Charles University in Prague

Faculty of Social Sciences

Institute of Economic Studies

MASTER THESIS

Behavioural Breaks in the Heterogeneous Agent Model

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Declaration of Authorship

The author hereby declares that he compiled this thesis independently, using only the listed resources and literature.

The author hereby declares that this thesis has not been used to obtain another university degree.

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Prague, May 19, 2011

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Acknowledgments

Abstract

This thesis merges the fields of Heterogeneous Agent Models (HAMs) and Behavioural Finance in order to bridge the main deficiencies of both approaches and to examine whether they can complement one another. Our approach suggests an alternative tool for examining HAM price dynamics and brings an original way of dealing with problematic empirical validation. First, we present the original model and discuss various extensions and attempts at empirical estimation. Next, we develop a unique benchmark dataset, covering five particularly turbulent U.S. stock market periods, and reveal an interesting pattern in this data. The main body applies a numerical analysis of the HAM extended with the selected Behavioural Finance findings: herding, overconfidence, and market sentiment. Using Wolfram Mathematica we perform Monte Carlo simulations of a developed algorithm. We show that the selected findings can be well modelled via the HAM and that they extend the original HAM considerably. Various HAM modifications lead to significantly different results and HAM is also able to partially replicate price behaviour during turbulent stock market periods.

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Abstrakt

Tato práce propojuje koncept modelů s heterogenními agenty (HAMů) s oblastí behaviorálních financí za účelem překlenutí hlavních nedostatků obou přístupů a ověření, zda se tyto mohou vzájemně vhodně doplnit. Náš přístup přináší alternativní metodu zkoumání dynamiky modelu a naznačuje, jak se lze originálně vypořádat s problematickou empirickou validací. Na začátku práce uvádíme původní model a diskutujeme jeho rozličné modifikace a snahy o empirické odhady. Dále představujeme unikátní dataset pokrývající pět značně neklidných období z historie akciových trhů v USA, ve kterém objevujeme zajímavé pravidelnosti v datech. Těžiště práce leží v numerické analýze modelu, který rozšiřujeme o vybrané poznatky z oblasti behaviorálních financí: stádní chování, nadmíru sebedůvěry a vliv tržního sentimentu. S použitím programu Wolfram Mathematica provádíme Monte Carlo simulace námi vyvinutého algoritmu. Ukazujeme, že pomocí HAMů lze vybrané poznatky velmi dobře modelovat a že tyto značně obohatí původní strukturu modelu. Rozličné modifikace modelu vedou k signifikantně rozdílným výsledkům a model je rovněž schopen částečně replikovat cenové výkyvy během neklidných období akciových trhů.

Bibliografická evidence

Sazba textu provedena v LaTeX 2_ε s použitím IES šablony od Tomáše Havránka.

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Acronyms

ABS	Adaptive Belief System
AR	Autoregressive
ARED	Adaptive Rational Equilibrium Dynamics
ARIMA	Autoregressive Integrated Moving Average
BF	Behavioural Finance
BP	Behavioural Parameter
BPD	Break Point Date
CARA	Constant Absolute Risk Aversion
CBS	Continuous Belief System
CD	Canadian Dollar
CRRA	Constant Relative Risk Aversion
DAX	Deutscher Aktien IndeX (German stock index)
DJIA	Dow Jones Industrial Average
DM	Deutsche Mark (former German currency)
EMH	Efficient Market Hypothesis
FF	French Franc (former French currency)
GARCH	Generalised Autoregressive Conditional Heteroskedasticity
GBP	British Pound (£)
HA	Heterogeneous Agent
HAM	Heterogeneous Agent Model
IID	Independent Identically Distributed
JY	Japanese Yen (¥)
LTL	Large Type Limit
NASDAQ	National Associaton of Securities Dealers Automated Quotations Composite Index

NYSE   New York Stock Exchange

PDF   Probability Distribution Function

RW   Random Walk

ST   Smart Trader

STS   Smart Traders and Sentiment

STAR   Smooth Transition Autoregressive

S&P500   Standard & Poor's 500 Index

UK   United Kingdom of Great Britain and Northern Ireland

USD   United States Dollar ($)

U.S.   United States

VAR   Vector Autoregression

VECM   Vector Error Correction Model

WOA   Worst Out Algorithm

Master Thesis Proposal

Author	Bc. Jiří Kukačka
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Proposed topic	Heterogeneous Agent Models with Behavioral Finance Findings

Topic characteristics   In recent academic finance, the Representative Agent Approach and the Efficient Market Hypothesis, which dominated the field in the past, are being replaced by more realistic agent based computational approaches. This thesis will focus on a subset of models of this kind, the Heterogeneous Agent Models, HAMs henceforth. The crucial idea of all HAMs is the abandonment of agents' rationality towards bounded, limited rationality. HAMs employ interacting groups of boundedly rational heterogeneous agents using simple but reasonable heuristics or rules of thumb to model the financial world and this methodology appears useful in explaining observed stylized facts in financial time series (e.g. excess trading volume, heavy tails, volatility clustering and others), as well as in fitting the real financial data.

Moreover, Behavioral Finance, BF henceforth, is another recently developing field, which deserves attention. BF argues that some phenomena observed in the financial world can be better explained using models with agents which are not fully rational. Additionally, BF suggests to employ the insight of behavioural sciences such as psychology and sociology into finance. There are many biases from rational behaviour well described by BF: loss aversion, overconfidence, herd behaviour and many others. However, there is still any relevant economic theory summarizing and explaining the most important conclusions of BF missing.

In our opinion, both approaches aim at the same point and can complement one another, which is also referred e.g. in LeBaron (2005). Therefore we would like to take the advantage of both approaches integrated together and explore

the possibilities of enriching the HAM methodology by BF findings. Considering HAM methodology, we plan to follow the Brock & Hommes (1998) approach and its extensions. From many of behavioural biases we plan to employ e.g. herd behaviour or market sentiment as these we find especially important during shocks and crisis periods.

Another area of our interest might be the HAMs behaviour during shock events. For this purpose, the high-frequency data analysis seems to be the optimal research method. The research question related might be whether HAMs are suitable to explain and model the price behaviour during these specific periods, where other known model approaches generally fail.

Hypotheses

1. HAMs fit the real financial data better than other competing approaches (e.g. ARIMA, GARCH 'family' etc.).
2. HAMs are able to explain stylized facts observed in real financial time series.
3. Selected BF findings can be well modelled via HAMs.
4. Incorporation of some of BF finding enriches the original HAM reasonably––it makes a good sense to consider BF in HA modelling.
5. Different HAM modifications lead to statistically significant differences in results/conclusions.
6. HAMs are able to explain/model the price behaviour during shock periods.

Methodology

1. Heterogeneous Agent Modelling based on Brock & Hommes (1998) approach and its extension.
   (a) Theoretical HAM design;
   (b) Model estimation and calibration using real financial data sets;
   (c) Computational simulations using Mathematica, MATLAB.
2. Time series analysis using standard approaches (e.g. ARIMA, GARCH 'family' etc.).
3. HAMs dynamics analysis during shock periods using high-frequency financial data.

4. Quantitative comparison and description of various HAM versions applying statistics and econometrics tools (distribution function and descriptive statistics, Hurst exponent, bifurcation diagram...).

5. Qualitative generalization of results.

Outline

1. Introduction
   (a) HAM Introduction and Literature Review
   (b) Main BF Findings

2. The HA Modelling
   (a) The General Setting
   (b) Incorporation of the BF Findings
   (c) Model Estimation and Calibration/Computational Simulation
   (d) Comparison, Description and Discussion of the Results

3. Conclusion
   (a) Interpretation and Generalization of the Results
   (b) Does It Make a Good Sense to Enrich HAMs by BF Findings?
   (c) Suggestions for Future Research

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Author   Supervisor

Chapter 1

Introduction

"It were not best that we should all think alike; it is difference of opinion that makes horse races."

Mark Twain, American author (1835–1910)

In recent academic financial literature, the representative agent approach and the Efficient Market Hypothesis (Fama 1970) together with the Rational Expectations Hypothesis (Muth 1961, Lucas 1972), which have dominated the field in the past, are being replaced by more realistic agent based computational approaches. We daresay that perhaps no individual could have ever believed that all people are alike and posses all available information as the above mentioned concepts state. On the other hand, if we open any economic textbook, theories based on such assumptions still largely dominate. These extremely simplified approaches need not necessarily be wrong and they certainly have their irreplaceable role in the system of economic science, serving as an optimal benchmark and defining theoretical boundaries. However, they surely do not reflect a matter of fact.

Indeed, Keynes (1936) points out that economic agents do not posses sufficient knowledge to form correct mathematical expectations and in his famous 'beauty contest' example he *"undoubtedly presupposes heterogeneous and hence non-rational expectations"* (Lux 2010, pg. 13). The No Trade Theorem by Milgrom & Stokey (1982) suggests that if the Efficient Market Hypothesis (EMH) was right, there would be no trade which is, obviously, at extreme odds with enormous trading volumes observed daily in financial markets. More recently, Chiarella & He (2003) mention a growing dissatisfaction with the Representa-

tive Agent Paradigm, and LeBaron (2005) states that at the end of the 20$^{\text{th}}$ century finance witnessed a revolution, in which he talks about the advent of the EMH, the Capital Asset Pricing Model, and the Black–Scholes Options Pricing Formula. De Grauwe & Grimaldi (2006) argue that the traditional rational expectations efficient market model has failed empirically as it requires agents to solve a mathematical problem to which mathematicians have as yet been unable to give a general solution. They also pointedly add that rational expectations models require agents to posses almost 'God-like capacities'. Hommes (2006) notices an important paradigm shift from a representative, rational agent approach towards a behavioural, agent-based approach, and more recently Frijns *et al.* (2010) mention a demise of the EMH.

The development of economic science naturally goes along with technological progress which inherently makes new scientific approaches possible. Not only in the field of economics, the use of personal computers enables almost revolutionary methods of scientific research. New practices and techniques, about which our teachers could only have dreamed when reading the science fiction literature, are now routinely available. New, more complex and more realistic theories are thus given a chance to be discovered.

To reflect these movements in economic thought, this thesis focuses on a subset of agent based models, the Heterogeneous Agent Models (HAMs). The most simple HAMs allow for analytical examination but if we want to move a model features a bit closer to real market conditions, the computational approach is necessary. The crucial idea of all HAMs is the abandonment of agents' full rationality. Agents do not become irrational but 'boundedly rational' (Simon 1955; 1957; Sargent 1993), they posses heterogeneous expectations and use simple forecasting rules to predict future development of market prices. The accuracy of their decisions is evaluated retroactively and based on this simple profitability analysis agents switch between several trading strategies — the parallel with the human learning process is more than apparent. Market fractions thus co-evolve over time and interactions between agents endogenously influence market prices which are no longer driven by exogenous news only.

Since De Long *et al.* (1990) shown that irrational 'noise traders' need not necessarily be driven out of the market and may even earn higher average returns in the long run, a growing body of academic literature concerning bounded rationality, the anti-EMH evidence and the presence of non-rational market agents has flourished. The evidence that both financial professionals and private investors rely on simple trading rules has been documented in many

studies (see Reitz & Westerhoff 2007 for an overview). In addition, the current wide dominance of computer trading favours the use of algorithmic solutions which implies that the technical analysis may help in predicting future price movements. Therefore, since the core assumptions of the noise trader approach have been verified (see e.g. Bange 2000, Sanders *et al.* 2000, and Chapter 6), there is good rationale to conduct further research in this area.

Moreover, it seems evident that psychology plays an important role in financial markets. Behavioural Finance (BF) is another recently developing field, which therefore deserves attention. BF can be viewed as another answer to the extremely unrealistic assumptions of the EMH and suggests to employ the insight of behavioural sciences such as psychology and sociology into finance. Its initiation dates back to the 1970s when influential psychological studies by Kahneman & Tversky (1974; 1979) were published. Just as HAMs do, BF also builds on the bounded rationality and argues that some phenomena observed in the financial world can be better explained using models with agents which are not fully rational. Some authors even suggest the *"behavioural origin of the stylised facts of financial returns"*, and of the *"statistical regularities of the data"* (Alfarano *et al.* 2005, pp. 19 & 39). Many significant biases from rational behaviour have been well described so far. These psychological finding may have significant impacts on the theory of stock trading as they directly violate EMH (Shiller 2003). However, any comprehensive economic theory summarising the most important conclusions of BF is still missing.

In addition, as Barberis & Thaler (2003, pg. 1099) have indicated, *"the costs of entering the stock market have fallen"* and almost everyone can simply become an amateur investor. Even more importantly, in many countries the obligatory retirement savings plans force people who have no economic education at all to make important investment decisions to which funds deposit their money. We need not be wrong saying that these people may be considerably more vulnerable to various behavioural biases than experienced financial practitioners.

In our opinion, both approaches — Heterogeneous Agent Models and Behavioural Finance — aim at the same point and can perfectly complement one another. We also expect that HAM methodology could serve as a very useful theoretical tool for BF verification. LeBaron (2005, pg. 41) argues that *"agent-based technologies are well suited for testing behavioural theories"* and anticipates that *"the connections between agent-based approaches and behavioural approaches will probably become more intertwined as both fields progress"*. More-

over, many studies have highlighted different behavioural patterns as an optimal way of motivating the underlying HAM assumptions of strategy switching and heterogeneous beliefs. Barberis & Thaler (2003) and Scheinkman & Xiong (2004) mention overconfidence, De Grauwe & Grimaldi (2006) and Boswijk *et al.* (2007) suggest market sentiment, and Chang (2007) and Chiarella *et al.* (2003) put stress on herding behaviour.

The *central idea* governing this thesis is therefore to *take advantage of both approaches* and to *interconnect* particular BF findings with heterogeneous expectations in an asset pricing framework in order to study resulting price dynamics. By doing so, we also investigate whether current HAM methodology can be reasonably *extended* by applying findings from the field of BF. Or conversely, whether HAMs can serve as a tool for BF theoretical verification. Financial crises and stock market crashes can be widely considered as periods when investors' rationality is restrained and where behavioural patterns are likely to emerge, strengthen and often play the dominant role. Hence, there is strong rationale to advance current research literature through an empirical verification of HAMs abilities and BF explanatory power, using data covering these periods of high-volatility.

Considering HAM methodology, we follow the Brock & Hommes (1998) model approach and its extensions. From the plethora of well documented behavioural biases we examine the impact of *herding, overconfidence*, and *market sentiment* as these are generally supposed to have a strong impact on traders' behaviour not only during turbulent periods. Standard statistical tools of data analysis together with computational simulations and time series examination techniques are employed. Specifically, this thesis aims to offer answers to following hypotheses:

1. HAMs are able to explain stylised facts observed in financial time series.

2. HAMs can be estimated on empirical data.

3. HAMs fit real financial data better than other competing approaches.

4. Selected BF findings can be well modelled via the HAM.

5. BF findings extend the original HAM considerably.

6. Different HAM modifications lead to significantly different results.

7. HAM is able to replicate price behaviour during turbulent stock market periods.

The thesis is structured as follows. After the introduction, two chapters dedicated to Heterogeneous Agent Models and Behavioural Finance provide the reader with information and literature overviews regarding two main fields of interest to this thesis. In Chapter 4 we offer a description of the Brock & Hommes (1998) heterogeneous model framework, which serves as a cornerstone of this work, and Chapter 5 discusses various model extensions emerging after the original article publication. The next part deals with three introductory hypotheses for which answers can be found in recent academic literature. Chapter 7 presents a unique dataset developed by authors especially for the purpose of this work and in the next chapter we pay our attention to the description of the numerical analysis and simulation techniques used in this thesis. Chapter 9 is devoted to the model results and economic interpretations and finally, Chapter 10 attempts to draw some overall conclusions.

Chapter 2

Heterogeneous Agent Modelling

> *"If we didn't have heterogeneity, there would be no trade."*
>
> Kenneth J. Arrow, American economist (1921)

The crucial idea of all Heterogeneous Agent Models (HAMs) is the abandonment of agents' full rationality towards bounded, limited rationality (Simon 1955; 1957; Sargent 1993). HAMs employ interacting groups of boundedly rational heterogeneous agents using simple but reasonable heuristics and extrapolation techniques to model the financial world and this methodology appears very successful in replicating stylised empirical features of financial markets and generating realistic time series (Lux 2010). Moreover, HAM methodology is able to produce models which are, considering the intuition of economic modelling, much closer to the real world then the 'efficient' models can ever be.

There are many different approaches to heterogeneous agent modelling and we mention some of them in Section 2.1 and Chapter 6. However, the spirit and the essence of the majority of all HAMs are similar. This thesis is based on an influential HAM introduced by Brock & Hommes (1998). This approach may be viewed as one of the cornerstones of heterogeneous agent modelling of financial returns. Some authors also point out the close relationship between this HAM and the field of BF. De Grauwe & Grimaldi (2005, pg. 693) state that *"by stressing the use of simple rules, this approach comes close to the one of behavioural finance"*; Gaunersdorfer & Hommes (2005, pg. 5) view the model as *"a simple formalization of general ideas from behavioural finance"*; and Wan & Kao (2009, pg. 1420) understand agents in the model as *"purely behaviouristic in nature"*. Chapter 4 is devoted to a detailed description of the

model framework, nonetheless, here we provide a simple explanation of how this HAM works to get the reader familiarised with this interesting concept.

Figure 2.1 outlines a stylised depiction of the operational process of the Brock & Hommes (1998) model algorithm. Two fundamental parts of the HAM are: a *market* and *traders*. We start with the latter. The population of traders can choose from several trading strategies (1). These are highly stylised and aim to represent distinctive types of real world market participants. In this specific HAM (as well as in the majority of other HAM approaches), two basic trading strategies are considered: *fundamentalists* and *chartists* (*technical analysts*). Fundamentalists believe that the asset price is determined solely by economic fundamentals according to the EMH (Fama 1970), and even if there is a short-term deviation, they expect that prices always converge to their fundamental values. On the other hand, chartists believe that future prices can be partially predicted using simple technical trading rules and extrapolation techniques. Accordingly, if the market price rises above its fundamental, fundamentalists

Figure 2.1: How the Brock & Hommes (1998) HAM Works
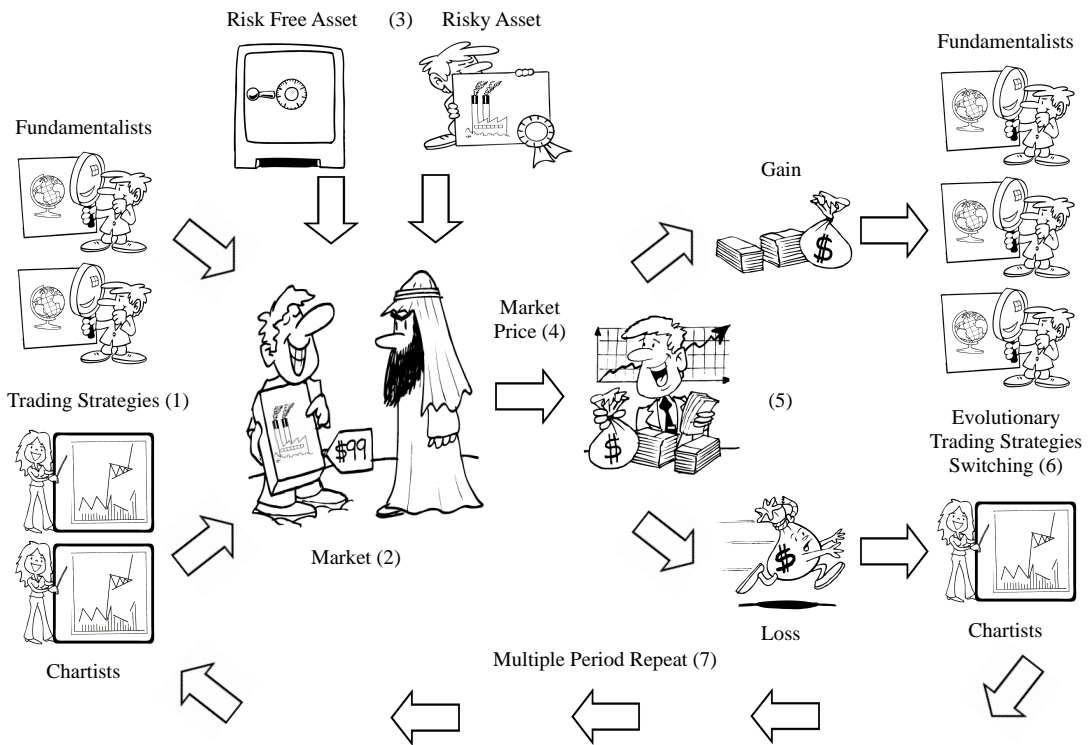
Source: The authors using timtim.com database.

predict a decrease, while chartists extrapolate further increase. Fundamentalists thus act as a stabilising force and chartists destabilise the market. In the model, various types of these two basic strategies may be implemented. All these trivial trader types can be understood as an absolute idealization of overreacting or underreacting investors, analysts and speculators, trend followers, optimists or pessimists, or other basic types of agents appearing in real markets. They can also be viewed as a consequence of various information sets in the market — fundamentalists can be understood as informed traders and conversely, chartists represent the uninformed part of the market.

Traders using various trading strategies meet in the *market* (2) where they can trade one *risk-free asset* and one *risky asset* (3). The risk-free asset offers a constant interest gain. Conversely, the price of the risky asset fluctuates according to demand and supply so that it clears the market. Traders who believe that the risky asset price increases in the next trading period buy, traders with the opposite opinion sell. If traders do not use all their wealth, the rest is invested into the risk-free instrument. After each 'day' of trading, the closing *market price* is determined as the weighted average of all different trader groups expectations (4). Traders hence discover whether their strategy was successful or whether they lost (5). According to observed *profitabilities*, traders have the chance to abandon their current strategy and *switch* to some more successful one. Therefore, the most profitable strategies attract more traders and vice versa (6). Next day, the process iterates with newly established market fractions and the *cycle multiply repeats* (7), creating the day-by-day market closing price time series.

More formally, the process described above is being executed via a set of mutually dependent equations (see Section 8.2) and is well suited for computational simulations.

2.1 Literature Review

This section aims to relate this work to some other recent academic literature on heterogeneous agent modelling.

Although the research interest in heterogeneous agent modelling is relatively new compared to other financial topics, the extensive literature has flourished in recent years. To get a general overview of the field and its development, Hommes (2006) and LeBaron (2005) offer excellent surveys. Other partial but very up-to-date reviews are offered by Lux (2008; 2010), Westerhoff (2009),

Hommes & Wagener (2009), or Chiarella *et al.* (2009). Glancing back at the most important historical milestones, Zeeman (1974) can be retrospectively considered as the pioneering article on heterogeneous agent modelling; in Frankel & Froot (1986) the idea of fundamentalists and chartists was first presented; De Grauwe *et al.* (1993) develop one of the first HAMs where market fractions are determined endogenously — the feature which seem almost routinely nowadays; Lux (1995) deals systematically with the notion of herd behaviour and switching among trading strategies and models these psychological factors explicitly; and Brock & Hommes (1998) have introduced an influential approach, which has offered a new framework for heterogeneous agent modelling and has served as a very basic starting point for a number of more complicated extensions.

This work is mainly focused on the framework introduced by Brock & Hommes (1997; 1998) and described in more detail in Chapter 4. As we introduce another perspective and application of this system, we are especially interested in other direct extensions of the original Brock & Hommes (1998) model. Most recently, this modelling approach and resulting research development is recapitulated in Hommes & Wagener (2009).

In this section, we present a short summary of this branch of the HAM literature only; Chapter 5 is devoted to a detailed description. Shortly after the Brock & Hommes (1998) article was published, concurrent studies began to emerge. Gaunersdorfer (2000b) suggests the concept of *time dependent conditional variance of returns* but comes to a conclusion that the quantitative features of the system are almost the same compared to the constant variance case. Gaunersdorfer (2000a), Gaunersdorfer & Hommes (2005), and Gaunersdorfer *et al.* (2008) then study the phenomenon of volatility clustering in the Brock & Hommes (1998) framework and extend the model via the *risk-adjusted performance measure* and the *correction term* in chartists' beliefs which hinders prices to depart far away from their fundamental values. Chiarella & He (2002b) considerably enrich the original model by relaxing the number of assumptions — allowing *different risk attitudes*, abandoning the assumption of constant conditional variance, and incorporating simple *learning schemes* into the process of beliefs and variance formation. Chiarella & He (2003) present an even more generalised version wherein the Walrasian market clearing scenario is replaced by a *market maker approach*. Articles by Chiarella & He (2002b; 2003) are then summarised in Chiarella *et al.* (2009). Vácha & Vošvrda (2002) and Vošvrda & Vácha (2003) extend the model with the notion of *memory* and

learning; Vácha & Vošvrda (2005) add the *stochastic formation* of beliefs making numerical simulations independent on arbitrary chosen parameter values; and Vácha & Vošvrda (2007a;b) introduce the *Worst Out Algorithm (WOA)*, which periodically replaces the worst trading strategies with the new ones. This feature gets the model closer to real market conditions. Baruník *et al.* (2009) incorporate the *Smart Traders (STs)* capable of estimating future trends via a simple AR(1) process and Vácha *et al.* (2011) extend the concept of STs by introducing the idea of *skilled traders* — agents capable of estimating not only trends, but also the bias in their expectations. In Vácha *et al.* (2009) the *market sentiment* — an interesting behavioural aspect — is studied. In a nutshell, the complete work of Vácha, Vošvrda and Baruník extends the original concept of the Brock & Hommes (1998) model in a different way than already introduced literature — their approach is to study noisy simulations.

We also mention other approaches which start from or adhere to the same essence as Brock & Hommes (1998) model but considerably depart from the original concept. Chiarella & He (2001; 2002a) develop and analyse HAMs in which the demand for the risky asset is derived from the Constant Relative Risk Aversion (CRRA) utility function instead of the Constant Absolute Risk Aversion (CARA) utility function in Brock & Hommes (1998). In Brock & Hommes (1998) all traders have knowledge about the fundamental price and their demand of the risky asset is independent on their wealth. In Chiarella & He (2001; 2002a) decisions of traders depend more realistically on their relative wealth. The wealth as well as the price process are thus growing. Chiarella *et al.* (2002; 2006) present HAMs derived from the CRRA utility function and with a market maker price setting. Moreover, similarly as in the Chiarella & He (2001) article, in the Chiarella *et al.* (2006) model, the demand is dependent on traders' wealth. The dividend process as well as the fundamental price trend are thus growing again. The interesting suggestion of this model is that in terms of wealth accumulation, chartists are generally more successful. Scheinkman & Xiong (2004) study a HAM with short sale constraints and suggest overconfidence as a potential source of heterogeneity. Anufriev & Bottazzi (2005) extend the model of Chiarella & He (2001) allowing for an arbitrarily large but finite number of different traders and not restricting the procedure of future price forecasting in any way. The article of Hommes *et al.* (2005) seems to be a reaction to the criticism of the Walrasian market equilibrium price scenario from Chiarella *et al.* (2002), Chiarella & He (2003), and others. Authors analyse a HAM directly based on Brock & Hommes (1998) but

extended in three important aspects. First, the market maker scenario is used. Second, the model allows for asynchronous updating of strategies, i.e. only a fixed proportion of traders update their strategies each particular period. Finally, a non-zero outside supply of the risky asset is considered. Despite these three significant changes, authors assert that *"the global picture of asset price dynamics is surprisingly similar as in Brock & Hommes (1998) ... and many global dynamic features are robust with respect to details of modelling market institution and evolutionary strategy switching"* (pg. 1045). Föllmer *et al.* (2005) present a stochastic HAM focused on the phenomenon of market bubbles, their appearance, growth, and bursting and De Grauwe & Grimaldi (2005; 2006) employ the *Adaptive Belief System (ABS)* introduced by Brock & Hommes (1998, pg. 1237) to develop and study exchange rate HAMs which are capable to reproduce many of empirical exchange rate 'puzzles'. Chang (2007) emphasises the impact of herding and argues that investors in fact lack any clear sense of the correct price of an asset, and thus their decision may be determined mainly socially. Therefore, he merges the Brock & Hommes (1998) approach with the framework of social interactions among agents and indicates that the herd behaviour is significantly determined also by endogenous social interactions generated by heterogeneous beliefs. Boswijk *et al.* (2007) reformulate the Brock & Hommes (1998) model in terms of price-to-cash-flow ratio and estimate their HAM employing the annual S&P500 data from 1871 to 2003. They found significant behavioural heterogeneity, substantial switching, and two different — 'mean reversion' and 'trend following' — regimes. Both findings fit well into the idea of heterogeneous agent modelling.

