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Finite Element Approximation of Problems in Non-Newtonian Fluid Mechanics

Mathematical Institute of Charles University

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I declare that I carried out this doctoral thesis independently, and only with the cited sources, literature and other professional sources.

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In Prague, December 15, 2011

Název práce: Finite Element Approximation of Problems in Non-Newtonian Fluid Mechanics
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Abstrakt: Práce se zabývá aproximací rovnic popisujících proudění jedné třídy ne-newtonovských tekutin metodou konečných prvků. Zaměřuje se zejména na nestlačitelné tekutiny, jejichž vazkost závisí nelineárně na rychlosti smyku a na tlaku. Rovnice popisující proudění jsou diskretizovány d-lineárními konečnými prvky stejného řádu, jež nesplňují podmínku inf-sup stability. Práce navrhuje stabilizaci v gradientu tlaku založenou na známé metodě lokální projekce (LPS). V případě vazkosti závisející pouze na rychlosti smyku jsou ukázány existence a jednoznačnost řešení stabilizované diskrétní úlohy a rovněž apriorní odhady chyby kvantifikující konvergenci metody. Pokud vazkost s rychlostí smyku klesá, dávají odvozené odhady řád konvergence optimální vzledem k regularitě řešení. Jak známo, Galerkinova metoda konečných prvků může vykazovat nestabilitu nejen následkem porušení diskrétní inf-sup podmínky, ale také díky dominující konvekci. Navržená stabilizace je proto vhodně rozšířena, aby se vypořádala s oběma původci nestability. Na konec je uvažována vazkost závisející na rychlosti smyku a na tlaku. Příslušná Galerkinova diskretizace je analyzována a konvergence diskrétních řešení je kvantifikována optimálními odhady chyby.
Klíčová slova: nenewtonovské tekutiny, vazkost závislá na tlaku a rychlosti smyku, metoda konečných prvků, apriorní odhady chyby

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Abstract: This dissertation is devoted to the finite element (FE) approximation of equations describing the motion of a class of non-Newtonian fluids. The main focus is on incompressible fluids whose viscosity nonlinearly depends on the shear rate and pressure. The equations of motion are discretized with equal-order d-linear finite elements, which fail to satisfy the inf-sup stability condition. In this thesis a stabilization technique for the pressure-gradient is proposed that is based on the well-known local projection stabilization (LPS) method. If the viscosity solely depends on the shear rate, the well-posedness of the stabilized discrete systems is shown and a priori error estimates quantifying the convergence of the method are proven. In the shear thinning case, the derived error estimates provide optimal rates of convergence with respect to the regularity of the solution. As is well-known, the Galerkin FE method may suffer from instabilities resulting not only from lacking inf-sup stability but also from dominating convection. The proposed LPS approach is then extended in order to cope with both instability phenomena. Finally, shear-rate- and pressure-dependent viscosities are considered. The Galerkin discretization of the governing equations is analyzed and the convergence of discrete solutions is quantified by optimal error estimates.
Keywords: Non-Newtonian fluids, shear-rate- and pressure-dependent viscosity, finite element method, a priori error estimates

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1 Introduction

The present thesis deals with the finite element approximation of equations describing the steady motion of incompressible fluids whose viscosity nonlinearly depends on the shear rate and pressure. Since the early formation of fluid mechanics it has been known that there is a large class of fluids which cannot adequately be described by the Navier-Stokes theory. Such fluids are referred to as non-Newtonian fluids. An important subclass of non-Newtonian fluids consists of those whose viscosity depends on the shear rate and/or the pressure. These fluids play an important role in various areas of application, for instance, in chemical engineering, blood rheology, and geology. Pressure-dependent viscosities appear in many industrial applications, such as in elastohydrodynamic lubrication, where very high pressures occur. The mathematical theory concerned with the self-consistency of the governing equations still is not fully developed but it has made good progress in recent years. Numerical simulations are frequently employed in engineering practice since real-world experiments of industrial processes can be complex, cost-intensive, and time-consuming. The finite element method (FEM) is often used for engineering simulations. Due to the complicated structure of the viscosity, the mathematical analysis of FEM is sophisticated and it offers many open questions. This thesis aims at closing the gap between mathematical theory and engineering simulations. It is devoted to the finite element (FE) approximation of the equations of motion and its mathematical analysis including error estimation.

First of all, we consider fluids whose viscosity solely depends on the shear rate. In particular, we focus on fluid models with p-structure that include the popular power-law and Carreau model. The parameter $p > 1$ stands for the power-law exponent. Such models are the most commonly used non-Newtonian fluid models and they capture typical non-Newtonian flow characteristics such as shear thinning or shear thickening behavior, which corresponds to exponents $p < 2$ or $p > 2$. The mathematical theory concerned with the self-consistency of the governing equations has been studied intensively since the 1960's, as the theory of monotone operators had developed. Details can be found in [MRR95, FMS03]. For p-structure models, the governing equations are referred to as the p-Navier-Stokes equations. In this thesis, we initially consider the p-Stokes system complemented with homogeneous Dirichlet boundary conditions and we analyze its finite element approximation. In contrast to the p-Navier-Stokes system, the p-Stokes system neglects inertial forces and, hence, we avoid mathematical difficulties caused by the convective term. Its FE approximation has been studied intensively in recent years and a priori error estimates quantifying the convergence of FEM have been proven (see [BN90, BL93b, BL94]). However, the existing results in literature are suboptimal in the sense that either the order of the error estimate is not optimal or the assumed regularity of the solution is too high and not realistic for general solutions. Optimal error estimates have been proven for the p-Laplace equation in Diening/Růžička [DR07]. In this thesis, we will derive a priori error estimates for p-Stokes systems that are optimal at least in the shear thinning case and that improve the error estimates established in [BN90, BL93b, BL94]. Related to the development of the present thesis, at the same time Belenki et al. have independently derived similar results which will be published in [BBDR10]. While their approach is based on finite elements satisfying the inf-sup (stability) condition of Babuška-Brezzi, in this thesis we analyze the approximation of the p-Stokes equations with equal-order finite elements that take advantage of a convenient implementation but that fail to fulfill the inf-sup condition. Due to the limited regularity of the solution, we mainly consider the low-order $\mathbb{Q}_1/\mathbb{Q}_1$ finite element which uses continuous isoparametric d-linear shape functions for both the velocity and pressure approximation. Since this equal-order discretization is not inf-sup stable, the pressure gradient needs to be stabilized. A popular stabilization technique is the local projection stabilization (LPS) method that was introduced for the Stokes problem in Becker/Braack [BB01]. The LPS method achieves stabilization of the pressure by adding appropriate stabilization terms to the standard Galerkin formulation which give a weighted L^2-control over the fluctuations of the pressure gradient (see [BB01]). For p-Stokes systems a priori error estimates have only been established for inf-sup stable elements so far because optimal error estimates have not yet been available and their derivation significantly complicates if additional stabilization is involved. In particular, the LPS method has not been studied in the context of p-Stokes systems up to now. If LPS is applied to problems with p-structure, the known LPS theory ensuring convergence for Stokes systems cannot simply be transferred to the p-Stokes problem due to its nonlinear nature. In this connection, the crucial question arises whether the solutions of the stabilized discrete equations actually converge to the exact solution of the p-Stokes system.

In this thesis, we analyze the LPS method applied to p-Stokes systems. We will propose a nonlinear stabilization term that is based on the known LPS method but that is adjusted to the p-structure of the problem: Since the pressure naturally belongs to the Lebesgue space $L^{p'}(\Omega)$ for $p' := p/(p-1)$, our idea consists in choosing a stabilization that controls fluctuations of the pressure gradient not in $L^2(\Omega)$ (as suggested for Stokes systems) but rather in $L^{p'}(\Omega)$. Our proposed stabilization term yields a weighted $L^{p'}$-control over fluctuations and it coincides with the well-known stabilization term used for Stokes systems in the case $p = 2$ (see [BB01]). We will show that the stabilized FE systems are well-posed. Let $\boldsymbol{v}$ and π be the exact velocity and pressure, and let $\boldsymbol{v}_h$ and π_h be the corresponding discrete approximations. As usual, h represents the maximum mesh size. For instance if $p \le 2$, then we will establish the following a priori error estimates,

$$\|\boldsymbol{v} - \boldsymbol{v}_h\|_{1,p} \le ch, \qquad \|\pi - \pi_h\|_{p'} \le ch^{\frac{2}{p'}}, \tag{1.1}$$

provided that the solution $(\boldsymbol{v}, \pi)$ is sufficiently smooth, see Theorem 4.11. Actually, the constants in (1.1) depend on the solution $(\boldsymbol{v}, \pi)$ through quantities that express the "natural regularity" of $(\boldsymbol{v}, \pi)$ which is available for sufficiently smooth data (cf. Ebmeyer [Ebm06]). It is well-known that, in order to derive sharp error bounds, one shall prove error estimates in terms of quasi-norms which naturally arise in degenerate problems of this type (cf. Barrett/Liu [BL94]). In order to derive (1.1), we combine both the quasi-norm technique and the well-known analysis of LPS for Stokes systems. Numerical experiments indicate that (1.1) is optimal with respect to the supposed regularity of the solution. The error estimates (1.1) remain valid for the classical power-law model. For $p > 2$ we will establish analog a priori error estimates (see Theorem 4.12) that, however, may be suboptimal concerning the order of convergence for the pressure. If the standard LPS method of [BB01] is applied to p-Stokes systems, we will only be able to derive (suboptimal) a priori error estimates which provide rates of convergence depending on the space dimension d. In contrast, our proposed stabilization allows error estimates independent of d, see (1.1).

It is well-known that for Navier-Stokes systems numerical instabilities result not only from lacking inf-sup stability of the FE ansatz but also from locally dominating convection in case of high Reynolds numbers. In Becker/Braack [BB04] it has been shown that the LPS approach of [BB01] can be extended to Navier-Stokes systems in order to cope with both instability phenomena. So far our studies have dealt with LPS for p-Stokes systems only. As the thesis continues, we will perform an analysis of LPS in the context of p-Navier-Stokes systems. We will extend the established LPS-theory for p-Stokes systems in order to properly treat dominating convection. For p-Oseen systems, we will prove optimal a priori error estimates that are similar to (1.1).

In the shear thinning case the classical power-law predicts an unbounded viscosity in the limit of zero shear rate. The corresponding equations of motion are then called *singular* power-law systems. The power-law is frequently used (see, e.g., [BCH75, SE86]). Since the extra stress tensor related to the singular power-law is not differentiable, numerical instabilities usually arise when the discrete power-law systems are solved via Newton's method (cf. [Deu04]). In this thesis, we will present a numerical method which enables the stable approximation of singular power-law systems. The proposed method is based on a simple regularization of the power-law viscosity. We will estimate the error resulting from regularization. The underlying regularization parameter is then coupled with the mesh size so that the error caused by regularization is of same order as the discretization error at least. Finally, we will demonstrate numerically that our regularized approximation method surpasses the non-regularized one regarding accuracy and numerical efficiency.

In this thesis, we also consider fluid models which are both shear thinning and pressure thickening. Similarly to Málek et al. [MNR02], the proposed structure of the viscosity allows a restricted sub-linear dependence on the pressure. The mathematical theory concerned with the self-consistency of the governing equations has emerged recently, see e.g. [FMR05, BMR07, Lan09, BMR09, LS11a]. The FE method has been studied extensively in the context of power-law/Carreau-type fluids whose viscosity only depends on the shear rate, but no FE analysis is available when the fluid's viscosity also depends on the pressure. In this thesis, we will extend the FE analysis performed for p-structure models in the sense that we will allow shear-rate- and pressure-dependent viscosities and that we will consider more general boundary conditions such as inhomogeneous Dirichlet or natural inflow/outflow boundary conditions. Due to the complex structure of the problem, we will restrict the mathematical analysis to inf-sup stable discretizations. We will show that the FE solutions $(\boldsymbol{v}_h, \pi_h)$ exist, that they are determined uniquely, and that they converge to the weak solution $(\boldsymbol{v}, \pi)$ strongly in $\boldsymbol{W}^{1,p}(\Omega) \times L^{p'}(\Omega)$, $p \in (1,2)$, for diminishing mesh size h. Finally, we will derive optimal a priori error estimates similar to (1.1), see Corollary 7.13. Note that Carreau-type models are covered as a special case.

Below we summarize all chapters of the thesis:

Theoretical Results: In Chapter 2 we formulate the incompressible p-(Navier-)Stokes equations. Instead of focusing on particular flow models, we state structural assumptions on the extra stress tensor (p-structure) that contain a large class of flow models. We discuss resulting properties of the extra stress tensor which play an important role in the FE analysis. Finally, we introduce the variational formulation of the p-(Navier-)Stokes equations, and we recall well-known theoretical results that ensure the existence and uniqueness of weak solutions and that deal with the regularity of weak solutions.

Finite Element Discretization: Chapter 3 deals with the FE discretization of the p-Navier-Stokes equations. Since we use an equal-order discretization, we need to stabilize the discrete Galerkin systems. We recall well-known stabilization methods such as local projection stabilization (LPS). Approximation properties of FE spaces can be characterized by estimates for interpolation errors. For the FE analysis of p-Stokes systems it is convenient to transfer the interpolation theory from Sobolev spaces to Orlicz-Sobolev spaces. In this chapter, we summarize important results on interpolation in Orlicz-Sobolev spaces that have been derived in [DR07]. Finally, we discuss implementational aspects.

Finite Element Approximation of the p-Stokes Equations: In Chapter 4, we rigorously analyze the discretization of the p-Stokes problem with equal-order bi- or tri-linear finite elements. First of all, we propose our stabilization method that is based on the LPS method and that is adjusted to the p-structure of the problem. Within the LPS framework, the pressure gradient is projected into an appropriate (possibly discontinuous) FE space that is supposed to satisfy a certain local inf-sup condition with respect to the original FE space. We show that there exists an interpolation operator of Scott-Zhang type that additionally features an orthogonality property with respect to the projection space and that satisfies an interpolation property in terms of quasi-norms. This modified interpolation operator enables us to prove a discrete analogon of the continuous inf-sup condition and, consequently, the well-posedness of the discrete stabilized systems. Then we derive a priori error estimates that quantify the convergence of the method. We confirm numerically that, at least in the shear thinning case, the derived error estimates are optimal with respect to the expected regularity of the solution. Furthermore, we establish a priori error estimates if the classical LPS scheme proposed in [BB01] is applied to p-Stokes systems. We also analyze the fully time-space discretization of non-steady p-Stokes systems. Finally, we present particular projection spaces that satisfy the abstract assumptions.

Approximation of the p-Navier-Stokes Equations: Chapter 5 is devoted to the finite element approximation of p-Navier-Stokes systems. First of all, we investigate the LPS method applied to the p-Oseen equations. The p-Oseen system appears within the solution of the non-steady p-Navier-Stokes system as an auxiliary problem if an A-stable time step method is employed. For it we are able to prove optimal a priori error estimates using methods from Chapter 4. Finally, we deal with a posteriori error estimation and adaptive mesh refinement. Generally, the numerical solution of the highly nonlinear p-Navier-Stokes equations can be cost-intensive and time-consuming. Hence adaptive methods are important since they enable us to reduce numerical costs without loss of accuracy. In the context of p-Navier-Stokes systems we discuss the well-known dual weighted residual (DWR) method (see [BR01]), which allows for both the quantitative assessment of the discretization errors and the adaptive refinement of the underlying meshes. In particular, we apply the DWR method to the p-Navier-Stokes equations for the computation of the drag coefficient.

Approximation of Singular Power-Law Systems: Chapter 6 deals with so-called singular power-law models which feature an unbounded viscosity in the limit of zero shear rate. We present a numerical method that enables the stable approximation of singular power-law systems. Finally, we prove a priori error estimates and we numerically validate them.

Fluids with Shear-Rate- and Pressure-Dependent Viscosity: Chapter 7 is dedicated to fluids whose viscosity depends on the shear rate and pressure. We analyze the Galerkin discretization of the governing equations. In particular, we show the well-posedness of the discrete systems. We prove that the discrete solutions converge to the solution of the original problem without any additional assumption on its regularity. We then derive a priori error estimates that provide optimal rates of convergence with respect to the supposed regularity. Finally, we illustrate the achieved results by numerical experiments.

Conclusion and Outlook: In Chapter 8, we summarize the derived results.

To sum up, the present thesis pursues the following aims:

- analyze the LPS-method in the context of p-Stokes systems
- derive optimal a priori estimates for the approximation error
- extend the established LPS-theory to p-Navier-Stokes systems
- apply the DWR method to the p-Navier-Stokes equations
- develop a stable approximation method for singular power-law systems
- analyze FEM for fluids with shear-rate- and pressure-dependent viscosity

- study general boundary conditions describing, e.g., a free outflow

Several results of the present thesis have already been published in (or have currently been submitted to) peer-reviewed journals while composing the thesis:

- Adrian Hirn, *Approximation of the p-Stokes equations with equal-order finite elements*, accepted for publication in J. Math. Fluid Mech. (2011), [Hir10].
- Adrian Hirn, Martin Lanzendörfer and Jan Stebel, *Finite element approximation of flow of fluids with shear rate and pressure dependent viscosity*, accepted for publication in IMA J. Numer. Anal. (2011), [HLS10].

2 Theoretical Results

In this chapter, we formulate the fundamental equations describing the motion of certain non-Newtonian fluids, and we deal with known theoretical results concerned with their well-posedness. First of all, in Section 2.1 we introduce our basic notation which is used throughout the thesis. Section 2.2 is dedicated to the derivation of the governing equations. Moreover, the practical relevance of the considered fluids is discussed. Instead of focusing on particular flow models, in Section 2.3 we state general structural assumptions on the extra stress tensor (p-structure) that allow for a large class of flow models. In Section 2.4 we derive resulting properties of the extra stress tensor that will be of relevance for the FE analysis of p-Stokes systems. In Sections 2.5, 2.6 we introduce the variational formulation of the p-(Navier-)Stokes equations which represents the basis for the FE discretization. Additionally, we recall well-known theoretical results that ensure the existence and uniqueness of weak solutions. In Section 2.6 we also discuss the regularity of weak solutions.

2.1 Notation and function spaces

The set of all positive real numbers is denoted by $\mathbb{R}^+$. Let $\mathbb{R}_0^+ := \mathbb{R}^+ \cup \{0\}$. For the Euclidean scalar product of two vectors $\boldsymbol{p}$, $\boldsymbol{q} \in \mathbb{R}^d$ we use the notation $\boldsymbol{p} \cdot \boldsymbol{q}$. The scalar product of $\boldsymbol{P}$, $\boldsymbol{Q} \in \mathbb{R}^{d \times d}$ is defined by $\boldsymbol{P} : \boldsymbol{Q} := \sum_{i,j=1}^{d} P_{ij} Q_{ij}$. We set $|\boldsymbol{Q}| := (\boldsymbol{Q} : \boldsymbol{Q})^{1/2}$. Often we use c as a generic constant, i.e., its value may change from line to line but does not depend on the important variables. We write $a \sim b$ if there exist positive constants c and C independent of all relevant quantities such that $cb \leq a \leq Cb$. Moreover, the notation $a \lesssim b$ is used for $a \leq Cb$ with a suitable constant $C > 0$.

Below we introduce function spaces, which will be used later on, and we recall their basic properties. Details and proofs can be found in the standard literature, e.g., in [Ada75, KJF77, Růž04]. Throughout the thesis let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a bounded domain with boundary $\partial\Omega$. If we do not provide further information, we will assume that $\partial\Omega$ is Lipschitz. The outer unit normal vector to $\partial\Omega$ is denoted by $\boldsymbol{n}$. For measurable $\omega \subset \Omega$, the d-dimensional Lebesgue measure of ω is denoted by $|\omega|$. For $\nu \in [1, \infty]$ we use the standard notations $L^\nu(\omega)$ for the Lebesgue space and $W^{m,\nu}(\omega)$ for the Sobolev space of order m: The Lebesgue space $L^\nu(\omega)$ consists of all measurable functions u on ω, for which

$$\|u\|_{\nu;\omega} := \|u\|_{L^\nu(\omega)} := \begin{cases} \Big(\int_\omega |u|^\nu \, \mathrm{d}\boldsymbol{x} \Big)^{1/\nu} & \text{if } \nu \in [1, \infty) \\ \operatorname{ess\,sup}_\omega |u| := \inf_{|N|=0} \sup_{\boldsymbol{x} \in \omega \setminus N} |u(\boldsymbol{x})| & \text{if } \nu = \infty \end{cases}$$

is finite. We identify two functions, u and v, that satisfy $\|u-v\|_{\nu;\omega}=0$. For $\nu\in[1,\infty]$ the functional $\|\cdot\|_{\nu;\omega}$ is a norm, and $(L^\nu(\omega),\|\cdot\|_{\nu;\omega})$ is a Banach space. Let $m\in\mathbb{N}_0:=\{0,1,2,\dots\}$, $\nu\in[1,\infty]$, and let $\omega\subset\Omega$ be open. Then the Sobolev space $W^{m,\nu}(\omega)$ contains all $u\in L^\nu(\omega)$ whose distributional derivatives $\partial^\alpha u\in L^\nu(\omega)$ exist for any α with $0\le|\alpha|\le m$. Here, $\alpha=(\alpha_1,\dots,\alpha_d)$ is a multi-index (each α_i is a nonnegative integer), $|\alpha|:=\sum_{i=1}^d\alpha_i$, and $\partial^\alpha:=\partial_1^{\alpha_1}\dots\partial_d^{\alpha_d}$ with $\partial_i^{\alpha_i}:=\partial^{\alpha_i}/\partial x_i^{\alpha_i}$. The symbols ∂^α are used for both the partial and distributional derivatives. Further synonyms for derivatives are given by $\nabla\boldsymbol{u}:=(\partial_j u_i)_{i,j=1}^d$ and $\nabla^m\boldsymbol{u}$. The latter one denotes the tensor of all partial derivatives of $\boldsymbol{u}$ up to the order m. The Sobolev space $W^{m,\nu}(\omega)$ is a Banach space with the norm

$$\|u\|_{m,\nu;\omega}:=\|u\|_{W^{m,\nu}(\omega)}:=\begin{cases}\left(\sum_{|\alpha|\le m}\|\partial^\alpha u\|^\nu_{L^\nu(\omega)}\right)^{1/\nu} & \text{if } 1\le\nu<\infty\\ \sum_{|\alpha|\le m}\operatorname{ess\,sup}_\omega|\partial^\alpha u| & \text{if } \nu=\infty\end{cases}.$$

Similarly, we define seminorms, $|u|_{m,\nu;\omega}:=\left(\sum_{|\alpha|=m}\|\partial^\alpha u\|^\nu_{\nu;\omega}\right)^{1/\nu}$. For functions $u\in L^1(\omega)$ with $|\omega|>0$, we denote the mean value of u over ω by $\langle u\rangle_\omega:=\fint_\omega u\,\mathrm{d}\boldsymbol{x}:=\frac{1}{|\omega|}\int_\omega u\,\mathrm{d}\boldsymbol{x}$. The symbol $L_0^\nu(\omega)$ stands for the subspace of $L^\nu(\omega)$ whose elements meet trivial mean-value, i.e., $L_0^\nu(\omega):=\{u\in L^\nu(\omega);\,\langle u\rangle_\omega=0\}$. For scalar functions u, v with $uv\in L^1(\omega)$, the notation $(u,v)_\omega$ is used for the integral $\int_\omega uv\,\mathrm{d}\boldsymbol{x}$. Spaces of $\mathbb{R}^d$-valued functions are denoted with boldface type, though no distinction is made in the notation of norms and inner products. Thus, the norm in $\boldsymbol{W}^{m,\nu}(\omega)\equiv[W^{m,\nu}(\omega)]^d$ is given by $\|\boldsymbol{w}\|_{m,\nu;\omega}=\left(\sum_{1\le i\le d}\sum_{0\le|\alpha|\le m}\|\partial^\alpha w_i\|^\nu_{\nu;\omega}\right)^{1/\nu}$. For vector-valued functions $\boldsymbol{u}$, $\boldsymbol{v}$ with $u_iv_i\in L^1(\omega)$, we set $(\boldsymbol{u},\boldsymbol{v})_\omega:=\int_\omega\boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}$. Analogously, for tensor-valued functions $\boldsymbol{U}$, $\boldsymbol{V}$ with $U_{ij}V_{ij}\in L^1(\omega)$, we define $(\boldsymbol{U},\boldsymbol{V})_\omega:=\int_\omega\boldsymbol{U}:\boldsymbol{V}\,\mathrm{d}\boldsymbol{x}$. In case of $\omega=\Omega$, we usually omit the index Ω. For $\nu\in[1,\infty)$ the notation $W_0^{1,\nu}(\Omega)$ is used for the Sobolev space with vanishing traces on $\partial\Omega$. It is well-known that for $\nu\in[1,\infty)$ there exists a continuous trace operator $\gamma:W^{1,\nu}(\Omega)\to L^\nu(\partial\Omega)$ with $u|_{\partial\Omega}:=\gamma(u)$. The space $W_0^{1,\nu}(\Omega)$ is then characterized by

$$W_0^{1,\nu}(\Omega):=\{u\in W^{1,\nu}(\Omega);\,u|_{\partial\Omega}=0\}.$$

We recall the Poincaré and generalized Korn inequality (see, e.g., Málek et al. [MNRR96]): For any $\nu\in(1,\infty)$ there exist constants c_1, $c_2>0$ only depending on ν and Ω such that

$$c_1\|\boldsymbol{w}\|_{1,\nu}\le\|\nabla\boldsymbol{w}\|_\nu\le c_2\|\boldsymbol{D}\boldsymbol{w}\|_\nu\qquad\forall\boldsymbol{w}\in\boldsymbol{W}_0^{1,\nu}(\Omega).\tag{2.1}$$

By virtue of (2.1), the seminorm $\|\nabla\cdot\|_\nu$ represents a norm on $W_0^{1,\nu}(\Omega)$ which is equivalent to the usual $W^{1,\nu}$-norm. The space $W_0^{1,\nu}(\Omega)$ is a closed subset of $W^{1,\nu}(\Omega)$. The dual space of $W_0^{1,\nu}(\Omega)$ is denoted by $W^{-1,\nu'}(\Omega)\equiv[W_0^{1,\nu}(\Omega)]^*$. It is a Banach space with the norm

$$\|g\|_{-1,\nu'}:=\sup_{w\in W_0^{1,\nu}(\Omega)}\frac{\langle g,w\rangle}{|w|_{1,\nu}}.$$

Here, $\langle\cdot,\cdot\rangle$ represents the duality pairing between $W^{-1,\nu'}(\Omega)$ and $W_0^{1,\nu}(\Omega)$. As usual, ν' stands for the dual exponent to ν defined by $1/\nu+1/\nu'=1$. The space $C^m(\Omega)$, $m\in\mathbb{N}$, denotes the space of all m-times differentiable functions on Ω whose derivatives up to order m are continuous. We set $C(\Omega) := C^0(\Omega)$ and we define

$$C^\infty(\Omega) := \bigcap_{m\in\mathbb{N}} C^m(\Omega).$$

The space $C^m(\overline{\Omega})$ consists of all functions from $C^m(\Omega)$ whose derivatives up to order m can be extended continuously onto $\overline{\Omega}$. It is a Banach space with the norm

$$\|u\|_{C^m(\overline{\Omega})} := \max_{|\alpha|\le m} \sup_{\boldsymbol{x}\in\overline{\Omega}} |\partial^\alpha u(\boldsymbol{x})|.$$

As usual, $\mathcal{D}(\Omega) := C_0^\infty(\Omega)$ denotes the set of all C^∞-functions with compact support in Ω. Its dual space is denoted by $\mathcal{D}'(\Omega)$. If $\partial\Omega$ is Lipschitz, then $W_0^{1,\nu}(\Omega)$ can be characterized as the closure of $C_0^\infty(\Omega)$ with respect to the $W_0^{1,\nu}$-norm. The following interpolation inequality is well-known: Let $f \in W^{1,s}(\Omega) \cap L^q(\Omega)$ with $1 \le q < \infty$. If $s < d$, then $f \in L^r(\Omega)$ with $r \le \frac{ds}{d-s}$ and for $q \le r \le \frac{ds}{d-s}$ there exists $C = C(\Omega, d, s, q, r) > 0$ so that

$$\|f\|_r \le C\|f\|_{1,s}^\alpha \|f\|_q^{1-\alpha}, \qquad \alpha \in [0,1], \qquad \frac{1}{r} = \alpha\Big(\frac{1}{s} - \frac{1}{d}\Big) + (1-\alpha)\frac{1}{q}. \tag{2.2}$$

If $s = d$ (If $s > d$), then (2.2) holds true for $q \le r < \infty$ (for $q \le r \le \infty$).

Let $(X, \|\cdot\|_X)$ be a Banach space. For $T > 0$ let $I := (0,T)$ be a bounded interval. The space $L^q(I;X)$ denotes the space of all Bochner measurable functions $u : I \to X$ such that

$$\|u\|_{L^q(I;X)} := \begin{cases} \Big(\int_0^T \|u(t)\|_X^q \,\mathrm{d}t\Big)^{1/q} & \text{if } q \in [1,\infty) \\ \operatorname{ess\,sup}_{t\in I} \|u(t)\|_X & \text{if } q = \infty \end{cases} \tag{2.3}$$

is finite. It is well-known that $(L^q(I;X), \|\cdot\|_{L^q(I;X)})$ is a Banach space. The space $C(\bar{I};X)$ consists of functions from $\bar{I}$ into X that are continuous on $\bar{I}$. This is a Banach space with the norm $\|u\|_{C(\bar{I};X)} := \sup_{t\in\bar{I}} \|u(t)\|_X$.

Let X, Y be Banach spaces. If a mapping $J : X \to Y$ is Gâteaux differentiable in $x \in X$, then $J'(x)(h)$ is referred to as the Gâteaux-derivative of J at $x \in X$ in direction $h \in X$,

$$J'(x)(h) := \frac{\mathrm{d}}{\mathrm{d}t} J(x+th)\Big|_{t=0} := \lim_{t\searrow 0} \frac{1}{t}\Big[J(x+th) - J(x)\Big].$$

Similarly, if $a(x)(y)$ is a semi-linear form, $a'(x)(h,y)$ denotes its directional derivative.

2.2 Non-Newtonian fluid models

Following the literature [MNR02, MR06], in this section we discuss several non-Newtonian fluid models, their physical properties and practical relevance. We introduce the governing equations that describe the motion of such fluids. Their derivation is based on physical conservation laws. The conservation of mass is equivalent to the continuity equation

$$\partial_t \varrho + \nabla \cdot (\varrho \boldsymbol{v}) = 0 \tag{2.4}$$

where the vector field $\boldsymbol{v} = (v_1, \dots, v_d)$ is the velocity and ϱ denotes the density of the fluid. The balance of linear and angular momentum leads to the momentum equations

$$\varrho \partial_t \boldsymbol{v} + \varrho [\boldsymbol{v} \cdot \nabla] \boldsymbol{v} - \nabla \cdot \boldsymbol{\mathcal{T}}^\top = \varrho \boldsymbol{f}, \qquad \text{and} \qquad \boldsymbol{\mathcal{T}} = \boldsymbol{\mathcal{T}}^\top. \tag{2.5}$$

Here, $\boldsymbol{\mathcal{T}} = (\mathcal{T}_{ij})_{i,j=1}^d$ is the Cauchy stress tensor, $\boldsymbol{\mathcal{T}}^\top$ denotes its transpose, and $\boldsymbol{f} = (f_1, \dots, f_d)$ describes an external body force. In case of incompressible fluids, the volume of subregions occupied by the fluid does not change in time. By means of Reynold's transport theorem, this leads to the incompressibility condition

$$\operatorname{div} \boldsymbol{v} \equiv \nabla \cdot \boldsymbol{v} = 0. \tag{2.6}$$

The fluid is called homogeneous if the density ϱ is constant in space. If the fluid under consideration is homogeneous and incompressible, then the density is also constant in time because of (2.4). For such fluids the equation (2.4) is automatically fulfilled.

The Cauchy stress tensor $\boldsymbol{\mathcal{T}}$ is expressed by a constitutive law. In classical fluid mechanics, it is usually assumed that the Cauchy stress tensor $\boldsymbol{\mathcal{T}}$ depends on the velocity gradient $\nabla \boldsymbol{v}$ and the density ϱ. It follows from the principle of material frame-indifference that the stress tensor $\boldsymbol{\mathcal{T}}$ depends on the velocity gradient $\nabla \boldsymbol{v}$ only through its symmetric part

$$\boldsymbol{D} := \boldsymbol{D}\boldsymbol{v} := \frac{1}{2}\Big(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^\top\Big) \qquad \text{with} \qquad D_{ij} := \frac{1}{2}\Big(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\Big). \tag{2.7}$$

We consider the following constitutive equation that relates the Cauchy stress $\boldsymbol{\mathcal{T}}$ to $\boldsymbol{D}$:

$$\boldsymbol{\mathcal{T}} = -\pi \boldsymbol{I} + \mu(\pi, |\boldsymbol{D}|^2)\boldsymbol{D}. \tag{2.8}$$

Since $\operatorname{tr} \boldsymbol{D} = \nabla \cdot \boldsymbol{v} = 0$, it follows from (2.8) that $-\pi = \frac{1}{3} \operatorname{tr} \boldsymbol{\mathcal{T}}$, i.e., π is the mean normal stress. Following [BMM10], we show that the constitutive equation (2.8) is consistent with the basic principles of continuum mechanics if an implicit relation between $\boldsymbol{\mathcal{T}}$ and $\boldsymbol{D}$,

$$\boldsymbol{F}(\boldsymbol{\mathcal{T}}, \boldsymbol{D}) = \boldsymbol{0},$$

is assumed. The principle of material frame-indifference implies that $\boldsymbol{F}$ satisfies

$$\boldsymbol{F}(\boldsymbol{Q}\boldsymbol{\mathcal{T}}\boldsymbol{Q}^\top, \boldsymbol{Q}\boldsymbol{D}\boldsymbol{Q}^\top) = \boldsymbol{Q}\boldsymbol{F}(\boldsymbol{\mathcal{T}}, \boldsymbol{D})\boldsymbol{Q}^\top \qquad \forall \boldsymbol{Q} \in \{\boldsymbol{Q} \in \mathbb{R}^{d\times d};\ \boldsymbol{Q}\boldsymbol{Q}^\top = \boldsymbol{Q}^\top\boldsymbol{Q} = \boldsymbol{I}\}$$

($\boldsymbol{F}$ is an isotropic second-order tensor). A representation theorem for such tensors yields

$$\begin{aligned}\alpha_0 \boldsymbol{I} + \alpha_1 \boldsymbol{\mathcal{T}} + \alpha_2 \boldsymbol{D} + \alpha_3 \boldsymbol{\mathcal{T}}^2 + \alpha_4 \boldsymbol{D}^2 + \alpha_5(\boldsymbol{D}\boldsymbol{\mathcal{T}} + \boldsymbol{\mathcal{T}}\boldsymbol{D}) + \alpha_6(\boldsymbol{\mathcal{T}}^2\boldsymbol{D} + \boldsymbol{D}\boldsymbol{\mathcal{T}}^2)&\\ + \alpha_7(\boldsymbol{\mathcal{T}}\boldsymbol{D}^2 + \boldsymbol{D}^2\boldsymbol{\mathcal{T}}) + \alpha_8(\boldsymbol{\mathcal{T}}^2\boldsymbol{D}^2 + \boldsymbol{D}^2\boldsymbol{\mathcal{T}}^2) = \boldsymbol{0}&\end{aligned}$$

where the functions α_i, $i = 0, \dots, 8$, depend on the invariants

$$\operatorname{tr} \boldsymbol{\mathcal{T}},\ \operatorname{tr} \boldsymbol{D},\ \operatorname{tr} \boldsymbol{\mathcal{T}}^2,\ \operatorname{tr} \boldsymbol{D}^2,\ \operatorname{tr} \boldsymbol{\mathcal{T}}^3,\ \operatorname{tr} \boldsymbol{D}^3,\ \operatorname{tr}(\boldsymbol{\mathcal{T}} \boldsymbol{D}),\ \operatorname{tr}(\boldsymbol{\mathcal{T}}^2 \boldsymbol{D}),\ \operatorname{tr}(\boldsymbol{\mathcal{T}} \boldsymbol{D}^2),\ \operatorname{tr}(\boldsymbol{\mathcal{T}}^2 \boldsymbol{D}^2).$$

Note that $\operatorname{tr} \boldsymbol{D}^2 = |\boldsymbol{D}|^2$. If we set $\alpha_0 = -\frac{1}{3} \operatorname{tr} \boldsymbol{\mathcal{T}}$, $\alpha_1 = 1$, $\alpha_2 = -\mu(-\frac{1}{3} \operatorname{tr} \boldsymbol{\mathcal{T}}, |\boldsymbol{D}|^2)$, and $\alpha_i = 0$ for $i \geq 3$, we arrive at (2.8). In particular, if $\mu \equiv \text{const}$, then the fluid belongs to the class of Newtonian fluids. Otherwise, the fluid is referred to as a non-Newtonian fluid. The extra stress tensor $\boldsymbol{\mathcal{S}} = (\boldsymbol{\mathcal{S}}_{ij})_{i,j=1}^d$ relates to the Cauchy stress tensor $\boldsymbol{\mathcal{T}}$ through

$$\boldsymbol{\mathcal{T}} = -\pi \boldsymbol{I} + \boldsymbol{\mathcal{S}}(\pi, \boldsymbol{D}\boldsymbol{v}). \tag{2.9}$$

It represents the viscous part of the Cauchy stress tensor which, e.g., describes shear stress. An important subclass of fluids is derived from (2.8) with $\mu = \mu(|\boldsymbol{D}|^2)$. Typical examples are the power-law model (2.11a) and the Carreau model (2.11b): Here, $\boldsymbol{\mathcal{S}}$ takes the form

$$\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}) = \mu(|\boldsymbol{D}\boldsymbol{v}|^2)\boldsymbol{D}\boldsymbol{v} \tag{2.10}$$

where, e.g., for fixed $\mu_0 > 0$, $p \in (1, \infty)$, and $\varepsilon \in [0, \infty)$ the function μ is given by

$$\mu(|\boldsymbol{D}\boldsymbol{v}|^2) := \mu_0 |\boldsymbol{D}\boldsymbol{v}|^{p-2} \qquad \text{“Power-law model” or} \tag{2.11a}$$

$$\mu(|\boldsymbol{D}\boldsymbol{v}|^2) := \mu_0 \Big(\varepsilon^2 + |\boldsymbol{D}\boldsymbol{v}|^2\Big)^{\frac{p-2}{2}} \qquad \text{“Carreau model”.} \tag{2.11b}$$

If two-dimensional simple shear flows are considered, i.e., $\boldsymbol{v} = (v_1(x_2), 0)^\mathsf{T}$, then the quantity $|v_1'(x_2)|$ $(= 2|\boldsymbol{D}\boldsymbol{v}|)$ is referred to as the shear rate. The function μ represents the generalized viscosity of the fluid. Thus, fluids constituted by (2.10) are also named *fluids with shear rate dependent viscosity*. Models of type (2.11) are by far the most commonly used non-Newtonian fluid models (see [MNR02]). Such models describe a plethora of materials in various areas of application: colloids and suspensions, biological fluids such as blood and synovial fluids, and lubricants. For an extensive discussion of such models we refer to [MRR95, MNRR96, GRRT08], and the references therein. If $1 < p < 2$, we observe lower apparent viscosities at higher shear rates. This property is called shear thinning. Most real fluids, which can be modeled by a constitutive law of type (2.11), show shear thinning behavior that corresponds to exponents $1 < p < 2$. The case $p > 2$ is less common, although there are some fluids with shear thickening behavior. For $p = 2$ the generalized viscosity μ is constant, and the fluid belongs to the class of Newtonian fluids.

Besides the class of models (2.10), we are interested in the wider class of models (2.8) itself. For many fluids, the variations in the fluid density are small whereas the variations in its viscosity may differ by many orders of magnitude due to significant changes in the pressure (see [MNR02]). Such fluids can effectively be modeled as incompressible fluids with pressure-dependent viscosity. Here we consider extra stress tensors of the form

$$\boldsymbol{\mathcal{S}}(\pi, \boldsymbol{D}\boldsymbol{v}) = \mu(\pi, |\boldsymbol{D}\boldsymbol{v}|^2)\boldsymbol{D}\boldsymbol{v}, \tag{2.12}$$

i.e., we deal with fluids whose viscosity depends on both the pressure and shear rate. The fluid models under consideration appear in various areas of application, for instance in elastohydrodynamic lubrication, geology and glaciology (see, e.g., [Hin98, BG06, SHH06, Sch07, Sze10]). Concerning the class of models (2.12), many details and extensive discussions can be found in Málek et al. [MR06, MR07].

Governing equations: Throughout this thesis the density ϱ of the fluid is assumed to be constant, $\varrho \equiv \varrho_0$. Inserting (2.9) into (2.5), dividing the result by ϱ_0, and taking into account the incompressibility constraint (2.6), we arrive at the equations of motion

$$\partial_t \boldsymbol{v} - \varrho_0^{-1}\nabla \cdot \boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}, \pi) + [\boldsymbol{v} \cdot \nabla]\boldsymbol{v} + \varrho_0^{-1}\nabla\pi = \boldsymbol{f}, \qquad \nabla \cdot \boldsymbol{v} = 0. \tag{2.13}$$

We relabel $\boldsymbol{\mathcal{S}}/\varrho_0$ and π/ϱ_0 again as $\boldsymbol{\mathcal{S}}$ and π. Hence, we always consider system (2.13) using the convention $\varrho_0 = 1$. The isothermal flow of an homogeneous incompressible viscous fluid in a bounded domain $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$ is then governed by the system of PDEs

$$\left.\begin{aligned} \partial_t \boldsymbol{v} - \nabla \cdot \boldsymbol{\mathcal{S}}(\pi, \boldsymbol{D}\boldsymbol{v}) + [\boldsymbol{v} \cdot \nabla]\boldsymbol{v} + \nabla\pi &= \boldsymbol{f} \\ \nabla \cdot \boldsymbol{v} &= 0 \end{aligned}\right\} \quad \text{in } (0,T) \times \Omega. \tag{2.14a}$$

Let $\hat{\boldsymbol{v}}$ be a given divergenceless initial velocity field. The relevant initial condition reads

$$\boldsymbol{v}(0, \boldsymbol{x}) = \hat{\boldsymbol{v}}(\boldsymbol{x}) \qquad \text{for almost all } \boldsymbol{x} \in \Omega. \tag{2.14b}$$

When the flow field does not change over time, then the flow is considered to be a steady flow. Steady flows occur in many situations. For instance, in industrial application, investigators are often interested in fully developed flows regardless of the flow history. If the flow is steady, then the system (2.14) reduces to the following system of PDEs:

$$\left.\begin{aligned} -\nabla \cdot \boldsymbol{\mathcal{S}}(\pi, \boldsymbol{D}\boldsymbol{v}) + [\boldsymbol{v} \cdot \nabla]\boldsymbol{v} + \nabla\pi &= \boldsymbol{f} \\ \nabla \cdot \boldsymbol{v} &= 0 \end{aligned}\right\} \quad \text{in } \Omega. \tag{2.15}$$

Remark 2.1. For models (2.10), we easily derive the identity

$$\begin{aligned} \sum_{i=1}^d \partial_i \mathcal{S}_{ij}(\boldsymbol{D}\boldsymbol{v}) &= \frac{1}{2}\sum_{i=1}^d \partial_i\Big(\mu(|\boldsymbol{D}\boldsymbol{v}|^2)[\partial_i v_j + v_j v_i]\Big) \\ &= \frac{1}{2}\mu(|\boldsymbol{D}\boldsymbol{v}|^2)\sum_i \partial_i[\partial_i v_j + \partial_j v_i] + \mu'(|\boldsymbol{D}\boldsymbol{v}|^2)\sum_i D_{ij}\partial_i|\boldsymbol{D}\boldsymbol{v}|^2 \\ &= \frac{1}{2}\mu(|\boldsymbol{D}\boldsymbol{v}|^2)(\Delta v_j + \partial_j \nabla \cdot \boldsymbol{v}) + 2\mu'(|\boldsymbol{D}\boldsymbol{v}|^2)\sum_{i,k,l} D_{ij}D_{kl}\partial_i\partial_k v_l. \end{aligned}$$

Here, we have assumed that $\boldsymbol{v}$ is sufficiently smooth. Hence, if $\mu \equiv \text{const}$ and $\nabla \cdot \boldsymbol{v} = 0$, then we recover the well-known Navier-Stokes model.

Often we neglect inertial forces in $(2.15)_1$ and, hence, we avoid mathematical difficulties related to the convective term. In this case, we arrive at the simplified system of PDEs

$$\left.\begin{aligned} -\nabla \cdot \boldsymbol{\mathcal{S}}(\pi, \boldsymbol{D}\boldsymbol{v}) + \nabla\pi &= \boldsymbol{f} \\ \nabla \cdot \boldsymbol{v} &= 0 \end{aligned}\right\} \quad \text{in } \Omega. \tag{2.16}$$

The following system (2.17) typically appears within the solution of problem (2.14) as an auxiliary problem if an A-stable implicit time step method is applied:

$$\left.\begin{aligned} -\nabla \cdot \boldsymbol{\mathcal{S}}(\pi, \boldsymbol{D}\boldsymbol{v}) + [\boldsymbol{b} \cdot \nabla]\boldsymbol{v} + \sigma\boldsymbol{v} + \nabla\pi &= \boldsymbol{f} \\ \nabla \cdot \boldsymbol{v} &= 0 \end{aligned}\right\} \quad \text{in } \Omega, \tag{2.17}$$

where the parameter $\sigma \in \mathbb{R}_0^+$ and the flow field $\boldsymbol{b} : \overline{\Omega} \to \mathbb{R}^d$ are given.

Similarity transformation: All quantities in (2.13) feature a physical dimension. In order to simulate real-life processes in practical applications, we have to employ a non-dimensionalized version of (2.13) which we derive below for the simple Carreau model (2.10) & (2.11b). To this end, we introduce the dimensionless variables

$$\tilde{\boldsymbol{x}} := \frac{\boldsymbol{x}}{L}, \qquad \tilde{\boldsymbol{v}} := \frac{\boldsymbol{v}}{U}, \qquad \tilde{t} := \frac{U}{L}t, \qquad \tilde{\pi} := \frac{\pi}{U^2 \varrho_0}$$

where L and U are characteristic length and bulk velocity respectively. Consequently,

$$\frac{\partial \tilde{\boldsymbol{v}}}{\partial \tilde{t}} = \frac{\partial \tilde{\boldsymbol{v}}}{\partial t}\frac{\partial t}{\partial \tilde{t}} = \frac{L}{U^2}\frac{\partial \boldsymbol{v}}{\partial t}, \qquad \tilde{\nabla}\tilde{\boldsymbol{v}} = \frac{L}{U}\nabla \boldsymbol{v}, \qquad \tilde{\boldsymbol{D}}\tilde{\boldsymbol{v}} = \frac{L}{U}\boldsymbol{D}\boldsymbol{v}.$$

Hence, in view of (2.11b) the momentum equations transform as follows

$$\frac{U^2}{L}\frac{\partial \tilde{\boldsymbol{v}}}{\partial \tilde{t}} - \frac{\mu_0}{\varrho_0 L}\tilde{\nabla}\cdot\left[\left(\varepsilon^2 + U^2 L^{-2}|\tilde{\boldsymbol{D}}\tilde{\boldsymbol{v}}|^2\right)^{\frac{p-2}{2}} U L^{-1}\tilde{\boldsymbol{D}}\tilde{\boldsymbol{v}}\right] + \frac{U^2}{L}[\tilde{\boldsymbol{v}}\cdot\tilde{\nabla}]\tilde{\boldsymbol{v}} + \frac{U^2}{L}\tilde{\nabla}\tilde{\pi} = \boldsymbol{f}.$$

The right-hand side $\boldsymbol{f}$ can be interpreted as given acceleration such as gravitational acceleration. Setting $\tilde{\boldsymbol{f}}(\tilde{\boldsymbol{x}}) := \frac{L}{U^2}\boldsymbol{f}(L\tilde{\boldsymbol{x}})$, we conclude the transformed equations of motion

$$\frac{\partial \tilde{\boldsymbol{v}}}{\partial \tilde{t}} - \mathrm{Re}_p^{-1}\,\tilde{\nabla}\cdot\left[\left(\tilde{\varepsilon}^2 + |\tilde{\boldsymbol{D}}\tilde{\boldsymbol{v}}|^2\right)^{\frac{p-2}{2}}\tilde{\boldsymbol{D}}\tilde{\boldsymbol{v}}\right] + [\tilde{\boldsymbol{v}}\cdot\tilde{\nabla}]\tilde{\boldsymbol{v}} + \tilde{\nabla}\tilde{\pi} = \tilde{\boldsymbol{f}}, \qquad \tilde{\nabla}\cdot\tilde{\boldsymbol{v}} = 0 \tag{2.18}$$

where $\tilde{\varepsilon} := \frac{L}{U}\varepsilon$ and the Reynolds number Re_p is given by

$$\mathrm{Re}_p := \frac{\varrho_0 U^{3-p} L^{p-1}}{\mu_0}. \tag{2.19}$$

Multiplying the momentum equation (2.18) with Re_p, relabeling $\mathrm{Re}_p\,\tilde{\pi}$ and $\mathrm{Re}_p\,\tilde{\boldsymbol{f}}$ again as $\tilde{\pi}$ and $\tilde{\boldsymbol{f}}$, in the steady case we finally arrive at the transformed equations of motion

$$-\tilde{\nabla}\cdot\left[\left(\tilde{\varepsilon}^2 + |\tilde{\boldsymbol{D}}\tilde{\boldsymbol{v}}|^2\right)^{\frac{p-2}{2}}\tilde{\boldsymbol{D}}\tilde{\boldsymbol{v}}\right] + \mathrm{Re}_p[\tilde{\boldsymbol{v}}\cdot\tilde{\nabla}]\tilde{\boldsymbol{v}} + \tilde{\nabla}\tilde{\pi} = \tilde{\boldsymbol{f}}, \qquad \tilde{\nabla}\cdot\tilde{\boldsymbol{v}} = 0. \tag{2.20}$$

Boundary conditions: We complement systems (2.14) – (2.17) by appropriate boundary conditions which are formulated below for the case of steady flows only. If the evolutionary model (2.14) is studied, then these boundary conditions are considered on $[0,T]\times\partial\Omega$.

- Homogeneous Dirichlet boundary conditions: Internal flows meet the condition

  $$\boldsymbol{v}\cdot\boldsymbol{n} = 0 \qquad \text{on } \partial\Omega. \tag{2.21}$$

  Usually, such flows are subjected to no-slip boundary conditions

  $$\boldsymbol{v_t} := \boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{n} = \boldsymbol{0} \qquad \text{on } \partial\Omega. \tag{2.22}$$

  If (2.21) holds, then $\boldsymbol{v_t} = \boldsymbol{v}$. Combining (2.21) and (2.22), we arrive at the condition

  $$\boldsymbol{v} = \boldsymbol{0} \qquad \text{on } \partial\Omega. \tag{2.23}$$

- Space periodic boundary conditions: We assume that Ω is a d-dimensional cube with sides of length $L > 0$, $\Omega := (0, L)^d$, and that $\boldsymbol{v}$, π are periodic with period L in each spatial variable x_j: for $\Gamma_j = \partial\Omega \cap \{x_j = 0\}$, $\Gamma_{j+d} = \partial\Omega \cap \{x_j = L\}$ we require that

$$\boldsymbol{v}|_{\Gamma_j} = \boldsymbol{v}|_{\Gamma_{j+d}}, \qquad \pi|_{\Gamma_j} = \pi|_{\Gamma_{j+d}}, \qquad j = 1, \ldots, d, \qquad \int_\Omega \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} = \boldsymbol{0}.$$

- Mixed boundary conditions: We assume that the boundary consists of two parts, $\partial\Omega = \Gamma \cup S$, $|S| > 0$. Then, we prescribe the boundary conditions

$$\boldsymbol{v} = \boldsymbol{v}_D \qquad \text{on } \Gamma, \tag{2.24a}$$
$$-\boldsymbol{\mathcal{S}}(\pi, \boldsymbol{D}\boldsymbol{v})\boldsymbol{n} + \pi\boldsymbol{n} = \boldsymbol{b} \qquad \text{on } S. \tag{2.24b}$$

  The function $\boldsymbol{v}_D$ represents a given velocity field on the boundary (non-homogeneous Dirichlet data) whereas the function $\boldsymbol{b}$ reflects a given force that acts on the boundary.

Constraint on the pressure: For the subsequent discussion we follow Hirn et al. [HLS10]. The absolute value of the pressure is naturally determined up to a constant. Taking into account the boundary conditions (2.24), we distinguish two cases:

(a) If we prescribe Dirichlet boundary conditions on the whole boundary, $\partial\Omega = \Gamma$, then we additionally fix the level of pressure by requiring

$$⨍_\Omega \pi \, \mathrm{d}\boldsymbol{x} = \pi_0 \in \mathbb{R}. \tag{2.25}$$

In case of evolutionary problems, for $\pi_0 : (0, T) \to \mathbb{R}$ we incorporate the condition

$$⨍_\Omega \pi(t, \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = \pi_0(t) \qquad \text{for all } t \in (0, T). \tag{2.26}$$

(b) If $|S| > 0$, then (2.24b) suffices to fix the level of pressure, i.e., it implicitly normalizes the pressure. In particular, the pressure is already uniquely determined without the mean value constraint. For models of class (2.12), this was shown in [LS11b, LS11a].

The constraint (2.25) requires some remarks: When the viscosity does not depend on the pressure, the constant π_0 that fixes the pressure is irrelevant. In this case, we may set $\pi_0 = 0$. However, it is a special feature of piezoviscous fluids that the number π_0 affects the whole solution through $\boldsymbol{\mathcal{S}}(\pi, \boldsymbol{D}\boldsymbol{v})$, including the velocity field. Hence, the non-physical constraint (2.25) comprises an important input parameter undeterminable by practical applications. By contrast, $\boldsymbol{b}$ in (2.24b) represents the force acting on the domain boundary and, hence, it reflects physically reasonable input data. Nevertheless, with no loss of generality we may assume that $\pi_0 = 0$. Indeed, since the structural assumptions on $\boldsymbol{\mathcal{S}}$, which will be formulated in Assumption 2.2 below, impose no constraint on the value of the pressure (they only control the derivative of $\boldsymbol{\mathcal{S}}$ with respect to the pressure), they are satisfied for $\boldsymbol{\mathcal{S}}(\pi - \pi_0, \cdot)$ with arbitrary π_0 provided that they are fulfilled for $\boldsymbol{\mathcal{S}}(\pi, \cdot)$.

Models such as (2.11) belong to the class of p-structure models (see Assumption 2.1). If the extra stress tensor $\boldsymbol{\mathcal{S}}$ exhibits p-structure, then the system (2.14) is referred to as the non-steady p-Navier-Stokes equations. For p-structure models the system (2.16) represents the steady p-Stokes equations whereas the system (2.17) is called the p-Oseen system.

In this thesis, we will analyze the finite element (FE) approximation of system (2.16). In doing so, we will distinguish between the following two cases:

(1) The fluid viscosity only depends on the shear rate.

(2) The viscosity depends on both the shear rate and pressure.

In case of (1), we complement system (2.16) by homogeneous Dirichlet boundary conditions. General boundary conditions of type (2.24) are discussed in the context of (2). Since we use equal-order discretizations (see Section 3.1), we violate the well-known discrete compatibility (or inf-sup stability) condition of Babuška-Brezzi. In order to overcome the instability of this discretization, we will propose a stabilization method based on local projections (see Section 4.1). For the analysis of our method, we will restrict ourselves to case (1). In Section 4.5 we will show a priori error estimates that quantify the convergence of the method. Then, in Chapter 5 we will extend our approach to the generalized p-Oseen problem (2.17) whose FE approximation may additionally suffer from dominating convection in case of high Reynolds numbers Re_p. Finally, in Chapter 7 we will investigate the FE approximation of problem (2.16) related to case (2). Due to the complex structure of the viscosity, here we will carry out the analysis for stable discretizations only.

2.3 Assumptions on the extra stress tensor

In this section we state structural assumptions on the extra stress tensor $\boldsymbol{\mathcal{S}}$ and we indicate how $\boldsymbol{\mathcal{S}}$ relates to N-functions. For this, we follow Section 2.1 in [BDR10]. We set $\mathbb{R}^{d\times d}_{\mathrm{sym}} := \left\{\boldsymbol{P} \in \mathbb{R}^{d\times d};\, \boldsymbol{P} = \boldsymbol{P}^\top\right\}$ and $\boldsymbol{P}^{\mathrm{sym}} := \frac{1}{2}\left(\boldsymbol{P} + \boldsymbol{P}^\top\right)$. Due to the principle of objectivity, $\boldsymbol{\mathcal{S}}$ depends on the velocity gradient $\nabla \boldsymbol{v}$ only through its symmetric part $\boldsymbol{D}\boldsymbol{v} := \frac{1}{2}\left(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^\top\right)$. Therefore, the extra stress tensor $\boldsymbol{\mathcal{S}} : \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ shall satisfy $\boldsymbol{\mathcal{S}}(\mathbf{0}) = \mathbf{0}$ and $\boldsymbol{\mathcal{S}}(\boldsymbol{P}) = \boldsymbol{\mathcal{S}}(\boldsymbol{P}^{\mathrm{sym}})$. Moreover, it is supposed to satisfy the following

Assumption 2.1 (extra stress tensor). *We assume that the extra stress tensor* $\boldsymbol{\mathcal{S}} : \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ *belongs to* $C^0(\mathbb{R}^{d\times d}, \mathbb{R}^{d\times d}_{\mathrm{sym}}) \cap C^1(\mathbb{R}^{d\times d} \setminus \{0\}, \mathbb{R}^{d\times d}_{\mathrm{sym}})$ *and satisfies* $\boldsymbol{\mathcal{S}}(\boldsymbol{Q}) = \boldsymbol{\mathcal{S}}(\boldsymbol{Q}^{\mathrm{sym}})$ *and* $\boldsymbol{\mathcal{S}}(\mathbf{0}) = \mathbf{0}$*. Furthermore, we assume that* $\boldsymbol{\mathcal{S}}$ *possesses* (p, ε)*-structure, i.e.,* *there exist* $p \in (1, \infty)$, $\varepsilon \in [0, \infty)$, *and constants* $\sigma_0, \sigma_1 > 0$ *such that the inequalities*[1]

$$\sum_{i,j,k,l=1}^{d} \partial_{kl} S_{ij}(\boldsymbol{Q}) P_{ij} P_{kl} \geq \sigma_0 (\varepsilon + |\boldsymbol{Q}^{\mathrm{sym}}|)^{p-2} |\boldsymbol{P}^{\mathrm{sym}}|^2, \tag{2.27}$$

$$|\partial_{kl} S_{ij}(\boldsymbol{Q})| \leq \sigma_1 (\varepsilon + |\boldsymbol{Q}^{\mathrm{sym}}|)^{p-2} \tag{2.28}$$

are satisfied for all $\boldsymbol{Q}$, $\boldsymbol{P} \in \mathbb{R}^{d\times d}$ *with* $\boldsymbol{Q}^{\mathrm{sym}} \neq \boldsymbol{0}$ *and all* $i, j, k, l \in \{1, \dots, d\}$.

Remark 2.2. Relevant examples satisfying Assumption 2.1 are the power-law model (2.11a) and the Carreau model (2.11b). For such models, the extra stress tensor $\boldsymbol{\mathcal{S}}$ is derived from a potential: Let us assume that there exists a convex function $\Phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with $\Phi \in C^1(\mathbb{R}_0^+) \cap C^2(\mathbb{R}^+)$ and $\Phi(0) = \Phi'(0) = 0$ such that for $i, j = 1, \dots, d$

$$\mathcal{S}_{ij}(\boldsymbol{Q}) = \partial_{ij} \Phi(|\boldsymbol{Q}^{\mathrm{sym}}|) = \Phi'(|\boldsymbol{Q}^{\mathrm{sym}}|) \frac{Q_{ij}^{\mathrm{sym}}}{|\boldsymbol{Q}^{\mathrm{sym}}|} \qquad \forall \boldsymbol{Q} \in \mathbb{R}^{d\times d} \setminus \{\boldsymbol{0}\}. \tag{2.29}$$

For many fluid models such as (2.11a) and (2.11b), the potential Φ exhibits (p, ε)-structure. This means that there exist $p \in (1, \infty)$, $\varepsilon \in [0, \infty)$, and c_0, $c_1 > 0$ such that

$$c_0 (\varepsilon + t)^{p-2} \leq \Phi''(t) \leq c_1 (\varepsilon + t)^{p-2} \qquad \forall t \in \mathbb{R}^+. \tag{2.30}$$

From (2.30) it follows (cf. [RD07], Section 6) that uniformly in $t \geq 0$ there hold

$$\Phi'(t) \sim \Phi''(t) t, \qquad \Phi(t) \sim \Phi'(t) t \tag{2.31}$$

where the constants only depend on p, c_0, and c_1. In view of (2.29) – (2.31), it can be shown similarly to Lemma 21 in [DE08] that Assumption 2.1 is satisfied.

Below we depict how the stress tensor relates to N-functions which are standard in the theory of Orlicz spaces. Details on Orlicz spaces can be found in [KR61] or [RD07].

Definition 2.1. *A continuous convex function* $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *is called* N*-function if* $\psi(0) = 0$, $\psi(t) > 0$ *for* $t > 0$, $\lim_{t\to 0+} \psi(t)/t = 0$ *and* $\lim_{t\to\infty} \psi(t)/t = \infty$.

Let ψ be as in Definition 2.1. Consequently, there exists the right derivative ψ' of ψ, which is non-decreasing and satisfies $\psi'(0) = 0$, $\psi'(t) > 0$ for $t > 0$, and $\lim_{t\to\infty} \psi'(t) = \infty$.

Definition 2.2. *Let* ψ *be an* N*-function. We define* $(\psi')^{-1} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *by* $(\psi')^{-1}(t) := \sup\{u \in \mathbb{R}_0^+;\ \psi'(u) \leq t\}$. *Then, the complementary function* $\psi^* : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *is defined by*

$$\psi^*(t) := \int_0^t (\psi')^{-1}(s)\,\mathrm{d}s := \int_0^t \sup\{u \in \mathbb{R}_0^+;\ \psi'(u) \leq s\}\,\mathrm{d}s.$$

[1] For functions $g : \mathbb{R}^{d\times d} \to \mathbb{R}$ we use the notation $\partial_{kl} g(\boldsymbol{Q}) := \frac{\partial g(\boldsymbol{Q})}{\partial Q_{kl}}$.

If ψ' is strictly increasing, then $(\psi')^{-1}$ is the inverse function of ψ'. In this case, ψ^* is again an N-function and $(\psi^*)'(t) = (\psi')^{-1}(t)$ for all $t > 0$. The complementary function ψ^* can be characterized by $\psi^*(t) \equiv \sup_{s\geq 0}(st - \psi(s))$ for all $t \geq 0$.

An important subclass of N-functions consists of those that satisfy the Δ_2-condition:

Definition 2.3. *An N-function ψ satisfies the Δ_2-condition if there exists $C > 0$ such that $\psi(2t) \leq C\psi(t)$ for all $t \geq 0$. Here, $\Delta_2(\psi)$ denotes the smallest such constant.*

Let ψ be an N-function. Since $\psi(t) \leq \psi(2t)$ for all $t \geq 0$, the Δ_2-condition is equivalent to $\psi(2t) \sim \psi(t)$ uniformly in $t \geq 0$. If $\Delta_2(\psi) < \infty$, then it holds $\psi(t) \sim \psi(ct)$ uniformly in $t \geq 0$ for any fixed $c > 0$. For a family $\{\psi_\lambda\}$ of N-functions we define $\Delta_2(\{\psi_\lambda\}) := \sup_\lambda \Delta_2(\psi_\lambda)$. Next we introduce the notion of *shifted N-functions*.

Definition 2.4. *Let ψ be an N-function with $\Delta_2(\{\psi, \psi^*\}) < \infty$. For all $a \geq 0$ we define the family of shifted functions $\{\psi_a\}_{a\geq 0}$ by*

$$\psi_a(t) := \int_0^t \psi_a'(s)\,\mathrm{d}s \qquad \textit{with} \qquad \psi_a'(t) := \psi'(a+t)\frac{t}{a+t}. \tag{2.32}$$

The following lemma ensures that $\{\psi_a\}_{a\geq 0}$ are again N-functions and satisfy the Δ_2-condition uniformly in $a \geq 0$ with Δ_2-constants only depending on $\Delta_2(\psi)$, $\Delta_2(\psi^*)$:

Lemma 2.1. *Let ψ and ψ_a be as in Definition 2.4. Then, for all $a \geq 0$ the shifted functions ψ_a and $(\psi_a)^*$ are again N-functions and they satisfy $\Delta_2(\{\psi_a, (\psi_a)^*\}_{a\geq 0}) < \infty$. The families ψ_a and $(\psi_a)^*$ satisfy the Δ_2-condition uniformly in $a \geq 0$ where the constants only depend on $\Delta_2(\{\psi, \psi^*\})$. Moreover, it holds $(\psi_a)^*(t) \sim (\psi^*)_{\psi'(a)}(t)$ uniformly in $a, t \geq 0$.*

Proof. See Lemma 23 in [DE08]. □

The following lemma provides Young-type inequalities, which will be a useful tool for the finite element analysis of (2.16):

Lemma 2.2 (Young-type inequalities). *Let ψ be an N-function with $\Delta_2(\{\psi, \psi^*\}) < \infty$. Then, for all $\delta > 0$ there exists a constant $c_\delta > 0$, so that for all $t, u \geq 0$ there hold*

$$tu \leq \delta\psi(t) + c_\delta\psi^*(u), \tag{2.33}$$

$$t\psi'(u) + \psi'(t)u \leq \delta\psi(t) + c_\delta\psi(u). \tag{2.34}$$

The constant c_δ only depends on δ and $\Delta_2(\{\psi, \psi^*\})$. Let ψ and ψ_a be given as in Lemma 2.1. Then, for all $\delta > 0$ there exists a constant $c_\delta > 0$, so that for all $a, t, u \geq 0$ there hold

$$tu \leq \delta\psi_a(t) + c_\delta(\psi_a)^*(u), \tag{2.35}$$

$$t\psi_a'(u) + \psi_a'(t)u \leq \delta\psi_a(t) + c_\delta\psi_a(u). \tag{2.36}$$

Proof. See Lemma 32 in [DE08]. □

The subsequent lemma depicts further properties of shifted N-functions.

