singular integral operators became a heavily studied branch of harmonic analysis. The *Coifman-Meyer multilinear operators* have the form

$$T(f_1,\dots,f_m)(x) = \mathrm{p.\,v.} \int_{\mathbb{R}^{mn}} f_1(x-y_1)\dots f_m(x-y_m) K(y_1,\dots,y_m)\, dy_1\dots dy_m, \tag{2.11}$$

where $f_1,\dots,f_m$ are n-dimensional Schwartz functions and K is an mn-dimensional singular integral kernel. Typically, we will assume that K is homogeneous of degree $-mn$ and its restriction Ω to the unit sphere $\mathbb{S}^{mn-1}$ has vanishing integral over $\mathbb{S}^{mn-1}$. Alternatively, one can also consider the closely related notion of *multilinear Fourier multiplier operators* and require K to be the inverse Fourier transform of a symbol that satisfies one of the standard assumptions of Fourier multiplier theorems, such as (2.6). In particular, taking K to be the Dirac delta measure at the origin, we see that (2.11) reduces to the pointwise product of the functions $f_1,\dots,f_m$. This operator is bounded from the product of Lebesgue spaces $L^{p_1}(\mathbb{R}^n)\times\dots\times L^{p_m}(\mathbb{R}^n)$ into another Lebesgue space $L^p(\mathbb{R}^n)$ if and only if the exponents satisfy the Hölder scaling condition

$$\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}. \tag{2.12}$$

We will assume that equation (2.12) is in place in what follows.

The situation when K is a homogeneous kernel that is smooth on the unit sphere falls within the scope of the classical Coifman-Meyer theory [18, 19], as long as $1 < p_1,\dots,p_m \le \infty$ and $p \ge 1$. We note that, however, unlike in the linear setting, the assumption $1 < p_1,\dots,p_m \le \infty$ and the Hölder scaling (2.12) do not imply that p has to fall into the Banach space regime $p \ge 1$. The corresponding boundedness results in the quasi-Banach setting when $p < 1$ were obtained by Grafakos and Torres [41] and Kenig and Stein [47]. At the present moment, boundedness properties of multilinear singular integral operators of the form (2.11), associated with a function Ω which does not possess any smoothness but merely belongs to $L^q(\mathbb{S}^{mn-1})$ for some

$q > 1$, are also well understood. The first paper systematically dealing with this problem was [38], which discussed the bilinear situation under the assumption that $q \geq 2$ and established bounds for the corresponding operator in a certain range of exponents by making use of a novel technique involving wavelet decomposition of the symbol. Further partial results include paper [F] of the present thesis, as well as [39, 42]. The paper [G] then proved boundedness for any $q > 1$, in the optimal open range of exponents.

2.4 Bilinear Hilbert transform and beyond

Going back in history, we next recall an approach that Calderón suggested in order to tackle the question of the boundedness of his first commutator. Even though Calderón himself managed to prove boundedness of the commutator by using a different strategy [7], his idea is still of interest as it initiated the development of new techniques in Fourier analysis.

Calderón observed that his first commutator can be obtained as a superposition of directional bilinear Hilbert transforms, namely operators of the form

$$H_\beta(f_1, f_2)(x) = \mathrm{p.\,v.} \int_{\mathbb{R}} f_1(x+y) f_2(x+\beta y) \frac{dy}{y}, \tag{2.13}$$

where f_1 and f_2 are one-dimensional Schwartz functions and $\beta \in (0,1)$. Bounds for Calderón's first commutator can thus be recovered from bounds for the operators H_β, as long as these bounds are uniform with respect to the parameter β. Nevertheless, it was not until the late 1990s when boundedness of the bilinear Hilbert transform was finally established by Lacey and Thiele [50, 51], who realized that the analysis of this operator is closely related to the problem of pointwise almost everywhere convergence of Fourier series. The desired uniform bounds for the bilinear Hilbert transform in a certain range of exponents sufficient to complete Calderón's original programme were later obtained by Grafakos and Li [40, 52]. This being said, we point out that unlike in the case of the Coifman-Meyer operators (2.11), the full range of exponents in which the bilinear Hilbert transform is bounded has not yet been established.