All HAMs mentioned above share the common feature of the mathematical tractability (at least to some extent) via considering only a restricted number — two to four — of trader types which might be viewed as an important aspect contributing to the economic theory. Nonetheless, with recent technological development, the computational approach (LeBaron 2005) to study complex HAMs with a large number of different interacting agents and/or richer sets of learning memory and updating schemes becomes the important part of the field. It is generally believed to get a model much more closer to real market conditions than any analytically tractable approach ever can. Chiarella & He (2003) mention Chen & Yeh (1997; 1999) or Lux & Marchesi (1999) as multi-agent models of this kind but similar in spirit to their model. Brock *et al.* (2005) look for an answer what might be the result of an aggregation of millions of different strategies, generalise the original ABS and suggest a rigorous

theoretical framework for evolutionary heterogeneous agent markets with many competing trader types. They introduce an idea of the *Large Type Limit (LTL)*, which is an approximation for the dynamics of these kind of markets and can be viewed as a bridge between the analytical and computational approach. According to Diks & Weide (2003; 2005), it is also the dispersion of beliefs among market participants, not only the aggregate values, that affects the behaviour of the market. They also argue that a small number of agents is inefficient to obtain a realistic market approximation. Therefore they provide a modification of the LTL framework and present the *Continuous Belief System (CBS)*. In contrast with the LTL approach, CBS uses a continuum of strategies available for any arbitrary number of traders and the *continuous logit model* to update market fractions. Authors suggest that their system can be used as an analytical alternative to many models based on the computational approach.

A really interesting experimental approach to heterogeneous agent research, which nonetheless goes in a similar vein with theoretical articles above, is introduced in Hommes *et al.* (2004) and Hommes (2011). Conducting controlled experiments in a laboratory environment, authors analyse whether the real human decision-making is consistent with the HAM assumptions and theoretical conclusions.

Finally, although many different HAMs have been developed and studied, surprisingly, not many attempts have been made to estimate a HAM on real market data. Section 6.2 is especially dedicated to a literature review of these sort of models.

Chapter 3

Behavioural Finance

> *"The economist may attempt to ignore psychology, but it is sheer*
> *impossibility for him to ignore human nature."*
>
> John M. Clark, American economist (1884–1963)

Behavioural Finance (BF) can be viewed as another answer to the extremely unrealistic assumptions of the EMH. To provide some suggestion as to how address this deficiency, BF proposes to employ the insight of behavioural sciences such as psychology and sociology into finance. To quote from Thaler (1999, pg. 15), *"we can enrich our understanding of financial markets by adding a human element"*. The 2002 Nobel Prize in Economics for Daniel Kahneman, one of the BF pioneers, can be perceived as recognition of the field and its acceptance into the 'wide economic family' of mainstream thoughts.

The literature on BF emphasises the role of boundedly rational traders using simple rules of thumb, the role of market psychology, and the impact of many cognitive biases. It has been documented in many studies that people widely and systematically depart from the notion of rational 'homo economicus'. Therefore, BF attempts to enrich the understanding of financial market processes by considering these aspects of human nature in financial models (Ricciardi & Simon 2000).

There are two main building blocks of BF. The first comprises *psychology*, research in *cognitive biases* and *Prospect Theory* (Kahneman & Tversky 1974; 1979). The second highlights the consequences of so called *'limits to arbitrage'*.

Prospect Theory offers a compact alternative to the traditional finance paradigm based on the Expected Utility Theory, EMH, rational expectations,

and Bayesian probabilities. Instead, Prospect Theory builds on psychological research in human decision making and judgment. It states that agents decide according to changes in their wealth rather than with respect to absolute value of their possessions. Human utility is therefore determined mainly by gains and losses. Additionally, people ascribe more significance to losses than to gains of the same magnitude. This well described behavioural bias is called Loss Aversion. Thaler (1999, pg. 15) claims that *"losses hurt roughly twice as much as gains feel good"* and according to Shefrin (2001, pg. 115) *"a loss has about two and a half times the impact of a gain"*. Furthermore, people are risk averse if they might gain, but they are willing to risk when they expect losses. The human utility function is thus asymmetric — more convex in the area of losses that it is concave in the area of gains. Finally, people tend to overweight small probabilities and underweight high probabilities. That is why lotteries are so popular and insurance companies are so profitable. This all leads to seemingly irrational situations where decision weights often do not match objective likelihoods. For instance, people seem to be highly influenced by the 'frame' in which the decision is introduced or how the question is presented. In a situation of a disaster concerning a group of 100 injured people, more people polled would opt to save 80 injured people than would choose to let 20 injured people die (Barberis & Thaler 2003; Baltussen 2009).

According to the traditional finance paradigm, markets are efficient, all available information is quickly reflected in prices and rational 'arbitrageurs' promptly correct possible misallocations caused by less than fully rational traders. Prices thus can not deviate from their fundamental values and if they do, it is only for a short-term until someone exploits such mispricing. However, BF argues that there are considerable *limits to arbitrage* in the real world. Strategies to exploit mispricing might be both costly and risky. Financial transactions are burdened by differences in fundamental risk, various fees and spreads, or restricted availability of relevant securities. Information demand, narrow time capacities, and limited human resources also play a very important role. Due to all these facts, mispricing can often survive even in the long run, rational arbitrageurs might not be able to exploit it and markets might move away from the notion of efficiency (Barberis & Thaler 2003; Baltussen 2009).

One of the crucial implications of BF is that, according to extensive psychological research, various judgment biases affecting market participants are prone to be systematic and persistent, therefore not cancelling one another

out randomly (Schleifer & Summers 1990). Tendencies described on the micro level can thus have considerable impact on the macro level. Despite all of these interesting and important findings, the main disadvantage of the field is the absence of any comprehensive economic theory summarising major BF conclusions. At this stage, therefore, BF can only serve as a useful complement to the other economic paradigms and approaches rather than as an independent theoretical concept.

3.1   Selected Findings

At the very beginning of this section, we would like to emphasise that we are widely aware of the evident fact that

> *"Real investors and markets are too complicated to be neatly summarised by a few selected biases and trading frictions."*

<div align="right">Baker & Wurgler (2007, pg. 130)</div>

At the same time, we are more than convinced that it makes good sense to exert efforts to understand some of these biases and frictions via economic modelling. Therefore, from the plethora of irregularities and seemingly irrational behavioural patterns we focus on three particular BF findings:

1. Herding;

2. Overconfidence;

3. Market Sentiment.

There are several good reasons why we should place special focus on these behavioural biases. First, they are *reasonably robust* and *well documented* in many studies. Second, they are generally supposed to have *strong impact* on traders' behaviour over the long run, not only during turbulent periods. Third, all three phenomena can be *well integrated* into the Brock & Hommes (1998) model framework which is rather compact and does not otherwise allow for major modifications without deviating from its overall structure.

Further, as it goes far beyond the scope of this work, we neither describe nor discuss here the plethora of remaining behavioural biases. To gain more information about this fascinating field of finance, please, let yourself be inspired in Section 3.2.

3.1.1 Herding

Herding or herd behaviour denotes a situation when many people make similar decisions based on a specific piece of information while ignoring other highly relevant facts. Yet again, Keynes (1936) comments on herding tendencies when he describes the stock market as a 'beauty contest'. For a financial market example, if stock prices go up, it is likely to attract public attention and allow for irrational enthusiasm that can develop into a market bubble in the end. High expectations of future prices are the reason for current high prices and vice versa, no matter that there is likely to be no real merit behind the expectations. As in a herd, people 'follow the crowd' in terms of both expectations and real investment decisions. *Momentum trading* or *positive feedback trading* can serve as good examples.

However, herding might be far from being irrational. On the individual level, imitating of actions of others — e.g. market leaders — might be an extremely cheap, easy, and effective way of learning and decision-making. Chang (2007) for instance understands herding as an evolutionary adaptation, which developed naturally as a cost-effective way of processing information. Moreover, for traders professing market psychology, moving against the herd may present an attractive investment strategy. On the other hand, for economy as a whole, herding is likely to decrease market efficiency and can even have disastrous impact.

Generally, people do not like uncertainty and behave according to observed patterns even if it is often hard to find any objective reasoning supporting such a strategy (Shiller 2003). People are also highly influenced by their environment and the world of finance is no exception. For animals, safety is one of fundamental reasons why they herd and for investors, professional money managers, or analysts the same holds in many situations (De Bondt & Thaler 1995). These interesting explanations are given by Diks & Weide (2005, pg. 750): *"being too different from the rest can be risky and might jeopardise career perspectives or reputation"* or *"younger analysts forecast closer to the average forecast"*, as *"they are more likely to be terminated when they deviate from the consensus"*.

Herding is sometimes considered as an opposite tendency to overconfidence regarding information efficiency. Bernardo & Welch (2001, pg. 326) for example argues that thanks to overconfident individuals, information *"that would be lost if rational individuals instead just followed the herd"* is preserved.

For brevity reasons, we do not offer a complete literature review on the topic of herding but do refer to several key contributions. For concerned readers, Hirshleifer (2001), Diks & Weide (2005), Alfarano *et al.* (2005), or Hommes (2006) are likely to serve as a good initial source of information. Modelling approaches to herding are offered by Kirman (1991; 1993), who study herding behaviour in ant colonies and might be considered as a 'promoter' of modelling of this mechanism. A HAM discussing herding behaviour is introduced by Chiarella & He (2002b), who reveal a tendency to herd when a particular strategy becomes significantly profitable. Chiarella *et al.* (2003) confirm the potential of imitating to be a rational strategy. Diks & Weide (2005) suggest that herding might increase market volatility.

3.1.2   Overconfidence

Many psychological studies indicate that people are generally overconfident. In a nutshell, overconfidence is a consistent tendency to overestimate own's skills and the accuracy of one's judgments. People often believe in their own superior knowledge and put much more weight on private information — especially if they are personally involved in gathering and assessing data — than on public signals — particularly when these are ambiguous. People also poorly estimate probabilities of future events and are too optimistic about future success — especially when it comes to challenging tasks. De Bondt & Thaler (1995, pg. 389) even comment on overconfidence as *"perhaps the most robust finding in the psychology of judgment"*.

The most famous example to illustrate overconfidence concerns driving abilities. From a sample of 81 U.S. students 82% believe they are in the top 30% of drivers in terms of driving safety and almost 93% find themselves above average in terms of driving skills (Svenson 1981) — both clearly mathematically impossible. Another interesting fact is that men have been found to be more overconfident than women and experts more overconfident than laymen (Barberis & Thaler 2003).

It is obvious that overconfidence is an extremely relevant topic for financial markets. Investors overconfident about their trading abilities are prone to pursue excessive trading (Odean 1998; 1999; Barber & Odean 2000; 2002), hold under-diversified portfolios (Goetzmann & Kumar 2008), or underestimate risk (De Bondt 1998). All these factors are likely to implicate higher transaction costs hand-in-hand with lower returns (Barber & Odean 2000). On the other

hand, one of positive implications of overconfident behaviour might be the reduced tendency to herd (Bernardo & Welch 2001).

Regarding further literature, we mention a few typical theoretical approaches to overconfidence modelling. Interested readers are, however, referred to Daniel *et al.* (1998), Hirshleifer (2001), or Barberis & Thaler (2003) to gain more information about the topic and other modelling approaches. Daniel *et al.* (1998) present a model of securities over- and underreaction based on overconfidence which is defined as the overestimation of private signals precision.[1] In their work they reveal a tendency of overconfidence to impact market prices and increase market volatility. In many studies, overconfidence is modelled as *"overestimation of the precision of one's information"* (Scheinkman & Xiong 2004, pg. 15). In another way of looking at it, Barberis & Thaler (2003) for instance suggest that overconfidence can be modelled as an underestimate of a variance.

3.1.3 Market Sentiment

Defined broadly, market sentiment refers to exaggeratedly pessimistic or optimistic and wishful beliefs about future market development, stock cash flows, and investment risks which are not fully justified by information at hand. To the best of our knowledge, market sentiment seems to be one of the most powerful driving forces on the stock market — as early as 1936 Keynes highlighted the role of sentiment as one of major determinants of investment decisions. This is particularly true during market crashes — de Jong *et al.* (2009b, pg. 1934) for instance point out that *"there is a clear shift in sentiment during extreme events"* and a study by Shiller (2000) shows that investor sentiment in terms of bubble expectation and investor confidence vary significantly through time. Market sentiment causes irrational shifts of aggregate demand or supply as the behaviour affected by market sentiment is correlated among traders (Schleifer & Summers 1990). These shifts might be triggered for various reasons. Investors might follow 'market gurus' or expert advice, use the same pricing models or sources of information (rating agencies), react rashly to signals they do not fully understand, or just follow the crowd.

The academic literature concerning market sentiment from various points of view is rather extensive; but, it is beyond the scope of this work to provide

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[1]Which is consistent with our description of overconfidence but in sharp contrast with the model of investor sentiment by Barberis *et al.* (1998) mentioned in Subsection 3.1.3 in which all information is public but subject to misinterpretation. However, the results of both models regarding the effect on market prices are comparable.

a complete review. Nonetheless, we again briefly refer to several interesting works by prominent BF authors. An article by Barberis *et al.* (1998) introduces a model of investor sentiment which is developed to reflect psychological as well as empirical evidence of overreaction and underreaction of stock prices. A study by Baker *et al.* (2005) briefly suggests several approaches to sentiment and overconfidence modelling. Baker & Wurgler (2007) develop the *Sentiment Index* — a methodology of measuring investor sentiment — and describe in detail which empirical proxies to employ for its creation. Boswijk *et al.* (2007) propose a simple theoretical framework of measuring the *average market sentiment* within the HAM framework.

Finally, in their article, Vácha *et al.* (2009) consider the simple form of market sentiment in the HAM framework (see Section 5.10) and conclude: *"the most interesting aspect would be to show the impact of different changes in sentiment on the market price"* (pg. 218). We examine this aspect deeply in Chapter 8.

3.2   Literature Review

The literature on BF is rather extensive and closely related to other social sciences such as sociology and psychology. The interdisciplinary approach is perhaps the fundamental feature distinguishing BF from many other fields of finance or economics in general. Another feature which is in sharp contrast to the literature on HAMs is the more or less descriptive or explanatory character of the majority of articles — so far the inevitable consequence of the inexplicit nature of the field. Although BF is not a central topic of this work, we offer the interested reader a basic overview.

At the beginning, it is important to note that it is almost impossible to specify the launch of the field. The first journal articles concerning BF as we understand it these days began to emerge in the 1990s, but its roots can be found decades before that. In our effort to name a breakthrough article or author we believe we will not be wrong if we point out Kahneman & Tversky (1974; 1979), two psychologists whose research in human decision making and judgment offered an alternative to the traditional finance paradigm assuming general rationality of all agents — Prospect Theory.[2] Their robust empirical

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[2]For his research in experimental economics and psychology as well as for his work in Prospect Theory, Daniel Kahneman was awarded the Nobel Memorial Prize in Economic Sciences in 2002.

findings summarising behavioural patterns and errors humans make when un-wittingly using heuristics or being victimised by various biases are still quoted today.

For novices, Baltussen (2009) provides an excellent introduction to the field and Shiller (2003) or Ricciardi (2005) add interesting historical summaries. Books by Thaler (1993) and Thaler *et al.* (2005) the offer a wide-ranging overview of the development of the field. For those who like a more popular form, Ricciardi & Simon (2000) might be a good initial choice. Subsequently, one may continue reading extensive surveys written by Hirshleifer (2001) sum-marising judgment and decision biases together with Barberis & Thaler (2003) and Malkiel (2003) resuming behavioural patterns and 'puzzles' observed in stock markets. For a more business-focused perspective, De Bondt & Thaler (1995), Shefrin (2001), or Baker *et al.* (2005) deal with the corporate finance and managerial implications.

For more thoughtful readers, Fama (1998) and Thaler (1999) present two essentially opposite opinion articles. On the one hand, Eugene Fama, the father and ever-ready defender of the EMH (Fama 1970), states that *"the evidence does not suggest that market efficiency should be abandoned, . . . the anomalies are chance results . . . and tend to disappear with reasonable changes in the way they are measured"*. On the other hand, Richard Thaler, one of the most important figures of the BF development foretells that *"in the not-too-distant future, the term 'behavioural finance' will be correctly viewed as a redundant phrase. What other kind of finance is there? In their enlightenment, economists will routinely incorporate as much 'behaviour' into their models as they observe in the real world. After all, to do otherwise would be irrational."*

Chapter 4

Brock & Hommes (1998) Model

"The purpose of science is not to analyse or describe but to make useful models of the world."

Edward de Bono, Maltese physician (1933)

In this chapter, we introduce the initial model presented in the influential article by Brock & Hommes (1998). This approach may be viewed as one of the cornerstones of heterogeneous agent modelling of financial returns and has served as a very basic starting point for a number of more complicated extensions. The model is a financial market application of the ABS and Adaptive Rational Equilibrium Dynamics (ARED) — the endogenous, evolutionary selection of heterogeneous expectation rules following the framework of Lucas (1978) and proposed in Brock & Hommes (1997; 1998). In comparison with the original version, our notation is slightly simplified following the model summaries in Hommes (2006) or Hommes & Wagener (2009) as we do not need the absolutely precise mathematical description for the purpose of this work. The reader interested in the mathematical background is advised to consult the original articles.

4.1 Model

We consider an asset pricing model with one risk free and one risky asset. The dynamics of the wealth is as follows:

$$W_{t+1} = RW_t + (p_{t+1} + y_{t+1} - Rp_t)z_t, \tag{4.1}$$

where W_{t+1} stands for the total wealth at time $t + 1$, p_t denotes the ex-dividend price per share of the risky asset at time t, and $\{y_t\}$ denotes its stochastic dividend process. The risk-free asset is perfectly elastically supplied at constant gross interest rate $R = 1 + r$, where r is the interest rate. Finally, z_t denotes the number of shares of the risky asset purchased at time t.

In this concept, the *CARA utility function* is employed. The type of utility function considered is essential for each economic model and determines its nature and dynamics. The utility for each *investor* (*trader* or *agent* alternatively) h is given by:

$$U(W) = -exp(-aW), \tag{4.2}$$

where $a > 0$ denotes the risk aversion, which is assumed to be equal for all investors.

In this model, the *Walrasian scenario* is assumed, i.e. investors are 'price takers' and price p_t is found when the sum of demand equals supply. This in fact means the price p_t at time t if derived employing information from time $t - 1$ and the expected utility for time $t + 1$.

Let E_t, V_t denote the conditional expectation and conditional variance operators, respectively, based on a publicly available information set consisting of past prices and dividends, i.e. on the information set

$$\mathcal{F}_t = \{p_t, p_{t-1}, \ldots ; y_t, y_{t-1}, \ldots \}. \tag{4.3}$$

Let $E_{h,t}$, $V_{h,t}$ denote the beliefs of *investor type h* (*trader type h* alternatively) about the conditional expectation and conditional variance. For analytical tractability, beliefs about the conditional variance of excess returns are assumed to be constant and the same for all investor types, i.e. $V_{h,t}(p_{t+1}+y_{t+1}-Rp_t) = \sigma^2$. Thus the conditional variance of total wealth $V_{h,t}(W_{t+1}) = z_t^2\sigma^2$.

Each investor is assumed to be a myopic[1] mean variance maximiser, so for each investor h the demand for the risky asset $z_{h,t}$ is the solution of:

$$max_{z_t} \left\{ E_{h,t}[W_{t+1}] - \frac{a}{2}V_{h,t}[W_{t+1}] \right\}. \tag{4.4}$$

Thus

$$E_{h,t}[p_{t+1} + y_{t+1} - Rp_t] - a\sigma^2 z_{h,t} = 0, \tag{4.5}$$

---

[1]To be 'myopic' means to have a lack of long run perspective in planning. Roughly speaking, it is the opposite expression to 'intertemporal' in economics modelling.

$$z_{h,t} = \frac{E_{h,t}[p_{t+1} + y_{t+1} - Rp_t]}{a\sigma^2}. \tag{4.6}$$

Let $n_{h,t}$ be the fraction of investors of type h at time t and its sum is one, i.e. $\sum_{h=1}^{H} n_{h,t} = 1$. Let $z_{s,t}$ be the overall supply of outside risky shares. The *Walrasian marker equilibrium* for demand and supply then yields:

$$\sum_{h=1}^{H} n_{h,t} z_{h,t} = \sum_{h=1}^{H} n_{h,t} \left\{ \frac{E_{h,t}[p_{t+1} + y_{t+1} - Rp_t]}{a\sigma^2} \right\} = z_{s,t}, \tag{4.7}$$

where H is the number of different investor types. In the simple case $H = 1$ we obtain the equilibrium pricing equation

$$Rp_t = E_{h,t}[p_{t+1} + y_{t+1}] - a\sigma^2 z_{s,t}. \tag{4.8}$$

We now rewrite Equation 4.8 for the specific case of zero supply of outside shares, i.e. $z_{s,t} = 0$ for all t. The market equilibrium then satisfies:

$$Rp_t = \sum_{h=1}^{H} n_{h,t} \{ E_{h,t}[p_{t+1} + y_{t+1}] \}. \tag{4.9}$$

In a completely rational market Equation 4.9 reduces to:

$$Rp_t = E_t[p_{t+1} + y_{t+1}], \tag{4.10}$$

and the price of the risky asset is completely determined by economic fundamentals and given by the discounted sum of its future dividend cash flow:

$$p_t^* = \sum_{k=1}^{\infty} \frac{E_t[y_{t+k}]}{(1+r)^k}, \tag{4.11}$$

where p_t^* depends upon the stochastic dividend process $\{y_t\}$ and denotes the *fundamental price* which serves as a benchmark for asset valuation based on economic fundamentals under rational expectations. In the specific case where the process $\{y_t\}$ is IID, $E_t\{y_{t+1}\} = \bar{y}$ which is a constant. The fundamental price, which all investors are able to derive, is then given by the simple formula:

$$p^* = \sum_{k=1}^{\infty} \frac{\bar{y}}{(1+r)^k} = \frac{\bar{y}}{r}. \tag{4.12}$$

It is also important to note a *transversality condition*:

$$\lim_{t \to \infty} \frac{E(p_t)}{R^t} = 0, \tag{4.13}$$

which excludes the existence of 'speculative bubble solutions' of Equation 4.10 growing constantly at rate R.[2] Therefore, the constant fundamental solution 4.12 is the only solution of Equation 4.10.

For the further analysis it is convenient to work not with the price levels, but with the deviation x_t from the fundamental price p_t^*:

$$x_t = p_t - p_t^*. \tag{4.14}$$

4.1.1 Heterogeneous Beliefs

Now we introduce the heterogeneous beliefs about the future prices. We follow the Brock & Hommes (1998) approach and assume the beliefs of individual trader types in the form:

$$E_{h,t}(p_{t+1} + y_{t+1}) = E_t(p_{t+1}^* + y_{t+1}) + f_h(x_{t-1}, \ldots, x_{t-L}), \quad \text{for all } h, t, \tag{4.15}$$

where p_{t+1}^* denotes the fundamental price (Equation 4.11), $E_t(p_{t+1}^* + y_{t+1})$ denotes the conditional expectation of the fundamental price based on the information set $\mathcal{F}_t = \{p_t, p_{t-1}, \ldots; y_t, y_{t-1}, \ldots\}$, $x_t = p_t - p_t^*$ is the deviation from the fundamental price (Equation 4.14), f_h is some deterministic function which can differ across trader types h and represents a 'h-type' model of the market, and L denotes the number of lags.

It is now important to be very precise about the class of beliefs. From the expression in Equation 4.15 it follows that beliefs about future dividends flow:

$$E_{h,t}(y_{t+1}) = E_t(y_{t+1}), \quad h = 1, \ldots H, \tag{4.16}$$

are the same for all trader types and equal to the true conditional expectation. In the case where the dividend process $\{y_t\}$ is IID, from Equation 4.12 we know that all trader types are able to derive the same fundamental price p_t^*.

On the other hand, traders' beliefs about future price abandon the idea of

---

[2]See Brock & Hommes (1998, pg. 1239) or Gaunersdorfer & Hommes (2005, pp. 6–7) for more detailed mathematical explanation.

perfect rationality and move the model closer to the real world. The form of this class of beliefs:

$$E_{h,t}(p_{t+1}) = E_t(p_{t+1}^*) + f_h(x_{t-1}, \ldots, x_{t-L}), \quad \text{for all } h, t, \tag{4.17}$$

allows prices to deviate from their fundamental value p_t^*, which is a crucial step in heterogeneous agent modelling. f_h allows individual trader types to believe that the market price will differ from its fundamental value p_t^*.

An important consequence of the assumptions above is that heterogeneous market equilibrium from Equation 4.9 can be reformulated in the deviations form, which can be conveniently used in empirical and experimental testing. We thus use Equation 4.14, 4.15 and the fact that $\sum_{h=1}^{H} n_{h,t} = 1$ to obtain:

$$Rx_t = \sum_{h=1}^{H} n_{h,t} E_{h,t}[x_{t+1}] = \sum_{h=1}^{H} n_{h,t} f_h(x_{t-1}, \ldots, x_{t-L}) \equiv \sum_{h=1}^{H} n_{h,t} f_{h,t}, \tag{4.18}$$

where $n_{h,t}$ is the value related to the beginning of period t, before the equilibrium price x_t has been observed.

4.1.2 Selection of Strategies

Beliefs of individual trader types are updated evolutionary following the *performance measure* (*fitness measure* or *fitness function* alternatively) according to Equation 4.21 or 4.22 below and thus create ABS, where the selection is controlled by endogenous market forces. It is actually an *expectation feedback system* as variables depend partly on the present values and partly on the future expectations. For details regarding this methodology of financial modelling consult Brock & Hommes (1997).

Before we introduce the performance measure equation, we must denote the excess return Ret_{t+1}[3] and define $Ret_{t+1} = p_{t+1} + y_{t+1} - Rp_t$ and $\rho_{h,t} = E_{h,t}[Ret_{t+1}]$. Now let us consider the goal function:

$$max_{z_t} \left\{ E_{h,t}[Ret_{t+1}]z_t - \frac{a}{2}z_t^2 V_{h,t}[Ret_{t+1}] \right\} = max_{z_t} \left\{ \rho_{h,t}z_t - \frac{a}{2}z_t^2\sigma^2 \right\}. \tag{4.19}$$

---

[3]Compared to the original Brock & Hommes (1998) article, we change the notation from R_{t+1} to Ret_{t+1} to not confuse the reader with the parallel notation of constant gross interest rate $R = 1 + r$.

As maximising expected utility of excess returns is essentially similar to maximising expected utility of wealth, Equation 4.19 is equivalent to Equation 4.4 up to constant. Therefore the solution — the demand of trader h for the risky shares — is the same and we denote it as $z(\rho_{h,t})$.

Using Equation 4.11 and 4.14, the realised excess return over period t to $t+1$ is computed as:

$$
\begin{aligned}
Ret_{t+1} &= p_{t+1} + y_{t+1} - Rp_t = x_{t+1} + p^*_{t+1} + y_{t+1} - Rx_t - Rp^*_t \\
&= x_{t+1} - Rx_t + \underbrace{p^*_{t+1} + y_{t+1} - E_t[p^*_{t+1} + y_{t+1}]}_{=\delta_{t+1}} + \underbrace{E_t[p^*_{t+1} + y_{t+1}] - Rp^*_t}_{=0} \\
&\equiv x_{t+1} - Rx_t + \delta_{t+1},
\end{aligned}
\qquad (4.20)
$$

where δ_{t+1} is a *Martingale Difference Sequence* w.r.t. $\mathcal{F}_t$, i.e. $E[\delta_{t+1} \mid \mathcal{F}_t] = 0$ for all t. δ_{t+1} can be interpreted as the financial market uncertainty, e.g. unexpected news about future dividends. Equation 4.20 can then be understood as a decomposition of the realised excess return Ret_{t+1} into the contribution of this theory ($x_{t+1} - Rx_t$) and EMH (δ_{t+1}).