Lemma 2.3 (change of shift). *Let ψ be an N-function with $\Delta_2(\{\psi, \psi^*\}) < \infty$. Then*

$$\psi_{|\boldsymbol{P}|}(|\boldsymbol{P}-\boldsymbol{Q}|) \sim \psi_{|\boldsymbol{Q}|}(|\boldsymbol{P}-\boldsymbol{Q}|) \quad \text{and} \quad \psi'_{|\boldsymbol{P}|}(|\boldsymbol{P}-\boldsymbol{Q}|) \sim \psi'_{|\boldsymbol{Q}|}(|\boldsymbol{P}-\boldsymbol{Q}|) \quad \forall \boldsymbol{P}, \boldsymbol{Q} \in \mathbb{R}^{d\times d}$$

where the constants only depend on $\Delta_2(\psi)$. For each $\delta > 0$ there exists $c(\delta) > 0$, which only depends on δ and $\Delta_2(\psi)$, such that for all $\boldsymbol{P}$, $\boldsymbol{Q} \in \mathbb{R}^{d\times d}$ and $t \geq 0$ there holds

$$\psi_{|\boldsymbol{P}|}(t) \leq c(\delta)\psi_{|\boldsymbol{Q}|}(t) + \delta\psi_{|\boldsymbol{Q}|}(|\boldsymbol{P}-\boldsymbol{Q}|).$$

Proof. The first statement is proven in Lemma 28 of Diening/Ettwein [DE08]. The second one is shown in Corollary 26 of Diening/Kreuzer [DK08]. □

Let us consider the following simple example: For $p > 1$ we introduce the convex functions

$$\varphi, \varphi^* \in C(\mathbb{R}_0^+, \mathbb{R}_0^+), \qquad \varphi(t) := \frac{1}{p}t^p, \qquad \varphi^*(t) := \frac{1}{p'}t^{p'} \tag{2.37}$$

with $p' = p/(p-1)$. Clearly, φ is an N-function and φ^* is the complementary function of φ. The shifted N-functions φ_a are then given by $\varphi_a(t) = \int_0^t (a+s)^{p-2} s \, ds$. Note that the family $\{\varphi_a\}_{a\geq 0}$ belongs to $C^1(\mathbb{R}_0^+) \cap C^2(\mathbb{R}^+)$ and that it satisfies

$$\min\{1, p-1\}(a+t)^{p-2} \leq \varphi''_a(t) \leq \max\{1, p-1\}(a+t)^{p-2}. \tag{2.38}$$

Hence, the inequalities (2.27) and (2.28) defining the (p, ε)-structure of $\boldsymbol{S}$ can be expressed equivalently in terms of the shifted N-functions φ_ε. In Remark 2.3 we summarize further properties of the shifted N-functions φ_a:

Remark 2.3. Let φ be given by (2.37) and let φ_a be defined as in (2.32). It can be shown easily that $\varphi'_a(t) \sim \varphi''_a(t)t$, $\varphi_a(t) \sim \varphi'_a(t)t$, and $\varphi_a(2t) \sim \varphi_a(t)$ uniformly in t, $a \geq 0$ where all constants only depend on p. Since φ_a satisfies the Δ_2-condition, it follows that $\varphi_a(t+s) \lesssim \varphi_a(\frac{t+s}{2}) \lesssim \varphi_a(t) + \varphi_a(s)$ uniformly in t, s, $a \in \mathbb{R}_0^+$ due to the convexity of φ_a. Later we will apply the above Lemmas 2.1–2.3 to the N-function $\psi := \varphi_\varepsilon$. In view of Definition 2.4 we realize that $\psi'_a(t) = \varphi'_{\varepsilon+a}(t)$ for all t, $a \geq 0$ and, consequently, $\psi_a(t) = \varphi_{\varepsilon+a}(t)$ for all t, $a \geq 0$. Therefore, from Lemma 2.3 we can infer the relation

$$\varphi_{\varepsilon+|\boldsymbol{P}|}(|\boldsymbol{P}-\boldsymbol{Q}|) \sim \varphi_{\varepsilon+|\boldsymbol{Q}|}(|\boldsymbol{P}-\boldsymbol{Q}|) \qquad \forall \boldsymbol{P}, \boldsymbol{Q} \in \mathbb{R}^{d\times d}.$$

As mentioned above, we also consider fluids with pressure dependent viscosities. The fluid models under consideration are similar to models with p-structure described in Assumption 2.1. Additionally, it is allowed that the viscosity depends on the pressure sublinearly.

Assumption 2.2. *We suppose that the extra stress tensor $\boldsymbol{S}$ belongs to the class* (2.12) *and satisfies the structural assumptions:*

(A1) *There exist constants* $\sigma_0, \sigma_1 > 0$ *such that for all* $\boldsymbol{P}, \boldsymbol{Q} \in \mathbb{R}^{d\times d}_{\text{sym}}$, $q \in \mathbb{R}$ *there holds*

$$\sigma_0(\varepsilon^2 + |\boldsymbol{P}|^2)^{\frac{p-2}{2}} |\boldsymbol{Q}|^2 \le \frac{\partial \boldsymbol{\mathcal{S}}(q, \boldsymbol{P})}{\partial \boldsymbol{P}} : (\boldsymbol{Q} \otimes \boldsymbol{Q}) \le \sigma_1(\varepsilon^2 + |\boldsymbol{P}|^2)^{\frac{p-2}{2}} |\boldsymbol{Q}|^2,$$

where $\mathbb{R}^{d\times d}_{\text{sym}} := \{\boldsymbol{P} \in \mathbb{R}^{d\times d};\ \boldsymbol{P} = \boldsymbol{P}^{\mathsf{T}}\}$ *and* $(\boldsymbol{Q} \otimes \boldsymbol{Q})_{ijkl} = Q_{ij}Q_{kl}$.

(A2) *For all* $\boldsymbol{P} \in \mathbb{R}^{d\times d}_{\text{sym}}$ *and* $q \in \mathbb{R}$ *there holds*

$$\left|\frac{\partial \boldsymbol{\mathcal{S}}(q, \boldsymbol{P})}{\partial q}\right| \le \gamma_0(\varepsilon^2 + |\boldsymbol{P}|^2)^{\frac{p-2}{4}}.$$

The following remark has already been formulated in Hirn et al. [HLS10]:

Remark 2.4. Models satisfying Assumptions **(A1)**–**(A2)** can approximate some real world liquids in a certain range of shear rates and pressures, see [MNR02, MR06, MR07] for examples and applications; see also Remark 7.7. Note that both assumptions are rather restrictive concerning the dependence of the viscosity on the pressure, which is usually considered as $\mu \sim \exp(\alpha\pi)$ in practical applications. The well-posedness for problems with super-linear dependence on the pressure is, however, an open problem, similarly as the limiting case $\varepsilon = 0$. For possible generalizations to unbounded viscosities see [BMR09]. An exemplary model that satisfies **(A1)**–**(A2)** with $p = 2$ can be found in [MR07].

Note that Assumption **(A1)** is equivalent to Assumption 2.1 if fluid models of class (2.12) are considered. Throughout this thesis, we suppose that the extra stress tensor $\boldsymbol{\mathcal{S}}$ satisfies either Assumption 2.1 in case of fluids with pressure-independent viscosity or Assumption 2.2 in case of fluids with pressure-dependent viscosity.

2.4 Properties of the extra stress tensor

Below we express several consequences of Assumptions 2.1 and 2.2 that will play a crucial role in the FE analysis. To this end, we define a nonlinear function $\boldsymbol{\mathcal{F}} : \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}_{\text{sym}}$ by

$$\boldsymbol{\mathcal{F}}(\boldsymbol{P}) := \Big(\varepsilon + |\boldsymbol{P}^{\text{sym}}|\Big)^{\frac{p-2}{2}} \boldsymbol{P}^{\text{sym}} \tag{2.39}$$

where p and ε are the same as in Assumption 2.1/2.2. The function $\boldsymbol{\mathcal{F}}$ is closely related to the extra stress tensor $\boldsymbol{\mathcal{S}}$ with (p, ε)-structure as depicted by the following lemma:

Lemma 2.4. *For $p \in (1,\infty)$ and $\varepsilon \in [0,\infty)$ let $\boldsymbol{\mathcal{S}}$ satisfy Assumption 2.1, let $\boldsymbol{\mathcal{F}}$ be defined by (2.39), and let φ be defined by (2.37). Then, uniformly for all $\boldsymbol{P}$, $\boldsymbol{Q} \in \mathbb{R}^{d\times d}$ there hold*

$$
\begin{aligned}
\Big(\boldsymbol{\mathcal{S}}(\boldsymbol{P}) - \boldsymbol{\mathcal{S}}(\boldsymbol{Q})\Big) : (\boldsymbol{P} - \boldsymbol{Q}) &\sim (\varepsilon + |\boldsymbol{P}^{\text{sym}}| + |\boldsymbol{Q}^{\text{sym}}|)^{p-2}|\boldsymbol{P}^{\text{sym}} - \boldsymbol{Q}^{\text{sym}}|^2 \\
&\sim \varphi_\varepsilon''(|\boldsymbol{P}^{\text{sym}}| + |\boldsymbol{Q}^{\text{sym}}|)|\boldsymbol{P}^{\text{sym}} - \boldsymbol{Q}^{\text{sym}}|^2 \\
&\sim \varphi_{\varepsilon+|\boldsymbol{P}^{\text{sym}}|}(|\boldsymbol{P}^{\text{sym}} - \boldsymbol{Q}^{\text{sym}}|) \\
&\sim |\boldsymbol{\mathcal{F}}(\boldsymbol{P}) - \boldsymbol{\mathcal{F}}(\boldsymbol{Q})|^2, \\
|\boldsymbol{\mathcal{S}}(\boldsymbol{P}) - \boldsymbol{\mathcal{S}}(\boldsymbol{Q})| &\sim (\varepsilon + |\boldsymbol{P}^{\text{sym}}| + |\boldsymbol{Q}^{\text{sym}}|)^{p-2}|\boldsymbol{P}^{\text{sym}} - \boldsymbol{Q}^{\text{sym}}| \\
&\sim \varphi'_{\varepsilon+|\boldsymbol{P}^{\text{sym}}|}(|\boldsymbol{P}^{\text{sym}} - \boldsymbol{Q}^{\text{sym}}|),
\end{aligned}
$$

where the constants only depend on σ_0, σ_1 and p. In particular, the constants are independent of $\varepsilon \geq 0$. Because of $\boldsymbol{\mathcal{S}}(\boldsymbol{0}) = \boldsymbol{\mathcal{F}}(\boldsymbol{0}) = \boldsymbol{0}$, we observe that $\boldsymbol{\mathcal{S}}(\boldsymbol{Q}) : \boldsymbol{Q} \sim |\boldsymbol{\mathcal{F}}(\boldsymbol{Q})|^2 \sim \varphi_\varepsilon(|\boldsymbol{Q}^{\text{sym}}|)$.

Proof. The lemma is proven in Diening/Ettwein [DE08]. □

Since below we will only insert symmetric tensors into $\boldsymbol{\mathcal{S}}$ and $\boldsymbol{\mathcal{F}}$, we drop the superscript “sym” in the above formulas and we restrict the admitted tensors to symmetric ones.

Remark 2.5. From Lemma 2.4 it easily follows that for all $\boldsymbol{Q} \in \mathbb{R}^{d\times d}_{\text{sym}}$ there hold:

$$
\begin{aligned}
&\boldsymbol{\mathcal{S}}(\boldsymbol{Q}) : \boldsymbol{Q} \gtrsim (|\boldsymbol{Q}|^p - \varepsilon^p) \quad \text{and} \quad |\boldsymbol{\mathcal{S}}(\boldsymbol{Q})| \lesssim |\boldsymbol{Q}|^{p-1} \quad \text{if} \quad p \in (1,2]; && (2.40)\\
&\boldsymbol{\mathcal{S}}(\boldsymbol{Q}) : \boldsymbol{Q} \gtrsim |\boldsymbol{Q}|^p \quad \text{and} \quad |\boldsymbol{\mathcal{S}}(\boldsymbol{Q})| \lesssim (\varepsilon + |\boldsymbol{Q}|)^{p-1} \quad \text{if} \quad p \in [2,\infty). && (2.41)
\end{aligned}
$$

All constants in (2.40) and (2.41) only depend on σ_0, σ_1, p.

As a further consequence of Lemma 2.4, we obtain the following result:

Lemma 2.5. *Under the assumptions of Lemma 2.4 for all $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$ there holds*

$$
\begin{aligned}
\int_\Omega \Big(\boldsymbol{\mathcal{S}}(\boldsymbol{Du}) - \boldsymbol{\mathcal{S}}(\boldsymbol{Dv})\Big) : (\boldsymbol{Du} - \boldsymbol{Dv}) \, \mathrm{d}\boldsymbol{x} &\sim \|\boldsymbol{\mathcal{F}}(\boldsymbol{Du}) - \boldsymbol{\mathcal{F}}(\boldsymbol{Dv})\|_2^2 \\
&\sim \int_\Omega \varphi_{\varepsilon+|\boldsymbol{Du}|}(|\boldsymbol{Du} - \boldsymbol{Dv}|) \, \mathrm{d}\boldsymbol{x}
\end{aligned}
$$

where the constants only depend on σ_0, σ_1 and p. In particular, they are independent of ε.

Lemma 2.5 highlights how the distance defined by $\boldsymbol{\mathcal{F}}$ connects to the *quasi-norm* introduced by Barrett/Liu [BL93a, BL94]. For $\boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{W}^{1,p}(\Omega)$ the quasi-norm is defined by

$$
|\boldsymbol{w}|^2_{(p,\boldsymbol{v})} := \int_\Omega (\varepsilon + |\boldsymbol{Dv}| + |\boldsymbol{Dw}|)^{p-2}|\boldsymbol{Dw}|^2 \, \mathrm{d}\boldsymbol{x}.
$$

The distance $|\cdot|_{(p,\boldsymbol{v})}$ is called quasi-norm, since $|\cdot|_{(p,\boldsymbol{v})}$ satisfies all properties of a norm except homogeneity. By means of Lemma 2.5, for all $\boldsymbol{v}, \boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ the equivalence

$$\begin{aligned}\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{u})\|_2^2 &\sim \int_\Omega \varphi_{\varepsilon+|\boldsymbol{D}\boldsymbol{v}|}(|\boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{u}|)\,\mathrm{d}\boldsymbol{x} \\ &\sim \int_\Omega \varphi'_{\varepsilon+|\boldsymbol{D}\boldsymbol{v}|}(|\boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{u}|)|\boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{u}|\,\mathrm{d}\boldsymbol{x} \\ &\sim \int_\Omega \frac{\varphi'(\varepsilon + |\boldsymbol{D}\boldsymbol{v}| + |\boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{u}|)}{\varepsilon + |\boldsymbol{D}\boldsymbol{v}| + |\boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{u}|}|\boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{u}|^2\,\mathrm{d}\boldsymbol{x} \sim |\boldsymbol{v} - \boldsymbol{u}|^2_{(p,\boldsymbol{v})}\end{aligned} \tag{2.42}$$

follows, where all constants only depend on σ_0, σ_1 and p. This relation ensures that all results below can also be expressed in terms of quasi-norms. The following lemma shows the connection between the quasi-norms and Sobolev norms:

Lemma 2.6. *For $p \in (1,\infty)$ and $\varepsilon \in (0,\infty)$ let $\boldsymbol{\mathcal{S}}$ satisfy Assumption 2.1, and let $\boldsymbol{\mathcal{F}}$ be defined by* (2.39)*.*

(i) Let $p \in (1,2]$. For functions $\boldsymbol{V}$, $\boldsymbol{U} \in L^p(\omega)^{d\times d}$ and for $\nu \in [1,2]$ there holds

$$\|\boldsymbol{V} - \boldsymbol{U}\|^2_{\nu;\omega} \lesssim \|\boldsymbol{\mathcal{F}}(\boldsymbol{V}) - \boldsymbol{\mathcal{F}}(\boldsymbol{U})\|^2_{2;\omega}\|(\varepsilon + |\boldsymbol{V}| + |\boldsymbol{U}|)^{2-p}\|_{\frac{\nu}{2-\nu};\omega} \tag{2.43}$$

provided that all terms are finite where the constant only depends on p, σ_0 and σ_1. If $\nu = 2$, then $\frac{\nu}{2-\nu} = \infty$. Moreover, for all $\boldsymbol{V}$, $\boldsymbol{U} \in L^p(\omega)^{d\times d}$ there holds

$$\|\boldsymbol{\mathcal{F}}(\boldsymbol{V}) - \boldsymbol{\mathcal{F}}(\boldsymbol{U})\|^2_{2;\omega} \lesssim \|\boldsymbol{V} - \boldsymbol{U}\|^p_{p;\omega}, \tag{2.44}$$

where the constant only depends on σ_0, σ_1 and p.

(ii) Let $p \in [2,\infty)$. Then, for all $\boldsymbol{V}$, $\boldsymbol{U} \in L^p(\omega)^{d\times d}$ there holds

$$\|\boldsymbol{V} - \boldsymbol{U}\|^p_{p;\omega} \lesssim \|\boldsymbol{\mathcal{F}}(\boldsymbol{V}) - \boldsymbol{\mathcal{F}}(\boldsymbol{U})\|^2_{2;\omega} \lesssim \|\varepsilon + |\boldsymbol{V}| + |\boldsymbol{U}|\|^{p-2}_{p;\omega}\|\boldsymbol{V} - \boldsymbol{U}\|^2_{p;\omega}, \tag{2.45}$$

where the constants only depend on σ_0, σ_1 and p.

Proof. From Lemma 2.4 it follows that $|\boldsymbol{\mathcal{F}}(\boldsymbol{V}) - \boldsymbol{\mathcal{F}}(\boldsymbol{U})|^\nu(\varepsilon + |\boldsymbol{V}| + |\boldsymbol{U}|)^{\frac{(2-p)\nu}{2}} \sim |\boldsymbol{V} - \boldsymbol{U}|^\nu$. Integrating this and applying Hölder's inequality with $\frac{\nu}{2} + \frac{2-\nu}{2} = 1$, we easily derive (2.43):

$$\int_\omega |\boldsymbol{V} - \boldsymbol{U}|^\nu\,\mathrm{d}\boldsymbol{x} \lesssim \Big(\int_\omega |\boldsymbol{\mathcal{F}}(\boldsymbol{V}) - \boldsymbol{\mathcal{F}}(\boldsymbol{U})|^2\,\mathrm{d}\boldsymbol{x}\Big)^{\frac{\nu}{2}}\Big(\int_\omega (\varepsilon + |\boldsymbol{V}| + |\boldsymbol{U}|)^{\frac{(2-p)\nu}{2-\nu}}\,\mathrm{d}\boldsymbol{x}\Big)^{\frac{2-\nu}{2}}.$$

For the proof of (2.44) we mention the following trivial inequalities

$$\frac{1}{2}(|\boldsymbol{P}_1| + |\boldsymbol{P}_2|) \le |\boldsymbol{P}_1| + |\boldsymbol{P}_1 - \boldsymbol{P}_2| \le 2(|\boldsymbol{P}_1| + |\boldsymbol{P}_2|) \qquad \forall \boldsymbol{P}_1, \boldsymbol{P}_2 \in \mathbb{R}^{d\times d}. \tag{2.46}$$

Using Lemma 2.4, (2.46), and the fact $p \le 2$, we infer (2.44) as follows:

$$\int_\omega |\boldsymbol{\mathcal{F}}(\boldsymbol{V}) - \boldsymbol{\mathcal{F}}(\boldsymbol{U})|^2\,\mathrm{d}\boldsymbol{x} \sim \int_\omega (\varepsilon + |\boldsymbol{V}| + |\boldsymbol{V} - \boldsymbol{U}|)^{p-2}|\boldsymbol{V} - \boldsymbol{U}|^2\,\mathrm{d}\boldsymbol{x} \lesssim \int_\omega |\boldsymbol{V} - \boldsymbol{U}|^p\,\mathrm{d}\boldsymbol{x}.$$

Finally, using the fact $p \geq 2$, and Hölder's inequality with $\frac{2}{p} + \frac{p-2}{p} = 1$, we conclude that

$$\int_\omega |\boldsymbol{V} - \boldsymbol{U}|^p \,\mathrm{d}\boldsymbol{x} \leq \int_\omega (\varepsilon + |\boldsymbol{V}| + |\boldsymbol{V} - \boldsymbol{U}|)^{p-2} |\boldsymbol{V} - \boldsymbol{U}|^2 \,\mathrm{d}\boldsymbol{x}$$
$$\lesssim \Big(\int_\omega (\varepsilon + |\boldsymbol{V}| + |\boldsymbol{U}|)^p \,\mathrm{d}\boldsymbol{x} \Big)^{\frac{p-2}{p}} \Big(\int_\omega |\boldsymbol{V} - \boldsymbol{U}|^p \,\mathrm{d}\boldsymbol{x} \Big)^{\frac{2}{p}}.$$

In view of Lemma 2.4, this proves (2.45). □

Lemma 2.7. *For $p \in (1, \infty)$ and $\varepsilon \in (0, \infty)$ let $\boldsymbol{\mathcal{S}}$ satisfy Assumption 2.1, and let $\boldsymbol{\mathcal{F}}$ be defined by* (2.39)*.*

(i) Let $p \in (1, 2]$. For all $\boldsymbol{V}, \boldsymbol{U} \in L^p(\omega)^{d\times d}$ there hold

$$\|\boldsymbol{\mathcal{S}}(\boldsymbol{V}) - \boldsymbol{\mathcal{S}}(\boldsymbol{U})\|_{2;\omega} \leq c\varepsilon^{\frac{p-2}{2}} \|\boldsymbol{\mathcal{F}}(\boldsymbol{V}) - \boldsymbol{\mathcal{F}}(\boldsymbol{U})\|_{2;\omega}, \tag{2.47}$$
$$\|\boldsymbol{\mathcal{S}}(\boldsymbol{V}) - \boldsymbol{\mathcal{S}}(\boldsymbol{U})\|_{p';\omega} \leq c\|\boldsymbol{\mathcal{F}}(\boldsymbol{V}) - \boldsymbol{\mathcal{F}}(\boldsymbol{U})\|_{2;\omega}^{\frac{2}{p'}} \tag{2.48}$$

where the constants only depend on σ_0, σ_1 and p.

(ii) Let $p \in [2, \infty)$. For all $\boldsymbol{V}, \boldsymbol{U} \in L^p(\omega)^{d\times d}$ there hold

$$\|\boldsymbol{\mathcal{S}}(\boldsymbol{V}) - \boldsymbol{\mathcal{S}}(\boldsymbol{U})\|_{p';\omega} \leq c\|\boldsymbol{\mathcal{F}}(\boldsymbol{V}) - \boldsymbol{\mathcal{F}}(\boldsymbol{U})\|_{2;\omega} \|\varepsilon + |\boldsymbol{V}| + |\boldsymbol{U}|\|_{p;\omega}^{\frac{p-2}{2}}, \tag{2.49}$$
$$\|\boldsymbol{\mathcal{S}}(\boldsymbol{V}) - \boldsymbol{\mathcal{S}}(\boldsymbol{U})\|_{p';\omega} \leq c\|\boldsymbol{V} - \boldsymbol{U}\|_{2;\omega} \|\varepsilon + |\boldsymbol{V}| + |\boldsymbol{U}|\|_{2p;\omega}^{p-2} \tag{2.50}$$

where the constants only depend on σ_0, σ_1 and p.

Proof. From Lemma 2.4 it follows that for $p \in (1, \infty)$ and $\nu \in \{2, p'\}$ it holds

$$\|\boldsymbol{\mathcal{S}}(\boldsymbol{V}) - \boldsymbol{\mathcal{S}}(\boldsymbol{U})\|_{\nu;\omega} \sim \Big(\int_\omega (\varepsilon + |\boldsymbol{V}| + |\boldsymbol{U}|)^{(p-2)\nu} |\boldsymbol{V} - \boldsymbol{U}|^\nu \,\mathrm{d}\boldsymbol{x} \Big)^{\frac{1}{\nu}}. \tag{2.51}$$

If $\nu = 2$ and $p \leq 2$, we immediately obtain inequality (2.47) using Lemma 2.4. If $\nu = p'$ and $p \leq 2 \Leftrightarrow p' \geq 2$, from (2.51) we easily deduce the following estimates:

$$\|\boldsymbol{\mathcal{S}}(\boldsymbol{V}) - \boldsymbol{\mathcal{S}}(\boldsymbol{U})\|_{p';\omega} \sim \Big(\int_\omega (\varepsilon + |\boldsymbol{V}| + |\boldsymbol{V} - \boldsymbol{U}|)^{(p-2)p'} |\boldsymbol{V} - \boldsymbol{U}|^{p'} \,\mathrm{d}\boldsymbol{x} \Big)^{\frac{1}{p'}}$$
$$\lesssim \Big(\int_\omega (\varepsilon + |\boldsymbol{V}| + |\boldsymbol{V} - \boldsymbol{U}|)^{p-2} |\boldsymbol{V} - \boldsymbol{U}|^2 \,\mathrm{d}\boldsymbol{x} \Big)^{\frac{1}{p'}}.$$

In view of Lemma 2.4, we arrive at (2.48). Finally, if $p \geq 2$, (2.51) with $\nu = p'$ implies

$$\|\boldsymbol{\mathcal{S}}(\boldsymbol{V}) - \boldsymbol{\mathcal{S}}(\boldsymbol{U})\|_{p';\omega} \sim \Big(\int_\omega (\varepsilon + |\boldsymbol{V}| + |\boldsymbol{U}|)^{\frac{(p-2)p'}{2}} (\varepsilon + |\boldsymbol{V}| + |\boldsymbol{U}|)^{\frac{(p-2)p'}{2}} |\boldsymbol{V} - \boldsymbol{U}|^{p'} \,\mathrm{d}\boldsymbol{x} \Big)^{\frac{1}{p'}}$$
$$\lesssim \Big(\int_\omega (\varepsilon + |\boldsymbol{V}| + |\boldsymbol{U}|)^p \mathrm{d}\boldsymbol{x} \Big)^{\frac{2-p'}{2p'}} \Big(\int_\omega (\varepsilon + |\boldsymbol{V}| + |\boldsymbol{U}|)^{p-2} |\boldsymbol{V} - \boldsymbol{U}|^2 \mathrm{d}\boldsymbol{x} \Big)^{\frac{1}{2}}.$$

Here, we have used Hölder's inequality with $\frac{p'}{2} + \frac{2-p'}{2} = 1$. By virtue of Lemma 2.4, the latter inequality yields the desired estimate (2.49). Similarly, (2.50) follows from (2.51):

$$\|\boldsymbol{\mathcal{S}}(\boldsymbol{V}) - \boldsymbol{\mathcal{S}}(\boldsymbol{U})\|_{p';\omega} \lesssim \Big(\int_\omega (\varepsilon + |\boldsymbol{V}| + |\boldsymbol{U}|)^{\frac{p-2}{2-p'}2p'} \,\mathrm{d}\boldsymbol{x} \Big)^{\frac{2-p'}{2p'}} \Big(\int_\omega |\boldsymbol{V} - \boldsymbol{U}|^2 \,\mathrm{d}\boldsymbol{x} \Big)^{\frac{1}{2}}.$$

Note that $\frac{p-2}{2-p'} = (p-1)$ and $\frac{2-p'}{2p'} = \frac{p-2}{2p}$. This completes the proof. □

Below we focus on fluids whose viscosity depends on the shear rate and pressure. We express several consequences of Assumptions **(A1)** and **(A2)**. The first lemma includes similar statements as presented by Lemma 2.4.

Lemma 2.8. *For given $p \in (1,\infty)$ and $\varepsilon \in [0,\infty)$ let $\boldsymbol{\mathcal{S}}$ satisfy* **(A1)**, *let $\boldsymbol{\mathcal{F}}$ be defined by* (2.39), *and let φ be defined by* (2.37). *Then, for all $\boldsymbol{P}$, $\boldsymbol{Q} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$, $q \in \mathbb{R}$, there hold*

$$\begin{aligned}\Big(\boldsymbol{\mathcal{S}}(q,\boldsymbol{P}) - \boldsymbol{\mathcal{S}}(q,\boldsymbol{Q})\Big) : (\boldsymbol{P} - \boldsymbol{Q}) &\sim (\varepsilon + |\boldsymbol{P}| + |\boldsymbol{Q}|)^{p-2}|\boldsymbol{P} - \boldsymbol{Q}|^2 \\ &\sim \varphi_{\varepsilon+|\boldsymbol{P}|}(|\boldsymbol{P} - \boldsymbol{Q}|) \sim |\boldsymbol{\mathcal{F}}(\boldsymbol{P}) - \boldsymbol{\mathcal{F}}(\boldsymbol{Q})|^2, \\ |\boldsymbol{\mathcal{S}}(q,\boldsymbol{P}) - \boldsymbol{\mathcal{S}}(q,\boldsymbol{Q})| &\sim \varphi'_{\varepsilon+|\boldsymbol{P}|}(|\boldsymbol{P} - \boldsymbol{Q}|),\end{aligned}$$

where the constants only depend on σ_0, σ_1 and p. In particular, they are independent of $\varepsilon \geq 0$. Moreover, the following estimates hold:

$$\boldsymbol{\mathcal{S}}(q,\boldsymbol{Q}) : \boldsymbol{Q} \geq \frac{\sigma_0}{2p}(|\boldsymbol{Q}|^p - \varepsilon^p) \qquad \textit{and} \qquad |\boldsymbol{\mathcal{S}}(q,\boldsymbol{Q})| \leq \frac{\sigma_1}{p-1}|\boldsymbol{Q}|^{p-1}. \tag{2.52}$$

Proof. See Lemma 2.4. The proof of (2.52) can be found in Málek et al. [MNRR96]. □

As a straightforward consequence of Assumptions **(A1)** and **(A2)** we also obtain

Lemma 2.9. *For given $p \in (1,\infty)$, $\varepsilon \in (0,\infty)$ and $\gamma_0 \in [0,\infty)$ let $\boldsymbol{\mathcal{S}}$ satisfy* **(A1)**, **(A2)**. *Then, for all $\boldsymbol{P}_0$, $\boldsymbol{P}_1 \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and $\pi, q \in \mathbb{R}$, denoting $\boldsymbol{P}_s := \boldsymbol{P}_0 + s(\boldsymbol{P}_1 - \boldsymbol{P}_0)$, there hold*

$$\frac{\sigma_0}{2}\int_0^1 (\varepsilon^2 + |\boldsymbol{P}_s|^2)^{\frac{p-2}{2}}|\boldsymbol{P}_1 - \boldsymbol{P}_0|^2 \mathrm{d}s \leq (\boldsymbol{\mathcal{S}}(\pi,\boldsymbol{P}_1) - \boldsymbol{\mathcal{S}}(q,\boldsymbol{P}_0)) : (\boldsymbol{P}_1 - \boldsymbol{P}_0) + \frac{\gamma_0^2}{2\sigma_0}|\pi - q|^2,$$

$$|\boldsymbol{\mathcal{S}}(\pi,\boldsymbol{P}_1) - \boldsymbol{\mathcal{S}}(q,\boldsymbol{P}_0)| \leq \sigma_1 \int_0^1 (\varepsilon^2 + |\boldsymbol{P}_s|^2)^{\frac{p-2}{2}}|\boldsymbol{P}_1 - \boldsymbol{P}_0| \,\mathrm{d}s + \gamma_0 \int_0^1 (\varepsilon^2 + |\boldsymbol{P}_s|^2)^{\frac{p-2}{4}}|\pi - q| \,\mathrm{d}s.$$

Proof. See, e.g., Bulíček et al. [BMR07], Lemma 1.4. □

In view of Lemma 2.9 we define the distance

$$d(\boldsymbol{v},\boldsymbol{u})^2 := \int_\Omega \int_0^1 (\varepsilon^2 + |\boldsymbol{D}\boldsymbol{u} + s(\boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{u})|^2)^{\frac{p-2}{2}}|\boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{u}|^2 \,\mathrm{d}s \,\mathrm{d}\boldsymbol{x} \tag{2.53}$$

for all $\boldsymbol{v}, \boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$. We arrive at

Corollary 2.10. *For $p \in (1,\infty)$, $\varepsilon \in (0,\infty)$ and $\gamma_0 \in [0,\infty)$ let $\boldsymbol{S}$ satisfy* **(A1)**, **(A2)**. *Let $d(\cdot,\cdot)$ be defined by (2.53). Then, for all $\boldsymbol{v},\boldsymbol{w} \in \boldsymbol{W}^{1,p}(\Omega)$ and $\pi, q \in L^2(\Omega)$ there holds:*

$$\frac{\sigma_0}{2} d(\boldsymbol{v},\boldsymbol{w})^2 \le (\boldsymbol{S}(\pi,\boldsymbol{Dv}) - \boldsymbol{S}(q,\boldsymbol{Dw}), \boldsymbol{Dv} - \boldsymbol{Dw})_\Omega + \frac{\gamma_0^2}{2\sigma_0}\|\pi - q\|_2^2. \tag{2.54}$$

For each $\delta > 0$ there exists a positive constant c_δ only depending on σ_1 and δ such that

$$(\boldsymbol{S}(\pi,\boldsymbol{Dv}) - \boldsymbol{S}(q,\boldsymbol{Dw}), \boldsymbol{Dv} - \boldsymbol{Dw})_\Omega \le c_\delta d(\boldsymbol{v},\boldsymbol{w})^2 + \delta\gamma_0^2\|\pi - q\|_2^2. \tag{2.55}$$

If $p < 2$, then for all $\boldsymbol{v},\boldsymbol{w} \in \boldsymbol{W}^{1,p}(\Omega)$ and all sufficiently smooth functions π, q there hold

$$\|\boldsymbol{S}(\pi,\boldsymbol{Dv}) - \boldsymbol{S}(q,\boldsymbol{Dw})\|_2 \le \sigma_1 \varepsilon^{\frac{p-2}{2}} d(\boldsymbol{v},\boldsymbol{w}) + \gamma_0 \varepsilon^{\frac{p-2}{2}}\|\pi - q\|_2, \tag{2.56}$$

$$\|\boldsymbol{S}(\pi,\boldsymbol{Dv}) - \boldsymbol{S}(q,\boldsymbol{Dw})\|_{p'} \le c d(\boldsymbol{v},\boldsymbol{w})^{\frac{2}{p'}} + \gamma_0 \varepsilon^{\frac{p-2}{2}}\|\pi - q\|_{p'}, \tag{2.57}$$

where $c = c(p,\sigma_1)$ is a positive constant.

Proof. Clearly, (2.54) and (2.55) directly follow from Lemma 2.9 and Young's inequality. Setting $\boldsymbol{D}_s := \boldsymbol{Dw} + s(\boldsymbol{Dv} - \boldsymbol{Dw})$, for $\nu \ge 1$ we infer from Lemma 2.9 that

$$\begin{aligned} \|\boldsymbol{S}(\pi,\boldsymbol{Dv}) - \boldsymbol{S}(q,\boldsymbol{Dw})\|_\nu \le \sigma_1 \Big(\int_\Omega \Big| \int_0^1 \big(\varepsilon^2 + |\boldsymbol{D}_s|^2\big)^{\frac{p-2}{2}} |\boldsymbol{Dv} - \boldsymbol{Dw}| \mathrm{d}s \Big|^\nu \mathrm{d}\boldsymbol{x} \Big)^{\frac{1}{\nu}} \\ + \gamma_0 \Big(\int_\Omega \Big| \int_0^1 \big(\varepsilon^2 + |\boldsymbol{D}_s|^2\big)^{\frac{p-2}{4}} |\pi - q| \mathrm{d}s \Big|^\nu \mathrm{d}\boldsymbol{x} \Big)^{\frac{1}{\nu}}. \end{aligned} \tag{2.58}$$

We immediately deduce (2.56) from (2.58) with $\nu = 2$ and Jensen's inequality. In order to derive (2.57), we recall the following well-known result (see [AF89], Lemma 2.1)

$$\big(\varepsilon^2 + (|\boldsymbol{P}_1| + |\boldsymbol{P}_2|)^2\big)^\alpha \sim \int_0^1 \big(\varepsilon^2 + |\boldsymbol{P}_2 + s(\boldsymbol{P}_1 - \boldsymbol{P}_2)|^2\big)^\alpha \mathrm{d}s \qquad \forall \boldsymbol{P}_1, \boldsymbol{P}_2 \in \mathbb{R}^{d\times d}_{\mathrm{sym}}, \tag{2.59}$$

which holds for each $\alpha > -1/2$ provided that $\varepsilon + |\boldsymbol{P}_1| + |\boldsymbol{P}_2| > 0$. The constants in (2.59) only depend on α. Using (2.59), (2.46), the fact $p < 2$, we conclude from (2.58) that

$$\begin{aligned} \|\boldsymbol{S}(\pi,\boldsymbol{Dv}) - \boldsymbol{S}(q,\boldsymbol{Dw})\|_{p'} &\le c \Big(\int_\Omega \big(\varepsilon^2 + (|\boldsymbol{Dw}| + |\boldsymbol{Dv} - \boldsymbol{Dw}|)^2\big)^{\frac{p-2}{2}p'} |\boldsymbol{Dv} - \boldsymbol{Dw}|^{p'} \mathrm{d}\boldsymbol{x} \Big)^{\frac{1}{p'}} \\ &\quad + \gamma_0 \Big(\int_\Omega \Big| \int_0^1 \big(\varepsilon^2 + |\boldsymbol{D}_s|^2\big)^{\frac{p-2}{4}} |\pi - q| \mathrm{d}s \Big|^{p'} \mathrm{d}\boldsymbol{x} \Big)^{\frac{1}{p'}} \\ &\le c \Big(\int_\Omega \big(\varepsilon^2 + (|\boldsymbol{Dw}| + |\boldsymbol{Dv} - \boldsymbol{Dw}|)^2\big)^{\frac{p-2}{2}} |\boldsymbol{Dv} - \boldsymbol{Dw}|^2 \mathrm{d}\boldsymbol{x} \Big)^{\frac{1}{p'}} \\ &\quad + \gamma_0 \varepsilon^{\frac{p-2}{2}} \Big(\int_\Omega |\pi - q|^{p'} \mathrm{d}\boldsymbol{x} \Big)^{\frac{1}{p'}}, \end{aligned}$$

where the constant c only depends on p and σ_1. This yields (2.57). □

The following lemma indicates that $d(\cdot,\cdot)$ is equivalent to the natural distance:

Lemma 2.11. *For $p \in (1,\infty)$, $\varepsilon \in (0,\infty)$ let $\boldsymbol{\mathcal{S}}$ satisfy* **(A1)**. *Let $d(\cdot,\cdot)$ be defined by* (2.53)*, and let $\boldsymbol{\mathcal{F}}$ be defined by* (2.39)*. For all $\boldsymbol{v}, \boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ and $\pi \in L^2(\Omega)$ there holds:*

$$d(\boldsymbol{v},\boldsymbol{u})^2 \sim \|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{u})\|_2^2 \sim (\boldsymbol{\mathcal{S}}(\pi,\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{S}}(\pi,\boldsymbol{D}\boldsymbol{u}), \boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{u})_\Omega. \tag{2.60}$$

All constants only depend on p, σ_0, σ_1.

Proof. In view of (2.59) and (2.46), the assertion follows from Lemma 2.8. □

2.5 The p-Stokes equations

In this section, we introduce the variational formulation of the p-Stokes equations which makes up the basis for the finite element discretization. Moreover we present well-known theoretical results that ensure the existence and uniqueness of weak solutions. We deal with incompressible fluids whose viscosity depends on the shear rate only. If not stated otherwise, we assume that for $p \in (1,\infty)$ and $\varepsilon \in [0,\infty)$ the extra stress tensor $\boldsymbol{\mathcal{S}}$ satisfies Assumption 2.1. Here, we may think of the stress tensors (2.11a) and (2.11b) as prototypes. We consider the p-Stokes equations complemented with homogeneous Dirichlet boundary conditions. As usual, $\mathcal{D}(\Omega) := C_0^\infty(\Omega)$ denotes the set of all smooth functions with compact support in Ω. We set $\boldsymbol{\mathcal{D}}_{\mathrm{div}}(\Omega) := \{\boldsymbol{w} \in \mathcal{D}(\Omega)^d;\ \nabla\cdot\boldsymbol{w} = 0\}$. Below we define the natural spaces for the velocity and pressure that are used throughout the thesis:

$$\begin{aligned}
\boldsymbol{\mathcal{H}}^q &:= \left\{\boldsymbol{w} \in L^q(\Omega)^d;\ \nabla\cdot\boldsymbol{w} = 0,\ \boldsymbol{w}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\right\} = \overline{\boldsymbol{\mathcal{D}}_{\mathrm{div}}(\Omega)}^{\|\cdot\|_q}\\
\boldsymbol{\mathcal{X}}^p &:= W_0^{1,p}(\Omega)^d = \overline{\mathcal{D}(\Omega)^d}^{\|\nabla\cdot\|_p}\\
\boldsymbol{\mathcal{V}}^p &:= \left\{\boldsymbol{w} \in W_0^{1,p}(\Omega)^d;\ \nabla\cdot\boldsymbol{w} = 0\right\} = \overline{\boldsymbol{\mathcal{D}}_{\mathrm{div}}(\Omega)}^{\|\nabla\cdot\|_p}\\
\mathcal{Q}^p &:= L_0^{p'}(\Omega) := \left\{q \in L^{p'}(\Omega);\ (q,1)_\Omega = 0\right\}
\end{aligned}$$

with $1/p + 1/p' = 1$. The variational formulation of the p-Stokes equations represents the model problem of the thesis. Basic concepts of the thesis including error estimation will be explained on the basis of this model problem, see Chapter 4.

The steady case: Let us consider the steady p-Stokes system (2.16) with homogeneous Dirichlet boundary conditions. The weak formulation of the p-Stokes system reads:

(P1) *For $\boldsymbol{f} \in (\boldsymbol{\mathcal{X}}^p)^* \equiv \boldsymbol{W}^{-1,p'}(\Omega)$ find $(\boldsymbol{v},\pi) \in \boldsymbol{\mathcal{X}}^p \times \mathcal{Q}^p$ such that*

$$(\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{w})_\Omega - (\pi, \nabla\cdot\boldsymbol{w})_\Omega = \langle \boldsymbol{f}, \boldsymbol{w}\rangle \qquad \forall \boldsymbol{w} \in \boldsymbol{\mathcal{X}}^p \tag{2.61a}$$

$$(\nabla\cdot\boldsymbol{v}, q)_\Omega = 0 \qquad \forall q \in \mathcal{Q}^p. \tag{2.61b}$$

The subsequent discussion is addressed to the well-posedness of Problem **(P1)**. Although we recall only known results, we sketch some proofs for sake of completeness. As is common practice in analysis, we reformulate Problem **(P1)** "hiding" the pressure:

(P2) *For* $\boldsymbol{f} \in (\boldsymbol{\mathcal{X}}^p)^*$ *find* $\boldsymbol{v} \in \boldsymbol{\mathcal{V}}^p$ *such that*

$$(\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{w})_\Omega = \langle \boldsymbol{f}, \boldsymbol{w} \rangle \qquad \forall \boldsymbol{w} \in \boldsymbol{\mathcal{V}}^p. \tag{2.62}$$

It is well-known that the two formulations are equivalent and that they are well-posed. In particular, it is well-established that there exists a unique solution to **(P2)**. One can infer the well-posedness of **(P2)** using the theory of monotone operators (see, e.g., [Růž04]).

Lemma 2.12. *There exists a unique solution* $\boldsymbol{v} \in \boldsymbol{\mathcal{V}}^p$ *to Problem* **(P2)** *which satisfies*

$$\|\boldsymbol{v}\|_{1,p} \leq c_1 \Big(\|\boldsymbol{f}\|_{-1,p'}^{\frac{1}{p-1}} + c_2 \varepsilon \Big) \tag{2.63}$$

where $c_1 > 0$ *only depends on* Ω, p, σ_0, σ_1 *and* $c_2 = 1$ *if* $p < 2$ *and* $c_2 = 0$ *otherwise.*

Proof. In view of Lemma 2.4 it is easy to see that the operator $\boldsymbol{v} \mapsto -\nabla \cdot \boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v})$ from $\boldsymbol{\mathcal{X}}^p$ to $(\boldsymbol{\mathcal{X}}^p)^*$ is strictly monotone, continuous and coercive. Since $\boldsymbol{\mathcal{V}}^p$ is a closed subset of $\boldsymbol{\mathcal{X}}^p$ and the operator is also strictly monotone and coercive on $\boldsymbol{\mathcal{V}}^p$, the Theorem of Browder&Minty implies (see, e.g., [Růž04]) that there exists a unique $\boldsymbol{v} \in \boldsymbol{\mathcal{V}}^p$ that satisfies (2.62). It remains to show that the solution $\boldsymbol{v}$ is bounded by a constant only depending on the data. Such an a priori estimate will play a crucial role in the finite element analysis of Problem **(P1)**. We restrict ourselves to the proof of (2.63) for $p < 2$ since we can derive (2.63) for $p \geq 2$ using exactly the same arguments. Setting $\boldsymbol{w} := \boldsymbol{v}$ in (2.62) and taking (2.40) into account, for $p < 2$ we conclude that for some $c = c(p, \sigma_0, \sigma_1) > 0$

$$\|\boldsymbol{f}\|_{-1,p'} \|\boldsymbol{v}\|_{1,p} \geq \langle \boldsymbol{f}, \boldsymbol{v} \rangle = (\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{v})_\Omega \geq c \Big(\|\boldsymbol{D}\boldsymbol{v}\|_p^p - \varepsilon^p |\Omega| \Big).$$

Using (2.1) and Young's inequality, we immediately arrive at (2.63). □

If $\boldsymbol{\mathcal{S}}$ is derived from a potential Φ with (p, ε)-structure (see Remark 2.2), we can introduce a functional $\mathcal{J} : \boldsymbol{\mathcal{X}}^p \to \mathbb{R}$ associated with Φ:

$$\mathcal{J}(\boldsymbol{u}) := \int_\Omega \Phi(|\boldsymbol{D}\boldsymbol{u}|) \, \mathrm{d}\boldsymbol{x} - \langle \boldsymbol{f}, \boldsymbol{u} \rangle \qquad \forall \boldsymbol{u} \in \boldsymbol{\mathcal{X}}^p. \tag{2.64}$$

It is easy to check that $\mathcal{J}$ is Gâteaux differentiable on $\boldsymbol{\mathcal{X}}^p$ and that its derivative is given by $\mathcal{J}'(\boldsymbol{u})(\boldsymbol{w}) = (\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{u}), \boldsymbol{D}\boldsymbol{w})_\Omega - \langle \boldsymbol{f}, \boldsymbol{w} \rangle$ for all $\boldsymbol{u}, \boldsymbol{w} \in \boldsymbol{\mathcal{X}}^p$. Since the operator $\boldsymbol{u} \mapsto -\nabla \cdot \boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{u})$ is strictly monotone on $\boldsymbol{\mathcal{X}}^p$, $\mathcal{J}'$ is strictly monotone on $\boldsymbol{\mathcal{X}}^p$ and, hence, $\mathcal{J}$ is strictly convex on $\boldsymbol{\mathcal{X}}^p$. In addition, $\mathcal{J}$ is coercive on $\boldsymbol{\mathcal{X}}^p$, i.e., $\mathcal{J}(\boldsymbol{u}) \to \infty$ for $\|\boldsymbol{D}\boldsymbol{u}\|_p \to \infty$. Because $\boldsymbol{\mathcal{V}}^p$ is a closed convex subset of $\boldsymbol{\mathcal{X}}^p$, it follows that there exists a unique solution to the minimization problem:

(P3) *For* $\boldsymbol{f} \in (\boldsymbol{\mathcal{X}}^p)^*$ *find* $\boldsymbol{v} \in \boldsymbol{\mathcal{V}}^p$ *such that*

$$\mathcal{J}(\boldsymbol{v}) \leq \mathcal{J}(\boldsymbol{w}) \qquad \forall \boldsymbol{w} \in \boldsymbol{\mathcal{V}}^p. \tag{2.65}$$

Furthermore, **(P3)** is equivalent to **(P2)**, its Euler equation.

Problem **(P2)** is sometimes referred to as the direct weak formulation whereas **(P1)** is called the mixed weak formulation. The question arises whether the weak formulation **(P2)** still keeps the information on the pressure. In fact, we will see that the mixed weak formulation **(P1)** is equivalent to **(P2)**≡**(P3)**. Below, we focus on the reconstruction of the pressure which is based on De Rahm's Theorem (see [Rah60]):

Lemma 2.13 (De Rahm). *Let Ω be any open subset of $\mathbb{R}^d$ and let $\boldsymbol{F}$ be a distribution of $\boldsymbol{\mathcal{D}}'(\Omega)$ that satisfies $\langle \boldsymbol{F}, \boldsymbol{w}\rangle = 0$ for all $\boldsymbol{w} \in \boldsymbol{\mathcal{D}}_{\mathrm{div}}(\Omega)$. Then, there exists a distribution $\pi \in \mathcal{D}'(\Omega)$ such that $\boldsymbol{F} = \nabla\pi$.*

In connection with De Rahm's Theorem, an important role is also played by

Lemma 2.14 (Nečas). *Let Ω be an open bounded subset of $\mathbb{R}^d$ with $\partial\Omega \in C^{0,1}$. Let $F \in \mathcal{D}'(\Omega)$. If $F, \frac{\partial F}{\partial x_i} \in W^{-1,q'}(\Omega) \equiv (W_0^{1,q}(\Omega))^*$ for some $q \in (1,\infty)$ and all $i = 1,\dots,d$, then there exists a function $\xi \in L^{q'}(\Omega)$, $q' = \frac{q}{q-1}$, such that*

$$\langle F, w\rangle = \int_\Omega \xi w \,\mathrm{d}\boldsymbol{x} \qquad \forall w \in \mathcal{D}'(\Omega).$$

Moreover, there exists a constant $c_N > 0$ such that

$$\|\xi\|_{q'} \le c_N\Big(\|F\|_{-1,q'} + \|\nabla F\|_{-1,q'}\Big).$$

Proof. We refer to Nečas [Neč66]. □

De Rahm's Theorem deals with arbitrary distributions whereas the p-Stokes problem involves distributions for which more information is known. As a result of Lemma 2.14, the pressure is then not only a distribution but also belongs to a Lebesgue space. The following lemma is a consequence of Lemma 2.14. Its proof can be found in [AG94].

Lemma 2.15. *Let Ω be a bounded domain of $\mathbb{R}^d$ with Lipschitz boundary. Let ν be any real number with $1 < \nu < \infty$. The gradient operator*[2] $\operatorname{grad} \in \mathcal{L}(L^\nu(\Omega); \boldsymbol{W}^{-1,\nu}(\Omega))$ *is defined by $\langle \operatorname{grad}\pi, \boldsymbol{w}\rangle := (\pi, -\nabla\cdot\boldsymbol{w})_\Omega$ for all $\boldsymbol{w} \in \boldsymbol{W}_0^{1,\nu'}(\Omega)$. Then, the range space of* grad *is a closed subspace of $\boldsymbol{W}^{-1,\nu}(\Omega)$. If in addition Ω is connected, there exists a constant $c > 0$, which only depends on Ω and ν, such that for all $\dot\pi \in L^\nu(\Omega)/\mathbb{R}$ there holds*

$$\|\dot\pi\|_{L^\nu(\Omega)/\mathbb{R}} \le c\|\nabla\pi\|_{W^{-1,\nu}(\Omega)}. \tag{2.66}$$

Remark 2.6. Note that there exists a constant $c > 0$ such that for all $\dot\pi \in L^\nu(\Omega)/\mathbb{R}$ the representative π with $\int_\Omega \pi \,\mathrm{d}\boldsymbol{x} = 0$ (mean-value zero) satisfies: $\|\pi\|_{L^\nu(\Omega)} \le c\|\dot\pi\|_{L^\nu(\Omega)/\mathbb{R}}$.

[2]For normed vector spaces X, Y we define $\mathcal{L}(X;Y) := \{F : X \to Y;\ F \text{ is continuous and linear}\}$.

Clearly, the operator $-\operatorname{grad} \in \mathcal{L}(L^{\nu'}(\Omega); \boldsymbol{W}^{-1,\nu'}(\Omega))$ is just the dual operator of $\operatorname{div} \in \mathcal{L}(\boldsymbol{W}_0^{1,\nu}(\Omega); L^{\nu}(\Omega))$. Since the range space of the gradient operator, R(grad), is a closed subspace of $\boldsymbol{W}^{-1,\nu'}(\Omega)$, the Closed Range Theorem implies that $\mathrm{R}(\mathrm{grad}) = (\mathrm{Ker}(\mathrm{div}))^\circ = (\boldsymbol{\mathcal{V}}^\nu)^\circ$ where $(\boldsymbol{\mathcal{V}}^\nu)^\circ := \{\boldsymbol{v}^* \in \boldsymbol{W}^{-1,\nu'}(\Omega);\ \langle \boldsymbol{v}^*, \boldsymbol{v}\rangle = 0 \text{ for all } \boldsymbol{v} \in \boldsymbol{\mathcal{V}}^\nu\}$. Thus, we arrive at

Lemma 2.16. *Let Ω be a bounded domain of $\mathbb{R}^d$ with Lipschitz boundary. Let ν be any real number with $1 < \nu < \infty$. A distribution $\boldsymbol{F} \in \boldsymbol{W}^{-1,\nu'}(\Omega)$ satisfies*

$$\langle \boldsymbol{F}, \boldsymbol{w}\rangle = 0 \qquad \forall \boldsymbol{w} \in \boldsymbol{\mathcal{V}}^\nu$$

if and only if there exists a $\pi \in L^{\nu'}(\Omega)$ such that $\boldsymbol{F} = \nabla\pi$, i.e.,

$$\langle \boldsymbol{F}, \boldsymbol{w}\rangle = \langle \nabla\pi, \boldsymbol{w}\rangle = -(\pi, \nabla\cdot\boldsymbol{w})_\Omega \qquad \forall \boldsymbol{w} \in \boldsymbol{W}_0^{1,\nu}(\Omega).$$

If in addition, the set Ω is connected, then π is defined uniquely by $\boldsymbol{F}$ up to an additive constant, and it exists a positive constant β, which only depends on ν and Ω, such that

$$\beta\|\dot{\pi}\|_{L^{\nu'}(\Omega)/\mathbb{R}} \le \|\nabla\pi\|_{W^{-1,\nu'}(\Omega)} = \sup_{\boldsymbol{w}\in\boldsymbol{W}_0^{1,\nu}(\Omega)} \frac{|(\pi, \nabla\cdot\boldsymbol{w})_\Omega|}{\|\nabla\boldsymbol{w}\|_\nu}. \tag{2.67}$$

Proof. We refer to [AG94]. □

Lemma 2.17. *Let $(X, \|\cdot\|_X)$, $(Q, \|\cdot\|_Q)$ be two reflexive Banach spaces and let $(X^*, \|\cdot\|_{X^*})$, $(Q^*, \|\cdot\|_{Q^*})$ be their corresponding dual spaces. Let $\mathcal{B} : X \to Q^*$ be a linear continuous operator and let $\mathcal{B}' : Q \to X^*$ be the dual operator of $\mathcal{B}$. Let $V := \mathrm{Ker}(\mathcal{B})$ be the kernel of $\mathcal{B}$. By $V^\circ \subset X^*$ we denote the polar set of V, i.e., $V^\circ := \{x^* \in X^*;\ \langle x^*, v\rangle = 0\, \forall v \in V\}$. By $\tilde{\mathcal{B}} : (X/V) \to Q^*$ we denote the quotient operator associated with $\mathcal{B}$.*

Then, the following statements (i)–(iii) are equivalent:

(i) there exists $\beta > 0$, such that

$$\inf_{q\in Q}\sup_{w\in X} \frac{\langle \mathcal{B}w, q\rangle}{\|q\|_Q\|w\|_X} \ge \beta.$$

(ii) $\mathcal{B}'$ is an isomorphism from Q onto V° and

$$\|\mathcal{B}'q\|_{X^*} \ge \beta\|q\|_Q \qquad \forall q \in Q.$$

(iii) $\tilde{\mathcal{B}}$ is an isomorphism from (X/V) onto Q^* and

$$\|\tilde{\mathcal{B}}\tilde{w}\|_{Q^*} \ge \beta\|\tilde{w}\|_{(X/V)} \qquad \forall \tilde{w} \in (X/V).$$

Proof. See [GR86]. □

Let us define an operator $\mathcal{B} \in \mathcal{L}(\boldsymbol{W}_0^{1,\nu}(\Omega); L_0^\nu(\Omega)^*)$ by $\langle \mathcal{B}\boldsymbol{v}, q\rangle := \int_\Omega q \nabla \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x}$ for all $\boldsymbol{v} \in \boldsymbol{W}_0^{1,\nu}(\Omega)$ and $q \in L_0^\nu(\Omega)^* \equiv L_0^{\nu'}(\Omega)$. By virtue of Lemma 2.16, the dual operator $\mathcal{B}'$ defined by $\langle \mathcal{B}'q, \boldsymbol{v}\rangle = \langle \mathcal{B}\boldsymbol{v}, q\rangle$ for all $\boldsymbol{v} \in \boldsymbol{W}_0^{1,\nu}(\Omega)$ and $q \in L_0^{\nu'}(\Omega)$ is an isomorphism from $L_0^{\nu'}(\Omega)$ to $(\boldsymbol{\mathcal{V}}^\nu)^\circ$ which additionally is continuous. Lemma 2.17 implies that the statement of Lemma 2.16 is equivalent to an "inf-sup" condition for the spaces $\boldsymbol{W}_0^{1,\nu}(\Omega) \times L_0^{\nu'}(\Omega)$ which will play an important role for the subsequent analysis:

Lemma 2.18 ("Inf-sup inequality"). *Let Ω be a bounded connected domain of $\mathbb{R}^d$ with Lipschitz boundary. Let ν be any real number with $1 < \nu < \infty$ and let ν' be its conjugate. Then, there exists a positive constant $\beta(\nu)$ such that*

$$\inf_{q \in L_0^{\nu'}(\Omega)} \sup_{\boldsymbol{w} \in \boldsymbol{W}_0^{1,\nu}(\Omega)} \frac{(q, \nabla \cdot \boldsymbol{w})_\Omega}{\|q\|_{\nu'} \|\boldsymbol{w}\|_{1,\nu}} \geq \beta(\nu). \tag{2.68}$$

Let us return to the p-Stokes problem. The following lemma is well-known:

Lemma 2.19. *There exists a unique solution $(\boldsymbol{v}, \pi) \in \boldsymbol{\mathcal{X}}^p \times \mathcal{Q}^p$ to Problem* **(P1)***. The velocity $\boldsymbol{v}$ satisfies the a priori estimate* (2.63)*.*

Proof. There exists a unique solution $\boldsymbol{v} \in \boldsymbol{\mathcal{V}}^p$ to Problem **(P2)**. It follows from taking $\boldsymbol{w} \in \boldsymbol{\mathcal{V}}^p \subset \boldsymbol{\mathcal{X}}^p$ in equation (2.61a) that $\boldsymbol{v}$, the solution to **(P2)**, is the unique $\boldsymbol{v}$ that solves (**P1**). Since $\boldsymbol{v} \in \boldsymbol{\mathcal{X}}^p$, there holds $\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}) \in L^{p'}(\Omega)^{d\times d}$ and, hence, $-\nabla \cdot \boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}) \in \boldsymbol{W}^{-1,p'}(\Omega)$. The force $\boldsymbol{f}$ satisfies $\boldsymbol{f} \in (\boldsymbol{\mathcal{X}}^p)^* \equiv \boldsymbol{W}^{-1,p'}(\Omega)$ as well. This implies that the operator $\boldsymbol{F} \equiv -\nabla \cdot \boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{f}$ belongs to $\boldsymbol{W}^{-1,p'}(\Omega)$ and fulfills $\langle \boldsymbol{F}, \boldsymbol{w}\rangle \equiv (\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{w})_\Omega - \langle \boldsymbol{f}, \boldsymbol{w}\rangle = 0$ for all $\boldsymbol{w} \in \boldsymbol{\mathcal{V}}^p$. By virtue of Lemma 2.16, there exists $\pi \in L^{p'}(\Omega)$ with $\boldsymbol{F} = \nabla\pi$, i.e., there holds $-\nabla \cdot \boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}) + \nabla\pi = \boldsymbol{f}$ in $(\boldsymbol{\mathcal{X}}^p)^*$. The pressure π is unique in $L^{p'}(\Omega)$ up to a constant and therefore unique in $\mathcal{Q}^p$. ☐

The non-steady case: For $T > 0$ let $I = (0, T)$. Before introducing the variational problem, we state two technical lemmas that deal with derivatives of functions $u : I \to X$ with values in a Banach space X. Their proofs can be found in [Tem01].

Lemma 2.20. *Let X be a Banach space and let u, g be two functions that belong to $L^1(I; X)$. Then the following three conditions are equivalent:*

(i) For each test function $w \in \mathcal{D}(I)$ it holds

$$\int_0^T u \partial_t w \, \mathrm{d}t = -\int_0^T g w \, \mathrm{d}t, \qquad \text{i.e., } g = \partial_t u \Big(= \frac{\mathrm{d}u}{\mathrm{d}t} \Big).$$

(ii) u is a.e. equal to a primitive function of g, i.e.,

$$u(t) = \xi + \int_0^t g(s) \, \mathrm{d}s, \qquad \xi \in X, \qquad \text{a.a. } t \in [0, T].$$

(iii) For each $\eta \in X^*$ it holds $\frac{\mathrm{d}}{\mathrm{d}t}\langle u, \eta\rangle = \langle g, \eta\rangle$ on $(0,T)$ in the sense of distributions.

If (i)–(iii) are satisfied, u is a.e. equal to a continuous function from $[0,T]$ to X.

Let X be a reflexive Banach space and let H be a Hilbert space such that $X \hookrightarrow_{\text{densly}} H$. Then, H^* can be identified with a dense subspace of X^*. Due to the Riesz representation theorem we can identify H and H^*. Hence, there hold the inclusions $X \hookrightarrow_{\text{densly}} H = H^* \hookrightarrow_{\text{densly}} X^*$. In this case the triple (X, H, X^*) is referred to as a Gelfand-triple.

Lemma 2.21. *Let (X, H, X^*) be a Gelfand-triple. For $1 < p < \infty$ let us define the space $W := \{u \in L^p(I;X);\ \frac{\mathrm{d}u}{\mathrm{d}t} \in L^{p'}(I;X^*)\}$. Then, there holds the continuous embedding $W \hookrightarrow C(\bar{I};H)$. Moreover, for all $u \in W$ there holds*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_H^2 = 2\langle \partial_t u, u\rangle_{X^*,X} \qquad \text{in } \mathcal{D}'(0,T).$$

For the weak formulation stated below we assume that the right-hand side $\boldsymbol{f}$ belongs to the space $L^{p'}(I;(\boldsymbol{\mathcal{X}}^p)^*)$ and that the initial data $\hat{\boldsymbol{v}}$ is an element of $\boldsymbol{\mathcal{H}}^2$.

(P4) *Find $\boldsymbol{v} \in L^\infty(I;\boldsymbol{\mathcal{H}}^2) \cap L^p(I;\boldsymbol{\mathcal{V}}^p)$ with $\partial_t \boldsymbol{v} \in L^{p'}(I;(\boldsymbol{\mathcal{V}}^p)^*)$ that satisfies*

$$\langle \partial_t \boldsymbol{v}, \boldsymbol{w}\rangle + (\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{w})_\Omega = \langle \boldsymbol{f}, \boldsymbol{w}\rangle \qquad \forall \boldsymbol{w} \in \boldsymbol{\mathcal{V}}^p \tag{2.69}$$

for almost all $t \in I$ and $\|\boldsymbol{v}(t) - \hat{\boldsymbol{v}}\|_2 \to 0$ for $t \searrow 0$.

The triple $(\boldsymbol{\mathcal{V}}^p, \boldsymbol{\mathcal{H}}^2, (\boldsymbol{\mathcal{V}}^p)^*)$ is a Gelfand triple for $p \geq \frac{2d}{d+2}$ since the continuous embedding $W^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$ holds for $p \geq \frac{2d}{d+2}$ due to Sobolev's embedding theorem. Consequently, the space $\boldsymbol{W} \equiv \{\boldsymbol{u} \in L^p(I;\boldsymbol{\mathcal{V}}^p);\ \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} \in L^{p'}(I;(\boldsymbol{\mathcal{V}}^p)^*)\}$ is continuously embedded in $C(\bar{I};\boldsymbol{\mathcal{H}}^2(\Omega))$. In particular, a function $\boldsymbol{u} \in \boldsymbol{W}$ is almost everywhere equal in $(0,T)$ to a continuous function from $[0,T]$ to $\boldsymbol{\mathcal{H}}^2$. Hence, the initial condition stated in Problem **(P4)**, i.e., the expression $\boldsymbol{v}(0)$, makes sense for functions $\boldsymbol{v} \in L^p(I;\boldsymbol{\mathcal{V}}^p)$ with $\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \in L^{p'}(I;(\boldsymbol{\mathcal{V}}^p)^*)$.

The following result is well-known (cf. [GGZ74, Růž04]):

Lemma 2.22. *For $p \geq \frac{2d}{d+2}$ there exists a unique solution $\boldsymbol{v}$ to Problem* **(P4)** *that satisfies*

$$\|\boldsymbol{v}\|^2_{L^\infty(I;L^2(\Omega))} + \|\boldsymbol{v}\|^p_{L^p(I;\boldsymbol{\mathcal{X}}^p)} \leq c\Big(\|\boldsymbol{f}\|^{p'}_{L^{p'}(I;(\boldsymbol{\mathcal{X}}^p)^*)} + \|\hat{\boldsymbol{v}}\|_2^2\Big). \tag{2.70}$$

Proof. We only recall that the proof of uniqueness does not require the *strict* monotonicity. Indeed, if $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ are two solutions to Problem **(P4)**, they satisfy the identity

$$\langle \partial_t \boldsymbol{v}_1 - \partial_t \boldsymbol{v}_2, \boldsymbol{w}\rangle + (\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}_1) - \boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}_2), \boldsymbol{D}\boldsymbol{w})_\Omega = 0 \qquad \forall \boldsymbol{w} \in \boldsymbol{\mathcal{V}}^p.$$

Setting $\boldsymbol{w} := \boldsymbol{v}_1 - \boldsymbol{v}_2$, using the monotonicity of $\boldsymbol{\mathcal{S}}$, and applying Lemma 2.21, we conclude

$$\langle \partial_t \boldsymbol{v}_1 - \partial_t \boldsymbol{v}_2, \boldsymbol{v}_1 - \boldsymbol{v}_2\rangle \leq 0 \qquad \Leftrightarrow \qquad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}_1(t) - \boldsymbol{v}_2(t)\|_2^2 \leq 0.$$

Integrating this inequality over $(0,t)$, we finally arrive at

$$\|\boldsymbol{v}_1(t) - \boldsymbol{v}_2(t)\|_2^2 \leq \|\boldsymbol{v}_1(0) - \boldsymbol{v}_2(0)\|_2^2 = 0.$$

Hence, we deduce that $\boldsymbol{v}_1(t) = \boldsymbol{v}_2(t)$ for each $t \in [0,T]$. □

Reconstruction of the pressure: Concerning the introduction of the pressure, there are significant differences between the steady and non-steady problem in case of no-slip boundary conditions. In general, time derivatives do not represent distributions and, hence, the pressure cannot simply be identified by means of De Rahm's theorem. In fact, time derivatives $\partial_t \boldsymbol{v}$ are elements of $L^{p'}(I;(\boldsymbol{\mathcal{V}}^p)^*)$, i.e., they belong to a dual space of divergence-free functions only. No information about $\partial_t \boldsymbol{v}$ is known in the space $L^{p'}(I;(\boldsymbol{\mathcal{X}}^p)^*)$.

