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Report on Habilitation of Dr. Mykhaylo Tyomkyn May 30, 2024

This report is an academic evaluation of the Habilitation thesis submitted by Dr. Mykhaylo Tyomkyn. The evaluator is Jacques Verstraete, who is a full professor in the Department of Mathematics at UCSD, specializing in combinatorics. The automatic plagiarism and originality check (Turnitin system report) showed no scientific error related to copying and it is clear that this is original work that only overlaps with existing literature. My overall assessment is that the thesis easily **meets the standard requirements** for a habilitation thesis, and I **strongly recommend the work for further progress** in the habilitation procedure.

Dr. Tyomkyn is an expert in mathematics, specializing in the area of extremal and probabilistic combinatorics and related areas. The submitted habilitation is an extensive compilation of some of the strongest research results, including a number of strong single-author works by Dr. Tyomkyn. These research results have been published in highly recognized mathematical journals, both in combinatorics and in general mathematics, including *Journal of the London Mathematical Society*, *Bulletin of the London Mathematical Society*, and *Proceedings of the American Mathematical Society*, as well as top journals in combinatorics including *Journal of Combinatorial Theory, Series B* and *Combinatorica*. The following material is a short appraisal of the main research results in this habilitation. Chapters 5, 7 and 8 will not be completely reviewed here, although these chapters do contain strong original results worthy of an excellent habilitation.

The first main result, discussed in Chapter 2 an d published in *Journal of the London Mathematical Society*, is to do with *hypergraph Langrangians*. For a given hypergraph H, one defines the Lagrangian (following Motzkin and Strauss) as

$$\lambda(H) = \max\{\sum_{e\in E(H)}\prod_{v\in e} y_v : \sum_{v\in V(H)} y(v) = 1 \wedge \forall v \in V(H), y_v \geq 0\}.$$

In other words, we put non-negative weights on the vertices and each edge of the hypergraph receives a weight which is the product of the vertex weights in that edge. The Lagrangian of a complete r-uniform hypergraph K_t^r on t vertices is $1/t^r\binom{t}{r}$. Frankl and Füredi conjectured that no r-uniform hypergraph with t vertices and $\binom{t}{r}$ edges has a larger Lagrangian, and more generally, that the Lagrangian of an r-uniform hypergraph with m edges is maximized by the hypergraph consisting of the first m sets in the colexicographic ordering of the r-element subsets of the positive integers. This is perhaps one of the most natural and important conjectures in the area. A number of notable prior research works studied this conjecture, obtaining partial results, notably due to Talbot for $r = 3$, but the strongest result to date is the result of Tyomkyn stated in this habilitation as Theorem 2.1.2, which solves the conjecture for each $r \geq 4$ for almost all values of $m \leq \binom{t}{r}$. This includes the breakthrough that for $m = \binom{t}{r}$ the conjecture is true, as well determining the size of the asymptotically largest blowup amongst all r-graphs of size m, which may indeed have applications in extremal combinatorics. These are impressive and original results, and likely to have a significant impact in the area.

The next chapter follows themes from Ramsey Theory. As is well-known from hypergraph Ramsey theory, every r-uniform n-vertex hypergraph contains a clique or independent set of size $\Theta((\log n)^{1/(r-1)}$ – we call these *homogeneous* subsets. The case $r = 2$ is the celebrated early application of the probabilistic method to diagonal Ramsey numbers by Erdős. A notoriously difficult conjecture of Erdős and Hajnal states that if H is an n-vertex graph that does not have an induced subgraph isomorphic to a given graph F, then in fact H contains a homogeneous subset of size $n^{\Theta(1)}$. This conjecture is generally wide open (some salient cases have only recently been solved in major works of structural graph theory), and has been studied very intensively in the last few decades. It is natural to consider the analog of this conjecture for hypergraphs: if an r-uniform n-vertex hypergraph H does not contain an induced subgraph isomorphic to a given hypergraph F, how large a homogeneous subset should H contain – we say H is *non-universal*? A subset X of H is called η*-homogeneous* if the number of edges induced by X is at least $(1-\eta)\binom{|X|}{r}$ or at most $\eta\binom{|X|}{r}$, so 0-homogeneous and homogeneous coincide. Tyomkyn, with coauthors Shapira and Amir, proved that for any $\eta > 0$, any n-vertex non-universal 3-uniform hypergraph H has an η-homogeneous set of size $\Omega_\eta(\log n)$, using hypergraph regularity and a number of new ideas. This bound is substantially larger than the bound of order $\sqrt{\log n}$ on the size of a 0-homogeneous set. An example of Rödl shows this is tight up to the implicit constant factor. This strong work was published in *Journal of Combinatorial Theory Series B*. Amir, Shapira and Tyomkyn also generalize Rödl's construction to an r-uniform n-vertex non-universal hypergraph where every η-homogeneous subset has size of order at most $(\log n)^{1/(r-2)}$, which may suggest (as conjectured by Tyomkyn) that non-universal r-uniform n-vertex hypergraphs contain η-homogeneous subsets of size at least about $(\log n)^{1/(r-2)}$. These results are of central important in hypergraph Ramsey theory, and the conjecture for r-uniform hypergraph is highly compelling.