A two-dimensional variant of the operator H_β having the form

$$T_B(f_1, f_2)(x) = \mathrm{p.\,v.} \int_{\mathbb{R}^2} f_1(x+y) f_2(x+B(y)) K(y)\, dy \tag{2.14}$$

was investigated in [22]. In equation (2.14), f_1 and f_2 are two-dimensional Schwartz functions, K is a two-dimensional singular integral kernel, in the sense that its Fourier transform satisfies (2.6), and $B : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear mapping. Similarly to the fact that the operator H_β becomes degenerate when β equals either 0 or 1, the properties of the operator T_B are dictated by the spectrum of B, and in particular by the fact which of the numbers 0 and 1 belongs to it. A case that is of particular interest to us is the one when the spectrum of B consists of both 0 and 1. This is the only case which could not be handled by the time-frequency analysis methods of [22], and bounds for this operator were established later by Kovač [48] by making use of a certain symmetrization procedure based on repeated applications of the

Cauchy-Schwarz inequality and telescoping identities. After a suitable change of variables, the operator considered in [48] can be written as

$$T(f_1, f_2)(x, y) = \text{p. v.} \int_{\mathbb{R}^2} f_1(x, y')f_2(x', y)K(x - x', y - y')\, dx'dy' \tag{2.15}$$

and is sometimes called the *twisted paraproduct*.

We next discuss the quadrilinear form

$$\begin{aligned} &\Lambda(f_1, f_2, f_3, f_4) \\ &= \text{p. v.} \int_{\mathbb{R}^4} f_1(x, y')f_2(x', y)f_3(x, y)f_4(x', y')K(x - x', y - y')\, dxdydx'dy', \end{aligned} \tag{2.16}$$

where f_1, f_2 and K are as above and f_3, f_4 are two-dimensional Schwartz functions. Setting formally $f_4 \equiv 1$ reduces (2.16) to a trilinear form dual to the twisted paraproduct (2.15). Boundedness properties of the form (2.16) in a certain range of exponents were established in [24, 25]. In connection with ergodic-theoretic applications, which will be discussed in more detail in the next subsection, the following variant of (2.16) also came into play [26]:

$$\text{p. v.} \int_{\mathbb{R}^4} f_1(x, y')f_2(x', y)f_3(x, y)f_4(x', y')K(y-x-x', y'-x-x')\, dxdydx'dy'. \tag{2.17}$$

To be able to deal with the above-mentioned multilinear forms in a more systematic way, it is useful to consider a common generalization of both (2.16) and (2.17) of the type

$$\text{p. v.} \int_{\mathbb{R}^4} f_1(x, y')f_2(x', y)f_3(x, y)f_4(x', y')K(\Pi(x, y, x', y'))\, dxdydx'dy', \tag{2.18}$$

where $\Pi : \mathbb{R}^4 \to \mathbb{R}^2$ is a linear surjection. A complete characterization of those projections Π for which (2.18) is bounded (up to a constant) by the product of the L^4-norms of the input functions f_1, f_2, f_3, f_4 was established in [27]. In fact, the paper [27] dealt with a higher-dimensional generalization of such an estimate, featuring 2^m input functions on $\mathbb{R}^m$ and one $2m$-dimensional singular integral kernel, and it proved bounds involving the L^{2^m}-norms of the input functions. While the optimal range of Lebesgue space estimates for such a form remains unknown, the paper [H] of the present thesis established boundedness properties of this form in an extended range of exponents, which is optimal at least in a certain weaker sense.

We finish this subsection by noting that the estimates above can be understood within the broader context of *singular Brascamp-Lieb inequalities*. We recall that the standard *Brascamp-Lieb inequalities* are estimates for multilinear forms consisting of integrating the tensor product of the input functions over a subspace of the direct sum of the domain spaces. Their validity was completely characterized in the work of Bennett, Carbery, Christ and Tao [3]. Singular Brascamp-Lieb inequalities are obtained by replacing some of the (originally integrable) input functions by singular integral kernels. The current level of understanding of singular Brascamp-Lieb inequalities is however very far from being able to establish a general theory mirroring the one from [3]; see also [28] for a survey on this topic.

2.5 Ergodic-theoretic applications

We next describe how bounds for the multilinear forms discussed above can be used to establish the convergence of certain ergodic averages. Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space and let $T : X \to X$ be a measure preserving transformation. The *single ergodic averages* are defined as

$$A_N f(x) = \frac{1}{N} \sum_{i=0}^{N-1} f(T^i x). \tag{2.19}$$

The classical *von Neumann mean ergodic theorem* [61] asserts that these averages converge in the L^2 norm, and *Birkhoff's pointwise ergodic theorem* [4] yields their convergence almost everywhere.