Lemma 2.23. *There exists a distribution π on $Q_T := \Omega \times (0,T)$ such that the distribution π and the function $\boldsymbol{v}$ given by Lemma 2.22 satisfy $\nabla \cdot \boldsymbol{v} = 0$ and*

$$\partial_t \boldsymbol{v} - \nabla \cdot \boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}) + \nabla \pi = \boldsymbol{f} \tag{2.71}$$

in the distribution sense in Q_T. It holds $\boldsymbol{v}(t) \to \hat{\boldsymbol{v}}$ in $L^2(\Omega)$ as $t \searrow 0$.

Proof. In [Tem01] the proof is carried out for Stokes systems. Here, we can follow the same line of arguments. In order to introduce the pressure, we define

$$\overline{\boldsymbol{\mathcal{S}}}(t) := \int_0^t \boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}(s))\,\mathrm{d}s, \qquad \boldsymbol{F}(t) := \int_0^t \boldsymbol{f}(s)\,\mathrm{d}s.$$

In view of Lemma 2.20, there holds $\overline{\boldsymbol{\mathcal{S}}} \in C([0,T];L^{p'}(\Omega)^{d\times d})$ and $\boldsymbol{F} \in C([0,T];(\boldsymbol{\mathcal{X}}^p)^*)$. Integrating (2.69) over $(0,t)$ and using Lemma 2.20, we conclude that

$$\langle \boldsymbol{v}(t) - \hat{\boldsymbol{v}}, \boldsymbol{w}\rangle + (\overline{\boldsymbol{\mathcal{S}}}(\boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{w})_\Omega = \langle \boldsymbol{F}(t), \boldsymbol{w}\rangle \qquad \forall t \in [0,T], \qquad \forall \boldsymbol{w} \in \boldsymbol{\mathcal{V}}^p.$$

This identity is equivalent to

$$\langle \boldsymbol{v}(t) - \hat{\boldsymbol{v}} - \nabla \cdot \overline{\boldsymbol{\mathcal{S}}}(\boldsymbol{D}\boldsymbol{v})(t) - \boldsymbol{F}(t), \boldsymbol{w}\rangle = 0 \qquad \forall t \in [0,T], \qquad \forall \boldsymbol{w} \in \boldsymbol{\mathcal{V}}^p.$$

Note that $\nabla \cdot \overline{\boldsymbol{\mathcal{S}}}(\boldsymbol{D}\boldsymbol{v}) \in C([0,T];(\boldsymbol{\mathcal{X}}^p)^*)$. From Lemma 2.16 we infer that for each $t \in [0,T]$ there exists a function $\Pi(t) \in L^{p'}(\Omega)$ such that

$$\boldsymbol{v}(t) - \hat{\boldsymbol{v}} - \nabla \cdot \overline{\boldsymbol{\mathcal{S}}}(\boldsymbol{D}\boldsymbol{v})(t) + \nabla \Pi(t) = \boldsymbol{F}(t) \qquad \forall t \in [0,T]. \tag{2.72}$$

Since $\nabla \Pi = \boldsymbol{F} + \nabla \cdot \overline{\boldsymbol{\mathcal{S}}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{v} + \hat{\boldsymbol{v}}$ and the right-hand side of this identity belongs to $C([0,T];\boldsymbol{W}^{-1,p'}(\Omega))$, we deduce that $\nabla \Pi \in C([0,T];\boldsymbol{W}^{-1,p'}(\Omega))$ and, consequently,

$$\Pi \in C([0,T];L^{p'}(\Omega)) \tag{2.73}$$

due to Lemma 2.15. This enables us to differentiate (2.72) with respect to the variable t in the distribution sense in $Q_T = \Omega \times (0,T)$. Setting

$$\pi = \frac{\partial \Pi}{\partial t}, \tag{2.74}$$

we just get (2.71). □

In general, we do not gain any information about π better than (2.73) – (2.74). We obtain higher regularity on π after assuming higher regularity on the data $\boldsymbol{f}$, $\hat{\boldsymbol{v}}$ and proving higher regularity for $\boldsymbol{v}$.

2.6 The p-Navier-Stokes equations

Besides Carreau-type models, we consider p-structure models that meet Assumption 2.1. In this section, we deal with the p-Navier-Stokes equations which, related to the p-Stokes system, come up with additional mathematical difficulties due to the convective term.

The steady case: Let us consider the steady p-Navier-Stokes system (2.15) complemented by homogeneous Dirichlet boundary conditions. The weak formulation of (2.15) reads:

(P5) *For* $\boldsymbol{f} \in (\boldsymbol{\mathcal{X}}^p)^*$ *find* $(\boldsymbol{v}, \pi) \in \boldsymbol{\mathcal{X}}^p \times \mathcal{Q}^p$ *such that*

$$(\boldsymbol{\mathcal{S}}(\boldsymbol{Dv}), \boldsymbol{Dw})_\Omega + ([\boldsymbol{v} \cdot \nabla]\boldsymbol{v}, \boldsymbol{w})_\Omega - (\pi, \nabla \cdot \boldsymbol{w})_\Omega = \langle \boldsymbol{f}, \boldsymbol{w} \rangle \qquad \forall \boldsymbol{w} \in \boldsymbol{\mathcal{X}}^p \tag{2.75}$$

$$(\nabla \cdot \boldsymbol{v}, q)_\Omega = 0 \qquad \forall q \in \mathcal{Q}^p. \tag{2.76}$$

Remark 2.7. Clearly, Problem **(P5)** is not well-posed for the full range of $p > 1$. From Sobolev's embedding theorem we deduce that the continuous embedding $W^{1,p}(\Omega) \hookrightarrow L^{2p'}(\Omega)$, $1/p + 1/p' = 1$, holds true for $p \geq 3d/(d+2)$. Hence, we realize that

$$([\boldsymbol{u} \cdot \nabla]\boldsymbol{v}, \boldsymbol{w})_\Omega \leq \|\boldsymbol{u}\|_{2p'} \|\nabla \boldsymbol{v}\|_p \|\boldsymbol{w}\|_{2p'} \leq c(d, p, \Omega) \|\boldsymbol{u}\|_{1,p} \|\boldsymbol{v}\|_{1,p} \|\boldsymbol{w}\|_{1,p} \tag{2.77}$$

provided that $p \geq 3d/(d+2)$. This means that for $\boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{\mathcal{X}}^p$ the term $([\boldsymbol{v} \cdot \nabla]\boldsymbol{v}, \boldsymbol{w})_\Omega$ is well-defined only if $p \geq 3d/(d+2)$. However, later we will suppose that $\boldsymbol{v}$, the solution to **(P5)**, belongs to better spaces so that the condition on p can be relaxed.

Remark 2.8. We state further properties of the convective term. Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} : \Omega \to \mathbb{R}^d$ be sufficiently smooth functions so that all subsequent integrals are well-defined. We assume that $\nabla \cdot \boldsymbol{u} = 0$ a.e. and that $\boldsymbol{w}$ possesses zero traces. Integration by parts yields

$$\left(\boldsymbol{u} \otimes \boldsymbol{v}, \nabla \boldsymbol{w}^{\mathsf{T}}\right)_\Omega = -\sum_{i,j=1}^{d} (\partial_i (u_i v_j), w_j)_\Omega = -([\boldsymbol{u} \cdot \nabla]\boldsymbol{v}, \boldsymbol{w})_\Omega. \tag{2.78}$$

Here, $(\boldsymbol{u} \otimes \boldsymbol{v})_{ij} := u_i v_j$. Using $\nabla \cdot \boldsymbol{u} = 0$ a.e. and integration by parts, we observe that

$$([\boldsymbol{u} \cdot \nabla]\boldsymbol{w}, \boldsymbol{v})_\Omega = \sum_{i,j} \int_\Omega v_j \partial_i (u_i w_j) = -\sum_{i,j} \int_\Omega u_i w_j \partial_i v_j = -([\boldsymbol{u} \cdot \nabla]\boldsymbol{v}, \boldsymbol{w})_\Omega \tag{2.79}$$

and, hence, $([\boldsymbol{u} \cdot \nabla]\boldsymbol{v}, \boldsymbol{v})_\Omega = 0$. This property is referred to as the skew symmetry.

The following lemma ensures the existence of weak solutions to Problem (2.15):

Lemma 2.24. *Let* $\Omega \subset \mathbb{R}^d$, $d \geq 2$, *be an open bounded set with* $\partial\Omega \in C^{1,1}$. *Let us consider the steady p-Navier-Stokes equations* (2.15) *complemented with homogeneous Dirichlet boundary conditions. For* $p > \frac{2d}{d+2}$ *and* $\varepsilon \in [0, \infty)$ *let the extra stress tensor* $\boldsymbol{\mathcal{S}}$ *satisfy Assumption 2.1. We assume that* $\boldsymbol{f}$ *belongs to* $\boldsymbol{W}^{-1,p'}(\Omega)$. *Then, there exists a weak solution* $\boldsymbol{v} \in \boldsymbol{\mathcal{V}}^p$ *to system* (2.15) *in the sense that* $\boldsymbol{v}$ *satisfies*

$$(\boldsymbol{\mathcal{S}}(\boldsymbol{Dv}), \boldsymbol{Dw})_\Omega - (\boldsymbol{v} \otimes \boldsymbol{v}, \boldsymbol{Dw})_\Omega = \langle \boldsymbol{f}, \boldsymbol{w} \rangle \qquad \forall \boldsymbol{w} \in \boldsymbol{\mathcal{D}}_{\text{div}}(\Omega). \tag{2.80}$$

Proof. We refer to [FMS03]. □

Remark 2.9. Note that $(\boldsymbol{v} \otimes \boldsymbol{v})_{ij} \equiv v_i v_j \in L^1(\Omega)$ for $p \geq \frac{2d}{d+2}$ due to Sobolev's embedding theorem. Lemma 2.24 is proven by means of the Lipschitz truncation method which allows to find a subsequence $(\boldsymbol{v}^{n_k}) \subset (\boldsymbol{v}^n)$ of conveniently introduced approximations $\boldsymbol{v}^n$ such that $\boldsymbol{D}\boldsymbol{v}^{n_k}$ converge almost everywhere to their weak limit $\boldsymbol{D}\boldsymbol{v}$.

Remark 2.10. Note that $v_i \partial_i v_j \in L^1(\Omega)$ for all $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$ provided that $p \geq \frac{2d}{d+1}$. Hence, in the case $p \geq \frac{2d}{d+1}$ the function $\boldsymbol{v}$ given by Lemma 2.24 solves

$$(\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{w})_\Omega + ([\boldsymbol{v} \cdot \nabla]\boldsymbol{v}, \boldsymbol{w})_\Omega = \langle \boldsymbol{f}, \boldsymbol{w} \rangle \qquad \forall \boldsymbol{w} \in \boldsymbol{\mathcal{D}}_{\mathrm{div}}(\Omega).$$

Reconstruction of the pressure: Let $p \geq \frac{3d}{d+2}$. It is well-known (cf. Lemma 2.24) that there exists a velocity field $\boldsymbol{v} \in \boldsymbol{\mathcal{V}}^p$ satisfying

$$(\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{w})_\Omega + ([\boldsymbol{v} \cdot \nabla]\boldsymbol{v}, \boldsymbol{w})_\Omega = \langle \boldsymbol{f}, \boldsymbol{w} \rangle \qquad \forall \boldsymbol{w} \in \boldsymbol{\mathcal{V}}^p. \tag{2.81}$$

Using $([\boldsymbol{v} \cdot \nabla]\boldsymbol{v}, \boldsymbol{w})_\Omega = -(\boldsymbol{v} \otimes \boldsymbol{v}, \boldsymbol{D}\boldsymbol{w})_\Omega$, we can rewrite (2.81) as

$$\int_\Omega (\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{v} \otimes \boldsymbol{v}) : \boldsymbol{D}\boldsymbol{w} \,\mathrm{d}\boldsymbol{x} - \langle \boldsymbol{f}, \boldsymbol{w} \rangle = 0 \qquad \forall \boldsymbol{w} \in \boldsymbol{\mathcal{V}}^p.$$

Clearly, there holds $\mathcal{S}_{ij}(\boldsymbol{D}\boldsymbol{v}) \in L^{p'}(\Omega)$. Since $p \geq \frac{3d}{d+2}$, due to Sobolev's embedding theorem there holds $(\boldsymbol{v} \otimes \boldsymbol{v})_{ij} \equiv v_i v_j \in L^{p'}(\Omega)$ as well. Consequently, the mapping

$$\boldsymbol{W}_0^{1,p}(\Omega) \ni \boldsymbol{w} \mapsto \int_\Omega (\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{v} \otimes \boldsymbol{v}) : \boldsymbol{D}\boldsymbol{w} \,\mathrm{d}\boldsymbol{x} - \langle \boldsymbol{f}, \boldsymbol{w} \rangle$$

is a linear continuous functional on $\boldsymbol{\mathcal{X}}^p \equiv \boldsymbol{W}_0^{1,p}(\Omega)$ that vanishes on $\boldsymbol{\mathcal{V}}^p$. By virtue of Lemma 2.16, there exists $\dot{\pi} \in L^{p'}(\Omega)/\mathbb{R}$ such that for any $\pi \in \dot{\pi}$

$$\int_\Omega (\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{v} \otimes \boldsymbol{v}) : \boldsymbol{D}\boldsymbol{w} \,\mathrm{d}\boldsymbol{x} - \langle \boldsymbol{f}, \boldsymbol{w} \rangle = \int_\Omega \pi \nabla \cdot \boldsymbol{w} \,\mathrm{d}\boldsymbol{x} \qquad \forall \boldsymbol{w} \in \boldsymbol{\mathcal{X}}^p.$$

Hence, the pairing $(\boldsymbol{v}, \pi)$ is a solution to Problem **(P5)**.

Regularity: The question arises whether weak solutions $\boldsymbol{v}$ to (2.15) are smoother as suggested by the variational formulation (2.81) (provided that the data are more regular). Below, we state theoretical results that deal with higher regularity of $\boldsymbol{v}$, i.e., that ensure the (local) existence of second derivatives of $\boldsymbol{v}$. If there exist second derivatives of $\boldsymbol{v}$ a.e., then it can usually be shown that there exist first derivatives of π a.e.. Higher regularity plays an essential role when, e.g., numerical methods such as finite element methods are analyzed since it allows us to quantify the convergence of the approximation. The following quantity is naturally involved with the derivation of higher regularity,

$$\mathcal{I}(\boldsymbol{v}) := \int_\Omega (\varepsilon + |\boldsymbol{D}\boldsymbol{v}|)^{p-2} |\nabla \boldsymbol{D}\boldsymbol{v}|^2 \,\mathrm{d}\boldsymbol{x}, \tag{2.82}$$

where p and ε are the same as in the assumption on the extra stress tensor. Roughly speaking, the term $\mathcal{I}(\boldsymbol{v})$ occurs when the term $-\nabla \cdot \boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v})$ is tested with $-\Delta \boldsymbol{v}$. The finiteness of $\mathcal{I}(\boldsymbol{v})$ is the main device that allows us to prove regularity concerning the existence of second-order derivatives of $\boldsymbol{v}$. We remark that due to the algebraic identity

$$\frac{\partial^2 v_i}{\partial x_j \partial x_k} = \frac{\partial D_{ik}(\boldsymbol{v})}{\partial x_j} + \frac{\partial D_{ij}(\boldsymbol{v})}{\partial x_k} - \frac{\partial D_{jk}(\boldsymbol{v})}{\partial x_i}$$

one can estimate $|\nabla^2 \boldsymbol{v}| \sim |\nabla \boldsymbol{D}\boldsymbol{v}|$ with constants only depending on d. The following lemma indicates that locally there exist second derivatives of the solution to (2.81).

Lemma 2.25 (Interior regularity). *Let $\Omega \subset \mathbb{R}^d$ be an open set. For $p \in (\frac{3d}{d+2}, 2]$ and $\varepsilon = 1$ let the extra stress tensor satisfy Assumption 2.1. Let $\boldsymbol{v} \in \boldsymbol{W}^{1,p}_{\mathrm{loc}}(\Omega)$ satisfy*

$$(\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{w})_\Omega + ([\boldsymbol{v} \cdot \nabla]\boldsymbol{v}, \boldsymbol{w})_\Omega = 0 \quad \forall \boldsymbol{w} \in \{\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega);\, \mathrm{supp}(\boldsymbol{u}) \subset\subset \Omega,\, \nabla \cdot \boldsymbol{u} = 0\}$$

and $\nabla \cdot \boldsymbol{v} = 0$ a.e. in Ω. Then, $\boldsymbol{v}$ satisfies

$$\int_{\Omega_0} (1 + |\boldsymbol{D}\boldsymbol{v}|)^{p-2} |\nabla \boldsymbol{D}\boldsymbol{v}|^2 \,\mathrm{d}\boldsymbol{x} < \infty \qquad \forall \Omega_0 \subset\subset \Omega.$$

In particular, there holds (cf. Lemma 4.10)

$$\boldsymbol{v} \in \boldsymbol{W}^{2,q}_{\mathrm{loc}}(\Omega) \qquad \forall q \in [1,2) \qquad \text{if } d = 2, \qquad \text{and} \qquad \boldsymbol{v} \in \boldsymbol{W}^{2,\frac{3p}{p+1}}_{\mathrm{loc}}(\Omega) \qquad \text{if } d = 3.$$

Proof. The proof can be found in [NW05]. □

For $d = 3$ and smooth Ω, global regularity (i.e., regularity up to the boundary) has been studied in Ebmeyer [Ebm06]. There, no-stick boundary conditions,

$$\boldsymbol{v} \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega, \quad \boldsymbol{n} \cdot [\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}) - \pi \boldsymbol{I}]\boldsymbol{t} = 0 \quad \text{on } \partial\Omega \quad \forall \boldsymbol{t} \in \{\boldsymbol{t} \in \mathbb{R}^3;\, \boldsymbol{t} \cdot \boldsymbol{n} = 0\},$$

have been considered. For $p \in (1, 2)$ and $\varepsilon \in \{0, 1\}$ let $\boldsymbol{\mathcal{S}}$ satisfy (p, ε)-structure (Assumption 2.1) and let $\boldsymbol{f}$ belong to $\boldsymbol{L}^{p'}(\Omega)$. For the p-Navier-Stokes problem it is proven in [Ebm06] that $\mathcal{I}(\boldsymbol{v})$ defined in (2.82) is finite provided that $p \in (\frac{9}{5}, 2)$. As in Lemma 2.25, the restriction on p stems from the low regularity of the convective term. By contrast, for the p-Stokes system $\mathcal{I}(\boldsymbol{v})$ is finite for each $p \in (1, 2)$, see [Ebm06]. In case of no-slip boundary conditions, global regularity results have been derived in [dV08]. Nevertheless, for such boundary conditions the regularity of weak solutions is a topic of current research.

Below we introduce the velocity space $\boldsymbol{\mathcal{V}}^p_{\mathrm{per}}$ adjusted to the setting of space-periodic functions. Let $\mathcal{D}_{\mathrm{per}}(\Omega)$ be the space of $C^\infty(\Omega)$-functions which are divergence-free and space-periodic with zero mean value. Then, the velocity space $\boldsymbol{\mathcal{V}}^p_{\mathrm{per}}$ is defined by

$$\boldsymbol{\mathcal{V}}^p_{\mathrm{per}} := \Big\{\text{closure of } \mathcal{D}_{\mathrm{per}}(\Omega)^d \text{ in } W^{1,p}(\Omega)\Big\}.$$

The following lemma ensures the existence of strong solutions to system (2.15) provided that space-periodic boundary conditions are considered:

Lemma 2.26. *Let $d = 3$. Let us consider the steady system* (2.15) *complemented with space-periodic boundary conditions. Let the extra stress tensor $\boldsymbol{\mathcal{S}}$ satisfy Assumption 2.1 with $p \in (\frac{9}{5}, 2]$ and $\varepsilon \in [0, \varepsilon_0]$ for some $\varepsilon_0 > 0$. We assume that $\boldsymbol{f} \in \boldsymbol{W}^{1,2}(\Omega)$. Then, there exists a strong solution $\boldsymbol{v} \in \boldsymbol{\mathcal{V}}^p_{\text{per}}$ to system* (2.15) *in the sense that $\boldsymbol{v}$ satisfies*

$$(\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{w})_\Omega + ([\boldsymbol{v} \cdot \nabla]\boldsymbol{v}, \boldsymbol{w})_\Omega = (\boldsymbol{f}, \boldsymbol{w})_\Omega \qquad \forall \boldsymbol{w} \in \boldsymbol{\mathcal{V}}^p_{\text{per}} \tag{2.83}$$

and

$$\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})\|_{1,2} \leq C = C(\varepsilon_0, p, \|\boldsymbol{f}\|_{1,2}, \Omega). \tag{2.84}$$

Moreover, for $\varepsilon > 0$ there exists a pressure π which satisfies

$$\pi \in W^{1,2}(\Omega), \qquad \|\pi\|_{1,2} \leq C' = C'(\varepsilon, p, \|\boldsymbol{f}\|_{1,2}, \Omega).$$

The constant C' may explode as $\varepsilon \to 0^+$.

Proof. We refer to [BDR10]. □

It is well-known that the regularity (2.84) is equivalent to $\mathcal{I}(\boldsymbol{v}) < \infty$ as depicted by

Lemma 2.27. *Let $p \in (1, \infty)$, $\varepsilon \in (0, \infty)$ and let $\mathcal{I}(\boldsymbol{v})$ be defined by* (2.82)*. Then for all sufficiently smooth $\boldsymbol{v}$ there holds*

$$\|\nabla \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})\|_2^2 \sim \mathcal{I}(\boldsymbol{v}), \tag{2.85}$$

where the constants only depend on p. In particular, they are independent of ε.

Proof. We refer to [BDR10]. □

The problem of **Hölder-regularity** has been studied, e.g., by Kaplický et al. [KMS97]. The authors considered stress tensors $\boldsymbol{\mathcal{S}}$ which for $p > 1$ and $\varepsilon > 0$ are derived from a potential with (p, ε)-structure, and they studied the two-dimensional space-periodic problem. They proved the following result: If $d = 2$, $p \in (1, 2)$, and $\boldsymbol{f} \in \boldsymbol{L}^{p'}(\Omega)$, then there exists a solution $(\boldsymbol{v}, \pi)$ to the p-Navier-Stokes system (2.15) complemented with space-periodic boundary conditions such that $\boldsymbol{v} \in \boldsymbol{W}^{2,p'}_{\text{loc}}(\mathbb{R}^2) \cap \boldsymbol{C}^{1,\alpha}(\overline{\Omega})$ and $\pi \in W^{1,p'}_{\text{loc}}(\mathbb{R}^2)$.

The unique global Hölder-regularity of solutions has been established for a two-dimensional Dirichlet boundary value problem by the same authors in [KMS02]. For $d = 2$ and Ω of class C^2, a global $C^{1,\alpha}$-solution $\boldsymbol{v}$ to Problem **(P5)** has been constructed. If $\boldsymbol{f} \in \boldsymbol{L}^{p'}(\Omega)$, then it is shown in [KMS02] that for $p > \frac{6}{5}$ there is a number $q > 2$ and a strong solution $(\boldsymbol{v}, \pi)$ to Problem **(P5)** such that $\boldsymbol{v} \in \boldsymbol{\mathcal{V}}^p \cap \boldsymbol{W}^{2,q}_{\text{loc}}(\Omega)$ and $\pi \in W^{1,q}_{\text{loc}}(\Omega)$. Moreover, there exist a number $q > 2$ and a strong solution to **(P5)** such that $\boldsymbol{v} \in \boldsymbol{W}^{2,q}(\Omega)$ and $\pi \in W^{1,q}(\Omega)$ provided that $p > \frac{3}{2}$. In particular, there holds the global regularity result $\boldsymbol{v} \in \boldsymbol{C}^{1,\alpha}(\overline{\Omega})$ and $\pi \in C^{0,\alpha}(\overline{\Omega})$ for some $\alpha > 0$. Finally, it is proven in [KMS02] that for $p \geq \frac{3}{2}$ the $C^{1,\alpha}$-solution is unique in the class of weak solutions provided that the data are small: If

$\boldsymbol{v}$ and $\boldsymbol{u}$ are strong and weak solutions to **(P5)**, respectively, then $\boldsymbol{v}$ and $\boldsymbol{u}$ coincide a.e. provided that $\|\boldsymbol{f}\|_{-1,p'} < \delta$ for sufficiently small δ.

In higher dimensions $d \geq 3$ Hölder-regularity up to the boundary has been proven by Crispo/Grisanti [CG08] for small data. The authors showed that if $p \in (1,2)$, $q > d$, $\Omega \in C^{0,\alpha_0}$ with $\alpha_0 = 1 - d/q$ and if the $L^q(\Omega)$-norm of $\boldsymbol{f}$ is bounded by a small constant, then there exists a unique weak solution $(\boldsymbol{v}, \pi)$ to the p-Navier-Stokes system (2.15) equipped with homogeneous Dirichlet boundary conditions such that $\boldsymbol{v} \in \boldsymbol{C}^{1,\alpha}(\overline{\Omega})$ and $\pi \in C^{0,\alpha}(\overline{\Omega})$ for all $\alpha < \alpha_0$. Note that, in the case $d \geq 3$, global Hölder-regularity for arbitrary data has not yet been resolved and remains an open problem.

The p-Oseen equations: Let us consider the steady p-Oseen system (2.17) complemented with homogeneous Dirichlet boundary conditions. The study of system (2.17) is motivated by the fact that it is needed for the error analysis of the time-discretized p-Navier-Stokes equations performed in [BDR09]. The weak formulation of the p-Oseen system (2.17) reads:

(P6) *For $\boldsymbol{f} \in (\boldsymbol{\mathcal{X}}^p)^*$ find $(\boldsymbol{v}, \pi) \in \boldsymbol{\mathcal{X}}^p \times \mathcal{Q}^p$ such that for all $(\boldsymbol{w}, q) \in \boldsymbol{\mathcal{X}}^p \times \mathcal{Q}^p$*

$$(\boldsymbol{\mathcal{S}}(\boldsymbol{Dv}), \boldsymbol{Dw})_\Omega + ([\boldsymbol{b} \cdot \nabla]\boldsymbol{v}, \boldsymbol{w})_\Omega + \sigma(\boldsymbol{v}, \boldsymbol{w})_\Omega - (\pi, \nabla \cdot \boldsymbol{w})_\Omega = \langle \boldsymbol{f}, \boldsymbol{w} \rangle \tag{2.86a}$$

$$(\nabla \cdot \boldsymbol{v}, q)_\Omega = 0. \tag{2.86b}$$

Here, we assume that the flow field $\boldsymbol{b}$ belongs to $\boldsymbol{W}^{1,\infty}(\Omega)$ and satisfies $\nabla \cdot \boldsymbol{b} = 0$ a.e..

Later we will require the existence of a strong solution to **(P6)**, i.e., we will assume that there exists $(\boldsymbol{v}, \pi) \in [\boldsymbol{\mathcal{X}}^p \cap \boldsymbol{W}^{2,q}(\Omega)] \times \mathcal{Q}^p$ with $q = \min\{2, p\}$ satisfying (2.86). If $\boldsymbol{v} \in \boldsymbol{W}^{2,q}$ with $q = \min\{2, p\}$, then the term $([\boldsymbol{b} \cdot \nabla]\boldsymbol{v}, \boldsymbol{w})_\Omega$ is well-defined for $p \geq \frac{2d}{d+2}$ due to Sobolev's embedding theorem. The next lemma deals with the existence of strong solutions:

Lemma 2.28. *Let $d = 3$. Let us consider the steady system* (2.17) *complemented with space-periodic boundary conditions. Let the extra stress tensor $\boldsymbol{\mathcal{S}}$ satisfy Assumption 2.1 with $p \in (\frac{7}{5}, 2]$ and $\varepsilon \in [0, \varepsilon_0]$ for some $\varepsilon_0 > 0$. Assume that $\boldsymbol{f} \in \boldsymbol{W}^{1,2}(\Omega)$ and $\boldsymbol{b} \in \boldsymbol{\mathcal{V}}^{3p}_{\text{per}}$ are given. Then, there exists a strong solution $\boldsymbol{v} \in \boldsymbol{\mathcal{V}}^p_{\text{per}}$ to* (2.17) *in the sense that $\boldsymbol{v}$ satisfies*

$$(\boldsymbol{\mathcal{S}}(\boldsymbol{Dv}), \boldsymbol{Dw})_\Omega + ([\boldsymbol{b} \cdot \nabla]\boldsymbol{v}, \boldsymbol{w})_\Omega + \sigma(\boldsymbol{v}, \boldsymbol{w})_\Omega = (\boldsymbol{f}, \boldsymbol{w})_\Omega \qquad \forall \boldsymbol{w} \in \boldsymbol{\mathcal{V}}^p_{\text{per}} \tag{2.87}$$

and

$$\|\nabla \boldsymbol{v}\|_2 + \|\boldsymbol{\mathcal{F}}(\boldsymbol{Dv})\|_{1,2} \leq C = C(\varepsilon_0, p, \boldsymbol{b}, \|\boldsymbol{f}\|_{1,2}, \Omega). \tag{2.88}$$

This solution is unique within the class $\boldsymbol{\mathcal{V}}^p_{\text{per}}$ for $p > \frac{3}{2}$ and it is unique within the class $\boldsymbol{\mathcal{V}}^{3p}_{\text{per}}$ for $p \geq \frac{7}{5}$. Moreover, for $\varepsilon > 0$ there exists a pressure π which satisfies

$$\pi \in W^{1,2}(\Omega), \qquad \|\pi\|_{1,2} \leq C' = C'(\varepsilon, p, \boldsymbol{b}, \|\boldsymbol{f}\|_{1,2}, \Omega).$$

The constant C' may explode as $\varepsilon \to 0^+$.

Proof. We refer to [BDR10]. □

The non-steady case: Let us introduce the weak formulation of the non-steady p-Navier-Stokes problem (2.14). For $T > 0$ let us set $I := (0,T)$ and $Q_T := I \times \Omega$. We assume that the right-hand side $\boldsymbol{f} \in L^{p'}(I; \boldsymbol{W}^{-1,p'}(\Omega))$ and the initial data $\hat{\boldsymbol{v}} \in \boldsymbol{\mathcal{H}}^2$ are given.

(P7) *Find* $\boldsymbol{v} \in L^\infty(I; \boldsymbol{L}^2(\Omega)) \cap L^p(I; \boldsymbol{\mathcal{V}}^p)$ *with* $\partial_t \boldsymbol{v} \in L^{p'}(I; (\boldsymbol{\mathcal{V}}^p)^*)$ *such that* $\boldsymbol{v}$ *satisfies*

$$\langle \partial_t \boldsymbol{v}, \boldsymbol{w} \rangle + (\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{w})_\Omega + ([\boldsymbol{v} \cdot \nabla]\boldsymbol{v}, \boldsymbol{w})_\Omega = \langle \boldsymbol{f}, \boldsymbol{w} \rangle \qquad \forall \boldsymbol{w} \in \boldsymbol{\mathcal{V}}^p \tag{2.89}$$

for almost all $t \in I$ *and* $\boldsymbol{v}(0) = \hat{\boldsymbol{v}}$ *in* $\boldsymbol{L}^2(\Omega)$.

Remark 2.11. For $\boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{\mathcal{V}}^p$ we observe that the mapping $\Omega \ni x \mapsto (v_j[\partial_j v_i]w_i)(x)$ belongs to $L^1(\Omega)$ if and only if $p \geq \frac{3d}{d+2}$. Choosing $\boldsymbol{w} \in L^p(I; \boldsymbol{\mathcal{V}}^p)$ and noting $\boldsymbol{v} \in L^p(I; \boldsymbol{\mathcal{V}}^p)$, we realize that the mapping $Q_T \ni (x,t) \mapsto (v_j[\partial_j v_i]w_i)(x,t)$ belongs to $L^1(Q_T)$ if and only if $p \geq 1 + \frac{2d}{d+2}$. Setting $\boldsymbol{w} := \boldsymbol{v}$ in (2.89) and using $([\boldsymbol{v} \cdot \nabla]\boldsymbol{v}, \boldsymbol{v})_\Omega = 0$, we conclude that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\boldsymbol{v}|^2 \, \mathrm{d}\boldsymbol{x} + (\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{v})_\Omega = \langle \boldsymbol{f}, \boldsymbol{v} \rangle.$$

Integrating this equation over $(0,t)$, we arrive at

$$\frac{1}{2} \int_\Omega |\boldsymbol{v}(.,t)|^2 \, \mathrm{d}\boldsymbol{x} + c \int_0^t (\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{v})_\Omega \, \mathrm{d}t = \int_0^t \langle \boldsymbol{f}, \boldsymbol{v} \rangle \, \mathrm{d}t + \frac{1}{2} \int_\Omega |\hat{\boldsymbol{v}}|^2 \, \mathrm{d}\boldsymbol{x}.$$

In view of (2.40) and (2.1), this demonstrates the a priori estimate

$$\|\boldsymbol{v}\|_{L^\infty(I;L^2(\Omega))} + \|\boldsymbol{v}\|_{L^p(I;\boldsymbol{\mathcal{V}}^p)} \leq C = C(\boldsymbol{f}, \hat{\boldsymbol{v}}, \Omega, p, \varepsilon_0).$$

We are interested in strong solutions to **(P7)**. This means that we look for a function

$$\boldsymbol{v} \in L^\infty(I; \boldsymbol{\mathcal{V}}^p) \cap L^q(I; \boldsymbol{W}^{2,q}(\Omega)) \quad \text{with} \quad q = \min\{2,p\}, \quad \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \in L^2(I; \boldsymbol{L}^2(\Omega)),$$

satisfying (2.89). The next lemma ensures the existence of strong solutions for $p \geq 1 + \frac{2d}{d+2}$.

Lemma 2.29. *Let us consider system* (2.14) *complemented with space-periodic boundary conditions. For* $p > 1$ *and* $\varepsilon = 1$ *let* $\boldsymbol{\mathcal{S}}$ *satisfy Assumption 2.1. We assume that* $\hat{\boldsymbol{v}} \in \boldsymbol{W}^{1,2}(\Omega)$ *and that* $\boldsymbol{f} \in L^{p'}(I; \boldsymbol{L}^{p'}(\Omega))$ *if* $p < 2$, *and* $\boldsymbol{f} \in L^2(I; \boldsymbol{L}^2(\Omega))$ *if* $p \geq 2$. *If* $p > \frac{3d}{d+2}$, *then there exists a solution* $\boldsymbol{v} \in \boldsymbol{\mathcal{V}}^p_{\mathrm{per}}$ *of Problem* **(P7)** *with* $\boldsymbol{\mathcal{V}}^p$ *replaced by* $\boldsymbol{\mathcal{V}}^p_{\mathrm{per}}$. *If* $p \geq 1 + \frac{2d}{d+2}$, *then* $\boldsymbol{v}$ *is unique and regular, i.e.,* $\boldsymbol{v} \in L^\infty(I; \boldsymbol{W}^{1,2}(\Omega)) \cap L^2(I; \boldsymbol{W}^{2,2}(\Omega))$.

Proof. The lemma is proven in [MNR93]. □

The following lemma shows the local in time existence of strong solutions:

Lemma 2.30. *Let $d = 3$. Let us consider system (2.14) complemented with space-periodic boundary conditions. For $p \in (7/5, 2]$ and $\varepsilon \in [0, \varepsilon_0]$ let $\boldsymbol{\mathcal{S}}$ satisfy Assumption 2.1. Let $\hat{\boldsymbol{v}} \in \boldsymbol{W}^{2,2}_{\mathrm{div}}(\Omega)$, $\nabla \cdot \boldsymbol{\mathcal{S}}(\boldsymbol{D}\hat{\boldsymbol{v}}) \in \boldsymbol{L}^2(\Omega)$, and $\boldsymbol{f} \in L^\infty(I; \boldsymbol{W}^{1,2}(\Omega)) \cap W^{1,2}(I; \boldsymbol{L}^2(\Omega))$. Then there exist a time $T' = T'(\varepsilon_0, p, \boldsymbol{f}, \hat{\boldsymbol{v}}, T, \Omega)$ with $0 < T' \leq T$ and a function $\boldsymbol{v} \in L^p(I'; \boldsymbol{\mathcal{V}}^p_{\mathrm{per}})$ with $I' = [0, T']$ that solves Problem* **(P7)** *with $\boldsymbol{\mathcal{V}}^p$ replaced by $\boldsymbol{\mathcal{V}}^p_{\mathrm{per}}$. The solution $\boldsymbol{v}$ satisfies*

$$\|\partial_t \boldsymbol{v}\|_{L^\infty(I';L^2(\Omega))} + \|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})\|_{W^{1,2}(I' \times \Omega)} + \|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})\|_{L^{2\frac{5p-6}{2-p}}(I';W^{1,2}(\Omega))} \leq C \tag{2.90}$$

where the constant C only depends on $\varepsilon_0, p, \boldsymbol{f}, \hat{\boldsymbol{v}}, T, \Omega$. In particular, there hold[3]

$$\boldsymbol{v} \in L^{p\frac{5p-6}{2-p}}(I'; \boldsymbol{W}^{2,\frac{3p}{p+1}}(\Omega)) \cap C(I'; \boldsymbol{W}^{1,s}(\Omega)), \qquad 1 \leq s < 6(p-1), \tag{2.91}$$

$$\boldsymbol{v}_t \in L^\infty(I'; \boldsymbol{L}^2(\Omega)) \cap L^{\frac{p(5p-6)}{(3p-2)(p-1)}}(I'; \boldsymbol{W}^{1,\frac{3p}{p+1}}(\Omega)). \tag{2.92}$$

Due to (2.91) and $p > 7/5$, it follows that $\boldsymbol{v} \in C(I'; \boldsymbol{W}^{1,\frac{12}{5}}(\Omega))$. The solution $\boldsymbol{v}$ is unique within the class $C(I'; \boldsymbol{W}^{1,\frac{12}{5}}(\Omega))$. For $\varepsilon > 0$ there exists a pressure π that satisfies

$$\nabla \pi \in L^{2\frac{5p-6}{2-p}}(I'; \boldsymbol{L}^2(\Omega)), \qquad \|\nabla \pi\|_{L^{2\frac{5p-6}{2-p}}(I';L^2(\Omega))} \leq C' = C'(\varepsilon, p, \boldsymbol{f}, \hat{\boldsymbol{v}}, T, \Omega).$$

The constant C' may explode as $\varepsilon \to 0^+$.

Proof. Lemma 2.30 is proven in Berselli et al. [BDR10]. □

[3]See Diening et al. [DPR02, DR05] and cf. Lemma 4.10.

3 Finite Element Discretization

In this chapter, we introduce the finite element (FE) discretization of the p-Navier-Stokes equations. Since we use an equal-order discretization, we need to stabilize the discrete Galerkin systems. In Section 3.2 we recall well-known stabilization methods such as local projection stabilization that are frequently used in computational fluid dynamics. In Section 3.3 we deal with interpolation in Orlicz-Sobolev spaces that will be crucial in the further course of the thesis. For it we basically follow the article [DR07]. Finally, Section 3.4 is dedicated to implementational aspects.

3.1 Finite element (FE) discretization

For ease of exposition, we assume that Ω is a polygonal ($d = 2$) or polyhedral ($d = 3$) domain. The finite element (FE) discretization is based on a decomposition of Ω. The domain Ω is subdivided into disjoint, open quadrilaterals or hexahedra K with diameter $h_K = \operatorname{diam}(K)$. All elements K together make up the triangulation $\mathbb{T}_h = \{K\}$ so that $\overline{\Omega} = \bigcup_{K\in\mathbb{T}_h} \overline{K}$. The mesh parameter h represents the maximum diameter of the cells, i.e., $h := \max\{h_K;\, K \in \mathbb{T}_h\}$. The symbol h also denotes the cell-wise constant function $h|_K = h_K$. Following the literature such as [Cia80], we formulate the definition of regular meshes: The mesh $\mathbb{T}_h = \{K\}$ is called regular if it satisfies the following conditions:

($M1$) $\overline{\Omega} = \bigcup_{K\in\mathbb{T}_h} \overline{K}$.

($M2$) $K \cap K' = \emptyset$ for all K, $K' \in \mathbb{T}_h$ with $K \neq K'$.

($M3$) Each face of a cell $K \in \mathbb{T}_h$ is either a face of another cell $K' \in \mathbb{T}_h$ or subset of the boundary $\partial\Omega$.

Sometimes the condition ($M3$) is weakened for two reasons (see [Sch10]): Firstly, we allow so-called hanging nodes in order to facilitate adaptive mesh refinement. Elements are allowed to possess nodes that are located in midpoints of faces or edges of neighboring cells. At most one hanging node is allowed on each face or edge. Secondly, we weaken condition ($M3$) in order to treat the case of non-polyhedral boundaries. In this case, we require that, instead of boundary-faces, only the vertices of such faces (and possibly some inner points) are subsets of the boundary. For the subsequent simulations we employ meshes that are organized in a patch-wise manner: We assume that the mesh $\mathbb{T}_h$ is generated by one uniform refinement of a coarser mesh $\mathbb{M}_h := \mathbb{T}_{2h}$. In particular, four ($d = 2$) or eight ($d = 3$) adjacent elements of $\mathbb{T}_h$ can be grouped together to form one element of $\mathbb{M}_h$.

Such macro-elements are called patches. This construction is of importance for particular stabilization methods and a posteriori error estimation, see Sections 3.2 and 5.6.

In case of quadrilaterals (or hexahedra), the finite elements are first defined on a reference element $\hat{K}$, and after they are transformed into functions (generally, no polynomials) defined on a physical element K. More precisely, on the reference element $\hat{K} := (-1,1)^d$ we introduce the spaces $\hat{\mathbb{Q}}_r(\hat{K})$ of tensor product polynomials up to degree $r \in \mathbb{N}_0$:

$$\hat{\mathbb{Q}}_r(\hat{K}) := \operatorname{span}\left\{ \prod_{i=1}^{d} \hat{x}_i^{\alpha_i},\ \alpha_i \in \{0,\dots,r\},\ \hat{\boldsymbol{x}} \in \hat{K} \right\}. \tag{3.1}$$

In case of $r = 1$, this space consists of all bi-linear ($d = 2$) or tri-linear ($d = 3$) functions. The mapping $\boldsymbol{F}_K : \hat{K} \to K$ denotes the transformation, which maps the reference cell $\hat{K}$ to the computational cell $K \in \mathbb{T}_h$. The local finite element space $\mathbb{Q}_r(K)$ is defined by

$$\mathbb{Q}_r(K) := \left\{ w : K \to \mathbb{R};\ w \circ \boldsymbol{F}_K \in \hat{\mathbb{Q}}_r(\hat{K}) \right\}. \tag{3.2}$$

In case of quadrilaterals (hexahedra), the mapping $\boldsymbol{F}_K$ is not affine linear in general. If the transformation $\boldsymbol{F}_K$ itself belongs to the space $\hat{\mathbb{Q}}_r(\hat{K})$, the resulting finite element ansatz is called isoparametric. The finite element space $X_{h,r}$ is characterized by

$$X_{h,r} := X_{h,r}(\mathbb{T}_h) := \left\{ w \in C(\overline{\Omega});\ w|_K \in \mathbb{Q}_r(K) \text{ for all } K \in \mathbb{T}_h \right\}. \tag{3.3}$$

Remark 3.1. The subsequent numerical analysis, which will be performed within the next chapters, also includes the case of *finite elements based on d-simplices.* For $r \in \mathbb{N}_0$ the space $\mathbb{P}_r(K)$ denotes the space of polynomials on K of degree less than or equal to r, i.e.,

$$\mathbb{P}_r(K) := \operatorname{span}\left\{ \prod_{i=1}^{d} x_i^{\alpha_i},\ 0 \le \alpha_i,\ 0 \le \sum_{i=1}^{d} \alpha_i \le r,\ \boldsymbol{x} \in K \right\}. \tag{3.4}$$

If $\mathbb{T}_h$ is based on d-simplices, then the finite element space $X_{h,r}$ is defined by (3.3) with $\mathbb{Q}_r(K)$ replaced by $\mathbb{P}_r(K)$. Note that, in this case, the finite element space $X_{h,r}$ need not to be defined by means of the reference mapping $\boldsymbol{F}_K$ although, in practice, polynomial spaces are usually defined on the reference element due to implementational aspects. Our software, which was employed for our numerical experiments, uses d-linear or d-quadratic finite elements based on quadrilateral meshes. Hence, in the present thesis we will mainly speak of $\mathbb{Q}_1$ or $\mathbb{Q}_2$ finite elements. However, we always keep in mind that the theoretical results, which will be derived below, remain valid for linear and higher-order ($r \ge 2$) finite elements based on d-simplices as defined above. For ease of exposition, we do not consider finite element spaces which are generated by local enrichment with bubble functions.

Let us briefly discuss the case of non-polygonal boundaries. Details can be found in [Sch10]. Regarding higher order elements ($r > 1$), there are degrees of freedom that are associated with points on edges or faces. In case of $r > 1$, the use of bi- or tri-linear transformations $\boldsymbol{F}_K$ may lead to a reduced accuracy along the boundary. In contrast, the use of isoparametric finite elements allows us to choose the transformation appropriately in the sense that the degrees of freedom related to nodes on edges or faces are located on the real boundary $\partial\Omega$. In this thesis, we only employ isoparametric finite elements. In particular, if $r = 1$, the mapping $\boldsymbol{F}_K$ is multilinear, i.e., $\boldsymbol{F}_K \in \mathbb{Q}_1(\hat{K})^d$.

In order to ensure approximation properties of the finite element spaces, we require additional conditions on the geometry of the elements. Following the literature (see Braess [Bra07] and Brenner/Scott [BS94]), we formulate the definition of nondegeneracy:

$(M4)$ For $K \in \mathbb{T}_h$ let B_K be the biggest ball inscribed in K. The family of meshes $\{\mathbb{T}_h;\, h \searrow 0\}$ is called nondegenerate if there exists a constant $\kappa_0 > 0$ such that

$$\frac{h_K}{\operatorname{diam}(B_K)} \leq \kappa_0 \qquad \forall K \in \bigcup_{h>0} \mathbb{T}_h. \tag{3.5}$$

Beyond that, the family of meshes $\{\mathbb{T}_h;\, h \searrow 0\}$ is called quasi-uniform if it holds

$$\min\{\operatorname{diam}(B_K);\, K \in \mathbb{T}_h\} \geq \frac{h}{\kappa_0} \qquad \forall h \in (0, 1]. \tag{3.6}$$

If the family is quasi-uniform, then it is nondegenerate, but not conversely. For general non-affine families of quadrilateral (or hexahedral) meshes, the usual shape regularity assumption (3.5) is not sufficient in order to ensure that the mapping $\boldsymbol{F}_K$ is bijective. Therefore, we suppose the shape regularity assumption given in [MT02, HS04]. Below we describe this assumption in detail. To this end, let $\boldsymbol{F}_T : \hat{T} \to T$ be the multilinear reference mapping that maps the reference hyper-cube $\hat{T} := (-1, 1)^d$ onto an arbitrary quadrilateral (hexahedra) T, i.e., let $\boldsymbol{F}_T \in \mathbb{Q}_1(\hat{T})^d$. In [MT02, HS04], multilinear transformations $\boldsymbol{F}_T$ have been investigated only. A Taylor expansion of $\boldsymbol{F}_T$ yields the representation

$$\boldsymbol{F}_T(\hat{\boldsymbol{x}}) = \boldsymbol{b}_T + \boldsymbol{B}_T\hat{\boldsymbol{x}} + \boldsymbol{g}_T(\hat{\boldsymbol{x}}) \tag{3.7}$$

where $\boldsymbol{b}_T := \boldsymbol{F}_T(\boldsymbol{0})$, $\boldsymbol{B}_T := \nabla\boldsymbol{F}_T(\boldsymbol{0})$, and $\boldsymbol{g}_T(\hat{\boldsymbol{x}}) := \boldsymbol{F}_T(\hat{\boldsymbol{x}}) - \boldsymbol{F}_T(\boldsymbol{0}) - \nabla\boldsymbol{F}_T(\boldsymbol{0})\hat{\boldsymbol{x}}$. Let $\hat{\Xi} \subset \hat{T}$ denote the d-simplex with vertices $(0, \ldots, 0)$, $(1, \ldots, 0)$, $\ldots$, $(0, \ldots, 1)$. Let Ξ_T be the image of $\hat{\Xi}$ under the affine mapping $\hat{\boldsymbol{x}} \mapsto \boldsymbol{B}_T\hat{\boldsymbol{x}} + \boldsymbol{b}_T$. For the simplices $\{\Xi_T : T \in \mathbb{T}_h\}$, we assume the usual shape regularity assumption (3.5):

$$\frac{h_{\Xi_T}}{\operatorname{diam}(B_{\Xi_T})} \leq \kappa_0 \qquad \forall T \in \mathbb{T}_h. \tag{3.8}$$

We recall that $|\cdot|$ also denotes the matrix norm induced by the Euclidean vector norm in $\mathbb{R}^d$. Then, for each element T, the distortion parameter γ_T is defined by

$$\gamma_T := \sup_{\hat{\boldsymbol{x}} \in \hat{T}} |\boldsymbol{B}_T^{-1}\nabla\boldsymbol{F}_T(\hat{\boldsymbol{x}}) - \boldsymbol{I}|. \tag{3.9}$$

The distortion parameter measures the deviation of T from a parallelogram (parallelepiped). For a parallelogram (parallelepiped) T, the reference mapping $\boldsymbol{F}_T$ is affine and $\gamma_T = 0$. For a family of uniformly refined meshes, there holds $\gamma_T \to 0$ as $h \to 0$. The definition of shape regularity can be formulated as follows:

($M5$) The mesh $\mathbb{T}_h$ consisting of quadrilateral (or hexahedral) elements is called shape-regular if the conditions (3.8) and $\gamma_T \leq \gamma_0 < 1$ for all $T \in \mathbb{T}_h$ are satisfied.

The shape regularity assumption ($M5$) imposes that the distortion of the quadrilateral (or hexahedral) elements from a parallelogram (or parallelepiped) is uniformly bounded. This guarantees that the mapping $\boldsymbol{F}_T : \hat{T} \to T$ is bijective. Moreover, it is shown in Lemma 2 in Matthies/Tobiska [MT02] that there exist c, $C > 0$ independent of h_T such that

$$\begin{aligned} cd!(1-\gamma_T)^d h_T^d \leq |\det(\nabla \boldsymbol{F}_T(\hat{\boldsymbol{x}}))| \leq Cd!(1+\gamma_T)^d h_T^d \qquad \forall \hat{\boldsymbol{x}} \in \hat{T}, \\ \sup_{\hat{\boldsymbol{x}}\in\hat{T}} |\nabla \boldsymbol{F}_T(\hat{\boldsymbol{x}})| \leq c(1+\gamma_T)h_T, \qquad \sup_{\boldsymbol{x}\in T} |\nabla \boldsymbol{F}_T^{-1}(\boldsymbol{x})| \leq C(1-\gamma_T)^{-1} h_T^{-1}. \end{aligned} \tag{3.10}$$

From (3.10) we can derive corresponding inequalities for the inverse $\boldsymbol{F}_T^{-1}$ using basic tools of analysis and linear algebra: For $\hat{\boldsymbol{x}} = \boldsymbol{F}_T^{-1}(\boldsymbol{x})$ it holds

$$\det(\nabla \boldsymbol{F}_T^{-1}(\boldsymbol{x})) = \det([\nabla \boldsymbol{F}_T(\hat{\boldsymbol{x}})]^{-1}) = [\det(\nabla \boldsymbol{F}_T(\hat{\boldsymbol{x}}))]^{-1}. \tag{3.11}$$

Throughout the thesis we assume that Assumptions ($M1$) – ($M3$), ($M5$) are satisfied.

Interpolation operators: The approximation properties of finite element spaces can be characterized by estimates for interpolation errors. Throughout the thesis, we use two types of interpolation operators: The point-wise Lagrange interpolation operator $i_h : C(\overline{\Omega}) \to X_{h,r}$ and the Scott-Zhang interpolation operator $j_h : W^{1,p}(\Omega) \to X_{h,r}$. For their precise definitions we refer to [BS94, SZ90]. The Lagrange interpolation operator is only defined for continuous functions. By contrast, the Scott-Zhang interpolation operator also interpolates non-smooth functions in $W^{1,p}(\Omega)$. Below we state important properties of the Scott-Zhang interpolation operator. To this end, we introduce some further notation. For $K \in \mathbb{T}_h$ we define the set of neighboring elements N_K and the neighborhood S_K by

$$N_K := \{K' \in \mathbb{T}_h : \overline{K'} \cap \overline{K} \neq \emptyset\}, \qquad S_K := \text{interior of} \bigcup_{K' \in N_K} \overline{K'}. \tag{3.12}$$

The sets S_K are connected and open. Furthermore, the non-degeneracy (3.5) of the mesh $\mathbb{T}_h$ implies the following two properties: For all $K \in \mathbb{T}_h$ there hold

$$\# N_K \leq N_0 \text{ for some } N_0 \in \mathbb{N}, \qquad |S_K| \sim |K| \text{ with constants independent of } h. \tag{3.13}$$

Below let $v \in W^{l,p}(\Omega)$ with $l \geq 1$ be arbitrary. For $1 \leq q \leq \infty$, $m \in \mathbb{N}_0$, and for all $K \in \mathbb{T}_h$ the stability of the Scott-Zhang interpolation operator,

$$\|j_h v\|_{m,q;K} \leq c \sum_{k=0}^{l} h_K^{k-m+\frac{d}{q}-\frac{d}{p}} |v|_{k,p;S_K}, \tag{3.14}$$

is proven in Scott/Zhang [SZ90]. Using the stability result (3.14) and the Bramble-Hilbert Lemma, for all $K \in \mathbb{T}_h$ we can conclude the local interpolation inequality (see [SZ90])

$$\|v - j_h v\|_{m,p;K} \leq C(d,r,\kappa_0) h_K^{l-m} |v|_{l,p;S_K} \qquad (0 \leq m \leq l \leq r+1). \tag{3.15}$$

In view of (3.13), from (3.15) we easily deduce the global estimate

$$\|v - j_h v\|_{m,p} \le C(d, r, \kappa_0) h^{l-m} \|v\|_{l,p} \qquad (0 \le m \le l \le r+1). \tag{3.16}$$

The results of [SZ90] are derived for finite element spaces $X_{h,r}$ based on d-simplices, see Remark 3.1. However, the Scott-Zhang interpolation operator can be generalized to quadrilateral (hexahedral) meshes (see [HS04]). Note that the Lagrange interpolation operator i_h satisfies an interpolation estimate which is similar to (3.16).

Remark 3.2. From (3.10), (3.11) it follows that for any $w \in W^{m,q}(K)$, $\hat{w}(\hat{\boldsymbol{x}}) := w(\boldsymbol{F}_K(\hat{\boldsymbol{x}}))$,

$$\begin{aligned} \|\nabla^m w\|_{q;K} &\lesssim \|\nabla \boldsymbol{F}_K^{-1}\|_{\infty;K}^m \|\det(\nabla \boldsymbol{F}_K)\|_{\infty,\hat{K}}^{\frac{1}{q}} \|\hat{\nabla}^m \hat{w}\|_{q;\hat{K}} \lesssim h_K^{-m+\frac{d}{q}} \|\hat{\nabla}^m \hat{w}\|_{q;\hat{K}}, \\ \|\hat{\nabla}^m \hat{w}\|_{q;\hat{K}} &\lesssim \|\nabla \boldsymbol{F}_K\|_{\infty;\hat{K}}^m \|\det(\nabla \boldsymbol{F}_K^{-1})\|_{\infty,K}^{\frac{1}{q}} \|\nabla^m w\|_{q;K} \lesssim h_K^{m-\frac{d}{q}} \|\nabla^m w\|_{q;K}. \end{aligned} \tag{3.17}$$

Let $m, l \ge 0$, $\nu, \mu \in [1, \infty)$. Assume that $m - \frac{d}{\mu} \ge l - \frac{d}{\nu}$ and $m \ge l$ so that $W^{m,\mu}(\hat{K}) \hookrightarrow W^{l,\nu}(\hat{K})$. Then (3.17) and (3.15) imply the following generalized interpolation inequality:

$$\begin{aligned} \|w - j_h w\|_{l,\nu;K} &\lesssim \sum_{k=0}^{l} h_K^{-k+\frac{d}{\nu}} \|\hat{\nabla}^k (\hat{w} - \widehat{j_h w})\|_{\nu;\hat{K}} \lesssim \sum_{k=0}^{m} h_K^{-l+\frac{d}{\nu}} \|\hat{\nabla}^k (\hat{w} - \widehat{j_h w})\|_{\mu;\hat{K}} \\ &\lesssim \sum_{k=0}^{m} h_K^{k-l+\frac{d}{\nu}-\frac{d}{\mu}} \|\nabla^k (w - j_h w)\|_{\mu;K} \lesssim h_K^{m-l+\frac{d}{\nu}-\frac{d}{\mu}} \|w\|_{m,\mu;S_K}. \end{aligned} \tag{3.18}$$

Inverse estimates: Below we discuss the relations among various norms on a finite element space. Let $\nu, \mu \in [1, \infty]$ and $0 \le m \le l$. Then, there holds (see [BS94])

$$\|w_h\|_{l,\nu;K} \le C h^{m-l+\frac{d}{\nu}-\frac{d}{\mu}} \|w_h\|_{m,\mu;K} \qquad \forall w_h \in X_{h,r} \qquad \forall K \in \mathbb{T}_h \tag{3.19}$$

for some $C = C(l, \nu, \mu, \gamma_0) > 0$. Next we state the global version of (3.19). If the family $\{\mathbb{T}_h\}$ is quasi-uniform, then for $\nu, \mu \in [1, \infty]$ and $0 \le m \le l$ there exists $C > 0$ such that

$$\|w_h\|_{l,\nu} \le C h^{m-l+\min(0, \frac{d}{\nu}-\frac{d}{\mu})} \|w_h\|_{m,\mu} \qquad \forall w_h \in X_{h,r}. \tag{3.20}$$

Galerkin discretization: We discuss the FE discretization of the p-Navier-Stokes equations. Let X_h and Q_h be appropriate FE spaces defined on $\mathbb{T}_h$ which satisfy $X_h \subset W^{1,\infty}(\Omega)$ and $Q_h \subset L^\infty(\Omega)$. Note that the inclusions $X_h \subset W^{1,\infty}(\Omega)$, $Q_h \subset L^\infty(\Omega)$ hold for all practical choices of X_h, Q_h. Then the FE spaces for the velocity and pressure are given by

$$\boldsymbol{\mathcal{X}}_h^p := \boldsymbol{X}_h \cap \boldsymbol{\mathcal{X}}^p, \qquad \boldsymbol{X}_h = [X_h]^d, \qquad \text{and} \qquad \mathcal{Q}_h^p := Q_h \cap \mathcal{Q}^p. \tag{3.21}$$

The Galerkin approximation of **(P5)** consists in replacing the Banach spaces $\boldsymbol{\mathcal{X}}^p$ and $\mathcal{Q}^p$ by the finite dimensional spaces $\boldsymbol{\mathcal{X}}_h^p$ and $\mathcal{Q}_h^p$: Find $\boldsymbol{u}_h \equiv (\boldsymbol{v}_h, \pi_h) \in \boldsymbol{\mathcal{X}}_h^p \times \mathcal{Q}_h^p$ such that

$$A(\boldsymbol{u}_h)(\boldsymbol{\omega}_h) = \langle \boldsymbol{f}, \boldsymbol{w}_h \rangle \qquad \forall \boldsymbol{\omega}_h \equiv (\boldsymbol{w}_h, q_h) \in \boldsymbol{\mathcal{X}}_h^p \times \mathcal{Q}_h^p \tag{3.22}$$

where for all $\boldsymbol{u} \equiv (\boldsymbol{v}, \pi)$ and $\boldsymbol{\omega} \equiv (\boldsymbol{w}, q)$ the semi-linear form A is defined by

$$A(\boldsymbol{u})(\boldsymbol{\omega}) := (\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{w})_\Omega + ([\boldsymbol{v} \cdot \nabla]\boldsymbol{v}, \boldsymbol{w})_\Omega - (\pi, \nabla \cdot \boldsymbol{w})_\Omega + (\nabla \cdot \boldsymbol{v}, q)_\Omega. \tag{3.23}$$

This discretization does not lead to a stable discretization unless the spaces $\boldsymbol{\mathcal{X}}_h^p$ and $\mathcal{Q}_h^p$ satisfy the inf-sup (Babuška-Brezzi) condition. This condition can be stated as follows:

(IS) *For any $\nu \in (1, \infty)$ there exists a positive constant $\tilde{\beta}(\nu)$ not depending on h such that*

$$\inf_{q_h \in \mathcal{Q}_h^\nu} \sup_{\boldsymbol{w}_h \in \boldsymbol{\mathcal{X}}_h^\nu} \frac{(q_h, \nabla \cdot \boldsymbol{w}_h)_\Omega}{\|q_h\|_{\nu'} \|\boldsymbol{w}_h\|_{1,\nu}} \geq \tilde{\beta}(\nu) > 0. \tag{3.24}$$

It is well-known that for Taylor-Hood elements (i.e., $X_h := X_{h,2}$, $Q_h := X_{h,1}$) the inf-sup condition **(IS)** is fulfilled. Mixed finite elements are extensively discussed in Brezzi&Fortin [BF91] and Girault&Raviart [GR86]. An *equal-order discretization* corresponds to the case when both the velocity and pressure are discretized with finite elements of same order (i.e., $Q_h := X_h$). Compared to Taylor-Hood elements, the equal-order discretization benefits from implementational advantages. However, for equal-order elements the discretization (3.22) is not stable. In particular, the discrete pressure may exhibit oscillations which do not reflect the physical objectivity. The instability is caused by the violation of the discrete inf-sup condition for the pair $\boldsymbol{\mathcal{X}}_h^p \times \mathcal{Q}_h^p$. In addition, the Galerkin formulation (3.22) may suffer from dominating convection in case of high Reynolds numbers.

The main focus of the thesis is on the equal-order discretization of the p-Navier-Stokes equations. In order to overcome the instabilities mentioned above, one may introduce appropriate stabilization terms $s_h(\boldsymbol{u}_h)(\boldsymbol{\omega}_h)$ depending on the discrete solution $\boldsymbol{u}_h$ and trial function $\boldsymbol{\omega}_h$ that are added to the standard Galerkin discretization (3.22). Many different stabilization methods such as local projection stabilization (LPS) have been proposed and investigated in the context of the Navier-Stokes equations (see Section 3.2). In this thesis, we aim at analyzing stabilization methods in the context of the p-Navier-Stokes equations. For p-Stokes systems we will propose a nonlinear stabilization term s_h based on the well-known LPS method that is adjusted to the p-structure of the problem and that leads to optimal convergence results (see Chapter 4). The stabilized discrete problem reads: Find $\boldsymbol{u}_h \equiv (\boldsymbol{v}_h, \pi_h) \in \boldsymbol{\mathcal{X}}_h^p \times \mathcal{Q}_h^p$ (the discrete solution) such that

$$A(\boldsymbol{u}_h)(\boldsymbol{\omega}_h) + s_h(\boldsymbol{u}_h)(\boldsymbol{\omega}_h) = \langle \boldsymbol{f}, \boldsymbol{w}_h \rangle \qquad \forall \boldsymbol{\omega}_h \equiv (\boldsymbol{w}_h, q_h) \in \boldsymbol{\mathcal{X}}_h^p \times \mathcal{Q}_h^p. \tag{3.25}$$

In order to be able to quantify the convergence of FEM, we need to know higher regularity of the exact solution $(\boldsymbol{v}, \pi)$. The availability of higher regularity usually requires that $\boldsymbol{f}$ belongs to a better space than $(\boldsymbol{\mathcal{X}}^p)^*$. For the remainder of the thesis we therefore assume that $\boldsymbol{f} \in \boldsymbol{L}^{p'}(\Omega)$. Since the available regularity of $(\boldsymbol{v}, \pi)$ is limited, we restrict ourselves to the case of low-order finite elements. If not stated otherwise, we consider the case $X_h = Q_h = X_{h,1}$.

3.2 Stabilization

As mentioned above, the standard Galerkin formulation (3.22) may suffer from instabilities resulting from violation of the discrete inf-sup condition (3.24) and dominating advection in case of high Reynolds numbers. In this section, we present two stabilization methods, which are frequently used in the context of Navier-Stokes equations: Residual based stabilization and local projection stabilization methods. For this we follow the survey article Braack et al. [BBJL07] which gives an overview of different stabilization methods.

