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Opinion on Mykhaylo Tyomkyn: Problems in Extremal Combinatorics, Habilitation Thesis.

This is an evaluation of the Habilitation Thesis of Mykhaylo Tyomkyn. The material of the thesis covers twelve papers of the candidate, written between 2016 and 2022. The topics of the papers is Extremal Combinatorics. Each of the papers is independent from the others. However, as in this field it is standard, the problems, conjectures and results have a center theme, and though there are a wide variety of tools, apriori one cannot predict in advance which one could be used for which type of problems. Hence, it is actually a focused research in the broad topics of Extremal Combinatorics.

As in a review it is not expected to list all the main results of the candidate, below I just sample some of the results that I liked, this does not necessarily mean that those are the best results of the thesis.

Chapter 2 is related to hypergraph Turán problems. For an r-uniform hypergraph H, its extremal function $ex(n, H)$ is the maximum number of hyperedges of an H-free n-vertex hypergraph. The classical area in extremal combinatorics is to determine this function for every H. While it is reasonable well-understood for the case $r = 2$, the $r > 2$ case seems to be more complex, and our understanding is limited. An efficient tool is the estimation of the Lagrangian of a hypergraph, which could be used to upper bound $ex(n, H)$. However, even the method of Langrangian is not well-understood. In 1989 Frankl and Füredi made a conjecture on the maximum Lagrangian of an r-graph on m edges. The main result of this chapter is the confirmation of the Frankl-Füredi Conjecture for 'most' values of m for every $r \geq 4$. The proof, among others uses the Kruskal-Katona theorem.

Section 3.1 investigates an extension of the famous Erdős-Hajnal conjecture, which claims that for every graph F there is a constant c such that if an n-vertex graph does not contain an induced copy of F, then it has a homogenous set (independent set or a complete subgraph) with n^c vertices. Despite lots of efforts, the Erdős-Hajnal conjecture is still open.

Theorem 3.1.1 is an interesting extension of the problem, which is the following:

For every 3-uniform hypergraph F and $\eta > 0$ there is a c such that every n-vertex 3-uniform hypergraph containing no induced copy of F has an η-homogenous set with $c \log n$ vertices. Here a set is η-homogenous, if the density of its edge set is at most η or at least $1 - \eta$. This result is best possible, upto the value of the constant c.

The proof builds on the methods used in other papers, in particular it uses the hypergraph regularity lemma. Additionally, the results has some consequences on hypergraph Ramsey theory.

Section 3.6 studies a variant of the Brown-Erdős-Sós conjecture. About 50 years ago they raised the following question:

Assume that H is an n-vertex, r-uniform linear hypergraph, where linear means that any pair of edges intersects in at most one vertex. Furthermore assume, that for a given v and k, every set of v vertices spans at most k hyperedges. The general question of Brown-Erdős-Sós is the following: At most how many hyperedges such hypergraph can have. The most famous case is when $r = 3, v = 6, k = 3$, which was solved by Ruzsa and Szemerédi, and it has many applications. However, many cases left open, and even now, it is considered as one of the central questions in combinatorics.

A recent related problem raised by Conlon and Nenadov is the following:

Prove that for every $r, k \geq 3$ and $c \geq 2$ and n sufficiently large, if G is a complete linear r-uniform hypergraph, then in every c-colouring of its edges there are k hyperedges of the same colour, which are spanned by at most $(r-2)k+3$ vertices. Here complete linear means that every pair of vertices belong to exactly one hyperedge.

Theorem 3.6.2 of the thesis solves this problem when r is sufficiently large. Theorem 3.6.3 let r be as small as 4, in case only 2 colors are used.

Chapter 4 is about a more modern topics in extremal graph theory, about the so-called inducibility: Given a graph F, what is the maximum number of induced copies of F in an n-vertex graph? This question is interesting when F contains an edge but it is not a clique. The general theory of inducibility is not well-understood. Even, the following question of Pippenger and Golumbic from 50 years ago is not solved: Is it true that maximum density of the induced k-cycle C_k is $k!/(k^k-k)$, when $k \geq 3$? When $k=3$, then we have C_3 as a clique, and for $k=4$ we have C_4, for which the extremal graph (i.e. the graph containing the maximum number of induced C_4's) is the complete bipartite graph. The $k=5$ case was settled by Balogh, Hu, Lidicky and Pfender, using among others the method of flag algebras. Theorem 4.1.3 gets close solving the conjecture, the provided upper bound is about a multiplicative factor 3 away from it. Such bound was out of reach using previous methods, in particular flag algebras.