Multiple ergodic averages arose from the work of Fürstenberg and his coauthors [33, 34, 35], which connected ergodic theory with arithmetic combinatorics. These averages are associated with m mutually commuting measure-preserving transformations $T_1, \ldots, T_m$ and have the form

$$M_N(f_1, \ldots, f_m) = \frac{1}{N} \sum_{i=0}^{N-1} f_1(T_1^i x) \ldots f_m(T_m^i x). \tag{2.20}$$

Their L^2-convergence in the case $m = 2$ was established by Conze and Lesigne [20] using techniques of ergodic theory and an analogous result for a general m was obtained by Tao [66] using combinatorial arguments. Ergodic-theoretic proofs of the results from [66] were established shortly afterwards in [1, 45].

The norm-convergence results mentioned above have a purely qualitative nature, in the sense that they do not provide any information on the rate of convergence. A quantitative norm-convergence estimate for the single ergodic averages was established in [46], showing that there is a positive constant C such that

$$\sum_{j=1}^{J} \|A_{n_j} f - A_{n_{j-1}} f\|_{L^2(X)}^2 \le C \|f\|_{L^2(X)}^2$$

holds for all sequences $n_0 < n_1 < \cdots < n_J$ of positive integers. Its analogue for the case of two commuting transformations was discussed in [26], where it was proved that

$$\sum_{j=1}^{J} \|M_{n_j}(f, g) - M_{n_{j-1}}(f, g)\|_{L^2(X)}^2 \le C \|f\|_{L^4(X)}^2 \|g\|_{L^4(X)}^2. \tag{2.21}$$

The proofs in [26] relied on harmonic analysis techniques, in particular on boundedness properties of the form (2.17) associated with a singular integral kernel K that does not satisfy standard symbol estimates (such as (2.6)) but instead features a certain multiparameter structure. It seems natural to conjecture that a result similar to (2.21) holds for any number of commuting transformations, namely that

$$\sum_{j=1}^{J} \|M_{n_j}(f_1, \ldots, f_m) - M_{n_{j-1}}(f_1, \ldots, f_m)\|_{L^2(X)}^2 \le C \prod_{\ell=1}^{m} \|f_\ell\|_{L^{2m}(X)}^2. \tag{2.22}$$

The validity of (2.22) is an open problem for any $m \geq 3$. The techniques of [26] do not apply when $m \geq 3$ as the transference to a harmonic-analytic problem would lead to a multilinear form that is not well understood at the moment, not even in the case of standard kernels. The paper [I] of the present thesis establishes a weaker variant of the estimate (2.22) for $m = 3$, which nevertheless still yields good quantitative bounds for the L^2-convergence of the associated triple ergodic averages.

We finish this introduction by pointing out that the pointwise a.e. convergence of the multiple ergodic averages (2.20) is a long-standing open problem, even in the case of two commuting transformations. Recent results involving the almost-everywhere convergence of different types of multiple ergodic averages include the papers [5, 12, 49].

3 Summary of the attached papers

3.1 [A] Fractional Orlicz-Sobolev embeddings

In this paper, we provide a complete characterization of embeddings of fractional Orlicz-Sobolev spaces $V_d^{s,A}(\mathbb{R}^n)$, consisting of those functions from $V^{s,A}(\mathbb{R}^n)$ that decay near infinity, into Orlicz spaces and, more generally, into rearrangement-invariant spaces. Our results work under the assumption that the Young function A has a *subcritical growth*, namely that $s \in (0,n) \setminus \mathbb{N}$ and

$$\int^{\infty} \left(\frac{t}{A(t)}\right)^{\frac{s}{n-s}} dt = \infty. \tag{3.1}$$

The assumption (3.1) corresponds to the condition $p \leq n/s$ when $A(t) = t^p$ near infinity. Remarkably, the results obtained in this paper are the exact analogues of the known results for Sobolev embeddings of integer order [13, 14, 15, 16], even though the methods of the proofs are substantially different. In particular, a crucial ingredient of our approach is an extension of the fractional Hardy inequality by Mazy'a and Shaposhnikova [54] into the Orlicz framework.