Residual based stabilization: In the context of Navier-Stokes equations, Brooks/Hughes [BH82] and Hughes/Franca/Balestra [HFB86] modified the standard Galerkin formulation (3.22) adding mesh-dependent residual terms. They include streamline-upwind stabilization for dominating convection, as well as pressure stabilization due to missing inf-sup stability. Hence, this method is referred to as the *streamline-upwind Petrov-Galerkin (SUPG) / pressure-stabilization Petrov-Galerkin (PSPG)* method. For $p = 2$ the Galerkin formulation (3.22) is modified by addition of

$$\begin{aligned} s_h^{\mathrm{SUPG}}(\boldsymbol{u}_h)(\boldsymbol{\omega}_h) := \sum_{K\in\mathbb{T}_h} \Big(&- \mu_0 \Delta \boldsymbol{v}_h + [\boldsymbol{v}_h \cdot \nabla]\boldsymbol{v}_h + \nabla \pi_h - \boldsymbol{f}, \\ &\alpha_K \nabla q_h + \varrho_K [\boldsymbol{v}_h \cdot \nabla]\boldsymbol{w}_h \Big)_K \end{aligned} \tag{3.26}$$

with $\boldsymbol{u}_h = (\boldsymbol{v}_h, \pi_h)$ and $\boldsymbol{\omega}_h = (\boldsymbol{w}_h, q_h)$. Here, the parameters α_K and ϱ_K are cell-wise parameters that depend on the local mesh size and on the particular choice of finite element spaces. In case of $X_h = Q_h = X_{h,1}$, the parameters α_K and ϱ_K are chosen as

$$\alpha_K := \alpha_0 \frac{h_K^2}{6\mu_0 + h_K \|\boldsymbol{v}_h\|_{\infty;K}} \qquad \text{and} \qquad \varrho_K := \varrho_0 \frac{h_K^2}{6\mu_0 + h_K \|\boldsymbol{v}_h\|_{\infty;K}} \tag{3.27}$$

with positive constants α_0 and ϱ_0. Concerning the choice of parameters, extensive discussions can be found in Braack et al. [BBJL07]. The method simultaneously stabilizes spurious oscillations that come from dominating convection and missing inf-sup stability. In particular, the term $(\nabla \pi_h, \alpha_K \nabla q_h)_K$ represents pressure stabilization whereas the term $([\boldsymbol{v}_h \cdot \nabla]\boldsymbol{v}_h, \varrho_K [\boldsymbol{v}_h \cdot \nabla]\boldsymbol{w}_h)_K$ reflects streamline diffusion. The remaining terms are present due to consistency of the method: If the continuous weak solution $\boldsymbol{u} = (\boldsymbol{v}, \pi)$ is smooth enough to be a strong solution, the stabilization part $s_h^{\mathrm{SUPG}}(\boldsymbol{u})(\boldsymbol{\omega}_h)$ vanishes for all $\boldsymbol{\omega}_h$.

Although the classical SUPG/PSPG method has successfully been applied to flow problems, it has been evaluated critically in recent years. Several drawbacks of the stabilization scheme (3.26) are well-known (cf. [BBJL07], [Sch10]):

- Boundary layers of the discrete pressure are introduced since the stabilized finite element system is equipped with the artificial Neumann boundary condition, $\partial_n \pi_h = 0$ on $\partial \Omega$, that arises from the stabilization term. This leads to a reduced accuracy near the boundary.

- The stabilization term exhibits a complicated algebraic structure: Artificial non-symmetric terms are introduced and artificial couplings between velocity and pressure are imposed.
- The evaluation of the stabilization term requires the computation of second derivatives, $\Delta \boldsymbol{v}_h|_K$, since for higher order trial functions ($r > 1$) the terms $\Delta \boldsymbol{v}_h|_K$ do not vanish. In case of $r = 1$, the terms $\Delta \boldsymbol{v}_h|_K$ vanish only if the reference mapping $\boldsymbol{F}_K$ is affine linear. The second derivatives are only needed for consistency. Their computation is cost-intensive because second derivatives of $\boldsymbol{F}_K^{-1}$ are needed. However, the neglect of the terms $\Delta \boldsymbol{v}_h|_K$ generally results in a decreased accuracy.

In case of non-Newtonian fluids ($p \neq 2$), the expression $\mu_0 \Delta \boldsymbol{v}_h$ in (3.26) has to be replaced by $\nabla \cdot \boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}_h)$. But then, in case of d-linear finite elements the terms $\nabla \cdot \boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}_h)|_K$ do not vanish even for affine linear reference mappings $\boldsymbol{F}_K$. However, the computation of second derivatives is cost-intensive. Due to the apparent drawbacks of the SUPG/PSPG-scheme, we do not analyze this classical stabilization method in the context of non-Newtonian fluids. By contrast, we deal with an alternative stabilization scheme based on local projections.

Local projection stabilization: In Becker/Braack [BB01], a stabilization technique was proposed that is based on local projections. The local projection stabilization (LPS) is designed for equal-order discretization of velocities and pressure ($X_h = Q_h = X_{h,r}$), and for stabilization of convective terms. It can also be applied to inf-sup stable discretizations (see Lube et al. [LRL07]). For its formulation we follow the lines of Braack/Lube [BL09]. Let us restrict ourselves to a certain class of meshes: We assume that the mesh $\mathbb{T}_h$ results from a coarser mesh $\mathbb{M}_h$ by one global refinement, i.e., $\mathbb{M}_h := \mathbb{T}_{2h}$. Hence, the mesh $\mathbb{T}_h$ consists of patches of elements. For instance in case of $d = 2$, four quadrilaterals can be grouped together in order to form one element of $\mathbb{M}_h$. There are variants of the LPS method for which this restriction can be omitted. Let $\hat{M} := (-1, 1)^d$ be the reference hyper-cube, and let $\boldsymbol{F}_M : \hat{M} \to M$ be the multilinear reference mapping. We introduce the space of patch-wise discontinuous finite elements of degree $r - 1$:

$$X_{2h,r-1}^{\text{disc}} := \{w \in L^2(\Omega);\, w|_M \circ \boldsymbol{F}_M \in \hat{\mathbb{Q}}_{r-1}(\hat{M})\ \forall M \in \mathbb{M}_h = \mathbb{T}_{2h}\}. \tag{3.28}$$

The L^2-projection $P_{2h,r-1} : L^2(\Omega) \to X_{2h,r-1}^{\text{disc}}$ is characterized by

$$(u - P_{2h,r-1}u, w)_\Omega = 0 \qquad \forall u \in L^2(\Omega) \qquad \forall w \in X_{2h,r-1}^{\text{disc}}. \tag{3.29}$$

Regarding $P_{2h,r-1}$, we define the fluctuation operator $\theta_h : L^2(\Omega) \to L^2(\Omega)$ by

$$\theta_h := \text{id} - P_{2h,r-1} \tag{3.30}$$

where id stands for the identity mapping. The operators $P_{2h,r-1}$ and θ_h are applied to vector-valued functions in a component-wise manner: $\boldsymbol{\theta}_h \boldsymbol{u} := (\theta_h u_1, \dots, \theta_h u_d)$. For the Navier-Stokes system, it was proposed in [BB04] that the stabilization term

$$s_h^{\text{LPS}}(\boldsymbol{u}_h)(\boldsymbol{\omega}_h) := \sum_{M \in \mathbb{M}_h} \Big\{ (\boldsymbol{\theta}_h \nabla \pi_h, \alpha_M \boldsymbol{\theta}_h \nabla q_h)_M + \Big(\boldsymbol{\theta}_h([\boldsymbol{v}_h \cdot \nabla]\boldsymbol{v}_h), \varrho_M \boldsymbol{\theta}_h([\boldsymbol{v}_h \cdot \nabla]\boldsymbol{w}_h) \Big)_M \Big\} \tag{3.31}$$

should be added to the standard Galerkin ansatz (3.22) in order to obtain a stable discretization. Similarly to the SUPG/PSPG method, this stabilization contains patch-wise parameters $\{\alpha_M\}$ and $\{\varrho_M\}$. They are chosen as in (3.27).

Remark 3.3. In case of Navier-Stokes systems ($p = 2$), it is well-known that an element-wise stabilization of the incompressibility constraint, $\nabla \cdot \boldsymbol{v} = 0$, can be important for the robustness of the discretization as soon as $0 < \mu_0 \ll 1$. The so-called grad-div stabilization can be achieved by addition of $(\theta_h(\nabla \cdot \boldsymbol{v}_h), \nu\theta_h(\nabla \cdot \boldsymbol{w}_h))_\Omega$ to (3.22). Here, ν is a patch-wise constant parameter that depends on the local mesh size: $\nu|_M \sim h_M^2/\varrho_M$ for all $M \in \mathbb{M}_h$.

In case of the Navier-Stokes equations, it is well-known that the stabilization (3.31) leads to a stable approximation of the continuous problem. A general convergence theory of local projection schemes is well-established. In particular, for the Oseen equations a priori error estimates providing optimal order of convergence have been proven, e.g., in [MST07]. However, no results are available in the context of non-Newtonian fluids. In this thesis, we study the local projection stabilization applied to p-Stokes/p-Oseen systems. This will be done in a more general framework in which the space $X_{2h,r-1}^{\text{disc}}$ will be replaced by an appropriate finite element space Y_h so that the pairing X_h/Y_h satisfies a certain local inf-sup condition (see Section 4.1). In the context of p-Stokes/p-Oseen systems, we do not only investigate the stabilization method (3.31) but also we propose a new modified version of the scheme (3.31) which is adjusted to the p-structure of the problem.

Let us discuss important variants of the LPS-scheme. For the first variant of (3.31), we introduce the global Lagrange interpolant onto the coarser mesh $\mathbb{M}_h = \mathbb{T}_{2h}$, $i_{2h,r} : X_{h,r} \to X_{2h,r} \subset X_{h,r}$. Instead of the fluctuation operator θ_h, the following filter can be used:

$$\bar{\theta}_h : X_{h,r} \to X_{h,r}, \qquad \bar{\theta}_h := \mathrm{id} - i_{2h,r}. \tag{3.32}$$

When we apply such filters, we achieve stabilization using the gradients of the fluctuations. The stabilization term, which is added to (3.22), reads (α_M, ϱ_M are chosen as in (3.27)):

$$s_h^{\mathrm{SGM}}(\boldsymbol{u}_h)(\boldsymbol{\omega}_h) := \sum_{M \in \mathbb{M}_h} \Big\{ (\nabla\bar{\theta}_h \pi_h, \alpha_M \nabla\bar{\theta}_h q_h)_M + ([\boldsymbol{v}_h \cdot \nabla]\bar{\boldsymbol{\theta}}_h \boldsymbol{v}_h, \varrho_M [\boldsymbol{v}_h \cdot \nabla]\bar{\boldsymbol{\theta}}_h \boldsymbol{w}_h)_M \Big\}. \tag{3.33}$$

For the second variant of (3.31), we restrict ourselves to the case of high-order finite elements ($r \geq 2$). Instead of using two different meshes $\mathbb{T}_h$ and $\mathbb{T}_{2h}$, we only employ the principal mesh $\mathbb{T}_h$, i.e., $\mathbb{M}_h := \mathbb{T}_h$. Similarly as above, we introduce the global Lagrange interpolant $i_{h,r-1} : X_{h,r} \to X_{h,r-1} \subset X_{h,r}$. Then, the following filter can be used:

$$\tilde{\theta}_h : X_{h,r} \to X_{h,r}, \qquad \tilde{\theta}_h := \mathrm{id} - i_{h,r-1}. \tag{3.34}$$

The stabilization term is given by (3.33) when $\bar{\theta}_h$ is replaced by $\tilde{\theta}_h$ and $\mathbb{M}_h = \mathbb{T}_h$ is used. This variant is very attractive from practical point of view. For Stokes systems this stabilization admits optimal a priori error estimates. However, for Oseen systems it becomes suboptimal because the term, which is responsible for stabilization of convection, ensures the convergence order of the space $X_{h,r-1}$ only (see [BBJL07]).

Remark 3.4. The stabilization (3.33) represents an important variant of the LPS scheme (3.31) which is based on subgrid modeling, see Section 5 in Matthies et al. [MST07]. The LPS method uses fluctuations of gradients $\boldsymbol{\theta}_h \nabla \pi_h$, whereas the subgrid modeling approach is based on gradients of fluctuations $\nabla \bar{\theta}_h \pi_h$. Subgrid modeling and LPS are closely related: Let us consider triangulations $\mathbb{T}_h$ made of d-simplices, and let $X_{h,1}$ and $X_{2h,1}$ denote the space of continuous, piecewise linear finite elements associated with $\mathbb{T}_h$ and $\mathbb{M}_h := \mathbb{T}_{2h}$, respectively. Let $i_{2h,1} : X_{h,1} \to X_{2h,1}$ be the Lagrange interpolant, and let $P_{2h,0}$ be the L^2-projection onto the space of piecewise constant functions on $\mathbb{M}_h$. It is shown in [MST07] that it holds $P_{2h,0}(\nabla w_h)|_M = \nabla i_{2h,1}(w_h|_M)$ for all $w_h \in X_{h,1}$ and $M \in \mathbb{M}_h$. Hence, the LPS method (3.31) and the subgrid modeling approach (3.33) coincide in this particular case. However, they may differ for $\mathbb{Q}_1/\mathbb{Q}_1$ finite elements in general.

Compared to the SUPG/PSPG-scheme, the algebraic structure of the LPS-term is easier: No artificial couplings between velocity and pressure are introduced. The computation of second derivatives is not necessary. However, the local projection stabilization is not fully consistent: The stabilization term does not vanish if the continuous solution is inserted. The LPS-schemes (3.31) and (3.33) have been designed taking into account that the consistency error is of same order as the discretization error.

3.3 Interpolation in Orlicz-Sobolev spaces

In context of the p-Laplace equation, optimal error estimates are well-known (see [DR07]). Their derivation is based on error estimation with respect to quasi-norms such as (2.42) which handle the non-degeneracy of the problem (see [EL05]). In this connection, interpolation errors occur with respect to quasi-norms. For their estimation it is convenient to transfer the interpolation theory from Sobolev spaces $W^{k,p}(\Omega)$ to Orlicz-Sobolev spaces $W^{k,\psi}(\Omega)$. In particular, for $1 \le j \le k$ the integral $\int |\nabla^j u|^p$ is replaced by $\int \psi(|\nabla^j u|)$ for some N-function ψ (see Definition 2.1). It is well-known that functions in Sobolev spaces can be approximated by "piecewise polynomials". In [DR07], the classical estimates for the interpolation error are generalized in the context of Orlicz-Sobolev spaces $W^{k,\psi}(\Omega)$. As in [DR07], we require the existence of an interpolation operator of Scott-Zhang type:

Assumption 3.1. *Let $l_0, r \in \mathbb{N}_0$. Let there exist an interpolation operator $\boldsymbol{j}_h : \boldsymbol{W}^{l_0,1}(\Omega) \to \boldsymbol{X}_{h,r}$ that satisfies the following properties: For $l \ge l_0$ and $m \in \mathbb{N}_0$ there holds*

$$\sum_{j=0}^{m} \fint_K |h_K^j \nabla^j \boldsymbol{j}_h \boldsymbol{v}| \,\mathrm{d}\boldsymbol{x} \le c(m,l) \sum_{k=0}^{l} h_K^k \fint_{S_K} |\nabla^k \boldsymbol{v}| \,\mathrm{d}\boldsymbol{x} \tag{3.35}$$

uniformly for all $K \in \mathbb{T}_h$ and $\boldsymbol{v} \in \boldsymbol{W}^{l,1}(\Omega)$. Furthermore there holds

$$\boldsymbol{j}_h \boldsymbol{v} = \boldsymbol{v} \qquad \forall \boldsymbol{v} \in \mathbb{P}_r(\Omega). \tag{3.36}$$

Here, the set S_K denotes a local neighborhood of K. For its definition see (3.12).

Remark 3.5. As mentioned in [DR07], the Clément interpolation operator satisfies Assumption 3.1. However, it does not preserve boundary values and, hence, it is not useful for our subsequent analysis. The Scott-Zhang interpolation operator fulfills Assumption 3.1. It is defined in such a way that it preserves homogeneous Dirichlet boundary conditions, i.e., $j_h : W_0^{1,1}(\Omega) \to X_{h,r} \cap W_0^{1,1}(\Omega)$. In this case, we have to choose $l_0 = 1$ in Assumption 3.1.

The following lemma generalizes the well-known interpolation estimates for Sobolev functions to the setting of Orlicz-Sobolev spaces[1] $W^{1,\psi}(\Omega)$:

Lemma 3.1. *Let ψ be an N-function that satisfies the Δ_2-condition. Let $\boldsymbol{j}_h$ and l be as in Assumption 3.1. Then, uniformly in $K \in \mathbb{T}_h$ and $\boldsymbol{v} \in \boldsymbol{W}^{1,\psi}(\Omega)$ there hold the following relations: (i) Orlicz-stability: There exists a constant $c = c(m, l, \Delta_2(\psi)) > 0$ such that*

$$\sum_{j=0}^{m} \fint_K \psi(h_K^j |\nabla^j \boldsymbol{j}_h \boldsymbol{v}|)\, \mathrm{d}\boldsymbol{x} \le c \sum_{k=0}^{l} \fint_{S_K} \psi(h_K^k |\nabla^k \boldsymbol{v}|)\, \mathrm{d}\boldsymbol{x}. \tag{3.37}$$

(ii) Orlicz-approximability: Let κ_0 be the constant in (3.5). If in addition $l \le r+1$, then there exists a positive constant $c = c(l, \Delta_2(\psi), \kappa_0)$ such that

$$\sum_{j=0}^{l} \fint_K \psi(h_K^j |\nabla^j (\boldsymbol{v} - \boldsymbol{j}_h \boldsymbol{v})|)\, \mathrm{d}\boldsymbol{x} \le c \fint_{S_K} \psi(h_K^l |\nabla^l \boldsymbol{v}|)\, \mathrm{d}\boldsymbol{x}. \tag{3.38}$$

(iii) Orlicz-continuity: If in addition $l \le r+1$, then there exists a constant $c = c(l, \Delta_2(\psi), \kappa_0)$:

$$\fint_K \psi(h_K^l |\nabla^l \boldsymbol{j}_h \boldsymbol{v}|)\, \mathrm{d}\boldsymbol{x} \le c \fint_{S_K} \psi(h_K^l |\nabla^l \boldsymbol{v}|)\, \mathrm{d}\boldsymbol{x}. \tag{3.39}$$

Proof. See [DR07]. Note that Lemma 3.1 is proven in [DR07] for finite element spaces based on simplices, $X_{h,r} = \{v \in L^1(\Omega);\, v|_K \in X_{h,r}(K)\}$ with $\mathbb{P}_r(K) \subset X_{h,r}(K) \subset \mathbb{P}_s(K)$ for $r \le s \in \mathbb{N}_0$. Since $X_{h,r}(K) \subset \mathbb{P}_s(K)$, there exists a constant $c = c(s)$ such that

$$\sup_{\boldsymbol{x} \in K} |\nabla^j \boldsymbol{w}_h(\boldsymbol{x})| \le c \fint_K |\nabla^j \boldsymbol{w}_h(\boldsymbol{y})|\, \mathrm{d}\boldsymbol{y} \qquad \forall \boldsymbol{w}_h \in \boldsymbol{X}_{h,r}, \qquad \forall K \in \mathbb{T}_h, \qquad j \in \mathbb{N}_0. \tag{3.40}$$

As depicted in [DR07], the Orlicz-stability (3.37) follows from (3.40), (3.35), and the properties of ψ. The Orlicz-approximability (3.38) results from $\mathbb{P}_r(K) \subset X_{h,r}(K)$, property (3.36), the Orlicz-stability (3.37), and Corollary 3.3 in [DR07] which generalizes the classical polynomial approximation theory in Sobolev spaces to the setting of Orlicz-Sobolev spaces (and whose proof is based on averaged Taylor polynomials, the non-degeneracy of $\mathbb{T}_h$ and the properties of ψ, see [DR07]). Clearly, (3.39) directly follows from (3.38) with $j = l$ and the triangle inequality (cf. Remark 2.3). □

[1] For an N-function ψ the classical Orlicz space $L^\psi(\Omega)$ and Orlicz-Sobolev space $W^{k,\psi}(\Omega)$, $k \in \mathbb{N}_0$, are defined as follows: $g \in L^\psi(\Omega)$ iff $\int_\Omega \psi(|g|)\, \mathrm{d}\boldsymbol{x} < \infty$, and $g \in W^{k,\psi}(\Omega)$ iff $\partial^\alpha g \in L^\psi(\Omega)$ for any multi-index α with $0 \le |\alpha| \le k$.

Remark 3.6. The question arises whether Lemma 3.1 remains valid for tensor product $\mathbb{Q}_r$-elements: The proof of Lemma 3.1 only requires the properties $\mathbb{P}_r(K) \subset X_{h,r}(K)$ and (3.40). In particular, if (3.40) is required, then the assumption $X_{h,r}(K) \subset \mathbb{P}_s(K)$, that in general is not satisfied for $\mathbb{Q}_r$-elements, can be relaxed. In case of $\mathbb{Q}_r$-elements, the space $X_{h,r}(K)$ is given by $X_{h,r}(K) \equiv \mathbb{Q}_r(K)$, see (3.2). In this case, the property (3.40) remains valid. Moreover, it holds $\mathbb{P}_r(K) \subset X_{h,r}(K)$ for all $K \in \mathbb{T}_h$ provided that the reference mapping $\boldsymbol{F}_K : \hat{K} \to K$ belongs to the space $\mathbb{Q}_1(\hat{K})^d$. This can be seen by the following argument: Let $P \in \mathbb{P}_r(K)$ be given. Since $\boldsymbol{F}_K$ is d-linear, $P(\boldsymbol{F}_K(\hat{\boldsymbol{x}}))$ is a polynomial of degree at most r in each variable $\hat{x}_1, \dots, \hat{x}_d$ separately, i.e., $P \circ \boldsymbol{F}_K \in \mathbb{Q}_r(\hat{K})$. This implies $P \in X_{h,r}(K)$. Hence, Lemma 3.1 holds for $\mathbb{Q}_r$-elements as long as $\boldsymbol{F}_K$ is d-linear.

Lemma 3.1 holds true for any fixed N-function ψ. All constants occurring in the local estimates of Lemma 3.1 depend on the Δ_2-constant of ψ, but they do not depend on the particular N-function ψ. This enables us to apply Lemma 3.1 to shifted N-functions ψ_a:

Corollary 3.2. *Let ψ and ψ_a be given as in Definition 2.4. Let $\boldsymbol{j}_h$ satisfy Assumption 3.1 with $l = 1$. Then, for all $a \geq 0$, $\boldsymbol{v} \in \boldsymbol{W}^{1,\psi}(\Omega)$, and $K \in \mathbb{T}_h$ there holds*

$$\fint_K \psi_a(|\nabla \boldsymbol{j}_h \boldsymbol{v}|)\,\mathrm{d}\boldsymbol{x} \leq c \fint_{S_K} \psi_a(|\nabla \boldsymbol{v}|)\,\mathrm{d}\boldsymbol{x} \tag{3.41}$$

where the constant c only depends on $\Delta_2(\psi)$ and κ_0.

Proof. See [DR07]. Due to Lemma 2.1, the Δ_2-constants of the shifted N-functions ψ_a are uniformly bounded with respect to $a \geq 0$. Then, the desired estimate follows from (3.39) with $l = 1$ applied to the function $h^{-1}\boldsymbol{v}$ for the family of shifted N-functions ψ_a. □

The well-known Corollary 3.2 will play an important role for the subsequent analysis. It will enable us to derive an interpolation inequality with respect to the natural distance.

Application to problems with p-structure: By means of Corollary 3.2 it is proven in [DR07] that interpolation operators of Scott-Zhang type satisfy the following local best approximation result: Let $\boldsymbol{\mathcal{F}}$ be defined by (2.39) with $\boldsymbol{P}^{\mathrm{sym}}$ replaced by $\boldsymbol{P}$. Let $\boldsymbol{j}_h$ satisfy Assumption 3.1 with $l = 1$ and $r \geq 1$. Then for all $K \in \mathbb{T}_h$ there holds

$$\fint_K |\boldsymbol{\mathcal{F}}(\nabla \boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\nabla \boldsymbol{j}_h \boldsymbol{v})|^2\,\mathrm{d}\boldsymbol{x} \leq c \inf_{\boldsymbol{Q} \in \mathbb{R}^{d\times d}} \fint_{S_K} |\boldsymbol{\mathcal{F}}(\nabla \boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{Q})|^2\,\mathrm{d}\boldsymbol{x} \qquad \forall \boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$$

where the constant c only depends on p and κ_0. From this result one can conclude

$$\fint_K |\boldsymbol{\mathcal{F}}(\nabla \boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\nabla \boldsymbol{j}_h \boldsymbol{v})|^2\,\mathrm{d}\boldsymbol{x} \leq c h_K^2 \fint_{S_K} |\nabla \boldsymbol{\mathcal{F}}(\nabla \boldsymbol{v})|^2\,\mathrm{d}\boldsymbol{x} \tag{3.42}$$

provided that $\boldsymbol{\mathcal{F}}(\nabla \boldsymbol{v}) \in W^{1,2}(\Omega)^{d\times d}$. Summing (3.42) over all $K \in \mathbb{T}_h$ and recalling the properties of the mesh (3.13), one can derive the following global version of (3.42):

$$\|\boldsymbol{\mathcal{F}}(\nabla \boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\nabla \boldsymbol{j}_h \boldsymbol{v})\|_2 \le ch \|\nabla \boldsymbol{\mathcal{F}}(\nabla \boldsymbol{v})\|_2. \tag{3.43}$$

Later we will show that (3.43) remains valid if the velocity gradient $\nabla \boldsymbol{v}$ is replaced by its symmetric part $\boldsymbol{D}\boldsymbol{v}$. Note that (3.43) also holds for higher order finite elements ($r \ge 2$) provided that the reference mapping $\boldsymbol{F}_K$ belongs to the space $\mathbb{Q}_1(\hat{K})^d$ (so that $\mathbb{P}_r(K) \subset \mathbb{Q}_r(K)$, see Remark 3.6). In the case $r \ge 2$, the convergence order of (3.43) is suboptimal. This can be easily observed if, e.g., the special case $p = 2$ is considered.

Following [DR07], with the help of (3.43) we can now derive a priori error estimates for equations with p-structure. Exemplarily, let us study the following p-Laplace system,

$$\begin{aligned} -\nabla \cdot \boldsymbol{\mathcal{S}}(\nabla \boldsymbol{v}) &= \boldsymbol{f} \quad \text{in } \Omega, \qquad \boldsymbol{\mathcal{S}}(\nabla \boldsymbol{v}) := \left(\varepsilon^2 + |\nabla \boldsymbol{v}|^2\right)^{\frac{p-2}{2}} \nabla \boldsymbol{v}, \\ \boldsymbol{v} &= \boldsymbol{v}_D \quad \text{on } \partial\Omega, \end{aligned} \tag{3.44}$$

and its discretization with $\mathbb{Q}_r$ finite elements. We assume that $\boldsymbol{v}_D$ is given as the trace of a globally defined function $\boldsymbol{v}_0 \in \boldsymbol{W}^{1,p}(\Omega)$. It is well-known (cf. [DR07]) that a conforming finite element discretization of (3.44) allows the following best approximation result: Let $\boldsymbol{v} \in \boldsymbol{v}_0 + \boldsymbol{W}_0^{1,p}(\Omega)$ be the weak solution to (3.44), and let $\boldsymbol{v}_h \in \boldsymbol{v}_{0,h} + \boldsymbol{\mathcal{X}}_{h,r}^p$ be its finite element approximation, where $\boldsymbol{v}_{0,h}$ denotes an appropriate approximation of the Dirichlet data, and $\boldsymbol{\mathcal{X}}_{h,r}^p$ is defined by $\boldsymbol{\mathcal{X}}_{h,r}^p := \boldsymbol{X}_{h,r} \cap \boldsymbol{W}_0^{1,p}(\Omega)$. Then there holds

$$\|\boldsymbol{\mathcal{F}}(\nabla \boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\nabla \boldsymbol{v}_h)\|_2 \le c \inf_{\boldsymbol{u}_h \in \boldsymbol{v}_{0,h} + \boldsymbol{\mathcal{X}}_{h,r}^p} \|\boldsymbol{\mathcal{F}}(\nabla \boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\nabla \boldsymbol{u}_h)\|_2 \tag{3.45}$$

for some $c = c(p) > 0$. Let $\boldsymbol{j}_h$ be an interpolation operator as in Assumption 3.1. Setting $\boldsymbol{v}_{0,h} := \boldsymbol{j}_h \boldsymbol{v}_0$, and combining (3.45) with (3.43), we arrive at the a priori error estimate

$$\|\boldsymbol{\mathcal{F}}(\nabla \boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\nabla \boldsymbol{v}_h)\|_2 \le ch \|\nabla \boldsymbol{\mathcal{F}}(\nabla \boldsymbol{v})\|_2, \tag{3.46}$$

that is optimal for $r = 1$ but suboptimal for $r \ge 2$. The derivation of optimal a priori error estimates for $r \ge 2$ is subject of current research of Prof. Lars Diening and the author.

Remark 3.7. For solutions $\boldsymbol{v}$ to the p-Laplace problem, the regularity $\boldsymbol{\mathcal{F}}(\nabla \boldsymbol{v}) \in W^{1,2}(\Omega)^{d\times d}$ is well-established. Formally, the term $\int |\nabla \boldsymbol{\mathcal{F}}(\nabla \boldsymbol{v})|^2 \,\mathrm{d}\boldsymbol{x}$ arises if the weak formulation is tested with $-\Delta \boldsymbol{v}$. Hence, the "natural" regularity for solutions of p-Laplace systems can be expressed via the quantity $\boldsymbol{\mathcal{F}}$. Note that the question about still higher regularity (existence of third derivatives) remains an open problem. In this connection the question arises which quantity (e.g., $\boldsymbol{\mathcal{F}}$ or $\boldsymbol{\mathcal{S}}$) is suitable to express still higher regularity.

For higher order finite elements ($r > 1$) we conjecture the following interpolation estimate (3.47), which can be seen as the straightforward extension of (3.43):

$$\|\boldsymbol{\mathcal{F}}(\nabla \boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\nabla \boldsymbol{j}_h \boldsymbol{v})\|_2 \lesssim h^{\min\{\beta, r\}} \|\nabla^\beta \boldsymbol{\mathcal{F}}(\nabla \boldsymbol{v})\|_2, \qquad \beta > 0. \tag{3.47}$$

In the case $\beta \neq 1$, estimate (3.47) is understood only formally and it is not justified from analytical point of view at all. Similarly to the derivation of (3.46), we would then obtain

$$\|\boldsymbol{\mathcal{F}}(\nabla \boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\nabla \boldsymbol{v}_h)\|_2 \lesssim h^{\min\{\beta,r\}} \|\nabla^\beta \boldsymbol{\mathcal{F}}(\nabla \boldsymbol{v})\|_2, \qquad \beta > 0. \tag{3.48}$$

For $\beta \neq 1$ the error estimate (3.48) represents a pure hypothesis. By means of the subsequent numerical simulations, we intend to shed some light on the hypothesis (3.48).

Numerical experiments: We would like to support our hypothesis (3.48) for $\beta > 1$ by numerical simulations. For $\beta \leq 1$, we will numerically validate (3.48) in Section 4.8. The following two experiments numerically demonstrate which convergence rates can be expected for $\mathbb{Q}_2$ finite elements ($r = 2$). They were accomplished by means of the software package Gascoigne [GAS]. The numerical algorithm solving the finite element systems and implementational aspects will be discussed in the forthcoming Section 3.4.

Example 1: For the approximation of (3.44) with $\mathbb{Q}_2$ elements, the obtained discretization errors and corresponding convergence rates are presented in Table 3.1. In this example, on the square $\Omega := (-0.5, 0.5)^2$ the exact solution $\boldsymbol{v} : \Omega \to \mathbb{R}^2$ to (3.44) was given by

$$\boldsymbol{v}(\boldsymbol{x}) = \begin{pmatrix} |x_1|^{p'} \\ 0 \end{pmatrix}, \qquad p' := \frac{p}{p-1}.$$

The discrete problem was then solved for $\boldsymbol{f} := -\nabla \cdot \boldsymbol{\mathcal{S}}(\nabla \boldsymbol{v})$ and $\boldsymbol{v}_D := \boldsymbol{v}|_{\partial\Omega}$. The parameter ε was set to $\varepsilon = 10^{-3}$. It is easy to check that, in the case $\varepsilon = 0$, $\boldsymbol{\mathcal{S}}(\nabla \boldsymbol{v})$ is linear in $\boldsymbol{x}$. As a result the right-hand side $\boldsymbol{f}$ reduces to a constant. In view of Table 3.1, we observe that the error measured in terms of $\boldsymbol{\mathcal{S}}$ behaves as $\mathcal{O}(h^{1+1/p'})$ for $p \geq 3/2$. We realize that the experimental order of convergence obtained for the $\boldsymbol{\mathcal{F}}$-distance amounts to $\min\{\frac{p'+1}{2}, 2\}$. In particular the error measured in terms of $\boldsymbol{\mathcal{F}}$ converges to zero with less than quadratic order as soon as $p > 3/2$. In contrast to $\boldsymbol{\mathcal{S}}(\nabla \boldsymbol{v})$, the quantity $\boldsymbol{\mathcal{F}}(\nabla \boldsymbol{v})$ is not smooth. An easy computation shows that $|\nabla^\beta \boldsymbol{\mathcal{F}}(\nabla \boldsymbol{v}(\boldsymbol{x}))| \sim |x_1|^{p'/2-\beta}$. We may ask for which values of $\beta > 0$ the requirement $|\nabla^\beta \boldsymbol{\mathcal{F}}(\nabla \boldsymbol{v})| \in L^2(\Omega)$ is satisfied. It turns out that this condition is fulfilled if and only if $\beta < (p'+1)/2$. Hence, Table 3.1 indicates that, for this particular example, the hypothesis (3.48) with $\beta \approx (p'+1)/2$ seems to be true. Note that the observed convergence rates for the error in $\boldsymbol{W}^{1,p}(\Omega)$ can be deduced from the ones for the $\boldsymbol{\mathcal{F}}$-distance taking into account Lemma 2.6. Finally we note that for $p = 2$ we observed quadratic convergence for all considered error quantities (which, for $p = 2$, actually coincide), but as soon as $p \neq 2$ we lost quadratic convergence.

Example 2: This example is in the same spirit as Example 1. In Table 3.2 we present convergence rates for the approximation of (3.44) with $\mathbb{Q}_2$ finite elements. The following experimental setup was considered: The analytical solution $\boldsymbol{v} : \Omega \to \mathbb{R}^2$ was given by

$$\boldsymbol{v}(\boldsymbol{x}) := |\boldsymbol{x}|^{a-1} \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}, \qquad a \in \mathbb{R}.$$

Table 3.1. Approximation of (3.44) with $\mathbb{Q}_2$ elements: $\boldsymbol{\mathcal{S}}(\nabla \boldsymbol{v})$ is smooth

		$\|\boldsymbol{\mathcal{F}}(\nabla \boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\nabla \boldsymbol{v}_h)\|_2$		$\|\boldsymbol{\mathcal{S}}(\nabla \boldsymbol{v}) - \boldsymbol{\mathcal{S}}(\nabla \boldsymbol{v}_h)\|_{p'}$		$\|\nabla \boldsymbol{v} - \nabla \boldsymbol{v}_h\|_p$	
p	#cells	error	conv.	error	conv.	error	conv.
1.1	16384	5.26e-05	2.00	2.84e-03	2.19	3.82e-06	1.99
	65536	1.32e-05	2.00	6.89e-04	2.05	9.56e-07	2.00
	262144	3.29e-06	2.00	1.71e-04	2.01	2.39e-07	2.00
1.3	16384	9.20e-05	2.00	2.03e-03	1.88	5.05e-05	2.00
	65536	2.30e-05	2.00	6.50e-04	1.64	1.26e-05	2.00
	262144	5.75e-06	2.00	1.64e-04	1.96	3.16e-06	2.00
1.5	16384	1.37e-04	1.62	1.45e-03	1.11	6.09e-05	1.81
	65536	4.26e-05	1.68	6.68e-04	1.12	1.62e-05	1.91
	262144	1.06e-05	2.01	2.64e-04	1.33	4.08e-06	1.99
1.7	16384	1.20e-04	1.70	4.63e-04	1.41	5.38e-05	1.90
	65536	3.68e-05	1.71	1.74e-04	1.41	1.43e-05	1.91
	262144	1.13e-05	1.71	6.52e-05	1.41	3.77e-06	1.92
1.8	16384	9.17e-05	1.62	1.84e-04	1.45	5.28e-05	1.77
	65536	2.98e-05	1.62	6.76e-05	1.44	1.54e-05	1.78
	262144	9.68e-06	1.62	2.48e-05	1.44	4.47e-06	1.78
1.9	16384	5.08e-05	1.55	6.81e-05	1.47	3.88e-05	1.63
	65536	1.73e-05	1.55	2.45e-05	1.47	1.25e-05	1.63
	262144	5.90e-06	1.55	8.83e-06	1.47	4.04e-06	1.63
1.999	16384	5.49e-07	1.50	5.51e-07	1.50	5.48e-07	1.50
	65536	1.94e-07	1.50	1.95e-07	1.50	1.94e-07	1.50
	262144	6.87e-08	1.50	6.90e-08	1.50	6.85e-08	1.50
2.1	16384	5.82e-05	1.45	4.61e-05	1.52	7.44e-05	1.38
	65536	2.12e-05	1.45	1.61e-05	1.52	2.85e-05	1.38
	262144	7.75e-06	1.45	5.59e-06	1.52	1.09e-05	1.39
2.3	16384	1.87e-04	1.38	1.04e-04	1.56	3.63e-04	1.20
	65536	7.15e-05	1.38	3.53e-05	1.56	1.57e-04	1.20
	262144	2.74e-05	1.38	1.20e-05	1.56	6.83e-05	1.20
2.5	16384	3.20e-04	1.33	1.38e-04	1.59	8.69e-04	1.07
	65536	1.27e-04	1.33	4.60e-05	1.59	4.15e-04	1.07
	262144	5.03e-05	1.33	1.52e-05	1.59	1.98e-04	1.07
3.0	16384	6.30e-04	1.25	1.82e-04	1.64	3.09e-03	0.83
	65536	2.65e-04	1.25	5.83e-05	1.65	1.73e-03	0.83
	262144	1.11e-04	1.25	1.85e-05	1.65	9.73e-04	0.83

All data were chosen as in Example 1. It is easy to check that $\boldsymbol{\mathcal{F}}(\nabla \boldsymbol{v}) \in W^{2,2}(\Omega)^{d\times d}$ if and only if $a > \frac{2}{p} + 1$. In this example we set $a = \frac{2}{p} + 1.01$. By means of Table 3.2 we observe that the discretization error measured in terms of $\boldsymbol{\mathcal{F}}$ behaves as $\mathcal{O}(h^2)$. Hence, Table 3.2 indicates that, for this particular example, the hypothesis (3.48) with $\beta = 2$ seems to be true. To sum up, we have numerically validated the hypothesis (3.48).

Table 3.2. Approximation of (3.44) with $\mathbb{Q}_2$ elements: $\boldsymbol{\mathcal{F}}(\nabla \boldsymbol{v})$ is regular

		$\Vert\boldsymbol{\mathcal{F}}(\nabla \boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\nabla \boldsymbol{v}_h)\Vert_2$		$\Vert\boldsymbol{\mathcal{S}}(\nabla \boldsymbol{v}) - \boldsymbol{\mathcal{S}}(\nabla \boldsymbol{v}_h)\Vert_{p'}$		$\Vert\nabla \boldsymbol{v} - \nabla \boldsymbol{v}_h\Vert_p$	
p	#cells	error	conv.	error	conv.	error	conv.
1.1	16384	1.73e-04	1.89	1.95e-01	0.40	6.96e-05	2.00
	65536	4.62e-05	1.91	1.46e-01	0.41	1.74e-05	2.00
	262144	1.22e-05	1.92	1.04e-01	0.48	4.35e-06	2.00
1.3	16384	1.45e-04	1.89	7.89e-03	0.93	6.76e-05	2.00
	65536	3.88e-05	1.90	4.15e-03	0.93	1.69e-05	2.00
	262144	1.03e-05	1.91	2.17e-03	0.93	4.22e-06	2.00
1.6	16384	1.21e-04	1.88	4.19e-04	1.50	7.33e-05	1.98
	65536	3.26e-05	1.90	1.48e-04	1.50	1.85e-05	1.99
	262144	8.66e-06	1.91	5.20e-05	1.51	4.63e-06	1.99
2.5	16384	1.05e-04	1.87	7.27e-05	1.97	2.34e-04	1.60
	65536	2.84e-05	1.89	1.84e-05	1.98	7.70e-05	1.60
	262144	7.56e-06	1.91	4.65e-06	1.99	2.53e-05	1.61
3.0	16384	1.08e-04	1.87	7.10e-05	1.98	5.09e-04	1.35
	65536	2.92e-05	1.89	1.78e-05	1.99	2.00e-04	1.35
	262144	7.76e-06	1.91	4.47e-06	2.00	7.87e-05	1.35

3.4 Implementational aspects

For the equal-order $\mathbb{Q}_1/\mathbb{Q}_1$ (or $\mathbb{Q}_2/\mathbb{Q}_2$) element we describe the numerical algorithm which solves the discrete system (3.25), and we discuss implementational aspects. The algorithm presented below has been employed within the software package Gascoigne [GAS] in order to generate our numerical simulations appearing throughout the thesis.

Linearization of the discrete problems: We deal with the numerical solution of the algebraic systems. Due to its nonlinear nature, system (3.25) needs to be linearized. Here, we apply Newton's method for linearization. In order to describe the algorithm, let $B(\cdot)(\cdot)$ be any semi-linear form. In the context of p-Navier-Stokes systems, the semi-linear form B is given by $B(\boldsymbol{u})(\boldsymbol{\omega}) := A(\boldsymbol{u})(\boldsymbol{\omega}) + s_h(\boldsymbol{u})(\boldsymbol{\omega})$ for all $\boldsymbol{u}$, $\boldsymbol{\omega} \in \boldsymbol{\mathcal{X}}^p \times \mathcal{Q}^p$ where A is defined in (3.23) and s_h stands for a stabilization term. In order to determine a solution $\boldsymbol{u}_h \in (\boldsymbol{v}_{0,h} + \boldsymbol{\mathcal{X}}_h^p) \times \mathcal{Q}_h^p$ of the discrete system

$$B(\boldsymbol{u}_h)(\boldsymbol{\omega}_h) - \langle \boldsymbol{f}, \boldsymbol{\omega}_h \rangle = 0 \qquad \forall \boldsymbol{\omega}_h \in \boldsymbol{\mathcal{X}}_h^p \times \mathcal{Q}_h^p, \tag{3.49}$$

we carry out Algorithm 3.1 (Newton's algorithm with step-size control). Here, $\boldsymbol{v}_{0,h}$ stands for an approximation (e.g., the Lagrange interpolant) of non-homogeneous Dirichlet data.

Algorithm 3.1. Newton's algorithm with step-size control

1: Choose an initial guess $\boldsymbol{u}_h^0 \in (\boldsymbol{v}_{0,h} + \boldsymbol{\mathcal{X}}_h^p) \times \mathcal{Q}_h^p$.
2: Compute $\boldsymbol{z}_h^k \in \boldsymbol{\mathcal{X}}_h^p \times \mathcal{Q}_h^p$, $k = 0, 1, \ldots$, from the linear equations

$$B'(\boldsymbol{u}_h^k)(\boldsymbol{z}_h^k, \boldsymbol{\omega}_h) = -B(\boldsymbol{u}_h^k)(\boldsymbol{\omega}_h) + \langle \boldsymbol{f}, \boldsymbol{\omega}_h \rangle \qquad \forall \boldsymbol{\omega}_h \in \boldsymbol{\mathcal{X}}_h^p \times \mathcal{Q}_h^p \tag{3.50}$$

where the directional derivative is given by

$$B'(\boldsymbol{u})(\boldsymbol{z}, \boldsymbol{\omega}) := \frac{\mathrm{d}}{\mathrm{d}\delta} B(\boldsymbol{u} + \delta \boldsymbol{z})(\boldsymbol{\omega}) \Big|_{\delta=0} := \lim_{\delta \to 0} \frac{1}{\delta} \Big\{ B(\boldsymbol{u} + \delta \boldsymbol{z})(\boldsymbol{\omega}) - B(\boldsymbol{u})(\boldsymbol{\omega}) \Big\}.$$

3: For given $\lambda \in (0, 1)$ determine minimal $l = 0, 1, \ldots$ for which

$$R(\boldsymbol{u}_{h,l}^{k+1}) < R(\boldsymbol{u}_h^k), \qquad \boldsymbol{u}_{h,l}^{k+1} := \boldsymbol{u}_h^k + \lambda^l \boldsymbol{z}_h^k,$$

and denote it by l^* where the nonlinear residual $R(\cdot)$ is defined by

$$R(\boldsymbol{u}_h) := \max_i \Big\{ B(\boldsymbol{u}_h)(\boldsymbol{\psi}_i) - \langle \boldsymbol{f}, \boldsymbol{\psi}_i \rangle \Big\} \qquad \forall \boldsymbol{u}_h \in \boldsymbol{\mathcal{X}}_h^p \times \mathcal{Q}_h^p. \tag{3.51}$$

Here, $\{\boldsymbol{\psi}_i\}$ denotes the nodal basis of $\boldsymbol{\mathcal{X}}_h^p \times \mathcal{Q}_h^p$.
4: Set $\boldsymbol{u}_h^{k+1} := \boldsymbol{u}_{h,l^*}^{k+1}$.

Step 3 of Algorithm 3.1 includes the step-size control which is crucial when highly nonlinear p-structure problems are solved via Newton's method. In general, the Newton update $\boldsymbol{z}_h^k$ is weighted by the relaxation parameter λ^l. The step-size control enables the globalization of Newton's method, i.e., the independence of the convergence with respect to the choice of $\boldsymbol{u}_h^0$. If $l^* = 0$, then Algorithm 3.1 performs one full Newton cycle.

If $B(\boldsymbol{u})(\boldsymbol{\omega}) := A(\boldsymbol{u})(\boldsymbol{\omega}) + s_h(\boldsymbol{u})(\boldsymbol{\omega})$, then the directional derivative $B'(\boldsymbol{u})(\boldsymbol{z}, \boldsymbol{\omega})$ looks like

$$B'(\boldsymbol{u})(\boldsymbol{z}, \boldsymbol{\omega}) = A'(\boldsymbol{u})(\boldsymbol{z}, \boldsymbol{\omega}) + s_h'(\boldsymbol{u})(\boldsymbol{z}, \boldsymbol{\omega}). \tag{3.52}$$

If the popular Carreau-type model (2.10) & (2.11b) is considered, then for $\boldsymbol{u} \equiv (\boldsymbol{v}, \pi)$, $\boldsymbol{z} \equiv$

$(\boldsymbol{\xi}, \eta)$, $\boldsymbol{\omega} \equiv (\boldsymbol{w}, q)$ the directional derivative $A'(\boldsymbol{u})(\boldsymbol{z}, \boldsymbol{\omega})$ is formally given by

$$\begin{aligned} A'(\boldsymbol{u})(\boldsymbol{z}, \boldsymbol{\omega}) = &\Big(\big(\varepsilon^2 + |\boldsymbol{D}\boldsymbol{v}|^2\big)^{\frac{p-2}{2}} \boldsymbol{D}\boldsymbol{\xi}, \boldsymbol{D}\boldsymbol{w}\Big)_\Omega \\ &+ (p-2)\Big(\big(\varepsilon^2 + |\boldsymbol{D}\boldsymbol{v}|^2\big)^{\frac{p-4}{2}} (\boldsymbol{D}\boldsymbol{v} : \boldsymbol{D}\boldsymbol{\xi})\boldsymbol{D}\boldsymbol{v}, \boldsymbol{D}\boldsymbol{w}\Big)_\Omega \\ &+ \Big((\boldsymbol{v}\cdot\nabla)\boldsymbol{\xi}, \boldsymbol{w}\Big)_\Omega + \Big((\boldsymbol{\xi}\cdot\nabla)\boldsymbol{v}, \boldsymbol{w}\Big)_\Omega - (\eta, \nabla\cdot\boldsymbol{w})_\Omega + (\nabla\cdot\boldsymbol{\xi}, q)_\Omega. \end{aligned} \tag{3.53}$$

In view of (3.53) we observe that in the case $p < 2$ and $\varepsilon = 0$ the directional derivative $A'(\boldsymbol{u}_h)(\boldsymbol{z}_h, \boldsymbol{\omega}_h)$ is not well-defined in general for all $\boldsymbol{u}_h \equiv (\boldsymbol{v}_h, \pi_h)$, $\boldsymbol{z}_h$, $\boldsymbol{\omega}_h \in \boldsymbol{\mathcal{X}}_h^p \times \mathcal{Q}_h^p$ if the critical set $\Omega_c := \{\boldsymbol{x} \in \Omega;\ \nabla\boldsymbol{v}_h(\boldsymbol{x}) \approx \boldsymbol{0}\}$ is not empty. For $\Omega_c \neq \emptyset$ Algorithm 3.1 generally suffers from instabilities. Note that the set Ω_c is not empty for typical solutions to (3.49). Hence, we often choose $\varepsilon > 0$ in order to stabilize Newton's method. The stabilization term s_h is given either by (3.33) or by the nonlinear variant

$$\begin{aligned} s_h(\boldsymbol{u}_h)(\boldsymbol{\omega}_h) := &\sum_{M\in\mathbb{M}_h} \Big((\tau + |\nabla\bar{\theta}_h\pi_h|)^{p'-2}\nabla\bar{\theta}_h\pi_h, \alpha_M\nabla\bar{\theta}_h q_h\Big)_M \\ &+ \sum_{M\in\mathbb{M}_h} ([\boldsymbol{v}_h\cdot\nabla]\bar{\boldsymbol{\theta}}_h\boldsymbol{v}_h, \varrho_M[\boldsymbol{v}_h\cdot\nabla]\bar{\boldsymbol{\theta}}_h\boldsymbol{w}_h)_M. \end{aligned} \tag{3.54}$$

Here, the fluctuation operator $\bar{\theta}$ is given by (3.32) in case of $\mathbb{Q}_1/\mathbb{Q}_1$-elements and by (3.34) in case of $\mathbb{Q}_2/\mathbb{Q}_2$-elements. The patch-wise parameters α_M and ϱ_M can be chosen, e.g., as in (3.27). The LPS-based stabilization (3.54) is particularly adjusted to the p-structure of the problem and it represents a novel approach for the approximation of the p-Navier-Stokes equations with equal-order finite elements. Note that the stabilization (3.54) is similar to the one proposed in Section 4.1 and analyzed in Sections 4.4, 4.5.

Solution of the linear subproblems: We deal with the solution of the linear systems of equations arising in each Newton step. The solution approach formulated below has already been described in [Sch10]. Let $\{\psi_i;\ i = 1, \dots, N\}$ be the nodal basis of X_h with $N = \dim(X_h)$. Since the finite element spaces $\boldsymbol{\mathcal{X}}_h^p$ and $\mathcal{Q}_h^p$ stem from an equal-order discretization, a basis of the space $\boldsymbol{\mathcal{X}}_h^p \times \mathcal{Q}_h^p$ is given by $\Big\{\boldsymbol{\psi}_i^{(\pi)}, \boldsymbol{\psi}_i^{(v_1)}, \dots, \boldsymbol{\psi}_i^{(v_d)};\ i = 1, \dots N\Big\}$ with $\boldsymbol{\psi}_i^{(\pi)} := (\psi_i, 0, \dots, 0)$, $\boldsymbol{\psi}_i^{(v_1)} := (0, \psi_i, 0, \dots, 0)$, and $\boldsymbol{\psi}_i^{(v_d)} := (0, \dots, 0, \psi_i)$ when the boundary condition on the velocity and the zero mean value constraint on the pressure are ignored (they are actually incorporated later). By virtue of the representation

$$\boldsymbol{z}_h^k = \sum_{i=1}^N \Big(\zeta_i^{(\pi)}\boldsymbol{\psi}_i^{(\pi)}, \zeta_i^{(v_1)}\boldsymbol{\psi}_i^{(v_1)}, \dots, \zeta_i^{(v_d)}\boldsymbol{\psi}_i^{(v_d)}\Big),$$

Newton's system (3.50) is equivalent to an algebraic system $\boldsymbol{B}\boldsymbol{\zeta} = \boldsymbol{c}$ for the unknowns $\boldsymbol{\zeta} = (\zeta_1, \dots, \zeta_N)^\mathsf{T} \in \mathbb{R}^{N(d+1)}$, $\zeta_i = \left(\zeta_i^{(\pi)}, \zeta_i^{(v_1)}, \dots, \zeta_i^{(v_d)}\right)$. The vector $\boldsymbol{c}$ is given by

$$\boldsymbol{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{pmatrix} \qquad \text{with entries} \qquad c_i = \begin{pmatrix} -B\left(\boldsymbol{u}_h^k\right)\left(\boldsymbol{\psi}_i^{(\pi)}\right) + \left\langle \boldsymbol{f}, \boldsymbol{\psi}_i^{(\pi)} \right\rangle \\ -B\left(\boldsymbol{u}_h^k\right)\left(\boldsymbol{\psi}_i^{(v_1)}\right) + \left\langle \boldsymbol{f}, \boldsymbol{\psi}_i^{(v_1)} \right\rangle \\ \vdots \\ -B\left(\boldsymbol{u}_h^k\right)\left(\boldsymbol{\psi}_i^{(v_d)}\right) + \left\langle \boldsymbol{f}, \boldsymbol{\psi}_i^{(v_d)} \right\rangle \end{pmatrix},$$

and the matrix $\boldsymbol{B} \in \mathbb{R}^{N(d+1)\times N(d+1)}$ exhibits the following block structure

$$\boldsymbol{B} := \begin{pmatrix} B_{11} & \cdots & B_{1N} \\ \vdots & \ddots & \vdots \\ B_{N1} & \cdots & B_{NN} \end{pmatrix}.$$

Each block B_{ij} represents a $(d+1) \times (d+1)$ matrix given by

$$B_{ij} := \begin{pmatrix} B'\left(\boldsymbol{u}_h^k\right)\left(\boldsymbol{\psi}_j^{(\pi)}, \boldsymbol{\psi}_i^{(\pi)}\right) & B'\left(\boldsymbol{u}_h^k\right)\left(\boldsymbol{\psi}_j^{(v_1)}, \boldsymbol{\psi}_i^{(\pi)}\right) & \cdots & B'\left(\boldsymbol{u}_h^k\right)\left(\boldsymbol{\psi}_j^{(v_d)}, \boldsymbol{\psi}_i^{(\pi)}\right) \\ B'\left(\boldsymbol{u}_h^k\right)\left(\boldsymbol{\psi}_j^{(\pi)}, \boldsymbol{\psi}_i^{(v_1)}\right) & B'\left(\boldsymbol{u}_h^k\right)\left(\boldsymbol{\psi}_j^{(v_1)}, \boldsymbol{\psi}_i^{(v_1)}\right) & \cdots & B'\left(\boldsymbol{u}_h^k\right)\left(\boldsymbol{\psi}_j^{(v_d)}, \boldsymbol{\psi}_i^{(v_1)}\right) \\ \vdots & \vdots & \ddots & \vdots \\ B'\left(\boldsymbol{u}_h^k\right)\left(\boldsymbol{\psi}_j^{(\pi)}, \boldsymbol{\psi}_i^{(v_d)}\right) & B'\left(\boldsymbol{u}_h^k\right)\left(\boldsymbol{\psi}_j^{(v_1)}, \boldsymbol{\psi}_i^{(v_d)}\right) & \cdots & B'\left(\boldsymbol{u}_h^k\right)\left(\boldsymbol{\psi}_j^{(v_d)}, \boldsymbol{\psi}_i^{(v_d)}\right) \end{pmatrix}.$$

The Dirichlet boundary conditions are enforced as follows: The degrees of freedom $\zeta_i^{(v_j)}$ on the boundary are eliminated by replacing the corresponding entries within the right-hand side $\boldsymbol{c}$ by zero and substituting the corresponding rows and columns within the matrix $\boldsymbol{B}$ by zero or one so that as a result $\zeta_i^{(v_j)} = 0$. Hence, all Newton updates $\boldsymbol{z}^k$ satisfy homogeneous Dirichlet boundary conditions. Consequently, the correct boundary conditions are recovered even in the case of non-homogeneous Dirichlet boundary conditions since the initial guess $\boldsymbol{u}_h^0$ already satisfies the prescribed boundary conditions.

We solve the linear preconditioned subproblems $\boldsymbol{M}\boldsymbol{B}\boldsymbol{\zeta} = \boldsymbol{M}\boldsymbol{c}$ applying the *Generalized Minimal Residual Method* (GMRES), see Saad [Saa03]. As preconditioner $\boldsymbol{M}$, we use the multigrid method. The smoother, which is used in the multigrid iteration, consists of a fix-point iteration based on a block ILU decomposition of $\boldsymbol{B}$. The incomplete LU factorization of $\boldsymbol{B}$ is based on the decomposition $\boldsymbol{B} = \boldsymbol{L}\boldsymbol{U} + \boldsymbol{H}$ where $\boldsymbol{L}$ is a lower and $\boldsymbol{U}$ is an upper triangle matrix. If $\boldsymbol{H} = \boldsymbol{0}$, then $\boldsymbol{B} = \boldsymbol{L}\boldsymbol{U}$ corresponds to a full LU decomposition. In this case, $\boldsymbol{B}$ and $\boldsymbol{U}$ are dense matrices. By contrast, in the incomplete version $\boldsymbol{B}$ and $\boldsymbol{U}$ exhibit the same structure as $\boldsymbol{B}$ so that as a result $\boldsymbol{H} \neq \boldsymbol{0}$. However, within the fixed-point iteration the matrix $\boldsymbol{H}$ is neglected. Thus, the fix-point iteration reads:

$$\boldsymbol{\zeta}^{k+1} = (\boldsymbol{I} - \boldsymbol{U}^{-1}\boldsymbol{L}^{-1}\boldsymbol{B})\boldsymbol{\zeta}^k + \boldsymbol{U}^{-1}\boldsymbol{L}^{-1}\boldsymbol{c}.$$

Compared to the classical ILU decomposition, the block ILU factorization is more cost-intensive but it leads to more robust smoother, see Hackbusch [Hac93].

In each iteration step of the linear solver we re-establish the zero mean value constraint on the pressure subtracting the mean value from the current pressure approximation.

4 Finite Element Approximation of the p-Stokes Equations

This chapter is dedicated to the finite element (FE) approximation of the p-Stokes problem **(P1)**. We discretize problem **(P1)** with equal-order d-linear finite elements ($X_h = Q_h = X_{h,1}$). Since this discretization is not inf-sup stable, we stabilize the Galerkin formulation by the local projection stabilization (LPS) method (see Becker/Braack [BB01]). Within the LPS framework for Stokes systems, one adds an appropriate linear stabilization term to the Galerkin formulation that gives a weighted L^2-control over the fluctuations of the pressure gradient. In contrast, we propose a nonlinear stabilization term of p'-Laplace type that yields a weighted $L^{p'}$-control over the fluctuations of the pressure gradient for $p' := p/(p-1)$. Our proposed stabilization term is adjusted to the p-structure of the problem since the pressure naturally belongs to $L^{p'}(\Omega)$. In this chapter, we perform a convergence analysis of LPS if either the classical stabilization (see [BB01]) or our modified version is used. In the latter case, for $p \in (1, 2]$ we establish the a priori error estimates

$$\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2 \leq ch, \qquad \|\pi - \pi_h\|_{p'} \leq ch^{\frac{2}{p'}}, \tag{4.1}$$

provided that the solution $(\boldsymbol{v}, \pi)$ satisfies the regularity assumption

$$\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) \in W^{1,2}(\Omega)^{d\times d}, \qquad \pi \in W^{1,p'}(\Omega), \tag{4.2}$$

where $\boldsymbol{\mathcal{F}}$ is defined by (2.39), see Theorem 4.11. For $p \in [2, \infty)$ we establish analog a priori error estimates, see Theorem 4.12. Note that Theorems 4.11 & 4.12 represent main results of the thesis. Numerical experiments indicate that, at least in the case $p \leq 2$, the derived a priori error estimates provide optimal rates of convergence with respect to the regularity. In contrast to its nonlinear counterpart, the classical LPS scheme of [BB01] does not allow an optimal convergence analysis meaning that, e.g. if $p < 2$, it does not lead to (4.1), unless slightly higher regularity than (4.2), such as $\boldsymbol{v} \in \boldsymbol{W}^{2,2}(\Omega)$, is assumed. For stable discretizations, the FE approximation of p-Stokes systems has been studied, e.g., by Barrett/Liu [BL93b, BL94]. However, their results are suboptimal in the sense that either the order of the error estimate is not optimal or the assumed regularity for the solution is too high and not realistic for general solutions. Hence, this thesis improves existing results in literature and, besides, it provides the first analytical investigation of LPS in the context of p-Stokes systems. The theoretical results of Sections 4.1 – 4.5 as well as the numerical experiments 1, 4, 5 in Section 4.8 have already been published in Hirn [Hir10].

In Section 4.1, we introduce our novel modified LPS method and, in Section 4.2, we study the structure of the proposed stabilization. In Section 4.3, we present a modified Scott-Zhang interpolation operator which enables the analysis of the stabilized FE method. As depicted in Section 4.4, this interpolation operator allows us to prove a discrete analogon of the continuous inf-sup condition. As a result, we show the well-posedness of the stabilized discrete systems. In Sections 4.5 and 4.6 we prove a priori error estimates quantifying the convergence of the LPS method. While in Section 4.5 we analyze the proposed nonlinear stabilization scheme, in Section 4.6 we focus on the classical LPS scheme (see Section 3.2) applied to p-Stokes systems. Section 4.7 deals with the time-space discretization of the non-steady p-Stokes problem **(P4)**. In Section 4.8, the derived a priori error estimates are illustrated by numerical experiments. Finally, in Section 4.9 we present some particular projection spaces satisfying the abstract assumptions of Section 4.1.

4.1 LPS in the context of p-Stokes systems

In this section we introduce the local projection stabilization (LPS) method following the literature (cf. Matthies et al. [MST07]). In contrast to Section 3.2, we study the LPS method in a general framework in which we do not specify the projection spaces. Within the LPS framework, we propose a stabilization term that is adjusted to the p-structure of the problem and that differs from the one introduced in [MST07].

In order to explain the stabilization method, we start with some additional notation. Let $\mathbb{M}_h$ be a non-overlapping, shape-regular decomposition of Ω constructed by coarsening $\mathbb{T}_h$ such that each $M \in \mathbb{M}_h$ with diameter h_M consists of one or more neighboring cells $K \in \mathbb{T}_h$ with $h_K \sim h_M$ for all $K \subset M$. For instance, one can imagine a two-level variant in which $\mathbb{T}_h$ results from the coarser mesh $\mathbb{M}_h$ by one global refinement: $\mathbb{M}_h = \mathbb{T}_{2h}$. We introduce the space Y_h as a finite element space defined on the macro partition $\mathbb{M}_h$, such that the pair X_h/Y_h satisfies the local inf-sup condition (Assumption 4.1) below. We denote the restriction of the space Y_h to $M \in \mathbb{M}_h$ by $Y_h(M) := \{q_h|_M;\, q_h \in Y_h\}$ and we define the auxiliary space $X_h^0(M)$ by $X_h^0(M) := \{w_h|_M;\, w_h \in X_h,\, w_h = 0 \text{ on } \Omega \setminus M\}$.

Assumption 4.1. *For $\nu \geq 1$ there exists $\bar{\beta} > 0$ independent of h such that*

$$\inf_{q \in Y_h(M)} \sup_{w \in X_h^0(M)} \frac{(w, q)_M}{\|w\|_{\nu;M} \|q\|_{\nu';M}} \geq \bar{\beta} > 0 \tag{4.3}$$

for all $h > 0$ and all $M \in \mathbb{M}_h$, where $\nu' := \nu/(\nu - 1)$. If $\nu = 1$, then $\nu' := \infty$.

Remark 4.1. Assumption 4.1 is similar to Assumption A3 in Matthies et al. [MST07]. However we changed the L^2-setting of [MST07] into an L^ν-setting with $\nu \geq 1$. For instance, one possible choice of Y_h is the discontinuous finite element space consisting of all piecewise constant functions on the coarser mesh $\mathbb{M}_h = \mathbb{T}_{2h}$. For such Y_h, (4.3) is shown in [MST07] in case of $\nu = 2$. We can easily prove (4.3) in the general case $\nu \geq 1$ by adjusting the proof of Lemma 3.2 in [MST07], see Section 4.9. We only have to replace the L^2-setting by an L^ν-setting. Further choices of X_h/Y_h are discussed in [MST07].

Let P_M be a local projection $P_M : L^\nu(M) \to Y_h(M)$. Clearly, P_M defines a global projection $P_h : L^\nu(\Omega) \to Y_h$ by $(P_h q)|_M := P_M(q|_M)$ for all $M \in \mathbb{M}_h$. Denoting the identity on $L^\nu(\Omega)$

by id, we define the associated fluctuation operator $\theta_h : L^\nu(\Omega) \to L^\nu(\Omega)$ by $\theta_h := \mathrm{id} - P_h$. These operators are applied to vector-valued functions in a component-wise manner, e.g., $\boldsymbol{P}_h : \boldsymbol{L}^\nu(\Omega) \to \boldsymbol{Y}_h$ with $\boldsymbol{P}_h \boldsymbol{q} := (P_h q_1, \ldots, P_h q_d)$. The following assumption on θ_h will ensure that the consistency error is small enough:

Assumption 4.2. *For $\nu > 1$ let the fluctuation operator θ_h satisfy*

$$\|\theta_h q\|_{\nu;M} \lesssim \|q\|_{\nu;M} \qquad \forall M \in \mathbb{M}_h, \qquad \forall q \in L^\nu(\Omega).$$

Remark 4.2. For instance, θ_h satisfies Assumption 4.2 if P_h is the L^2-projection onto Y_h and $Y_h(M)$ contains the space of constant functions, see [MST07].

Then we modify the discrete problem (3.22) by adding the stabilization term

$$s_h(\pi)(q) := \sum_{M \in \mathbb{M}_h} \alpha_M \Big((\tau + |\boldsymbol{\theta}_h \nabla \pi|)^{p'-2} \boldsymbol{\theta}_h \nabla \pi, \boldsymbol{\theta}_h \nabla q \Big)_M \qquad \text{with} \qquad \alpha_M := \alpha_0 h_M^s, \tag{4.4}$$

where $\alpha_0 > 0$, s, $\tau \geq 0$, and p is the same as in Assumption 2.1. For $p \neq 2$ the stabilization term s_h is nonlinear in its first argument. The appropriate choice of s will be determined by the convergence analysis of the method. The stabilized discrete system reads:

(P1$_h$) *Find $(\boldsymbol{v}_h, \pi_h) \in \boldsymbol{\mathcal{X}}_h^p \times \mathcal{Q}_h^p$ such that*

$$(\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}_h), \boldsymbol{D}\boldsymbol{w}_h)_\Omega - (\pi_h, \nabla \cdot \boldsymbol{w}_h)_\Omega = (\boldsymbol{f}, \boldsymbol{w}_h)_\Omega \qquad \forall \boldsymbol{w}_h \in \boldsymbol{\mathcal{X}}_h^p \tag{4.5a}$$

$$s_h(\pi_h)(q_h) + (\nabla \cdot \boldsymbol{v}_h, q_h)_\Omega = 0 \qquad \forall q_h \in \mathcal{Q}_h^p. \tag{4.5b}$$

Note that our proposed method recovers the standard LPS scheme for Stokes systems in the particular case $p = 2$. In fact, the semilinear form s_h defined in (4.4) coincides with the classical LPS term introduced in Becker/Braack [BB01] for Stokes systems in the case $p = 2$ and $s = 2$. Below we always assume that Assumptions 4.1 and 4.2 are satisfied. The following sections will show stability and convergence of the method.