The *performance measure* is then given by realised profits for strategy h and is defined as:

$$
\pi_{h,t} = \pi(Ret_{t+1}, \rho_{h,t}) = Ret_{t+1} z(\rho_{h,t}) = (x_{t+1} - Rx_t + \delta_{t+1}) z(\rho_{h,t}).
\qquad (4.21)
$$

In general, realised profits depend upon stochastic dividend process, thus $Ret_{t+1} = x_{t+1} - Rx_t + \delta_{t+1}$ (Equation 4.20). Brock & Hommes (1998) consider two alternatives:

1. the deterministic nonlinear asset pricing dynamics with $\delta_{t+1} = 0$ for all t and constant dividend $\bar{y}$ per time period. For this particular setting Equation 4.21 can be simplified and rewritten in deviations from the fundamental with slightly rearranged understanding of the time notation (Hommes 2006, pg. 49) as:

$$
\pi_{h,t} = (x_t - Rx_{t-1}) \frac{f_{h,t-1} - Rx_{t-1}}{a\sigma^2};
\qquad (4.22)
$$

2. stochastic dividend process $y_t = \bar{y} + \varepsilon_t$, with ε_t IID having a uniform

distribution on a small interval $[-\omega, \omega]$.[4] Under such conditions $\delta_{t+1} = \varepsilon_{t+1}$.

The additional memory can be introduced into the performance measure Equation 4.21 by employing a weighted average of past realised profits:

$$U_{h,t} = \pi_{h,t} + \eta U_{h,t-1}, \tag{4.23}$$

where $0 \leq \eta \leq 1$ represents the *'dilution parameter'*[5] of the past memory in the performance measure.[6]

Market fractions of trader types $n_{h,t}$ are then given by the discrete choice probability — the *multinomial logit model*:

$$n_{h,t} = \frac{\exp(\beta U_{h,t-1})}{Z_t} \overset{\eta=0}{=} \frac{\exp(\beta \pi_{h,t-1})}{Z_t}, \tag{4.24}$$

$$Z_t \equiv \sum_{h=1}^{H} \exp(\beta U_{h,t-1}) \overset{\eta=0}{=} \sum_{h=1}^{H} \exp(\beta \pi_{h,t-1}), \tag{4.25}$$

where the one-period-lagged timing of $U_{h,t-1}$ or $\pi_{h,t-1}$ ensures that all information for the market fraction $n_{h,t}$ updating are available at the beginning of period t, β is the *intensity of choice* parameter measuring how fast traders are willing to switch between different strategies, and $\eta = 0$ denotes the case in which a *model without memory* is considered and thus $U_{h,t} = \pi_{h,t}$. Z_t is then just normalization ensuring $\sum_{h=1}^{H} n_{h,t} = 1$.

Considering the updating timing we need to understand perfectly its time structure with a special attention to the inner structure of period t. The updating time line can be depicted as follows:

$$n_{h,t-1} \longrightarrow x_{t-1} \longrightarrow \pi_{h,t-1} \longrightarrow n_{h,t} \ldots \tag{4.26}$$

In other words, the price deviation of 'yesterday' from the fundamental value x_{t-1} depends on market fractions of yesterday $n_{h,t-1}$ of particular strategies $f_{h,t-1}$ (Equation 4.18) observed at the beginning of period $t-1$. Consequently it is required (together with its lagged values) for the determination

---

[4]Compared to the original Brock & Hommes (1998) article we change the interval notation from ε to ω to not confuse the reader with the parallel notation of ε_t.

[5]Gaunersdorfer & Hommes (2005) use the upper threshold $\eta \leq 1 + r$.

[6]Nonetheless, in the majority of case studies, Brock & Hommes (1998) use $\eta = 0$ to keep the analysis simple and analytically tractable. This in fact means authors mostly work with a model without memory where $U_{h,t} = \pi_{h,t}$. Thus, instead of $U_{h,t}$ we could directly use $\pi_{h,t}$ in Equations 4.24 and 4.25.

of yesterday performance measures $\pi_{h,t-1}$ (Equation 4.22). These lagged performance measures enter Equation 4.24 to update todays market fractions $n_{h,t}$ and the entire cycle repeats.[7]

Next, β can also represent the amount of uncertainty in traders' choice — the more uncertainty, the lower β. Thus, if $\beta = +\infty$, all traders homogeneously choose the best strategy, i.e. the strategy with the highest performance measure. Conversely, in the case of $\beta = 0$ traders are confused and have no motivation to adapt their strategies so they distribute themselves randomly, evenly across the set of available trader types — market fractions are constant over time and equal to $1/H$. For positive finite β traders behave in a boundedly rational manner according to the actual fitness of particular strategies. The crucial feature of Equation 4.24 is that the higher performance measure of a particular strategy h, the more traders will choose that strategy in the near future.

4.1.3   Basic Belief Types

In the original paper by Brock & Hommes (1998), authors analyse the behaviour of the artificial market consisting of a few simple *belief types* (*trader types* or *strategies*). The aim of investigating the model with only two, three, or four belief types is to describe the role of each particular belief type in deviation from fundamental price and to investigate the complexity of the simple model dynamics with the help of the bifurcation theory.

All beliefs have the simple linear form:

$$f_{h,t} = g_h x_{t-1} + b_h, \tag{4.27}$$

where g_h denotes the trend and b_h is the bias of trader type h. This form comes from the argument that only a very simple forecasting rules can have a real impact on equilibrium prices as complicated rules are unlikely to be learned and followed by sufficient number of traders. Hommes (2006) also notices another important feature of Equation 4.27, which is that x_{t-1} is used to forecast x_{t+1}, because Equation 4.7 has not revealed equilibrium p_t yet when p_{t+1} forecast is estimated.

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[7]Kouwenberg & Zwinkels (2010, pg. 6) add an assumption generally concerning the costs of updating: *"Since all investors compare the performance of the forecasting rules, we assume they have the necessary knowledge and skill to use them. As such, we can assume without loss of generality that traders can switch between rules without any costs."*

The first belief type are *fundamentalists* or *rational 'smart money' traders*. They believe that the asset price is determined solely by economic fundamentals according to the EMH introduced in Fama (1970) and computed as the present value of the discounted future dividends flow. Fundamentalists believe that prices always converge to their fundamental values. In the model, fundamentalist comprise the special case of Equation 4.27 where $g_h = b_h = f_{h,t} = 0$. It is important to note that fundamentalists' demand also reflects market actions of other trader types. Fundamentalists have all past market prices and dividends in their information set $\mathcal{F}_{h,t}$, but they are not aware of the fractions $n_{h,t}$ of other trader types. So they are not perfectly rational as the behave as if all traders were fundamentalists too. Fundamentalists might pay costs $C \geq 0$ to understand how the market and fundamentals work and to obtain necessary information for their computation.

Chartists or *technical analysts*, sometimes called *'noise traders'* represent another belief type. They believe the asset price is not determined by economic fundamentals only, but it can be partially predicted using simple technical trading rules, extrapolation techniques or taking various patterns observed in the past prices into account. If $b_h = 0$, trader h is called a pure *trend chaser* if $0 < g_h \leq R$ and a strong trend chaser if $g_h > R$. Additionally, if $-R \leq g_h < 0$, the trader h is called *contrarian* or strong contrarian if $g_h < -R$.

Next, if $g_h = 0$ trader h is considered to be *purely upward biased* if $b_h > 0$ or *purely downward biased* if $b_h < 0$.

Finally, *rational traders* with perfect foresight and computational ability are considered as well.[8] Their belief is defined as:

$$f_{R,t} = x_{t+1}, \tag{4.28}$$

as at each date they know all past prices, past dividends, all fraction and market equilibrium equation. Thus they are able to compute x_{t+1} perfectly. For obtaining such information to compute rational expectations, however, they pay costs $C \geq 0$.

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[8]Gaunersdorfer & Hommes (2005, pg. 5) comment on this hereby: *"A convenient feature of our model is that the traditional benchmark rational expectations model is nested as a special case within the heterogeneous framework. Our model thus provides a link between the traditional theory and the new behavioral approach to finance."*

4.2 Main findings

The paper of Brock & Hommes (1998) aims to contribute to the decades-lasting academic debate concerning the market role of irrational traders and to the crucial question of whether irrational traders can survive in the market or whether they would inevitably be driven out of the market. This section aims to refer briefly to the research methodology and the main findings of the paper.

Put simply, authors employ numerical tools such as *phase diagrams*, *bifurcation diagrams*, *Lyapunov characteristic exponents*, or *fractal dimensions* to analyse dynamics of the model, stability of steady states or on the other hand, cyclical behaviour of the system under various settings. For a brief explanation of the methodology and additional references see pages 1242–1245.

> *"Overreacting investors and/or securities analysts would be driven out of the market in an infinite memory world where rational expectations are costlessly available. But . . . since such investors are present in real markets we should study what kinds of relaxations of perfect rationality can lead to survival of 'boundedly rational' traders in equilibrium."*

> Brock & Hommes (1998, pg. 1246)

For the analysis of the interaction between *two* belief types the difference in fractions m_t is defined as:

$$m_t = n_{1,t} - n_{2,t}. \tag{4.29}$$

The main findings then are:

1. In the *rational traders vs. trend chasers case* with $\eta < 1$:

   - When $0 < g < R$ and $C = 0$, the fundamental steady state with $x = 0$ is the unique one and traders are divided 'half-and-half' between both trader types for any β.

   - When $g > 2R - 1$, two additional non-fundamental steady states arise.

   - The setting with $R < g < 2R - 1$ produces more complicated behaviour.

- Therefore even if information for obtaining rational expectation is costless and η is close to 1, agents with perfect foresight do not drive out strong trend chasers.

- Nonetheless, as argued in Hommes (2006, pg. 49), *"global dynamics in such an example is difficult to handle, because the system is only implicitly defined. Such implicitly defined evolutionary systems cannot be solved explicitly."*

2. In the *fundamentalists vs. trend chasers case* with $\eta = 0$ for both trader types and $C \geq 0$ for fundamentalists:

   - When $0 < g < R$ and $C = 0$, the situation is the very same as in the previous case with a unique fundamental steady state with $x = 0$ and no difference in profits. If $C > 0$, the proportion of trend chasers increases to 1 as C or $\beta \to \infty$. In other words, no individual is willing to pay extra costs if it does not bring any extra profit.

   - When $g > 2R$, two additional non-fundamental steady states occur.

   - The setting with $R < g < 2R$ produces more complicated behaviour.

   - Without *noise* and with higher β, time series of x_t and m_t exhibit relatively regular, weakly chaotic switching but the model is very sensitive to noise.

   - With a *noise* added to the dividend process, irregular switching between price close to fundamental price and periods of 'optimism' and 'pessimism' strongly prevails.

3. In the *fundamentalists vs. contrarians case* a high β leads to chaotic price dynamics with irregular chaotic fluctuations around fundamental price.

4. To sum up, *"the presence trend chasers or contrarians may lead to market instability and chaos"* (pg. 1258).

The main findings of the analysis of the interaction between *three* belief types — *fundamentalists and two types of purely biased traders*: type 2 upward biased and type 3 downward biased — with $\eta = 0$ and $C = 0$ are:

1. For low β, system is stable with prices converging to the fundamental value.

2. When β is high, the fundamental steady state becomes unstable and fundamentalists cannot drive oppositely purely biased traders out of the market when their biases are balanced in terms of adding up to zero ($b_2 = -b_3$) even if there are no information costs for them. Therefore all three trader types co-exist together in the market, their fractions vary over time and the market price fluctuates around the fundamental value. Authors interpret this result in the sense that centralised market institutions can protect biased traders who are therefore not eliminated from the market.

3. If β tends to infinity, the system converges to globally stable 4-periods cycle.

4. This particular setting may lead to perpetual oscillations of x_t and m_t time series but cannot produce chaotic behaviour.

The main findings of the analysis of the interaction between *four* belief types — *fundamentalists and trend chasers with upward bias, trend chasers with downward bias and strong, pure trend chasers* — with $\eta = 0$ and $C = 0$ are:

1. For low β a unique fundamental steady state with $x = 0$ arises. With rising β the steady state becomes unstable with periodic or quasi-periodic fluctuations. When β further increases, chaotic fluctuation of both x_t and $n_{h,t}$ and an irregular switching between stable (with the market price close to the fundamental value) and unstable (upward trends with most traders of the second type followed with a sudden drops) phases occurs. When β exceeds a very high level of 94, some points occur where almost the entire market is governed by fundamentalists.

2. Adding noise to the system leads to more chaotic fluctuations around fundamental price with more frequent deviations as well as to more frequent temporary speculative bubbles. In both noise-free as well as noisy case the start and the direction of a bubble is hard to predict. However, in the noise-free case, the burst of particular bubbles seems to be reasonably forecastable.

3. Irregular and unpredictable switching is caused by trend chasers or contrarians. Oppositely biased traders alone (the three belief types case) are responsible for cyclical behaviour, but do not trigger chaotic fluctuations.

4. Again, even if there are no information costs, fundamentalist are not able to drive other trader types out and stabilise prices.

The main summarising findings of the article therefore suggest the dynamics of the resulting nonlinear system[9] is mostly governed by:

1. The specific mixture of trader types.

2. Changes in the intensity of choice β to switch between particular strategies. Generally, when β increases, the fundamental steady state becomes unstable.

Authors finally conclude that the answer to the question from the very beginning of the article, i.e. whether irrational traders can survive in the market or whether they would inevitably be driven out of the market, is, viewed in the light of this research, *"not as obvious as one might have guessed"* (pg. 1267).

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[9]Chiarella & He (2002b, pg. 2) pointedly remark that the system is *"capable of generating the entire 'zoo' of complex behaviour from local stability to high order cycles and chaos."*

Chapter 5

Model Extensions

"Creativity is a natural extension of our enthusiasm."

Earl Nightingale, American motivational speaker (1921–1989)

As this thesis introduces a different perspective and application of the Brock & Hommes (1998) framework, we are particularly interested in other *direct extensions* of the original model, and thus we present the most interesting ones in this chapter.

5.1 Time Dependent Variance and the Stabilising Force

Gaunersdorfer (2000b) suggests the concept of *time dependent conditional variance of returns*. This idea seems natural, as in real markets agent surely do estimate all important variables. A detailed bifurcation analysis is provided and several numerical tools, such as bifurcation diagrams or Lyapunov characteristic exponents, are employed. The author also works with a model extended via the *stabilising force* in chartists' beliefs which hinders price to depart far away from its fundamental value and the *risk-adjusted performance measure*. We will comment on the latter of these two additional concepts in more detail in Section 5.2.

Traders update their homogeneous beliefs about conditional variance[1] in

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[1] The idea of heterogeneous beliefs about conditional variance in a HAM is elaborated e.g. in De Grauwe & Grimaldi (2006).

each period as an exponentially decreasing weighted average of squared market returns observed in the past. In comparison with Brock & Hommes (1998), where beliefs about conditional variance are fixed as $V_{h,t}(p_{t+1}+y_{t+1}-Rp_t) = \sigma^2$, here the generalised form looks as follows:

$$\tilde{\sigma}_t^2 = w_\sigma \tilde{\sigma}_{t-1}^2 + (1 - w_\sigma)(x_{t-1} - Rx_{t-2} - \tilde{\mu}_{t-1})^2,$$
$$\tilde{\mu}_t = w_\mu \tilde{\mu}_{t-1} + (1 - w_\mu)(x_{t-1} - Rx_{t-2}), \tag{5.1}$$

where weights w_σ and $w_\mu \in \langle 0, 1 \rangle$ and $\tilde{\mu}_t$ defines exponential moving averages of returns.

The stabilising force is then introduced via a modified fundamentalists' performance measure and is defined as:

$$\tilde{U}_{h,t} = U_{h,t} - C - \alpha x_t^2, \tag{5.2}$$

where $C \geq 0$ denotes the cost of obtaining fundamental information and $\alpha \geq 0$ defines an exogenous stabilising force supporting fundamentalist belief and driving prices back to the fundamental value if they deviate too much. Thus, market fractions are partially determined by market conditions.

The analysis is focused only on the interaction of two trader types: fundamentalists and trend chasers, and the incorporation of time dependent conditional variance leads to a five-dimensional system in contrast to the Brock & Hommes (1998) system, which is only three-dimensional. The system has one fundamental steady state and two mutually opposite non-fundamental steady states which all are the same as in the original case with constant beliefs about conditional variance of returns. Moreover, as in the original Brock & Hommes (1998) article, high β produces chaotic dynamics. On the other hand, authors find that in the case of time dependent variance of returns, β has to be higher compared to constant beliefs to get prices back to the fundamental value. The reason is that total variance in the performance measure is larger, which mathematically has the same effect as a fall in β.

Finally, however, the author comes to a conclusion that *"global qualitative features of the price dynamics are similar to the case with constant beliefs about variances"* and that the *"analysis gives a justification to concentrate on the more tractable model with constant beliefs about variances..."* (pg. 821). Therefore, this study might be viewed as one of the 'blind-ending branches' of

the Brock & Hommes (1998) model extensions.

5.2   Risk-Adjusted Performance Measure and the Correction Term

Gaunersdorfer (2000a), Gaunersdorfer & Hommes (2005), and Gaunersdorfer *et al.* (2008) study the phenomenon of volatility clustering in the Brock & Hommes (1998) framework[2] and extend the model by adding the *risk-adjusted performance measure* and the *correction term*. As the discussion of stylised facts in finance — one of the main topics of all three papers — goes beyond the scope of this thesis and Section 6.1 is devoted to a summary of this research area, we will only briefly comment on some interesting extensions of the Brock & Hommes (1998) model.

First, the past-realised risk adjusted profit for strategy h is considered[3] within the performance measure. The aim is to incorporate the risk taken to achieve a particular profit into decision-making of agents. Equation 4.21 is thus extended to the form:

$$\pi_{h,t} = \pi(Ret_{t+1}, \rho_{h,t}) = Ret_{t+1} z(\rho_{h,t}) = Ret_{t+1} z(\rho_{h,t}) - \frac{a}{2}\sigma^2 z(\rho_{h,t})^2. \quad (5.3)$$

However, the authors use a slightly different notation in their model and work directly with expected prices, instead of deviations from the fundamental price, which may cause some difficulties in practical comparison of both models. Moreover, biased traders are not considered at all and the performance measure is defined in terms of differences $U_{h,t} = \pi_{h,t-1} - \pi_{t-1} + \eta U_{h,t-1}$, where $\pi_t = \pi(Ret_{t+1}, Ret_{t+1})$ denotes the profits of investors with perfect foresight.

Second, the authors incorporate a 'two-step' updating process of market fractions. The first step remains the same as in the original model (Equation 4.24), but the second step introduces a correction term, which conditions the market fractions on the deviation from the fundamental price. This idea is similar to that of the stabilising force in Gaunersdorfer (2000b) (see Section 5.1), but makes a slightly more sense regarding the economic intuition

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[2]The authors call it a 'deterministic skeleton' as exogenous shocks are not considered: $\delta_{t+1} = \varepsilon_{t+1} = 0$.

[3]The authors also notice the general case of risk-adjusted dividends $y_{t+1}^{ra} = y_{t+1} - a\sigma^2 z_{s,t}$ (Gaunersdorfer & Hommes 2005, pg. 6).

which is as follows: *"If prices are too high or too low, technical traders might get nervous and do not believe that price trend ... will go on any longer and a correction to the fundamental is about to occur. That is, traders believe that temporary speculative bubble may arise, but these bubbles will not last forever."* (Gaunersdorfer 2000a, pg. 10). The form of the market fractions for the two trader types case is then:[4]

$$\hat{n}_{2,t} = n_{2,t}\exp\left[\frac{-(p_{t-1} - p_t)^2}{\alpha^{ct}}\right], \quad \alpha^{ct} > 0,$$

$$\hat{n}_{1,t} = 1 - \hat{n}_{2,t}, \tag{5.4}$$

where α^{ct} is the correction term parameter.

Thus, market fractions are determined almost in the same way as in the original model, as long as prices do not deviate too much. But as the gap increases, the correction term becomes smaller, and most of the chartists switch to fundamentalists' strategy again. This adaptation has a similar effect to the transversality condition (see Section 4.1) — it allows for temporary bubbles, but at the same time it inhibits the emergence of unrealistically large deviations.

Finally, the authors suggest another possible extension — to consider a more realistic stochastic dividend process and its consequences in the model framework. However, this approach was left for future research.

The main findings of these articles are either comparable to the original Brock & Hommes (1998) model or related to the significant presence of endogenously arising volatility clustering and autocorrelation patterns — topics beyond the scope of this thesis. Noise is interpreted as adding a small fraction of randomly behaving noise traders to the market clearing Equation 4.7. Finally, the authors state that their model is able to fit real data surprisingly well as *"the simulated return series is qualitatively similar to the S&P 500 daily return series"* (Gaunersdorfer & Hommes 2005, pg. 21).

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[4]Compared to the original articles by Gaunersdorfer (2000a), Gaunersdorfer & Hommes (2005), and Gaunersdorfer *et al.* (2008), we reverse $\hat{n}$ and n, as in the original Brock & Hommes (1998) model framework as well as in this thesis n denotes the final resulting market fractions.

5.3 Different Risk Attitudes, Learning and Variance Estimation

Four year after the publication of the original Brock & Hommes (1998) article, Chiarella & He (2002b) present a generalised version of the model with the aim to study the model behaviour after relaxing a number of assumptions, especially homogeneous risk aversion.

As a different notion of risk is generally viewed as one of the main reasons to why people trade, the authors allow agents to have *different risk attitudes*. They do so by generalising Equation 4.4 to the form:

$$max_{z_t} \left\{ E_{h,t}[W_{t+1}] - \frac{a_h}{2} V_{h,t}[W_{t+1}] \right\}, \tag{5.5}$$

where $a_h > 0$ denotes the 'h-trader' risk aversion coefficient, which now differs among particular traders.[5]

Moreover, as *different notions of volatility* are one of the attributes of financial markets, the authors also aim to incorporate it into the model. They do so by transforming Equation 4.15 to the form:

$$\begin{aligned} E_{h,t}(p_{t+1} + y_{t+1}) &= E_t(p_{t+1}^* + y_{t+1}) + f_h(x_{t-1}, \dots, x_{t-L}), \quad \text{for all } h, t, \\ V_{h,t}(p_{t+1} + y_{t+1}) &= V_t(p_{t+1}^* + y_{t+1}) + d_h(x_{t-1}, \dots, x_{t-L}) \\ &= \sigma_{ch}^2 + d_h(x_{t-1}, \dots, x_{t-L}), \quad \text{for all } h, t, \end{aligned} \tag{5.6}$$

where d_h is some deterministic function[6] which can differ across trader types h and represents a 'h-type' model of market variance and σ_{ch} denotes the common part of individual beliefs about conditional variance of excess returns.[7]

Finally, the authors incorporate simple *learning schemes* into the process of beliefs and variance formation via generalisation[8] of the simple linear beliefs form in Equation 4.27 to:

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[5]Subsequent equations related to this one are generalised in the similar way.

[6]Compared to the original Chiarella & He (2002b) paper, we change the notation from g_h to d_h not to confuse the reader, as in the original Brock & Hommes (1998) article as well as in this thesis g_h denotes the trend parameter. Moreover, we change the original σ to σ_{ch} and a to a_{ch} for the same reason.

[7]Subsequent equations related to this one are generalised in a similar way.

[8]We again slightly change the notation of the following equations in order to keep consistency within this work.

$$f_{h,t} = g_h \bar{x}_t, \qquad \bar{x}_t = \frac{1}{L} \sum_{p=1}^{L} x_{t-p}, \tag{5.7}$$

where L is the number of lags, which is in each particular case constant for all traders. On the other hand, it is interesting that in contrast to the original model as well as to other extensions, Chiarella & He (2002b) do not consider biased traders at all.

Opinions about variance then follow a nonlinear learning process:

$$\bar{\sigma}_t^2 = \frac{1}{L} \sum_{p=1}^{L} (x_{t-p} - \bar{x}_t)^2,$$

$$d_{h,t} = \sigma_{ch}^2 \mu \left\{ 1 - \frac{1}{(1 + \bar{\sigma}_t^2)\xi} \right\}, \tag{5.8}$$

where constants $\mu, \xi \geq 0$ and the form of the equation guarantees some bounds for variance estimates.

The paper is focused mainly on a study of two beliefs systems. As smart money investors are generally believed to be more risk averse than noise traders, the relative risk ratio a_{ch} is defined as $a_{ch} = \frac{a_2}{a_1}$ to capture different risk attitudes. Typically, fundamentalists are expected to be more risk averse than chartists and thus $a_{ch} < 1$. The summarised findings are:

1. In the *fundamentalists (1) vs. pure trend chasers (2, $d_{2,t} = 0$) case* with $L = 1$:

   - $0 < g < R$ leads to a unique fundamental globally stable steady state E_1 with $x = 0$ regardless the level of traders' risk attitudes.

   - On the other hand, $g > R$ causes that the stability of E_1 depends on the relative risk ratio a_{ch}. Particularly, for very strong trend chasers with $g > (a_{ch} + 1)R$, equilibrium E_1 becomes unstable and two new non-zero steady states occur and for $R < g < (a_{ch} + 1)R$, equilibrium E_1 remains stable for large a_{ch}, i.e. when trend chasers become more risk averse.

   A numerical simulation shows how noise contributes to irregular dynamics of the model and how this model processes the external noise in a substantially more complicated way than linear models:

- For $a_{ch} < 1$, decreasing a_{ch} triggers weakly chaotic fluctuations around the fundamental price with temporary upward and downward trends.

- For larger $a_{ch} > 1$, system without noise is stable and the market is dominated by fundamentalist with the price converging to its fundamental value. However, noise has a significant effect and is able to destabilise the entire otherwise stable system.

- Relative risk ratio a_{ch} has a stronger impact on the dynamics of the system than the variance σ_{ch}.

When $L \geq 2$, the equilibrium is similar as in the case with $L = 1$. However, as L increases, the system becomes more complicated but the external noise has a less significant effect. In addition, it affects more systems with high a_{ch}. In other words, when the system is chaotic even without noise, adding noise is likely to have no significant effect. But when the system without noise is stable, noise might have destabilising effect.

2. In the *fundamentalists (1) vs. contrarians (2) case* with $L = 1$:

- The results and the model dynamics are the same as in Brock & Hommes (1998).

- A numerical simulation reveals that noise has a weak impact and small a_{ch} leads to irregular fluctuations of the price around the fundamental value, i.e., when fundamentalists are more risk averse, market becomes more chaotic.

When $L \geq 2$, numerical simulations show that adding noise has a small effect when a_{ch} is large, but the opposite is true when a_{ch} is small.

To conclude, relaxing some assumptions of the original Brock & Hommes (1998) model leads to a markedly enriched system with some significant differences. On the other hand, many of the original results are robust enough with regard to suggested generalisations.

5.4 Market Maker

The paper by Chiarella & He (2003) follows closely the paper by Chiarella & He (2002b) (see Section 5.3), in which they point out the question of the appropriateness of the Walrasian auctioneer scenario, in which the equilibrium price is set so that overall demand equals overall supply, as the market clearing mechanism in price determination. Namely, in the Walrasian scenario, a desired level of holdings of the risky asset is determined at each particular p_t, regardless of the information whether p_t is the market clearing price. Hence, traders in fact do not face any real market price that someone else would be willing to pay which in itself might be an important information. Thus, the authors view this scenario as an unsatisfactory explanation of how financial markets work and extend the model by Chiarella & He (2002b) by introducing the *non-Walrasian market maker scenario* as the procedure generating the market clearing price.