3.2 [B] Boundedness of functions in fractional Orlicz-Sobolev spaces

This paper complements the paper [A] by dealing with the missing case of embeddings of fractional Orlicz-Sobolev spaces $V_d^{s,A}(\mathbb{R}^n)$ into rearrangement-invariant spaces in the *supercritical* growth regime, namely when the Young function A grows so fast near infinity that

$$\int^{\infty} \left(\frac{t}{A(t)}\right)^{\frac{s}{n-s}} dt < \infty. \tag{3.2}$$

The assumption $s \in (0,n) \setminus \mathbb{N}$ is still imposed throughout, and it is shown that such an assumption is in fact necessary, in the sense that the fractional Orlicz-Sobolev space $V_d^{s,A}(\mathbb{R}^n)$ is not embedded into any rearrangement-invariant space if $s > n$. The main contribution of the paper is the proof that condition (3.2) is necessary and sufficient for the space $V_d^{s,A}(\mathbb{R}^n)$ to be embedded into $L^\infty(\mathbb{R}^n)$ and, in turn, also to the space of continuous functions. Nevertheless, the space $L^\infty(\mathbb{R}^n)$ only captures the local properties of functions from $V_d^{s,A}(\mathbb{R}^n)$. In order to describe also the global behavior of these functions, we improve the above-mentioned result by determining the best possible Orlicz target spaces as well as the best possible rearrangement-invariant target spaces for the embeddings of the spaces $V_d^{s,A}(\mathbb{R}^n)$. In this way, we again obtain the expected fractional analogues of the known integer-order results. Our proofs are based on a reduction to the subcritical growth regime, discussed in the paper [A], deriving thus the optimal supercritical embeddings from the subcritical ones.

3.3 [C] A sharp version of the Hörmander multiplier theorem

In this paper, we establish a variant of the multiplier theorem by Calderón and Torchinski [8] (see condition (2.9)) in which the fractional Sobolev space $L^r_s(\mathbb{R}^n)$ is replaced by the fractional Lorentz-Sobolev space $L^{\frac{n}{s},1}_s(\mathbb{R}^n)$. This is the best possible result one can expect within the framework of the fractional Sobolev spaces of the form (2.5). Namely, the symbol of each Fourier multiplier operator necessarily needs to be bounded, and the Lorentz space $L^{\frac{n}{s},1}(\mathbb{R}^n)$ is locally the largest rearrangement-invariant space for which the associated s-th order Sobolev space, defined as in (2.5), is embedded into $L^\infty(\mathbb{R}^n)$, see [17, 64]. Our proofs rely crucially on the above-mentioned sharp Sobolev embedding $L^{\frac{n}{s},1}_s(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ but also require a refinement of other classical tools of Fourier analysis into the Lorentz framework.

Connecting the result of the article [C] to the research conducted in [A] and [B], it would be of interest to investigate whether analogous multiplier theorems can be formulated by making use of Sobolev spaces from the V-scale. One principle obstacle towards achieving this goal is that it is not entirely clear what the correct definition of the fractional Lorentz-Sobolev space should be in this setting.

3.4 [D] On the failure of the Hörmander multiplier theorem in a limiting case

This paper deals with the question whether the version of the Hörmander multiplier theorem obtained by Calderón and Torchinski [8] holds in the limiting case

$$\frac{s}{n} = \left|\frac{1}{p} - \frac{1}{2}\right|. \tag{3.3}$$

We answer this question negatively, and our counterexample in fact works not only for the multiplier theorem formulated via the condition (2.9), but also for its variant involving the Lorentz-Sobolev space $L^{r,q}_s(\mathbb{R}^n)$ with arbitrary parameters $r \in (1,\infty)$ and $q \in [1,\infty]$. Our proof uses the randomization technique in the spirit of [68, Chapter 4], which was further developed in [36] and [F].

It is worth recalling that condition (2.9) is well known *not* to be sufficient for the L^p-boundedness of the associated Fourier multiplier operator if s/n is strictly smaller than $|1/p-1/2|$, see [36, 43, 56, 57, 67]. However, the validity of the theorem in the limiting case (3.3) remained unknown and the corresponding Lorentz-Sobolev variant of this question was mentioned as an open problem in [63]. At the same time, a number of questions regarding the limiting behavior of Fourier multiplier operators still remains unresolved, see the discussion in [63, Appendix A].