Remark 4.3. If the standard stabilization proposed in [BB01] is applied to p-Stokes systems, convergence of the method can also be expected and will be quantified in Section 4.6. However, for the classical LPS method we will only be able to establish suboptimal a priori error estimates whose order depends on the space dimension d, see Corollaries 4.14, 4.15, 4.18. In contrast, our modified stabilization (4.4) will enable us to derive optimal a priori error estimates independent of d, see Theorem 4.11.

4.2 Properties of the stabilization term

Below let s_h be defined by (4.4). In this section, we highlight the structure of s_h and we show some resulting properties. To this end, we introduce a nonlinear function $\boldsymbol{\mathcal{G}} : \mathbb{R}^d \to \mathbb{R}^d$,

$$\boldsymbol{\mathcal{G}}(\boldsymbol{q}) := (\tau + |\boldsymbol{q}|)^{\frac{p'-2}{2}} \boldsymbol{q} \qquad (\boldsymbol{q} \neq \boldsymbol{0}), \qquad \boldsymbol{\mathcal{G}}(\boldsymbol{0}) := \boldsymbol{0}, \tag{4.6}$$

where p' and τ are the same as in (4.4). The following two lemmas can easily be proven by adapting the results of Section 2.4. Their proofs are based on a vector-valued version of Lemma 2.4 (see [DE08]). Lemma 4.1 depicts how the distance induced by $\boldsymbol{\mathcal{G}}$ relates to the standard $L^{p'}$-norm, whereas Lemma 4.2 clarifies the connection between $\boldsymbol{\mathcal{G}}$ and s_h.

Lemma 4.1. *Let $U \subset \Omega$ be a measurable subset of Ω and let $\boldsymbol{g}, \boldsymbol{q} \in \boldsymbol{L}^{p'}(\Omega)$. If $p \in (1,2]$, then there exist constants c, $C > 0$ only depending on p such that*

$$c\|\boldsymbol{g}-\boldsymbol{q}\|_{p';U}^{p'} \leq \|\boldsymbol{\mathcal{G}}(\boldsymbol{g})-\boldsymbol{\mathcal{G}}(\boldsymbol{q})\|_{2;U}^2 \leq C\|\varepsilon+|\boldsymbol{g}|+|\boldsymbol{q}|\|_{p';U}^{p'-2}\|\boldsymbol{g}-\boldsymbol{q}\|_{p';U}^2. \tag{4.7}$$

If $p \in [2,\infty)$, then there exist constants c, $C > 0$ only depending on p such that

$$\|\boldsymbol{g}-\boldsymbol{q}\|_{p';U} \leq c\|\boldsymbol{\mathcal{G}}(\boldsymbol{g})-\boldsymbol{\mathcal{G}}(\boldsymbol{q})\|_{2;U}\|\tau+|\boldsymbol{g}|+|\boldsymbol{q}|\|_{p';U}^{\frac{2-p'}{2}}, \tag{4.8}$$

$$\|\boldsymbol{\mathcal{G}}(\boldsymbol{g})-\boldsymbol{\mathcal{G}}(\boldsymbol{q})\|_{2;U}^2 \leq C\|\boldsymbol{g}-\boldsymbol{q}\|_{p';U}^{p'}. \tag{4.9}$$

Proof. The proof is similar to the one of Lemma 2.6. A vector-valued version of Lemma 2.4 (see Diening/Ettwein [DE08]) shows that for $p \in (1,\infty)$ there holds

$$|\boldsymbol{\mathcal{G}}(\boldsymbol{g})-\boldsymbol{\mathcal{G}}(\boldsymbol{q})|^2 \sim |\boldsymbol{g}-\boldsymbol{q}|^2(\tau+|\boldsymbol{g}|+|\boldsymbol{q}|)^{p'-2} \tag{4.10}$$

a.e. in U where the constants only depend on p. First of all let $p \in (1,2]$, i.e., $p' \in [2,\infty)$. Since $p' \geq 2$ and $|\boldsymbol{g}|+|\boldsymbol{q}| \geq \frac{1}{2}(|\boldsymbol{g}|+|\boldsymbol{g}-\boldsymbol{q}|)$, we realize that $|\boldsymbol{g}-\boldsymbol{q}|^{p'} \lesssim |\boldsymbol{\mathcal{G}}(\boldsymbol{g})-\boldsymbol{\mathcal{G}}(\boldsymbol{q})|^2$ a.e. in U. Integrating this over U, we arrive at $(4.7)_1$. Integrating (4.10) over U, and using Hölder's inequality with $\frac{2}{p'}+\frac{p'-2}{p'}=1$, we obtain $(4.7)_2$. Now let $p \in [2,\infty)$, i.e., $p' \in (1,2]$. The relation (4.10) implies that $|\boldsymbol{g}-\boldsymbol{q}|^{p'} \sim |\boldsymbol{\mathcal{G}}(\boldsymbol{g})-\boldsymbol{\mathcal{G}}(\boldsymbol{q})|^{p'}(\tau+|\boldsymbol{g}|+|\boldsymbol{q}|)^{(2-p')p'/2}$ a.e. in U. Integrating this over U and applying Hölder's inequality with $\frac{p'}{2}+\frac{2-p'}{2}=1$, we easily deduce (4.8). From (4.10) and $p' \leq 2$ it follows that $|\boldsymbol{\mathcal{G}}(\boldsymbol{g})-\boldsymbol{\mathcal{G}}(\boldsymbol{q})|^2 \lesssim |\boldsymbol{g}-\boldsymbol{q}|^{p'}$ a.e.. Integrating this over U, we finally get (4.9). This completes the proof. □

Lemma 4.2. *Let φ^* be defined by (2.37). For all $\pi, q \in W^{1,p'}(\Omega)$ there holds*

$$\begin{aligned} s_h(\pi)(\pi-q)-s_h(q)(\pi-q) &\sim \sum_{M\in\mathbb{M}_h} \alpha_M\|\boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h\nabla\pi)-\boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h\nabla q)\|_{2;M}^2 \\ &\sim \sum_{M\in\mathbb{M}_h} \alpha_M \int_M (\varphi^*)_{\tau+|\boldsymbol{\theta}_h\nabla\pi|}(|\boldsymbol{\theta}_h\nabla\pi-\boldsymbol{\theta}_h\nabla q|)\,\mathrm{d}\boldsymbol{x} \end{aligned}$$

where the constants only depend on p.

Proof. Using the definition of s_h and a vector-valued version of Lemma 2.4 with p, ε, $\boldsymbol{\mathcal{F}}$, φ replaced by p', τ, $\boldsymbol{\mathcal{G}}$, φ^*, we obtain the equivalence stated in the assertion. □

Lemma 4.3. *For $p \in (1,\infty)$ let s_h be defined by (4.4). For all $\delta > 0$ there exists a constant $c = c(\delta, p, \alpha_0) > 0$ such that for all $\pi, q \in W^{1,p'}(\Omega)$ there holds*

$$|s_h(\pi)(\pi-q)| \leq ch^s\|\tau+|\nabla\pi|\|_{p';\Omega}^{p'} + \delta\sum_{M\in\mathbb{M}_h}\alpha_M\|\boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h\nabla\pi)-\boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h\nabla q)\|_{2;M}^2. \tag{4.11}$$

Proof. Let φ and φ^* be given by (2.37). Using the vector-valued version of Lemma 2.4 (with p, ε, φ replaced by p', τ, φ^*), we estimate

$$\begin{aligned}|s_h(\pi)(\pi - q)| &= \Big| \sum_{M\in\mathbb{M}_h} \alpha_M \Big((\tau + |\boldsymbol{\theta}_h \nabla\pi|)^{p'-2} \boldsymbol{\theta}_h\nabla\pi, \boldsymbol{\theta}_h\nabla\pi - \boldsymbol{\theta}_h\nabla q \Big)_M \Big| \\ &\lesssim \sum_{M\in\mathbb{M}_h} \alpha_M \int_M (\varphi^*)'_{\tau+|\boldsymbol{\theta}_h\nabla\pi|}(|\boldsymbol{\theta}_h\nabla\pi|)|\boldsymbol{\theta}_h\nabla\pi - \boldsymbol{\theta}_h\nabla q| \,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

Applying Young's inequality (2.36), Lemma 4.2, the change-of-shift Lemma 2.3, and the stability of $\boldsymbol{\theta}_h$ (Assumption 4.2), for arbitrary $\delta > 0$ we obtain

$$\begin{aligned}|s_h(\pi)(\pi - q)| &\lesssim c_\delta \sum_{M\in\mathbb{M}_h} \alpha_M \int_M (\varphi^*)_{\tau+|\boldsymbol{\theta}_h\nabla\pi|}(|\boldsymbol{\theta}_h\nabla\pi|) \,\mathrm{d}\boldsymbol{x} \\ &\quad + \delta \sum_{M\in\mathbb{M}_h} \alpha_M \int_M (\varphi^*)_{\tau+|\boldsymbol{\theta}_h\nabla\pi|}(|\boldsymbol{\theta}_h\nabla\pi - \boldsymbol{\theta}_h\nabla q|) \,\mathrm{d}\boldsymbol{x} \\ &\lesssim c_\delta h^s \int_\Omega (\varphi^*)_{\tau+|\boldsymbol{\theta}_h\nabla\pi|}(|\boldsymbol{\theta}_h\nabla\pi|) \,\mathrm{d}\boldsymbol{x} + \delta \sum_{M\in\mathbb{M}_h} \alpha_M \|\boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h\nabla\pi) - \boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h\nabla q)\|_{2;M}^2 \\ &\lesssim c_\delta h^s \int_\Omega (\varphi^*)(\tau + |\boldsymbol{\theta}_h\nabla\pi|) \,\mathrm{d}\boldsymbol{x} + \delta \sum_{M\in\mathbb{M}_h} \alpha_M \|\boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h\nabla\pi) - \boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h\nabla q)\|_{2;M}^2 \\ &\lesssim c_\delta h^s \|\tau + |\nabla\pi|\|_{p'}^{p'} + \delta \sum_{M\in\mathbb{M}_h} \alpha_M \|\boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h\nabla\pi) - \boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h\nabla q)\|_{2;M}^2\end{aligned}$$

where c_δ only depends on p, α_0 and δ. □

Remark 4.4. Let $p \in [2, \infty)$. Then, we may modify the proof of Lemma 4.3 as follows:

$$\begin{aligned}|s_h(\pi)(\pi - q)| &\lesssim \sum_{M\in\mathbb{M}_h} \alpha_M \int_M (\varphi^*)'_{|\boldsymbol{\theta}_h\nabla\pi|}(|\boldsymbol{\theta}_h\nabla\pi|)|\boldsymbol{\theta}_h\nabla\pi - \boldsymbol{\theta}_h\nabla q| \,\mathrm{d}\boldsymbol{x} \\ &\lesssim c_\delta h^s \sum_{M\in\mathbb{M}_h} \|\boldsymbol{\theta}_h\nabla\pi\|_{p';M}^{p'} + \delta \sum_{M\in\mathbb{M}_h} \alpha_M \|\boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h\nabla\pi) - \boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h\nabla q)\|_{2;M}^2.\end{aligned}$$

Hence, for $p \in [2, \infty)$ in (4.11) the expression $\|\tau + |\nabla\pi|\|_{p';\Omega}^{p'}$ can be replaced by $\|\nabla\pi\|_{p';\Omega}^{p'}$.

4.3 Modified interpolation operator

The key idea in the error analysis consists in the construction of an interpolant into X_h which exhibits an additional orthogonality property with respect to the space Y_h. This interpolant shall also feature appropriate approximation properties with respect to the quasi-norm. The subsequent lemma generalizes Theorem 2.2 in Matthies et al. [MST07] and Theorem 5.7 in Diening/Růžička [DR07]. In the latter one, the interpolation estimate

$$\fint_M |\boldsymbol{\mathcal{F}}(\nabla\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\nabla\boldsymbol{j}_h\boldsymbol{v})|^2 \,\mathrm{d}\boldsymbol{x} \le c h_M^2 \fint_{S_M} |\nabla\boldsymbol{\mathcal{F}}(\nabla\boldsymbol{v})|^2 \,\mathrm{d}\boldsymbol{x} \tag{4.12}$$

has been proven for finite elements based on simplices provided that the interpolation operator $\boldsymbol{j}_h$ satisfies Assumption 3.1 with $r_0 \geq 1$. Below we will prove (4.12) with ∇ replaced by $\boldsymbol{D}$ for d-linear finite elements.

Lemma 4.4. *Let $\nu \geq 1$ and let X_h/Y_h satisfy Assumption 4.1. We set $\boldsymbol{\mathcal{X}}_h^\nu := \boldsymbol{X}_h \cap \boldsymbol{W}_0^{1,\nu}(\Omega)$ and $\boldsymbol{Y}_h = [Y_h]^d$. Then, there exist interpolation operators $j_h : W^{1,\nu}(\Omega) \to X_h$ and $\boldsymbol{j}_h : \boldsymbol{W}_0^{1,\nu}(\Omega) \to \boldsymbol{\mathcal{X}}_h^\nu$, which satisfy the following properties:*
(i) Orthogonality with respect to Y_h, $\boldsymbol{Y}_h$: For all $w \in W^{1,\nu}(\Omega)$ and $\boldsymbol{w} \in \boldsymbol{W}_0^{1,\nu}(\Omega)$:

$$(w - j_h w, q_h)_\Omega = 0 \qquad \forall q_h \in Y_h, \tag{4.13}$$

$$(\boldsymbol{w} - \boldsymbol{j}_h \boldsymbol{w}, \boldsymbol{q}_h)_\Omega = 0 \qquad \forall \boldsymbol{q}_h \in \boldsymbol{Y}_h. \tag{4.14}$$

(ii) Let $1 \leq l \leq 2$. Then for all $M \in \mathbb{M}_h$, $w \in W^{l,\nu}(\Omega)$ and $\boldsymbol{w} \in \boldsymbol{W}^{l,\nu}(\Omega) \cap \boldsymbol{W}_0^{1,\nu}(\Omega)$:

$$\|w - j_h w\|_{\nu;M} + h_M \|\nabla(w - j_h w)\|_{\nu;M} \lesssim h_M^l \|w\|_{l,\nu;S_M}, \tag{4.15}$$

$$\|\boldsymbol{w} - \boldsymbol{j}_h \boldsymbol{w}\|_{\nu;M} + h_M \|\nabla(\boldsymbol{w} - \boldsymbol{j}_h \boldsymbol{w})\|_{\nu;M} \lesssim h_M^l \|\boldsymbol{w}\|_{l,\nu;S_M}. \tag{4.16}$$

(iii) For $p \in (1,\infty)$ and $\varepsilon \in [0,\infty)$ let $\boldsymbol{\mathcal{F}}$ be defined by (2.39). If $\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{w}) \in [W^{1,2}(\Omega)]^{d\times d}$, then for all $M \in \mathbb{M}_h$ there holds

$$\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{w}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{w})\|_{2;M} \lesssim h_M \|\nabla \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{w})\|_{2;S_M}. \tag{4.17}$$

Here, S_M denotes a local neighborhood of M as defined in (3.12) which appears in the definition of interpolation operators for non-smooth functions (cf. Brenner/Scott [BS94]).

Proof. The construction of an interpolant satisfying the properties (i) and (ii) with $\nu = 2$ is well-known and was accomplished for the analysis of the LPS method in the context of the Stokes/Oseen equations. For this we refer to Matthies et al. [MST07]. Here we construct the interpolant similarly utilizing Assumption 4.1. We show the additional interpolation property (4.17) following the arguments of Diening/Růžička [DE08].

(i)+(ii): We follow the proof of Theorem 2.2 in Matthies et al. [MST07]. First of all we note that the Scott-Zhang interpolation operator i_h can be extended to vector-valued functions (in a component-wise manner) and to quadrilateral meshes. Furthermore, i_h is defined in such a way that it preserves homogeneous Dirichlet boundary conditions. Hence, $i_h : W^{1,\nu}(\Omega) \to X_h$ and $\boldsymbol{i}_h : \boldsymbol{W}_0^{1,\nu}(\Omega) \to \boldsymbol{\mathcal{X}}_h^\nu$ with $\boldsymbol{i}_h \boldsymbol{w} := (i_h w_1, \ldots, i_h w_d)$. Let $Y_h(M)^*$ denote the dual space of $Y_h(M)$, $Z_h(M) := \{w_h \in X_h^0(M);\ (w_h, q_h)_M = 0\, \forall q_h \in Y_h(M)\}$, and let $Z_h(M)^\perp$ be the L^2-orthogonal complement of $Z_h(M)$ in $X_h^0(M)$. The linear continuous operator $B_h : X_h^0(M) \to Y_h(M)^*$ defined by

$$\langle B_h w_h, q_h \rangle := (w_h, q_h)_M \qquad \forall w_h \in X_h^0(M), \qquad \forall q_h \in Y_h(M),$$

is an isomorphism from $Z_h(M)^\perp$ onto $Y_h(M)^*$ with

$$\bar{\beta} \|w_h\|_{\nu;M} \leq \|B_h w_h\|_{Y_h(M)^*} \qquad \forall w_h \in Z_h(M)^\perp$$

(note $Z_h(M) = \mathrm{Ker}(B_h)$) if and only if (4.3) holds true (see Lemma 2.17). Consequently, for each $w \in W^{1,\nu}(\Omega)$ there exists a unique $z_h(w) \in Z_h(M)^\perp$ such that

$$\langle B_h z_h(w), q_h\rangle = (z_h(w), q_h)_M = (w - i_h w, q_h)_M \qquad \forall q_h \in Y_h(M), \tag{4.18}$$

$$\|z_h(w)\|_{\nu;M} \le \frac{1}{\beta}\|w - i_h w\|_{\nu;M}, \tag{4.19}$$

where i_h is the Scott-Zhang interpolation operator. We set $j_h w|_M := i_h w|_M + z_h(w)$ for all $M \in \mathbb{M}_h$. Due to $\bigoplus_{M\in\mathbb{M}_h} Z_h(M)^\perp \subset \bigoplus_{M\in\mathbb{M}_h} X_h^0(M) \subset X_h$, this defines a global interpolant $j_h : W^{1,\nu}(\Omega) \to X_h$. The orthogonality property (4.13) follows from (4.18), whereas the interpolation property (4.15) results from (4.19) and the properties of i_h (cf. Theorem 2.2 in [MST07]). Indeed, recalling (3.15), we deduce that

$$\|w - j_h w\|_{\nu;M} \le \left(1 + \frac{1}{\beta}\right)\|w - i_h w\|_{\nu;M} \lesssim h_M^l \|w\|_{l,\nu;S_M} \tag{4.20}$$

for all $w \in W^{l,\nu}(\Omega)$ and $1 \le l \le 2$ where S_M denotes a local neighborhood of M which appears in the definition of the Scott-Zhang operator. In order to show the approximation property in the $W^{1,\nu}$-semi-norm, we use the inverse inequality (3.19) and (4.19):

$$\|\nabla z_h(w)\|_{\nu;M} \le C h_M^{-1}\|z_h(w)\|_{\nu;M} \le C h_M^{-1}\|w - i_h w\|_{\nu;M} \lesssim h_M^{l-1}\|w\|_{l,\nu;S_M}.$$

Consequently, using the triangle inequality, we conclude that

$$\|\nabla(w - j_h w)\|_{\nu;M} \le \|\nabla(w - i_h w)\|_{\nu;M} + \|\nabla z_h(w)\|_{\nu;M} \lesssim h_M^{l-1}\|w\|_{l,\nu;S_M}.$$

Using the definition $\boldsymbol{j}_h\boldsymbol{w} := \boldsymbol{i}_h\boldsymbol{w} + \boldsymbol{z}_h(\boldsymbol{w})$, the mapping property $\boldsymbol{i}_h : \boldsymbol{W}_0^{1,\nu}(\Omega) \to \boldsymbol{\mathcal{X}}_h^\nu$, and $\boldsymbol{z}_h(\boldsymbol{w})|_{\partial\Omega} = 0$, we deduce $\boldsymbol{j}_h : \boldsymbol{W}_0^{1,\nu}(\Omega) \to \boldsymbol{\mathcal{X}}_h^\nu$, (4.14), and (4.16).

(iii): For $w \in L^1(U)$ with $|U| > 0$ we denote the mean value of w over U by $\langle w\rangle_U := \fint_U w \,\mathrm{d}\boldsymbol{x} := \frac{1}{|U|}\int_U w(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$. The interpolation estimate (4.16) with $\nu = 1$ implies

$$\int_M |\boldsymbol{j}_h\boldsymbol{w}|\,\mathrm{d}\boldsymbol{x} \le \int_M |\boldsymbol{j}_h\boldsymbol{w} - \boldsymbol{w}|\,\mathrm{d}\boldsymbol{x} + \int_M |\boldsymbol{w}|\,\mathrm{d}\boldsymbol{x} \lesssim \sum_{k=0}^{1}\int_{S_M} h_M^k|\nabla^k\boldsymbol{w}|\,\mathrm{d}\boldsymbol{x}$$

$$\int_M h_M|\nabla\boldsymbol{j}_h\boldsymbol{w}|\,\mathrm{d}\boldsymbol{x} \le \int_M h_M|\nabla(\boldsymbol{j}_h\boldsymbol{w} - \boldsymbol{w})|\,\mathrm{d}\boldsymbol{x} + \int_M h_M|\nabla\boldsymbol{w}|\,\mathrm{d}\boldsymbol{x} \lesssim \sum_{k=0}^{1}\int_{S_M} h_M^k|\nabla^k\boldsymbol{w}|\,\mathrm{d}\boldsymbol{x}.$$

Since the mesh is non-degenerate, there holds $|M| \sim |S_M|$ with constants independent of M. Thus, the interpolation operator $\boldsymbol{j}_h$ satisfies the following $W^{1,1}$-stability,

$$\sum_{j=0}^{1}\fint_M h_M^j|\nabla^j\boldsymbol{j}_h\boldsymbol{w}|\,\mathrm{d}\boldsymbol{x} \le c\sum_{k=0}^{1}\fint_{S_M} h_M^k|\nabla^k\boldsymbol{w}|\,\mathrm{d}\boldsymbol{x} \qquad \forall \boldsymbol{w} \in \boldsymbol{W}^{1,1}(\Omega). \tag{4.21}$$

We recall that the Scott-Zhang operator $\boldsymbol{i}_h$ is a projection: $\boldsymbol{i}_h\boldsymbol{w}_h = \boldsymbol{w}_h$ for all $\boldsymbol{w}_h \in \boldsymbol{X}_h$. Consequently, the interpolation operator $\boldsymbol{j}_h$ is a projection as well,

$$\boldsymbol{j}_h\boldsymbol{w}_h = \boldsymbol{i}_h\boldsymbol{w}_h + \boldsymbol{z}_h(\boldsymbol{w}_h) = \boldsymbol{w}_h \qquad \forall \boldsymbol{w}_h \in \boldsymbol{X}_h, \tag{4.22}$$

since $\boldsymbol{z}_h(\boldsymbol{w}_h) = \mathbf{0}$ due to (4.19). Next, we observe (cf. Remark 3.6) that $\mathbb{P}_1(K) \subset X_h(K)$ for all $K \in \mathbb{T}_h$, where $\mathbb{P}_1(K)$ is the space of linear polynomials and $X_h(K) := \{w : K \to \mathbb{R};\ w \circ \boldsymbol{F}_K \in \mathbb{Q}_1(\hat{K})\}$. This implies $\mathbb{P}_1(\Omega) \subset X_h$. Recalling (4.22), we realize that

$$\boldsymbol{j}_h \boldsymbol{w} = \boldsymbol{w} \qquad \forall \boldsymbol{w} \in \mathbb{P}_1(\Omega)^d. \tag{4.23}$$

For $p \in (1, \infty)$ and $\varepsilon \in [0, \infty)$ let $\boldsymbol{\mathcal{F}}$ be defined by (2.39), and let φ be given by (2.37). For $a \geq 0$ let the shifted N-functions φ_a be defined by (2.32). Since $\boldsymbol{j}_h$ satisfies (4.21) and (4.23), by virtue of Corollary 3.2 there exists $c > 0$ only depending on p such that

$$\fint_M \varphi_a(|\nabla \boldsymbol{j}_h \boldsymbol{w}|)\,\mathrm{d}\boldsymbol{x} \leq c \fint_{S_M} \varphi_a(|\nabla \boldsymbol{w}|)\,\mathrm{d}\boldsymbol{x} \qquad \forall \boldsymbol{w} \in \boldsymbol{W}^{1,p}(\Omega) \tag{4.24}$$

for all $a \geq 0$, $M \in \mathbb{M}_h$. It is crucial that the constant in (4.24) does not depend on the shift $a \geq 0$. In order to derive (4.17), we exploit some arguments of [DE08]. Let $\boldsymbol{q} \in \mathbb{P}_1(\Omega)^d$ be an arbitrary linear polynomial. Using Lemma 2.4, Lemma 2.3, adding the identity $\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{q} - \boldsymbol{D}\boldsymbol{q} = \mathbf{0}$ and recalling Remark 2.3, we estimate

$$\begin{aligned}
\fint_M |\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{w}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{w})|^2\,\mathrm{d}\boldsymbol{x} &\sim \fint_M \varphi_{\varepsilon+|\boldsymbol{D}\boldsymbol{w}|}(|\boldsymbol{D}\boldsymbol{w} - \boldsymbol{D}\boldsymbol{j}_h\boldsymbol{w}|)\,\mathrm{d}\boldsymbol{x} \\
&\lesssim \fint_M \varphi_{\varepsilon+|\boldsymbol{D}\boldsymbol{q}|}(|\boldsymbol{D}\boldsymbol{w} - \boldsymbol{D}\boldsymbol{j}_h\boldsymbol{w}|)\,\mathrm{d}\boldsymbol{x} + \fint_M \varphi_{\varepsilon+|\boldsymbol{D}\boldsymbol{q}|}(|\boldsymbol{D}\boldsymbol{w} - \boldsymbol{D}\boldsymbol{q}|)\,\mathrm{d}\boldsymbol{x} \\
&\lesssim \fint_M \varphi_{\varepsilon+|\boldsymbol{D}\boldsymbol{q}|}(|\boldsymbol{D}\boldsymbol{j}_h(\boldsymbol{q} - \boldsymbol{w})|)\,\mathrm{d}\boldsymbol{x} + \fint_M \varphi_{\varepsilon+|\boldsymbol{D}\boldsymbol{q}|}(|\boldsymbol{D}\boldsymbol{w} - \boldsymbol{D}\boldsymbol{q}|)\,\mathrm{d}\boldsymbol{x} =: I_1 + I_2.
\end{aligned} \tag{4.25}$$

Applying (4.24) to the term I_1, we conclude that

$$I_1 \lesssim \fint_M \varphi_{\varepsilon+|\boldsymbol{D}\boldsymbol{q}|}(|\nabla \boldsymbol{j}_h(\boldsymbol{q} - \boldsymbol{w})|)\,\mathrm{d}\boldsymbol{x} \lesssim \fint_{S_M} \varphi_{\varepsilon+|\boldsymbol{D}\boldsymbol{q}|}(|\nabla \boldsymbol{w} - \nabla \boldsymbol{q}|)\,\mathrm{d}\boldsymbol{x}.$$

Since $\boldsymbol{q}$ is an arbitrary linear polynomial, we can choose $\nabla \boldsymbol{q} := \langle \nabla \boldsymbol{w} \rangle_{S_M} \in \mathbb{R}^{d \times d}$. This particular choice of $\nabla \boldsymbol{q}$ allows us to apply the N-function-version of Korn's inequality whose proof can be found in Diening et al. [DRS10]. Hence, we obtain the estimate

$$I_1 \lesssim \fint_{S_M} \varphi_{\varepsilon+|\boldsymbol{D}\boldsymbol{q}|}(|\nabla \boldsymbol{w} - \langle \nabla \boldsymbol{w} \rangle_{S_M}|)\,\mathrm{d}\boldsymbol{x} \lesssim \fint_{S_M} \varphi_{\varepsilon+|\boldsymbol{D}\boldsymbol{q}|}(|\boldsymbol{D}\boldsymbol{w} - \langle \boldsymbol{D}\boldsymbol{w} \rangle_{S_M}|)\,\mathrm{d}\boldsymbol{x}. \tag{4.26}$$

Noting the identity $\boldsymbol{D}\boldsymbol{q} = \frac{1}{2}(\nabla \boldsymbol{q} + (\nabla \boldsymbol{q})^{\mathsf{T}}) = \frac{1}{2}(\langle \nabla \boldsymbol{w} \rangle_{S_M} + \langle (\nabla \boldsymbol{w})^{\mathsf{T}} \rangle_{S_M}) = \langle \boldsymbol{D}\boldsymbol{w} \rangle_{S_M}$, combining (4.25) and (4.26), and applying Lemma 2.4, we arrive at

$$\fint_M |\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{w}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{w})|^2\,\mathrm{d}\boldsymbol{x} \lesssim \fint_{S_M} |\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{w}) - \boldsymbol{\mathcal{F}}(\langle \boldsymbol{D}\boldsymbol{w} \rangle_{S_M})|^2\,\mathrm{d}\boldsymbol{x} =: I_3, \tag{4.27}$$

where we have also used the estimate $I_2 \lesssim I_3$. By means of Lemma 2.4 the equivalence

$$I_3 \sim \fint_{S_M} \Big(\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{w}) - \boldsymbol{\mathcal{S}}(\langle \boldsymbol{D}\boldsymbol{w} \rangle_{S_M})\Big) : (\boldsymbol{D}\boldsymbol{w} - \langle \boldsymbol{D}\boldsymbol{w} \rangle_{S_M})\,\mathrm{d}\boldsymbol{x}$$

follows where $\boldsymbol{\mathcal{S}}$ is given as in Assumption 2.1. Since $\fint_{S_M} \boldsymbol{D}\boldsymbol{w} - \langle \boldsymbol{D}\boldsymbol{w}\rangle_{S_M} \mathrm{d}\boldsymbol{x} = \boldsymbol{0}$ and $\boldsymbol{\mathcal{S}}(\langle \boldsymbol{D}\boldsymbol{w}\rangle_{S_M})$ as well as $\boldsymbol{\mathcal{S}}(\boldsymbol{\mathcal{F}}^{-1}(\langle \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{w})\rangle_{S_M}))$ are constant, we conclude that

$$I_3 \sim \fint_{S_M} \Big(\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{w}) - \boldsymbol{\mathcal{S}}(\boldsymbol{\mathcal{F}}^{-1}(\langle \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{w})\rangle_{S_M}))\Big) : (\boldsymbol{D}\boldsymbol{w} - \langle \boldsymbol{D}\boldsymbol{w}\rangle_{S_M})\, \mathrm{d}\boldsymbol{x}.$$

Applying Lemma 2.4 and Young's inequality (2.36), for arbitrary $\delta > 0$ we obtain

$$\begin{aligned} I_3 &\lesssim \fint_{S_M} \varphi'_{\varepsilon+|\boldsymbol{D}\boldsymbol{w}|}(|\boldsymbol{D}\boldsymbol{w} - \boldsymbol{\mathcal{F}}^{-1}(\langle \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{w})\rangle_{S_M})|)|\boldsymbol{D}\boldsymbol{w} - \langle \boldsymbol{D}\boldsymbol{w}\rangle_{S_M}|\, \mathrm{d}\boldsymbol{x} \\ &\lesssim \delta \fint_{S_M} \varphi_{\varepsilon+|\boldsymbol{D}\boldsymbol{w}|}|\boldsymbol{D}\boldsymbol{w} - \langle \boldsymbol{D}\boldsymbol{w}\rangle_{S_M}|\, \mathrm{d}\boldsymbol{x} + c_\delta \fint_{S_M} \varphi_{\varepsilon+|\boldsymbol{D}\boldsymbol{w}|}(|\boldsymbol{D}\boldsymbol{w} - \boldsymbol{\mathcal{F}}^{-1}(\langle \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{w})\rangle_{S_M})|)\, \mathrm{d}\boldsymbol{x} \\ &\sim \delta I_3 + c_\delta \fint_{S_M} |\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{w}) - \langle \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{w})\rangle_{S_M}|^2\, \mathrm{d}\boldsymbol{x} \end{aligned}$$

where $c_\delta > 0$ only depends on p and δ. Choosing $\delta > 0$ sufficiently small, we deduce that

$$\fint_{S_M} |\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{w}) - \boldsymbol{\mathcal{F}}(\langle \boldsymbol{D}\boldsymbol{w}\rangle_{S_M})|^2\, \mathrm{d}\boldsymbol{x} \lesssim \fint_{S_M} |\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{w}) - \langle \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{w})\rangle_{S_M}|^2\, \mathrm{d}\boldsymbol{x}. \tag{4.28}$$

Combining (4.27) and (4.28), we arrive at

$$\fint_M |\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{w}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{w})|^2\, \mathrm{d}\boldsymbol{x} \lesssim \fint_{S_M} |\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{w}) - \langle \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{w})\rangle_{S_M}|^2\, \mathrm{d}\boldsymbol{x}.$$

Then the assertion follows from Poincaré's inequality applied to $\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{w}) \in L^2(S_M)^{d\times d}$.□

Remark 4.5. (i) From (4.16) we deduce that the interpolation operator $\boldsymbol{j}_h$ is $W^{1,\nu}$-stable:

$$\|\boldsymbol{j}_h\boldsymbol{w}\|_{1,\nu;M} \lesssim \|\boldsymbol{w}\|_{1,\nu;S_M} \qquad \forall \boldsymbol{w} \in \boldsymbol{W}^{1,\nu}(\Omega). \tag{4.29}$$

(ii) By setting $q_h = 1$ in (4.13) we conclude that $j_h : \mathcal{Q}^p \cap W^{1,p'}(\Omega) \to \mathcal{Q}_h^p$. Consequently, j_h is also an appropriate interpolation operator for the pressure.

4.4 Well-posedness of the stabilized systems

In this section we show that solutions to $(\mathbf{P1}_h)$ exist and that they are uniquely determined. We prove that the solutions to $(\mathbf{P1}_h)$ are uniformly bounded with respect to their natural norms. The well-posedness of $(\mathbf{P1}_h)$ is based on the following lemma that can be seen as the discrete analogon of the inf-sup stability condition (2.68):

Lemma 4.5. *Let $\nu \in (1,\infty)$ and $\nu' := \nu/(\nu-1)$. Then for all $q_h \in \mathcal{Q}_h^\nu$ there holds*

$$\tilde{\beta}(\nu)\|q_h\|_{\nu'} \le \sup_{\boldsymbol{w}_h \in \boldsymbol{\mathcal{X}}_h^\nu} \frac{(\nabla \cdot \boldsymbol{w}_h, q_h)_\Omega}{\|\nabla \boldsymbol{w}_h\|_\nu} + \Big(\sum_{M\in\mathbb{M}_h} h_M^{\nu'}\|\boldsymbol{\theta}_h(\nabla q_h)\|_{\nu';M}^{\nu'}\Big)^{\frac{1}{\nu'}}, \tag{4.30}$$

where $\tilde{\beta}(\nu) > 0$ is a constant independent of h.

Proof. We know that the pair $\boldsymbol{W}_0^{1,\nu}(\Omega) \times L_0^{\nu'}(\Omega)$ satisfies the inf-sup condition (2.68): there exists a positive constant $\beta(\nu)$ such that

$$\beta(\nu)\|q\|_{\nu'} \le \sup_{\boldsymbol{w}\in \boldsymbol{W}_0^{1,\nu}(\Omega)} \frac{(\nabla\cdot \boldsymbol{w}, q)_\Omega}{\|\nabla \boldsymbol{w}\|_\nu} \qquad \forall q \in L_0^{\nu'}(\Omega).$$

Since $\mathcal{Q}_h^\nu \subset L_0^{\nu'}(\Omega)$ it follows that for all $q_h \in \mathcal{Q}_h^\nu$ there holds

$$\beta(\nu)\|q_h\|_{\nu'} \le \sup_{\boldsymbol{w}\in \boldsymbol{W}_0^{1,\nu}(\Omega)} \frac{(\nabla\cdot \boldsymbol{j}_h\boldsymbol{w}, q_h)_\Omega\|\nabla \boldsymbol{j}_h\boldsymbol{w}\|_\nu}{\|\nabla \boldsymbol{j}_h\boldsymbol{w}\|_\nu\|\nabla \boldsymbol{w}\|_\nu} + \sup_{\boldsymbol{w}\in \boldsymbol{W}_0^{1,\nu}(\Omega)} \frac{(\nabla\cdot (\boldsymbol{w}-\boldsymbol{j}_h\boldsymbol{w}), q_h)_\Omega}{\|\nabla \boldsymbol{w}\|_\nu}, \tag{4.31}$$

where $\boldsymbol{j}_h : \boldsymbol{W}_0^{1,\nu}(\Omega) \to \boldsymbol{\mathcal{X}}_h^\nu$ is the interpolation operator of Lemma 4.4. Using integration by parts, the orthogonality of $\boldsymbol{j}_h$ with respect to $\boldsymbol{Y}_h$ (note $\boldsymbol{P}_h\nabla q_h \in \boldsymbol{Y}_h$), and Hölder's inequality, we deduce that

$$\begin{aligned}
|(\nabla\cdot(\boldsymbol{w}-\boldsymbol{j}_h\boldsymbol{w}), q_h)_\Omega| &= |(\boldsymbol{w}-\boldsymbol{j}_h\boldsymbol{w}, \nabla q_h)_\Omega| = |(\boldsymbol{w}-\boldsymbol{j}_h\boldsymbol{w}, \nabla q_h - \boldsymbol{P}_h\nabla q_h)_\Omega| \\
&\le \sum_{M\in\mathbb{M}_h} h_M^{-1}\|\boldsymbol{w}-\boldsymbol{j}_h\boldsymbol{w}\|_{\nu;M} h_M \|\boldsymbol{\theta}_h(\nabla q_h)\|_{\nu';M} \\
&\le c\Big(\sum_{M\in\mathbb{M}_h} h_M^{-\nu}\|\boldsymbol{w}-\boldsymbol{j}_h\boldsymbol{w}\|_{\nu;M}^\nu\Big)^{\frac{1}{\nu}}\Big(\sum_{M\in\mathbb{M}_h} h_M^{\nu'}\|\boldsymbol{\theta}_h(\nabla q_h)\|_{\nu';M}^{\nu'}\Big)^{\frac{1}{\nu'}}.
\end{aligned}$$

Due to the interpolation property of $\boldsymbol{j}_h$, the inequality

$$|(\nabla\cdot(\boldsymbol{w}-\boldsymbol{j}_h\boldsymbol{w}), q_h)_\Omega| \le c\|\nabla \boldsymbol{w}\|_\nu \Big(\sum_{M\in\mathbb{M}_h} h_M^{\nu'}\|\boldsymbol{\theta}_h(\nabla q_h)\|_{\nu';M}^{\nu'}\Big)^{\frac{1}{\nu'}} \tag{4.32}$$

follows. Using the stability of the interpolation operator (4.29), we conclude that the first term on the right-hand side of (4.31) can be estimated by

$$\sup_{\boldsymbol{w}\in \boldsymbol{W}_0^{1,\nu}(\Omega)} \frac{(\nabla\cdot \boldsymbol{j}_h\boldsymbol{w}, q_h)_\Omega\|\nabla \boldsymbol{j}_h\boldsymbol{w}\|_\nu}{\|\nabla \boldsymbol{j}_h\boldsymbol{w}\|_\nu\|\nabla \boldsymbol{w}\|_\nu} \le c \sup_{\boldsymbol{w}_h\in \boldsymbol{\mathcal{X}}_h^\nu} \frac{(\nabla\cdot \boldsymbol{w}_h, q_h)_\Omega}{\|\nabla \boldsymbol{w}_h\|_\nu}. \tag{4.33}$$

Combining (4.31), (4.32), (4.33), we get the desired estimate (4.30). □

Now, we are in a position to show the well-posedness of the discrete system. In the case of stable discretizations ($s_h \equiv 0$), the existence of unique solutions to the finite element equations can be proven similarly as in the continuous case. In our situation however, the proof of existence requires a different approach since it is not possible to decouple the nonlinear finite element system by restricting to discrete divergence-free test functions.

Lemma 4.6. *For $p \in (1,\infty)$ and $\varepsilon \in [0,\varepsilon_0]$ let $\boldsymbol{S}$ satisfy Assumption 2.1. Let s_h be defined by* (4.4) *with $s \in [0,p']$ and $\tau \in [0,\tau_0]$. Then there exists a solution to* $(\mathbf{P1}_h)$. *Any such solution $(\boldsymbol{v}_h,\pi_h) \in \boldsymbol{\mathcal{X}}_h^p \times \mathcal{Q}_h^p$ satisfies the a priori estimate*

$$\begin{aligned}
\|\boldsymbol{v}_h\|_{1,p}^p + \|\boldsymbol{S}(\boldsymbol{D}\boldsymbol{v}_h)\|_{p'}^{p'} + s_h(\pi_h)(\pi_h) &\le C_1(\Omega,p,\varepsilon_0,\sigma_0,\sigma_1,\boldsymbol{f}), \\
\tilde{\beta}(p)\|\pi_h\|_{p'} &\le C_2(\Omega,p,\varepsilon_0,\sigma_0,\sigma_1,\boldsymbol{f},\alpha_0,\tau_0),
\end{aligned} \tag{4.34}$$

where $\tilde{\beta}(p) > 0$ *is the constant appearing in* (4.30)*. The constants* C_1 *and* C_2 *only depend on the data quoted within the brackets. If* $p \le 2$ *then the constant* C_2 *does not depend on* τ_0*, whereas if* $p > 2$ *the constant* C_1 *does not depend on* ε_0*.*

Proof. In order to show the existence of a solution we consider the following auxiliary problem: for $\delta > 0$ find $(\boldsymbol{v}_h^\delta, \pi_h^\delta) \in \boldsymbol{\mathcal{X}}_h^p \times \mathcal{Q}_h^p$ such that

$$\begin{aligned} (\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}_h^\delta), \boldsymbol{D}\boldsymbol{w}_h)_\Omega - (\pi_h^\delta, \nabla \cdot \boldsymbol{w}_h)_\Omega &= (\boldsymbol{f}, \boldsymbol{w}_h)_\Omega \qquad && \forall \boldsymbol{w}_h \in \boldsymbol{\mathcal{X}}_h^p \\ (\nabla \cdot \boldsymbol{v}_h^\delta, q_h)_\Omega + s_h(\pi_h^\delta)(q_h) + \delta(\pi_h^\delta, q_h)_\Omega &= 0 && \forall q_h \in \mathcal{Q}_h^p. \end{aligned} \tag{4.35}$$

The additional term $\delta(\pi_h^\delta, q_h)_\Omega$ ensures that the nonlinear operator associated with the left-hand side of (4.35) is coercive in $\boldsymbol{\mathcal{X}}_h^p \times \mathcal{Q}_h^p$. Due to Lemma 2.4 and Lemma 4.2, this operator is strictly monotone and continuous in $\boldsymbol{\mathcal{X}}_h^p \times \mathcal{Q}_h^p$. Applying the theory of monotone operators, we conclude the existence of a unique solution $(\boldsymbol{v}_h^\delta, \pi_h^\delta) \in \boldsymbol{\mathcal{X}}_h^p \times \mathcal{Q}_h^p$ to (4.35) for each $\delta > 0$. Next we show that this solution satisfies an a priori bound independent of δ. We begin with the case $p \le 2$. Setting $\boldsymbol{w}_h := \boldsymbol{v}_h^\delta$ and $q_h := \pi_h^\delta$ in (4.35), summing both equations, using (2.40), Hölder's, Poincaré's, Korn's and Young's inequality, we conclude

$$\|\boldsymbol{v}_h^\delta\|_{1,p}^p + \|\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}_h^\delta)\|_{p'}^{p'} + s_h(\pi_h^\delta)(\pi_h^\delta) + \delta\|\pi_h^\delta\|_2^2 \le C_1 = C_1(\Omega, p, \sigma_0, \sigma_1, \varepsilon_0, \boldsymbol{f}). \tag{4.36}$$

Utilizing the discrete inf-sup inequality (4.30), equation $(4.35)_1$, the condition $p' \ge 2$, we can estimate the discrete pressure π_h^δ with respect to the $L^{p'}(\Omega)$-norm as follows:

$$\begin{aligned} \tilde{\beta}(p)\|\pi_h^\delta\|_{p'} &\le \sup_{\boldsymbol{w}_h \in \boldsymbol{\mathcal{X}}_h^p} \frac{(\nabla \cdot \boldsymbol{w}_h, \pi_h^\delta)_\Omega}{\|\nabla \boldsymbol{w}_h\|_p} + \Big(\sum_{M \in \mathbb{M}_h} h_M^{p'} \|\boldsymbol{\theta}_h \nabla \pi_h^\delta\|_{p';M}^{p'} \Big)^{\frac{1}{p'}} \\ &\le \sup_{\boldsymbol{w}_h \in \boldsymbol{\mathcal{X}}_h^p} \frac{(\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}_h^\delta), \boldsymbol{D}\boldsymbol{w}_h)_\Omega - (\boldsymbol{f}, \boldsymbol{w}_h)_\Omega}{\|\nabla \boldsymbol{w}_h\|_p} \\ &\quad + C h^{1-\frac{s}{p'}} \Big(\sum_{M \in \mathbb{M}_h} \alpha_M \int_M (\tau + |\boldsymbol{\theta}_h \nabla \pi_h^\delta|)^{p'-2} |\boldsymbol{\theta}_h \nabla \pi_h^\delta|^2 \,\mathrm{d}\boldsymbol{x} \Big)^{\frac{1}{p'}} \\ &\le C\Big(\|\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}_h^\delta)\|_{p'} + \|\boldsymbol{f}\|_{p'} + h^{1-\frac{s}{p'}} s_h(\pi_h^\delta)(\pi_h^\delta)^{\frac{1}{p'}} \Big), \end{aligned}$$

where $C = C(\Omega, p, \sigma_0, \sigma_1, \alpha_0)$. Since we have assumed $s \le p'$, the a priori $L^{p'}(\Omega)$-bound

$$\tilde{\beta}(p)\|\pi_h^\delta\|_{p'} \le C_2 = C_2(\Omega, p, \sigma_0, \sigma_1, \varepsilon_0, \boldsymbol{f}, \alpha_0) \tag{4.37}$$

follows. The constants C_1 and C_2 do not depend on δ. In the case $p \in (2, \infty)$, the proof of (4.36) is similar whereas the proof of (4.37) requires slightly different arguments. Below we depict the proof of (4.37) for $p > 2$. Using Lemma 4.1 and Hölder's inequality with

$\frac{p'}{2} + \frac{2-p'}{2} = 1$, we deduce that for all $q_h \in \mathcal{Q}_h^p$

$$\begin{aligned} A &:= \Big(\sum_{M \in \mathbb{M}_h} h_M^{p'} \|\boldsymbol{\theta}_h(\nabla q_h)\|_{p';M}^{p'} \Big)^{\frac{1}{p'}} \\ &\lesssim \Big(\sum_{M \in \mathbb{M}_h} h_M^{\frac{p's}{2}} \|\boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h \nabla q_h)\|_{2;M}^{p'} h_M^{\frac{p'(2-s)}{2}} \Big(\tau_0 |M|^{\frac{1}{p'}} + \|\boldsymbol{\theta}_h \nabla q_h\|_{p';M}\Big)^{\frac{2-p'}{2}p'} \Big)^{\frac{1}{p'}} \\ &\lesssim \Big[\sum_M h_M^s \|\boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h \nabla q_h)\|_{2;M}^2 \Big]^{\frac{1}{2}} \Big[\sum_M h_M^{\frac{p'(2-s)}{2-p'}} \Big(\tau_0^{p'} |M| + \|\boldsymbol{\theta}_h \nabla q_h\|_{p';M}^{p'}\Big) \Big]^{\frac{2-p'}{2p'}} . \end{aligned}$$

Since $s \le p' \Leftrightarrow \frac{p'(2-s)}{2-p'} \ge p'$, in view of Lemma 4.2 we arrive at

$$A \lesssim s_h(q_h)(q_h)^{\frac{1}{2}} \Big(h\tau_0 |\Omega|^{\frac{1}{p'}} + A\Big)^{\frac{2-p'}{2}} \qquad \forall q_h \in \mathcal{Q}_h^p.$$

If $A \ge h\tau_0|\Omega|^{\frac{1}{p'}}$, then $A \lesssim s_h(q_h)(q_h)^{\frac{1}{2}} A^{\frac{2-p'}{2}}$ and, hence, $A \lesssim s_h(q_h)(q_h)^{\frac{1}{p'}}$. As a result, we conclude that there exists a constant $c = c(p, \alpha_0) > 0$ such that

$$\Big(\sum_{M \in \mathbb{M}_h} h_M^{p'} \|\boldsymbol{\theta}_h(\nabla q_h)\|_{p';M}^{p'} \Big)^{\frac{1}{p'}} \le c s_h(q_h)(q_h)^{\frac{1}{p'}} + h\tau_0 |\Omega|^{\frac{1}{p'}} \tag{4.38}$$

for all $q_h \in \mathcal{Q}_h^p$. Using (4.30), (4.38), (4.35), Lemma 2.4, the Hölder and Poincaré inequality, we can estimate the discrete pressure π_h^δ as follows:

$$\begin{aligned} \tilde{\beta} \|\pi_h^\delta\|_{p'} &\le \sup_{\boldsymbol{w}_h \in \boldsymbol{\mathcal{X}}_h^p} \frac{(\nabla \cdot \boldsymbol{w}_h, \pi_h^\delta)_\Omega}{\|\nabla \boldsymbol{w}_h\|_p} + \Big(\sum_{M \in \mathbb{M}_h} h_M^{p'} \|\boldsymbol{\theta}_h \nabla \pi_h^\delta\|_{p';M}^{p'} \Big)^{\frac{1}{p'}} \\ &\le C\Big(\|\varepsilon_0 + |\boldsymbol{D}\boldsymbol{v}_h^\delta|\|_p^{p-1} + \|\boldsymbol{f}\|_{p'} + s_h(\pi_h^\delta)(\pi_h^\delta)^{\frac{1}{p'}} + \tau_0 \Big) \end{aligned}$$

for some $C = C(\Omega, p, \alpha_0) > 0$. We finally arrive at (4.37) for $p > 2$. We note that the constant C_2 in (4.37) additionally depends on τ_0 in this case.

For $\{\delta_k\}_{k \in \mathbb{N}}$ with $\delta_k \searrow 0$ let $(\boldsymbol{v}_h^{\delta_k}, \pi_h^{\delta_k})$ be the solutions to (4.35). Due to the uniform a priori estimates (4.36) and (4.37) there exists $(\boldsymbol{v}_h, \pi_h) \in \boldsymbol{\mathcal{X}}_h^p \times \mathcal{Q}_h^p$ such that a subsequence (which is again denoted by $(\boldsymbol{v}_h^{\delta_k}, \pi_h^{\delta_k})$) converges to $(\boldsymbol{v}_h, \pi_h)$ strongly:

$$\boldsymbol{D}\boldsymbol{v}_h^{\delta_k} \to \boldsymbol{D}\boldsymbol{v}_h \text{ in } L^p(\Omega) \qquad \text{and} \qquad \pi_h^{\delta_k} \to \pi_h \text{ in } L^{p'}(\Omega) \text{ for } k \to \infty.$$

By passing to the limit in (4.35), we show that $(\boldsymbol{v}_h, \pi_h)$ is the solution of $(\mathbf{P1}_h)$. We can pass to the limit in the nonlinear term $(\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}_h^{\delta_k}), \boldsymbol{D}\boldsymbol{w}_h)_\Omega$ using the following arguments: first of all, we observe that we can find a further subsequence of $\{\boldsymbol{v}_h^{\delta_k}\}$ (for simplicity we do not change the notation) such that $\boldsymbol{D}\boldsymbol{v}_h^{\delta_k} \to \boldsymbol{D}\boldsymbol{v}_h$ almost everywhere in Ω for $k \to \infty$. Thus, $\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}_h^{\delta_k}) \to \boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}_h)$ almost everywhere in Ω for $k \to \infty$ since $\boldsymbol{\mathcal{S}}$ is continuous. In view of $(2.40)_2$, Vitali's theorem then implies that

$$\int_\Omega \boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}_h^{\delta_k}) : \boldsymbol{D}\boldsymbol{w}_h \,\mathrm{d}\boldsymbol{x} \;\to\; \int_\Omega \boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}_h) : \boldsymbol{D}\boldsymbol{w}_h \,\mathrm{d}\boldsymbol{x} \qquad (k \to \infty).$$

In the stabilization term $s_h(\pi_h^{\delta_k})(q_h)$ we can pass to the limit using exactly the same arguments. This is possible since the inverse inequality (3.19) implies $\nabla\pi_h^{\delta_k}|_M \to \nabla\pi_h|_M$ in $L^{p'}(M)$ and, consequently, $\boldsymbol{\theta}_h(\nabla\pi_h^{\delta_k})|_M \to \boldsymbol{\theta}_h(\nabla\pi_h)|_M$ in $L^{p'}(M)$ $(k\to\infty)$ for all $M \in \mathbb{M}_h$ due to the continuity of the fluctuation operator $\boldsymbol{\theta}_h$. In the remaining terms we can pass to the limit using standard arguments. Consequently, the limit $(\boldsymbol{v}_h, \pi_h) \in \boldsymbol{\mathcal{X}}_h^p \times \mathcal{Q}_h^p$ solves system (4.5). As a result, $(\boldsymbol{v}_h, \pi_h)$ is a solution to $(\mathbf{P1}_h)$ and it satisfies (4.34). $\square$

Lemma 4.7 (Uniqueness). *For $p \in (1,\infty)$ and $\varepsilon \in [0,\varepsilon_0]$ let $\boldsymbol{\mathcal{S}}$ satisfy Assumption 2.1 and let s_h be defined by (4.4). If a solution $(\boldsymbol{v}_h, \pi_h) \in \boldsymbol{\mathcal{X}}_h^p \times \mathcal{Q}_h^p$ to Problem $(\mathbf{P1}_h)$ exists, then $(\boldsymbol{v}_h, \pi_h)$ is uniquely determined.*

Proof. Assume that $(\boldsymbol{v}_h^i, \pi_h^i) \in \boldsymbol{\mathcal{X}}_h^p \times \mathcal{Q}_h^p$, $i \in \{1,2\}$, are two solutions to Problem $(\mathbf{P1}_h)$. Setting $\boldsymbol{\xi}_h := (\boldsymbol{v}_h^1 - \boldsymbol{v}_h^2)$ and $\eta_h := (\pi_h^1 - \pi_h^2)$, we observe that

$$
\begin{aligned}
(\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}_h^1) - \boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}_h^2), \boldsymbol{D}\boldsymbol{w}_h)_\Omega - (\eta_h, \nabla\cdot\boldsymbol{w}_h)_\Omega + (\nabla\cdot\boldsymbol{\xi}_h, q_h)_\Omega& \\
+ s_h(\pi_h^1)(q_h) - s_h(\pi_h^2)(q_h) = 0&
\end{aligned}
\tag{4.39}
$$

for all $(\boldsymbol{w}_h, q_h) \in \boldsymbol{\mathcal{X}}_h^p \times \mathcal{Q}_h^p$. Testing (4.39) with $\boldsymbol{w}_h := \boldsymbol{\xi}_h$ and $q_h := \eta_h$, we conclude that

$$
(\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}_h^1) - \boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}_h^2), \boldsymbol{D}\boldsymbol{\xi}_h)_\Omega = 0 \qquad \text{and} \qquad s_h(\pi_h^1)(\eta_h) - s_h(\pi_h^2)(\eta_h) = 0,
$$

and, hence, $\boldsymbol{v}_h^1 = \boldsymbol{v}_h^2$ and $\boldsymbol{\theta}_h\nabla\pi_h^1 = \boldsymbol{\theta}_h\nabla\pi_h^2$ due to the strict monotonicity. Utilizing Lemma 4.5 (with $q_h = \eta_h$), (4.39), $\boldsymbol{v}_h^1 = \boldsymbol{v}_h^2$, and $\boldsymbol{\theta}_h\nabla\eta_h = 0$, we arrive at $\pi_h^1 = \pi_h^2$. $\square$

4.5 Error estimates for the proposed stabilization scheme

In this section we derive a priori error estimates such as (4.1) which quantify the convergence of the LPS method. Numerical experiments will indicate that the derived error estimates are optimal at least in the shear thinning case. Note that for p-Laplace systems optimal error estimates have been proven in Diening/Růžička [DR07]. The key idea in the analysis is to estimate the approximation error with respect to quasi-norms that naturally arise in degenerate problems of this type (cf. Barrett/Liu [BL94]). In order to derive our sharp error estimates, we combine both the quasi-norm technique and the well-known analysis of LPS for Stokes systems. For derivation of error estimates, we distinguish between the cases $p \le 2$ (Theorem 4.11) and $p \ge 2$ (Theorem 4.12) due to technical reasons.

Remarks on the regularity of the solution: In order to derive a priori error estimates, we need to require additional regularity of the solution. In particular, we will assume the natural regularity $\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) \in W^{1,2}(\Omega)^{d\times d}$ and $\pi \in W^{1,p'}(\Omega)$ which is available for sufficiently smooth data. The question arises which information on the second derivatives of $\boldsymbol{v}$ can be extracted from the first derivatives of $\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})$? This will be answered by the following known lemmas. Although we do not need all of the following results for the purpose of this section, we will use them in the further course of the thesis and, hence, we will present them here for sake of completeness.

Lemma 4.8. *For $p \in (1,2]$ and $\varepsilon \in (0,\infty)$ let $\mathcal{I}$ be defined by (2.82). Then, for all $q \in [1,2]$ and for all sufficiently smooth $\boldsymbol{v}$ there holds*

$$\|\nabla^2 \boldsymbol{v}\|_q \le c\mathcal{I}(\boldsymbol{v})^{\frac{1}{2}} \|(\varepsilon + |\boldsymbol{D}\boldsymbol{v}|)^{\frac{2-p}{2}}\|_{\frac{2q}{2-q}} \tag{4.40}$$

where $\frac{2q}{2-q} = \infty$ for $q = 2$. The constant c only depends on p.

Proof. We refer to [DR05] and [BDR10]. □

Lemma 4.9. *Let $p \in (1,2)$ and $\varepsilon \in (0,\infty)$. There exists $c = c(p) > 0$ such that*

$$\|\nabla^2 \boldsymbol{v}\|_p^p \le c\Big(\|\nabla \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})\|_2^2 + \|\varepsilon + |\boldsymbol{D}\boldsymbol{v}|\|_p^p\Big). \tag{4.41}$$

Proof. Cf. [DR05]. Setting $\Omega_0 := \Big\{\boldsymbol{x} \in \Omega;\ (\varepsilon + |\boldsymbol{D}\boldsymbol{v}(\boldsymbol{x})|) \le |\nabla^2 \boldsymbol{v}(\boldsymbol{x})|\Big\}$, we estimate

$$\begin{aligned}
\int_\Omega |\nabla^2 \boldsymbol{v}|^p \,\mathrm{d}\boldsymbol{x} &= \int_{\Omega_0} |\nabla^2 \boldsymbol{v}|^{p-2} |\nabla^2 \boldsymbol{v}|^2 \,\mathrm{d}\boldsymbol{x} + \int_{\Omega\setminus\Omega_0} |\nabla^2 \boldsymbol{v}|^p \,\mathrm{d}\boldsymbol{x} \\
&\le \int_{\Omega_0} (\varepsilon + |\boldsymbol{D}\boldsymbol{v}|)^{p-2} |\nabla^2 \boldsymbol{v}|^2 \,\mathrm{d}\boldsymbol{x} + \int_{\Omega\setminus\Omega_0} (\varepsilon + |\boldsymbol{D}\boldsymbol{v}|)^p \,\mathrm{d}\boldsymbol{x} \le c\mathcal{I}(\boldsymbol{v}) + \|\varepsilon + |\boldsymbol{D}\boldsymbol{v}|\|_p^p.
\end{aligned}$$

Then, (4.41) follows from Lemma 2.27. □

Lemma 4.10. *Let $p \in (1,2)$ and $\varepsilon \in (0,\infty)$. Then for all sufficiently smooth $\boldsymbol{v}$ there holds*

$$\|\boldsymbol{v}\|_{2,q}^p \le c\Big(\|\nabla \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})\|_2^2 + \|\nabla \boldsymbol{v}\|_p^p + \varepsilon^p\Big), \tag{4.42}$$

where $q = 2 - \delta$ for arbitrary $\delta \in (0, 2-p]$ if $d = 2$ and $q = \frac{3p}{p+1}$ if $d = 3$. The constant c only depends on p and Ω. In particular, it is independent of ε.

Proof. We refer to [DR05]. There, the assertion is proven for $d = 3$, $\varepsilon = 1$, and for all $\boldsymbol{v}$ with $(\boldsymbol{v}, 1)_\Omega = 0$. For the proof of (4.42) we apply Lemma 4.8:

$$\|\nabla^2 \boldsymbol{v}\|_q \le c\mathcal{I}(\boldsymbol{v})^{\frac{1}{2}} \|(\varepsilon + |\boldsymbol{D}\boldsymbol{v}|)^{\frac{2-p}{2}}\|_{\frac{2q}{2-q}} = c\mathcal{I}(\boldsymbol{v})^{\frac{1}{2}} \|(\varepsilon + |\boldsymbol{D}\boldsymbol{v}|)\|_{\frac{(2-p)q}{2-q}}^{\frac{2-p}{2}}.$$

In order to ensure $W^{1,q}(\Omega) \hookrightarrow L^{\frac{(2-p)q}{2-q}}(\Omega)$, we have to require that $1 - \frac{d}{q} \ge -\frac{d(2-q)}{(2-p)q}$. This condition is satisfied if $q = 2 - \delta$ ($d = 2$) and $q = \frac{3p}{p+1}$ ($d = 3$). For such q, we get

$$\|\nabla^2 \boldsymbol{v}\|_q \le c\mathcal{I}(\boldsymbol{v})^{\frac{1}{2}} \Big(\varepsilon + \|\boldsymbol{D}\boldsymbol{v}\|_{\frac{(2-p)q}{2-q}}\Big)^{\frac{2-p}{2}} \le c\mathcal{I}(\boldsymbol{v})^{\frac{1}{2}} \Big(\varepsilon + \|\boldsymbol{D}\boldsymbol{v}\|_q + \|\nabla \boldsymbol{D}\boldsymbol{v}\|_q\Big)^{\frac{2-p}{2}}.$$

If $\|\nabla \boldsymbol{D}\boldsymbol{v}\|_q \ge \varepsilon + \|\boldsymbol{D}\boldsymbol{v}\|_q$, the latter inequality implies $\|\nabla^2 \boldsymbol{v}\|_q^p \le c\mathcal{I}(\boldsymbol{v})$. Otherwise, due to Sobolev's embedding theorem, for $q = 2 - \delta$ ($d = 2$) and $q = \frac{3p}{p+1}$ ($d = 3$) we conclude that

$$\begin{aligned}
\|\nabla \boldsymbol{D}\boldsymbol{v}\|_q &\le \varepsilon + \|\boldsymbol{D}\boldsymbol{v}\|_q \le c\Big(\varepsilon + \|\boldsymbol{D}\boldsymbol{v}\|_p + \|\nabla \boldsymbol{D}\boldsymbol{v}\|_p\Big) \\
&\le c\Big(\varepsilon + \|\boldsymbol{D}\boldsymbol{v}\|_p + c(\mathcal{I}(\boldsymbol{v}) + \|\varepsilon + |\boldsymbol{D}\boldsymbol{v}|\|_p^p)^{\frac{1}{p}}\Big).
\end{aligned}$$

For the latter estimate we have used (4.41). Summing up we arrive at

$$\|\nabla^2 \boldsymbol{v}\|_q^p \le c\Big(\varepsilon^p + \mathcal{I}(\boldsymbol{v}) + \|\varepsilon + |\boldsymbol{D}\boldsymbol{v}|\|_p^p\Big).$$

This proves the lemma. □

Since all constants appearing in Lemmas 4.8 – 4.10 do not depend on $\varepsilon \in (0,\infty)$, one can show that the inequalities (4.40) – (4.42) remain true for $\varepsilon = 0$.

A priori error estimates - case $p < 2$: In the shear thinning case the following theorem provides a priori error estimates which improve previous results concerning the rate of convergence or the assumed regularity of the solution (cf. Barrett/Liu [BL93b] and [BL94]). Numerical experiments indicate that these error estimates are optimal.

Theorem 4.11. *For $p \in (1,2]$ and $\varepsilon \in [0,\varepsilon_0]$ let $\boldsymbol{\mathcal{S}}$ satisfy Assumption 2.1 and let $\boldsymbol{\mathcal{F}}$ be defined by* (2.39)*. Let $(\boldsymbol{v},\pi) \in \boldsymbol{\mathcal{X}}^p \times \mathcal{Q}^p$ be the unique solution to* **(P1)** *and let $(\boldsymbol{v}_h,\pi_h) \in \boldsymbol{\mathcal{X}}_h^p \times \mathcal{Q}_h^p$ be the unique solution to* **(P1$_h$)** *where the stabilization term s_h is defined by* (4.4)*. We assume the additional regularity $\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) \in W^{1,2}(\Omega)^{d\times d}$ and $\pi \in W^{1,p'}(\Omega)$ with $1/p + 1/p' = 1$. Then, for $\alpha_M := \alpha_0 h_M^s$ with $\alpha_0 > 0$ and $s = 2$ the error of approximation is estimated in terms of $h := \max\{h_M;\, M \in \mathbb{M}_h\}$ as follows:*

$$\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2 \le C_v h, \qquad \|\boldsymbol{v} - \boldsymbol{v}_h\|_{1,p} \le C_v' h, \tag{4.43}$$

$$\|\pi - \pi_h\|_{p'} \le C_\pi h^{\frac{2}{p'}}. \tag{4.44}$$

The constants C_v, C_v', $C_\pi > 0$ only depend on p, ε_0, σ_0, σ_1, Ω, $\boldsymbol{f}$, α_0, τ_0, $\|\nabla\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})\|_2$, $\|\pi\|_{1,p'}$, and C_π additionally depends on $\tilde\beta(p)$.