The model follows the Chiarella & He (2002b) structure, but there are three classes of market participants: fundamentalists, chartists, and a *market maker*. The market maker receives the buy and sell offers for the risky asset at the beginning of each period and determines the excess demand. Then he settles all short and long positions and announces the price for the next period as a function of the excess demand. Technically, this is done through the *excess demand function $z_{e,t}$*:

$$z_{e,t} = \sum_{h=1}^{H} n_{h,t} z_{h,t}, \tag{5.9}$$

and the *speed of the price adjustment ν*:[9]

$$x_{t+1} = x_t + \nu z_{e,t}. \tag{5.10}$$

The role of the market maker is thus: take short (if $z_{e,t} > 0$) or long (if $z_{e,t} < 0$) position to clear the market.

However, although this extension seems reasonable in terms of bringing the model closer to the real market, it is not optimal for the purpose of this thesis. The reason is twofold. First, the model is further enriched even in the comparison with Chiarella & He (2002b), which brings more complicated dynamics, challenging interpretation of results, as well as more variables and

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[9]Compared to the original article, we change the notation from μ to ν not to confuse the reader, as in the original Chiarella & He (2003) article as well as in this thesis μ has already been used as a parameter in Equation 5.8.

combinations which effect can be studied. Second, because of the complexity of the model it is more problematic to compare its results with other related works. Therefore, here we present only the general description and findings. Interested readers are advised to consult the original article.

Compared to the Chiarella & He (2002b) approach, the model is enriched and generalised in several aspects. First, Equation 5.6, 5.7, and 5.8 consider various numbers of lags and Equation 5.7 uses various values of bias for each trader type h. Moreover, Equation 5.8 is also generalised to reflect different knowledge of fundamentalists and chartists about the variance of excess returns. The main findings then are:

1. As in Chiarella & He (2002b), prices are affected by the relative risk ratio. For different cases the impact varies.

2. Speed of the price adjustment ν influences significantly the stability of the steady states — namely if the speed increases and contrarians are involved. It may even have a more significant effect than the extrapolation rate g_h. The difference between the two market clearing scenarios is markedly larger when the speed of the price adjustment ν increases.

3. The dynamics of the market exhibits some significant differences from the Walrasian scenario. Put simply, the incorporation of the market maker matters.

4. Generally, with homogeneous beliefs, the longer the learning process, the more stable the Walrasian equilibrium is. With heterogeneous beliefs, this depends on the particular case — in some cases (fundamentalists vs. trend chasers), the stability conditions are independent of the number of lags, in other cases, the impact is positive (fundamentalists vs. contrarians). However, with the market maker, increasing lag length may even deteriorate the fundamental stability conditions.

5. Even a small group of traders with considerably divergent expectations can destabilise the entire market. Conversely, individual forecasting rules which alone lead to divergence may 'cancel out' producing local stability.

6. Noise has significant effects and may lead to highly irregular fluctuations which confirms the conclusions of Chiarella & He (2002b).

In a nutshell, the approach of Chiarella & He (2003) not only considerably enriches the Brock & Hommes (1998) model and even the Chiarella & He (2002b) model, but also exhibits some interesting and robust different characteristics developed through abandoning the Walrasian scenario.

5.5 Memory and Learning

Vácha & Vošvrda (2002) and Vošvrda & Vácha (2003) examine the influence of agents' memory and a learning process on the level of agents' profitability. Memory and learning are introduced to the original model through the process of strategy selection. In this extension, however, not only the last period profitability, but also previous periods are considered. To introduce memory into the market, related equations are slightly changed — they contain a weighted sum of past values of the performance measure instead of one recent past value only. Equation 4.24 and 4.25 are then redefined as:

$$n_{h,t} = \frac{\exp(\beta \sum_{p=1}^{m} \eta_{h,t} \pi_{h,t-p})}{Z_t},$$

(5.11)

$$Z_t = \sum_{h=1}^{H} \exp\left(\beta \sum_{p=1}^{m} \eta_{h,t} \pi_{h,t-p}\right),$$

(5.12)

where m is the memory length and $\eta_{h,t}$ is the vector of memory weights, which follows a similar logic as (but is not mathematically identical with) η from the original Brock & Hommes (1998) model (see Equation 4.23).

Moreover, to add the notion of learning into the system, in the 2002 article all belief types contain a learning process with various lag lengths. Equation 4.27 is then redefined as:

$$f_{h,t} = g_h \frac{1}{L_h} \sum_{p=1}^{L_h} x_{t-p} + b_h,$$

(5.13)

where L_h is the number of lags which are considered within strategy h. This version of learning can be viewed as an upgrade of Equation 5.7 of the Chiarella & He (2002b) system.

At first sight, one may think that these relatively small modifications cannot change the behaviour of the system significantly, but the contrary is the case. Authors show that *"there are significant differences in profitability of trader's*

strategies as memory length is changed and learning process is implemented to the beliefs of traders" (Vácha & Vošvrda 2002, pg. 19).

The numerical analysis of the Vácha & Vošvrda (2002) article is focused on the market[10] consisting of four trader types: fundamentalists and three types of chartists. Table 5.1 summarises the parameters of the system.

Table 5.1: Parameters of the Vácha & Vošvrda (2002) System

Sign	Parameters		Type
$N1$	$g_1 = 0$	$b_1 = 0$	Fundamentalists
$N2$	$g_2 = 1.1$	$b_2 = 0.2$	Trend chasers with upward bias
$N3$	$g_3 = 0.9$	$b_3 = -0.2$	Trend chasers with downward bias
$N4$	$g_4 = 1$	$b_4 = 0$	Pure trend chasers

Source: Vácha & Vošvrda (2002, pp. 19–20).

The main findings are as follows:[11]

1. Without memory ($m = 1$ in 5.11 and 5.12), for values smaller than $\beta = 90$ fundamentalists ($N1$) dominate the market with the market share ranging from 1/4 to 1/3. However, for $\beta > 90$, chaotic price fluctuations arise and the strategy $N2$ becomes dominant gaining control about roughly 50% of the market at average.

2. With longer memory for all strategies ($m = 20$), in comparison with the previous case, fundamentalists ($N1$) become the dominant strategy and with rising β their market share steadily rises. Moreover, the system is more stable with lower price volatility and smaller amplitude.

3. If longer memory ($m = 20$) is preserved and the learning process is added into the beliefs of $N4$ ($L_4 = 30$), the profitability of $N4$ increases considerably and exceeds both strategies $N2$ and $N3$. Fundamentalists still lead the market.

4. The effect of learning is even more apparent when learning process is implemented into beliefs of $N2$ and $N3$ ($m = 20$, $L_2 = L_3 = 30$). In

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[10]Beside the already mentioned modifications of the system, noise with a uniform distribution $\langle -0.005, 0.005 \rangle$ is added to the dividend process.

[11]The interested reader is encouraged to a deeper examination of figures in original paper (pp. 20–21), which clearly depict the development of particular profitabilities and market shares.

this case, both strategies along take the lead of the market as β is rising. Conversely, fundamentalists lose their profits against the previous scenarios.

The paper by Vošvrda & Vácha (2003) numerically examines the effect of different memory lengths among particular models or even among various trading strategies within the same model.

1. In the model with fundamentalists and three types of trend chasers, the stabilising effect of memory is shown. Without memory we can observe chaotic price fluctuations for larger values of β, but when memory is added ($m = 2$), fundamentalists become the dominant strategy. Moreover, for $m = 18$, the market becomes stable with no price fluctuations. Results from other simulations with different values of β reveal a rising profitability of fundamentalist strategy with rising memory.

2. In the model with fundamentalists and three types of contrarians, without memory we can observe a complicated dynamics and an insignificant role of fundamentalists. However, when the memory is added ($m = 20$), the system becomes more stable and prices less volatile. Consecutively, for higher values of β, fundamentalists become the dominant strategy.

3. The model with fundamentalists, trend chasers, and contrarians also discovers a positive effect of rising memory length to the fundamentalists' profitability. Moreover, increased memory also helps contrarians to outperform other market participants.

To sum up, the work of Vácha & Vošvrda (2002) suggests the strong advantage of strategies which contain the learning process. Moreover, the system with implemented memory is shown as more stable and more favourable to fundamentalists' profits. Additionally, the paper by Vošvrda & Vácha (2003) confirms the positive effect of memory to the stability of the system and to fundamentalist' profitability.

5.6 Stochastic Formation of Beliefs

An article by Vácha & Vošvrda (2005) employs a *stochastic formation* of beliefs and memory lengths in the performance measure and proposes an alternative approach to examination of the heterogeneous agent systems. Put simply, the paper aims to reveal the effect of the memory length and its probability distribution on the persistence of the simulated price time series. The theoretical base of the model is the same as we introduce in Section 4.1 and naturally, memory in the performance measure is implemented (Vácha & Vošvrda 2002). Nonetheless, neither the learning process nor memory weights are introduced.

The paper is mainly based on a robust non-parametric methodology of *rescaled range analysis*, *R/S analysis* alternatively. This methodology is used for detecting the long-run dependency and persistence of trends or for distinguishing between random and non-random systems as well as random and fractal time series. The basic tools for this type of analysis were proposed by Hurst (1951) who studied the time series of water flows of the Nile River in 1950s. See e.g. Samorodintsky (2007) for an excellent treatment and mathematical background.

One of the central terms is so called *Hurst exponent H*:

$$H \in \langle 0, 1 \rangle . \tag{5.14}$$

An IID system of random variables has $H = 0.5$. The values of the Hurst exponent $H < 0.5$ suggest an anti-persistent process, which reverts itself more frequently than a random process. On the other hand, $H > 0.5$ indicates a persistent process with the long-memory effects.

In the simulations, 20 trader types are used, 4000 observation are generated for each setting, and $\beta = 80$. Stochastic formation of beliefs and memory lengths is implemented. Trend g_h and bias b_h are generated by a random number generator from the normal distribution with parameters $N(0, 0.16)$ and $N(0, 0.09)$, respectively and memory length is randomly generated using various distribution (normal, uniform, Weibull, fixed) and various means (5, 10, 20, 40).

Authors conclude that a short memory length of the performance measure, which can be understood as a short investment horizon of particular agents, cause more volatile prices, but *"by values of the Hurst coefficients, there exist possibilities of the price predictions due to the persistence of the fundamental strategy structures"* (pg. 169). On the contrary, longer memory lengths cause

more stable market. Although using different methodology, these findings are in accordance with the previous research (see Section 5.5) of Vácha & Vošvrda (2002) which supports their robustness.

5.7   Worst Out Algorithm

Vácha & Vošvrda (2007a) introduce a very intriguing idea of a periodic replacement of the trading strategy with the lowest performance measures by a new one randomly chosen from a given set. According to authors, this should get the model closer to real market conditions. Moreover, the learning process, memory in the performance measure and the stochastic formation of beliefs and lags are implemented. In this sense, authors build on their previous research and enrich the model by another feature.

The aim of the paper is to compare the behaviour of the system while using two different distribution functions controlling the length of the learning process for all belief types. Equation 4.27 is then redefined as:

$$f_{h,t} = g_h \sum_{p=1}^{L_h} x_{t-p} + b_h, \tag{5.15}$$

where L_h is the number of lags, g_h denotes the trend and b_h is the bias of trader type h. Length L_h as well as parameters g_h and b_h are generated randomly. For L_h, two distribution functions are used — normal $F_N(L) \sim N(20, 25)$ and uniform $F_U(L) \sim U(1, 40)$. Memory in the performance measure has the same length $m_h = L_h$ (Equation 5.11) with the same weights — there is no memory fading in the process. Trend g_h and bias b_h are realizations from the normal distribution $N(0, 0.16)$ and $N(0, 0.09)$, respectively. Same distributions are applied for adding a new strategy in the WOA algorithm.

In the simulations, 40 trader types are used, 10 600 observation are generated for both normal and uniform distribution function, and $\beta = 120$. WOA changes the structure of market agents after every 50 iterations — thus 212 replacements are done during the simulation which means each agent is replaced four times at average. Hurst exponent (see Section 5.6) is used for the analysis of the correlation structures of the generated series.

Simulations reveal considerable differences between the outcomes of the models where normal and uniform distributions are used. The normal distribution returns series are closer to the real market according to basic descriptive

statistics (mean is closer to zero and kurtosis is considerably higher). In the normal case the time series of returns appears slightly persistent as $H = 0.519$ which gives a possibility of some predictions. On the contrary, the uniform case with $H = 0.412$ suggest an anti-persistent process. The effect of the WOA is opposite in this two cases. In the normal case, introduction of the WOA leads to the more persistent returns series — H increases from $H = 0.423$ in the first 3600 observations to $H = 0.589$ in the last 3600 observations. The opposite trend holds in the uniform case where WOA leads to the more anti-persistent returns series — H decreases from $H = 0.369$ in the first 3600 observations to $H = 0.286$ in the last 3600 observations.

Table 5.2: Selected Descriptive Statistics of the Initial and Final Trading Strategies in Vácha & Vošvrda (2007a)

	Normal case		Uniform case	
	Mean	Kurtosis	Mean	Kurtosis
Initial g	0.0120	-0.1890	-0.0160	-0.0084
Final g	-0.2620	-0.3650	-0.4520	0.9650
Initial b	0.0200	-0.1310	-0.0048	-0.1810
Final b	0.0058	1.6860	0.0150	-0.4720
Initial L	19.7880	-0.0190	19.483	-1.1930
Final L	18.9830	-0.0539	18.213	-1.4680

Source: Vácha & Vošvrda (2007a, pp. 62–63).

The most interesting results are, however, connected with a question of what strategies can survive in the market. Thus the WOA effect on the change of the trading strategies set is broadly examined. Authors focus especially on the beginning and the end of the experiment. If the normal distribution is used, there is a clear shift towards contrarian strategies (see the development of mean in Table 5.2) and a radical tendency to zero bias (substantially higher kurtosis of the final b in Table 5.2). Moreover, strategies with longer learning horizons seem to be less successful as the final mean of the learning length L is slightly lower than in the initial set. There is also evident variance decrease as WOA contributes to market learning by eliminating the worst-profitable strategies. The uniform distribution indicates even stronger shift to contrarians' strategies and a substantial disadvantage of the strategies with longer memory length which are being eliminated by the strategies with a very short memory. This tendency contributes to increasing returns variance.

In a nutshell, the paper suggests that WOA causes the stabilising and risk-

decreasing market role in the normal case, while it increases the market risk when the uniform distribution for generating of new strategies is used. Additionally, WOA appears more favourable to contrarians then to trend chasers and reveals a possibility of a prediction under the condition of normally distributed lengths of the learning process.

5.8   Updated WOA & Wavelet Decomposition

The paper by Vácha & Vošvrda (2007b) updates the WOA methodology of Vácha & Vošvrda (2007a) by enriching the WOA replacing ability. The original WOA replaces only the worst-performing strategy while the updated version of the algorithm eliminates from zero up to eight strategies with the lowest performance measure which represents more than 50% replacing ability. Authors also suggest another alternative approach for examining the heterogeneous agent systems — a wavelet decomposition.

The model setting is identical to the Vácha & Vošvrda (2007a) in Section 5.7 which facilitates the convenient comparison of results. The updated WOA replaces zero (0WOA), one (1WOA), two (2WOA), three (3WOA), four (4WOA), five (5WOA), six (6WOA), and eight (8WOA) strategies with the lowest performance measure by the same number of new strategies. In the simulations, 15 trader types are used — the replacement ratio thus ranges from 0% (0WOA) to $53.\bar{3}\%$ (8WOA). The higher the replacement ratio is, the more dramatic changes in the system occur after every replacement period and the more likely turbulences in prices emerge.

New strategies are generated stochastically from the same set as initial strategies, i.e. trend g_h and bias b_h are generated using the normal distribution — $N(0, 0.16)$ and $N(0, 0.09)$, respectively. Length of the learning process L_h has a uniform distribution on a range of integers $1, 2 \ldots 100$ — $U(1, 100)$. 8192 observation are generated with $\beta = 120$. WOA eliminates the unsuccessful strategies and adds new ones after every 40 iterations — 204 replacements cycles are done during a simulation. Hurst exponent H (see Section 5.6) is again used for the analysis of the correlation structures of the generated series.

The simulations reveal an interesting behaviour pattern, which can be given a reasonable economic interpretation. First, the analysis of the long-run memory shows a strong learning effect causing a persistence of returns after introducing WOA irrespective the type of the WOA used. This, however, might be in conflict with Vácha & Vošvrda (2007a), where the introduction of the WOA

leads to a more anti-persistent return series if the length of the learning process
has the uniform distribution.

0WOA has $H \doteq 0.438$, which is relatively close to $H = 0.5$, which would
suggest a random system. This is an anticipated situation as strategies are ran-
domly generated and no strategy is replaced during the simulation. Table 5.3
depicts all values of the Hurst exponent for a particular type of the WOA. 2WOA
has the highest level of persistence $H = 0.732$ and one can clearly see the de-
creasing tendency of persistence when WOAs with higher replacement rate are
employed. This phenomenon can be explained by the dilution of the above men-
tion learning process as large number of randomly generated strategies replace
the eliminated trader types. The same reasoning could be successfully used for
the decreasing trend of kurtosis from its highest value (1WOAs) as WOAs with
higher replacement rates are employed. Moreover, one can see the increasing
trend of variance as the WOA replacement rate rises. This supports the results
of Vácha & Vošvrda (2007a), where in the uniform case the implementation of
the WOAs also leads to higher volatility of price returns. The dilution of the
learning process can explain also this observed fact.

Table 5.3: Hurst Exponent, Kurtosis and Variance of the Simulated
Time Series Price Returns in Vácha & Vošvrda (2007b)

	0WOA	1WOA	2WOA	3WOA	4WOA	5WOA	6WOA	8WOA
Hurst Exp.	0.438	0.714	0.732	0.693	0.724	0.687	0.605	0.589
Kurtosis	-1.2	56.5	27.7	7.9	21.7	10.0	4.8	4.2
Variance	0.016	0.017	0.024	0.034	0.025	0.036	0.066	0.064

Source: Vácha & Vošvrda (2007b, pg. 44).

Authors subsequently apply the wavelet decomposition on simulated re-
turns time series. The wavelet decomposition is a relatively new technique but
complementary to other existing techniques of time series analysis. It is a con-
venient tool for activity and frequency detection in the financial time series.
The continuous wavelet transform and the discrete wavelet transform are two
main versions of the analysis. The key-stone is the decomposition of such a
time series into wavelets of different scales. As the wavelet analysis is based on
the time-scale domain, it is much more suitable for price time series than the
'original' Fourier analysis based on the time-frequency domain. Moreover, as
the financial time series are defined over the discontinuous domain, the discrete
wavelet transform appears natural for its examination. Nonetheless, it is not

the aim of this section to introduce the theory of the wavelet decomposition — the interested reader is referred to the original paper with many supplementary references.

In the paper, the example comparison of 1WOA, 6WOA, and 8WOA is presented. Generally, the main findings reveal the higher occurrence of price turbulences in the higher WOA cases compared to the lower WOA cases. Moreover, the wavelet variance decomposition suggests energy allocation at the lower scales (higher frequencies) and the phenomenon of higher activity (volatility) levels at all scales if the WOA replacement rate rises.

5.9   Smart Traders

The paper by Baruník *et al.* (2009) can serve as a perfect evidence of how a simple idea can bring very appealing results. In the paper the idea of *Smart Traders (STs)* — agents capable to estimate future price movements — is introduced. STs use the a simple AR(1) model based on the set of past deviations and the maximum likelihood estimation to forecast the future price change.

The model follows the original Brock & Hommes (1998) setting, nonetheless the approach to the performance measure and the market fractions updating is slightly reformulated and simplified regarding the form at once — Equation 4.21 and 4.23 are modified as we can see below:

$$U_{h,t} = \frac{1}{m_h} \sum_{l=0}^{m_h-1} \left[(x_{t-l} - Rx_{t-1-l}) \frac{f_{h,t-1-l} - Rx_{t-1-l}}{a\sigma^2} \right], \qquad (5.16)$$

where $U_{h,t}$ is the performance measure of strategy h evaluated at the beginning of t, which follows the logic of (but is not mathematically identical with) $U_{h,t}$ in Equation 4.23 and m_h denotes the memory length for trading strategy h. In contrast to previous papers of these authors, Equation 4.24 and 4.25 defining the market fractions then keep their original form as memory is already implemented in the performance measure directly:

$$n_{h,t} = \frac{\exp(\beta U_{h,t-1})}{Z_t}, \qquad (5.17)$$

$$Z_t \equiv \sum_{h=1}^{H} \exp(\beta U_{h,t-1}). \qquad (5.18)$$

Moreover, a simple rule defining the basic belief types in the original setting

(Equation 4.27) is enriched by the assumption that deviations x_t follow an AR(1) process. STs base their forecasts of x_{t+1} on the information set $\mathcal{F}_t = \{x_t, x_{t-1}, \ldots, x_{t-k-1}\}$. Beliefs of STs then has the form:

$$f^1_{h,t+1} = \hat{f}_{h,t} = \phi_1 x_{t-1}, \qquad (5.19)$$

where ϕ_1 denotes the estimated trend $\hat{g}_{h,t}$ and k is the length of the information set of $\mathcal{F}$ which varies among various types of STs used in the simulations. From Equation 5.19 we can clearly see that STs have zero bias parameter, i.e. $b_h = 0$.

In the simulations, STs' beliefs $f^1_{h,t}$ are employed together with the second group of strategies $f^2_{h,t}$ generated stochastically (see Section 5.6) using WOA selection method during simulation (see Section 5.7). WOA is set to replace the four worst performing $f^2_{h,t}$ strategies by four randomly chosen new ones after every 40 iterations. Trend g_h and bias b_h are generated randomly using $N(0, 0.16)$ and $N(0, 0.09)$, respectively. Same distributions are applied for adding new strategy in the WOA algorithm. The memory length m_h of the strategy $f^2_{h,t}$ is a realization from the uniform distribution $U(1, 100)$. When $m_h = 1$ for all types h, we get the original Brock & Hommes (1998) model.

In the simulations, the initial model without STs (0ST) is compared with models with 1 (1STs), 2 (2STs), 3 (3STs), 4 (4STs), 5 (5STs) and 10 (10STs). In each simulation 40 trading strategies is considered — the particular number of STs is supplemented by the appropriate number of stochastically generated $f^2_{h,t}$ strategies. The length k of the information set varies among individual ST strategies, e.g. for 1ST $k = 40$ or for 5STs $k_{1i=1}^5 = \{80, 60, 40, 20, 5\}$ — for complete setting consult the original paper (2009, pg. 169). Other parameters are fixed: $\beta = 300$, the number of iterations $N = 15000$, $a\sigma = 1$, and $R = 1.1$. Moreover, to gain sufficiently robust result, each model is simulated 45 times.

The simulations results reveal some strong effects of STs implementation in the model outcomes. First, the variance of price deviations x_t remains more or less similar with the increasing number of STs. This can be interpreted that the presence of STs has no effect to the market risk and uncertainty. Second, the shape of the Probability Distribution Functions (PDFs) of x_t varies significantly in terms of skewness and kurtosis when different number of STs is used. To capture this, Figure 5.1 compares the selected PDFs of the models with 0ST, 5STs and 10STs. Models without STs or with only one ST produce a platykurtic distribution while models with two and more STs produce a leptokurtic dis-

Figure 5.1: Empirical PDFs of x_t for Simulated Models
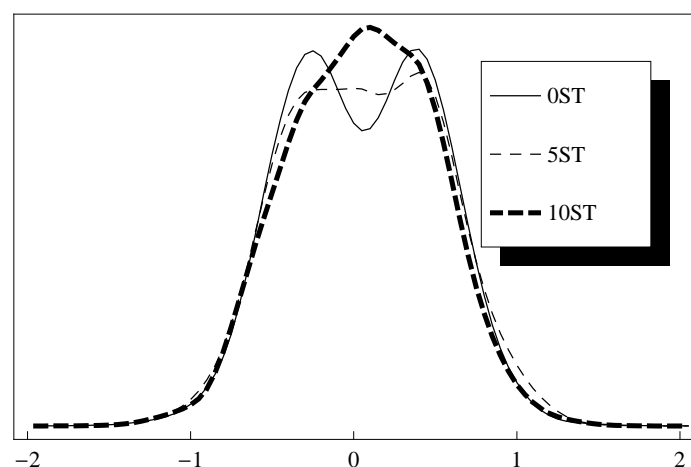
Source: Baruník *et al.* (2009, pg. 170).

tribution. Statistical significance of the mutual difference of particular PDFs has been additionally confirmed by the Kruskal-Wallis test[12] (Kruskal & Wallis 1952). Moreover, the increasing number of STs changes the shape of the PDF from bimodal to unimodal. Both effects suggest that STs modify the x_t distribution of the simulated market more closer to the distribution of real financial market returns.[13]

Next, the implementation of STs increases the Hurst exponent (see Section 5.6) and thus the persistence of the simulated market significantly. There is even a growing trend with the increasing number of STs. Finally, the increasing number of STs has a strong impact to distribution of the trend parameter g_h. With increasing number of STs we can observe significant increase of kurtosis of the PDF. The cases of 0ST, 5STs and 10STs are depicted in Figure 5.2.

In a nutshell, *"the concept of smart traders improves the model so it can better approximate real markets"* (pg. 171), which is exactly the ultimate goal HAMs aspire to achieve.

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[12]Authors consequently use the same test for statistical confirmation of all following assertions.

[13]These results are, however, in a sharp contrast to Benčík (2010, pg. 32), who develops a considerably more robust and less time demanding algorithm in $C++$ for the same model setup with 10STs only and concludes that results *"look nearly identically"* for 0ST, 5STs and regardless of the WOA setting. Author states that *"the only explanation that comes to mind is the low number of runs realised in the original paper that led to improper data obtained"* (pg. 32).

Figure 5.2: Empirical PDFs of the Trend Parameters g_h

5.10 Sentiment Patterns

Vácha *et al.* (2009) examine the impact of *STs and market sentiment (STS)* added to the original Brock & Hommes (1998) model and thus bring a notion of the market psychology into HAMs. The idea of STs is introduced in Baruník *et al.* (2009) — the conclusions of both papers regarding the effect of STs is the same. For details, please, consult Section 5.9.

However, the idea of investigating the market sentiment impact is new indeed and closely related to the topic of this thesis. The changes in the market sentiment are defined as a shifts (jumps) of the beliefs about the trend parameter of newly incoming strategies in the WOA selection process — the trend-following and contrarian strategies are modelled.

The model is again an extension of the original Brock & Hommes (1998) framework and the setting is the same as in Baruník *et al.* (2009). The only difference is the number of simulations which is 36.

Three variant of the model are considered: the model with 0 smart traders (0ST), with 5 smart traders (5STs) and a model with 5 STs and sentiment changes (5STS) in the second group of strategies $f_{h,t}^2$. Market sentiment is then modelled as jumps of trend parameter g_h between realizations from the normal distributions $N(0.04, 0.16)$ and $N(-0.04, 0.16)$ after every 4000 iterations.

Figure 5.3 depicts the PDFs of the 0ST, 5STs, and 5STS models. As we have already mentioned above, the paper reveals a very similar impact of STs (con-

Figure 5.3: Empirical PDFs of x_t for Simulated Models without STs,
with 5STs, and with 5STs and Sentiment (STS)
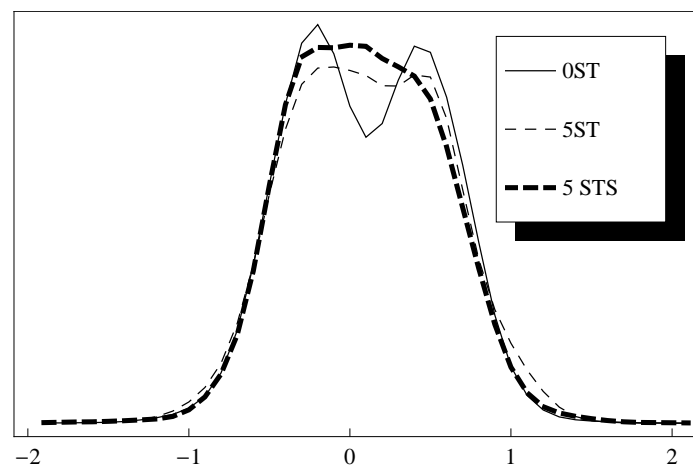
firmed by the Kruskal-Wallis test) as Baruník *et al.* (2009) — see Section 5.9.
What is new is the modification of the model by market sentiment. Authors
conclude that variance of x_t does not change after adding the market sentiment
into the model. Moreover, considering changes in the distribution of x_t, mar-
ket sentiment changes the PDF significantly. On the other hand, comparing
the effect of the market sentiment to the values of the Hurst exponent in 5ST
and 5STS models, Kruskal-Wallis test does not reject the hypothesis of equal
medians which suggests that market sentiment in this particular form does not
affect the persistence of the price deviations.