3.5 [E] A sharp variant of the Marcinkiewicz theorem with multipliers in Sobolev spaces of Lorentz type

In this paper, we establish a multiparameter variant of the result from [C], which can also be understood as a sharp version of the classical Marcinkiewicz multiplier theorem [53]. We first recall the fractional variant of the Marcinkiewicz theorem established by Carbery and Seeger [9], which asserts that the condition

$$\sup_{(k_1,\dots,k_n)\in\mathbb{Z}^n} \|(I-\partial_1^2)^{\frac{s}{2}}\dots(I-\partial_n^2)^{\frac{s}{2}}[\psi(\xi_1)\dots\psi(\xi_n)m(2^{k_1}\xi_1,\dots,2^{k_n}\xi_n)]\|_{L^r(\mathbb{R}^n)} < \infty \tag{3.4}$$

guarantees the L^p-boundedness of the associated Fourier multiplier operator T_m as long as $1 < p < \infty$ and

$$s > \left|\frac{1}{p} - \frac{1}{2}\right| = \frac{1}{r}.$$

Here, $(I-\partial_i^2)^{\frac{s}{2}}$ stands for the fractional Laplace operator in the i-th variable, given by multiplication by $(1+4\pi^2\xi_i^2)^{\frac{s}{2}}$ on the Fourier transform side. Further, ψ is a smooth function on $\mathbb{R}$ supported in the union of the intervals $[-2,-1/2]$ and $[1/2,2]$. In analogy with the paper [C], we provide an improvement of the Marcinkiewicz multiplier theorem in which the Lebesgue space $L^r(\mathbb{R}^n)$ in condition (3.4) is replaced by a suitable Lorentz space. Notably, the variant of (3.4) with $L^r(\mathbb{R}^n)$ replaced by $L^{\frac{1}{s},1}(\mathbb{R}^n)$ fails to be sufficient for the L^p-boundedness of T_m due to the failure of the corresponding multiparameter Sobolev embedding into $L^\infty(\mathbb{R}^n)$. Nevertheless, we establish a positive result by making use of a slightly smaller Lorentz-type space whose norm is equivalent to the functional

$$\int_0^\infty f^*(t)t^s \log^\beta\left(e+\frac{1}{t}\right)\frac{dt}{t},$$

where $s \in (0,1/2]$ and $\beta > (1-s)n$. We note that the function space that should be expected to come into play in view of a sharp multiparameter Sobolev embedding into $L^\infty(\mathbb{R}^n)$ has the form as above with $\beta = (1-s)n$. It remains an open problem whether the corresponding multiplier theorem holds in this limiting case.

3.6 [F] $L^2 \times L^2 \to L^1$ boundedness criteria

In this paper, we establish a sharp criterion for the $L^2 \times L^2 \to L^1$ boundedness of bilinear Fourier multiplier operators of the form (2.11) with $m = 2$, associated with a kernel K whose Fourier transform has bounded partial derivatives of all orders. Our condition requires the symbol $\widehat{K}$ to belong to the Lebesgue space $L^r(\mathbb{R}^n)$ for some $r < 4$, the L^4-integrability being no longer sufficient. This optimal range should be compared with $r = \infty$ in the well-known Plancherel criterion for the L^2-boundedness of linear Fourier multipliers. As an application, we establish the $L^2 \times L^2 \to L^1$ boundedness of bilinear singular integral operators associated with homogeneous kernels whose restriction to the unit sphere belongs to $L^q(\mathbb{S}^{2n-1})$ for $q > 4/3$. The sufficiency part of our proofs is based on a refinement of the wavelet decomposition technique introduced in [38] while the constructions showing the sharpness of the results are inspired by those from [37].

3.7 [G] Multilinear singular integrals with homogeneous kernels near L^1

In this paper, we obtain the optimal open range of $L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_m}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ bounds for m-linear singular integral operators of the form (2.11) associated with homogeneous kernels whose restriction Ω to the unit sphere $\mathbb{S}^{mn-1}$ belongs to $L^q(\mathbb{S}^{mn-1})$ for some $q > 1$. This improves various earlier results, which required the lower bound $q > 4/3$ in the bilinear setting [F], [42], or $q \geq 2$ in the multilinear setting [39]. Our proofs build on the earlier estimates from [39, 42] and combine them, via complex interpolation, with a new estimate obtained by making use of boundedness properties of shifted maximal and square functions. Bounds for the latter two operators have been applied before in connection with similar problems; see, e.g., Muscalu's alternative proof of the boundedness of Calderón's commutators [59, 60].