Proof. Let j_h and $\boldsymbol{j}_h$ be the interpolation operators of Lemma 4.4. We begin with the proof of (4.43). We split the error $(\boldsymbol{v} - \boldsymbol{v}_h)$ in an interpolation part and a projection part:

$$\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2^2 \lesssim \|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{v})\|_2^2 + \|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2^2.$$

According to Lemma 4.4 the desired estimate holds for the interpolation error:

$$\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{v})\|_2^2 \lesssim h^2 \|\nabla\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})\|_2^2. \tag{4.45}$$

Thus it is sufficient to estimate the projection error $\boldsymbol{\xi}_h := (\boldsymbol{j}_h\boldsymbol{v} - \boldsymbol{v}_h)$ and $\eta_h := (j_h\pi - \pi_h)$ with respect to the following quantity,

$$|(\boldsymbol{\xi}_h,\eta_h)|_{\text{lps}}^2 := \|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2^2 + \sum_{M\in\mathbb{M}_h} \alpha_M \|\boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h\nabla j_h\pi) - \boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h\nabla\pi_h)\|_{2;M}^2, \tag{4.46}$$

where $\boldsymbol{\mathcal{G}}$ is defined by (4.6). Applying Lemma 2.4 and Lemma 4.2, we conclude

$$|(\boldsymbol{\xi}_h,\eta_h)|_{\text{lps}}^2 \sim (\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{v}) - \boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}_h), \boldsymbol{D}\boldsymbol{\xi}_h)_\Omega + s_h(j_h\pi)(\eta_h) - s_h(\pi_h)(\eta_h).$$

Adding the following trivial identity

$$0 = -(\pi - \pi_h, \nabla \cdot \boldsymbol{\xi}_h)_\Omega + (\nabla \cdot (\boldsymbol{v} - \boldsymbol{v}_h), \eta_h)_\Omega - (j_h\pi - \pi, \nabla \cdot \boldsymbol{\xi}_h)_\Omega + (\nabla \cdot (\boldsymbol{j}_h\boldsymbol{v} - \boldsymbol{v}), \eta_h)_\Omega$$

and using the disturbed Galerkin orthogonality

$$(\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}_h), \boldsymbol{D}\boldsymbol{w}_h)_\Omega - (\pi - \pi_h, \nabla \cdot \boldsymbol{w}_h)_\Omega + (\nabla \cdot (\boldsymbol{v} - \boldsymbol{v}_h), q_h)_\Omega = s_h(\pi_h)(q_h)$$

for all $\boldsymbol{w}_h \in \boldsymbol{\mathcal{X}}_h^p$ and $q_h \in \mathcal{Q}_h^p$, we obtain

$$\begin{aligned} |(\boldsymbol{\xi}_h, \eta_h)|^2_{\text{lps}} \sim s_h(j_h\pi)(\eta_h) + (\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{v}) - \boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{\xi}_h)_\Omega - (j_h\pi - \pi, \nabla \cdot \boldsymbol{\xi}_h)_\Omega \\ + (\nabla \cdot (\boldsymbol{j}_h\boldsymbol{v} - \boldsymbol{v}), \eta_h)_\Omega =: I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{4.47}$$

We consider the terms of (4.47) separately. Applying Lemma 4.3, and the stability of the interpolation operator j_h, for arbitrary $\delta_1 > 0$ we can bound the first term I_1 by

$$\begin{aligned} I_1 &\leq c_{\delta_1} h^s \|\tau + |\nabla j_h\pi|\|_{p'}^{p'} + \delta_1 \sum_{M\in\mathbb{M}_h} \alpha_M \|\boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h\nabla j_h\pi) - \boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h\nabla\pi_h)\|^2_{2;M} \\ &\leq c_{\delta_1} h^s \Big[\tau_0^{p'}|\Omega| + \|\pi\|_{1,p'}^{p'}\Big] + \delta_1 |(\boldsymbol{\xi}_h, \eta_h)|^2_{\text{lps}} \end{aligned} \tag{4.48}$$

where c_{δ_1} only depends on p, α_0 and δ_1. Let φ and φ_a be defined by (2.37) and (2.32). Applying Lemma 2.4 twice, using the Young-type inequality (2.36) and interpolation inequality (4.45), for arbitrary $\delta_2 > 0$ we estimate the second term I_2 in (4.47) as follows,

$$\begin{aligned} I_2 &\leq c \int_\Omega \varphi'_{\varepsilon+|\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{v}|}(|\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{v} - \boldsymbol{D}\boldsymbol{v}|)|\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{v} - \boldsymbol{D}\boldsymbol{v}_h| \,\mathrm{d}\boldsymbol{x} \\ &\leq \delta_2 c \int_\Omega \varphi_{\varepsilon+|\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{v}|}(|\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{v} - \boldsymbol{D}\boldsymbol{v}_h|) \,\mathrm{d}\boldsymbol{x} + c_{\delta_2} \int_\Omega \varphi_{\varepsilon+|\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{v}|}(|\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{v} - \boldsymbol{D}\boldsymbol{v}|) \,\mathrm{d}\boldsymbol{x} \\ &\sim \delta_2 c \|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2^2 + c_{\delta_2}\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})\|_2^2 \\ &\leq \delta_2 c |(\boldsymbol{\xi}_h, \eta_h)|^2_{\text{lps}} + c_{\delta_2} h^2 \|\nabla\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})\|_2^2 \end{aligned} \tag{4.49}$$

where c_{δ_2} only depends on p, σ_0, σ_1 and δ_2. Next we estimate I_3. Using Hölder's and Young's inequality, we deduce that for each $\delta_3 > 0$ there exists a constant c_{δ_3} such that

$$I_3 \leq \Big|(\pi - j_h\pi, \nabla \cdot \boldsymbol{\xi}_h)_\Omega\Big| \leq \|\pi - j_h\pi\|_{p'}\|\nabla\boldsymbol{\xi}_h\|_p \leq \delta_3\|\nabla(\boldsymbol{j}_h\boldsymbol{v} - \boldsymbol{v}_h)\|_p^2 + c_{\delta_3}\|\pi - j_h\pi\|_{p'}^2.$$

Utilizing Korn's inequality, Lemma 2.6 (i) and Lemma 2.4, we conclude that

$$I_3 \leq \delta_3 c\Big[\varepsilon_0|\Omega|^{\frac{1}{p}} + \|\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{v}\|_p + \|\boldsymbol{D}\boldsymbol{v}_h\|_p\Big]^{2-p}\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2^2 + c_{\delta_3}\|\pi - j_h\pi\|_{p'}^2.$$

From (4.29), (2.63), and (4.34) it follows that the expression within the square brackets is uniformly bounded by a constant $c = c(\Omega, p, \varepsilon_0, \boldsymbol{f})$. Consequently, we get the estimate

$$I_3 \leq \delta_3 c |(\boldsymbol{\xi}_h, \eta_h)|^2_{\text{lps}} + c_{\delta_3} h^2 \|\pi\|_{1,p'}^2. \tag{4.50}$$

In order to estimate the term I_4, we use integration by parts (the discrete pressure is continuous), the orthogonality property of $\boldsymbol{j}_h$ with respect to $\boldsymbol{Y}_h$, Hölder's inequality, Young's inequality with $\delta_4 > 0$, Lemma 4.1, and the interpolation property of $\boldsymbol{j}_h$, such that

$$\begin{aligned}
I_4 &\leq \left|(\nabla\cdot(\boldsymbol{j}_h\boldsymbol{v}-\boldsymbol{v}),\eta_h)_\Omega\right| = \left|(\boldsymbol{j}_h\boldsymbol{v}-\boldsymbol{v},\nabla\eta_h)_\Omega\right| = \left|(\boldsymbol{j}_h\boldsymbol{v}-\boldsymbol{v},\boldsymbol{\theta}_h(\nabla\eta_h))_\Omega\right| \\
&\leq \sum_{M\in\mathbb{M}_h} \alpha_M^{-\frac{1}{p'}}\|\boldsymbol{j}_h\boldsymbol{v}-\boldsymbol{v}\|_{p;M}\alpha_M^{\frac{1}{p'}}\|\boldsymbol{\theta}_h(\nabla\eta_h)\|_{p';M} \\
&\leq c_{\delta_4}\sum_{M\in\mathbb{M}_h} \alpha_M^{-(p-1)}\|\boldsymbol{j}_h\boldsymbol{v}-\boldsymbol{v}\|_{p;M}^p + \delta_4\sum_{M\in\mathbb{M}_h}\alpha_M\|\boldsymbol{\theta}_h(\nabla\eta_h)\|_{p';M}^{p'} \\
&\leq c_{\delta_4}\sum_{M\in\mathbb{M}_h} h_M^{2p-s(p-1)}\|\boldsymbol{v}\|_{2,p;S_M}^p + \delta_4 c\sum_{M\in\mathbb{M}_h}\alpha_M\|\boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h\nabla j_h\pi)-\boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h\nabla\pi_h)\|_{2;M}^2 \\
&\leq c_{\delta_4}h^{2p-s(p-1)}\|\boldsymbol{v}\|_{2,p}^p + \delta_4 c|(\boldsymbol{\xi}_h,\eta_h)|_{\mathrm{lps}}^2,
\end{aligned} \tag{4.51}$$

where the constant c_{δ_4} only depends on p, α_0 and δ_4. Combining (4.47) – (4.51), choosing $\delta_1,\ldots,\delta_4$ sufficiently small, and absorbing the involved terms into the left-hand side of (4.47), we conclude that there exists a constant $c = c(p,\varepsilon_0,\boldsymbol{f},\sigma_0,\sigma_1,\alpha_0,\Omega) > 0$ such that

$$\begin{aligned}
|(\boldsymbol{\xi}_h,\eta_h)|_{\mathrm{lps}}^2 \leq c\Big(& h^2\|\nabla\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})\|_2^2 + h^{2p-s(p-1)}\|\boldsymbol{v}\|_{2,p}^p \\
&+ h^2\|\pi\|_{1,p'}^2 + h^s\Big[\tau_0^{p'}|\Omega| + \|\pi\|_{1,p'}^{p'}\Big]\Big).
\end{aligned} \tag{4.52}$$

According to (4.41), the L^p-norm of $\nabla^2\boldsymbol{v}$ can be estimated by the L^2-norm of $\nabla\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})$. In order to ensure the optimal rate of convergence, we have to choose $s = 2$. This proves $(4.43)_1$. Using Poincaré's and Korn's inequality, Lemma 2.6, the uniform a priori estimates (2.63) and (4.34), we finally arrive at

$$\|\boldsymbol{v}-\boldsymbol{v}_h\|_{1,p} \leq c\|\boldsymbol{D}(\boldsymbol{v}-\boldsymbol{v}_h)\|_p \leq c\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})-\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2$$

for some $c = c(p,\varepsilon_0,\Omega,\boldsymbol{f}) > 0$. In view of $(4.43)_1$ this implies $(4.43)_2$.

It remains to prove the pressure-estimate (4.44). We split the discretization error $(\pi-\pi_h)$ in an interpolation part and a projection part:

$$\|\pi-\pi_h\|_{p'} \leq \|\pi-j_h\pi\|_{p'} + \|j_h\pi-\pi_h\|_{p'}.$$

Because the desired result holds for the interpolation error, it is sufficient to estimate the projection error $\eta_h := (j_h\pi-\pi_h)$. From Lemma 4.5 the inequality

$$\tilde{\beta}\|\eta_h\|_{p'} \leq \sup_{\boldsymbol{w}_h\in\boldsymbol{\mathcal{X}}_h^p}\frac{|(\nabla\cdot\boldsymbol{w}_h,\eta_h)_\Omega|}{\|\nabla\boldsymbol{w}_h\|_p} + \Big(\sum_{M\in\mathbb{M}_h} h_M^{p'}\|\boldsymbol{\theta}_h(\nabla\eta_h)\|_{p';M}^{p'}\Big)^{\frac{1}{p'}} =: J_1 + J_2 \tag{4.53}$$

follows. Firstly we estimate J_1. From **(P1)** and **(P1$_h$)** we conclude the identity

$$(j_h\pi-\pi_h,\nabla\cdot\boldsymbol{w}_h)_\Omega = (\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v})-\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}_h),\boldsymbol{D}\boldsymbol{w}_h)_\Omega + (j_h\pi-\pi,\nabla\cdot\boldsymbol{w}_h)_\Omega$$

for all $\boldsymbol{w}_h \in \boldsymbol{\mathcal{X}}_h^p$. Consequently, we obtain the inequality

$$J_1 \leq \sup_{\boldsymbol{w}_h \in \boldsymbol{\mathcal{X}}_h^p} \frac{|(\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}_h), \boldsymbol{D}\boldsymbol{w}_h)_\Omega|}{\|\nabla \boldsymbol{w}_h\|_p} + \sup_{\boldsymbol{w}_h \in \boldsymbol{\mathcal{X}}_h^p} \frac{|(j_h\pi - \pi, \nabla \cdot \boldsymbol{w}_h)_\Omega|}{\|\nabla \boldsymbol{w}_h\|_p}. \tag{4.54}$$

Using Hölder's inequality, (2.48), and the interpolation property of j_h, we deduce that

$$J_1 \lesssim \|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2^{\frac{2}{p'}} + \|j_h\pi - \pi\|_{p'} \lesssim \|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2^{\frac{2}{p'}} + h\|\pi\|_{1,p'}. \tag{4.55}$$

Next we estimate J_2. Recalling inequality (4.52) ($s = 2$), we observe that $|(\boldsymbol{\xi}_h, \eta_h)|_{\text{lps}}^2 = \mathcal{O}(h^2)$. Consequently, by means of Lemma 4.1 we obtain the estimate

$$\begin{aligned} J_2 &= \Big(\sum_{M \in \mathbb{M}_h} h_M^{p'} \|\boldsymbol{\theta}_h \nabla \eta_h\|_{p';M}^{p'}\Big)^{\frac{1}{p'}} \leq c h^{1-\frac{2}{p'}} \Big(\sum_{M \in \mathbb{M}_h} \alpha_M \|\boldsymbol{\theta}_h \nabla j_h\pi - \boldsymbol{\theta}_h \nabla \pi_h\|_{p';M}^{p'}\Big)^{\frac{1}{p'}} \\ &\leq c h^{1-\frac{2}{p'}} \Big(\sum_{M \in \mathbb{M}_h} \alpha_M \|\boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h \nabla j_h \pi) - \boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h \nabla \pi_h)\|_{2;M}^2\Big)^{\frac{1}{p'}} \leq c h^{1-\frac{2}{p'}} |(\boldsymbol{\xi}_h, \eta_h)|_{\text{lps}}^{\frac{2}{p'}} \leq ch. \end{aligned} \tag{4.56}$$

Combining (4.53), (4.55), (4.56), and (4.43), we get the desired estimate (4.44). □

A priori error estimates - case $p > 2$: In this paragraph we derive related a priori error estimates for $p \in (2, \infty)$. Actually, the case $p \in (2, \infty)$ differs from the case $p \in (1, 2]$ only slightly. Hence, we restrict ourselves to highlight the differences between the two cases.

Theorem 4.12. *For $p \in [2, \infty)$ and $\varepsilon \in [0, \varepsilon_0]$ let $\boldsymbol{\mathcal{S}}$ satisfy Assumption 2.1 and let $\boldsymbol{\mathcal{F}}$ be defined by (2.39). Let $(\boldsymbol{v}, \pi) \in \boldsymbol{\mathcal{X}}^p \times \mathcal{Q}^p$ be the unique solution of* **(P1)** *and let $(\boldsymbol{v}_h, \pi_h) \in \boldsymbol{\mathcal{X}}_h^p \times \mathcal{Q}_h^p$ be the unique solution of* **($\mathbf{P1}_h$)***, where the stabilization term s_h is defined by (4.4). We assume the additional regularity $\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) \in W^{1,2}(\Omega)^{d \times d}$, $\boldsymbol{v} \in \boldsymbol{W}^{2,p}(\Omega)$, $\pi \in W^{1,p'}(\Omega)$ where $1/p + 1/p' = 1$. Then, for $\alpha_M := \alpha_0 h_M^s$ with $\alpha_0 > 0$ and $s = p'$:*

$$\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2 \leq C_v h^{\frac{p'}{2}}, \qquad \|\boldsymbol{v} - \boldsymbol{v}_h\|_{1,p} \leq C_v' h^{\frac{1}{p-1}}, \tag{4.57}$$

$$\|\pi - \pi_h\|_{p'} \leq C_\pi h^{\frac{p'}{2}}. \tag{4.58}$$

The constants C_v, C_v', $C_\pi > 0$ only depend on p, ε_0, σ_0, σ_1, Ω, $\boldsymbol{f}$, α_0, τ_0, $\|\nabla \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})\|_2$, $\|\boldsymbol{v}\|_{2,p}$, $\|\pi\|_{1,p'}$, and C_π additionally depends on $\tilde{\beta}(p)$.

Proof. The proof of Theorem 4.12 differs from the proof of Theorem 4.11 only slightly. Hence, we restrict ourselves to clarify the differences. Again it is sufficient to estimate the quantity $|(\boldsymbol{\xi}_h, \eta_h)|_{\text{lps}}^2$ defined by (4.46). As above, we obtain $|(\boldsymbol{\xi}_h, \eta_h)|_{\text{lps}}^2 \sim I_1 + I_2 + I_3 + I_4$ where $I_1, \ldots, I_4$ are defined by (4.47). In view of Remark 4.4, for arbitrary $\delta_1 > 0$ and $\delta_2 > 0$ the terms I_1 and I_2 are estimated analogously to the proof of Theorem 4.11:

$$I_1 \leq c_{\delta_1} h^s \|\pi\|_{1,p'}^{p'} + \delta_1 |(\boldsymbol{\xi}_h, \eta_h)|_{\text{lps}}^2, \qquad I_2 \leq c_{\delta_2} h^2 \|\nabla \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})\|_2^2 + \delta_2 |(\boldsymbol{\xi}_h, \eta_h)|_{\text{lps}}^2.$$

Using Hölder's and Young's inequality, for any $\delta_3 > 0$ we estimate the term I_3 by

$$I_3 \le \Big|(\pi - j_h\pi, \nabla\cdot \boldsymbol{\xi}_h)_\Omega\Big| \le \|\pi - j_h\pi\|_{p'}\|\nabla\boldsymbol{\xi}_h\|_p \le c_{\delta_3}\|\pi - j_h\pi\|_{p'}^{p'} + \delta_3\|\nabla(\boldsymbol{j}_h\boldsymbol{v} - \boldsymbol{v}_h)\|_p^p.$$

Applying Korn's inequality, Lemma 2.6 (ii) and the interpolation property of j_h, we conclude

$$I_3 \lesssim c_{\delta_3}\|\pi - j_h\pi\|_{p'}^{p'} + \delta_3\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2^2 \lesssim c_{\delta_3}h^{p'}\|\pi\|_{1,p'}^{p'} + \delta_3|(\boldsymbol{\xi}_h, \eta_h)|_{\mathrm{lps}}^2.$$

Using integration by parts, the orthogonality of $\boldsymbol{j}_h$ with respect to $\boldsymbol{Y}_h$, Hölder's inequality, (4.8), Young's inequality, and the stability of $\boldsymbol{\theta}_h$ and j_h, for arbitrary $\delta_4 > 0$ we estimate

$$\begin{aligned} I_4 &\le |(\boldsymbol{j}_h\boldsymbol{v} - \boldsymbol{v}, \nabla\eta_h)_\Omega| = |(\boldsymbol{j}_h\boldsymbol{v} - \boldsymbol{v}, \boldsymbol{\theta}_h\nabla\eta_h)_\Omega| \le \sum_{M\in\mathbb{M}_h}\|\boldsymbol{j}_h\boldsymbol{v} - \boldsymbol{v}\|_{p;M}\|\boldsymbol{\theta}_h\nabla\eta_h\|_{p';M} \\ &\le c_{\delta_4}\sum_{M\in\mathbb{M}_h}\alpha_M^{-1}\|\boldsymbol{j}_h\boldsymbol{v} - \boldsymbol{v}\|_{p;M}^2\Big(\tau|M|^{\frac{1}{p'}} + \|\pi\|_{1,p';S_M} + \|\nabla\pi_h\|_{p';M}\Big)^{2-p'} \\ &\quad + \delta_4\sum_{M\in\mathbb{M}_h}\alpha_M\|\boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h\nabla j_h\pi) - \boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h\nabla\pi_h)\|_{2;M}^2 \end{aligned}$$

where the constant c_{δ_4} only depends on p and δ_4. Using the local inverse estimate (3.19), the interpolation property of $\boldsymbol{j}_h$, and Hölder's inequality with $\frac{2}{p} + \frac{p-2}{p} = 1$, we arrive at

$$\begin{aligned} I_4 &\le c_{\delta_4}\sum_{M\in\mathbb{M}_h}h_M^{4-s+p'-2}\|\boldsymbol{v}\|_{2,p;S_M}^2\Big(\tau|M|^{\frac{1}{p'}} + \|\pi\|_{1,p';S_M} + \|\pi_h\|_{p';M}\Big)^{2-p'} \\ &\quad + \delta_4\sum_{M\in\mathbb{M}_h}\alpha_M\|\boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h\nabla j_h\pi) - \boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h\nabla\pi_h)\|_{2;M}^2 \\ &\le c_{\delta_4}h^{4-s+p'-2}\Big(\sum_{M\in\mathbb{M}_h}\|\boldsymbol{v}\|_{2,p;S_M}^p\Big)^{\frac{2}{p}}\Big(\sum_{M\in\mathbb{M}_h}\Big(\tau|M|^{\frac{1}{p'}} + \|\pi\|_{1,p';S_M} + \|\pi_h\|_{p';M}\Big)^{p'}\Big)^{\frac{p-2}{p}} \\ &\quad + \delta_4\sum_{M\in\mathbb{M}_h}\alpha_M\|\boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h\nabla j_h\pi) - \boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h\nabla\pi_h)\|_{2;M}^2 \\ &\le c_{\delta_4}h^{4-s+p'-2}\|\boldsymbol{v}\|_{2,p}^2\Big(\tau_0|\Omega|^{\frac{1}{p'}} + \|\pi\|_{1,p'} + \|\pi_h\|_{p'}\Big)^{2-p'} + \delta_4|(\boldsymbol{\xi}_h, \eta_h)|_{\mathrm{lps}}^2 \end{aligned}$$

(we note that $2 - p' = p'(p-2)/p$). Combining all estimates above and choosing $\delta_1, \ldots, \delta_4$ sufficiently small, we easily deduce that there exists $c = c(p, \sigma_0, \sigma_1, \alpha_0, \Omega) > 0$ such that

$$\begin{aligned} |(\boldsymbol{\xi}_h, \eta_h)|_{\mathrm{lps}}^2 \le c\Big(&h^s\|\pi\|_{1,p'}^{p'} + h^2\|\nabla\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})\|_2^2 + h^{p'}\|\pi\|_{1,p'}^{p'} \\ &+ h^{2-s+p'}\|\boldsymbol{v}\|_{2,p}^2\Big(\tau_0|\Omega|^{\frac{1}{p'}} + \|\pi\|_{1,p'} + \|\pi_h\|_{p'}\Big)^{2-p'}\Big). \end{aligned} \tag{4.59}$$

Due to (4.34), π_h is uniformly bounded in $L^{p'}(\Omega)$. In order to ensure the optimal rate of convergence, we have to choose $s = p'$. This proves $(4.57)_1$. Using Poincaré's and Korn's inequality, Lemma 2.6 (ii), we finally arrive at

$$\|\boldsymbol{v} - \boldsymbol{v}_h\|_{1,p} \le c\|\boldsymbol{D}(\boldsymbol{v} - \boldsymbol{v}_h)\|_p \le c\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2^{\frac{2}{p}}$$

for some $c = c(p) > 0$. By virtue of $(4.57)_1$, this implies $(4.57)_2$.

In order to verify (4.58), we use similar arguments as in the derivation of (4.44). Again it is sufficient to estimate the terms J_1 and J_2 defined by (4.53). From (4.54) and (2.49) we deduce that the term J_1 can be estimated by

$$J_1 \lesssim \Big[\varepsilon|\Omega|^{\frac{1}{p}} + \|\boldsymbol{D}\boldsymbol{v}\|_p + \|\boldsymbol{D}\boldsymbol{v}_h\|_p\Big]^{\frac{p-2}{2}} \|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2 + h\|\pi\|_{1,p'}.$$

From (2.63) and (4.34) it follows that the expression within the square brackets is uniformly bounded by a constant $c = c(\Omega, p, \varepsilon_0, \boldsymbol{f})$. We estimate the term J_2 as follows: Employing estimate (4.8), the stability of $\boldsymbol{\theta}_h$ and j_h, and the inverse estimate (3.19), we realize that

$$\begin{aligned} J_2^{p'} &\lesssim \sum_{M\in\mathbb{M}_h} h_M^{p'} \|\boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h\nabla j_h\pi) - \boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h\nabla\pi_h)\|_{2;M}^{p'} \|\tau + |\boldsymbol{\theta}_h\nabla j_h\pi| + |\boldsymbol{\theta}_h\nabla\pi_h|\|_{p';M}^{\frac{2-p'}{2}p'} \\ &\lesssim \sum_{M\in\mathbb{M}_h} h_M^{\frac{p'}{2}p'} \|\boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h\nabla j_h\pi) - \boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h\nabla\pi_h)\|_{2;M}^{p'} \Big(\tau|M|^{\frac{1}{p'}} + \|\pi\|_{1,p';S_M} + \|\pi_h\|_{p';M}\Big)^{\frac{2-p'}{2}p'}. \end{aligned}$$

Raising this to the power $1/p'$, using Hölder's inequality with $\frac{p'}{2} + \frac{2-p'}{2} = 1$, the uniform a priori bound for $\|\pi_h\|_{p'}$, and recalling (4.59) with $s = p'$, we finally conclude that

$$\begin{aligned} J_2 &\lesssim \Big(\sum_{M\in\mathbb{M}_h} h_M^{p'} \|\boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h\nabla j_h\pi) - \boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h\nabla\pi_h)\|_{2;M}^2\Big)^{\frac{1}{2}} \\ &\quad\times\Big(\sum_{M\in\mathbb{M}_h}\Big(\tau|M|^{\frac{1}{p'}} + \|\pi\|_{1,p';S_M} + \|\pi_h\|_{p';M}\Big)^{p'}\Big)^{\frac{2-p'}{2p'}} \\ &\lesssim |(\boldsymbol{0},\eta_h)|_{\mathrm{lps}}\Big(\tau_0|\Omega|^{\frac{1}{p'}} + \|\pi\|_{1,p'} + \|\pi_h\|_{p'}\Big)^{\frac{2-p'}{2}} \lesssim h^{\frac{p'}{2}}. \end{aligned}$$

Summing up, we obtain (4.58) in view of (4.57). This completes the proof. □

Remark 4.6. Since $p > 2$, by means of Lemma 2.4 we conclude the useful inequality

$$\begin{aligned} \|\boldsymbol{D}\boldsymbol{u} - \boldsymbol{D}\boldsymbol{v}\|_2^2 &\le \varepsilon^{2-p}\int_\Omega (\varepsilon + |\boldsymbol{D}\boldsymbol{u}| + |\boldsymbol{D}\boldsymbol{v}|)^{p-2}|\boldsymbol{D}\boldsymbol{u} - \boldsymbol{D}\boldsymbol{v}|^2\,\mathrm{d}\boldsymbol{x} \\ &\le c\varepsilon^{2-p}\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{u}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})\|_2^2 \qquad \forall \boldsymbol{u},\boldsymbol{v}\in\boldsymbol{W}^{1,p}(\Omega) \end{aligned} \tag{4.60}$$

provided that $\varepsilon > 0$. Inequality (4.60) implies an a priori error estimate for the velocity in $\boldsymbol{W}^{1,2}(\Omega)$ that is of same order as the related error estimate expressed by the $\boldsymbol{\mathcal{F}}$-distance.

Remark 4.7. The regularity assumption $\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) \in W^{1,2}(\Omega)^{d\times d}$ of Theorem 4.12 is redundant, since $\boldsymbol{v} \in \boldsymbol{W}^{2,p}(\Omega)$ already implies $\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) \in W^{1,2}(\Omega)^{d\times d}$ for $p \ge 2$. In order to see this, we define the measurable set $\Omega_0 := \{\boldsymbol{x}\in\Omega;\ (\varepsilon + |\boldsymbol{D}\boldsymbol{v}(\boldsymbol{x})|) \le |\nabla^2\boldsymbol{v}(\boldsymbol{x})|\}$ and we recall the definition of $\mathcal{I}$ in (2.82). According to $\|\nabla\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})\|_2^2 \sim \mathcal{I}(\boldsymbol{v})$ (see Lemma 2.27) it suffices to prove $\mathcal{I}(\boldsymbol{v}) < \infty$. Using $|\nabla \boldsymbol{D}\boldsymbol{v}| \sim |\nabla^2 \boldsymbol{v}|$, for $p \geq 2$ we estimate

$$\begin{aligned}\mathcal{I}(\boldsymbol{v}) &= \int\limits_{\Omega_0} (\varepsilon + |\boldsymbol{D}\boldsymbol{v}|)^{p-2} |\nabla \boldsymbol{D}\boldsymbol{v}|^2 \, \mathrm{d}\boldsymbol{x} + \int\limits_{\Omega \setminus \Omega_0} (\varepsilon + |\boldsymbol{D}\boldsymbol{v}|)^{p-2} |\nabla \boldsymbol{D}\boldsymbol{v}|^2 \, \mathrm{d}\boldsymbol{x} \\ &\lesssim \int\limits_{\Omega_0} |\nabla^2 \boldsymbol{v}|^p \, \mathrm{d}\boldsymbol{x} + \int\limits_{\Omega \setminus \Omega_0} (\varepsilon + |\boldsymbol{D}\boldsymbol{v}|)^p \, \mathrm{d}\boldsymbol{x} < \infty,\end{aligned}$$

since $\boldsymbol{v} \in \boldsymbol{W}^{2,p}(\Omega)$ and Ω is bounded. Consequently, we get $\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) \in W^{1,2}(\Omega)^{d\times d}$.

Remark 4.8. Considering the proof of Theorem 4.12, we observe that the regularity assumption $\boldsymbol{v} \in \boldsymbol{W}^{2,p}(\Omega)$ is only needed for the estimation of the term I_4. Hence, we may attempt to estimate I_4 differently: First of all we recall that

$$I_4 \leq \sum_{M \in \mathbb{M}_h} \int\limits_M h_M^{-1} |\boldsymbol{j}_h \boldsymbol{v} - \boldsymbol{v}| h_M |\boldsymbol{\theta}_h \nabla \eta_h| \, \mathrm{d}\boldsymbol{x}.$$

Lemma 2.2 implies that for each $\delta_4 > 0$ there exists a constant $c_{\delta_4} > 0$ such that

$$\begin{aligned}I_4 \leq c_{\delta_4} &\sum_{M \in \mathbb{M}_h} \int\limits_M \Big((\varphi^*)_{\tau + |\boldsymbol{\theta}_h \nabla j_h \pi|}\Big)^* (h_M^{-1} |\boldsymbol{j}_h \boldsymbol{v} - \boldsymbol{v}|) \, \mathrm{d}\boldsymbol{x} \\ &+ \delta_4 \sum_{M \in \mathbb{M}_h} \int\limits_M (\varphi^*)_{\tau + |\boldsymbol{\theta}_h \nabla j_h \pi|} (h_M |\boldsymbol{\theta}_h \nabla \eta_h|) \, \mathrm{d}\boldsymbol{x}.\end{aligned}$$

From the properties of shifted N-functions we deduce that $\varphi_a(t) \sim (a+t)^{p-2} t^2$ uniformly in $a, t \geq 0$. As a consequence, for $p \in (1, \infty)$ and any $h \in [0,1]$ the inequality $\varphi_a(ht) \lesssim h^{\min\{p,2\}} \varphi_a(t)$ can be shown. Applying this and Lemma 2.1, we arrive at

$$\begin{aligned}I_4 &\lesssim c_{\delta_4} \sum_{M \in \mathbb{M}_h} \int\limits_M \varphi_{(\varphi^*)'(\tau + |\boldsymbol{\theta}_h \nabla j_h \pi|)} (h_M^{-1} |\boldsymbol{j}_h \boldsymbol{v} - \boldsymbol{v}|) \, \mathrm{d}\boldsymbol{x} \\ &\quad + \delta_4 \sum_{M \in \mathbb{M}_h} h_M^{p'} \int\limits_M (\varphi^*)_{\tau + |\boldsymbol{\theta}_h \nabla j_h \pi|} (|\boldsymbol{\theta}_h \nabla \eta_h|) \, \mathrm{d}\boldsymbol{x} \\ &\lesssim c_{\delta_4} \sum_{M \in \mathbb{M}_h} \int\limits_M \varphi_{(\varphi^*)'(\tau + |\boldsymbol{\theta}_h \nabla j_h \pi|)} (h_M^{-1} |\boldsymbol{j}_h \boldsymbol{v} - \boldsymbol{v}|) \, \mathrm{d}\boldsymbol{x} + \delta_4 |(\boldsymbol{0}, \eta_h)|_{\mathrm{lps}} =: I_5 + \delta_4 |(\boldsymbol{0}, \eta_h)|_{\mathrm{lps}}.\end{aligned}$$

Using $\varphi_a(t) \sim (a+t)^{p-2} t^2$ and Hölder's inequality with $\frac{2}{p} + \frac{p-2}{p} = 1$, we conclude that

$$\begin{aligned}I_5 &\lesssim c_{\delta_4} \sum_{M \in \mathbb{M}_h} h_M^{-2} \|\boldsymbol{j}_h \boldsymbol{v} - \boldsymbol{v}\|_{p;M}^2 \bigg(\int\limits_M \Big((\varphi^*)'(\tau + |\boldsymbol{\theta}_h \nabla j_h \pi|) + h_M^{-1} |\boldsymbol{j}_h \boldsymbol{v} - \boldsymbol{v}|\Big)^p \, \mathrm{d}\boldsymbol{x} \bigg)^{\frac{p-2}{p}} \\ &\lesssim c_{\delta_4} \sum_{M \in \mathbb{M}_h} h_M^{-2} \|\boldsymbol{j}_h \boldsymbol{v} - \boldsymbol{v}\|_{p;M}^2 \Big(\|\tau + |\boldsymbol{\theta}_h \nabla j_h \pi|\|_{p';M}^{p'} + h_M^{-p} \|\boldsymbol{j}_h \boldsymbol{v} - \boldsymbol{v}\|_{p;M}^p \Big)^{\frac{p-2}{p}}.\end{aligned}$$

Compared to the proof of Theorem 4.12, I_5 does not allow a better convergence order with respect to the supposed regularity. In fact, I_5 leads to the same convergence rate. However, the estimation of I_5 does not require an inverse estimate for finite element functions.

Remark 4.9. The question arises whether the regularity assumption $\boldsymbol{v} \in \boldsymbol{W}^{2,p}(\Omega)$ stated in Theorem 4.12 may be relaxed and confined to the requirement $\boldsymbol{\mathcal{F}}(\boldsymbol{Dv}) \in W^{1,2}(\Omega)^{d\times d}$ which seems to be more natural. In order to shed some light on that issue, first of all we introduce the Nikol'skiĭ spaces $\mathcal{N}^{s,p}(\Omega)$ (see, e.g., Kufner et al. [KJF77]): Let $m \geq 0$ be an integer, $0 < \sigma < 1$, $s = m + \sigma$, $\boldsymbol{z} \in \mathbb{R}^d$, $\Omega_\delta := \{\boldsymbol{x} \in \Omega;\ \operatorname{dist}(\boldsymbol{x}, \partial\Omega) \geq \delta\}$, and $1 \leq p < \infty$. The space $\mathcal{N}^{s,p}(\Omega)$ consists of all functions $g : \Omega \to \mathbb{R}$ for which the norm

$$\|g\|_{\mathcal{N}^{s,p}(\Omega)} = \left(\|g\|^p_{L^p(\Omega)} + \sum_{|\alpha|=m} \sup_{\delta>0,\ 0<|\boldsymbol{z}|<\delta} \int_{\Omega_\delta} \frac{|\partial^\alpha g(\boldsymbol{x}+\boldsymbol{z}) - \partial^\alpha g(\boldsymbol{x})|^p}{|\boldsymbol{z}|^{\sigma p}} \,\mathrm{d}\boldsymbol{x}\right)^{\frac{1}{p}} \tag{4.61}$$

is finite. Below we suppose that the velocity $\boldsymbol{v}$ belongs to the Nikol'skiĭ space $\boldsymbol{\mathcal{N}}^{1+2/p,p}(\Omega)$. The regularity assumption $\boldsymbol{v} \in \boldsymbol{\mathcal{N}}^{1+2/p,p}(\Omega)$ seems to be reasonable because it is well-established (see [DER07]) that $\boldsymbol{\mathcal{F}}(\nabla \boldsymbol{v}) \in W^{1,2}(\Omega)^{d\times d}$ implies $\boldsymbol{v} \in \boldsymbol{\mathcal{N}}^{1+2/p,p}(\Omega)$.

The regularity assumption $\boldsymbol{v} \in \boldsymbol{W}^{2,p}(\Omega)$ stated in Theorem 4.12 can be relaxed to $\boldsymbol{\mathcal{F}}(\boldsymbol{Dv}) \in W^{1,2}(\Omega)^{d\times d}$ and $\boldsymbol{v} \in \boldsymbol{\mathcal{N}}^{1+2/p,p}(\Omega)$ provided that $\nabla\pi_h$ is uniformly bounded in $L^{p'}(\Omega)$. This can be seen as follows: From an embedding theorem (see [KJF77]) we deduce that $\boldsymbol{v} \in \boldsymbol{\mathcal{N}}^{1+2/p,p}(\Omega)$ implies $\boldsymbol{v} \in \boldsymbol{W}^{1+2/p-\delta,p}(\Omega)$ for all $\delta > 0$. Reminding the proof of Theorem 4.12, and assuming that $h_M \sim h$ for all $M \in \mathbb{M}_h$, we then estimate term I_4 by

$$\begin{aligned} I_4 &\leq c_{\delta_4} h^{-s} \|\boldsymbol{j}_h \boldsymbol{v} - \boldsymbol{v}\|_p^2 \Big(\tau + \|\nabla\pi\|_{p'} + \|\nabla\pi_h\|_{p'}\Big)^{2-p'} + \delta_4 c |(\boldsymbol{\xi}_h, \eta_h)|^2_{\mathrm{lps}} \\ &\leq c_{\delta_4} h^{-s+2(1+\frac{2}{p}-\delta)} \|\boldsymbol{v}\|^2_{1+\frac{2}{p}-\delta,p} + \delta_4 c |(\boldsymbol{\xi}_h, \eta_h)|^2_{\mathrm{lps}} \end{aligned} \tag{4.62}$$

due to $\|\nabla\pi_h\|_{p'} \leq C$. Although $1+\frac{2}{p}-\delta$ is not an integer in general, the stated approximation property of $\boldsymbol{j}_h$ holds true according to the real method of interpolation for Sobolev spaces (see [BS94]). The error estimate (4.57) remains valid provided that $-s + 2(1 + \frac{2}{p} - \delta) \geq p'$. Since $s := p'$ and δ is arbitrarily small, the latter condition amounts to $1 + \frac{2}{p} > p' \Leftrightarrow p + 2 > p + p' \Leftrightarrow p > 2$. Finally, we remark that $J_2 = \mathcal{O}(h)$ because of our assumption.

If the pressure belongs to $W^{2,p'}(\Omega)$, then the estimates of Theorem 4.12 can be improved:

Corollary 4.13. *Let the assumptions of Theorem 4.12 be satisfied. For $\nu > 1$ and $k \in \{0,1\}$ let the fluctuation operator θ_h satisfy the property $\|\theta_h w\|_{\nu;M} \leq C h_M^k \|\nabla^k w\|_{\nu;M}$ for all $w \in W^{k,\nu}(\Omega)$ and $M \in \mathbb{M}_h$, where $C > 0$ is a constant independent of h. If additionally $\pi \in W^{2,p'}(\Omega)$, then for $\alpha_M := \alpha_0 h_M^s$ with $s \in [1, p']$ the error of approximation is estimated by (the choice $s := 1$ is asymptotically optimal)*

$$\|\boldsymbol{\mathcal{F}}(\boldsymbol{Dv}) - \boldsymbol{\mathcal{F}}(\boldsymbol{Dv}_h)\|_2 + \|\pi - \pi_h\|_{p'} \leq Ch, \qquad \|\boldsymbol{v} - \boldsymbol{v}_h\|_{1,p} \leq C' h^{\frac{2}{p}}.$$

Proof. In the proof of Theorem 4.12 we estimate the terms I_1 and I_3 as follows: Using

Remark 4.4, the stability of θ_h, the approximation property of θ_h and j_h, we obtain

$$\begin{aligned} I_1 &\lesssim c_{\delta_1} h^s \sum_{M\in\mathbb{M}_h} \|\boldsymbol{\theta}_h \nabla j_h \pi\|_{p';M}^{p'} + \delta_1 \sum_{M\in\mathbb{M}_h} \alpha_M \|\boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h \nabla j_h \pi) - \boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h \nabla \pi_h)\|_{2;M}^2 \\ &\lesssim c_{\delta_1} h^s \sum_{M\in\mathbb{M}_h} \Big(\|\boldsymbol{\theta}_h \nabla (j_h \pi - \pi)\|_{p';M}^{p'} + \|\boldsymbol{\theta}_h \nabla \pi\|_{p';M}^{p'} \Big) + \delta_1 |(\boldsymbol{\xi}_h, 0)|_{\text{lps}}^2 \\ &\lesssim c_{\delta_1} h^{s+p'} \|\pi\|_{2,p'}^{p'} + \delta_1 |(\mathbf{0}, \eta_h)|_{\text{lps}}^2. \end{aligned}$$

Similarly as in the proof of Theorem 4.12, we conclude that

$$I_3 \lesssim c_{\delta_3} h^{2p'} \|\pi\|_{2,p'}^{p'} + \delta_3 |(\boldsymbol{\xi}_h, 0)|_{\text{lps}}^2.$$

Following the proof of Theorem 4.12, for $s \in [1, p']$ we hence arrive at $|(\boldsymbol{\xi}_h, \eta_h)|_{\text{lps}} = \mathcal{O}(h)$. As a result, we get $J_2 = \mathcal{O}(h)$ and we can easily complete the proof. □

4.6 Error estimates for the classical LPS method

Theorems 4.11 and 4.12 can be seen as generalizations of the LPS method to fluid models with p-structure. Note that in the context of linear Stokes systems ($p = 2$) the LPS method is well studied, see Becker/Braack [BB01]. For Stokes systems the bilinear form

$$s_h(\pi)(q) := \sum_{M\in\mathbb{M}_h} \alpha_M (\boldsymbol{\theta}_h \nabla \pi, \boldsymbol{\theta}_h \nabla q)_M \tag{4.63}$$

has been used in order to stabilize the discretized equations of motion. Stabilization methods such as (4.63) can also be applied to p-Stokes systems as depicted below.

Case $p \le 2$: The next Corollary is motivated by our subsequent numerical experiments:

Corollary 4.14. *Let $d = 2$. For $p \in (1, 2]$ and $\varepsilon \in [0, \varepsilon_0]$ let $\boldsymbol{\mathcal{S}}$ satisfy Assumption 2.1 and let $\boldsymbol{\mathcal{F}}$ be defined by (2.39). Suppose that $\mathbb{M}_h$ is quasi-uniform. Let $(\boldsymbol{v}, \pi)$ be the solution to* **(P1)**, *and let $(\boldsymbol{v}_h, \pi_h)$ be the solution to* **(P1$_h$)**, *where s_h is defined by (4.63) with $\alpha_M := \alpha_0 h_M^2$. Assume that $(\boldsymbol{v}, \pi)$ satisfies the regularity $\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) \in W^{1,2}(\Omega)^{d\times d}$, $\boldsymbol{v} \in \boldsymbol{W}^{1,\infty}(\Omega)$ and $\pi \in W^{1,2}(\Omega)$. Then the approximation error is estimated by*

$$\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2 \le C_v h, \qquad \|\boldsymbol{v} - \boldsymbol{v}_h\|_{1,p} \le C_v' h, \tag{4.64}$$

$$\|\pi - \pi_h\|_{p'} \le C_\pi h^{\frac{2}{p'}}. \tag{4.65}$$

The constants C_v, C_v', C_π only depend on p, ε_0, Ω, α_0, $\|\nabla \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})\|_2$, $\|\pi\|_{1,2}$, $\|\boldsymbol{v}\|_{1,\infty}$, and C_π additionally depends on $\tilde{\beta}(p)$.

Remark 4.10. Compared to Theorem 4.11, Corollary 4.14 avoids the $W^{1,p'}$-regularity assumption on the pressure and confines it to $\pi \in W^{1,2}(\Omega)$ provided that the velocity additionally satisfies $\boldsymbol{v} \in \boldsymbol{W}^{1,\infty}(\Omega)$. Note that, in case of $d = 2$, $C^{1,\alpha}$-regularity of the velocity is well-established: For space-periodic boundary conditions $C^{1,\alpha}$-regularity has been proven in [KMS97], whereas for homogeneous Dirichlet boundary conditions it has been shown in [KMS02]. Corollary 4.14 provides the same order of convergence as Theorem 4.11. Note that Corollary 4.14 includes the singular case $\varepsilon = 0$.

Proof of Corollary 4.14. The proof differs from the proof of Theorem 4.11 only slightly. Once again it is sufficient to estimate the projection errors $\boldsymbol{\xi}_h := \boldsymbol{j}_h\boldsymbol{v} - \boldsymbol{v}_h$ and $\eta := j_h\pi - \pi_h$. Since s_h is linear in both arguments, we may replace the distance $|\cdot|_{\text{lps}}$ defined in (4.46) by

$$|(\boldsymbol{\xi}_h, \eta_h)|^2_{\text{lps}} := \|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2^2 + s_h(\eta_h)(\eta_h). \tag{4.66}$$

We have to estimate the terms $I_1, \dots, I_4$ which arise in (4.47). The term I_1 is bounded by

$$I_1 := s_h(j_h\pi)(\eta_h) \le s_h(j_h\pi)(j_h\pi)^{\frac{1}{2}} s_h(\eta_h)(\eta_h)^{\frac{1}{2}} \le c_{\delta_1} h^2 \|\nabla\pi\|_2^2 + \delta_1 |(\boldsymbol{0}, \eta_h)|^2_{\text{lps}}.$$

Similarly to the proof of Theorem 4.11, we estimate the term I_2 by

$$I_2 \le c_{\delta_2} h^2 \|\nabla \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})\|_2^2 + \delta_2 |(\boldsymbol{\xi}_h, 0)|^2_{\text{lps}}.$$

Before we proceed with I_3, we depict that $\boldsymbol{j}_h$ is $\boldsymbol{W}^{1,\infty}$-stable: We know from (4.29) that $\boldsymbol{j}_h$ is locally $\boldsymbol{W}^{1,1}$-stable, i.e., there holds $\|\boldsymbol{j}_h\boldsymbol{w}\|_{1,1;M} \lesssim \|\boldsymbol{w}\|_{1,1;S_M}$ for all $\boldsymbol{w} \in \boldsymbol{W}^{1,1}(\Omega)$ and $M \in \mathbb{M}_h$. Moreover, since $X_h(M)$ is finite dimensional, there holds $|\nabla^i \boldsymbol{j}_h\boldsymbol{w}(\boldsymbol{y})| \lesssim \fint_M |\nabla^i \boldsymbol{j}_h \boldsymbol{w}| \,\mathrm{d}\boldsymbol{x}$, $i \in \{0,1\}$, for all $\boldsymbol{y} \in M$ and $M \in \mathbb{M}_h$. Due to the non-degeneracy of $\mathbb{M}_h$ it follows that $\|\boldsymbol{j}_h\boldsymbol{w}\|_{1,\infty;M} \lesssim \|\boldsymbol{w}\|_{1,\infty;S_M}$ for all $\boldsymbol{w} \in \boldsymbol{W}^{1,\infty}(\Omega)$. This yields

$$\|\boldsymbol{j}_h\boldsymbol{w}\|_{1,\infty;\Omega} \lesssim \|\boldsymbol{w}\|_{1,\infty;\Omega} \qquad \forall \boldsymbol{w} \in \boldsymbol{W}^{1,\infty}(\Omega). \tag{4.67}$$

Using (2.43) with $\nu = 2$ and the $\boldsymbol{W}^{1,\infty}$-stability of $\boldsymbol{j}_h$, we modify estimate (4.50) as follows:

$$\begin{aligned} I_3 &:= (\pi - j_h\pi, \nabla\cdot\boldsymbol{\xi}_h)_\Omega \le \|\boldsymbol{D}(\boldsymbol{j}_h\boldsymbol{v} - \boldsymbol{v}_h)\|_2 \|\pi - j_h\pi\|_2 \\ &\le (\varepsilon + \|\nabla \boldsymbol{j}_h\boldsymbol{v}\|_\infty + \|\nabla\boldsymbol{v}_h\|_\infty)^{\frac{2-p}{2}} \|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2 \|\pi - j_h\pi\|_2 \\ &\le \delta_3 |(\boldsymbol{\xi}_h, 0)|^2_{\text{lps}} + c_{\delta_3} h^2 (\varepsilon_0 + \|\boldsymbol{v}\|_{1,\infty} + \|\nabla\boldsymbol{v}_h\|_\infty)^{2-p} \|\pi\|^2_{1,2}. \end{aligned}$$

Since $p < 2$ and $\boldsymbol{v} \in \boldsymbol{W}^{1,\infty}(\Omega)$, $\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) \in W^{1,2}(\Omega)^{d\times d}$ implies $\boldsymbol{v} \in \boldsymbol{W}^{2,2}(\Omega)$ by virtue of

$$\begin{aligned} \int_\Omega |\nabla^2\boldsymbol{v}|^2 \,\mathrm{d}\boldsymbol{x} &\le (\varepsilon + \|\boldsymbol{D}\boldsymbol{v}\|_\infty)^{2-p} \int_\Omega (\varepsilon + |\boldsymbol{D}\boldsymbol{v}|)^{p-2} |\nabla^2\boldsymbol{v}|^2 \,\mathrm{d}\boldsymbol{x} \\ &\le c(\varepsilon + \|\boldsymbol{D}\boldsymbol{v}\|_\infty)^{2-p} \int_\Omega |\nabla \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})|^2 \,\mathrm{d}\boldsymbol{x} < \infty. \end{aligned}$$

Using integration by parts, the orthogonality of $\boldsymbol{j}_h$ with respect to $\boldsymbol{Y}_h$, Hölder's and Young's inequality, and the approximation property of $\boldsymbol{j}_h$, we estimate the term I_4 by

$$\begin{aligned} I_4 &:= (\nabla\cdot(\boldsymbol{j}_h\boldsymbol{v} - \boldsymbol{v}), \eta_h)_\Omega \le \sum_{M\in\mathbb{M}_h} \|\boldsymbol{j}_h\boldsymbol{v} - \boldsymbol{v}\|_{2;M} \|\boldsymbol{\theta}_h \nabla\eta_h\|_{2;M} \\ &\le c_{\delta_4} \sum_{M\in\mathbb{M}_h} \alpha_M^{-1} \|\boldsymbol{j}_h\boldsymbol{v} - \boldsymbol{v}\|^2_{2;M} + \delta_4 \sum_{M\in\mathbb{M}_h} \alpha_M \|\boldsymbol{\theta}_h\nabla\eta_h\|^2_{2;M} \le c_{\delta_4} h^2 \|\boldsymbol{v}\|^2_{2,2} + \delta_4 |(\boldsymbol{0}, \eta_h)|^2_{\text{lps}} \end{aligned}$$

(cf. [MST07]). Collecting all estimates above and choosing $\delta_1, \dots, \delta_4$ sufficiently small, we easily deduce that the projection error $|(\boldsymbol{\xi}_h, \eta_h)|_{\mathrm{lps}}$ is bounded by (w.l.o.g. $\varepsilon_0 \geq 1$)

$$|(\boldsymbol{\xi}_h, \eta_h)|_{\mathrm{lps}} \leq Ch(\varepsilon_0 + \|\nabla \boldsymbol{v}\|_\infty + \|\nabla \boldsymbol{v}_h\|_\infty)^{\frac{2-p}{2}}, \tag{4.68}$$

where the constant $C > 0$ only depends on $\|\nabla \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})\|_2$, $\|\pi\|_{1,2}$, $\|\boldsymbol{v}\|_{1,\infty}$, p, ε_0, Ω. We depict that $\boldsymbol{v}_h$ is uniformly bounded in $\boldsymbol{W}^{1,\infty}(\Omega)$. Using the inverse inequality (3.20) with $d = 2$, the $\boldsymbol{W}^{1,\infty}$-stability of $\boldsymbol{j}_h$, Korn's inequality, Lemma 2.6 (i) with $\nu = 2$, we estimate

$$\begin{aligned}
\|\boldsymbol{v}_h\|_{1,\infty} &\leq \|\boldsymbol{v}_h - \boldsymbol{j}_h\boldsymbol{v}\|_{1,\infty} + \|\boldsymbol{j}_h\boldsymbol{v}\|_{1,\infty} \\
&\leq c\Big[h^{-1}\|\boldsymbol{v}_h - \boldsymbol{j}_h\boldsymbol{v}\|_{1,2} + \|\boldsymbol{v}\|_{1,\infty}\Big] \\
&\leq c\Big[h^{-1}\|\boldsymbol{D}\boldsymbol{v}_h - \boldsymbol{D}\boldsymbol{j}_h\boldsymbol{v}\|_{2} + \|\boldsymbol{v}\|_{1,\infty}\Big] \\
&\leq c\Big[h^{-1}\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{v})\|_{2}\Big(\varepsilon_0 + \|\nabla \boldsymbol{v}_h\|_\infty + \|\nabla \boldsymbol{v}\|_\infty\Big)^{\frac{2-p}{2}} + \|\boldsymbol{v}\|_{1,\infty}\Big].
\end{aligned} \tag{4.69}$$

Combining (4.69) and (4.68), we conclude that

$$\|\boldsymbol{v}_h\|_{1,\infty} \leq C = C\Big(\|\nabla \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})\|_2, \|\pi\|_{1,2}, \|\boldsymbol{v}\|_{1,\infty}\Big). \tag{4.70}$$

The constant C in (4.70) also depends on p, ε_0, σ_0, σ_1, Ω. However, C is independent of h. In view of (4.70), (4.68) yields the desired error estimate $(4.64)_1$. Clearly, $(4.64)_2$ follows from (2.43) and $(4.64)_1$. It remains to prove the error estimate for the pressure. In order to derive (4.65), we consult Lemma 4.5 which applied to the projection error η_h reads

$$\tilde{\beta}(p)\|\eta_h\|_{p'} \leq \sup_{\boldsymbol{w}_h \in \boldsymbol{\mathcal{X}}_h^p} \frac{|(\nabla \cdot \boldsymbol{w}_h, \eta_h)_\Omega|}{\|\nabla \boldsymbol{w}_h\|_p} + \Big(\sum_{M \in \mathbb{M}_h} h_M^{p'}\|\boldsymbol{\theta}_h \nabla \eta_h\|_{p';M}^{p'}\Big)^{\frac{1}{p'}} =: J_1 + J_2.$$

Interpolating $L^{p'}(\Omega)$ between $L^2(\Omega)$ and $W^{1,2}(\Omega)$, and recalling the interpolation property (4.15), and the $W^{1,2}$-stability of j_h, for $p > \frac{2d}{d+2}$ and $\lambda := \frac{d}{2} - \frac{d}{p'}$ we obtain the estimate

$$\|\pi - j_h\pi\|_{p'} \leq c\,\|\pi - j_h\pi\|_{1,2}^{\lambda}\|\pi - j_h\pi\|_2^{1-\lambda} \leq ch^{1+\frac{d}{p'}-\frac{d}{2}}\|\pi\|_{1,2}. \tag{4.71}$$

Now the term J_1 can be estimated as follows: Using similar arguments as in the proof of Theorem 4.11 and the interpolation property (4.71) with $d = 2$, we conclude that

$$J_1 \leq \|\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}_h)\|_{p'} + \|j_h\pi - \pi\|_{p'} \leq c\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2^{\frac{2}{p'}} + ch^{\frac{2}{p'}}\|\pi\|_{1,2}.$$

Finally the term J_2 can be estimated by means of the inverse inequality (3.19),

$$\begin{aligned}
J_2 &\equiv \Big(\sum_{M \in \mathbb{M}_h} h_M^{p'}\|\boldsymbol{\theta}_h \nabla \eta_h\|_{p';M}^{p'}\Big)^{\frac{1}{p'}} \leq c\Big(\sum_{M \in \mathbb{M}_h} h_M^{p'} h_M^{(\frac{d}{p'}-\frac{d}{2})p'}\|\boldsymbol{\theta}_h \nabla \eta_h\|_{2;M}^{p'}\Big)^{\frac{1}{p'}} \\
&\leq c\Big(h^{\frac{2d}{p'}-d}\sum_{M \in \mathbb{M}_h} h_M^2\|\boldsymbol{\theta}_h \nabla \eta_h\|_{2;M}^2\Big)^{\frac{1}{2}} \leq ch^{\frac{d}{p'}-\frac{d}{2}}|(\boldsymbol{0}, \eta_h)|_{\mathrm{lps}}.
\end{aligned} \tag{4.72}$$

Recalling $|(\boldsymbol{\xi}_h, \eta_h)|_{\mathrm{lps}} = \mathcal{O}(h)$, we easily complete the proof of (4.65). □

Corollary 4.15. *Let $d \geq 2$. For $p \in (1,2]$, $\varepsilon \in [0, \varepsilon_0]$ let $\boldsymbol{\mathcal{S}}$ satisfy Assumption 2.1 and let $\boldsymbol{\mathcal{F}}$ be defined by* (2.39). *Let $(\boldsymbol{v}, \pi)$ be the solution to* **(P1)**, *and let $(\boldsymbol{v}_h, \pi_h)$ be the solution to* $(\mathbf{P1}_h)$, *where s_h is defined by* (4.63) *with $\alpha_M := \alpha_0 h_M^2$. Assume that $(\boldsymbol{v}, \pi)$ satisfies the regularity $\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) \in W^{1,2}(\Omega)^{d\times d}$, $\boldsymbol{v} \in \boldsymbol{W}^{2,2}(\Omega)$ and $\pi \in W^{1,p'}(\Omega)$. Then there hold*

$$\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2 \leq C_v h, \qquad \|\boldsymbol{v} - \boldsymbol{v}_h\|_{1,p} \leq C_v' h, \tag{4.73}$$

$$\|\pi - \pi_h\|_{p'} \leq C_\pi h^{1+\frac{d}{p'}-\frac{d}{2}}. \tag{4.74}$$

The constants C_v, C_v', C_π only depend on p, ε, Ω, α_0, $\|\nabla \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})\|_2$, $\|\pi\|_{1,2}$, $\|\boldsymbol{v}\|_{2,2}$, and C_π additionally depends on and $\tilde{\beta}(p)$.

Proof. The proof combines the proofs of Theorem 4.11 and Corollary 4.14. Once again it is sufficient to estimate the projection errors $\boldsymbol{\xi}_h := \boldsymbol{j}_h\boldsymbol{v} - \boldsymbol{v}_h$ and $\eta := j_h\pi - \pi_h$ with respect to the distance $|(\boldsymbol{\xi}_h, \eta_h)|_{\text{lps}}$ defined in (4.66). Similarly to the proofs of Theorem 4.11 and Corollary 4.14, we can estimate the terms $I_1, \ldots, I_4$ that arise in (4.47) as follows:

$$I_1 \leq c_{\delta_1} h^2 \|\nabla \pi\|_2^2 + \delta_1 |(\mathbf{0}, \eta_h)|_{\text{lps}}^2, \qquad I_2 \leq c_{\delta_2} h^2 \|\nabla \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})\|_2^2 + \delta_2 |(\boldsymbol{\xi}_h, 0)|_{\text{lps}}^2,$$

$$I_3 \leq c_{\delta_3} h^2 \|\pi\|_{1,p'}^2 + \delta_3 |(\boldsymbol{\xi}_h, 0)|_{\text{lps}}, \qquad I_4 \leq c_{\delta_4} h^2 \|\boldsymbol{v}\|_{2,2}^2 + \delta_4 |(\mathbf{0}, \eta_h)|_{\text{lps}}^2.$$

Combining (4.47) with the above estimates for $I_1, \ldots, I_4$ and choosing $\delta_1, \ldots, \delta_4$ sufficiently small, we easily conclude $(4.73)_1$. The estimate $(4.73)_2$ follows from (2.43), $(4.73)_1$, (2.63) and $(4.34)_1$. Note that $(4.34)_1$ also holds for s_h as in (4.63). Finally, the pressure-error estimate (4.74) follows from the combination of (4.53), (4.55), (4.72), and $(4.73)_1$. ☐

Remark 4.11. If we relax the assumption $\boldsymbol{v} \in \boldsymbol{W}^{2,2}(\Omega)$ in Corollary 4.15, then we would obtain a priori error estimates for the velocity that, compared to (4.73), provide lower rates of convergence which additionally depend on the space dimension d. In contrast, the stabilization scheme, which has been proposed in Section 4.1, allows an order of convergence independent of d since it is adjusted to the p-structure of the problem.

Case $p \geq 2$: In this paragraph we prove related error estimates for $p \geq 2$. First of all we restrict ourselves to the nondegenerate case $\varepsilon > 0$. The requirement $\varepsilon > 0$ enables us to derive an a priori error estimate for the pressure with respect to the $L^2(\Omega)$-norm. If the pressure belongs to $W^{1,2}(\Omega)$, then the application of the method (4.63) is justified by

Corollary 4.16. *Let $d = 2$. For $p \in [2, \infty)$ and $\varepsilon \in (0, \varepsilon_0]$ let $\boldsymbol{\mathcal{S}}$ satisfy Assumption 2.1 and let $\boldsymbol{\mathcal{F}}$ be defined by* (2.39). *We suppose that $\mathbb{M}_h$ is quasi-uniform. Let $(\boldsymbol{v}, \pi)$ be the solution to* **(P1)**, *and let $(\boldsymbol{v}_h, \pi_h)$ be the solution to* $(\mathbf{P1}_h)$, *where s_h is defined by* (4.63) *with $\alpha_M := \alpha_0 h_M^2$. Assume that the solution $(\boldsymbol{v}, \pi)$ satisfies $\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) \in W^{1,2}(\Omega)^{d\times d}$ and $\pi \in W^{1,2}(\Omega)$. Then, the velocity-error is estimated by*

$$\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2 \leq C_v h, \qquad \|\boldsymbol{v} - \boldsymbol{v}_h\|_{1,p} \leq C_v' h^{\frac{2}{p}}. \tag{4.75}$$

If additionally $\boldsymbol{v} \in \boldsymbol{W}^{1,\infty}(\Omega)$, then the pressure-error in $L^2(\Omega)$ is estimated by

$$\|\pi - \pi_h\|_2 \leq C_\pi h. \tag{4.76}$$

The constants C_v, C_v', $C_\pi > 0$ only depend on p, ε, Ω, α_0, $\|\nabla \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})\|_2$, $\|\pi\|_{1,2}$, and C_π additionally depends on $\|\boldsymbol{v}\|_{1,\infty}$ and $\tilde{\beta}(2)$.

Remark 4.12. The velocity-error estimate (4.75) holds for arbitrary space dimension $d \geq 2$. Compared to (4.57), estimate (4.75) provides better rates of convergence. Its proof requires higher regularity of π but less regularity of $\boldsymbol{v}$. The pressure-error estimate (4.76) predicts a better convergence order than (4.58). Its proof requires the extra assumption $\boldsymbol{v} \in \boldsymbol{W}^{1,\infty}(\Omega)$ which, in general, is satisfied for $d = 2$ (see [KMS02]). Note that Corollary 4.16 does not include the case $\varepsilon = 0$. Estimate (4.76) does not represent a "surprising" result since, due to $\varepsilon > 0$ and $\boldsymbol{v} \in \boldsymbol{W}^{1,\infty}(\Omega)$, the generalized viscosity remains bounded from below and above and, hence, (2.16) can basically be interpreted as a Stokes system.