In short, the effect of the market sentiment might seem ambiguous. It is
important to state that these results are only the first attempt to model the
market sentiment and its changes and different forms and approaches need to
be examined to get more familiar with this concept.

5.11   Skilled Traders

In a forthcoming paper, Vácha *et al.* (2011) extend the concept of STs intro-
duced in Baruník *et al.* (2009) and establish the concept of *skilled traders* —
agents with an advanced capability to forecast future price movements. Skilled
trader is a new strategy type, that can predict both the trend parameter g_h

and bias b_h, while assuming price deviations follow an AR(1) process. The maximum likelihood method is used for estimation of coefficients.

Three different settings are studied with the proposed expectation that with more skilled traders, the persistence of prices increases — i.e. the market becomes less efficient:

1. Skilled traders who use simple linear for predictions the trend parameter g_h only, the bias b_h is generated randomly;

2. Skilled traders who predict the trend parameter g_h and the bias is fixed to $b_h = 0$;

3. Skilled traders who predict both the trend parameter g_h as well as the bias b_h.

For strategies with stochastic beliefs, trend g_h and bias b_h are generated randomly using $N(0, 0.16)$ and $N(0, 0.09)$, respectively. The memory length m_h is then a realization from the uniform distribution $U(1, 100)$.

Skilled traders base their forecasts of x_{t+1} on the information set $\mathcal{F}_t = \{x_t, x_{t-1}, \dots, x_{t-k-1}\}$. The length of k_h for each skilled trader type h is a random realization from the uniform distribution $U(5, 50)$. The skilled trader strategy is then defined as:

$$f^1_{h,t+1} = \hat{f}_{h,t} = \hat{g}_{h,t} x_{t-1} + \hat{b}_{h,t}, \tag{5.20}$$

where $\hat{g}_{h,t}$ and $\hat{b}_{h,t}$ are the estimates of the trend parameter a the bias parameter, respectively. The memory length of skilled traders $m_h \sim U(1, 100)$.

In the simulations, the model with $\beta = 500$ and 40 traders are considered. While the number of the particular type of skilled traders is gradually increasing, the impact on variance, the Hurst exponent (see Section 5.6), and descriptive statistics is examined. The result are as follows:

1. In the model with a randomly generated bias b_h, a growing number of skilled traders cause a rapid and almost steady increase of variance as well as the Hurst exponent after the ratio of skilled trader exceeds 3/4. This means the market becomes more persistent, but also more volatile. Kurtosis also increases and converts from platykurtic to leptokurtic with the growing number of skilled traders which moves the simulated returns closer to real market data. The highest values can be observed when

skilled traders strongly dominate the market. Lastly, the mean of the trend g_h is also increasing with the growing ratio of skilled traders, which coheres with the variance increase.

2. In the model with $b_h = 0$, the impact of a growing number of skilled traders is opposite to the first case. Authors explain this feature by decreasing heterogeneity in the model as the growing number of traders has zero bias. The Hurst exponent does not reveal any considerable trend and returns remain platykurtic irrespective the market structure. What is interesting is mean of the trend parameter g_h, which is slowly increasing to positive values with the growing number of skilled traders, but starts to decrease after the number of skilled traders exceeds 28. Moreover, when there are 35 skilled traders in the market, mean of the trend parameter g_h falls close to zero, which means all market participants are close to the fundamentalist belief at average. Then it continues to drop even below zero, converting the market participants to contrarians in average.

3. In the model where skilled traders are able to predict the bias parameter b_h, the results are similar to the second case (also b_h estimates are close to zero), except for the Hurst exponent, which grows almost linearly with the increasing number of skilled traders. It reaches a value close to 0.9 in the case of the market full of skilled traders. It is therefore shown that skilled traders of this type contribute considerably to the system inefficiency.

The forthcoming paper by Vácha *et al.* (2011) therefore shows that skilled traders influence the market structure in all examined cases and change the market structure considerably when the ratio of skilled traders reaches a particular point.

Chapter 6

Introductory Hypotheses

"The great tragedy of science — the slaying of a beautiful hypothesis by an ugly fact."

Thomas H. Huxley, English biologist (1825–1895)

This chapter offers several answers to the first three 'introductory hypotheses' mentioned in Chapter 1. These hypotheses are widely general, not easy to answer unambiguously, and not very suitable for statistical testing. Therefore we have looked for answers in recent academic literature and here we present an overview giving, to the best of our knowledge, as satisfactory conclusions as possible.

6.1 Are HAMs Able to Explain stylised Facts Observed in Financial Time Series?

The general ability to replicate the important stylised facts of financial returns time series — the essential empirical patterns observed in real financial data — is perhaps the most celebrated and highlighted feature of HAMs. Indeed, almost every paper concerning any HAM topic mentions this fact. Important stylised facts and their relation to HAMs are summarised e.g. in Hommes (2006)[1] or Lux

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[1]Here we also mention several recent papers not referred in Hommes (2006) dealing to some extent with this topic. The following examples do not aim to offer any extensive list, but together with Hommes (2006) and Lux (2008) can serve as a sufficient response to the general question in the title: Gaunersdorfer (2000a), Alfarano *et al.* (2005), Föllmer *et al.* (2005), De Grauwe & Grimaldi (2006), Gaunersdorfer *et al.* (2008), Lux (2010), Franke (2009), Franke (2009), Bauer *et al.* (2009), de Jong *et al.* (2009b).

(2008). We present a list containing the most important stylised facts below:

- High trading volume;

- Excess volatility;

- Volatility clustering;

- Excess kurtosis and fat tails in the returns distribution;

- Noise amplification;

- Temporary bubbles and trend following;

- Sudden crashes and mean reversion;

- Positive correlations of returns at short horizons and negative correlation of returns at long horizons;

- Unpredictability of returns at daily horizon and mean reversion of returns at long horizon;

- Persistence of asset prices — they follow a near unit root process;

- Long memory.

6.2   Can HAMs Be Estimated on Empirical Data?

While many different HAMs have been developed and studied, surprisingly, not many attempts have been made to estimate a HAM on real market data. Moreover, only few of those have been compared in terms of forecasting power performance or in terms of fitting real financial data with alternative 'competing' approaches such as ARIMA, GARCH 'family' or alternative 'rational' models. For the reason, one has to bear in mind that *"although the heterogeneity of agents approach is intellectually satisfying, the heterogeneity model has hardly been estimated with empirical financial data because of the non-linear nature of the model that mainly arises from the existence of the mechanism that governs the switching between beliefs"* (de Jong *et al.* 2010, pg. 1653). Moreover, Westerhoff & Reitz (2005, pg. 642) draw attention to the fact that *"one has to sacrifice certain real-life market details. If the setup is too complicated, econometric analysis is precluded."* However, since the complexity of HAMs often prevents

an analytical solution, the empirical validation of such systems remains one of the most important tools of analysis.

As the question in the title is thus hardly to be answered generally, we offer examples where HAMs make a 'good job'. Therefore, this section and Section 6.3 summarise briefly the current state and divide the recent academic empirical literature on HAMs into three groups: foreign exchange market HAMs, commodity market HAMs and stock market HAMs.

Exchange market HAMs seem to be the most popular for the empirical estimation. Vigfusson (1997) rewrites the chartist-and-fundamentalist exchange rate forecasting model introduced by Frankel & Froot (1986; 1990; 1991) as a regime-switching model and he estimates it using the Canada–U.S. daily exchange rates in the period 1983–1992. The author reveals an empirical evidence to support this model and he also suggests that chartist regime seems much more important in explaining the data then the fundamentalist regime. Winker & Gilli (2001) and Gilli & Winker (2003) build on the exchange rate HAM introduced by Kirman (1991; 1993) and stress the importance as well as the high complexity of the empirical estimation, calibration, and validation of HAMs in general. They hence develop and present some computational algorithms to tackle this difficulty and use them for the estimation of the model employing the daily DM/USD exchange rates from the period 1991–2000. The authors infer that *"the foreign exchange market can be better characterised by switching moods of the investors than by assuming that the mix of fundamentalists and chartists remains rather stable over time"* (pg. 310). A further exchange rate HAM is introduced by Westerhoff & Reitz (2003). The authors estimate a STAR-GARCH model using the set of daily rates of main world currencies to the USD between 1980 and 1996. The results favour the presence of chartist- and fundamentalist-driven exchange rate dynamics and reveal substantial fluctuations of market fractions with fundamentalists leaving the market when the deviation from the fundamental value increases. This leads to a very opposite effect than the often proposed stabilising impact of the fundamentalists' market presence. The paper by De Grauwe & Grimaldi (2006) employs another exchange rate HAM to replicate the empirical 'puzzles' and anomalies such as the 'disconnect puzzle' or the 'excess volatility puzzle'. They also conclude that chartists are generally more profitable and that the notion of the self-fulfilling character and dynamics of chartist profitability are present in their model. In other words, in the world where chartists set the rules, it pays off to be a chartist. Manzan & Westerhoff (2007) estimate a model based on

the fundamentalist-chartist approach using the monthly rates of the five major world currencies (DM, JY, CD, FF, and GBP) against the USD from the January 1974 until December 1998. The model has significant explanatory power for the in-sample estimation but fits significantly only two of currencies (JY and FF) when it comes to the out-of-sample prediction. Additionally, for three currencies (DM, JY, and FF) the model outperforms the Random Walk (RW) model in terms of forecasting power. The authors conclude that their model supports the notion of short-term unpredictability and long-term predictability of exchange rate markets. An original as well as interesting application of the heterogeneous agent approach appears in de Jong *et al.* (2009a; 2010) who examine the function of the European Monetary System between its launch in March 1979 and December 1998, when it was replaced by the European Exchange Rate Mechanism II. As in the de Jong *et al.* (2009b) paper, data is taken from *DataStream*. The dataset contains several exchange rate series versus DM and includes 102 weakly and 238 monthly observations. The model works significantly better than the RW model and gives a significant evidence of the behavioural heterogeneity and agents' switching among trading strategies. The authors conclude that it thus provides a possible explanation for the target zone exchange rate dynamics. Bauer *et al.* (2009) cover almost a similar topic but study the model from the theoretical viewpoint and asses its ability to replicate empirical stylised facts. Wan & Kao (2009) improve the Westerhoff & Reitz (2003) model and offer the empirical evidence for the contrarians' (see Subsection 4.1.3) presence. Using the daily exchange rates on DM, JY, and GBP over periods January 1980 – December 1996 and April 1991 – December 2004 and the STAR-GARCH approach they extend the results by Westerhoff & Reitz (2003) and confirm the existence of contrarian strategies in the foreign exchange markets.

Commodity price series are also well suited for the HAM verification. Shiller (1984) considers the investment in speculative assets as a social activity, stresses the importance of social psychology and the influence of social movements and thus suggests that investors' behaviour and market prices would be influenced by 'fashion'. In 1984, such economic ideas were rather sporadic as EMH strongly dominated the field. The author proposes an alternative model where smart money investors with rational expectations interact with ordinary investors[2]

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[2]An interesting feature of the model is that the author does not make any assumptions about the behaviour of ordinary investors at all, but only define their total demand of stocks (pp. 477–478).

— in fact an extremely simple HAM, compares it with the efficient market model, estimates fractions of the two types of investors using S&P500 data from 1990 to 1983, and finds a substantial fluctuations — the evidence of heterogeneity and social movements influence. Finally, he shows how the efficient market model fails to forecast stock price movements as well as the magnitude of volatility. Baak (1999) develops a methodology to discover the presence of boundedly rational agents, introduces a bounded rationality model of the U.S. beef cattle market between 1900 and 1990 with two types of ranchers — in fact a HAM again, and estimates approximately one-third fraction of boundedly rational ranchers. Consequently, the comparison with the basal rational equilibrium model shows that the specification test supports the bounded rationality model. The new model also produces lower mean-squared errors of one period ahead forecasting associated with three out of four observed variables (pg. 1538). Another paper focused on the U.S. beef market by Chavas (2000) empirically examines the nature of expectations among market participants in the period of 1948–1992. The model detects a statistically significant presence of heterogeneous expectations and estimates particular proportions: 18.3% of beef producers behave according to rational expectations, 35% use quasi-rationally trend extrapolation techniques, and the rest, 46.7%, behave in a 'naive' manner when simply expecting the last observed price. Thus, the author provides another justification of the heterogeneous agent approach in market modelling. Westerhoff & Reitz (2005) introduce a basic HAM to explain cycles in commodity prices. Appling the STAR–GARCH procedure to the U.S. corn price index over the period from May 1973 to May 2003, i.e. 360 monthly observations obtained by the U.S. Department of Labour they show that technical trading can to a great extent explain commodity market cycles. Reitz & Westerhoff (2007) propose another commodity HAM. Monthly USD prices for cotton, soybeans, lead, sugar, rice, and zinc covering the period from January 1973 to May 2003 are employed together with the STAR–GARCH approach, which generally outperforms ARIMA or some other competing approaches according to various studies. Accordingly, the model can help to explain the cyclical behaviour of the commodity prices. Cyclicality of oil price is considered in Reitz & Slopek (2009) who develop an empirical oil market HAM and estimate it on the West Texas Intermediate crude oil prices taken from the International Monetary Fund International Financial Statistics database. The dataset consists of 252 monthly observations between January 1986 to December 2006 and the model gives significant evidence of the presence of oil market

speculators represented by chartists, who might thus be responsible for price movements' amplification in recent years.

Last but not least, stock market data are the next from the natural candidates to give some economical interpretation to theoretical HAMs. Alfarano *et al.* (2005) empirically estimate a HAM with the asymmetric herding properties[3] based on Kirman (1991; 1993). A two-step estimation and the maximum likelihood method are used. They employ several datasets containing daily returns of gold prices from 1974 to 1998, daily stock prices of the Deutsche Bank and Siemens in the period 1974–2001, and daily variation of DAX over the period 1959–1998. The authors conclude the model is able to reproduce the crucial stylised facts of financial returns and therefore the incorporation of asymmetric herding appears worthwhile. Boswijk *et al.* (2007) reformulate the Brock & Hommes (1998) model in terms of price-to-cash-flow ratio and estimate their HAM employing the annual S&P500 data from 1871 to 2003. The authors reveal significant evidence of two trading groups and state that their *"paper may be seen as one of the first attempts to estimate a behavioural model with heterogeneous agents on stock market data"* (pg. 7) and that it provides an explanation of the unprecedented stock price growth in the late nineties. Franke (2009) estimates the model by Manzan & Westerhoff (2005) using the controversial method of simulated moments as usual methods such as the maximum likelihood estimation becomes unfeasible for this sort of models. The model is estimated on empirical data including S&P500 from January 1980 to March 2007, DJIA, DAX, Nikkei index, USD/DM exchange rates, and finally USD/JY exchange rates and as it is concluded, it *"performs reasonably well"* in terms of the selected criterion and *"has some meaningful explanatory power"* (pg. 814). The paper by de Jong *et al.* (2009b) originally employs the heterogeneous agent approach to analyse the shift-contagion during the Asian crisis. The authors continue in the tradition of Brock & Hommes (1998) but they add a new belief type — *internationalists*, allow for multiple asset trading, and introduce a two-market — domestic and foreign — model. The model is estimated for the Thai stock exchange — Bangkok S.E.T., and the Hong Kong stock exchange — the Hang Seng data between 1980 and 2007 obtained from *DataStream*. A special focus is devoted to the crisis period 1997 and 1998 and the results are consistent with those of Westerhoff & Reitz (2003), Boswijk *et al.* (2007), de Jong *et al.* (2009a; 2010), and Frijns *et al.* (2010) —

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[3]In contrast to Winker & Gilli (2001) and Gilli & Winker (2003) who analysed the symmetric herding model.

the authors find an empirical evidence supporting the heterogeneity of traders as well as evolutionary switching between trading strategies. An innovative use of the heterogeneous agent modelling appears in Frijns *et al.* (2010) who develop a HAM for the volatility trading and pricing of options. Daily closing DAX prices covering the entire year 2000 obtained from the European Futures and Options Exchange are employed to estimate the model. Results support the hypothesised heterogeneity of active traders and the evidence for switching is even stronger for the option market than for the stock market (Boswijk *et al.* 2007).

6.3 Can HAMs Fit Real Financial Data Better than Other Competing Approaches?

Albeit the empirical literature critically discussing and comparing the forecasting abilities of HAMs with other time series modelling approaches is extremely scarce yet, we can point out the first pioneering articles on this research topic.

First outcomes of this specific sort of HAM research are briefly reported by Manzan & Westerhoff (2007) and de Jong *et al.* (2010) who simply state that their models outperform the RW model without providing any good evidence. To the best of our knowledge, the most considerable results appeared, however, not before 2010. Ellen & Zwinkels (2010) design a simple HAM for the oil market and estimate it on the Brent and West Texas Intermediate Cushing oil monthly USD prices from January 1984 to August 2009 obtained by *DataStream*, i.e. the very similar set as Reitz & Slopek (2009) used for the purposes of their work. Results then suggest a significant price effect of both fundamentalists and chartists as well as switching between trading strategies which is in accordance with Reitz & Slopek (2009) work and many preceding papers. What is new, indeed, is a serious attempt to compare the out-of-sample forecasting power of the model with the RW and VAR(1,1) models. In such test, the HAM outperforms both alternative models in terms of mean error, mean squared error, and mean absolute error in all but 2 from 60 cases comprising the 1, 3, 6, 9, and 12 month forecasting horizon. The only two cases where other model is superior are 1 month VAR(1,1) horizons. On the other hand, HAM reveals the best results in the 6 month forecast. The authors conclude that their model provides *"a parsimonious model with economic interpretation, which is able to outperform standard econometric models"*. As the literature on HAMs is comprised by

theoretical approaches, we find that this application is particularly contributing to the development of the economic interpretation of HAMs and its policy use.

Another example of a similar approach is introduced by Kouwenberg & Zwinkels (2010) who are the first who estimate a HAM for the U.S. housing market. The dataset[4] used covers quarterly observations on prices and rents for the owner occupied stock of housing ranging from 1960 to 2009, i.e. 197 records. The model is estimated for the periods 1961–1994 and 1961–2009. In both cases a significant presence of both trader types and the changing of market proportions with the dominance of chartists between 1992–2005 is reported. Finally, the model is compared according to forecasting ability with the VECM and ARIMA(4,0,0) approach. The authors employ the in-sample period 1962–2000 and the out-of-sample period 2001–2009 to compare results on the basis of mean error, mean squared error, and mean absolute error. In only 1 in 24 cases that contains the 1, 2, 3, and 4 quarters forecasting horizons HAM performs less accurately than the VECM, for the rest HAM 'wins the game'. A notable fact is that the only exception embodies the same characters as the two exceptions in the forecasting superiority of the HAM in Ellen & Zwinkels (2010) — it concerns the shortest one-period forecast horizon of the VECM. From both papers it thus seems HAMs might be especially preferable for longer forecast periods.

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[4]The dataset is uniquely available online at the *Lincoln Institute of Land Policy* web pages: `http://www.lincolninst.edu`.

Chapter 7

Empirical Benchmark Sample

"I had some of the students in my finance class actually do some empirical work on capital structures, to see if we could find any obvious patterns in the data, but we couldn't see any."

Merton Miller, American economist (1923–2000)

In this chapter, we present a unique dataset developed by the authors especially for the purpose of this thesis. Its aim is to compare and contrast the outcomes (see Chapter 9) of the HAM extended with findings from the field of BF (see Chapter 8) with empirical stock market data.

7.1 Data Choice Specification

In Chapter 8 and 9, we analyse the dynamics of the model around the *Break Point Date (BPD)* where new behavioural elements are 'injected' into the model. Therefore we find more than valuable to employ an empirical benchmark to verify or contradict our findings. Financial crises and stock market crashes can be widely considered as periods when investors' rationality is restrained and where behavioural patterns are likely to emerge, strengthen and often play the dominant role.

The benchmark dataset therefore consists of all publicly accessible individual Dow Jones Industrial Average (DJIA) stocks covering five particularly turbulent stock market periods. The era we consider starts with Black Monday 1987, the largest one-day stock market drop in the history, and terminates with the Lehman Brothers Holdings bankruptcy in 2008, one of the milestones of the

Financial crisis of 2007–2010. To support this idea, I would quote e.g. Malkiel (2003, pg. 72–76) who perceives periods surrounding Black Monday 1987 and Dot-com Bubble 2000 as typical examples of *behavioural market crashes* where not rational, but *"psychological considerations must have played the dominant role"*. We do not study, however, these events in depth and do not offer any description or explanation of them — that would be beyond the scope of this thesis. In a nutshell, Table 7.1 gives a summary of particular events and related BPDs and Figure 7.1 then graphically depicts DJIA prices and returns 20 working days before and after each of the selected BPDs.

Table 7.1: Financial Crises 1987–2011 Considered

Event	BPD	Description
Black Monday 1987	October 19, 1987 (Mo)	the historical largest one-day DJIA drop: -22.61%
Ruble Devaluation 1998	August 17, 1998 (Mo)	Russian government announced the ruble devaluation and ruble-denominated debts restructuring; important point in the Russian financial crisis
Dot-com Bubble Burst	October 10, 2000 (Fri)	NASDAQ reached its historical intra-day peak 5132.52 USD and closed at 5048.62; on Monday 13 it opened, however, at 4879.03, i.e. circa the 3% decline
WTC 9/11 Attack in NY	September 11, 2001 (Tue)	NYSE consequently closed and reopened again not before September 17 (Mo)
Lehman Brothers Holdings Bankruptcy	September 15, 2008 (Mo)	'Chapter 11' bankruptcy protection publicly announced; one of the milestones of the Financial crisis of 2007–2010

Source: The authors using various public sources.

The choice of the events is necessarily partially arbitrary and based on some economic intuition. Nonetheless, we try to minimise the arbitrariness by considering two reasonable criteria which chosen events ought to satisfy. First, the BPD should be well determinable. A good example is e.g. September 11, 2001. Second, when looking back, the event should have had a direct and substantial impact on the U.S. stock market. Therefore, e.g. Black Wednesday in the UK in 1992 is not considered.

Several financial crisis periods such as well known Scandinavian banking crises in 1990s, already mentioned Black Wednesday in 1992, Asian financial

crisis around 1997, and others do not satisfy some of these criteria well and therefore we omit them from our analysis. Moreover, to obtain a relatively mutually consistent data, we have purposely decided for the DJIA, one of the oldest, closely watched, and most stable index in the world. The need for certain consistency also restricts how deep into history we can immerse. With regard to historical changes among DJIA components, March 12, 1987 has been assessed as the historical border line as since then half of the index components have been gradually substituted. For more detail about DJIA components see Section 7.2. Additionally, if there are more potential BPD during a particular crisis period, we opt for the first one as we suppose this one is likely to have the greatest psychological impact and even if not, it fits best into the model we consider in Chapter 8.

7.2 Dataset Description

Before we start with the dataset description, we would like to remind that we are aware of the possible arbitrariness of the dataset elements choice. At the same time we have to add that this feature seems to be unavoidable. To the best of our knowledge, it is a very relevant, statistically reasonably large, and consistent benchmark sample. On the other hand, this thesis is not dependent on the theoretical accuracy of the dataset itself. Therefore, unlike trying to develop the most accurate sample, we tried to develop at least some sharing several key features. The main aim is only to have a benchmark to compare and contrast the outcomes of the extended HAM with, not to estimate the model on.

The sample consists of the price differences $(p_t - p_{t-1})$ of the daily closing prices of all publicly accessible individual DJIA stocks covering five particularly turbulent stock market periods. The price differences have been chosen instead of usually analysed stock prices or returns as they empirically resemble the outcomes of the model the most — after repeated simulations the average statistics are mutually comparable, especially in the short-run.

The sample is rather large and consists of 5519 observations. All data were collected on April 10, 2011 using the *Wolfram Mathematica* software *Financial-Data* function for data downloading. The key idea of the dataset creation is as follows. In Section 7.1 we have chosen several events of interest and the specific BPDs have been determined. According to historical development of the DJIA components we have updated the list of components for each period of interest.

Figure 7.1: Financial Crises — DJIA Prices (USD) and Returns (%)

(a) Black Monday 1987 — Prices

(b) Black Monday 1987 — Returns

(c) Ruble Devaluation 1998 — Prices

(d) Ruble Devaluation 1998 — Returns

(e) Dot-com Bubble 2000 — Prices

(f) Dot-com Bubble 2000 — Returns

(g) WTC 9/11 Attack 2001 — Prices

(h) WTC 9/11 Attack 2001 — Returns

(i) Lehman Brothers 2008 — Prices

(j) Lehman Brothers 2008 — Returns
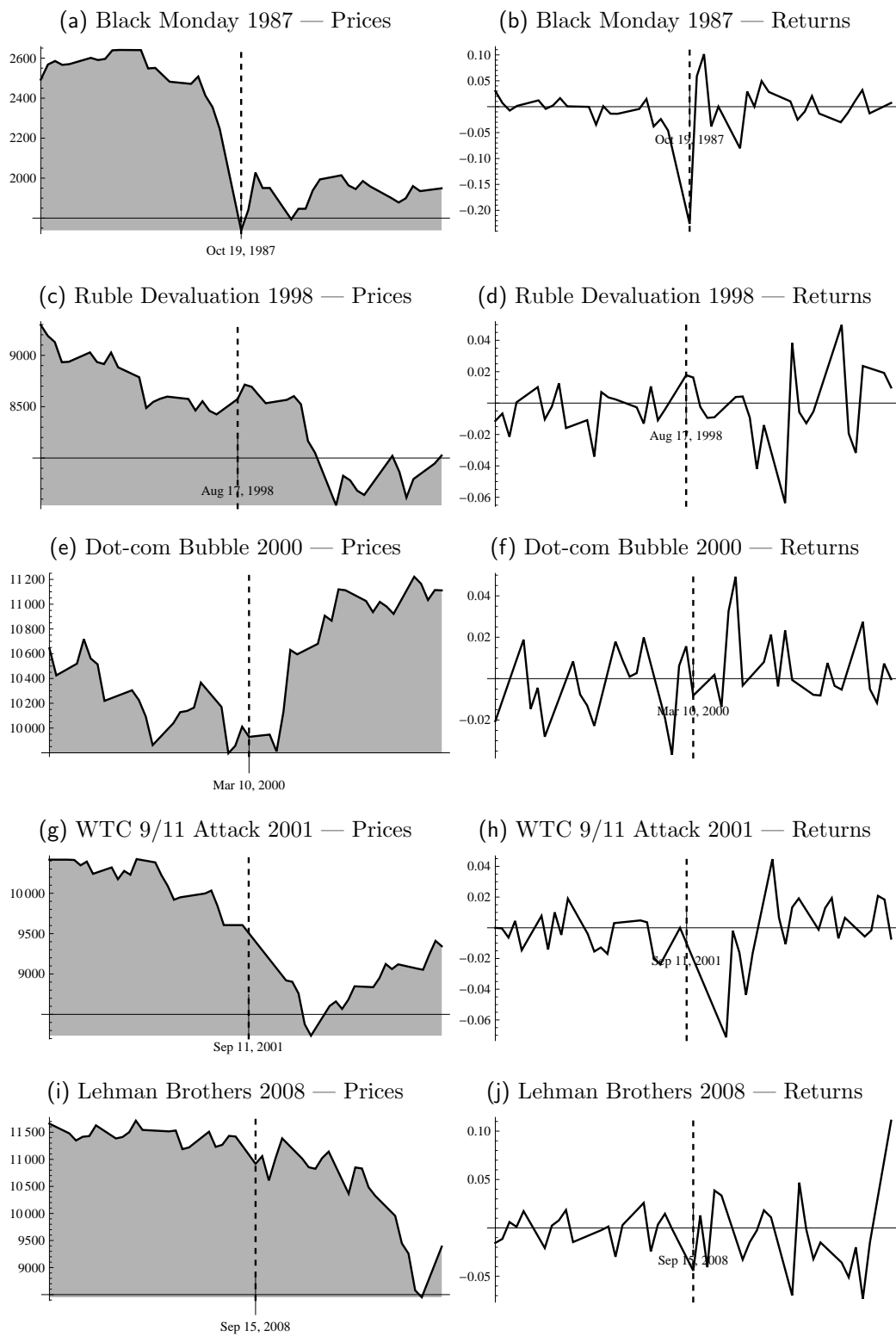
Note: The time frame of all x-axes is +/- 20 working days from the depicted BPD.

