Review of the Habilitation Thesis

Operators of harmonic analysis, related function spaces and applications

by Dr. Lenka Slavíková

The candidate, Dr. Lenka Slavíková, has submitted an impressive collection of papers as her habilitation. To be more precise, the habilitation thesis consists of a preface, an introduction, summaries of the attached papers and the papers itself.

The first major subject in the collection of papers is given by *Fractional Sobolev spaces and their generalizations.* It consists of the two papers A and B.
I'm rather impressed by these two papers. Whereas Orlicz-Sobolev spaces are well understood in many directions, the theory of fractional Orlicz-Sobolev spaces is almost at its beginning. Only very few papers are dealing with this subject. Here one of the problems is an appropriate definition. There are several reasonable possibilities. From the modern theory of function spaces one knows the Slobodeckii spaces as well as the Bessel potential spaces and their homogeneous variants. In connection with both families also the name Sobolev spaces of fractional order of smoothness is common. For both scales there exist a great number of equivalent characterizations, e.g., by differences, by oszillations, by atoms and wavelets, by approximation, by interpolation and in a Littlewood-Paley form. Dr. Slavikova and her co-authers decided for Slobodeckii spaces characterized by differences as the point of departure. In principal this way has been used earlier by Bonder, Salort (2019), Bahrouni, Ounaies, Tavares (2020), Azroul, Benkirane, Srati (2020), Bahrouni, Ounaies (2020) and De Napoli, Bonder, Salort (2021). Let $0 < s < 1$ and let A be a Young function. All these contributions use as the basic homogeneous ingredient the term

$$|u|_{s,A,\mathbb{R}^n} := \inf \left\{ \lambda > 0 : \quad \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A\Big(\frac{|u(x) - u(y)|}{\lambda\, |x-y|^s}\Big) \frac{dx\, dy}{|x-y|^n} \leq 1 \right\} .$$

Traditionally this is complemented by adding $\|u\|_{L^A(\mathbb{R}^n)}$ leading to the so-called inhomogeneous spaces. This is done in the quoted references. But the authors prefer to work with the following definition

$$V_d^{s,A}(\mathbb{R}^n) := \{u \in \mathcal{M}(\mathbb{R}^n) : |u|_{s,A,\mathbb{R}^n} < \infty\,, \ |\{x \in \mathbb{R}^n : \ |u(x)| > t\}| < \infty \quad \text{for all } t > 0\}\,.$$

Clearly, these spaces $V_d^{s,A}(\mathbb{R}^n)$ are larger than the classical inhomogeneous spaces and they are easier to handle than purely homogeneous spaces where one has to deal with equivalence classes modulo constants (polynomials of a certain degree if $s \geq 1$). In some sense, the condition $|\{x \in \mathbb{R}^n : \ |u(x)| > t\}| < \infty \quad \text{for all } t > 0$ supplements long term investigations within the theory of homogeneous distribution spaces on how to define that a distribution should vanish near infinity, we refer to works of Bourdaud (1988), Bahouri, Chemin, Danchin (2011) and Cobbs (2023).

If one is dealing with new classes of functions, then for applications as well as for a better understanding of these new classes, embeddings into simpler and better known spaces are of importance. Under minimal restrictions, Dr. Slaviková and her co-authors are able to compute the optimal target space within Orlicz spaces and the optimal target space within rearrangement invariant target spaces. This improves all previously known results for fractional Orlicz-Sobolev spaces. It has pushed forward the general theory of fractional Orlicz-Sobolev spaces.

The second major subject is given by *Fourier multiplier operators.* It consists of the three papers C,D, and E.

Fourier multipliers are an indispensible tool in various branches of mathematics. Central results are the theorems of Marcinkiewicz, Mikhlin and Hörmander. The first two papers deal with interesting limiting situations of the Theorem of Hörmander. Let ϕ be a smooth function supported in an annulus in $\mathbb{R}^n$ and such that

$$\sum_{k=-\infty}^{\infty} \phi(2^k\xi) = 1 \qquad \text{for all} \quad \xi \neq 0.$$