Proof of Corollary 4.16. The proof is based on the proofs of Theorem 4.11 and Corollary 4.14. Once again it is sufficient to estimate the projection errors $\boldsymbol{\xi}_h := \boldsymbol{j}_h\boldsymbol{v} - \boldsymbol{v}_h$ and $\eta := j_h\pi - \pi_h$ with respect to the distance $|(\boldsymbol{\xi}_h, \eta_h)|_{\text{lps}}$ given by (4.66). As above we estimate the terms $I_1, \ldots, I_4$ that arise in (4.47). Recalling the proofs of Corollary 4.14 and Theorem 4.11, we observe that the terms I_1 and I_2 are bounded by

$$I_1 \leq c_{\delta_1} h^2 \|\nabla\pi\|_2^2 + \delta_1 |(\boldsymbol{0}, \eta_h)|_{\text{lps}}^2, \qquad I_2 \leq c_{\delta_2} h^2 \|\nabla\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})\|_2^2 + \delta_2 |(\boldsymbol{\xi}_h, 0)|_{\text{lps}}^2.$$

Since $p > 2$ and $\varepsilon > 0$, by virtue of (4.60) the estimate (4.50) can be modified as follows:

$$\begin{aligned} I_3 &:= (\pi - j_h\pi, \nabla\cdot\boldsymbol{\xi}_h)_\Omega \leq \|\boldsymbol{D}(\boldsymbol{j}_h\boldsymbol{v} - \boldsymbol{v}_h)\|_2 \|\pi - j_h\pi\|_2 \\ &\leq \delta_3 \|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2^2 + c_{\varepsilon,\delta_3} \|\pi - j_h\pi\|_2^2 \leq \delta_3 |(\boldsymbol{\xi}_h, 0)|_{\text{lps}}^2 + c_{\varepsilon,\delta_3} h^2 \|\pi\|_{1,2}^2. \end{aligned}$$

In case of $p > 2$ and $\varepsilon > 0$, $\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) \in W^{1,2}(\Omega)^{d\times d}$ implies $\boldsymbol{v} \in \boldsymbol{W}^{2,2}(\Omega)$ because of

$$\int_\Omega |\nabla^2\boldsymbol{v}|^2 \,\mathrm{d}\boldsymbol{x} \leq \varepsilon^{2-p} \int_\Omega (\varepsilon + |\boldsymbol{D}\boldsymbol{v}|)^{p-2} |\nabla^2\boldsymbol{v}|^2 \,\mathrm{d}\boldsymbol{x} \leq c\varepsilon^{2-p} \int_\Omega |\nabla\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})|^2 \,\mathrm{d}\boldsymbol{x} < \infty.$$

Thus we can estimate the term I_4 just like in the proof of Corollary 4.14:

$$I_4 \leq c_{\delta_4} h^2 \|\boldsymbol{v}\|_{2,2}^2 + \delta_4 |(\boldsymbol{0}, \eta_h)|_{\text{lps}}^2.$$

Collecting all estimates above and choosing $\delta_1, \ldots, \delta_4$ sufficiently small, we easily deduce that $|(\boldsymbol{\xi}_h, \eta_h)|_{\text{lps}} = \mathcal{O}(h)$. As a result we arrive at $(4.75)_1$. Estimate $(4.75)_2$ follows from (2.45) and $(4.75)_1$. It remains to prove the error estimate for the pressure. First of all, we depict that $\boldsymbol{v}_h$ is uniformly bounded in $\boldsymbol{W}^{1,\infty}(\Omega)$ provided that $\boldsymbol{v} \in \boldsymbol{W}^{1,\infty}(\Omega)$. Using the inverse inequality (3.20) with $d = 2$, the $\boldsymbol{W}^{1,\infty}$-stability of $\boldsymbol{j}_h$, Korn's inequality and (4.60), we may estimate the $\boldsymbol{W}^{1,\infty}$-norm of $\boldsymbol{v}_h$ as follows

$$\begin{aligned} \|\boldsymbol{v}_h\|_{1,\infty} &\leq \|\boldsymbol{j}_h\boldsymbol{v} - \boldsymbol{v}_h\|_{1,\infty} + \|\boldsymbol{j}_h\boldsymbol{v}\|_{1,\infty} \\ &\leq c\Big[h^{-1}\|\boldsymbol{j}_h\boldsymbol{v} - \boldsymbol{v}_h\|_{1,2} + \|\boldsymbol{v}\|_{1,\infty}\Big] \\ &\leq c\Big[h^{-1}\|\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{v} - \boldsymbol{D}\boldsymbol{v}_h\|_2 + \|\boldsymbol{v}\|_{1,\infty}\Big] \\ &\leq c\Big[h^{-1}\varepsilon^{\frac{2-p}{2}}\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2 + \|\boldsymbol{v}\|_{1,\infty}\Big]. \end{aligned}$$

Recalling $|(\boldsymbol{\xi}_h, \eta_h)|_{\text{lps}} = \mathcal{O}(h)$, we realize that the right-hand side can be estimated independently of h and, hence, $\boldsymbol{v}_h$ is uniformly bounded in $\boldsymbol{W}^{1,\infty}(\Omega)$. In order to derive (4.76), we consult Lemma 4.5 that applied to the projection error η_h reads

$$\tilde{\beta}(2)\|\eta_h\|_2 \le \sup_{\boldsymbol{w}_h \in \boldsymbol{\mathcal{X}}_h^2} \frac{|(\nabla \cdot \boldsymbol{w}_h, \eta_h)_\Omega|}{\|\nabla \boldsymbol{w}_h\|_2} + \Big(\sum_{M \in \mathbb{M}_h} h_M^2 \|\boldsymbol{\theta}_h \nabla \eta_h\|_{2;M}^2 \Big)^{\frac{1}{2}} =: J_1 + J_2.$$

Similarly to the proof of Theorem 4.11 we deduce that the term J_1 can be bounded by

$$J_1 \le \|\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2 + \|j_h \pi - \pi\|_2. \tag{4.77}$$

Using Lemma 2.4 and the fact $p \ge 2$, we conclude that

$$\begin{aligned} \|\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2 &\lesssim \Big(\int_\Omega (\varepsilon + |\boldsymbol{D}\boldsymbol{v}| + |\boldsymbol{D}\boldsymbol{v}_h|)^{2(p-2)} |\boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{v}_h|^2 \,\mathrm{d}\boldsymbol{x} \Big)^{\frac{1}{2}} \\ &\lesssim \Big(\varepsilon + \|\nabla \boldsymbol{v}\|_\infty + \|\nabla \boldsymbol{v}_h\|_\infty \Big)^{p-2} \|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2. \end{aligned} \tag{4.78}$$

We recall that $\boldsymbol{v}_h$ is uniformly bounded in $\boldsymbol{W}^{1,\infty}(\Omega)$. Inserting (4.78) into (4.77), we get

$$J_1 \le c\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2 + h\|\pi\|_{1,2}.$$

Since $J_2 \equiv s_h(\eta_h)(\eta_h)^{\frac{1}{2}}/\sqrt{\alpha_0}$ and $|(\boldsymbol{\xi}_h, \eta_h)|_{\mathrm{lps}} = \mathcal{O}(h)$, we easily infer estimate (4.76). □

Compared to Theorem 4.12, Corollary 4.16 leads to improved a priori error estimates with respect to the order of convergence. Note that Corollary 4.16 requires different assumptions on the regularity and that it only includes the nondegenerate case $\varepsilon > 0$. The subsequent Corollary 4.17 depicts that we can get rid of the condition $\varepsilon > 0$ without losing convergence rate if we employ the following stabilization term suggested in [BBJL07, MST07],

$$s_h\Big((\boldsymbol{v}, \pi)\Big)\Big((\boldsymbol{w}, q)\Big) := \sum_{M \in \mathbb{M}_h} \Big(\alpha_M (\boldsymbol{\theta}_h \nabla \pi, \boldsymbol{\theta}_h \nabla q)_M + \nu_M (\theta_h \nabla \cdot \boldsymbol{v}, \theta_h \nabla \cdot \boldsymbol{w})_M \Big). \tag{4.79}$$

The patch-wise constants α_M and ν_M are specified in Corollary 4.17 below. For the remainder of this section we assume that the fluctuation operator θ_h does not only satisfy stability as in Assumption 4.2 but also approximability: We suppose that for $\nu > 1$, $k \in \{0, 1\}$, it holds $\|\theta_h w\|_{\nu;M} \lesssim h_M^k \|\nabla^k w\|_{\nu;M}$ for all $w \in W^{k,\nu}(\Omega)$ and $M \in \mathbb{M}_h$. The term (4.79) stabilizes not only the pressure gradient but also the incompressibility constraint.

Corollary 4.17. *Let $d \ge 2$. For $p \in [2, \infty)$ and $\varepsilon \in [0, \varepsilon_0]$ let $\boldsymbol{\mathcal{S}}$ satisfy Assumption 2.1 and let $\boldsymbol{\mathcal{F}}$ be defined by (2.39). Let $(\boldsymbol{v}, \pi)$ be the solution to* **(P1)***, and let $(\boldsymbol{v}_h, \pi_h)$ be the solution to* **(P1$_h$)** *where the stabilization s_h is defined by (4.79) with $\alpha_M := \alpha_0 h_M^2$ and $\nu_M := \nu_0 > 0$ for all $M \in \mathbb{M}_h$. We assume that the solution $(\boldsymbol{v}, \pi)$ satisfies $\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) \in W^{1,2}(\Omega)^{d \times d}$, $\boldsymbol{v} \in \boldsymbol{W}^{2,2}(\Omega)$, and $\pi \in W^{1,2}(\Omega)$. Then, the error of approximation is estimated by*

$$\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2 \le C_v h, \qquad \|\boldsymbol{v} - \boldsymbol{v}_h\|_{1,p} \le C_v' h^{\frac{2}{p}}, \tag{4.80}$$

$$\|\pi - \pi_h\|_{p'} \le C_\pi h. \tag{4.81}$$

The constants C_v, C_v', C_π only depend on p, Ω, α_0, ν_0, $\|\nabla \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})\|_2$, $\|\boldsymbol{v}\|_{2,2}$, $\|\pi\|_{1,2}$, and C_π additionally depends on $\tilde{\beta}(p)$.

Remark 4.13. If $p > 2$ and $\varepsilon > 0$, $\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) \in W^{1,2}(\Omega)^{d\times d}$ implies $\boldsymbol{v} \in \boldsymbol{W}^{2,2}(\Omega)$ due to

$$\int\limits_\Omega |\nabla^2 \boldsymbol{v}|^2 \,\mathrm{d}\boldsymbol{x} \leq \varepsilon^{2-p} \int\limits_\Omega (\varepsilon + |\boldsymbol{D}\boldsymbol{v}|)^{p-2} |\nabla^2 \boldsymbol{v}|^2 \,\mathrm{d}\boldsymbol{x} \leq c\varepsilon^{2-p} \int\limits_\Omega |\nabla \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})|^2 \,\mathrm{d}\boldsymbol{x} < \infty.$$

Hence, the assumptions of Corollary 4.16 are more restrictive than the ones of Corollary 4.17.

Proof. The proof is based on the proofs of Theorem 4.11 and Corollary 4.16. It is sufficient to estimate the projection errors $\boldsymbol{\xi}_h := \boldsymbol{j}_h \boldsymbol{v} - \boldsymbol{v}_h$ and $\eta := j_h \pi - \pi_h$ with respect to

$$|(\boldsymbol{\xi}_h, \eta_h)|^2_{\mathrm{lps}} := \|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{j}_h \boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2^2 + s_h\Big((\boldsymbol{\xi}_h, \eta_h)\Big)\Big((\boldsymbol{\xi}_h, \eta_h)\Big). \tag{4.82}$$

Similarly to the proof of Theorem 4.11, we obtain the equivalence

$$\begin{aligned} |(\boldsymbol{\xi}_h, \eta_h)|^2_{\mathrm{lps}} \sim s_h\Big((\boldsymbol{j}_h \boldsymbol{v}, j_h \pi)\Big)\Big((\boldsymbol{\xi}_h, \eta_h)\Big) + (\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{j}_h \boldsymbol{v}) - \boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{\xi}_h)_\Omega \\ - (j_h \pi - \pi, \nabla \cdot \boldsymbol{\xi}_h)_\Omega + (\nabla \cdot (\boldsymbol{j}_h \boldsymbol{v} - \boldsymbol{v}_h), \eta_h)_\Omega =: I_1, \dots, I_4. \end{aligned} \tag{4.83}$$

We estimate the terms $I_1, \dots, I_4$ separately. Using the stability of θ_h and the approximation property of θ_h and $\boldsymbol{j}_h$, for arbitrary $\delta_1 > 0$ we estimate the term I_1 by (cf. [MST07])

$$\begin{aligned} I_1 &\leq c_{\delta_1} s_h\Big((\boldsymbol{j}_h \boldsymbol{v}, j_h \pi)\Big)\Big((\boldsymbol{j}_h \boldsymbol{v}, j_h \pi)\Big) + \delta_1 s_h\Big((\boldsymbol{\xi}_h, \eta_h)\Big)\Big((\boldsymbol{\xi}_h, \eta_h)\Big) \\ &\leq c_{\delta_1} \sum_M \Big[\alpha_M \|\nabla j_h \pi\|^2_{2;M} + \nu_M \|\nabla \cdot (\boldsymbol{j}_h \boldsymbol{v} - \boldsymbol{v})\|^2_{2;M} + \nu_M \|\theta_h \nabla \cdot \boldsymbol{v}\|^2_{2;M}\Big] + \delta_1 |(\boldsymbol{\xi}_h, \eta_h)|^2_{\mathrm{lps}} \\ &\leq c_{\delta_1} \Big(h^2 \|\pi\|^2_{1,2} + h^2 \|\boldsymbol{v}\|^2_{2,2}\Big) + \delta_1 |(\boldsymbol{\xi}_h, \eta_h)|^2_{\mathrm{lps}} \end{aligned}$$

where c_{δ_1} only depends on α_0, ν_0, δ_1. Recalling the proof of Thm. 4.11, we observe that

$$I_2 \leq c_{\delta_2} h^2 \|\nabla \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})\|_2^2 + \delta_2 |(\boldsymbol{\xi}_h, 0)|^2_{\mathrm{lps}}.$$

Using the orthogonality property of j_h, we estimate the term I_3 as follows (cf. [MST07]):

$$\begin{aligned} I_3 &:= (\pi - j_h \pi, \nabla \cdot \boldsymbol{\xi}_h)_\Omega = (\pi - j_h \pi, \theta_h \nabla \cdot \boldsymbol{\xi}_h)_\Omega \\ &\leq c_{\delta_3} \sum_{M \in \mathbb{M}_h} \nu_M^{-1} \|\pi - j_h \pi\|^2_{2;M} + \delta_3 \sum_{M \in \mathbb{M}_h} \nu_M \|\theta_h \nabla \cdot \boldsymbol{\xi}_h\|^2_{2;M} \\ &\leq c_{\delta_3} h^2 \|\pi\|^2_{1,2} + \delta_3 |(\boldsymbol{\xi}_h, 0)|^2_{\mathrm{lps}}. \end{aligned}$$

As in the proof of Corollary 4.14, for arbitrary $\delta_4 > 0$ the term I_4 is bounded by

$$I_4 \leq c_{\delta_4} h^2 \|\boldsymbol{v}\|^2_{2,2} + \delta_4 |(\boldsymbol{0}, \eta_h)|^2_{\mathrm{lps}}.$$

Collecting all estimates above and choosing $\delta_1, \dots, \delta_4$ sufficiently small, we easily deduce that $|(\boldsymbol{\xi}_h, \eta_h)|_{\mathrm{lps}} = \mathcal{O}(h)$. As a result we arrive at $(4.80)_1$. Estimate $(4.80)_2$ follows from (2.45) and $(4.80)_1$. It remains to prove the pressure-estimate (4.81). To this end, we consult Lemma 4.5 that applied to the projection error η_h reads

$$\tilde{\beta}(p) \|\eta_h\|_{p'} \leq \sup_{\boldsymbol{w}_h \in \boldsymbol{\mathcal{X}}_h^p} \frac{|(\nabla \cdot \boldsymbol{w}_h, \eta_h)_\Omega|}{\|\nabla \boldsymbol{w}_h\|_p} + \Big(\sum_{M \in \mathbb{M}_h} h_M^{p'} \|\boldsymbol{\theta}_h \nabla \eta_h\|^{p'}_{p';M}\Big)^{\frac{1}{p'}} =: J_1 + J_2.$$

Similarly to the proof of Thm. 4.12 it follows that for some $c = c(\Omega, p, \varepsilon_0, \boldsymbol{f}) > 0$

$$J_1 \le c(\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2 + h\|\pi\|_{1,2}). \tag{4.84}$$

Using Hölder's inequality with $\frac{p'}{2} + \frac{2-p'}{2} = 1$ twice, we conclude that

$$J_2 \le \Big(\sum_{M\in\mathbb{M}_h} h_M^{p'}\|\boldsymbol{\theta}_h\nabla\eta_h\|_{2;M}^{p'}|M|^{\frac{2-p'}{2}}\Big)^{\frac{1}{p'}} \le \Big(\sum_{M\in\mathbb{M}_h} h_M^{2}\|\boldsymbol{\theta}_h\nabla\eta_h\|_{2;M}^{2}\Big)^{\frac{1}{2}}\Big(\sum_{M\in\mathbb{M}_h}|M|\Big)^{\frac{2-p'}{2}}$$

and, hence, $J_2 \le |(\mathbf{0}, \eta_h)|_{\mathrm{lps}}|\Omega|^{\frac{2-p'}{2}}/\sqrt{\alpha_0}$. Since $|(\boldsymbol{\xi}_h, \eta_h)|_{\mathrm{lps}} = \mathcal{O}(h)$, we arrive at (4.81). □

Refinement of Corollary 4.14: Using the stabilization (4.79), for $p \in (1,2]$ we can also confine Corollary 4.14 in the sense that we can replace the regularity assumption $\boldsymbol{v} \in \boldsymbol{W}^{1,\infty}(\Omega)$ by the less restrictive one $\boldsymbol{v} \in \boldsymbol{W}^{2,2}(\Omega)$. Note that, indeed, for $p \in (1,2]$ the conditions $\boldsymbol{v} \in \boldsymbol{W}^{1,\infty}(\Omega)$ and $\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) \in W^{1,2}(\Omega)^{d\times d}$ imply $\boldsymbol{v} \in \boldsymbol{W}^{2,2}(\Omega)$ due to

$$\int_\Omega |\nabla^2\boldsymbol{v}|^2 \,\mathrm{d}\boldsymbol{x} \le (\varepsilon + \|\boldsymbol{D}\boldsymbol{v}\|_\infty)^{2-p}\int_\Omega (\varepsilon+|\boldsymbol{D}\boldsymbol{v}|)^{p-2}|\nabla^2\boldsymbol{v}|^2 \,\mathrm{d}\boldsymbol{x} < \infty$$

(see Lemma 2.27). We end up with the following version of Corollary 4.14:

Corollary 4.18. *Let $d \ge 2$. For $p \in (1,2]$ and $\varepsilon \in [0,\varepsilon_0]$ let $\boldsymbol{\mathcal{S}}$ satisfy Assumption 2.1 and let $\boldsymbol{\mathcal{F}}$ be defined by* (2.39)*. Suppose that $\mathbb{M}_h$ is quasi-uniform. Let $(\boldsymbol{v}, \pi)$ be the solution to* **(P1)***, and let $(\boldsymbol{v}_h, \pi_h)$ be the solution to* **(P1$_h$)***, where the stabilization s_h is defined by* (4.79) *with $\alpha_M := \alpha_0 h_M^2$ and $\nu_M := \nu_0 > 0$ for all $M \in \mathbb{M}_h$. Assume that $(\boldsymbol{v}, \pi)$ satisfies the regularity $\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) \in W^{1,2}(\Omega)^{d\times d}$, $\boldsymbol{v} \in \boldsymbol{W}^{2,2}(\Omega)$ and $\pi \in W^{1,2}(\Omega)$. Then there hold*

$$\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2 \le C_v h, \qquad \|\boldsymbol{v} - \boldsymbol{v}_h\|_{1,p} \le C_v' h, \tag{4.85}$$

$$\|\pi - \pi_h\|_{p'} \le C_\pi h^{1+\frac{d}{p'}-\frac{d}{2}}. \tag{4.86}$$

If additionally $\varepsilon > 0$, then the pressure-error in $L^2(\Omega)$ is estimated by

$$\|\pi - \pi_h\|_2 \le C_\pi' h. \tag{4.87}$$

The constants C_v, C_v', C_π only depend on p, ε_0, Ω, α_0, ν_0, $\|\nabla\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})\|_2$, $\|\pi\|_{1,2}$, $\|\boldsymbol{v}\|_{2,2}$, and C_π additionally depends on $\tilde\beta(p)$. The constant C_π' only depends on p, ε, Ω, α_0, ν_0, $\tilde\beta(2)$, $\|\nabla\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})\|_2$, $\|\boldsymbol{v}\|_{2,2}$, $\|\pi\|_{1,2}$, and it may explode as $\varepsilon \to 0^+$.

Remark 4.14. Note that for $d = 2$ the condition $\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) \in W^{1,2}(\Omega)^{d\times d}$ implies $\boldsymbol{v} \in \boldsymbol{W}^{2,2-\delta}(\Omega)$ for all $\delta > 0$ due Lemma 4.10.

Proof of Corollary 4.18. The proof combines the proofs of Corollaries 4.17 and 4.14. Once again we have to estimate the terms $I_1, \dots, I_4$ that arise in (4.83). We estimate the terms I_1, I_3 as in Corollary 4.17 whereas we estimate the terms I_2, I_4 just like in Corollary 4.14. Following the proof of Corollary 4.17, we consequently arrive at $(4.85)_1$. We derive (4.86) recapitulating the proof of (4.65) for $d \ge 2$. Finally, we obtain (4.87) if we follow the proof of (4.65) while carrying out all arguments in an L^2-setting and using (2.47) for $\varepsilon > 0$. □

4.7 Non-steady p-Stokes equations

In this section we investigate the time-space discretization of non-steady p-Stokes systems. Concerning time discretization, optimal a priori error estimates have recently been derived in [BDR09], in which a semi-implicit Euler scheme applied to the p-Navier-Stokes system (2.14) has been considered. In order to assess the approximation error caused by temporal and spatial discretization, we generalize previous results established in Sections 4.5, 4.6.

Time discretization: For $T > 0$ let $I := [0, T]$ be a time interval. We discretize **(P4)** in time. To this end, for $N \in \mathbb{N}$ we introduce the time step size $k := T/N > 0$ and the corresponding net $\mathbb{I}_N := \{t_n\}_{n=0}^N$ with $t_n := nk$. We consider the implicit Euler scheme:

($\mathbf{P4}^{\boldsymbol{k}}$) *Let $\boldsymbol{v}^0 := \hat{\boldsymbol{v}}$. For $n = 1, \dots, N$ find $(\boldsymbol{v}^n, \pi^n) \in \boldsymbol{\mathcal{X}}^p \times \mathcal{Q}^p$ such that*

$$\begin{aligned} (d_t\boldsymbol{v}^n, \boldsymbol{w})_\Omega + (\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}^n), \boldsymbol{D}\boldsymbol{w})_\Omega - (\pi^n, \nabla \cdot \boldsymbol{w})_\Omega &= (\boldsymbol{f}, \boldsymbol{w})_\Omega \qquad \forall \boldsymbol{w} \in \boldsymbol{\mathcal{X}}^p \\ (\nabla \cdot \boldsymbol{v}^n, q)_\Omega &= 0 \qquad \forall q \in \mathcal{Q}^p \end{aligned} \tag{4.88}$$

where the discrete time derivative is defined by

$$d_t\boldsymbol{v}^n := \frac{\boldsymbol{v}^n - \boldsymbol{v}^{n-1}}{k}. \tag{4.89}$$

Remark 4.15. Testing $(4.88)_1$ with $\boldsymbol{w} := \boldsymbol{v}^n$ and using (2.40), we observe that

$$\max_{1\le n\le N} \|\boldsymbol{v}^n\|_2^2 + k\sum_{n=1}^N \|\boldsymbol{v}^n\|_{1,p}^p \le C = C(\boldsymbol{f}, \hat{\boldsymbol{v}}, p, \varepsilon_0, \Omega). \tag{4.90}$$

The time discretization of p-structure systems has been studied intensively in recent years (cf. [DPR02, DPR06]). In [BDR09], Berselli et al. analyzed the p-Navier-Stokes equations (2.14) complemented with space-periodic boundary conditions and its time discretization with a semi-implicit Euler scheme. They derived optimal error estimates as depicted by

Lemma 4.19. *For $d = 3$ let us consider system* (2.14) *complemented with space-periodic boundary conditions. For $p \in (3/2, 2]$ and $\varepsilon \in [0, \varepsilon_0]$ let the extra stress tensor $\boldsymbol{\mathcal{S}}$ satisfy Assumption 2.1. We assume that $\boldsymbol{f} \in C(I; \boldsymbol{W}^{1,2}(\Omega)) \cap W^{1,2}(I; \boldsymbol{L}^2(\Omega))$ and that $\hat{\boldsymbol{v}} \in \boldsymbol{W}^{2,2}_{\text{div}}(\Omega)$ with $\nabla \cdot \boldsymbol{\mathcal{S}}(\boldsymbol{D}\hat{\boldsymbol{v}}) \in \boldsymbol{L}^2(\Omega)$. Let $\boldsymbol{v}$ be the strong solution to Problem* **(P7)** *with $\boldsymbol{\mathcal{V}}^p$ replaced by $\boldsymbol{\mathcal{V}}^p_{\text{per}}$ as in Lemma 2.30. In particular, $\boldsymbol{v}$ satisfies the regularity* (2.90). *We set $\boldsymbol{v}^0 := \hat{\boldsymbol{v}}$. For $n = 1, 2, \dots$ let $\boldsymbol{v}^n$ be the solution to the system*

$$\left.\begin{aligned} d_t\boldsymbol{v}^n - \nabla \cdot \boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}^n) + [\boldsymbol{v}^{n-1} \cdot \nabla]\boldsymbol{v}^n + \nabla\pi^n &= \boldsymbol{f}(t_n) \\ \nabla \cdot \boldsymbol{v}^n &= 0 \end{aligned}\right\} \quad \text{in } \Omega \tag{4.91}$$

endowed with space-periodic boundary conditions. Here, $d_t\boldsymbol{v}^n$ is defined as in (4.89). *Then there exists a time-step size $k_0 > 0$ such that for $k \in (0, k_0)$ there holds*

$$\max_{0\le n\le N} \|\boldsymbol{v}(t_n) - \boldsymbol{v}^n\|_2^2 + k\sum_{n=0}^N \|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}(t_n)) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}^n)\|_2^2 \le Ck^2 \tag{4.92}$$

where the constant C and k_0 only depend on ε_0, p, $\boldsymbol{f}$, $\hat{\boldsymbol{v}}$, T, Ω.

In [BDR09], Berselli et al. proved additional regularity of the semi-discrete solutions:

Lemma 4.20. *Let $d = 3$. For $p \in (3/2, 2]$ and $\varepsilon \in [0, \varepsilon_0]$ let $\boldsymbol{\mathcal{S}}$ satisfy Assumption 2.1. We assume that $\boldsymbol{f} \in C(I; \boldsymbol{W}^{1,2}(\Omega))$ and that $\hat{\boldsymbol{v}} \in \boldsymbol{W}^{2,2}_{\mathrm{div}}(\Omega)$. Then there exists $k' = k'(p, \varepsilon_0, \boldsymbol{f}, \hat{\boldsymbol{v}}, T, \Omega)$ such that for $k \in (0, k')$ the solution $\boldsymbol{v}^n$ to system* (4.91) *satisfies*

$$\max_{0 \le n \le N} \|d_t \boldsymbol{v}^n\|_2^2 + k \sum_{n=0}^{N} \|\nabla \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}^n)\|_2^{2\frac{5p-6}{2-p}} + k \sum_{n=0}^{N} \|d_t \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}^n)\|_2^2 \le C \tag{4.93}$$

where the constant C only depends on p, ε_0, $\boldsymbol{f}$, $\hat{\boldsymbol{v}}$, T, Ω. Moreover, $\nabla \pi^n$ belongs to the space $l^{2\frac{5p-6}{2-p}}(\mathbb{I}_N; \boldsymbol{L}^2(\Omega))$ and its corresponding norm is bounded by a constant that only depends on p, ε, $\boldsymbol{f}$, $\hat{\boldsymbol{v}}$, T, Ω and that may explode as $\varepsilon \to 0^+$.

As a consequence of (4.93), Berselli et al. showed in [BDR09] that the strong solution $\boldsymbol{v}^n$ to system (4.91) even belongs to $l^\infty(\mathbb{I}_N; \boldsymbol{W}^{1,r}(\Omega))$ with $1 \le r < 6(p-1)$ and that its corresponding norm is bounded by a constant which only depends on p, ε_0, $\boldsymbol{f}$, $\hat{\boldsymbol{v}}$, T, Ω. This result follows from the following well-known inequality (see Diening et al. [DPR02, BDR09]): For $1 \le r < 6(p-1)$ there exists a constant $c = c(p, \Omega, r)$ such that

$$\max_{0 \le n \le N} \|\nabla \boldsymbol{v}^n\|_r^p \le c\Big(\varepsilon^p + k \sum_{n=0}^{N} \Big(\|\nabla \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}^n)\|_2^{2\frac{5p-6}{2-p}} + \|d_t \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}^n)\|_2^2\Big)\Big). \tag{4.94}$$

Note that $(5p-6)/(2-p) > 1$ in (4.93) for the considered range of p. The following well-known lemma depicts that, consequently, the strong solution $\boldsymbol{v}^n$ to system (4.91) belongs to $l^2(\mathbb{I}_N; \boldsymbol{W}^{2,\frac{4}{4-p}}(\Omega))$ and its norm is bounded independently of k.

Lemma 4.21. *Let $p \le 2$. For all sufficiently smooth $\boldsymbol{w}^n \in l^\infty(\{t_n\}_{n=l}^m; \boldsymbol{L}^2(U))$ there holds*

$$k \sum_{n=l}^{m} \|\boldsymbol{w}^n\|_{2,\frac{4}{4-p};U}^2 \le c \sup_{l \le n \le m} \|\varepsilon + |\nabla \boldsymbol{w}^n|\|_{2;U}^{2-p} \Big[k \sum_{n=l}^{m} \|\nabla \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{w}^n)\|_{2;U}^2\Big]$$

where the constant $c > 0$ only depends on p.

Proof. See Lemma 4.2 in [DER07]. Actually, the desired estimate appears within the proof of Lemma 4.2 in [DER07] and it is shown with $\boldsymbol{D}$ replaced by ∇. The proof of Lemma 4.21 follows the same arguments. Note that the assertion holds for arbitrary $d \ge 2$. □

Knowledge about the regularity of $(\boldsymbol{v}^n, \pi^n)$, as provided by Lemma 4.20, enables the derivation of error estimates for the space discretization, as depicted in the next paragraph.

Space-time discretization: The semi-discrete Problem $(\mathbf{P4}^k)$ is discretized in space by equal-order d-linear $\mathbb{Q}_1/\mathbb{Q}_1$ finite elements. For pressure-stabilization, we apply the LPS method introduced in Section 3.2. The fully discretized problem reads:

($\mathbf{P4}_h^k$) *Let $\boldsymbol{v}_h^0 := \boldsymbol{j}_h \boldsymbol{v}^0$. For $n = 1, \ldots, N$ find $\boldsymbol{u}_h^n \equiv (\boldsymbol{v}_h^n, \pi_h^n) \in \boldsymbol{\mathcal{X}}_h^p \times \mathcal{Q}_h^p$ such that*

$$
\begin{aligned}
(d_t \boldsymbol{v}_h^n, \boldsymbol{w}_h)_\Omega + &(\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}_h^n), \boldsymbol{D}\boldsymbol{w}_h)_\Omega - (\pi_h^n, \nabla \cdot \boldsymbol{w}_h)_\Omega + (\nabla \cdot \boldsymbol{v}_h^n, q_h)_\Omega \\
&+ s_h(\boldsymbol{u}_h^n)(\boldsymbol{\omega}_h) = (\boldsymbol{f}, \boldsymbol{w}_h)_\Omega \qquad \forall \boldsymbol{\omega}_h \equiv (\boldsymbol{w}_h, q_h) \in \boldsymbol{\mathcal{X}}_h^p \times \mathcal{Q}_h^p
\end{aligned}
\tag{4.95}
$$

where s_h stands for a stabilization term such as (4.79).

Remark 4.16. Testing (4.95) with $\boldsymbol{\omega}_h := (\boldsymbol{v}_h^n, \pi_h^n)$ and using (2.40), we observe that

$$
\max_{1 \le n \le N} \|\boldsymbol{v}_h^n\|_2^2 + k \sum_{n=1}^N \|\boldsymbol{v}_h^n\|_{1,p}^p \le C = C(\boldsymbol{f}, \hat{\boldsymbol{v}}, p, \varepsilon_0, \Omega). \tag{4.96}
$$

The following theorem measures the error between the solution $\boldsymbol{v}^n$ of the semi-discrete Problem ($\mathbf{P4}^k$) and the solution $\boldsymbol{v}_h^n$ of the fully-discrete Problem ($\mathbf{P4}_h^k$). For its proof we combine methods from [DER07] and Section 4.6.

Theorem 4.22. *For $p \in (1, 2]$ and $\varepsilon \in [0, \varepsilon_0]$ let the extra stress tensor $\boldsymbol{\mathcal{S}}$ satisfy Assumption 2.1. Let $(\boldsymbol{v}^n, \pi^n)$ be the solution to Problem ($\mathbf{P4}^k$), and let $(\boldsymbol{v}_h^n, \pi_h^n)$ be the solution to Problem ($\mathbf{P4}_h^k$), where the stabilization term s_h is defined by (4.79). For $\nu > 1$, $k \in \{0, 1\}$ let the fluctuation operator θ_h satisfy $\|\theta_h w\|_{\nu;M} \lesssim h_M^k \|\nabla^k w\|_{\nu;M}$ for all $w \in W^{k,\nu}(\Omega)$, $M \in \mathbb{M}_h$. We assume that there exists a constant $C > 0$ independent of k so that*

$$
\sup_{1 \le n \le N} \|\nabla \boldsymbol{v}^n\|_2^2 + k \sum_{n=1}^N \|\nabla \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}^n)\|_2^2 + k \sum_{n=1}^N \|\nabla \pi^n\|_2^2 \le C. \tag{4.97}
$$

Moreover, we suppose that $\boldsymbol{v}^0 = \hat{\boldsymbol{v}} \in \boldsymbol{W}_0^{1,2}(\Omega)$. If $h^{2-\frac{d(2-p)}{2}} \le ck$ with some $c > 0$, then for $\alpha_M := \alpha_0 h^2$ and $\nu_M := \nu_0$ the error of approximation is estimated by

$$
\begin{aligned}
\sup_{1 \le n \le N} \|\boldsymbol{v}^n - \boldsymbol{v}_h^n\|_2^2 &+ k \sum_{n=1}^N \|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}^n) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h^n)\|_2^2 \\
&+ k \sum_{n=1}^N s_h\Big((\boldsymbol{\xi}_h^n, \eta_h^n)\Big)\Big((\boldsymbol{\xi}_h^n, \eta_h^n)\Big) \le C' h^{2-\frac{d(2-p)}{2}}.
\end{aligned}
\tag{4.98}
$$

Here, $\boldsymbol{\xi}_h^n := \boldsymbol{j}_h \boldsymbol{v}^n - \boldsymbol{v}_h^n$ and $\eta_h^n := j_h \pi^n - \pi_h^n$, where j_h is the interpolation operator of Lemma 4.4. The constant $C' > 0$ only depends on C, $\hat{\boldsymbol{v}}$, $\boldsymbol{f}$, p, ε_0, Ω, α_0, ν_0.

Remark 4.17. The smallness-assumption on the mesh-size is less restrictive than the Courant-Friedrichs-Lewy (CFL) condition. It also appears in the article [DER07] in which the temporal and spatial discretization of parabolic p-structure systems is analyzed. Such p-structure systems correspond to our p-Stokes systems if the pressure and the incompressibility constraint are omitted. Concerning the time-space discretization of p-structure systems, Diening et al. [DER07] established the optimal a priori error estimate

$$
\sup_{n \in \{1, \ldots, N\}} \|\boldsymbol{v}(t_n) - \boldsymbol{v}_h^n\|_2^2 + k \sum_{n=1}^N \|\boldsymbol{\mathcal{F}}(\nabla \boldsymbol{v}(t_n)) - \boldsymbol{\mathcal{F}}(\nabla \boldsymbol{v}_h^n)\|_2^2 \le c(h^2 + k^2) \tag{4.99}
$$

provided that $h^{2-\frac{d(2-p)}{2}} \leq ck$. Hence, the error estimate (4.98) seems to be suboptimal with respect to the convergence order. Compared to (4.99), the reduced convergence rate of (4.98) results from the low regularity of the semi-discrete pressure π^n. In particular, in Theorem 4.22 it is only assumed that π^n belongs to $l^2(\mathbb{I}_N; W^{1,2}(\Omega))$ and that its norm is uniformly bounded. In order to derive an optimal error estimate similar to (4.99), we need to assume that π^n remains uniformly bounded in $l^{p'}(\mathbb{I}_N; W^{1,p'}(\Omega))$ (see Corollary 4.23 below). However, we are not able to show this stronger regularity of π^n. Note that the regularity-assumption (4.97) is satisfied at least in the case of space periodic boundary conditions (see Lemma 4.20 and its subsequent discussion).

Proof of Theorem 4.22. We proceed similarly to the article [DER07], in which parabolic p-structure systems and their time-space discretizations are studied. We define $\boldsymbol{e}_h^n := \boldsymbol{v}^n - \boldsymbol{v}_h^n$. Taking the difference between (4.88) and (4.95), we observe that

$$\begin{aligned}(d_t \boldsymbol{e}_h^n, \boldsymbol{w}_h)_\Omega + (\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}^n) - \boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}_h^n), \boldsymbol{D}\boldsymbol{w}_h)_\Omega - (\pi^n - \pi_h^n, \nabla\cdot \boldsymbol{w}_h)_\Omega \\ + (\nabla\cdot \boldsymbol{e}_h^n, q_h)_\Omega = s_h\Big((\boldsymbol{v}_h^n, \pi_h^n)\Big)\Big((\boldsymbol{w}_h, q_h)\Big) \qquad \forall (\boldsymbol{w}_h, q_h) \in \boldsymbol{\mathcal{X}}_h^p \times \mathcal{Q}_h^p.\end{aligned} \tag{4.100}$$

Setting $\boldsymbol{\xi}_h^n := \boldsymbol{j}_h \boldsymbol{v}^n - \boldsymbol{v}_h^n$ and $\eta_h^n := j_h \pi^n - \pi_h^n$, we define the quantity

$$\begin{aligned}E := \frac{1}{2}\|\boldsymbol{e}_h^m\|_2^2 + \frac{k^2}{2}\sum_{n=1}^m \|d_t \boldsymbol{e}_h^n\|_2^2 + k\sum_{n=1}^m (\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}^n) - \boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}_h^n), \boldsymbol{D}\boldsymbol{e}_h^n)_\Omega \\ + k\sum_{n=1}^m \Big\{ s_h\Big((\boldsymbol{j}_h \boldsymbol{v}^n, j_h \pi^n)\Big)\Big((\boldsymbol{\xi}_h^n, \eta_h^n)\Big) - s_h\Big((\boldsymbol{v}_h^n, \pi_h^n)\Big)\Big((\boldsymbol{\xi}_h^n, \eta_h^n)\Big)\Big\}.\end{aligned} \tag{4.101}$$

We notice that $(d_t \boldsymbol{e}_h^n, \boldsymbol{e}_h^n)_\Omega = (d_t \boldsymbol{e}_h^n, \boldsymbol{j}_h \boldsymbol{e}_h^n)_\Omega + (d_t \boldsymbol{e}_h^n, \boldsymbol{v}^n - \boldsymbol{j}_h \boldsymbol{v}^n)_\Omega$ due to $\boldsymbol{j}_h \boldsymbol{v}_h^n = \boldsymbol{v}_h^n$ and

$$\begin{aligned} k\sum_{n=1}^m (d_t \boldsymbol{e}_h^n, \boldsymbol{e}_h^n)_\Omega &= \sum_{n=1}^m \{\|\boldsymbol{e}_h^n\|_2^2 - (\boldsymbol{e}_h^{n-1}, \boldsymbol{e}_h^n)_\Omega\} \\ &= \frac{1}{2}\sum_{n=1}^m \|\boldsymbol{e}_h^n\|_2^2 + \frac{1}{2}\sum_{n=1}^m \|\boldsymbol{e}_h^{n-1}\|_2^2 - \sum_{n=1}^m (\boldsymbol{e}_h^{n-1}, \boldsymbol{e}_h^n)_\Omega - \frac{1}{2}\|\boldsymbol{e}_h^0\|_2^2 + \frac{1}{2}\|\boldsymbol{e}_h^m\|_2^2 \\ &= \frac{1}{2}\sum_{n=1}^m \|\boldsymbol{e}_h^n - \boldsymbol{e}_h^{n-1}\|_2^2 - \frac{1}{2}\|\boldsymbol{e}_h^0\|_2^2 + \frac{1}{2}\|\boldsymbol{e}_h^m\|_2^2 \\ &= \frac{k^2}{2}\sum_{n=1}^m \|d_t \boldsymbol{e}_h^n\|_2^2 - \frac{1}{2}\|\boldsymbol{e}_h^0\|_2^2 + \frac{1}{2}\|\boldsymbol{e}_h^m\|_2^2.\end{aligned}$$

Using this, we can rewrite E as follows:

$$\begin{aligned} E = k\sum_{n=1}^m (d_t \boldsymbol{e}_h^n, \boldsymbol{v}^n - \boldsymbol{v}_h^n)_\Omega + k\sum_{n=1}^m (\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}^n) - \boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}_h^n), \boldsymbol{D}\boldsymbol{v}^n - \boldsymbol{D}\boldsymbol{v}_h^n)_\Omega \\ - k\sum_{n=1}^m (\pi^n - \pi_h^n, \nabla\cdot(\boldsymbol{v}^n - \boldsymbol{v}_h^n))_\Omega + k\sum_{n=1}^m (\nabla\cdot(\boldsymbol{v}^n - \boldsymbol{v}_h^n), \pi^n - \pi_h^n)_\Omega \\ + k\sum_{n=1}^m \Big\{ s_h\Big((\boldsymbol{j}_h \boldsymbol{v}^n, j_h \pi^n)\Big)\Big((\boldsymbol{\xi}_h^n, \eta_h^n)\Big) - s_h\Big((\boldsymbol{v}_h^n, \pi_h^n)\Big)\Big((\boldsymbol{\xi}_h^n, \eta_h^n)\Big)\Big\} + \frac{1}{2}\|\boldsymbol{e}_h^0\|_2^2.\end{aligned}$$

Using the disturbed Galerkin orthogonality (4.100), we consequently arrive at

$$\begin{aligned}
E &= k\sum_{n=1}^{m}(d_t\boldsymbol{e}_h^n, \boldsymbol{v}^n - \boldsymbol{j}_h\boldsymbol{v}^n)_\Omega + k\sum_{n=1}^{m}(\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}^n) - \boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}_h^n), \boldsymbol{D}\boldsymbol{v}^n - \boldsymbol{D}\boldsymbol{j}_h\boldsymbol{v}^n)_\Omega \\
&\quad - k\sum_{n=1}^{m}(\pi^n - j_h\pi^n, \nabla\cdot(\boldsymbol{v}^n - \boldsymbol{j}_h\boldsymbol{v}^n))_\Omega - k\sum_{n=1}^{m}(j_h\pi^n - \pi_h^n, \nabla\cdot(\boldsymbol{v}^n - \boldsymbol{j}_h\boldsymbol{v}^n))_\Omega \\
&\quad + k\sum_{n=1}^{m}(\nabla\cdot(\boldsymbol{v}^n - \boldsymbol{v}_h^n), \pi^n - j_h\pi^n)_\Omega + k\sum_{n=1}^{m} s_h\Big((\boldsymbol{j}_h\boldsymbol{v}^n, j_h\pi^n)\Big)\Big((\boldsymbol{\xi}_h^n, \eta_h^n)\Big) + \frac{1}{2}\|\boldsymbol{e}_h^0\|_2^2 \\
&=: F_1 + F_2 + F_3 + F_4 + F_5 + F_6 + F_7. \qquad (4.102)
\end{aligned}$$

By means of Lemma 2.4, for some $c = c(p) > 0$ we estimate the quantity E from below by

$$\begin{aligned}
E \geq \frac{1}{2}\|\boldsymbol{e}_h^m\|_2^2 &+ \frac{k^2}{2}\sum_{n=1}^{m}\|d_t\boldsymbol{e}_h^n\|_2^2 + ck\sum_{n=1}^{m}\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}^n) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h^n)\|_2^2 \\
&+ k\sum_{n=1}^{m} s_h\Big((\boldsymbol{\xi}_h^n, \eta_h^n)\Big)\Big((\boldsymbol{\xi}_h^n, \eta_h^n)\Big).
\end{aligned}$$

Below we estimate the terms $F_1, \ldots, F_7$ defined in (4.102) separately. Using Young's inequality, we conclude that for each $\delta_1 > 0$ there exists $c_{\delta_1} > 0$ such that

$$F_1 \leq \delta_1 k^2\sum_{n=1}^{m}\|d_t\boldsymbol{e}_h^n\|_2^2 + c_{\delta_1}\sum_{n=1}^{m}\|\boldsymbol{v}^n - \boldsymbol{j}_h\boldsymbol{v}^n\|_2^2.$$

Using the properties of $\boldsymbol{j}_h$, and applying Lemma 4.21, we deduce that

$$\begin{aligned}
F_1 &\leq \delta_1 k^2\sum_{n=1}^{m}\|d_t\boldsymbol{e}_h^n\|_2^2 + c_{\delta_1}h^{4-\frac{d(4-p)}{2}+d}\sum_{n=1}^{m}\|\boldsymbol{v}^n\|_{2,\frac{4}{4-p}}^2 \\
&\leq \delta_1 k^2\sum_{n=1}^{m}\|d_t\boldsymbol{e}_h^n\|_2^2 + c_{\delta_1}h^{4-\frac{d(2-p)}{2}}\sup_{n\in\{1,\ldots,m\}}\|\varepsilon + |\nabla\boldsymbol{v}^n|\|_2^{2-p}\Big[\sum_{n=1}^{m}\|\nabla\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}^n)\|_2^2\Big].
\end{aligned}$$

Assuming $h^{2-\frac{d(2-p)}{2}} \leq ck$, we obtain $h^{4-\frac{d(2-p)}{2}} = h^2h^{2-\frac{d(2-p)}{2}} \leq ch^2k$ and, hence,

$$F_1 \leq \delta_1 k^2\sum_{n=1}^{m}\|d_t\boldsymbol{e}_h^n\|_2^2 + c_{\delta_1}h^2\sup_{n\in\{1,\ldots,m\}}\|\varepsilon + |\nabla\boldsymbol{v}^n|\|_2^{2-p}\Big[k\sum_{n=1}^{m}\|\nabla\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}^n)\|_2^2\Big].$$

Using Lemma 2.4 and Lemma 2.2, for arbitrary $\delta_2 > 0$ we easily derive the inequality

$$\begin{aligned}
F_2 &\leq c_{\delta_2}k\sum_{n=1}^{m}\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}^n) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{v}^n)\|_2^2 + \delta_2 k\sum_{n=1}^{m}\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}^n) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h^n)\|_2^2 \\
&\leq c_{\delta_2}h^2k\sum_{n=1}^{m}\|\nabla\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}^n)\|_2^2 + \delta_2 k\sum_{n=1}^{m}\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}^n) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h^n)\|_2^2
\end{aligned}$$

where c_{δ_2} only depends on p and δ_2. Using the orthogonality of j_h with respect to Y_h, Young's inequality, we deduce that for arbitrary $\delta_3 > 0$ the term $F_3 + F_5$ is estimated by

$$\begin{aligned} F_3 + F_5 &= k \sum_{n=1}^{m} (\pi^n - j_h \pi^n, \nabla \cdot \boldsymbol{\xi}_h^n)_\Omega = k \sum_{n=1}^{m} (\pi^n - j_h \pi^n, \theta_h \nabla \cdot \boldsymbol{\xi}_h^n)_\Omega \\ &\le c_{\delta_3} k \sum_{n=1}^{m} \sum_{M \in \mathbb{M}_h} \nu_M^{-1} \|\pi^n - j_h \pi^n\|_{2;M}^2 + \delta_3 k \sum_{n=1}^{m} \sum_{M \in \mathbb{M}_h} \nu_M \|\theta_h \nabla \cdot \boldsymbol{\xi}_h^n\|_{2;M}^2 \\ &\le c_{\delta_3} h^2 k \sum_{n=1}^{m} \|\pi^n\|_{1,2}^2 + \delta_3 k \sum_{n=1}^{m} s_h\Big((\boldsymbol{\xi}_h^n, 0)\Big)\Big((\boldsymbol{\xi}_h^n, 0)\Big). \end{aligned}$$

Using integration by parts (functions in $\mathcal{Q}_h^p$ are continuous and $\boldsymbol{v}^n$, $\boldsymbol{j}_h \boldsymbol{v}^n$ belong to $\boldsymbol{W}_0^{1,p}(\Omega)$), the orthogonality of $\boldsymbol{j}_h$ with respect to $\boldsymbol{Y}_h$, Young's inequality, the interpolation property of $\boldsymbol{j}_h$, and Lemma 4.21, we conclude that for each $\delta_4 > 0$ there exists $c_{\delta_4} > 0$ such that

$$\begin{aligned} F_4 &= k \sum_{n=1}^{m} (\boldsymbol{v}^n - \boldsymbol{j}_h \boldsymbol{v}^n, \nabla \eta_h^n)_\Omega = k \sum_{n=1}^{m} (\boldsymbol{v}^n - \boldsymbol{j}_h \boldsymbol{v}^n, \boldsymbol{\theta}_h \nabla \eta_h^n)_\Omega \\ &\le c_{\delta_4} k \sum_{n=1}^{m} \sum_{M \in \mathbb{M}_h} \alpha_M^{-1} \|\boldsymbol{v}^n - \boldsymbol{j}_h \boldsymbol{v}^n\|_{2;M}^2 + \delta_4 k \sum_{n=1}^{m} \sum_{M \in \mathbb{M}_h} \alpha_M \|\boldsymbol{\theta}_h \nabla \eta_h^n\|_{2;M}^2 \\ &\lesssim c_{\delta_4} k \sum_{n=1}^{m} \sum_{M \in \mathbb{M}_h} \alpha_M^{-1} h_M^{4 - \frac{d(4-p)}{2} + d} \|\boldsymbol{v}^n\|_{2, \frac{4}{4-p}; S_M}^2 + \delta_4 k \sum_{n=1}^{m} s_h\Big((\mathbf{0}, \eta_h^n)\Big)\Big((\mathbf{0}, \eta_h^n)\Big) \\ &\lesssim c_{\delta_4} h^{2 - \frac{d(2-p)}{2}} \sup_{n \in \{1, \dots, m\}} \|\varepsilon + |\nabla \boldsymbol{v}^n|\|_2^{2-p} k \sum_{n=1}^{m} \|\nabla \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}^n)\|_2^2 + \delta_4 k \sum_{n=1}^{m} s_h\Big((\mathbf{0}, \eta_h^n)\Big)\Big((\mathbf{0}, \eta_h^n)\Big). \end{aligned}$$

Using Young's inequality, the interpolation property of θ_h, $\boldsymbol{j}_h$, and Lemma 4.21, we realize that for each $\delta_5 > 0$ there exists a constant $c_{\delta_5} > 0$ only depending on p and δ_5 such that

$$\begin{aligned} F_6 &\le c_{\delta_5} k \sum_{n=1}^{m} \sum_{M \in \mathbb{M}_h} \Big\{ \alpha_M \|\boldsymbol{\theta}_h \nabla j_h \pi^n\|_{2;M}^2 + \nu_M \Big(\|\theta_h \nabla \cdot (\boldsymbol{j}_h \boldsymbol{v}^n - \boldsymbol{v}^n)\|_{2;M}^2 + \|\theta_h \nabla \cdot \boldsymbol{v}^n\|_{2;M}^2 \Big) \Big\} \\ &\quad + \delta_5 k \sum_{n=1}^{m} \sum_{M \in \mathbb{M}_h} \Big\{ \alpha_M \|\boldsymbol{\theta}_h \nabla \eta_h^n\|_{2;M}^2 + \nu_M \|\theta_h \nabla \cdot \boldsymbol{\xi}_h^n\|_{2;M}^2 \Big\} \\ &\le c_{\delta_5} k \sum_{n=1}^{m} \sum_{M \in \mathbb{M}_h} \Big\{ \alpha_M \|\pi^n\|_{1,2;S_M}^2 + \nu_M h_M^{2 - \frac{d(2-p)}{2}} \|\boldsymbol{v}^n\|_{2, \frac{4}{4-p}; S_M}^2 \Big\} \\ &\quad + \delta_5 k \sum_{n=1}^{m} s_h\Big((\boldsymbol{\xi}_h^n, \eta_h^n)\Big)\Big((\boldsymbol{\xi}_h^n, \eta_h^n)\Big) \\ &\le c_{\delta_5} \Big(h^2 k \sum_{n=1}^{m} \|\pi^n\|_{1,2}^2 + h^{2 - \frac{d(2-p)}{2}} \sup_{n \in \{1, \dots, m\}} \|\varepsilon + |\nabla \boldsymbol{v}^n|\|_2^{2-p} k \sum_{n=1}^{m} \|\nabla \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}^n)\|_2^2 \Big) \\ &\quad + \delta_5 k \sum_{n=1}^{m} s_h\Big((\boldsymbol{\xi}_h^n, \eta_h^n)\Big)\Big((\boldsymbol{\xi}_h^n, \eta_h^n)\Big). \end{aligned}$$

Finally, since $\boldsymbol{v}^0 = \hat{\boldsymbol{v}}$ and $\hat{\boldsymbol{v}} \in \boldsymbol{W}^{1,2}(\Omega)$, the term F_7 can be estimated by

$$F_7 \equiv \frac{1}{2}\|\boldsymbol{v}^0 - \boldsymbol{j}_h\boldsymbol{v}^0\|_2^2 \le ch^2\|\hat{\boldsymbol{v}}\|_{1,2}^2.$$

Collecting all estimates above, choosing $\delta_1, \dots, \delta_5$ sufficiently small, absorbing the terms with $\delta_1, \dots, \delta_5$ into the left-hand side, taking the supremum over $m = 1, \dots, N$, and recalling (4.97), we can easily complete the proof. □

Since the regularity assumption (4.97) is satisfied (see Lemma 4.20), we can combine Theorem 4.22 and Lemma 4.19 so that we arrive at an a priori estimate for the overall discretization error $\boldsymbol{v}(t_n) - \boldsymbol{v}_h^n$ which provides an optimal convergence order with respect to k but a suboptimal convergence rate with respect to h. (We believe that an estimate for $\boldsymbol{v}(t_n) - \boldsymbol{v}^n$ similar to (4.92) remains valid for the simplified p-Stokes system at least in the case of space-periodic boundary conditions.) Note that in Theorem 4.22 we would obtain an optimal a priori error estimate with respect to the convergence order if we suppose the following stronger regularity of the semi-discrete velocity: $\|\boldsymbol{v}^n\|_{l^2(\mathbb{I}_N;W^{2,2}(\Omega))} \le C$. Alternatively, the following Corollary 4.23 shows that we obtain an optimal a priori error estimate if we assume stronger regularity of the semi-discrete pressure π^n. In particular, we require that π^n remains uniformly bounded in $l^{p'}(\mathbb{I}_N; W^{1,p'}(\Omega))$ with $p' \ge 2$. However, we are not able to show the supposed regularity and, hence, we are not allowed to state it as an assumption since we are considering an approximative system which is discretized in time. As a result, the following Corollary can only be understood in a formal sense.

Corollary 4.23. *For $p \in (1, 2]$ and $\varepsilon \in [0, \varepsilon_0]$ let the extra stress tensor $\boldsymbol{\mathcal{S}}$ satisfy Assumption 2.1. Let $(\boldsymbol{v}^n, \pi^n)$ be the solution to Problem* $(\mathbf{P4}^k)$, *and let $(\boldsymbol{v}_h^n, \pi_h^n)$ be the solution to Problem* $(\mathbf{P4}_h^k)$ *where the stabilization term s_h is defined by* (4.4). *We assume that there exists a constant $C > 0$ independent of k so that*

$$\sup_{1\le n\le N}\|\nabla\boldsymbol{v}^n\|_2^2 + k\sum_{n=1}^N\|\nabla\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}^n)\|_2^2 + k\sum_{n=1}^N\|\nabla\pi^n\|_{p'}^{p'} \le C. \tag{4.103}$$

Moreover, we suppose that $\boldsymbol{v}^0 = \hat{\boldsymbol{v}} \in \boldsymbol{W}_0^{1,2}(\Omega)$. If $h^{2-\frac{d(2-p)}{2}} \le ck$ for some $c > 0$ and $\alpha_M := \alpha_0 h^s$ with $s = 2$, then the error of approximation can be estimated by

$$\sup_{1\le n\le N}\|\boldsymbol{v}^n - \boldsymbol{v}_h^n\|_2^2 + k\sum_{n=1}^N\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}^n) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h^n)\|_2^2$$
$$+ k\sum_{n=1}^N\sum_{M\in\mathbb{M}_h}\alpha_M\|\boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h\nabla j_h\pi^n) - \boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h\nabla\pi_h^n)\|_{2;M}^2 \le C'h^2$$

where j_h is the interpolation operator of Lemma 4.4. The constant $C' > 0$ only depends on C, $\hat{\boldsymbol{v}}$, $\boldsymbol{f}$, p, ε_0, Ω, α_0.

Proof of Corollary 4.23. We modify the proof of Theorem 4.22 appropriately. Following the proof of Theorem 4.22, we similarly arrive at (4.102) and we aim at estimating the terms $E, F_1, \dots, F_5$ defined in (4.101) and (4.102).

By means of Lemma 2.4 and Lemma 4.2, we estimate the quantity E from below by

$$E \geq \frac{1}{2}\|\boldsymbol{e}_h^m\|_2^2 + \frac{k^2}{2}\sum_{n=1}^m \|d_t \boldsymbol{e}_h^n\|_2^2 + c_1 k \sum_{n=1}^m \|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}^n) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h^n)\|_2^2 \\ + c_2 k \sum_{n=1}^m \sum_{M\in\mathbb{M}_h} \alpha_M \|\boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h \nabla j_h \pi^n) - \boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h \nabla \pi_h^n)\|_{2;M}^2$$

where the constants c_1 and c_2 only depend on p. The terms F_1 and F_2 are estimated exactly as in the proof of Theorem 4.22. Using Hölder's inequality, the interpolation property of $\boldsymbol{j}_h$, Young's inequality, and applying Lemma 4.9, we estimate the term F_3 as follows:

$$F_3 \leq k \sum_{n=1}^m \|\pi^n - j_h \pi^n\|_{p'} \|\nabla \boldsymbol{v}^n - \nabla \boldsymbol{j}_h \boldsymbol{v}^n\|_p \leq ch^2 k \sum_{n=1}^m \|\pi^n\|_{1,p'} \|\boldsymbol{v}^n\|_{2,p} \\ \leq ch^2 k \sum_{n=1}^m \|\pi^n\|_{1,p'}^{p'} + ch^2 k \sum_{n=1}^m \|\nabla \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}^n)\|_2^2 + ch^2 k \sum_{n=1}^m \|\varepsilon + |\boldsymbol{D}\boldsymbol{v}^n|\|_p^p.$$

Applying integration by parts ($\boldsymbol{v}^n$, $\boldsymbol{j}_h \boldsymbol{v}^n$ belong to $\boldsymbol{W}_0^{1,p}(\Omega)$), using the orthogonality property of $\boldsymbol{j}_h$ with respect to $\boldsymbol{Y}_h$, we estimate the term F_4 by

$$F_4 = k \sum_{n=1}^m (\boldsymbol{v}^n - \boldsymbol{j}_h \boldsymbol{v}^n, \nabla \eta_h^n)_\Omega = k \sum_{n=1}^m (\boldsymbol{v}^n - \boldsymbol{j}_h \boldsymbol{v}^n, \boldsymbol{\theta}_h \nabla \eta_h^n)_\Omega \\ \leq k \sum_{n=1}^m \sum_{M\in\mathbb{M}_h} \alpha_M^{-\frac{1}{p'}} \|\boldsymbol{v}^n - \boldsymbol{j}_h \boldsymbol{v}^n\|_{p;M} \alpha_M^{\frac{1}{p'}} \|\boldsymbol{\theta}_h \nabla \eta_h^n\|_{p';M}.$$

Applying Young's inequality, we deduce that for each $\delta_3 > 0$ there exists $c_{\delta_3} > 0$ such that

$$F_4 \leq c_{\delta_3} k \sum_{n=1}^m \sum_{M\in\mathbb{M}_h} \alpha_M^{-(p-1)} \|\boldsymbol{v}^n - \boldsymbol{j}_h \boldsymbol{v}^n\|_{p;M}^p + \delta_3 k \sum_{n=1}^m \sum_{M\in\mathbb{M}_h} \alpha_M \|\boldsymbol{\theta}_h \nabla \eta_h^n\|_{p';M}^{p'} \\ \lesssim c_{\delta_3} k \sum_{n=1}^m \sum_{M\in\mathbb{M}_h} \alpha_M^{1-p} h_M^{2p} \|\boldsymbol{v}^n\|_{2,p;S_M}^p + \delta_3 k \sum_{n=1}^m \sum_{M\in\mathbb{M}_h} \alpha_M \|\boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h \nabla j_h \pi^n) - \boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h \nabla \pi_h^n)\|_{2;M}^2 \\ \lesssim c_{\delta_3} h^{2p+s(1-p)} k \sum_{n=1}^m \|\boldsymbol{v}^n\|_{2,p}^p + \delta_3 k \sum_{n=1}^m \sum_{M\in\mathbb{M}_h} \alpha_M \|\boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h \nabla j_h \pi^n) - \boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h \nabla \pi_h^n)\|_{2;M}^2.$$

Using (2.43) and Young's inequality, for any $\delta_4 > 0$ we estimate the term F_5 as follows:

$$F_5 \leq ck \sum_{n=1}^m \|\pi^n - j_h \pi^n\|_{p'} \|\varepsilon + |\boldsymbol{D}\boldsymbol{v}^n| + |\boldsymbol{D}\boldsymbol{v}_h^n|\|_p^{\frac{2-p}{2}} \|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}^n) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h^n)\|_2 \\ \leq c_{\delta_4} k \sum_{n=1}^m \|\pi^n - j_h \pi^n\|_{p'}^2 \|\varepsilon + |\boldsymbol{D}\boldsymbol{v}^n| + |\boldsymbol{D}\boldsymbol{v}_h^n|\|_p^{2-p} + \delta_4 k \sum_{n=1}^m \|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}^n) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h^n)\|_2^2.$$

Using the properties of j_h and Young's inequality (with $\frac{2}{p'} + \frac{2-p}{p} = 1$), we arrive at

$$
\begin{aligned}
F_5 &\leq c_{\delta_4} h^2 k \sum_{n=1}^{m} \|\pi^n\|_{1,p'}^2 \|\varepsilon + |\boldsymbol{D}\boldsymbol{v}^n| + |\boldsymbol{D}\boldsymbol{v}_h^n|\|_p^{2-p} + \delta_4 k \sum_{n=1}^{m} \|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}^n) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h^n)\|_2^2 \\
&\leq c_{\delta_4} h^2 k \sum_{n=1}^{m} \Big\{ \|\pi^n\|_{1,p'}^{p'} + \|\varepsilon + |\boldsymbol{D}\boldsymbol{v}^n| + |\boldsymbol{D}\boldsymbol{v}_h^n|\|_p^p \Big\} + \delta_4 k \sum_{n=1}^{m} \|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}^n) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h^n)\|_2^2.
\end{aligned}
$$

Applying Lemma 4.3 and the $W^{1,p'}$-stability of j_h, we realize that for each $\delta_5 > 0$ there exists a constant $c_{\delta_5} > 0$ only depending on p and δ_5 such that the term F_6 is bounded by

$$
F_6 \leq c_{\delta_5} h^s k \sum_{n=1}^{m} \|\tau + |\nabla \pi^n|\|_{p'}^{p'} + \delta_5 k \sum_{n=1}^{m} \sum_{M \in \mathbb{M}_h} \alpha_M \|\boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h \nabla j_h \pi^n) - \boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h \nabla \pi_h^n)\|_{2;M}^2.
$$

As in the proof of Thm. 4.22 we obtain $F_7 = \mathcal{O}(h^2)$. In view of $s = 2$ we can complete the proof following the proof of Thm. 4.22 and taking into account (4.90), (4.96), (4.103). □

If we suppose that we are allowed to combine Theorem 4.22 and Lemma 4.19, then we would arrive at an a priori estimate for the discretization error $\boldsymbol{v}(t_n) - \boldsymbol{v}_h^n$ which would provide an optimal convergence order with respect to k and h.

4.8 Numerical experiments

In this section we present numerical experiments which illustrate the established a priori error estimates. All computations were performed for the Carreau-type model (2.10), (2.11b). If not stated otherwise, the parameters were set to $\mu_0 := 1$ and $\varepsilon := 10^{-5}$. Problem **(P1)** was discretized with equal-order d-linear ($\mathbb{Q}_1/\mathbb{Q}_1$) finite elements based on quadrilateral meshes. Since the considered discretization is not stable, the LPS-based stabilization methods of Sections 4.1 and 3.2 were applied. The algebraic equations were solved by Newton's method, the linear subproblems by the GMRES method. The multigrid method was applied as a preconditioner. Details on the numerical solver and information about its realization within the software package Gascoigne [GAS] can be found in Section 3.4. In the following experiments we measure the error of approximation for the quantities

$$
\begin{aligned}
&E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{F}}} := \|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2, \quad && E_{\boldsymbol{v}}^{1,\nu} := \|\nabla(\boldsymbol{v} - \boldsymbol{v}_h)\|_\nu, \quad E_{\boldsymbol{v}}^{\nu} := \|\boldsymbol{v} - \boldsymbol{v}_h\|_\nu, \\
&E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{S}}} := \|\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}_h)\|_{p'}, && E_{\pi}^{\nu} := \|\pi - \pi_h\|_\nu,
\end{aligned}
\tag{4.104}
$$

and we depict the experimental order of convergence (EOC) with respect to the number of elements (under global mesh refinement). As usual, $(\boldsymbol{v}, \pi)$ denotes the (continuous) solution to **(P1)** and $(\boldsymbol{v}_h, \pi_h)$ is referred to as the (discrete) solution to **(P1$_h$)**. The order of convergence is determined by the standard formula $\log(E(h)/E(h/2))/\log(2)$ where $E(h)$ stands for one of the quantities in (4.104). In this section, we aim at answering the question whether the order of convergence predicted by our theoretical results coincides with the rate of convergence observed by numerical experiments. First of all, by means of Examples 1–3 we numerically confirm the a priori error estimates of Theorem 4.11 for different values of $p \le 2$. Then, in Example 4 we demonstrate the optimality of the error estimates (4.43) and (4.44) with respect to the supposed regularity of $(\boldsymbol{v}, \pi)$. Via Examples 5–6 we numerically validate Theorem 4.12 for different values of $p \ge 2$. In Example 7 we discuss super-approximation effects that usually occur for $\mathbb{Q}_1/\mathbb{Q}_1$ elements provided that a smooth solution is approximated on a sequence of regular meshes. By means of Example 8 we illustrate the a priori error estimates of Corollaries 4.14 and 4.16 which deal with the standard LPS method proposed in [BB01] and its application to p-Stokes systems. Note that the above experiments are performed in two space dimensions. Finally, via Example 9 we verify the derived a priori error estimates in three space dimensions.