Source: The authors' own computations via *Wolfram Mathematica*.

Figure 7.2: DJIA Price Differences (USD) Distributions

(a) Black Monday 1987

(b) Ruble Devaluation 1998

(c) Dot-com Bubble 2000

(d) WTC 9/11 Attack 2001

(e) Lehman Brothers 2008
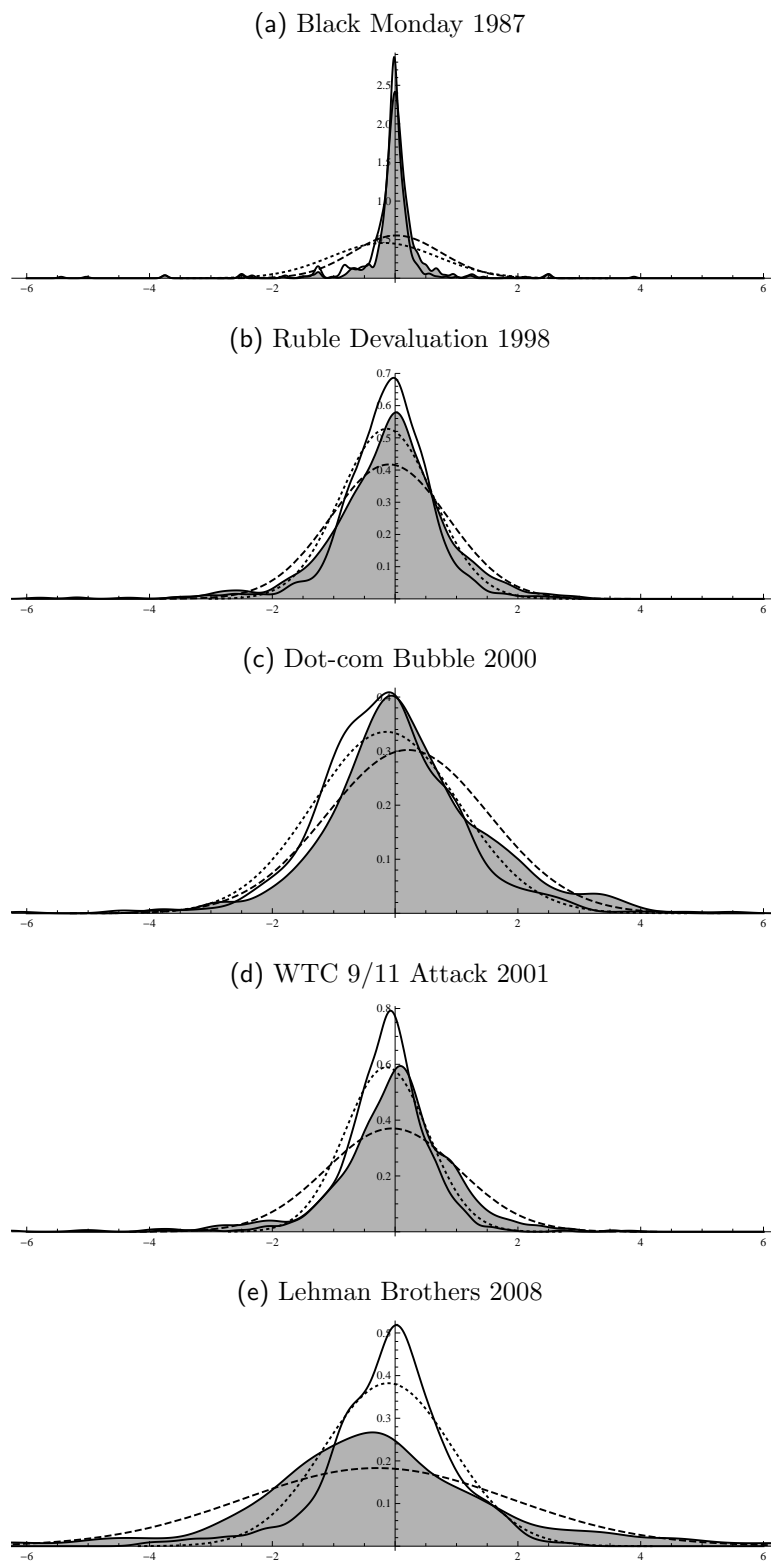
Note: Full lines depict empirical PDFs of data before and after (gray filling) the BPD.
Discontinuous lines depict fits of $N(\mu, \sigma^2)$ before (dotted) and after (dashed) the BPD.

Source: The authors' own computations via *Wolfram Mathematica*.

Then we have downloaded the price differences data for all available stocks
for two periods: 20 working days before and 20 working days after the BPD.
20 working day, i.e. 4 working weeks or 1 stylised working month. We have
decided for this rather short period for two reasons. First, the dynamics of the
model stabilises relatively fast. Second, from the psychological aspect, we ex-
pect to observe the most interesting changes in behavioural patterns especially
very close to the BPD.

Some of the historical data are, nonetheless, unavailable for particular
stocks. The reason can basically be twofold. Either the company ceased to
exist (a bankruptcy, a merge, or an acquisition) or a particular stock trading
was suspended for some time. In both cases the historical data of a particular
stock are unavailable.[1] There is one specific case of the American International
Group (AIG). This stock was removed from the index and replaced by Kraft
Food (KFT) on September 22, 2008 — i.e. during the 'Lehman Brothers 2008'
period — after a series of enormous drops around September 15, 2008. To keep
sample consistency we omit this stock as an outlier.

DJIA consists of 30 stock and the index base has changed 48 times during its
history starting in 1896. Therefore the structure of the partial samples varies
across time and due to unavailability of some data, the partial samples are not
all of the same magnitude. However, for the purpose of our analysis this is
not a problem. What we only need is a consistent set of as many observations
as possible to extract some interesting patterns of data dynamics around the
BPDs, not the very similar sets for each period. Table 7.2 outlines the changes
in the historical structure of the index and shows which historical data are
available.

Figure 7.2 depicts PDFs of distributions of the price differences $(p_t - p_{t-1})$
for each partial sample before and after the BPD. For comparison, the PDFs of
the normal distribution based on each particular sample mean and standard
deviation is depicted in the same picture for each partial sample. Excess kurto-
sis and heavy tails (see Section 6.1) are distinct at first glance. Table 7.3 then
offers a summary of important descriptive statistics keeping the same logic as
Figure 7.2. Here we would like to point out in advance that in Table 7.3 we
work with percentual changes. However, as the notion of percentual change
does not seem to have sense for negative numbers (see e.g. *The Math Forum*

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[1]If a company changed its name, it usually keeps its ticker symbol and the historical data
remain available.

for mathematical explanation[2] and suggested solution which seems, however, not very beneficial in our situation). Therefore we mention percentual changes only for non-negative descriptive statistics, i.e. variance and kurtosis.

To check the robustness of our findings, we have also developed three analogous datasets to compare the results of the primal sample. For the first control dataset we consider the very same structure of stocks but the observed period is twice as long., i.e. 40 working days. Therefore the sample consists of double amount of data. The second and third control sets comprise only those stock that 'survived' in the index for the entire observed period (AA, AXP, BA, DD, GE, GM, IBM, KO, MCD, MMM, MRK, PG, T, UTX, XOM) and we again analyse the 20 and 40 working days before and after the BPD. Given the very same set of stocks considered in each period all partial subsamples then have the same magnitude, i.e. 300 and 600 observations, respectively.

The 'global picture' of the comparison of all four datasets does not reveal, however, anything considerably different from the original dataset (see Table 7.3). There are, of course, some minor differences. The most notable seems to be the case of kurtosis — in the primal sample in four of five periods kurtosis decreases after the BPD, in the control samples the decrease holds for only three of five periods. Table A.1 in Appendix A summarises all comparisons. However, it is not obvious to what extent we should attribute those rather minor changes to twice larger observation periods, in which interesting behavioural patterns may disappear or to smaller and thus statistically less testified samples.

7.3   What Can We Infer from Data

At first glance, our data are not normal, which is strongly confirmed by the *Jarque–Bera Test* for normality of distribution (see Table 7.3). Typical stylised facts of financial returns (see Section 6.1) such as excess kurtosis or heavy tails are also fulfilled. In Table 7.3 one can clearly see *interesting shifts* of mean, variance, skewness and kurtosis between the 'before' and 'after' periods. The first three of these four descriptive statistics increase in four of five analysed

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[2]Available from: `http://mathforum.org/library/drmath/view/55720.html` [Accessed April 15, 2011]. The core part of the explanation: *"Percent change is a meaningless statistic when the underlying quantity can be positive or negative (or zero). The actual change means something, but dividing it by a number that may be zero or of the opposite sign does not convey any meaningful information, because the amount by which a profit changes is not proportional to its previous value."*

Table 7.2: DJIA Components 1987–2008

Event	Period	Stocks (ticker symbols)	#
Black Monday	Sept 21 – Oct 19, 1987	AA, AXP, BA, CVX, DD, EK, FL, GE, GM, GT, HON, IBM, IP, KO, MCD, MMM, MO, MRK, NAV, PG, T, UTX, XOM; N/A: BS, PRI, 'old' S, TX, UK, 'old' WX, Xn	460
	Oct 19 – Nov 16, 1987	–||–	460
Ruble Devaluation	Jul 20 – Aug 17, 1998	AA, AXP, BA, C, CAT, CVX, DD, DIS, EK, GE, GM, GT, HON, HPQ, IBM, IP, JNJ, JPM, KO, MCD, MMM, MO, MRK, PG, T, UTX, WMT, XOM; N/A: 'old' S, UK	560
	Aug 17 – Sep 15, 1998	–||–	560
Dot-com Bubble	Feb 10 – Mar 10, 2000	AA, AXP, BA, C, CAT, DD, DIS, EK, GE, GM, HD, HON, HPQ, IBM, INTC, IP, JNJ, JPM, KO, MCD, MMM, MO, MRK, MSFT, PG, T, UTX, WMT, XOM; N/A: IP only for Feb 23, 2000, SBC	579
	Mar 10 – Apr 7, 2000	–||–	580
WTC 9/11 Attack	Aug 10 – Sep 10, 2001	AA, AXP, BA, C, CAT, DD, DIS, EK, GE, GM, HD, HON, HPQ, IBM, INTC, IP, JNJ, JPM, KO, MCD, MMM, MO, MRK, MSFT, PG, T, UTX, WMT, XOM; N/A: SBC	580
	Sep 10 – Oct 12, 2001	–||–	580
Lehman Brothers	Aug 15 – Sep 15, 2008	AA, AXP, BA, BAC, C, CAT, CVX, DD, DIS, GE, GM, HD, HPQ, IBM, INTC, JNJ, JPM, KO, MCD, MMM, MRK, MSFT, PFE, PG, T, UTX, VZ, WMT, XOM; omitted as an outlier: AIG	580
	Sep 15 – Oct 13, 2008	–||–, but omitted AIG was replaced by KFT on September 22, 2008 — we omit KFT as well to keep dataset consistency	580

Note: # stands for number of observations. N/A stands for stock not available for a particular period.
'old' denotes companies which historical ticker symbol is nowadays used for another company.

Source: The authors using the DJIA official web pages http://www.djindexes.com.

Table 7.3: DJIA Partial Samples Descriptive Statistics

Event	B/A	Mean (USD)	Δ	Variance (USD)	Δ (%)	Skew.	Δ	Kurt.	Δ (%)	Min. (USD)	Max. (USD)	JB (p-val.)
Black Monday	B	-0.181		0.762		-6.689		72.130		-11.25	2.50	0.000
	A	0.022	↑***	0.484	↓*** 31.9	0.484	↑	37.126	↓ 48.5	-6.25	6.25	0.000
Ruble Devaluation	B	-0.136		0.571		-1.017		8.716		-5.17	2.67	0.000
	A	-0.096	↑	0.914	↑*** 60.0	-0.721	↑	6.566	↓ 24.7	-5.77	2.92	0.000
Dot-com Bubble	B	-0.143		1.413		-0.967		12.406		-10.24	4.29	0.000
	A	0.208	↑***	1.744	↑*** 23.4	0.178	↑	4.710	↓ 62.0	-6.12	5.50	0.000
WTC 9/11 Attack	B	-0.142		0.458		-0.376		7.554		-3.76	3.54	0.000
	A	-0.046	↑**	1.162	↑*** 153.6	-1.471	↓	10.328	↑ 36.7	-7.25	3.82	0.000
Lehman Brothers	B	-0.112		1.090		-0.986		7.234		- 6.97	2.86	0.000
	A	-0.295	↓**	4.744	↑*** 335.3	0.273	↑	6.603	↓ 8.7	-8.63	11.02	0.000
Summary			4/5 ↑		4/5 ↑***		4/5 ↑		4/5 ↓			

Note: All figures except Min. and Max. are rounded to 3 decimal digits, however, for percentage calculation original values are employed.

B/A denotes 'Before/After' the BPD. Δ is the change in magnitude B/A: ↑ means increase in the Set of Real Numbers $\Re$; ↓ means decrease in $\Re$.

Skew. is skewness, Kurt. is kurtosis, JB denotes the *Jarque–Bera Test* for normality of distribution performed via the *JarqueBeraALMTest* function.

Asterisks denote statistical significances of equal means xor variances B/A: *** denote hypothesis rejected at 1% significance level, ** at 5% level, * at 10% level.

Tests of equal means xor variances B/A are performed via the *MeanDifferenceTest* function and the *VarianceRatioTest* function.

Source: The authors' own computations via *Wolfram Mathematica*.

periods, while kurtosis decreases in the same ratio of cases. These findings are not only interesting from the statistical point of view. What really matters is the economic interpretation beyond.

Let us start with *mean* which rather increases. One of the possible explanations might be that after a sudden market crash the resulting short-run tendency is to compensate the huge drop in prices by several increases. Speculators might play a substantial role in this situation. These increases then statistically exceed the huge drop at the beginning which leads to the overall picture of a higher mean. Another rationale might be as follows. The market crash is a climax when the negative trend of market prices culminates. After that, therefore, there is no more space for drops and the market naturally increases.

The case of increasing *variance* is a classical textbook example of increasing market risk and unpredictability. This is one of the accompaniments of all turbulent market periods.

The most challenging issue for an economic interpretation is the increasing *skewness* tendency. This means that the mass of the distribution shifts from right to left, the right tail becomes longer and high values become more scarce. All of these features are likely to be related to general crisis tendencies but their interpretation with regard to the increasing mean does not seem unambiguous.

Finally, for the decreasing trend of *kurtosis* several straightforward explanations may apply. In the crisis period after a market crash extreme observations (both negative and positive) become more likely and, in the contrary, observations close to mean are less probable. Tails of related distribution thus become heavier and kurtosis decreases.

The most interesting part now will be to compare these considerable empirical results with the outcomes of the model simulations presented in Chapter 9.

Chapter 8

Computational Simulations

"Computers are useless. They can only give you answers."

Pablo R. Picasso, Spanish painter (1881–1973)

This chapter describes the 'heart of this thesis' — a numerical analysis and simulation techniques incorporating selected BF finding into a HAM based on the Brock & Hommes (1998) approach. In particular, we analyse the dynamics of the model around the BPD where behavioural elements are 'injected' into the system and compare it to our empirical benchmark sample based on stock market data surrounding five particularly turbulent stock market periods (see Chapter 7). Behavioural patterns are thus embedded into an asset pricing framework, which allows to examine their direct impact. The simplicity of this approach also enables us to keep the impact of behavioural modifications as clear as possible and extend the model in a variety of ways. The occurrence of behavioural patterns at the BPD is obviously not perceived as 'out of the blue' because the behaviour of market participants is subject to many biases at all times. We interpret it rather as a 'considerable amplification' of their impact during market crashes and crisis periods. From the plethora of biases, irregularities or seemingly irrational behavioural patterns studied within the field of BF we focus on three particular findings and extend the model in the following behavioural directions:

1. Herding;

2. Overconfidence;

3. Market sentiment.

The economic motivation, description, and reasons for the specific choice of these three behavioural biases have already been mentioned in Section 3.1. Most importantly, all three phenomena can be well integrated into the Brock & Hommes (1998) model framework which is rather compact and does not otherwise allow for major modifications without deviating from its overall structure.

8.1   Hypotheses

The main goal of this chapter is to provide evidence in order to confirm or reject remaining hypotheses defined at the beginnig of this thesis in Chapter 1. Chapter 9 then attempts to offer interpretations of results and discuss following statements:

- Selected BF findings can be well modelled via the HAM.

- BF findings extend the original HAM considerably.

- Different HAM modifications lead to significantly different results.

- HAM is able to replicate price behaviour during turbulent stock market periods.

8.2   Joint Setup

To be able to mutually compare all different model setups, we define a joint setup which has been used for all simulations.

Adaptive Belief System Our algorithm is based on the Hommes (2006, pg. 50) model summary. ABS in Hommes (2006) is simplified a little and rewritten in deviations from the fundamental value with slightly rearranged understanding of the time notation compared to Brock & Hommes (1998) presented in Chapter 4. Nonetheless, the logic and the essence of the model remain the same. The main reason for this choice is that the same approach has already been used, as well as considerably enriched in the most recent works of Baruník *et al.* (2009), Vácha *et al.* (2009), Vácha *et al.* (2011), and Benčík (2010) and we therefore have a great opportunity to link this thesis to the previous research. ABS is then compactly described by the three mutually dependent equations:

$$Rx_t = \sum_{h=1}^{H} n_{h,t} f_{h,t} + \epsilon_t \equiv \sum_{h=1}^{H} n_{h,t}(g_h x_{t-1} + b_h) + \epsilon_t, \qquad (8.1)$$

$$n_{h,t} = \frac{\exp(\beta U_{h,t-1})}{\sum_{h=1}^{H} \exp(\beta U_{h,t-1})}, \qquad (8.2)$$

$$U_{h,t-1} = (x_{t-1} - Rx_{t-2})\frac{f_{h,t-2} - Rx_{t-2}}{a\sigma^2}$$

$$\equiv (x_{t-1} - Rx_{t-2})\frac{g_h x_{t-3} + b_h - Rx_{t-2}}{a\sigma^2}, \qquad (8.3)$$

where ϵ_t substitutes $\delta_t = \varepsilon_t$ (see Page 27) and generally denotes the noise term representing the market uncertainty and unpredictable occasions. For details of the notation, please, consult Chapter 4.

Fixed Parameters The inevitable feature of all HAMs are too many degrees of freedom together with a large number of parameters which can be modified and studied. Therefore we need to fix several variables to be able to analyse particular changes of the model *ceteris paribus*. With reference to Baruník *et al.* (2009) and Vácha *et al.* (2009) we set the constant gross interest rate $R = 1 + r = 1.1$; and the linear term $1/a\sigma^2$ consisting of the risk aversion coefficient $a > 0$ and the constant conditional variance of excess returns σ^2 is fixed to 1. In addition to that, we use relatively small number of traders, $H = 5$ and neither memory nor learning process are implemented in the algorithm (see Section 5.5) to keep the impact of the behavioural modifications as clear as possible.

Monte Carlo Method The only reasonable option to computationally examine the impact of suggested changes on the model outcomes is to perform simple *Monte Carlo simulations* (from many e.g. Metropolis & Ulam 1949 or Rubinstein & Kroese 2008). Within this method, we repeatedly stochastically generate crucial variables using different random number generator settings and consequently run the model employing generated values. All the simulated data is then put together and this sample represents the 'true' distribution of model outcomes. The sufficient number of runs is therefore very important to obtain statistically valid and reasonably robust sample. Based on the comparison of the model outcomes for 1000, 500, 300, and 100 runs we discovered that 100 runs is enough for the purpose of this work and to show the impact of the behavioural

modifications in the model framework. The random number generator setting is then as follows:

- The *trend parameter* g_h is drawn from the normal distr. $N(0, 0.16)$;
- The *bias parameter* b_h is drawn from the normal distr. $N(0, 0.09)$;
- The *noise term* ϵ_t is drawn from the uniform distr. $U(-0.05, 0.05)$.

The proper choice of the setup values is essential for the meaningful function of the model. Bearing this in mind, with regard to the trend and bias parameters we refer to Baruník *et al.* (2009), Vácha *et al.* (2009), and Vácha *et al.* (2011) who use the same setting. Additionally, the magnitude of noise has to be considered even more carefully. Noise is an inevitable part of real world market data and any model aiming to approximate stylised reality has to include some notion of market uncertainty. Noise can be interpreted as the presence of irrational noise traders, whose behaviour can hardly be modelled and thus has to be added endogenously, or the occurrence of unexpected news. Furthermore, noise has also its irreplaceable role in the derivation and examination of the theoretical model (see Section 4.2, 5.3, or 5.4) and expresses that the model is still extremely simplified to reflect all market dynamics. Most importantly, we need to assure that the magnitude of noise does not 'overshadow' the effect of analysed modifications and be still able to distinguish what is the impact of random noise and what is the impact of our changes in the model framework. Benčík (2010) examines the effect of various noise settings, namely $\epsilon_t \in U(-0.02, 0.02)$, $\epsilon_t \in U(-0.05, 0.05)$ and $\epsilon_t \in U(-0.1, 0.1)$ and concludes that although different noise variance causes some minor changes in model outcomes (pg. 36), all models across different noises embody major similarities. In his own thesis, Benčík consequently uses only the 'golden mean' option $\epsilon_t \in U(-0.05, 0.05)$ and we use the same solution within this work.

8.3 Algorithm Description

In this section, we introduce the computational algorithm developed in *Wolfram Mathematica*. The algorithm consists of three embedded cycles. In the 'inside' cycle all computations are carried out and two 'outside' cycles step values of the intensity of choice β and the *intensity of the behavioural element* in single

decimal steps of the overall range so that we get results of 100 different run settings of β and the *Behavioural Parameter (BP)* after each simulation.

Glossary of Key Terms As the structure of the algorithm and terms used might be in some cases subjects of ambiguities, we provide a small glossary of key terms at the beginning of the algorithm description.

- *Setup combination* is a specific setting for each simulation — type of the BP has to be chosen and defined. *An example:* overconfidence affecting the bias parameter b_h only.

- *Simulation* means that a specific setup combination is executed in *Wolfram Mathematica* software.

- *Run* is a part of the simulation. Each simulation has 100 runs. For each run, different run setting, i.e. the combination of the intensity of choice β and the intensity of the BP is set. *An example:* within the simulation from the previous example the run with $\beta = 225$ and BP $= 0.25$ is executed at the moment.

- *Repeat cycle* is a part of the run. Within a run the process of random parameters generation is repeated 100 times to obtain reasonably large datasets. *An example:* within the run from the previous example the repeat cycle with randomly generated noise, trend parameters g_h and, bias parameters b_h is executed at the moment.

- *Iteration* is a part of the repeat cycle. The number of iterations defines the length of generated time series. *An example:* within each repeat cycle a time series with 250 iterations, i.e. 250 'days' long, is generated.

- *Subsamples* are outcomes of each repeat cycle. The complete subsample consists on 100 observations and the '20 day' subsample consists of 20 observations (see details below).

- *Samples* are outcomes of each run. They are generated via gradual data holding within the run process. The complete sample consists on 10000 observations and the '20 day' sample consists of 2000 observations. The algorithm produces four samples defined below.

- *Results* are obtained via statistical examination of samples after each run. After the simulation of each setup combination we thus obtain results of 100 different run settings in an aggregate form.

Stepping and Parameter Ranges For the intensity of choice β no general consensus[1] exists about any 'optimal' value to use. Moreover, to estimate β using real market data seems almost impossible[2] in practice because of the non-linear nature of the model. Hence β still remains a rather theoretical concept. However, larger β implies higher willingness of traders to switch between strategies based on their profitability — the best strategies at each specific period are chosen by more agents — and reversals in the price development are thus more likely. To comprise the large variety of possible values we use the range $\beta \in \langle 5, \dots 500 \rangle$ with single steps of 55, i.e. $\beta = \{5, 60, 115, 170, 225, 280, 335, 390, 445, 500\}$. On the other hand, for each BP the range varies and we describe its logic in further sections.

Iterations For each of 100 different repeat cycle settings, the number of iterations — the length of the generated time series — is 250 'days', i.e. $t = \{0, \dots 250\}$. The first 10% of observations are not considered because the model might need some 'time' to achieve stable behaviour. The crucial idea of our modelling approach is that we change the dynamics of the model in the middle of generated time series, i.e. from the 126. itteration further, via 'injecting' a new behavioural element and we study the dynamics of the system closely before and closely after this change. The creation of the empirical benchmark sample reflect exactly this procedure (see Chapter 7). The last 10% of observations are then not considered as well to obtain samples of the same magnitude.

Samples Having final time series simulated, we focus on *four subseries*. These comprise the first half of the whole series (100 observations) and last 20 observations of the first half, both without any behavioural element having impact to the model outcomes. Next, we separately save the first 20 observations of the second half of the whole series and the second half of the whole series (100 observations again), both effected by the behavioural element presence. During the process all 100 different run

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[1]Vácha & Vošvrda (2002) and Vošvrda & Vácha (2003) show the impact of rising β in various setups; Vácha & Vošvrda (2005) use $\beta = 80$; Vácha & Vošvrda (2007a;b) employ $\beta = 120$; Baruník *et al.* (2009) and Vácha *et al.* (2009) work with $\beta = 300$; Vácha *et al.* (2011) use $\beta = 500$; and finally Benčík (2010) employs $\beta \in \langle 0, \dots 449 \rangle$ and $\beta \in \langle 0, \dots 1000 \rangle$.

[2]Nonetheless, it is more than interesting to enquire after some (but extremely scarce) efforts to reach a conclusion about the β magnitude in related empirical studies. E.g. Frijns *et al.* (2010, pg. 2281) found that the intensity of choice *"is positive and of considerable magnitude throughout the sample"* of daily closing DAX prices covering the entire year 2000 obtained from the European Futures and Options Exchange.

setting we gradually store data of these four subsamples together and at the end we get four relatively large samples:

- The complete 'before' sample (10000 observations);
- The '20 day before' sample (2000 observations);
- The '20 day after' sample (2000 observations);
- The complete 'after' sample (10000 observations).

'Anti-overflow' Condition For the case when simulated series starts to diverge, the 'anti-overflow' condition is included in the algorithm. In such situations, the current run is interrupted and not counted into the total of 100, no data is stored and a new run with newly generated values follows.

Fundamentalists by Default There is one more configurable option in the algorithm — the default presence of fundamentalists with $g_1 = b_1 = 0$. In the theoretical model (see Chapter 4) fundamentalists play the crucial role and the algorithm thus enable the user to set whether fundamentalist strategy is generated by default for each run or whether it has the same probability of occurrence as other trend and bias combinations. When the option is 'off', the trader/strategy no. 1 is generated randomly, for 'on' the trader/strategy no. 1 is always the fundamentalist.

Hypotheses Testing Finally, for all hypotheses testing, the levels of significance 5% is set by default if not stated otherwise.