While the paper [G] largely settles the question of the strong-type boundedness of the operators (2.11) under the assumptions discussed above, some questions still remain open regarding weak endpoint bounds. In addition, the situation when Ω does not belong to any Lebesgue space $L^q(\mathbb{S}^{mn-1})$ for $q > 1$, but merely to some Orlicz space close to $L^1(\mathbb{S}^{mn-1})$, such as $L\log L(\mathbb{S}^{mn-1})$, is not yet well understood.

3.8 [H] Local bounds for singular Brascamp-Lieb forms with cubical structure

In this paper, we study boundedness properties of the 2^m-linear form

$$\text{p. v.} \int_{\mathbb{R}^{2m}} \prod_{j\in\mathcal{C}} F_j(\Pi_j x) K(\Pi x)\, dx. \tag{3.5}$$

Here, $\mathcal{C}$ is the set of functions $j : \{1,\dots,m\} \to \{0,1\}$, Π_j is the projection from $\mathbb{R}^{2m}$ to $\mathbb{R}^m$ given by

$$\Pi_j(x_1^0,\dots,x_m^0,x_1^1,\dots,x_m^1)^T = (x_1^{j(1)},\dots,x_m^{j(m)})^T$$

and $\Pi : \mathbb{R}^{2m} \to \mathbb{R}^m$ is a generic surjection. We note that the form (3.5) reduces to (2.18) when $m = 2$. We also point out that each function $j \in \mathcal{C}$ can be identified with a corner of the m-dimensional unit cube $[0,1]^m$, explaining thus the name *cubical structure*.

As the main result, we establish the bound

$$\left|\text{p. v.} \int_{\mathbb{R}^{2m}} \prod_{j\in\mathcal{C}} F_j(\Pi_j x) K(\Pi x)\, dx\right| \leq C \prod_{j\in\mathcal{C}} \|F_j\|_{L^{p_j}(\mathbb{R}^m)}, \tag{3.6}$$

where $p_j > 2^{m-1}$ for each $j \in \mathcal{C}$ and $\sum_{j\in\mathcal{C}} 1/p_j = 1$. This result is obtained as a consequence of suitable local bounds involving a new type of a strong maximal function. This operator is a hybrid case between the strong maximal function and the standard Hardy-Littlewood maximal function, featuring the multiparameter structure of the former while preserving the endpoint boundedness properties of the latter.

We note that the lower bound $p_j > 2^{m-1}$ is optimal, in the sense that the same condition with 2^{m-1} replaced by any lower number is no longer sufficient for the validity of the estimate (3.6). However, the optimal range of exponents for which (3.6) holds remains unknown, as it may be possible to lower some of the exponents p_j at the expense of adding additional constraints on the remaining exponents.

3.9 [I] Norm-variation of triple ergodic averages for commuting transformations

In this paper, we establish the following norm-variation estimate for triple ergodic averages with respect to three commuting transformations:

$$\sum_{j=1}^{J} \|M_{n_j}(f,g,h) - M_{n_{j-1}}(f,g,h)\|_{L^2(X)}^r \le C\|f\|_{L^6(X)}^r \|g\|_{L^6(X)}^r \|h\|_{L^6(X)}^r. \tag{3.7}$$

Here, $n_0 < n_1 < \cdots < n_J$ is a sequence of positive integers and $r > 4$. By a standard transference, this result reduces to estimating the six-linear singular Brascamp-Lieb form

$$\begin{aligned} \text{p. v.} \int_{\mathbb{R}^5} & f_1(x_3^0, x_1, x_2) f_2(x_3^1, x_1, x_2) f_3(x_0, x_3^0, x_2) f_4(x_0, x_3^1, x_2) f_5(x_0, x_1, x_3^0) f_6(x_0, x_1, x_3^1) \\ & \times K(x_3^0 - x_0 - x_1 - x_2, x_3^1 - x_0 - x_1 - x_2)\, dx_0 dx_1 dx_2 dx_3^0 dx_3^1. \end{aligned} \tag{3.8}$$

Unlike its quadrilinear variant (2.17), this form no longer has a cubical structure and any bounds for it appear to be out of reach of current techniques. This fact prevents us from proving the estimate (3.7) with $r = 2$, the conjectured optimal bound. However, when the kernel K consists of only finitely many scales then we are able to estimate the form (3.8), up to a certain controlled loss in the number of scales, by the eight-linear form as in (3.5) with $m = 3$, leading to the bound (3.7) with $r > 4$. We also note that it is still an open problem whether any r-variation estimates with $r < \infty$ hold for four or more commuting transformations.

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