By $L^r_s(\mathbb{R}^n)$ we denote the Bessel potential space with smoothness s and integrability r. Then

$$\sup_{k\in\mathbb{Z}} \|\phi(\xi)m(2^k\xi)\|_{L^r_s(\mathbb{R}^n)} < \infty$$

and

$$\frac{s}{n} > \left|\frac{1}{p} - \frac{1}{2}\right| = \frac{1}{r}$$

guarantee that m is a multiplier for $L^p(\mathbb{R}^n)$, see Calderón, Torchinski (1977). Slavíková and Grafakos showed that replacing $L^r_s(\mathbb{R}^n)$ by $L^{n/s,1}_s(\mathbb{R}^n)$, the Bessel potential space built on the Lorentz space $L^{n/s,1}(\mathbb{R}^n)$, then $\frac{s}{n} = \left|\frac{1}{p} - \frac{1}{2}\right| = \frac{1}{r}$ is sufficient for m to be a multiplier for $L^p(\mathbb{R}^n)$. Beside Seegers contribution for H^1 in 1988 this is the only positive result known in the limiting situation. These results have been supplemented by Slavíková herself. She proved that $\frac{s}{n} = \left|\frac{1}{p} - \frac{1}{2}\right| = \frac{1}{r}$ and

$$\sup_{k\in\mathbb{Z}} \|\phi(\xi)m(2^k\xi)\|_{L^r_s(\mathbb{R}^n)} < \infty$$

is not enough for m to be a multiplier for $L^p(\mathbb{R}^n)$. Both papers together represent a nice contribution to the theory of Fourier multipliers. In a transparenht way it improves the understanding of the Hörmander Theorem.

In the paper with Grafakos and Mastylo, Slavíková proved an interesting multiparameter variant of the limiting result obtained in paper [C].

The remaining three major subjects are given by *Singular integral operators: from linear to multilinear theory, Bilinear Hilbert transform and beyond* and *Ergodic-theoretic applications.* They consist of the four papers F, G, H and I.

In these last chapters Slavíková switches to investigations of the boundedness of multilinear operators. In the paper with Grafakos and He sufficient conditions for a class of bilinear multiplier operators are derived which are almost necessary.

In the paper with Dosidis multilinear singular integrals of the type

$$T^m_\Omega(f_1, \dots, f_m)(x) := pv \int_{\mathbb{R}^{mn}} K(x - y_1, \dots, x - y_m) \prod_{i=1}^{m} f_i(y_i)\, dy$$

are considered, where

$$K(x) = \frac{\Omega(x')}{|x|^{mn}}, \qquad x \in \mathbb{R}^{mn} \setminus \{0\}, \quad x' = \frac{x}{|x|}$$

$\Omega \in L^q(\mathbb{S}^{mn-1})$ for some $q > 1$ and $\int_{\mathbb{S}^{mn-1}} \Omega(\theta)\, d\sigma(\theta) = 0$. Let $1 < p_1, \dots, p_m < \infty$, $1/p := \sum_{i=1}^m 1/p_i$ be such that $(1/p_1, \dots, 1/p_m) \in \mathcal{H}_q$, where $\mathcal{H}_q$ is the set of all vectors $(1/p_1, \dots, 1/p_m)$ such that

$$\Big(\sum_{i=1}^m \frac{\alpha_i}{p_i}\Big) + \frac{|\alpha| - 1}{q} < |\alpha| \qquad \text{for all} \quad \alpha \in \{0,1\}^m .$$

Then T_Ω^m maps $L^{p_1}(\mathbb{R}^n) \times \ldots \times L^{p_1}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$. It is known that the set $\mathcal{H}_q$ is the largest open set for which such a result is true. Partial negative results, also obtained by Dosidis and Slavíková, are known even for the boundary $\partial\mathcal{H}_q$.
Bilinear and multilinear singular integrals have found more and more applications. In my opinion they are of increasing importance, e.g., for Kato-Ponce inequalities, to mention one which is of interest for me personally.

In her collaboration with Durcik and Thiele, Slavíková investigates singular Brascamp-Lieb forms and some applications. A singular Brascamp-Lieb form with cubical structure is a principal value integral

$$pv \int_{\mathbb{R}^{2m}} \prod_{j \in C} F_j(\Pi_j x)\, K(\Pi x)\, dx ,$$

where Π_j, Π are linear surjections and K is a Calderón-Zygmund kernel. The main result consists in a boundedness assertion which improves earlier work of Durcik and Thiele (2020).

Let me summarize my opinion. The Introduction of the thesis is short but well written. The short subsections 3.1 - 3.9 reflect the main contents of the collected papers A-I. The value of the collected papers is enormous. Almost all have appeared in very good journals. More important, all contain serious contributions to different fields of mathematics. These papers bring new scientific knowledge in at least two directions, new results and new or improved methods for proving them. The general list of publications of Dr. Slavíková shows that she is a productive talented young mathematician with a wide knowledge.
The results of the computer based analysis of the habilitation thesis show no significant overlap with other works.

The development of the research of Dr. Slavíková is quite interesting. It starts with function spaces in Prague, continues with Fourier multipliers and singular integral operators in Missouri and with Brascamp-Lieb inequalities in Bonn. In all her places of work she concentrated on the local specialities which requires a high degree of ability and flexibility.

This Habilitation Thesis is most probably the best I had to review up to now. In my opinion it meets all standard requirements. Therefore I **strongly recommend** the acceptance of this thesis.

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