Chapter 4 of the habilitation considers the *inducibility* of graphs, based on the publications of Hefetz and Tyomkyn, and Alon, Hefetz, Krivelevich and Tyomkyn in *Journal of Combinatorial Theory Series B* and *Combinatorics , Probability and Computing*. For a given graph F, the *inducibility of* F, denoted $\mathrm{ind}(F)$, is defined to be the asymptotic maximum proportion of subsets of an n-vertex graph of size $|V(F)|$ which induce F. The value of $\mathrm{ind}(F)$ appears to be very difficult to determine for most graphs F. The modern approach uses the method of *flag algebras* introduced by Razborov. An important conjecture in the area is

$$\mathrm{ind}(C_k) = \frac{k!}{k^k - k}$$

where C_k is the cycle of length k. If one takes the blowup of a cycle C_k, with n/k vertices in each part, one obtains the lower bound

$$\mathrm{ind}(C_k) \geq \frac{(n/k)^k}{\binom{n}{k}} \geq \frac{(n/k)^k}{n^k/k!} = \frac{k!}{k^k}.$$

This is not quite the bound in the conjecture, but the bound in the conjecture is obtained by iteratively placing a blowup of C_5 in each independent set in the previous construction. This motivates the above conjecture. The special case $k = 5$ has received considerable attention, and was solved by Balogh, Hu, Lidický and Pfender, using the method of flag algebras and the stability method. In fact the extreme examples are indeed iterated blowups of 5-cycles. The conjecture remains open, in fact even $\mathrm{ind}(C_k) = (1 + o_k(1))k!/k^k$ is not known as $k \to \infty$. The current state of the art by Tyomkyn is exposited in this chapter, where it is shown $\mathrm{ind}(C_k) \le 128ek!/81k^k$. This is a solid step towards the inducibility conjecture for cycles. In the same chapter, the inducibility problem for general graphs is considered, and in particular an attractive conjecture is made on the limit superior of $\mathrm{ind}(F)$ over all graphs F which are neither complete nor empty: the limit value should be $1/e$. This is motivated by the simple observation that the probability that if F consists of a single edge on k vertices, then for large n the expected number of induced copies of F in the random graph $G_{n,p}$ with $p = 1/\binom{k}{2}$ is asymptotic to $(1/e + o_k(1))\binom{n}{k}$ as $k \to \infty$. This is a special case of the so-called Large Inducibility Conjecture. Towards this conjecture, it is shown in this chapter that $\limsup_F \mathrm{ind}(F)$ is indeed bounded away from 1, and various other bounds are given when F is further away from complete or empty.

Chapter 6 discusses a result on monochromatic circuits in two-edge-colorings of K_n. Specifically, a *circuit* in a graph is a sequence $(v_0, v_1, \dots, v_{k-1}, v_0)$ of vertices such that $\{v_i, v_{i+1}\}$ is an edge of the graph (subscripts mod k) and such that no edge $\{v_i, v_{i+1}\}$ is repeated (though vertices may be repeated). The *length* of the circuit is the number of edges in the circuit. In this chapter, Tyomkyn and coauthors prove that in every 2-coloring of K_n, there is a monochromatic circuit of length at least $\frac{2}{9}n^2 + O(n^{3/2})$, which is asymptotically tight for the simple reason that if we take the first color to be the complete bipartite graph $K_{n/3,2n/3}$ (with n divisible by 3) then every monochromatic circuit has size at most $2n^2/9$. This strong result solves the problem of long monochromatic circuits in two-edge-colorings of K_n, and leaves a number of open questions regarding colorings with more colors (results can be deduced for four-edge-colorings from this chapter). This was published in *Journal of Graph Theory* with Conlon.

This reviewer has read the habilitation of Dr. Tyomkyn in full, and reviewed the results in this appraisal. The results are very strong, new and original, and published in the highest quality journals. As stated above, my overall assessment is that the thesis meets the standard requirements for a habilitation, and I strongly recommend the work for further progress in the habilitation procedure.

Sincerely yours,

Jacques Verstraete