Chapter 7 is another modern part of extremal combinatorics, though it has connection to classical results. The main theme is that assume that the edge set of the complete graph is properly colored, i.e., each color class forms a matching. Then what can be said about rainbow patterns, i.e. about subgraphs, whose edges are all colored using different colors. This section investigates that under what conditions one can find repeated copies of the same structure. Two copies of a graph H in a colouring of K_n are colour-isomorphic, if there exists an isomorphism between them preserving the colours. To be more precise, define $f_k(n,H)$ to be the smallest integer C such that there is a proper edge-coloring of the complete graph K_n with C colors containing no k vertex-disjoint colour-isomorphic copies of H.

Theorem 7.1.2 is a general characterization of $f_2(n,H)$: It claims that this function is quadratic if H is a forest, otherwise its growth rate is at most n^{c_H} for some constant $c_H < 2$, and if H is not bipartite then $f_2(n,H) \leq n+1$.

Theorem 7.1.2 claims that for arbitrary non-forest H there is a constant k that $f_k(n,H)$ is at most linear. The proof, somewhat surprisingly, uses the random algebraic method, one of the modern deep tools of combinatorics.

Chapter 8 contributes new results toward the theory of weak-saturation. The concept of weak saturation was introduced by Bollobás more than 50 years ago. A (hyper)graph G is weakly H-saturated, if all there is an ordering of the pair of vertices of the complement of G that adding them as an edge, one by one, to G in this order, in each step we create a new copy of H. The function $wsat_r(n,H)$ is the minimum number of hyperedges that an n vertex r-uniform weakly H-saturated hypergraph must have. The extremal question is that what is the minimum number of edges of an n vertex weakly H-saturated graph. This innocent looking problem was open for more than 10 years, and deep algebraic tools, including exterior algebras was used, to settle the case when H is a complete graph.

One of the main results of Chapter 8 is Theorem 8.1.1., which determines the minimum number of edges needed, when both G and H are complete multipartite graphs (and the process is slightly different). The proof of Theorem 8.1.1 combines exterior algebra techniques with a new ingredient: the use of the coluorful exterior algebra

Section 8.6 considers the non-weak version of the saturation problem, which was also introduced by Bollobás in the same paper. An r-uniform hypergraph G is H-saturated (where H is also an r-uniform hypergraph), if $H \not\subset G$ and for every f r-tuple of the vertices of G, either f is a hyperedge of G, or adding f to G creates an H. The function $sat_r(n,H)$ is the minimum number of hyperedges that an n vertex r-uniform H-saturated hypergraph must have. While it was proved in the 70s that for $r=2$ and H to be a complete graph the sat and $wsat$ functions are equal, the general case was wide open. Determining the asymptotic of $wsat$ became famous as "Tuza's" conjecture (here I do not state the precise statement). Theorem 8.6.1

solves Tuza's problem. The proof is again deep and highly technical.

Overall, the collection of the results are impressive, they are sufficiently broad, and the depth of the proofs shows the competence of the author.

I was asked to come up with a clear statement as to whether the thesis meets the standard requirements for a habilitation thesis.

The quality of the thesis exceeds the standard requirements for a habilitation thesis.

I was asked to come up with an explicit recommendation or non-recommendation of work for further progress in the habilitation procedure.

I strongly recommend for further progress in the habilitation procedure.

I was asked to check "Turnitin" report about overlapping of the Thesis with other papers. I confirm, that the only significant overlaps are coming from the papers authored by Tyomkyn. All other overlaps are minor, and the result of having similar introduction to papers of similar flavors, but all the results are new (in case not, no one is aware of them, and any coincidence is incidental.) Hence, I am confident that all the results claimed in the thesis are new and due to (upto joint work with some collaborators) of Tyomkyn.

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