8.4   Modelling of Behavioural Patterns

This section describes our approach to the modelling of stylised behavioural patterns. For a general description and an economic motivation of selected behavioural findings, please, see Section 3.1. The model framework of Hommes (2006) is rather compact and does not otherwise allow for major modifications without deviating from its overall structure. Therefore it offers only a limited 'space' to incorporate behavioural modification into the system of Equations 8.1, 8.2, and 8.3. For instance, the well-known behavioural bias called Loss Aversion (see Chapter 3), which is based on Prospect Theory (Kahneman & Tversky 1979) is theoretically incompatible as the model is derived from the mainstream Expected Utility Theory.

Regarding particular ranges of the behavioural element intensity, which are set unavoidably arbitrary, we use two rationales to support our decisions. The common sense suggesting 'reasonable' values which seem not to be markedly far away from possible reality is guarded by the approximate 10% treshold for average number of overflows in the case of the most divergent settings (i.e. those where terminal values are employed). With a large number of overflows we might obtain filtered data which does not represent the 'true' sample distribution.

Herding As a certain notion of herding is naturally included[3] in the evolutionary adaptive system of strategy switching, in this thesis we present another original modelling approach to herding. The examination of herding patterns in HAMs is always based on short-run profitabilities of individual strategies and herding is detected via the evolution of market fractions $n_{h,t}$. This concept of herding is hence more or less (boundedly) rational. Therefore we introduce a concept of rather irrational *'blind' herding* which is based on public information and aims to imitate traders' behaviour during large stocks sell-offs after a market crash. In this approach one of trading strategies ($h = 5$) does not behave in the traditional way but copies the behaviour of the most successful trades of the previous day. At time t the strategy primarily evaluates its own performance measure, then compares the performance measures of all other strategies and for the next period $t + 1$ ti adjusts its beliefs about the trend g_5 and bias g_5 parameters, so that they mimic the last period's most profitable strategy. The mimicking effect is thus one period lagged and this delay represents the reaction of less informed market participant who just follow the crowd. Actually, at each moment, only four effective strategies are present within the model but from the mathematical point of view (Equations 4.24 & 4.25 or 8.2), outcomes vary contrasted to a model without the mimicking strategy, i.e. with 4 traders only. Including the fifth mimicking strategy can lead to substantially different results in favour of the strategy which is currently being imitated, especially when the intensity of choice β is small. For herding, no range of the BP needs to be set but we also simulate 100 setup combinations to obtain robust and comparable results.

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[3]See e.g. Chiarella & He (2002b), Chiarella *et al.* (2003), De Grauwe & Grimaldi (2006), or Hommes (2006).

Overconfidence Behavioural overconfidence can be modelled as a routine tendency to overestimate the accuracy of own judgments. Trying to incorporate this into the model framework, we are left with no other choice than work with the trend g_h and bias b_h parameters. However, this makes perfect sense and we model overconfidence as an overestimation of generated values. Roughly speaking, an overconfident trend chasing trader behaves even more surely and follows the observed trend strongly than in a normal (randomly generated) situation. He also expects the price to rise or drop even more than according to his (randomly generated) premises. The range of the 'overconfidence parameter' is $\langle 0.05, \ldots 0.5 \rangle$ and one can imagine this as the representation of the excess assurance in percentage terms — from 5 to 50%. *Three options* are examined: overconfidence affecting the trend parameter g_h only, overconfidence affecting the bias parameter b_h only, and overconfidence affecting both parameters. As overconfidence is possitive 'from definition', negative values are not considered.

Market Sentiment We model the market sentiment as shifts of the mean values of probability distributions from which the trend parameter g_h and the bias parameter b_h are generated. Both impacts of the 'positive' and the 'negative' sentiment are examined. Vácha *et al.* (2009) model the market sentiment as jumps of the trend parameter g_h between realizations from the normal distributions $N(0.04, 0.16)$ and $N(-0.04, 0.16)$. We generally consider *four options*: sentiment affecting the trend parameter g_h only (the Vácha *et al.* 2009-like case), sentiment effecting the bias parameter b_h only, sentiment effecting both parameters, and so called 'mix' case where the positive sentiment affecting bias b_h is combined with the negative sign of the trend parameter g_h and vice versa. The interpretation of both effects is, however, considerably different. If we decrease the mean of the trend parameter g_h, the contrarian strategies are more likely to be generated. Nonetheless, this does not tell much about the type of sentiment we have set — whether the sentiment is intended to be positive or negative — we primarily adjust the response of agents to actual price development. On the other hand, manipulating with the mean of the bias parameter b_h we directly set the trend — decreasing the mean we model negative market sentiment ('price rather drops tomorrow') and vice versa. Next, as we aim to model the behaviour in extremely turbulent times, we

assume higher shifts than 'only' one tenth of the standard deviation as in Vácha *et al.* (2009). The range for the 'positive sentiment parameter' is $\langle 0.04, \ldots 0.4 \rangle$ for trend and $\langle 0.03, \ldots 0.3 \rangle$ for bias, i.e. the minimum is one tenth of the standard deviation and the maximum equals one standard deviation of related distributions. For degree of negative sentiment, the opposite values $\langle -0.4, \ldots -0.04 \rangle$ for trend and $\langle -0.3, \ldots -0.03 \rangle$ for bias are used. At first sight, this approach might seem a little similar to the overconfidence modelling, but the opposite is true. Although both appertain to the trend and bias parameters, in the overconfidence case we only symmetrically strengthen traders' responses to the current market development, while in the market sentiment case we asymmetrically deflect the market behaviour and traders' beliefs.

Example Outcome Series To capture the impact of the behavioural elements incorporation on the model outcomes, Figure 8.1 depicts three pairs of randomly generated series for a single repeat cycle — one for herding, one for overconfidence affecting both parameters and one for sentiment affecting both parameters — with $\beta = 225$ and BP $= 0.25$ for overconfidence and 0.2 (g_h) & 0.15 (b_h) for the market sentiment. One can clearly see the structural change at the BPD. Regarding the single series properties, these are strongly influenced by the random combination of generated parameters. Thus, once the series might exhibit an upward shift, once it might be biased down, but in total the resulting distribution indeed exhibits properties which are to large extent comparable with real market data.

Figure 8.1: Example Outcomes of One Repeat Cycle

(a) Original Series $\beta = 225$

(b) Herding $\beta = 225$

(c) Original Series $\beta = 225$

(d) Overconfidence $\beta = 225$, BP $= 0.25$

(e) Original Series $\beta = 225$

(f) Market Sentiment $\beta = 225$, BP $= 0.2$ & 0.15
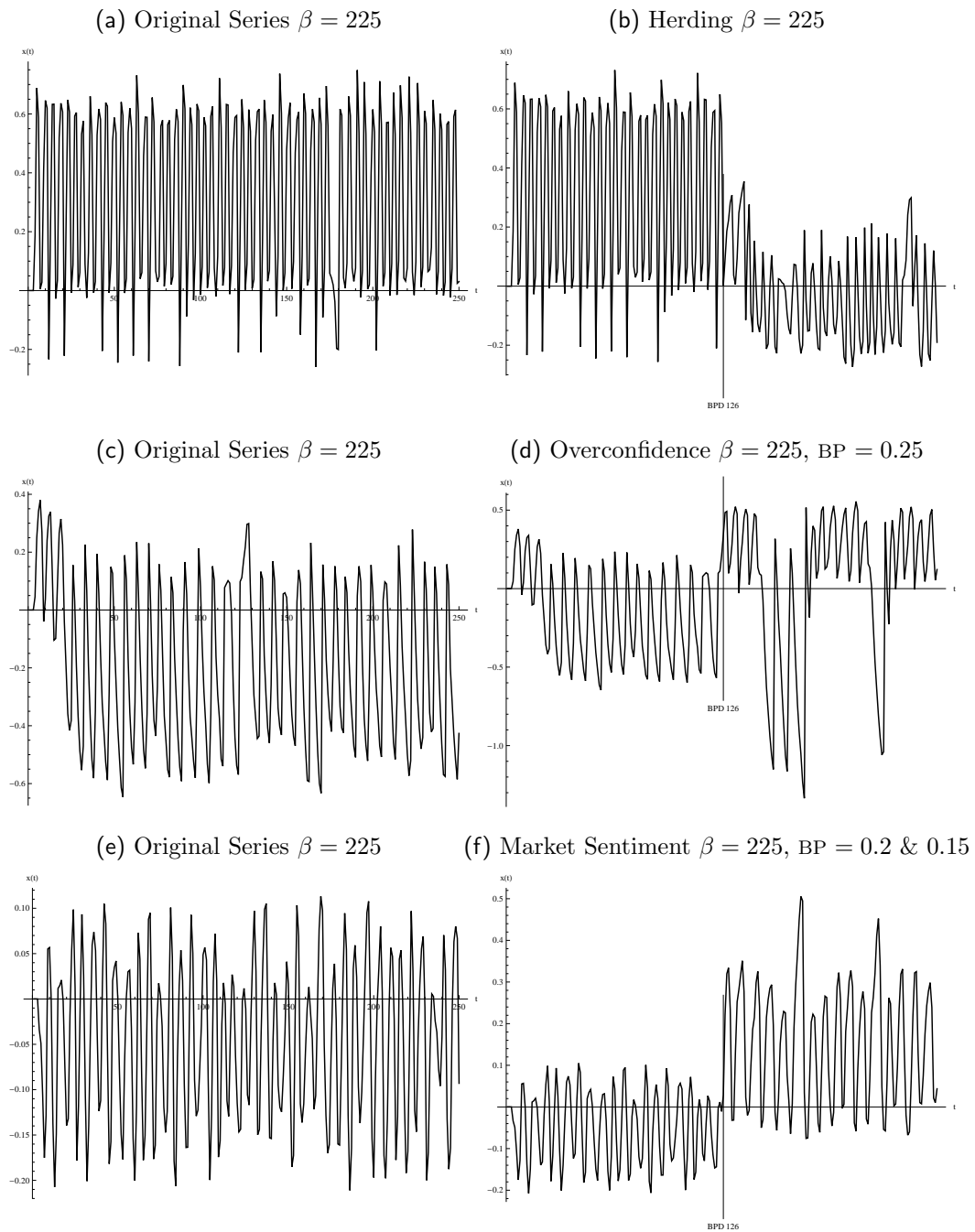
Source: The authors' own computations via *Wolfram Mathematica*.

Chapter 9

Results and Interpretations

"Results? Why, man, I have gotten a lot of results. I know
several thousand things that won't work."

Thomas A. Edison, American inventor (1847–1931)

In this chapter, after simulating and analyzing of more than 100 million observations, we present the most interesting findings.

9.1 Simulations Results

We simulate 13 different setup combinations with the 'fundamentalists by default' option 'off' including: no behavioural impact, herding, 3 setup combinations for overconfidence, and 8 setup combinations for market sentiment (see Section 8.4 for more setup details). The overview of the aggregate results is summarised in Table 9.1 and Table A.2 in Appendix A then depicts the aggregate descriptive statistics.

Within each simulation, we keep tracking many features. First, we evaluate the same pattern which has been revealed within the empirical benchmark sample (see Chapter 7), i.e. the shifts of mean, variance, skewness, and kurtosis between the 'before' and 'after' periods. The first three of these four descriptive statistics increase in four of five analysed periods, while kurtosis decreases in the same ratio of cases. We also mention the arithmetic average of the percentual magnitude changes before and after the BPD for variance and kurtosis. Second, employing the *Cramér–von Mises Test* for equal distributions we observe whether there are statistically significant differences among particular samples,

Table 9.1: Simulations Results — Overview

Sample	Mean ↑ (out of 100 runs)	Var. ↑ (out of 100 runs)	∅Δ (%)	Skew. ↑ (out of 100 runs)	Kurt. ↓ (out of 100 runs)	∅Δ (%)	Cramér–von Mises T.				Jarque–Bera T.			
							B-b	b-a	a-A	A-B	B	b	a	A
No Behavioural Impact	48	53	9.3	50	48	18.5	100	100	100	100	2	7	7	2
Herding	78	39	24.0	60	44	44.0	100	61	100	4	1	1	3	1
Overconfidence (bias)	47	97	81.8	51	54	41.7	100	18	100	2	1	4	7	0
Overconfidence (trend)	52	94	145.9	48	8	557.1	100	98	100	58	1	7	0	0
Overconfidence	53	99	548.7	53	4	695.5	100	14	100	1	4	13	1	0
M. Sentiment+ (bias)	100	79	33.6	96	36	209.9	100	3	100	0	1	8	5	2
M. Sentiment+ (trend)	47	99	251.9	47	9	641.7	100	46	100	17	1	16	2	0
M. Sentiment+ (mix)	100	3	-20.0	51	74	-20.7	100	1	100	0	1	6	16	4
M. Sentiment+	100	99	443.3	86	10	740.4	100	4	100	0	3	7	0	0
M. Sentiment- (bias)	2	75	32.9	10	40	56	100	3	100	0	1	6	6	0
M. Sentiment- (trend)	44	2	-25.5	46	86	-24.5	100	70	100	22	2	3	18	5
M. Sentiment- (mix)	0	97	429.1	18	13	713.4	100	1	100	0	2	9	2	2
M. Sentiment-	0	2	-21.6	32	78	-20.7	100	3	100	0	1	3	10	3

Note: Percents are rounded to 1 decimal digit. Var. is variance, Skew. is skewness, Kurt. is kurtosis, +/- denote 'positive/negative' market sentiment.

B/A denotes 'Before/After' the BPD. Caps stand for the complete samples, small letters for the '20 days' samples (see Section 8.3).

∅Δ is the arithmetic average of % magnitude changes B/A the BPD for variance and kurtosis: ↑ means increase in the Set of Real Numbers $\Re$; ↓ means decrease in $\Re$.

Cramér–von Mises Test for equal distributions B/A (H_0) is performed via the *CramerVonMisesTest* func. and the number of non-rejections at 5% sig. level is counted.

Jarque–Bera Test for normality of distribution (H_0) is performed via the *JarqueBeraALMTest* function and the number of non-rejections at 5% signif. level is counted.

Source: The authors' own computations via *Wolfram Mathematica.*

namely we compare: the complete 'before' sample (B) and the '20 day before' sample (b), the '20 day before' sample (b) and the '20 day after' sample (a), the '20 day after' sample (a) and the complete 'after' sample (A), and finally the complete 'after' sample (A) and the the complete 'before' sample (A). We generally expect B & b and A & a to be largely similar and, on the other hand, b & a and A & B to exhibit strong dissimilarities. Third, using the *Jarque–Bera Test* for normality of distribution we examine the non-normality of particular samples. We expect samples to be non-normal and strongly *leptokurtic*, some exceptions may appear within the 20 days samples because of only a limited number of observations after each run.

For each descriptive statistic we compute the arithmetic average value. In some cases, the results in Table 9.1 and Table A.2 might appear at odds that this is caused by different methodology of collecting results — in Table 9.1 we just count how many setup combinations out of total 100 fulfill particular feature regardless their magnitude while figures in Table A.2 depicting the arithmetic averages are more vulnerable to be influenced by several markedly small or large numbers. We always depict values for the '20 days before' sample and '20 day after' sample and for variance and kurtosis we compute the percentual change in the average magnitude before and after the BPD. Finally, we depict maximum and minimum values and generally expect higher fluctuations in the 'after' samples. Below we comment on the most interesting findings step by step.

Normality According to the *Jarque–Bera Test* for normality of distribution, simulated samples are largely non-normal with substantial excess kurtosis, thus likely to exhibit leptokurtic properties and confirm our prediction. This fundamental finding is thus consistent with the empirical benchmark sample where all subsamples exhibit excess kurtosis. For the complete samples, there is only several cases of normality, perhaps those with low BPs. However, several 20 day samples — especially the market sentiment cases — reach considerable values. It seems that market sentiment affecting the trend g_h only and the 'mixed' case (also trend-affecting) produce more normal samples than all other setup combinations. Apparently, modifications affecting trend g_h when the behavioural 'sentiment parameter' is included have certain tendency to offset the ability to produce real market-like leptokurtic distributions — one of the most highlighted features of the model. This finding is, however, partially consistent with

the benchmark sample tendency to exhibit decreases in kurtosis after the BPDs.

Empirical Pattern Fitting According to the logic of this thesis, the empirical benchmark pattern fitting is the most important feature to examine. Viewed in this light, we will comment on results of each of the three behavioural modifications individually.

- *Herding* seems to affect the model structure the least. It exhibits more or less an average effect on all descriptive statistics with prevailing effect of kurtosis increase which goes counter the empirical findings. Although the presence of the herding towards the most profitable strategy produces some minor differences, namely mean shifts and dissimilarities in distributions, herding effect is comparatively rather similar to the case without any behavioural impact.

- *Overconfidence* has a middle impact of the three behavioural modifications. All three overconfidence setups increase variance in the large majority of cases, which is not only comparable with the empirical findings but also with conclusions of Daniel *et al.* (1998) (see Subsection 3.1.2). The results of mean and skewness shifts are roughly '50:50', i.e. comparable with the case without any behavioural impact. An intriguing feature is the rapid variance and kurtosis increase after the BPD in both cases with the trend overconfidence. The *'winner'* is therefore the setup combination of overconfidence affecting the bias parameter b_h only, which reveals substantially higher ability to fit the decreasing kurtosis empirical pattern. At the same time it does not produce such extensive variance and especially kurtosis increases. When thinking about an economic interpretation, one can understand this specific setup as a situation when all market participant strictly use similar pricing models and thus their trend extrapolation is not a subject to any bias, while their personal feelings and expectations of the market future development are highly impacted by overconfidence.

- *Market sentiment* seems to be — compared with the previous modifications — the most promising behavioural change of the model structure. At first glance, effects of the positive and the negative market sentiment generate roughly inverse results. The puzzling fact

is that the positive market sentiment fits the empirical benchmark sample pattern considerably better, whereas the negative market sentiment is able only to mimic decreases in kurtosis (2 of 4 cases) and variance increases (2 of 4 cases). This might be explained in a similar way as we offer for the positive mean shifts within the empirical benchmark sample in Chapter 7 (see Page 75).

Further on we therefore focus on the positive market sentiment only. Market sentiment affecting trend g_h seems to be a weak modification as it exhibits average values for the mean and skewness shifts and has very low performance in the case of kurtosis. Mixed sentiment cancels the important variance shift almost entirely out but is able to mimic the kurtosis decrease. Again, we observe excessive variance and especially kurtosis upward jumps when the trend affecting sentiment is introduced. Sentiment changes affecting either both trend g_h and bias b_h or bias b_h only seem to be the most successful modifications and we again announce the market sentiment affecting bias parameter b_h only the *'winner of the contest'*. This particular modification embodies higher performance in the kurtosis decrease fitting and, most importantly, it exhibits much more reasonable percentual changes of variance and kurtosis in comparison with the empirical values.

Behavioural Modifications Strength Employing the *Cramér–von Mises Test* for equal distributions we study the the ability of particular behavioural modifications to produce significantly different data distributions before and after the BPD. From this perspective, herding and all trend affecting setup combinations generally seem to be the weakest modifications with overconfidence affecting trend g_h fore. On the other hand, almost all non-trend sentiment modification exhibit excellent results from this point of view. Our expectation that the 20 day samples and the complete samples from the same period come from the same distribution has been 100% affirmed as no single deviation appeared.

Volatility Increases of variance seem to be the most robust results of the majority of the overconfidence and market sentiment setup combinations. This finding confirms the conclusions of Daniel *et al.* (1998) and Diks & Weide (2005) (see Subsection 3.1.2). The tendency of the minimum and maximum values then also more or less confirms our expectation of

higher volatility after the BPD. Nonetheless, the minimum and maximum values are rather chance results as they only represent a single extreme value of all 100 runs.

9.2   Model Extensions

In this section, we compare the primary results of Section 9.1 with several possible model extensions and examine the effect of distinctive model structure changes on the robustness of related results. The same setup combinations except those with the negative market sentiment, which generate not very applicable result, are investigated. The overviews of the aggregate results are summarised in Table 9.2 & 9.3 and Table A.3 & A.4 in Appendix A then depict the aggregate descriptive statistics. The structure of the tracked information is the same as in Section 9.1.

9.2.1   Fundamentalists by Default

As fundamentalists play the crucial role in the theoretical model framework, we now turn the 'fundamentalists by default' option 'on'. Therefore, the fundamentalist strategy with $g_1 = b_1 = 0$ is always present in the model.

To sum up, fundamentalists' default presence seems to have only a minor and non-systematic impact on the model outcomes. It affects mainly kurtosis and moves only two setup combinations slightly closer to the empirical benchmark sample. The effect on kurtosis is also evident from the results of the *Jarque–Bera Test* which rejects normality in more cases than within the primary results.

- In the *herding case*, fundamentalists cause less similar distributions of the 'before' and 'after' 20 day samples.

- In the *overconfidence case*, fundamentalists strongly decrease variance but only for the setup combination where both trend g_h and bias b_h are affected. For the same setup kurtosis strongly increases in the 'before' sample and strongly decreases in the 'after' sample. In this specific case fundamentalists thus balance the kurtosis magnitudes before and after the BPD.

- In the *market sentiment case*, we can also observe decreasing fundamentalists' impact on kurtosis, but only in the 'mix' case.

Table 9.2: Extensions Results I. — Overview

Extension & Sample	Mean ↑ (out of 100 runs)	Var. ↑ (out of 100 runs)	∅Δ (%)	Skew. ↑ (out of 100 runs)	Kurt. ↓ (out of 100 runs)	∅Δ (%)	Cramér–von Mises T.				Jarque–Bera T.			
							B-b	b-a	a-A	A-B	B	b	a	A
Fundamentalists:														
Herding	76	29	14.0	73	37	30.2	100	38	100	1	0	1	0	0
Overconfidence (bias)	51	99	86.3	46	52	57.5	100	19	100	4	1	6	5	0
Overconfidence (trend)	45	98	127.8	42	6	419.8	100	99	100	70	2	9	2	1
Overconfidence	52	98	103.8	53	66	50.1	100	20	100	4	0	4	2	1
M. Sentiment+ (bias)	99	78	32.9	93	32	107.4	100	3	100	0	2	3	3	2
M. Sentiment+ (trend)	56	98	241.7	60	6	524.7	100	49	100	21	0	9	0	0
M. Sentiment+ (mix)	100	2	-27.8	65	84	-27.0	100	2	100	0	0	2	14	2
M. Sentiment+	100	98	394.4	87	8	606.0	100	3	100	0	1	3	1	0
Stochastic Parameters:														
Herding	80	36	-5.1	47	50	56.5	100	76	100	13	1	7	7	1
Overconfidence (bias)	41	98	74.8	52	51	27.2	100	0	100	0	2	6	5	1
Overconfidence (trend)	50	99	167.8	52	2	765.8	100	100	100	54	3	11	0	0
Overconfidence	58	100	317.1	56	3	636.4	100	0	100	0	4	8	1	0
M. Sentiment+ (bias)	100	86	20.7	94	37	61.4	100	0	100	0	2	7	1	0
M. Sentiment+ (trend)	51	99	316.0	54	1	957.8	100	18	100	0	0	4	0	0
M. Sentiment+ (mix)	100	3	-21.7	44	89	-22.8	100	0	100	0	0	6	24	5
M. Sentiment+	100	100	356.4	98	3	946.0	100	0	100	0	3	5	0	0

Note: Percents are rounded to 1 decimal digit. Var. is variance, Skew. is skewness, Kurt. is kurtosis, +/– denote 'positive/negative' market sentiment.

B/A denotes 'Before/After' the BPD. Caps stand for the complete samples, small letters for the '20 days' samples (see Section 8.3).

∅Δ is the arithmetic average of % magnitude changes B/A the BPD for variance and kurtosis: ↑ means increase in the Set of Real Numbers $\Re$; ↓ means decrease in $\Re$.

Cramér–von Mises Test for equal distributions B/A (H$_0$) is performed via the *CramerVonMisesTest* func. and the number of non-rejections at 5% sig. level is counted.

Jarque–Bera Test for normality of distribution (H$_0$) is performed via the *JarqueBeraALMTest* function and the number of non-rejections at 5% signif. level is counted.

Table 9.3: Extensions Results II. — Overview

Extension & Sample	Mean ↑ (out of 100 runs)	Var. ↑ (out of 100 runs)	∅Δ (%)	Skew. ↑ (out of 100 runs)	Kurt. ↓ (out of 100 runs)	∅Δ (%)	Cramér–von Mises T.				Jarque–Bera T.			
							B-b	b-a	a-A	A-B	B	b	a	A
Combinations:														
Herd. & Over. (bias)	88	49	52.7	62	33	137.9	100	72	100	8	3	8	4	1
Herd. & Sent.+ (bias)	99	58	111.3	87	29	156.2	100	0	100	0	1	8	1	0
Over. (b.) & Sent.+ (b.)	100	97	112.7	91	31	80.8	95	1	100	0	1	10	3	2
H. & O. (b.) & S.+ (b.)	100	66	58.6	86	33	122.9	100	0	100	0	1	10	2	0
Memory:														
Herding	85	69	70.9	65	43	60.1	97	33	95	10	0	1	0	0
Overconfidence (bias)	49	95	135.2	47	53	101.9	100	11	98	3	1	2	2	1
Overconfidence (trend)	41	95	96.2	46	50	28.7	97	15	95	3	0	1	2	1
Overconfidence	51	99	362.1	52	8	400.4	99	16	96	3	0	1	1	0
M. Sentiment+ (bias)	99	64	122.2	77	48	161.3	100	2	37	0	0	4	2	0
M. Sentiment+ (trend)	55	95	299.2	59	18	345.3	100	25	96	10	0	0	1	0
M. Sentiment+ (mix)	100	4	-30.1	67	76	-15.7	99	1	34	0	0	2	0	0
M. Sentiment+	98	97	489.5	84	14	506.6	98	1	31	0	1	2	0	0

Note: Percents are rounded to 1 decimal digit. Var. is variance, Skew. is skewness, Kurt. is kurtosis, +/- denote 'positive/negative' market sentiment.

B/A denotes 'Before/After' the BPD. Caps stand for the complete samples, small letters for the '20 days' samples (see Section 8.3).

∅Δ is the arithmetic average of % magnitude changes B/A the BPD for variance and kurtosis: ↑ means increase in the Set of Real Numbers $\Re$; ↓ means decrease in $\Re$.

Cramér–von Mises Test for equal distributions B/A (H_0) is performed via the *CramerVonMisesTest* func. and the number of non-rejections at 5% sig. level is counted.

Jarque–Bera Test for normality of distribution (H_0) is performed via the *JarqueBeraALMTest* function and the number of non-rejections at 5% signif. level is counted.

Source: The authors' own computations via *Wolfram Mathematica*.

9.2.2 Stochastic Formation of Parameters

It is perhaps more realistic to assume that the levels of the market sentiment or overconfidence as well as the tendency to switch between particular strategies vary both among traders and in time. In an effort to reflect this sensible idea, we introduce the stochastic formation of the intensity of choice β as well as the BP intensity. Both parameters are newly generated randomly for each run, which means we employ 10000 different random combinations of these two model inputs for each setup combinations. Parameter ranges remain the same, i.e.:

- The *intensity of choice β* is drawn from the uniform distribution $U(5, 500)$;

- The *intensity of overconfidence* is drawn from the uniform distribution $U(0.05, 0.5)$;

- The *intensity of the market sentiment* is drawn from the uniform distribution $U(0.04, 0.4)$ for trend g_h and from $U(0.03, 0.3)$ for bias b_h.

As we expected, stochastic formation of parameters does not bring any considerable difference to model outcomes. In fact, this modification does not change the structure of the model at all, it only ensures that inputs are more randomly generated. Thus the only notable impact is the tendency to less similarities among 'before' and 'after' 20 day samples which is probably caused by the non-deterministic nature of parameters.

9.2.3 Combinations

We also consider the possible combinations of more behavioural modifications in the model and examine whether combinations of impacts lead rather to a consolidation of results or to cancelling the effects out. We combine four 'winning' setups and hence following combinations are analysed:

- *Herding & overconfidence* affecting the bias parameter b_h only;

- *Herding & market sentiment* affecting the bias parameter b_h only;

- *Overconfidence & market sentiment* affecting the bias parameter b_h only;

- *Combination of all* these three modifications.

Results of simulations seem ambiguous and at first glance it is not clear which component dominates. The combinations of various modifications lead more likely to cancelling the effects out than to a consolidation of results. Market sentiment seems to be the governing element, but its effect is comparatively smaller when it is combined with other modifications than when it works alone. Herding follows and outperforms the impact of overconfidence. Compared with the primary findings, no strong tendencies are apparent, new results are most often either the same or somewhere 'half-way' between original results.

9.2.4   Memory

The effect of memory is undoubtedly one of the most interesting features which can be studied within the heterogeneous agent modelling. The real-life parallel with the human memory and learning is apparent. As Hommes (2006, pg. 56) points out, *"another important issue is how memory in the fitness measure affects stability of evolutionary adaptive systems and survival of technical trading"*. Memory effect has been widely examined in various studies (see Chapter 5) and we also incorporate this notion into our system. Memory process is the most distinctive modification of the model structure from all four considered extensions and for its implementation we use the similar approach as Baruník *et al.* (2009) and Vácha *et al.* (2009). More formally, Equation 8.3 is extended to the form:

$$
\begin{aligned}
U_{h,t-1} &= \frac{1}{m_h} \sum_{l=0}^{m_h-1} \left[(x_{t-1-l} - Rx_{t-2-l}) \frac{f_{h,t-2-l} - Rx_{t-2-l}}{a\sigma^2} \right] \\
&\equiv \frac{1}{m_h} \sum_{l=0}^{m_h-1} \left[(x_{t-1-l} - Rx_{t-2-l}) \frac{g_h x_{t-3-l} + b_h - Rx_{t-2-l}}{a\sigma^2} \right], \quad (9.1)
\end{aligned}
$$

where m_h denotes the memory length of each particular strategy h. In simulations, *memory lengths* m_h are the generated randomly from the uniform distribution $U(0, 20)$. This range has been defined with regard to computational time necessary for the memory implementation (which is in this particular setting more than 2.5 times higher than with zero memory) and also considering the length of the observed periods before and after the BPD. In Figure 8.1 and 9.1 we depict three pairs of randomly generated series for a single repeat cycle with memory — one for herding, one for overconfidence affecting both parameters, and one for the market sentiment affecting both parameters with

the same parameter setting. Again, one can clearly see how memory changes the structure of the whole series both before and after the BPD.

Figure 9.1: Example Outcomes of One Repeat Cycle with Memory

(a) Original Series $\beta = 225$ (b) Herding $\beta = 225$

(c) Original Series $\beta = 225$ (d) Overconfidence $\beta = 225$, BP $= 0.25$

(e) Original Series $\beta = 225$ (f) Market Sentiment $\beta = 225$, BP $= 0.2$ & 0.15
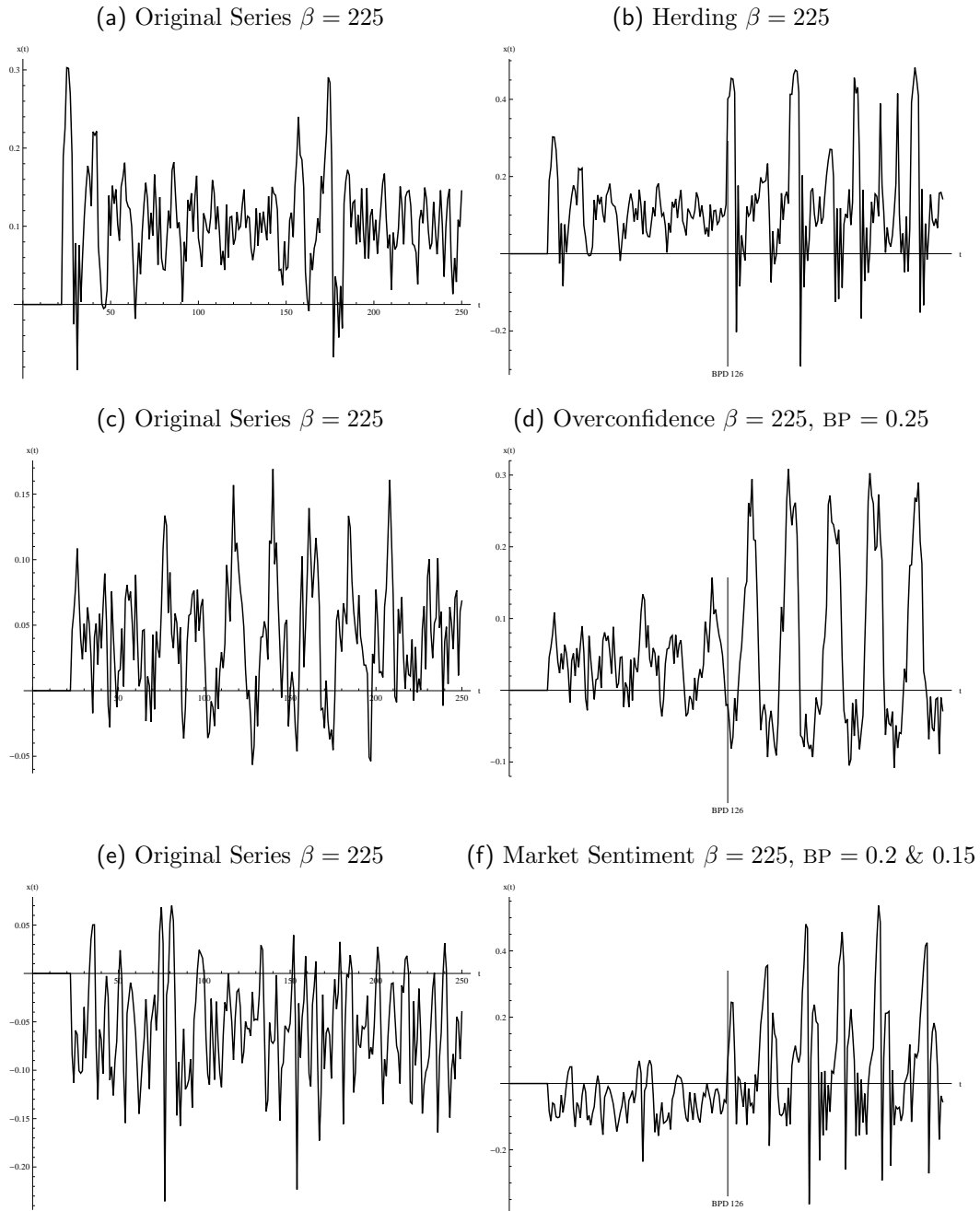
Source: The authors' own computations via *Wolfram Mathematica*.

Although memory has surely the most evident impact on the model outcomes, it still does not cause major dissimilarities compared to the primary results. The presence of memory impacts mainly the structure of data distributions and make them substantially mutually distinct and non-normal as is evident from the aggregate results of the *Cramér–von Mises* and *Jarque–Bera* Tests — the number of rejections of equal distributions rises almost for all setup combinations and the same holds for normality. Moreover, memory enormously affects the structure of the '20 day after' sample which does no longer share the same distribution with the 'complete after' sample for the majority of runs. This result suggests a strong memory effect around the BPD. Memory also influences kurtosis and moves the ratio of grows and declines closer to the empirical benchmark sample values for some setup combinations. Considering the *'winning'* modification — the case of the *market sentiment* affecting the bias b_h parameter only — memory slightly reinforces its ability to fit the benchmark sample pattern. Memory perhaps also partially affects skewness and variance, but these deviations are largely non-systematic and might be rather chance results.

9.3   Interpretation and Generalization of Results

In this section, we would like to answer four remaining hypotheses form the beginning of this work. By doing so, we recapitulate important results and offer the reader a 'synthesised' summary.

Selected BF findings can be well modelled via the HAM. To recap, in Chapter 8 we model herding, overconfidence, and market sentiment and extend the model via these three behavioural elements. *Herding* is modelled so that one of strategies copies the behaviour of the most successful trades of the previous day. One period delay thus represents the reaction of traders who just follow the crowd. *Overconfidence* is modelled as an overestimation of randomly generated values. An overconfident agent behaves even more surely than in a normal situation. Finally, we model the *market sentiment* as shifts of the mean values of probability distributions from which agents' beliefs are generated. Agents are thus more optimistic or pessimistic when forecasting future prices. All three modelling approaches perfectly follow the economic intuition and we can resume that HAM is well suited for their implementation.

BF findings extend the original HAM considerably. In Table 9.1 one can see how the behavioural elements 'injected' into the system change the structure of generated data. The impact on descriptive statistics is more than apparent. The *Cramér–von Mises Test* confirms that data distributions before and after the BPD differ considerably and statistically significantly. For graphical illustration we refer to Figure 8.1 and 9.1 capturing the impact of all three behavioural elements on the model outcomes. Although we depict only three randomly generated series, the structural change at the BPD is evident.

Different HAM modifications lead to significantly different results. Table 9.1 also illustrates how particular modifications affect the structure of the HAM. *Cramér–von Mises Test* again confirms the statistical significance of mutual differences among data distributions. Moreover, we analyse the impact of the four additional model extensions. These comprise the default presence of *fundamentalists*, *stochastic formation* of parameters which are in the basic setup defined discretely, *combinations* of the 'winning' setups and *memory*. Impacts of particular extensions on the model outcomes vary but generally they cause more or less only minor differences. However, *memory* exhibits the most evident and statistically significant influence on the model structure.

HAM is able to replicate price behaviour during turbulent stock market periods. As financial crises and stock market crashes can be widely considered as periods when investors' rationality is restrained and where behavioural patterns often play the dominant role, we compare the model outcomes with the empirical benchmark dataset developed especially for the purpose of this thesis. The sample consists of the individual DJIA stocks covering five particularly turbulent stock market periods starting with Black Monday 1987 and terminating with the Lehman Brothers bankruptcy in 2008. Most importantly, we reveal an interesting pattern among this data. When data before and after the BPD are compared, mean, variance, and skewness increase in four of five analysed periods, while kurtosis decreases in the same ratio of cases. These findings are not only interesting from the statistical point of view but may suggest a more general tendency. Table 9.1 summarises how different modifications of the model fit this pattern. In particular, the case of the *market sentiment* affecting the bias parameter b_h only exhibits a very good fit in three of four observed

aspects. When *memory* is added into the system (see Table 9.3), some minor changes are observed and the ability of this particular modification to fit the benchmark pattern is even reinforced.

9.4 Suggestions for Future Research

During the writing of this thesis, several new ideas have appeared which might be an inspiration for future research. To quote form LeBaron (2000): *"The field is only in its infancy, and much remains to be done."*

First, it would be useful to investigate the robustness of the model from different aspects — more traders might be considered, they might choose from more than one risky asset (see Hommes 2006, pg. 56), or costs to obtain information might be included. Also various modifications of the model allowing for different additional features (see Chapter 5) might be employed. Large datasets for each run, or more generally a higher number of runs might be generated. However, that would need to anticipate more extensive computational capacity and time — yet simulations analysed within this thesis already need more than 50 hours to be executed on a standard personal computer.

Second, other feasible behavioural patterns might be identified and examined. On the other hand, a different methodology might be applied to behavioural patterns considered in this thesis. More 'standard' tools of a single time series analysis mentioned in Chapter 5 such as autocorrelation functions, the Hurst exponent, bifurcation diagrams, and market fractions $n_{h,t}$ tracking in time might reveal other interesting features. But as this research direction is more focused on the pure impact analysis of a specific modification on the model behaviour and has been accomplished many times in the past, this thesis has intentionally avoid it.

A rather challenging task would be to redesign the model or to develop a model of the same spirit which would be compatible with Prospect Theory (Kahneman & Tversky 1979). That would enable the study of other behavioural phenomena, such as Loss Aversion, which are not theoretically based on the Expected Utility Theory and thus can not be examined within the Brock & Hommes (1998) asset pricing framework.

Finally, there is strong rationale to establish an interdisciplinary cooperation with academic colleagues from the fields of sociology, psychology, and computer science to conduct further laboratory experiments on interconnect stylised theoretical conclusions with real people behaviour.

Chapter 10

Conclusion

"If all the economists were laid end to end, they'd never reach a conclusion."

George B. Shaw, Irish playwright (1856–1950)

This thesis merges two approaches recently developing within the field of financial economics — Heterogeneous Agent Models (HAMs) and Behavioural Finance (BF) — and examines whether they can complement one another. The crucial idea of both approaches is the abandonment of agents' full rationality. Both approaches then emphasise the role of boundedly rational traders using simple forecasting techniques and rules of thumb. On the one hand, Heterogeneous Agent Models face serious difficulties with their empirical validation and, on the other hand, Behavioural Finance lacks any comprehensive economic theory summarising its most important conclusions. Hence, this thesis takes advantage of both approaches and interconnects them together in order to consider two underlying questions: whether current HAM methodology can be reasonably extended by applying findings from the field of BF, or conversely, whether HAMs can serve as a tool for BF theoretical verification.

We introduce a different perspective and application of the Brock & Hommes (1998) HAM approach. We also offer a complete description of the original model and widely discuss various model extensions and empirical attempts to estimate HAMs on real market data. From the plethora of well documented behavioural biases we examine the impact of herding, overconfidence, and market sentiment. Behavioural patterns are embedded into an asset pricing framework in order to study resulting price dynamics.

As financial crises and stock market crashes can be widely considered as periods when investors' rationality is restrained to a great extent, we advance current research literature through an empirical verification of the HAM abilities and BF's explanatory power, using data covering these periods of high-volatility. In Chapter 7 we present a unique benchmark dataset developed by authors especially for the purpose of this thesis. The dataset consists of all currently publicly accessible individual DJIA stocks covering five particularly turbulent stock market periods. The era we consider starts with Black Monday 1987 and terminates with the Lehman Brothers bankruptcy in 2008. Most importantly, we reveal an interesting pattern among this data. When data before and after the BPD are compared, mean, variance, and skewness increase in four of five analysed periods, while kurtosis decreases in the same ratio of cases.

From a theoretical point of view, we show that selected BF findings can be well modelled via the HAM. Herding is modelled so that one of strategies copies the behaviour of the most successful trades of the previous day. Overconfidence is modelled as the overestimation of randomly generated values. Finally, we model the market sentiment as shifts of the mean values of probability distributions from which agents' beliefs are generated.

The 'heart of this thesis' is a numerical analysis of the HAM extended with the selected findings from the field of BF. We analyse the dynamics of the model around the Break Point Date (BPD), where behavioural elements are 'injected' into the system, and compare it to our empirical benchmark sample. Using *Wolfram Mathematica* software we perform simple Monte Carlo simulations of a developed algorithm. Computational approach allows us to investigate a large number of setup combinations, to conduct large volumes of runs, and to analyse more than 100 million observations.

We also show that BF findings extend the original HAM considerably and different HAM modifications lead to different outcomes. Both results are statistically significant and are confirmed by the Cramér–von Mises Test. From this perspective, herding and all trend-affecting setup combinations generally seem to be the weakest modifications. On the other hand, almost all non-trend sentiment modifications exhibit excellent results. Moreover, we analyse the impact of four additional model extensions. These comprise the default presence of fundamentalists, stochastic formation of parameters, combinations of the 'winning' setups, and memory. Generally, the first three modifications cause more or less only minor differences, while memory exhibits the most evident and statistically significant influence on the model structure. Results further

indicate that HAM is able to partially replicate price behaviour during turbulent stock market periods. In particular, the market sentiment case affecting the bias parameter only exhibits a very good fit when compared to the empirical benchmark sample. When memory is added into the system, a strong impact around the BPD is observed. Memory also reinforces the ability to fit the benchmark pattern for particular setup combinations.

To conclude, our approach suggests an alternative tool for examining the dynamics of changes in the HAM structure. To the best of our knowledge, this work also offers a first attempt to really match the fields of HAMs and BF in order to bridge the main deficiencies of both approaches. In this thesis, we study the impact of 'behavioural breaks' but other phenomena such as interest rate shocks or new supplies of information might be implemented. Moreover, we also suggest an original way how to simply deal with the biggest deficiency of the HAM methodology — the empirical verification of results. As such, it is a step forward and an intriguing extension. To give an overall summary, Behavioural Finance matters — at least in the Heterogeneous Agent Model. We show that it makes sense to consider Behavioural Finance within the heterogeneous agent modelling and that both approaches can desirably complement one another.

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Appendix A

Supplementary Tables

On the following pages, a few supplementary tables are provided.

Table A.1: DJIA Control Samples Comparison

Event	Δ Mean			Δ Variance (%)			Δ Skewness			Δ Kurtosis (%)		
	1.	2.	3.	1.	2.	3.	1.	2.	3.	1.	2.	3.
Black Monday	↑***	↑***	↑***	↓*** 57.9	↓*** 22.9	↓*** 50.6	↑	↑	↑	↓ 79.1	↓ 54.9	↓ 84.3
Ruble Devaluation	↑	↓	↓	↑*** 59.7	↑*** 62.5	↑*** 64.4	↑	↓	↑	↓ 22.0	↓ 25.4	↓ 42.1
Dot-com Bubble	↑***	↑***	↑***	↓** 18.1	↑*** 30.0	↑ 0.9	↑	↑	↑	↓ 75.2	↓ 41.8	↓ 60.7
WTC 9/11 Attack	↑	↑***	↑**	↑*** 172.4	↑*** 98.3	↑*** 101.2	↓	↓	↓	↑ 52.5	↑ 69.6	↑ 91.9
Lehman Brothers	↓**	↓***	↓**	↑*** 310.0	↑*** 292.0	↑*** 313.7	↑	↑	↑	↑ 37.9	↑ 5.3	↑ 29.5
Summary	10/15 ↑			11/15 ↑			11/15 ↑			9/15 ↓		

Note: The order of samples in each column: 1. control sample (40 days), 2. control sample (15 stocks/20 days), 3. control sample (15 stocks/40 days).

B/A denotes 'Before/After' the BPD. Δ is the change in magnitude B/A: ↑ means increase in the Set of Real Numbers ℜ; ↓ means decrease in ℜ.

Asterisks denote statistical significances of equal means xor variances B/A: *** denote hypothesis rejected at 1% significance level, ** at 5% level, * at 10% level.

Tests of equal means xor variances B/A are performed via the *MeanDifferenceTest* function and the *VarianceRatioTest* function.

Source: The authors' own computations via *Wolfram Mathematica.*

Table A.2: Simulations Results — Values

Sample	Ø Mean		Ø Variance		Δ (%)	Ø Skewness		Ø Kurtosis		Δ (%)	Min.		Max.	
	b	a	b	a		b	a	b	a		b	a	b	a
No Behavioural Impact	0.001	0.002	0.210	0.222	5.3	-0.286	-0.176	17.4	18.3	4.9	-32.9	-18.7	14.9	37.2
Herding	0.002	0.018	0.249	0.178	-28.4	0.104	0.434	14.4	14.9	-3.2	-24.0	-7.1	53.3	17.2
Overconfidence (bias)	0.001	0.001	0.194	0.324	66.8	0.126	0.156	14.1	16.4	16.5	-18.3	-19.4	15.6	24.7
Overconfidence (trend)	-0.002	-0.005	0.147	0.381	159.1	-0.104	-0.523	7.1	30.8	334.5	-12.5	-23.1	6.3	40.6
Overconfidence	-0.003	0.002	0.326	0.737	125.9	-0.149	0.096	6.7	33.4	399.7	-75.7	-27.4	56.6	42.3
M. Sentiment+ (bias)	-0.002	0.174	0.191	0.246	28.5	-0.070	1.389	18.5	23.9	29.4	-17.5	-14.8	30.9	25.6
M. Sentiment+ (trend)	-0.001	0.001	0.144	0.533	271.0	-0.004	-0.061	4.9	31.8	551.3	-5.3	-25.9	4.1	57.8
M. Sentiment+ (mix)	-0.001	0.129	0.168	0.130	-23.0	-0.185	0.038	10.8	5.5	-48.6	-12.4	-6.4	7.1	5.7
M. Sentiment+	0.003	0.278	0.161	0.877	445.7	0.074	2.754	7.7	40.3	422.5	-10.5	-34.6	17.2	57.1
M. Sentiment- (bias)	0.004	-0.171	0.375	0.490	30.5	-0.377	-1.220	29.5	25.7	-12.8	-24.8	-40.8	61.8	78.6
M. Sentiment- (trend)	0.000	-0.001	0.189	0.129	-32.0	-0.068	-0.320	13.8	15.3	11.0	-22.1	-22.1	16.9	7.9
M. Sentiment- (mix)	-0.002	-0.274	0.175	0.829	373.1	-0.003	-2.258	7.6	37.2	389.6	-11.4	-60.5	11.7	22.1
M. Sentiment-	0.004	-0.126	0.193	0.149	-23.0	0.145	0.022	14.2	10.4	-26.7	-13.8	-14.4	13.3	14.0

Note: All figures are rounded to 1 or 3 decimal digits, however, for percentage calculation original values are employed. +/- denote 'positive/negative' market sentiment.

B/A denotes 'Before/After' the BPD. Caps stand for the complete samples, small letters for the '20 days' samples (see Section 8.3).

Δ is the change in average magnitude B/A the BPD for variance and kurtosis: ↑ means increase in $\Re$; ↓ means decrease in $\Re$.

Source: The authors' own computations via *Wolfram Mathematica.*

Table A.3: Extensions Results I. — Values

Sample	∅ Mean		∅ Variance		Δ (%)	∅ Skewness		∅ Kurtosis		Δ (%)	Min.		Max.	
	b	a	b	a		b	a	b	a		b	a	b	a
Fundamentalists:														
Herding	-0.001	0.018	0.332	0.170	-48.6	-0.137	0.302	17.5	16.7	-4.7	-81.2	-24.4	18.8	17.6
Overconfidence (bias)	0.004	0.002	0.164	0.312	90.7	0.229	-0.024	16.5	18.2	10.6	-11.1	-17.9	20.2	37.7
Overconfidence (trend)	-0.001	-0.000	0.126	0.307	143.5	0.105	-0.015	6.2	25.6	314.4	-4.1	-28.1	9.5	32.2
Overconfidence	0.001	-0.001	0.175	0.324	84.8	0.244	0.157	14.1	14.6	4.1	-14.3	-39.6	22.3	22.9
M. Sentiment+ (bias)	0.003	0.159	0.315	0.237	-24.8	-0.042	1.347	25.8	25.9	0.005	-32.6	-13.4	72.1	31.3
M. Sentiment+ (trend)	0.002	0.002	0.137	0.476	248.6	0.057	0.193	6.3	31.9	405.2	-8.5	-34.4	8.2	33.9
M. Sentiment+ (mix)	-0.009	0.117	0.283	0.119	-57.9	-0.303	-0.040	25.3	18.6	-26.4	-50.5	-35.9	24.1	15.1
M. Sentiment+	0.001	0.256	0.143	0.708	394.3	-0.030	2.439	7.8	36.1	362.2	-8.0	-22.0	8.7	32.9
Stochastic Parameters:														
Herding	0.007	0.018	0.340	0.213	-37.4	0.678	0.600	22.2	25.9	16.6	-15.9	-17.0	66.6	38.2
Overconfidence (bias)	-0.003	-0.003	0.281	0.469	66.6	-0.327	-0.218	24.6	21.4	-12.7	-40.2	-38.6	30.8	42.6
Overconfidence (trend)	-0.004	-0.004	0.151	0.406	170.0	-0.067	-0.251	5.9	39.0	560.2	-7.2	-29.6	5.9	31.7
Overconfidence	0.000	0.000	0.157	0.664	323.1	0.077	0.080	6.0	35.1	488.3	-6.4	-27	5.4	31.2
M. Sentiment+ (bias)	-0.000	0.174	0.247	0.338	36.6	-0.071	1.232	18.8	19.9	6	-22.2	-28.0	23.6	30.1
M. Sentiment+ (trend)	-0.002	0.001	0.152	0.637	318.0	-0.069	-0.089	5.3	47.7	793.1	-5.7	-24.8	5.9	40.4
M. Sentiment+ (mix)	0.000	0.132	0.311	0.140	-55.1	-0.389	0.115	22.8	5.7	-75.1	-57.0	-5.2	13.4	6.6
M. Sentiment+	0.000	0.274	0.161	0.727	350.5	0.006	3.92	6.8	48.0	601.9	-12.3	-21.4	7.0	30.2

Note: All figures are rounded to 1 or 3 decimal digits, however, for percentage calculation original values are employed. +/- denote 'positive/negative' market sentiment.

B/A denotes 'Before/After' the BPD. Caps stand for the complete samples, small letters for the '20 days' samples (see Section 8.3).

Δ is the change in average magnitude B/A the BPD for variance and kurtosis: ↑ means increase in the Set of Real Numbers ℜ; ↓ means decrease in ℜ.

Source: The authors' own computations via *Wolfram Mathematica.*

Table A.4: Extensions Results II. — Values

Sample	∅ Mean		∅ Variance		Δ (%)	∅ Skewness		∅ Kurtosis		Δ (%)	Min.		Max.	
	b	a	b	a		b	a	b	a		b	a	b	a
Combinations:														
Herd. & Over. (bias)	0.001	0.018	0.322	0.192	-40.4	0.202	0.620	23.7	26.9	13.4	-37.0	-13.2	45.4	31.0
Herd. & Sent.+ (bias)	0.001	0.187	0.283	0.289	2.5	0.208	1.294	20.7	24.4	17.6	-36.3	-35.3	48.4	37.6
O. (b.) & S.+ (b.)	0.009	0.237	0.212	0.425	100.1	-0.209	1.001	13.6	16.5	21.3	-18.9	-19.4	17.9	40.4
H. & O. (b.) & S.+ (b.)	-0.004	0.187	0.523	0.263	-49.7	-0.188	0.942	20.0	24.8	24.3	-75.9	-46.0	54.0	33.4
Memory:														
Herding	-0.000	0.020	0.164	0.286	74.5	-0.003	0.219	31.2	33.0	5.7	-22.9	-56.8	32.0	51.5
Overconfidence (bias)	-0.001	0.000	0.107	0.203	90.2	0.100	0.230	15.8	19.4	22.6	-14.3	-13.9	27.8	28.0
Overconfidence (trend)	0.003	-0.001	0.130	0.284	118.5	0.547	0.437	22.1	25.4	15.0	-11.6	-40.1	24.4	35.8
Overconfidence	-0.001	-0.004	0.083	0.378	352.7	0.056	0.115	7.8	32.7	321.2	-6.6	-37.5	6.6	17.2
M. Sentiment+ (bias)	0.000	0.148	0.142	0.355	150.0	0.040	1.718	26.7	35.5	32.9	-13.2	-41.6	32.6	69.4
M. Sentiment+ (trend)	0.001	0.000	0.082	0.329	300.2	-0.076	-0.125	10.2	39.2	284.7	-4.6	-29.4	6.2	32.9
M. Sentiment+ (mix)	0.000	0.115	0.110	0.061	-44.9	-0.093	0.395	21.5	8.9	-58.8	-14.6	-4.8	23.2	5.7
M. Sentiment+	-0.002	0.225	0.089	0.488	447.9	-0.430	2.522	13.7	46.5	239.3	-15.1	-21.1	4.0	30.2

Note: All figures are rounded to 1 or 3 decimal digits, however, for percentage calculation original values are employed. +/- denote 'positive/negative' market sentiment.

B/A denotes 'Before/After' the BPD. Caps stand for the complete samples, small letters for the '20 days' samples (see Section 8.3).

Δ is the change in average magnitude B/A the BPD for variance and kurtosis: ↑ means increase in the Set of Real Numbers ℜ; ↓ means decrease in ℜ.

Source: The authors' own computations via *Wolfram Mathematica.*

Appendix B

Content of Enclosed DVD

There is a DVD enclosed with this thesis, which contains an electronic version of this work, source codes, empirical data used, and results:

- Folder 1: electronic version of this thesis in .pdf format.

- Folder 2: *Wolfram Mathematica* source codes in .nb format.

- Folder 3: empirical data in .txt format.

- Folder 4: results in .pdf format.