Table 4.1. Numerical verification of Theorem 4.11 for $p < 2$

(a) $p = 1.1$

	$E_\pi^{p'}$		$E_{\boldsymbol{v}}^{1,p}$	
#cells	error	conv.	error	conv.
1024	4.28e-03	0.15	5.61e-04	1.05
4096	3.85e-03	0.15	2.74e-04	1.03
16384	3.42e-03	0.17	1.36e-04	1.02
65536	3.02e-03	0.18	6.73e-05	1.01
262144	2.66e-03	0.18	3.35e-05	1.01
expected		0.18		1.00

(b) $p = 1.2$

	$E_\pi^{p'}$		$E_{\boldsymbol{v}}^{1,p}$	
#cells	error	conv.	error	conv.
1024	1.84e-03	0.34	5.67e-04	1.02
4096	1.47e-03	0.32	2.81e-04	1.01
16384	1.17e-03	0.33	1.40e-04	1.01
65536	9.29e-04	0.33	6.99e-05	1.00
262144	7.36e-04	0.34	3.49e-05	1.00
expected		0.33		1.00

(c) $p = 1.3$

	$E_\pi^{p'}$		$E_{\boldsymbol{v}}^{1,p}$	
#cells	error	conv.	error	conv.
1024	8.58e-04	0.50	5.85e-04	1.00
4096	6.25e-04	0.46	2.93e-04	1.00
16384	4.54e-04	0.46	1.46e-04	1.00
65536	3.29e-04	0.46	7.32e-05	1.00
262144	2.38e-04	0.46	3.66e-05	1.00
expected		0.46		1.00

(d) $p = 1.5$

	$E_\pi^{p'}$		$E_{\boldsymbol{v}}^{1,p}$	
#cells	error	conv.	error	conv.
1024	2.20e-04	0.80	6.36e-04	0.97
4096	1.37e-04	0.69	3.22e-04	0.98
16384	8.58e-05	0.67	1.62e-04	0.99
65536	5.38e-05	0.67	8.17e-05	0.99
262144	3.38e-05	0.67	4.10e-05	0.99
expected		0.67		1.00

Example 1: First of all we deal with the shear thinning case. We numerically validate Theorem 4.11, see Table 4.1. As a first designed experiment, we chose the computational domain $\Omega := (-0.5, 0.5) \times (-0.5, 0.5)$ and we prescribed the exact solution to **(P1)** by

$$\boldsymbol{v}(\boldsymbol{x}) := |\boldsymbol{x}|^{a-1} \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} \qquad \text{and} \qquad \pi(\boldsymbol{x}) := x_1 x_2 + (x_1 x_2)^3. \tag{4.105}$$

Problem $(\mathbf{P1}_h)$ was solved for the following data: The right-hand side $\boldsymbol{f}$ was given by $\boldsymbol{f} := -\nabla \cdot \boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}) + \nabla \pi$, and Dirichlet boundary conditions with $\boldsymbol{v}_D := \boldsymbol{v}|_{\partial\Omega}$ were prescribed on the whole boundary $\partial\Omega$. The stabilization term s_h was chosen similarly to the one in (4.4) with $\alpha_M = \alpha_0 h_M^2$ and $\tau = 1$. However, instead of $\boldsymbol{\theta}_h \nabla \pi_h$ as in (4.4), the gradient of fluctuations $\nabla \bar{\theta}_h \pi_h$ was used where the filter $\bar{\theta}_h$ is defined by (3.32). Although we analyzed LPS schemes based on fluctuations of gradients $\boldsymbol{\theta}_h \nabla \pi_h$ only and we have $\boldsymbol{\theta}_h \nabla \pi_h \neq \nabla \bar{\theta}_h \pi_h$ in general, we believe that the choice $\nabla \bar{\theta}_h \pi_h$ allows a similar convergence analysis and the same a priori error estimates (cf. Remark 3.4, [BB01, MST07]). The stabilization parameter α_0 was set to $\alpha_0 = 0.3$. Note that in all examples the stabilization method was less sensitive with respect to α_0. Clearly, the regularity of $\boldsymbol{v}$ is controlled by the choice of $a \in \mathbb{R}$. We easily compute that $\nabla \cdot \boldsymbol{v} = 0$, $|\nabla \boldsymbol{v}(\boldsymbol{x})| \sim |\boldsymbol{x}|^{a-1}$, and $|\nabla \boldsymbol{\mathcal{F}}(\boldsymbol{Dv}(\boldsymbol{x}))| \sim |\boldsymbol{x}|^{\frac{(a-1)p}{2}-1}$ for $\varepsilon = 0$. Hence, it holds $\boldsymbol{\mathcal{F}}(\boldsymbol{Dv}) \in W^{1,2}(\Omega)^{d\times d}$ provided that $\frac{(a-1)p}{2} - 1 > -1 \Leftrightarrow a > 1$. In this example we set $a = 1.01$. According to (4.43) and (4.44) we expect the convergence rates 1 for the velocity in $\boldsymbol{W}^{1,p}(\Omega)$ and $\frac{2}{p'}$ for the pressure in $L^{p'}(\Omega)$. Considering Table 4.1, we realize that the numerical results agree with the theoretical ones very well. In particular, Examples 4.1(a) – 4.1(d) reflect that the order of convergence for the pressure depends on the choice of p as predicted by (4.44).

Table 4.2. Numerical verification of Theorem 4.11 for $p < 2$

(a) $p = 1.1$

#elements	$E_{\boldsymbol{v}}^{1,p}$	$E_{\boldsymbol{v}}^{p}$	$E_{\boldsymbol{v}}^{\boldsymbol{S}}$	$E_{\pi}^{p'}$	E_{π}^{2}
1024	1.08	1.90	0.18	0.19	0.95
4096	1.05	1.95	0.18	0.17	0.99
16384	1.03	1.98	0.18	0.17	1.01
65536	1.01	1.99	0.18	0.17	1.01
262144	1.01	1.99	0.18	0.17	1.01
expected	1.00			0.18	1.00

(b) $p = 1.2$

#elements	$E_{\boldsymbol{v}}^{1,p}$	$E_{\boldsymbol{v}}^{p}$	$E_{\boldsymbol{v}}^{\boldsymbol{S}}$	$E_{\pi}^{p'}$	E_{π}^{2}
1024	1.02	1.94	0.34	0.36	1.01
4096	1.02	1.97	0.34	0.32	1.00
16384	1.01	1.98	0.34	0.33	1.01
65536	1.01	1.99	0.34	0.33	1.01
262144	1.00	1.99	0.34	0.33	1.01
expected	1.00			0.33	1.00

(c) $p = 1.3$

#elements	$E_{\boldsymbol{v}}^{1,p}$	$E_{\boldsymbol{v}}^{p}$	$E_{\boldsymbol{v}}^{\boldsymbol{S}}$	$E_{\pi}^{p'}$	E_{π}^{2}
1024	1.00	1.95	0.46	0.59	1.11
4096	1.00	1.97	0.46	0.46	1.03
16384	1.00	1.98	0.46	0.46	1.02
65536	1.00	1.99	0.46	0.46	1.01
262144	1.00	1.99	0.46	0.46	1.01
expected	1.00			0.46	1.00

(d) $p = 1.5$

#elements	$E_{\boldsymbol{v}}^{1,p}$	$E_{\boldsymbol{v}}^{p}$	$E_{\boldsymbol{v}}^{\boldsymbol{S}}$	$E_{\pi}^{p'}$	E_{π}^{2}
1024	0.97	1.95	0.66	1.42	1.51
4096	0.98	1.96	0.67	0.80	1.22
16384	0.99	1.98	0.67	0.68	1.08
65536	0.99	1.98	0.67	0.67	1.03
262144	0.99	1.99	0.67	0.67	1.01
expected	1.00			0.67	1.00

Example 2: The following experiments are in the same spirit as the ones in Example 1. Here we do not only demonstrate Theorem 4.11 but also we determine the experimental order of convergence with respect to further quantities such as $E_{\boldsymbol{v}}^{\boldsymbol{S}}$. We chose the computational domain Ω as in Example 1 and we prescribed the exact solution to **(P1)** by

$$\boldsymbol{v}(\boldsymbol{x}) := |\boldsymbol{x}|^{a-1} \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} \qquad \text{and} \qquad \pi(\boldsymbol{x}) := |\boldsymbol{x}|^{b} x_1 x_2, \qquad a, b \in \mathbb{R}. \tag{4.106}$$

The data $\boldsymbol{f}$ and $\boldsymbol{v}_D$, for which Problem $(\mathbf{P1}_h)$ was solved, were chosen similarly as in Example 1. There the patch-wise constant α_M, that arises in the stabilization term s_h, was given by $\alpha_M := \alpha_0 h_M^2/\mu_0$. In view of the LPS-theory for Stokes systems, the choice $\alpha_M := \alpha_0 h_M^2/\mu(|\boldsymbol{D}\boldsymbol{v}_h|^2)$ would however be more natural where $\mu(|\boldsymbol{D}\boldsymbol{v}_h|^2)$ is the generalized viscosity defined in (2.11b). Numerical experiments indicate that both forms of α_M influence neither the stability of the discrete pressure nor the order of convergence (see Example 3). In this example the stabilization term s_h is chosen as in Example 1 but its patch-wise constant α_M is set to $\alpha_M := \alpha_0 h_M^2/\mu(|\boldsymbol{D}\boldsymbol{v}_h|^2)$. The requirement $\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) \in W^{1,2}(\Omega)^{d\times d}$ and $\pi \in W^{1,p'}(\Omega)$ amounts to the condition $a > 1$ and $b > -\frac{2}{p'} - 1$. Table 4.2 depicts the rates of convergence that were obtained for $a = 1.01$ and $b = 2$. Since π is smooth, in (4.54) the interpolation error $(\pi - j_h\pi)$ is of higher order than the quantity $E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{S}}}$. Consequently, in view of (4.53)–(4.56) the order of convergence for $E_\pi^{p'}$ should basically be determined by the one for $E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{S}}}$. Studying Table 4.2, we observe that $E_\pi^{p'}$ is exactly of same order as $E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{S}}}$. Recalling Theorem 4.11, we realize that the numerical results agree with the theoretical ones very well. In particular, the rate of convergence for $E_\pi^{p'}$ depends on the parameter p as predicted by Theorem 4.11. We also observe that $E_{\boldsymbol{v}}^p$ behaves as $\mathcal{O}(h^2)$. Hence we are allowed to conjecture that a duality argument, which is similar to the one described in [BS94], may be applicable here.

Table 4.3. Numerical verification of Theorem 4.11: $p = 1.1$, $a = 1.01$, $b = -1.17$

#elements	$E_{\boldsymbol{v}}^{1,p}$	$E_{\boldsymbol{v}}^{p}$	$E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{S}}}$	$E_\pi^{p'}$	E_π^2
1024	1.08	1.90	0.18	1.01	1.54
4096	1.05	1.95	0.18	0.92	1.33
16384	1.03	1.98	0.18	0.33	1.16
65536	1.02	1.99	0.18	0.19	1.07
262144	1.01	1.99	0.18	0.18	1.03
expected	1.00			0.18	1.00

Table 4.3 shows the experimental rates of convergence that were obtained for $a = 1.01$ and $b = -\frac{2}{p'} - 0.99$. In contrast to the previous experiment, neither the velocity $\boldsymbol{v}$ nor the pressure π were smooth functions but they satisfy the condition $\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) \in W^{1,2}(\Omega)^{d\times d}$ and $\pi \in W^{1,p'}(\Omega)$. In view of Table 4.3 the experimental order of convergence coincides with the theoretical rate of convergence predicted by Theorem 4.11. It should be pointed out that $E_\pi^2 \approx \mathcal{O}(h)$ is expected as long as $\varepsilon > 0$, cf. (4.87), (5.32). To sum up, the numerical observations agree with Theorem 4.11. The quantities $E_\pi^{p'}$ and $E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{S}}}$ converge with same order. Since $E_{\boldsymbol{v}}^p$ behaves as $\mathcal{O}(h^2)$, a duality argument seems to be applicable here.

Example 3: In Examples 1–2 we observed that both patch-wise stabilization parameters $\alpha_M := \alpha_0 h_M^2/\mu_0$ and $\alpha_M := \alpha_0 h_M^2/\mu(|\boldsymbol{D}\boldsymbol{v}_h|^2)$ lead to the same convergence order. However the second choice seems to be more suitable from numerical point of view as depicted by the following experiment (see Table 4.4). The analytical solution $(\boldsymbol{v}, \pi)$ was given by (4.106) with $a = 1.01$ and $b = 2$. Hence, the velocity $\boldsymbol{v}$ satisfies $\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) \in W^{1,2}(\Omega)^{d\times d}$ and

Table 4.4. Validation of Theorem 4.11 for $p = 1.2$ and different versions of α_M

(a) $\alpha_M := \alpha_0 h_M^2/\mu_0$

	$E_{\boldsymbol{v}}^{1,p}$		$E_\pi^{p'}$		E_π^2		Numerical costs	
#cells	error	conv.	error	conv.	error	conv.	#Newt.-It.	#GMRES-It.
256	1.16e-03	–	2.58e-03	–	1.44e-03	–	7 (2)	1;1;1;1;1;1;1
1024	5.72e-04	1.02	2.01e-03	0.36	7.14e-04	1.01	7 (2)	1;1;1;1;1;1;1
4096	2.83e-04	1.02	1.61e-03	0.32	3.57e-04	1.00	6 (2)	*;*;*;35;29;17
16384	1.41e-04	1.01	1.28e-03	0.33	1.78e-04	1.00	6 (2)	*;*;*;39;33;21
65536	7.00e-05	1.01	1.01e-03	0.33	8.90e-05	1.00	6 (2)	*;*;*;*;37;23
262144	3.49e-05	1.00	8.04e-04	0.33	4.44e-05	1.00	6 (2)	*;*;*;*;39;23

(b) $\alpha_M := \alpha_0 h_M^2/\mu(|\boldsymbol{D}\boldsymbol{v}_h|^2)$

	$E_{\boldsymbol{v}}^{\mathcal{F}}$		$E_\pi^{p'}$		E_π^2		Numerical costs	
#cells	error	conv.	error	conv.	error	conv.	#Newt.-It.	#GMRES-It.
256	1.16e-03	–	2.53e-03	–	1.49e-03	–	7 (2)	1;1;1;1;1;1;1
1024	5.72e-04	1.02	1.96e-03	0.36	7.37e-04	1.01	7 (2)	1;1;1;1;1;1;1
4096	2.83e-04	1.02	1.57e-03	0.32	3.68e-04	1.00	6 (2)	8;8;7;7;6;4
16384	1.41e-04	1.01	1.25e-03	0.33	1.83e-04	1.01	6 (2)	8;9;8;7;6;4
65536	7.00e-05	1.01	9.92e-04	0.33	9.10e-05	1.01	6 (2)	8;9;8;7;6;5
262144	3.49e-05	1.00	7.90e-04	0.33	4.51e-05	1.01	6 (2)	8;15;12;9;7;6

the pressure π is smooth. In Table 4.4, we depict the absolute errors and corresponding convergence rates. We also compare the numerical complexity. The discrete nonlinear problem was solved by means of Newton's method with step-size control, see Algorithm 3.1. The numerical costs were measured by the number of iterations that were performed by Newton's algorithm in order to reduce the (nonlinear) residual up to the prescribed tolerance $TOL = 10^{-11}$. Here, the number within the brackets exhibits the total number of iterations performed by the step-size control. The linear system of equations, that arises in each Newton step, was solved by the GMRES method. As a preconditioner, we applied 2 iterations of the multigrid method with W-cycle. Within the W-cycle, we performed 4 pre-smoothing/post-smoothing steps. In case of grids with less than 2000 elements, the linear systems of equations were solved directly. In Table 4.4, for each Newton step we depict the number of iterations that were performed by the GMRES algorithm in order to reduce the (linear) residual up to the prescribed tolerance $TOL = 10^{-12}$. The symbol "*" indicates that the tolerance was not reached within 40 iterations of GMRES. In view of Table 4.4, the two versions of α_M lead to similar order of convergence for the pressure. If we compare the number of iterations performed by GMRES, we realize that the choice $\alpha_M = \alpha_0 h_M^2/\mu(|\boldsymbol{D}\boldsymbol{v}_h|^2)$ requires less iterations of GMRES and, hence, it allows less computational effort than $\alpha_M = \alpha_0 h_M^2/\mu_0$. As a result, if $\alpha_M = \alpha_0 h_M^2/\mu(|\boldsymbol{D}\boldsymbol{v}_h|^2)$ is used, then the linear systems of equations arising from Newton iteration seem to be better-conditioned. Hence, we use $\alpha_M := \alpha_0 h_M^2/\mu(|\boldsymbol{D}\boldsymbol{v}_h|^2)$ for the following simulations.

Table 4.5. Optimality of the a priori error estimates: Case $p = 1.4$

	$a = 0.99$		$a = 0.70$		$a = 0.40$		$a = 0.10$	
#cells	$E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{F}}}$	$E_{\pi}^{p'}$	$E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{F}}}$	$E_{\pi}^{p'}$	$E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{F}}}$	$E_{\pi}^{p'}$	$E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{F}}}$	$E_{\pi}^{p'}$
1024	0.88	0.95	0.75	0.56	0.57	0.45	0.37	0.34
4096	0.90	0.59	0.76	0.45	0.57	0.33	0.37	0.21
16384	0.91	0.57	0.77	0.45	0.58	0.33	0.37	0.21
65536	0.92	0.57	0.77	0.45	0.58	0.33	0.37	0.20
262144	0.93	0.57	0.78	0.45	0.58	0.33	0.37	0.20
expected	0.99	0.57	0.79	0.45	0.58	0.33	0.37	0.21

Example 4: The numerical results shown in Table 4.5 indicate that the a priori estimates (4.43), (4.44) are optimal with respect to the required regularity of the solution. Once again, the exact solution $(\boldsymbol{v}, \pi)$ was given by (4.106) with $a \in \mathbb{R}$ and fixed $b = 2$. In this example we investigated the convergence of the method regarding the regularity of $\boldsymbol{v}$. Table 4.5 depicts the EOC for $E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{F}}}$ and $E_{\pi}^{p'}$. As expected, in case of $E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{F}}}$ we lose linear convergence as soon as $\boldsymbol{\mathcal{F}}(\boldsymbol{Dv})$ no more belongs to $W^{1,2}(\Omega)^{d\times d}$. More precisely, we observe that $E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{F}}} \approx ch^{\beta}\|\nabla^{\beta}\boldsymbol{\mathcal{F}}(\boldsymbol{Dv})\|_2$ with $\beta \approx (a-1)\frac{p}{2} + 1$ noting that $|\nabla^{\beta}\boldsymbol{\mathcal{F}}(\boldsymbol{Dv})| \in L^2(\Omega)$ iff $\beta < (a-1)\frac{p}{2} + 1$. Moreover, we realize that $E_{\pi}^{p'}$ is of order $\{(a-1)\frac{p}{2} + 1\}\frac{2}{p'}$. In view of (4.55), the numerical observations agree with our expectations.

Table 4.6. Numerical verification of Theorem 4.12 for $p > 2$

(a) $p = 3.0$, $a = 1.34$, $b = -0.32$

	$E_{\pi}^{p'}$		$E_{\boldsymbol{v}}^{1,p}$	
#cells	error	conv.	error	conv.
1024	1.02e-02	1.02	8.55e-02	0.52
4096	5.04e-03	1.01	5.97e-02	0.52
16384	2.49e-03	1.01	4.18e-02	0.52
65536	1.24e-03	1.01	2.93e-02	0.51
262144	6.12e-04	1.01	2.06e-02	0.51
expected		0.75		0.50

(b) $p = 3.5$, $a = 1.44$, $b = -0.42$

	$E_{\pi}^{p'}$		$E_{\boldsymbol{v}}^{1,p}$	
#cells	error	conv.	error	conv.
1024	1.65e-02	1.00	1.78e-01	0.42
4096	8.21e-03	1.01	1.34e-01	0.41
16384	4.09e-03	1.01	1.00e-01	0.41
65536	2.03e-03	1.01	7.54e-02	0.41
262144	1.01e-03	1.01	5.67e-02	0.41
expected		0.70		0.40

Example 5: For the proposed LPS-based stabilization scheme (4.4) we numerically verify the derived a priori error estimates of Theorem 4.12 in the case $p \geq 2$, see Table 4.6. Here the exact velocity $\boldsymbol{v}$ was given by $(4.106)_1$ and the exact pressure π was prescribed by $\pi(\boldsymbol{x}) := |\boldsymbol{x}|^b - ⨍_{\Omega} |\boldsymbol{x}|^b \mathrm{d}\boldsymbol{x}$. The data $\boldsymbol{f}$, $\boldsymbol{v}|_D$ and the stabilization s_h were chosen as in Example 1 but the patch-wise constant α_M, which appears in the definition of s_h, was set to $\alpha_M := \alpha_0 h_M^{p'}/\mu(|\boldsymbol{Dv}_h|^2)$. The availability of the error estimates (4.57) and (4.58) requires the regularity $\boldsymbol{v} \in \boldsymbol{W}^{2,p}(\Omega)$ and $\pi \in W^{1,p'}(\Omega)$, which is equivalent to the conditions $a > 2 - 2/p$ and $b > 1 - 2/p'$. Considering Tables 4.6(a) and 4.6(b), we realize

that the error $E_{\boldsymbol{v}}^{1,p}$ behaves as $\mathcal{O}(h^{1/(p-1)})$ and, hence, it converges as predicted by Theorem 4.12. However, we observe linear convergence for the pressure in $L^{p'}(\Omega)$ although we would expect the rate of convergence $p'/2$. Hence, the a priori estimate (4.58) may be suboptimal or the observed convergence rate for $E_{\pi}^{p'}$ may be caused by super-approximation effects which, however, generally occur in case of smooth solutions only. Further investigations are necessary, and they are carried out in Examples 6 and 7.

Table 4.7. Numerical verification of Theorem 4.12 for $p > 2$

(a) $p = 2.5$; $b = -2.19$

#elements	$E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{F}}}$	$E_{\boldsymbol{v}}^{1,p}$	$E_{\boldsymbol{v}}^{p}$	$E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{S}}}$	$E_{\pi}^{p'}$	E_{π}^{2}
1024	0.83	0.67	1.66	1.00	1.00	0.80
4096	0.84	0.67	1.67	1.00	1.00	0.81
16384	0.84	0.67	1.67	1.01	1.01	0.81
65536	0.84	0.67	1.67	1.01	1.01	0.81
262144	0.84	0.67	1.67	1.01	1.01	0.81
expected	0.83	0.67			0.83	

(b) $p = 3$; $b = -2.32$

#elements	$E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{F}}}$	$E_{\boldsymbol{v}}^{1,p}$	$E_{\boldsymbol{v}}^{p}$	$E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{S}}}$	$E_{\pi}^{p'}$	E_{π}^{2}
1024	0.75	0.50	1.46	1.00	0.99	0.67
4096	0.75	0.50	1.45	1.00	1.00	0.67
16384	0.76	0.50	1.50	1.01	1.01	0.68
65536	0.76	0.51	1.51	1.01	1.01	0.68
262144	0.76	0.51	1.51	1.01	1.01	0.68
expected	0.75	0.50			0.75	

(c) $p = 3.5$; $b = -2.42$

#elements	$E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{F}}}$	$E_{\boldsymbol{v}}^{1,p}$	$E_{\boldsymbol{v}}^{p}$	$E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{S}}}$	$E_{\pi}^{p'}$	E_{π}^{2}
1024	0.69	0.40	1.36	0.99	0.99	0.57
4096	0.70	0.40	1.29	1.00	1.00	0.58
16384	0.70	0.40	1.13	1.00	1.00	0.58
65536	0.70	0.40	1.35	1.01	1.01	0.58
262144	0.71	0.40	1.39	1.01	1.01	0.58
expected	0.70	0.40			0.70	

Example 6: We illustrate the a priori error estimates (4.57) & (4.58) for less regular velocity as required in Theorem 4.12, and we determine the EOC with respect to further quantities such as $E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{S}}}$, see Table 4.7. Here, the analytical solution was given by (4.106) with $a = 1.01$ and $b = -2/p' - 0.99$ so that $\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) \in W^{1,2}(\Omega)^{d\times d}$ and $\pi \in W^{1,p'}(\Omega)$ is fulfilled. The data $\boldsymbol{f}$, $\boldsymbol{v}|_D$ and the stabilization s_h were chosen as in Example 5. In view of Tables 4.7(a) – 4.7(b), we realize that $E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{F}}} = \mathcal{O}(h^{p'/2})$ and $E_{\boldsymbol{v}}^{1,p} = \mathcal{O}(h^{1/(p-1)})$ although $\boldsymbol{v} \notin \boldsymbol{W}^{2,p}(\Omega)$. Note that Theorem 4.12 predicts the observed convergence provided that

$\boldsymbol{v} \in \boldsymbol{W}^{2,p}(\Omega)$. Hence, for this particular example the assumption $\boldsymbol{\mathcal{F}}(\boldsymbol{Dv}) \in W^{1,2}(\Omega)^{d\times d}$ and $\pi \in W^{1,p'}(\Omega)$ seems to be sufficient to ensure (4.57). According to Remark 4.9, we can expect convergence as in (4.57) & (4.58) if $\nabla\pi_h$ is uniformly bounded in $\boldsymbol{L}^{p'}(\Omega)$. Indeed, we numerically observed that the $L^{p'}$-norm of $\nabla(\pi - \pi_h)$ behaves as $\mathcal{O}(1)$. Concerning the pressure convergence, Tables 4.7(a) – 4.7(b) indicate that, similarly to Example 5, $E_\pi^{p'} \approx \mathcal{O}(h)$ in the case $p > 2$. Although we have not been able to verify $E_\pi^{p'} = \mathcal{O}(h)$ analytically, by virtue of (4.53) and (4.54) we may explain this convergence behavior referring to the apparent EOC for $E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{S}}}$. Using the inverse inequality (3.20), the interpolation inequality (3.18), we can easily derive the following relation between $E_\pi^{p'}$ and E_π^2:

$$\|\pi - \pi_h\|_2 \lesssim h^{1-\frac{d}{p'}+\frac{d}{2}}\|\pi\|_{1,p'} + h^{-\frac{d}{p'}+\frac{d}{2}}\|\pi - \pi_h\|_{p'} \qquad (p \geq 2).$$

Hence, for $d = 2$ we deduce from $E_\pi^{p'} = \mathcal{O}(h)$ that the pressure error in $L^2(\Omega)$ converges with order $2 - 2/p' = 2/p$. By Tables 4.7(a) – 4.7(b) the behavior $E_\pi^2 = \mathcal{O}(h^{2/p})$ is well reflected. To sum up, we observed that the experimental convergence order for the velocity agrees with the theoretical one. The pressure converges linearly in $L^{p'}(\Omega)$ for all considered $p > 2$ and, hence, its convergence is better than expected from (4.58). As a result the error estimate (4.58) may be suboptimal. If we compare the experimental order of convergence for $E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{S}}}$ and $E_\pi^{p'}$, we realize that both quantities are of same order. We recall that we made the same observation in the case $p \leq 2$. Consequently we conjecture that, in order to derive sharp pressure-error estimates in the case $p > 2$, we should estimate the quantity $E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{S}}}$ directly and we should not relate it to the natural distance $E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{F}}}$, see Lemma 2.7.

Table 4.8. Verification of Corollary 4.13 for a smooth solution: Case $p = 3$

(a) $\mathbb{Q}_1/\mathbb{Q}_1$ elements

#elements	$E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{F}}}$	$E_{\boldsymbol{v}}^{1,p}$	$E_{\boldsymbol{v}}^{p}$	$E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{S}}}$	$E_\pi^{p'}$	E_π^2
1280	0.94	0.99	2.12	0.94	0.87	0.60
5120	0.99	1.00	2.04	0.99	1.59	1.44
20480	1.00	1.00	2.00	1.00	1.73	1.63
81920	1.00	1.00	2.00	1.00	1.76	1.62
327680	1.00	1.00	2.00	1.00	1.76	1.58
1310720	1.00	1.00	2.00	1.00	1.75	1.55
expected	1.00	0.67			1.00	

(b) $\mathbb{Q}_2/\mathbb{Q}_2$ elements

#elements	$E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{F}}}$	$E_{\boldsymbol{v}}^{1,p}$	$E_{\boldsymbol{v}}^{p}$	$E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{S}}}$	$E_\pi^{p'}$	E_π^2
1280	1.96	2.04	3.14	1.95	2.04	2.03
5120	1.98	2.02	3.05	1.98	2.01	2.01
20480	1.99	2.01	3.01	1.99	2.01	2.00
81920	2.00	2.00	3.00	2.00	2.00	2.00
327680	2.00	2.00	3.00	2.00	2.00	2.00
supposed	2.00	1.33			2.00	

Example 7: In the context of $\mathbb{Q}_1/\mathbb{Q}_1$ elements, we numerically investigate the role of super-approximation. For low-order elements, super-approximation effects are well-studied (see [BLR86]). They may occur if, e.g., uniform triangulations are employed and the solution is sufficiently smooth. In this example a smooth solution $(\boldsymbol{v}, \pi)$ was prescribed by (4.106) with $a = 3$ and $b = 2$. The data $\boldsymbol{f}$, $\boldsymbol{v}|_D$ were chosen accordingly as in Example 1, and the stabilization s_h was given by (3.54) with $\alpha_M := \alpha_0 h_M^2/\mu(|\boldsymbol{D}\boldsymbol{v}|^2)$ and $\varrho_M := 0$. In view of Corollary 4.13, we expect that $E_{\boldsymbol{v}}^{\mathcal{F}} = \mathcal{O}(h)$, $E_{\boldsymbol{v}}^{1,p} = \mathcal{O}(h^{2/p})$ and $E_{\pi}^{p'} = \mathcal{O}(h)$. The numerical results of Table 4.8(a) were obtained for the $\mathbb{Q}_1/\mathbb{Q}_1$-discretization, whereas the results of Table 4.8(b) were generated with $\mathbb{Q}_2/\mathbb{Q}_2$ finite elements. In Table 4.8(a), the pressure converges better than predicted by Corollary 4.13. The convergence rates for $E_{\pi}^{p'}$ are even better than the ones for $E_{\boldsymbol{v}}^{\mathcal{S}}$. Note that in all previous examples $E_{\pi}^{p'}$ was of same order as $E_{\boldsymbol{v}}^{\mathcal{S}}$. Here, the convergence rates for $E_{\pi}^{p'}$ cannot be explained by the apparent convergence rates for $E_{\boldsymbol{v}}^{\mathcal{S}}$. By contrast, as depicted in Table 4.8(b) the $\mathbb{Q}_2/\mathbb{Q}_2$-discretization yields convergence rates for $E_{\pi}^{p'}$ which rather agree with our expectations. At least $E_{\pi}^{p'}$ converges with same order as $E_{\boldsymbol{v}}^{\mathcal{S}}$. Hence, in Table 4.8(a) the improved convergence order for $E_{\pi}^{p'}$ seems to be a special feature of the $\mathbb{Q}_1/\mathbb{Q}_1$-discretization. For both discretizations, the velocity-error $E_{\boldsymbol{v}}^{1,p}$ behaves better than predicted by Corollary 4.13. Below we numerically investigate whether the improved convergence is caused by super-approximation.

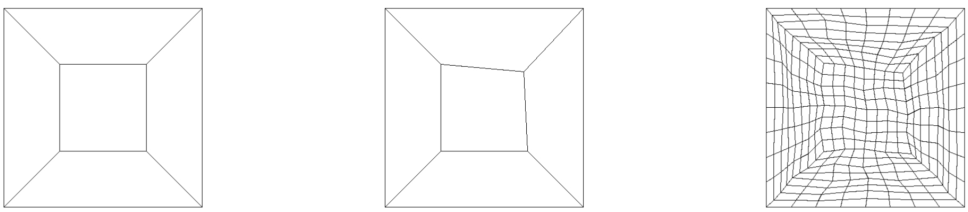
Table 4.9. Super approximation for a smooth solution: Case $p = 3$

(a) Regular meshes

	$E_{\boldsymbol{v}}^{1,p}$		$E_{\pi}^{p'}$	
#cells	error	conv.	error	conv.
320	7.68e-02	–	2.39e-03	–
1280	3.88e-02	0.99	1.31e-03	0.87
5120	1.94e-02	1.00	4.36e-04	1.59
20480	9.71e-03	1.00	1.31e-04	1.73
81920	4.85e-03	1.00	3.88e-05	1.76
327680	2.43e-03	1.00	1.14e-05	1.76
1310720	1.21e-03	1.00	3.40e-06	1.75
expected		0.67		1.00

(b) Distorted meshes

	$E_{\boldsymbol{v}}^{1,p}$		$E_{\pi}^{p'}$	
#cells	error	conv.	error	conv.
320	7.85e-02	–	2.52e-03	–
1280	4.01e-02	0.97	1.42e-03	0.83
5120	1.99e-02	1.01	5.13e-04	1.47
20480	1.00e-02	0.99	1.82e-04	1.49
81920	5.00e-03	1.00	7.02e-05	1.37
327680	2.50e-03	1.00	3.06e-05	1.19
1310720	1.25e-03	1.00	1.43e-05	1.10
expected		0.67		1.00



(a) Regular initial mesh (b) Distorted initial mesh (c) Distorted mesh with 320 el.

Figure 4.1. Distorted mesh (c) with apparent patch-structure based on (b)

In all previous examples, the mesh was refined uniformly. In particular, in each refinement step one quadrilateral is uniformly subdivided in four quadrilaterals of same size. By contrast, in Table 4.9 we numerically solved the same problem as in Table 4.8(a) for disturbed grids: On each refinement level every inner node of the grid, which is not located next to the boundary, was randomly displaced up to $0 < \delta_x < 0.15h$ in x-direction and $0 < \delta_y < 0.15h$ in y-direction (see Figure 4.1). Table 4.9 indicates a similar convergence of the velocity for both regular and distorted meshes. As a result, this observation does not allow us to state whether the improved convergence order for $E_{\boldsymbol{v}}^{1,p}$ can be explained by super-approximation effects. By contrast, Table 4.9 reveals a reduced convergence rate for the pressure on distorted meshes. In fact, in view of Table 4.9(b) we realize that $E_\pi^{p'}$ almost behaves as $\mathcal{O}(h)$. Hence, super-approximation seems to be involved.

Table 4.10. Super approximation for a smooth solution: Case $p = 2$

(a) Discretization errors

	$E_{\boldsymbol{v}}^{1,2}$		E_π^2	
#cells	error	conv.	error	conv.
256	4.66e-02	–	2.67e-03	–
1024	2.33e-02	1.00	9.33e-04	1.52
4096	1.16e-02	1.00	3.23e-04	1.53
16384	5.82e-03	1.00	1.13e-04	1.52
65536	2.91e-03	1.00	3.94e-05	1.52
262144	1.46e-03	1.00	1.38e-05	1.51
expected		1.00		1.00

(b) Projection errors

	$\|\nabla \boldsymbol{\xi}_h\|_2$		$\|\eta_h\|_2$	
#cells	error	conv.	error	conv.
256	2.26e-03	–	2.69e-03	–
1024	8.04e-04	1.49	9.36e-04	1.52
4096	2.81e-04	1.51	3.24e-04	1.53
16384	9.84e-05	1.52	1.13e-04	1.52
65536	3.45e-05	1.51	3.94e-05	1.52
262144	1.21e-05	1.51	1.38e-05	1.51

In Table 4.8 we observed an improved convergence rate for $E_\pi^{p'}$ (and E_π^2) larger than one although we would expect from (4.53) and (4.54) that the convergence rate for $E_\pi^{p'}$ (and E_π^2) is restricted to one for $\mathbb{Q}_1$ elements. The improved convergence order for $E_\pi^{p'}$ is not related to the p-structure of the problem but it is rather caused by super-approximation in the context of $\mathbb{Q}_1/\mathbb{Q}_1$ elements. This statement is supported by Table 4.10 which shows a computation for $p = 2$. Note that the case $p = 2$ corresponds to the linear Stokes equations. In Table 4.10 we solved the above problem for $p = 2$ using a sequence of regular meshes. Table 4.10(a) depicts the obtained discretization errors $E_{\boldsymbol{v}}^{1,2}$ and E_π^2. We expect linear convergence for E_π^2 but we observe the improved convergence rate 3/2. In this connection, we also measured the projection errors. Table 4.10(b) presents the projection errors $\boldsymbol{\xi}_h := (\boldsymbol{j}_h \boldsymbol{v} - \boldsymbol{v}_h)$ and $\eta_h := (j_h \pi - \pi_h)$. Comparing Tables 4.10(a) and 4.10(b), we realize that the convergence rates for η_h agree with the ones for $(\pi - \pi_h)$. The velocity error $\boldsymbol{\xi}_h$ in $\boldsymbol{W}^{1,2}(\Omega)$ converges with same order as the the pressure error η_h in $L^2(\Omega)$. For $p = 2$ the pressure-estimates (4.53) and (4.54) can be expressed as follows:

$$\tilde{\beta}\|\eta_h\|_2 \leq \sup_{\boldsymbol{w}_h \in \boldsymbol{\mathcal{X}}_h^2} \frac{(\nabla \boldsymbol{v} - \nabla \boldsymbol{j}_h \boldsymbol{v}, \nabla \boldsymbol{w}_h)_\Omega}{\|\nabla \boldsymbol{w}_h\|_2} + \sup_{\boldsymbol{w}_h \in \boldsymbol{\mathcal{X}}_h^2} \frac{(\nabla \boldsymbol{\xi}_h, \nabla \boldsymbol{w}_h)_\Omega}{\|\nabla \boldsymbol{w}_h\|_2} + \frac{s_h(\eta_h)(\eta_h)^{\frac{1}{2}}}{\sqrt{\alpha_0}} + \mathcal{O}(h^2).$$

The first term on the right-hand side is known to be of quadratic order (see Blum [Blu91]) whereas the second one is estimated by the quantity $\|\nabla \boldsymbol{\xi}_h\|_2$ whose behavior is numerically illustrated in Table 4.10(b). The convergence order for η_h is basically determined by the one for $\nabla \boldsymbol{\xi}_h$. In fact, both quantities converge with same order. We believe that this observation can be shown analytically at least in the case $p = 2$ by means of the following procedure: Similarly to the derivation of (4.47), we easily obtain the identity

$$\|\nabla \boldsymbol{\xi}_h\|_2^2 + s_h(\eta_h)(\eta_h) \sim (\nabla \boldsymbol{j}_h \boldsymbol{v} - \nabla \boldsymbol{v}, \nabla \boldsymbol{\xi}_h)_\Omega - (j_h\pi - \pi, \nabla \cdot \boldsymbol{\xi}_h)_\Omega \\ + (\nabla \cdot (\boldsymbol{j}_h \boldsymbol{v} - \boldsymbol{v}), \eta_h)_\Omega + s_h(j_h\pi - \pi)(\eta_h) + s_h(\pi)(\eta_h).$$

One has to show that the terms on the right-hand side are of higher order. Since the theory of super-approximation is not topic of the thesis, we do not proceed further in this direction. To sum up, we conjecture that the improved convergence of $E_\pi^{p'}$, which was observed in Table 4.8, is caused by super-approximation due to the smoothness of $(\boldsymbol{v}, \pi)$.

Table 4.11. Stabilization by classical LPS. Verification of Corollary 4.14

(a) $p = 1.1$

#elements	$E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{F}}}$	$E_{\boldsymbol{v}}^{1,p}$	$E_{\boldsymbol{v}}^{p}$	$E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{S}}}$	$E_\pi^{p'}$	E_π^2
1024	0.91	1.06	1.90	0.18	0.19	1.02
4096	0.92	1.04	1.97	0.18	0.19	1.02
16384	0.93	1.03	1.92	0.18	0.19	1.01
65536	0.94	1.02	1.68	0.18	0.19	1.01
262144	0.94	1.01	1.40	0.18	0.19	1.01
expected	1.00	1.00			0.18	1.00

(b) $p = 1.3$

#elements	$E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{F}}}$	$E_{\boldsymbol{v}}^{1,p}$	$E_{\boldsymbol{v}}^{p}$	$E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{S}}}$	$E_\pi^{p'}$	E_π^2
1024	0.97	1.22	1.85	0.47	0.47	1.01
4096	0.98	1.17	1.65	0.47	0.47	1.01
16384	0.98	1.12	1.51	0.47	0.47	1.01
65536	0.98	1.09	1.44	0.47	0.47	1.01
262144	0.98	1.06	1.42	0.47	0.47	1.01
expected	1.00	1.00			0.46	1.00

Example 8: We numerically verify the a priori error estimates of Corollaries 4.14 and 4.16 which quantify the convergence of the standard LPS method proposed in [BB01] in the context of p-Stokes systems. The exact solution $(\boldsymbol{v}, \pi)$ to Problem **(P1)** was prescribed by (4.106) with $a = 1.01$ and $b = -1.99$ so that $\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) \in W^{1,2}(\Omega)^{d\times d}$ and $\pi \in W^{1,2}(\Omega)$ is satisfied as required in Corollaries 4.14 and 4.16. The data $\boldsymbol{f}$, $\boldsymbol{v}|_D$ were chosen accordingly as in Example 1. The stabilization s_h was given by (4.63) with $\alpha_M = \alpha_0 h_M^2/\mu(|\boldsymbol{D}\boldsymbol{v}|^2)$, but in (4.63) the fluctuation of the gradient $\boldsymbol{\theta}_h \nabla \pi_h$ was replaced by the gradient of the fluctuation $\nabla \bar{\theta}_h \pi_h$ and the filter $\bar{\theta}_h$ was chosen as in (3.32). First of all, by means of Table 4.11 we numerically confirm the a priori error estimates of Corollary 4.14 in the case $p \le 2$. Since $\boldsymbol{v} \in \boldsymbol{W}^{1,\infty}(\Omega)$ for $a > 1$, Corollary 4.14 predicts that $E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{F}}} = \mathcal{O}(h)$, $E_{\boldsymbol{v}}^{1,p} = \mathcal{O}(h)$, and $E_\pi^{p'} = \mathcal{O}(h^{2/p'})$. Considering Table 4.11, we observe that the experimental order of

convergence agrees with the expected one very well. By means of Table 4.12 we illustrate the a priori error estimates of Corollary 4.16 in the case $p \geq 2$. Compared to Example 6 and Table 4.7, the better regularity of π should lead to improved convergence rates for both the velocity and pressure. Theoretically we expect that $E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{F}}} = \mathcal{O}(h)$, $E_{\boldsymbol{v}}^{1,p} = \mathcal{O}(h^{2/p})$, and $E_{\pi}^{2} = \mathcal{O}(h)$. In view of Table 4.12, we realize good agreement of the numerical results with the theoretical ones. Once again we observe that $E_{\pi}^{p'}$ is of same order as $E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{S}}}$.

Table 4.12. Stabilization by classical LPS. Verification of Corollary 4.16

(a) $p = 2.5$; $b = -1.99$

#elements	$E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{F}}}$	$E_{\boldsymbol{v}}^{1,p}$	$E_{\boldsymbol{v}}^{p}$	$E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{S}}}$	$E_{\pi}^{p'}$	E_{π}^{2}
1024	1.01	0.81	1.79	1.21	1.20	1.01
4096	1.01	0.81	1.80	1.21	1.21	1.01
16384	1.01	0.81	1.81	1.21	1.21	1.01
65536	1.01	0.81	1.81	1.21	1.21	1.01
262144	1.01	0.81	1.81	1.21	1.21	1.01
expected	1.00	0.80				1.00

(b) $p = 3$; $b = -1.99$

#elements	$E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{F}}}$	$E_{\boldsymbol{v}}^{1,p}$	$E_{\boldsymbol{v}}^{p}$	$E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{S}}}$	$E_{\pi}^{p'}$	E_{π}^{2}
1024	1.01	0.67	1.56	1.34	1.33	1.01
4096	1.01	0.67	1.57	1.34	1.34	1.01
16384	1.01	0.67	1.66	1.34	1.34	1.01
65536	1.01	0.67	1.67	1.34	1.34	1.01
262144	1.01	0.67	1.67	1.34	1.34	1.01
expected	1.00	0.67				1.00

Example 9: We perform some numerical experiments in three space dimensions which are in the same spirit as the previous two-dimensional experiments. In Tables 4.13, 4.14 we demonstrate Theorems 4.11, 4.12 for $d = 3$ and in Table 4.15 we validate Corollary 4.18 for $d = 3$. Here, on the cube $\Omega := (-0.5, 0.5)^3$ the exact solution of **(P1)** was given by

$$\boldsymbol{v}(\boldsymbol{x}) := |\boldsymbol{x}|^{a-2} \begin{pmatrix} x_2 x_3 \\ -0.5 x_1 x_3 \\ -0.5 x_1 x_2 \end{pmatrix} \qquad \text{and} \qquad \pi(\boldsymbol{x}) := |\boldsymbol{x}|^{b} x_1 x_2 x_3. \tag{4.107}$$

In Table 4.13 the parameters a and b have been chosen so that $\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) \in W^{1,2}(\Omega)^{d\times d}$ and $\pi \in W^{1,p'}(\Omega)$. We easily compute that $\nabla \cdot \boldsymbol{v} = 0$ and $|\nabla \boldsymbol{v}(\boldsymbol{x})| \sim |\boldsymbol{x}|^{a-1}$. For $\varepsilon = 0$ we observe $|\nabla \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}(\boldsymbol{x}))| \sim |\boldsymbol{x}|^{\frac{(a-1)p}{2}-1}$. Hence, it holds $\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) \in W^{1,2}(\Omega)^{3\times 3}$ if and only if $(a-1)p - 2 > -3$. This condition is equivalent to $a > \frac{p-1}{p}$. For Table 4.13 we set $a = \frac{p-1}{p} + 0.01$ and $b = 2$. Due to Theorem 4.11 we expect that $E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{F}}} = \mathcal{O}(h)$, $E_{\boldsymbol{v}}^{1,p} = \mathcal{O}(h)$, and $E_{\pi}^{p'} = \mathcal{O}(h^{2/p'})$. In view of Table 4.13, we realize that for all considered quantities the experimental order of convergence agrees with the theoretical one predicted by the a priori error estimates. In Table 4.14 the parameters a and b have been chosen so that

Table 4.13. Numerical verification of Theorem 4.11 for $d = 3$ and $p < 2$

(a) $p = 1.1$, $a = 0.10$

#elements	$E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{F}}}$	$E_{\boldsymbol{v}}^{1,p}$	$E_{\boldsymbol{v}}^{p}$	$E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{S}}}$	$E_{\pi}^{p'}$
4096	0.84	0.99	1.72	0.18	1.12
32768	0.88	1.02	1.84	0.18	0.20
262144	0.90	1.02	1.91	0.18	0.18
expected	1.00	1.00			0.18

(b) $p = 1.2$, $a = 0.17$

#elements	$E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{F}}}$	$E_{\boldsymbol{v}}^{1,p}$	$E_{\boldsymbol{v}}^{p}$	$E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{S}}}$	$E_{\pi}^{p'}$
4096	0.84	0.98	1.82	0.34	1.72
32768	0.87	1.00	1.91	0.33	0.64
262144	0.89	1.00	1.95	0.33	0.34
expected	1.00	1.00			0.33

$\boldsymbol{v} \in \boldsymbol{W}^{2,p}(\Omega)$ and $\pi \in W^{1,p'}(\Omega)$. Note that these requirements are equivalent to the conditions $a > -3/p + 2$ and $b > -3(p-1)/p - 2$. For Table 4.14 we set $a = -3/p + 2.01$ and $b = -3(p-1)/p - 1.99$. By virtue of Theorem 4.12 we expect that $E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{F}}} = \mathcal{O}(h^{p'/2})$, $E_{\boldsymbol{v}}^{1,p} = \mathcal{O}(h^{1/(p-1)})$, and $E_{\pi}^{p'} = \mathcal{O}(h^{p'/2})$. In view of Table 4.14, we realize that the experimental convergence order for the velocity agrees with the theoretical one. But we observe linear convergence for the pressure although we expect the convergence rate $p'/2$ only. Note that we made similar observations in the case $d = 2$ (see Examples 5 and 6). As in the above examples, the quantity $E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{S}}}$ converges with same order as $E_{\pi}^{p'}$ and the apparent convergence rate for $E_{\pi}^{p'}$ may be explained by the one for $E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{S}}}$. We mention that if we prescribe the regularity $\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) \in W^{1,2}(\Omega)^{d\times d}$ instead of $\boldsymbol{v} \in \boldsymbol{W}^{2,p}(\Omega)$ only, then we observe similar convergence rates as in Table 4.14. Hence we conjecture that for this particular example the regularity assumption $\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) \in W^{1,2}(\Omega)^{d\times d}$ and $\pi \in W^{1,p'}(\Omega)$ is sufficient to ensure the availability of the error estimates stated in Theorem 4.12.

Table 4.14. Numerical verification of Theorem 4.12 for $d = 3$ and $p > 2$

(a) $p = 3.0$, $a = 1.01$, $b = -3.99$

#elements	$E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{F}}}$	$E_{\boldsymbol{v}}^{1,p}$	$E_{\boldsymbol{v}}^{p}$	$E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{S}}}$	$E_{\pi}^{p'}$
4096	0.66	0.36	0.99	0.93	1.02
32768	0.72	0.42	1.26	1.00	1.01
262144	0.74	0.46	1.38	1.01	1.01
expected	0.75	0.50			0.75

(b) $p = 3.5$, $a = 1.15$, $b = -4.13$

#elements	$E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{F}}}$	$E_{\boldsymbol{v}}^{1,p}$	$E_{\boldsymbol{v}}^{p}$	$E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{S}}}$	$E_{\pi}^{p'}$
4096	0.62	0.28	0.80	0.94	1.03
32768	0.69	0.37	1.24	1.02	1.01
262144	0.70	0.40	1.35	1.02	1.01
expected	0.70	0.40			0.70

In Example 4.15 the parameters a and b have been chosen so that $\boldsymbol{v} \in \boldsymbol{W}^{2,2}(\Omega)$ and $\pi \in W^{1,2}(\Omega)$. Note that the assumptions $\boldsymbol{v} \in \boldsymbol{W}^{2,2}(\Omega)$ and $\pi \in W^{1,2}(\Omega)$ amount to the conditions $a > 0.5$ and $b > -3.5$. In Example 4.15 we set $a = 0.51$ and $b = -3.49$. According to Corollary 4.18, we expect that $E_{\boldsymbol{v}}^{1,p} = \mathcal{O}(h)$ and $E_{\pi}^{2} = \mathcal{O}(h)$. Moreover, Corollary 4.18 predicts the convergence rate $\frac{3}{p'} - \frac{1}{2}$ for $E_{\pi}^{p'}$. Considering Table 4.15, we observe a good agreement of the numerical results with the theoretical ones.

Table 4.15. Numerical verification of Corollary 4.18 for $d = 3$ and $p < 2$

(a) $p = 1.1$

#elements	$E_{\boldsymbol{v}}^{1,p}$	$E_{\boldsymbol{v}}^{p}$	$E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{S}}}$	$E_{\pi}^{p'}$	E_{π}^{2}
512	0.99	1.81	0.21	-0.21	1.13
4096	1.02	1.86	0.22	-0.22	1.05
32768	1.02	1.94	0.21	-0.22	1.01
262144	1.01	1.97	0.20	-0.22	1.00
expected	1.00			-0.22	1.00

(b) $p = 1.2$

#elements	$E_{\boldsymbol{v}}^{1,p}$	$E_{\boldsymbol{v}}^{p}$	$E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{S}}}$	$E_{\pi}^{p'}$	E_{π}^{2}
512	0.98	1.83	0.37	0.13	1.11
4096	1.02	1.89	0.39	0.01	1.04
32768	1.01	1.95	0.38	0.01	1.01
262144	1.01	1.97	0.37	0.01	1.01
expected	1.00			0.00	1.00

(c) $p = 1.3$

#elements	$E_{\boldsymbol{v}}^{1,p}$	$E_{\boldsymbol{v}}^{p}$	$E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{S}}}$	$E_{\pi}^{p'}$	E_{π}^{2}
512	0.97	1.84	0.50	0.45	1.09
4096	1.01	1.91	0.52	0.20	1.04
32768	1.01	1.96	0.52	0.20	1.01
262144	1.01	1.97	0.51	0.20	1.01
expected	1.00			0.19	1.00

(d) $p = 1.5$

#elements	$E_{\boldsymbol{v}}^{1,p}$	$E_{\boldsymbol{v}}^{p}$	$E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{S}}}$	$E_{\pi}^{p'}$	E_{π}^{2}
512	0.93	1.86	0.65	0.78	1.06
4096	0.98	1.91	0.70	0.51	1.03
32768	0.99	1.95	0.71	0.51	1.01
262144	1.00	1.97	0.72	0.51	1.01
expected	1.00			0.50	1.00

Conclusion: In this chapter we proposed the novel LPS-based stabilization (4.4) which was particularly designed for the approximation of p-Stokes systems with equal-order finite elements. For low-order d-linear elements, we derived a priori error estimates which quantify the convergence of the method (see Theorems 4.11 and 4.12). In the case $p \leq 2$ the derived error estimates provide optimal rates of convergence with respect to the supposed regularity of the solution. They improve existing results in literature, see [BN90, BL93b, BL94]. Note that the results of [BN90, BL93b, BL94] are suboptimal in the sense that either the rate of convergence is not optimal or the assumed regularity of the solution is too high and not realistic for general solutions. In the case $p \geq 2$ our a priori error estimates yield an optimal convergence rate for the velocity and a possibly suboptimal convergence order for the pressure provided that the velocity satisfies slightly more regularity than its natural one. Our numerical experiments indicate that the pressure error $E_{\pi}^{p'}$ converges with same order as $E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{S}}}$. This observation was also made in Belenki et al. [BBDR10]. In order to obtain sharp error estimates for $E_{\pi}^{p'}$ in the case $p \geq 2$, one should therefore attempt to estimate $E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{S}}}$ directly, and one should not relate it to the natural distance $E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{F}}}$ via Lemma 2.7. Note that in the proof of Theorem 4.12 the quantity $E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{S}}}$ was estimated by $E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{F}}}$ as suggested by Lemma 2.7. If the pressure gradient is stabilized with the standard LPS method for Stokes systems as introduced in [BB01], then similar a priori error estimates were derived (see Corollaries 4.14 and 4.16). They are optimal with respect to the rate of convergence in case of $d = 2$, but their derivation requires either additional regularity assumptions on the solution $(\boldsymbol{v}, \pi)$ or the restriction to the case $\varepsilon > 0$. Their rate of convergence depends on the space dimension d. By contrast, the LPS-based stabilization proposed in (4.4) allows a priori error estimates which, at least for $p \leq 2$, provide optimal rates of convergence for arbitrary space dimension $d \geq 2$. Moreover we observed super approximation for the pressure whenever we approximated a smooth solution on a sequence of uniformly refined meshes. This improved convergence supports the usage of LPS-based stabilization: If the pressure is smooth, artificial terms such as $h^s \sum_{M\in\mathbb{M}_h} \|\boldsymbol{\theta}_h \nabla\pi\|_{p';M}^{p'}$ (see, e.g., Remark 4.4) resulting from stabilization are usually of higher order than s since they involve fluctuations $\boldsymbol{\theta}_h$ satisfying $\|\boldsymbol{\theta}_h \nabla\pi\|_{p';M} \lesssim h_M \|\nabla^2\pi\|_{p';M}$. In contrast let us imagine a simplified version of (4.4) for which the fluctuations of gradients $\boldsymbol{\theta}_h\nabla\pi$ are replaced by gradients $\nabla\pi$. For such simplified stabilization, the order of convergence would be restricted to s so that an improved convergence due to super approximation would not be possible.

4.9 Final remarks on LPS

We close the chapter with some remarks on the LPS scheme proposed in Section 4.1. In Assumption 4.1 we required that the pairing X_h/Y_h between the original FE space X_h and the projection space Y_h satisfies a certain local inf-sup condition. In fact Assumption 4.1 can be satisfied for several choices of Y_h. In this section we exemplarily verify Assumption 4.1 for particular pairings X_h/Y_h following the literature [MST07]. Let $\hat{M} := (-1,1)^d$ be the reference hyper-cube with vertices $\hat{a}_i$, $i = 1,\dots,2^d$, and the barycenter $\hat{a}_0$ and let $\boldsymbol{F}_M : \hat{M} \to M$ be the multilinear reference mapping. Let $\hat{M}$ be refined into 2^d congruent cubes $\hat{K}_i$, $i = 1,\dots,2^d$. This induces a refinement of M into 2^d cells. The union of all these cells forms the principal mesh $\mathbb{T}_h = \cup_{M\in\mathbb{M}_h} \Big\{ \boldsymbol{F}_M(\hat{K}_i);\, i = 1,\dots,2^d \Big\}$. We define

$$
\begin{aligned}
X_{h,r} &:= \{ w \in C(\overline{\Omega});\, w|_K \circ \boldsymbol{F}_K \in \mathbb{Q}_r(\hat{K}) \ \forall K \in \mathbb{T}_h \}, \\
X_{2h,r-1}^{\text{disc}} &:= \{ w \in L^2(\Omega);\, w|_M \circ \boldsymbol{F}_M \in \mathbb{Q}_{r-1}(\hat{M}) \ \forall M \in \mathbb{M}_h = \mathbb{T}_{2h} \}.
\end{aligned}
$$

Actually the spaces $X_{h,r}$ and $X_{2h,r-1}^{\text{disc}}$ have already been introduced in (3.3) and (3.28).

Lemma 4.24 (Local inf-sup condition). *Let $\mathbb{M}_h$ satisfy the mesh-property $(M5)$, i.e., let the distortion parameter γ_M defined in (3.9) fulfill $\gamma_M \le \gamma_0 < 1$. Let the local projection scheme be defined for the pair $X_h/Y_h = X_{h,r}/X_{2h,r-1}^{\text{disc}}$ with a fixed polynomial degree $r \in \mathbb{N}$. As in Section 4.1, we set $Y_h(M) := \{ q_h|_M;\, q_h \in Y_h \}$ and*

$$
X_h^0(M) := \{ w_h|_M;\, w_h \in X_h,\, w_h = 0 \text{ on } \Omega \setminus M \}.
$$

Then for $\nu \ge 1$ there exists $\bar{\beta} = \bar{\beta}(\gamma_0) > 0$ independent of h such that

$$
\inf_{q\in Y_h(M)} \sup_{w\in X_h^0(M)} \frac{(w,q)_M}{\|w\|_{\nu;M} \|q\|_{\nu';M}} \ge \bar{\beta} > 0
$$

for all $h > 0$ and all $M \in \mathbb{M}_h$, where $\nu' := \nu/(\nu-1)$. If $\nu = 1$, then $\nu' := \infty$. If $\gamma_0 \to 1$, then the constant $\bar{\beta}$ may degenerate, i.e., $\bar{\beta} \to 0$.

Proof. We follow the proof of Lemma 3.2 in [MST07]. There the desired result has been proven in a Hilbert space setting. Here we can use the same arguments. First of all let $\nu \in (1,\infty)$. From (3.10) it follows that (note $\boldsymbol{x} = \boldsymbol{F}_M(\hat{\boldsymbol{x}})$, $\hat{q}(\hat{\boldsymbol{x}}) := q(\boldsymbol{x})$)

$$
\|q\|_{\nu';M}^{\nu'} = \int_{\hat{M}} |\hat{q}(\hat{\boldsymbol{x}})|^{\nu'} |\det(\nabla \boldsymbol{F}_M(\hat{\boldsymbol{x}}))| \,\mathrm{d}\hat{\boldsymbol{x}} \le C d! (1+\gamma_M)^d h_M^d \|\hat{q}\|_{\nu';\hat{M}}^{\nu'} \tag{4.108}
$$

for all $q \in Y_h(M)$. Let $\hat{b} : \hat{M} \to \mathbb{R}$ be the piecewise multilinear hat function associated with $\hat{a}_0$, i.e., let $\hat{b}(\hat{a}_0) = 1$, $\hat{b}(\hat{a}_i) = 0$ for $i = 0, \dots, 2^d$. For arbitrary $q \in Y_h(M)$ we choose $w(\boldsymbol{x}) := (\hat{q} \cdot \hat{b}) \circ \boldsymbol{F}_M^{-1}(\boldsymbol{x})$. Note that $\hat{q} \in \mathbb{Q}_{r-1}(\hat{M})$. Because $\hat{q} \cdot \hat{b}$ is continuous on the closure of $\hat{M}$, $(\hat{q} \cdot \hat{b})|_{\hat{K}_i} \in \mathbb{Q}_r(\hat{K}_i)$ for $i = 1, \dots, 2^d$, and $\hat{b}|_{\partial\hat{M}} = 0$, we can conclude that

$$\hat{w}(\hat{\boldsymbol{x}}) := \hat{q}(\hat{\boldsymbol{x}})\hat{b}(\hat{\boldsymbol{x}}) \in \Big\{\hat{u} \in C(\text{closure of } \hat{M});\ \hat{u}|_{\partial\hat{M}} = 0,\ \hat{u}|_{\hat{K}_i} \in \mathbb{Q}_r(\hat{K}_i),\ i = 1, \dots, 2^d\Big\}.$$

Hence, we realize that $w \in X_h^0(M)$ (note that $\boldsymbol{x} \in \partial M \Rightarrow \boldsymbol{F}_M^{-1}(\boldsymbol{x}) \in \partial\hat{M} \Rightarrow w(\boldsymbol{x}) = \hat{w}(\boldsymbol{F}_M^{-1}(\boldsymbol{x})) = 0$). Recalling (3.10), we observe that

$$\begin{aligned}(q, w)_M &= \int_M q(\boldsymbol{x})w(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_{\hat{M}} \hat{q}(\hat{\boldsymbol{x}})\hat{w}(\hat{\boldsymbol{x}})|\det(\nabla\boldsymbol{F}_M(\hat{\boldsymbol{x}}))|\,\mathrm{d}\hat{\boldsymbol{x}} \\ &= \int_{\hat{M}} \hat{q}(\hat{\boldsymbol{x}})\hat{q}(\hat{\boldsymbol{x}})\hat{b}(\hat{\boldsymbol{x}})|\det(\nabla\boldsymbol{F}_M(\hat{\boldsymbol{x}}))|\,\mathrm{d}\hat{\boldsymbol{x}} \geq C d!(1-\gamma_M)^d h_M^d \int_{\hat{M}} \hat{q}(\hat{\boldsymbol{x}})^2\hat{b}(\hat{\boldsymbol{x}})\,\mathrm{d}\hat{\boldsymbol{x}}.\end{aligned}$$

Since the space $\mathbb{Q}_{r-1}(\hat{M})$ is finite dimensional, all norms on $\mathbb{Q}_{r-1}(\hat{M})$ are equivalent. Hence

$$\Big\|\hat{q} \cdot \hat{b}^{\frac{1}{2}}\Big\|_{2;\hat{M}} \geq C\|\hat{q}\|_{2;\hat{M}} \qquad \forall \hat{q} \in \mathbb{Q}_{r-1}(\hat{M})$$

for some $C > 0$. As a result, we arrive at

$$(q, w)_M \geq C d!(1-\gamma_M)^d h_M^d \|\hat{q}\|_{2;\hat{M}}^2. \tag{4.109}$$

For all $\hat{\boldsymbol{x}} \in \hat{M}$ it holds $|\hat{b}(\hat{\boldsymbol{x}})| \leq 1$. Consequently, in view of (3.10) we obtain the estimate

$$\|w\|_{\nu;M}^\nu \leq \int_{\hat{M}} |\hat{q}(\hat{\boldsymbol{x}})|^\nu |\det(\nabla\boldsymbol{F}_M(\hat{\boldsymbol{x}}))|\,\mathrm{d}\hat{\boldsymbol{x}} \leq C d!(1+\gamma_M)^d h_M^d \|\hat{q}\|_{\nu;\hat{M}}^\nu. \tag{4.110}$$

Using (4.108), (4.110), and the equivalence of norms on $\mathbb{Q}_{r-1}(\hat{M})$, we conclude that

$$\begin{aligned}\|w\|_{\nu;M}\|q\|_{\nu';M} &\leq C\Big(d!(1+\gamma_M)^d h_M^d\Big)^{1/\nu}\|\hat{q}\|_{\nu;\hat{M}}\Big(d!(1+\gamma_M)^d h_M^d\Big)^{1/\nu'}\|\hat{q}\|_{\nu';\hat{M}} \\ &\leq C d!(1+\gamma_M)^d h_M^d \|\hat{q}\|_{2;\hat{M}}^2.\end{aligned} \tag{4.111}$$

If $\nu = 1$, then $\nu' = \infty$ and $\|q\|_{\infty;M} \leq \|\hat{q}\|_{\infty;\hat{M}}$. Therefore, for $\nu = 1$ we obtain an analog estimate that is similar to (4.111). Combining (4.109) and (4.111), we deduce that for all $\nu \geq 1$ and for all $q \in Y_h(M)$ there exists $w \in X_h^0(M)$ such that

$$\frac{(q, w)_M}{\|w\|_{\nu;M}\|q\|_{\nu';M}} \geq C\Big(\frac{1-\gamma_M}{1+\gamma_M}\Big)^d \geq C\Big(\frac{1-\gamma_0}{1+\gamma_0}\Big)^d =: \bar{\beta}.$$

This yields the assertion. □

5 Approximation of the p-Navier-Stokes Equations

This chapter is devoted to the finite element (FE) discretization of the p-Navier-Stokes problem **(P5)**. The standard Galerkin finite element method (FEM) may suffer from numerical instabilities resulting not only from violation of the inf-sup stability condition but also from dominating advection in case of high Reynolds numbers (cf. [BL09]). The local projection stabilization (LPS) method can be applied to handle both instability phenomena. In this chapter, we will extend the LPS approach proposed in Section 4.1 to the generalized p-Oseen problem **(P6)**. Such p-Oseen problems usually appear as an auxiliary problem when the non-steady p-Navier-Stokes system is discretized with an implicit A-stable time step method (cf. [BDR09, BL09]). In the shear thinning case, we will show optimal a priori error estimates that ensure the convergence of the method and that are similar to those established in Theorem 4.11. Finally, we will study the FE approximation of **(P5)**. In this connection we will discuss a posteriori error estimation. From practical point of view, a posteriori error estimation plays an important role since it allows to assess the actual discretization error numerically. In contrast, a priori error estimation yields upper bounds for the discretization error that depend on the unknown exact solution and that cannot be evaluated numerically. The dual weighted residual (DWR) method has been developed particularly for goal-oriented a posteriori error estimation. It also allows adaptive mesh refinement which enables to reduce numerical costs without loss of accuracy. In this chapter, we will apply the DWR method to the p-Navier-Stokes equations.

In Section 5.1, we introduce the LPS method in the context of p-Oseen systems. In Section 5.2 we summarize resulting properties of the stabilization term and we discuss the well-posedness of the stabilized discrete systems. In Section 5.3 we analyze the LPS method applied to the p-Oseen equations. In particular, we derive a priori error estimates by extending the basic concepts of Chapter 4. The established results are motivated by the time-discretization of the p-Navier-Stokes equations in Section 5.4, whereas they are numerically validated in Section 5.5. Following the literature [BR03], in Section 5.6 we introduce the DWR method. Finally, in Section 5.7 we apply the DWR method to the steady p-Navier-Stokes equations for the computation of the drag coefficient.

5.1 LPS in the context of p-Oseen systems

In this section, we consider the p-Oseen system (2.17) complemented with homogeneous Dirichlet boundary conditions and we study its discretization with equal-order $\mathbb{Q}_1/\mathbb{Q}_1$ finite elements. The investigation of system (2.17) is motivated by the fact that it is needed for the error analysis of the time-discretized non-steady p-Navier-Stokes equations if an A-stable semi-implicit Euler scheme is applied (see Section 5.4 or Berselli et al. [BDR09]). System (2.17) corresponds to the steady ($\sigma = 0$) and the non-steady ($\sigma > 0$) time-discretized p-Navier-Stokes-system with linearized convective term. For ease of presentation, for all $\boldsymbol{u} \equiv (\boldsymbol{v}, \pi)$ and $\boldsymbol{\omega} \equiv (\boldsymbol{w}, q)$ we introduce the semi-linear form

$$A(\boldsymbol{u})(\boldsymbol{\omega}) := (\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{w})_\Omega + \Big((\boldsymbol{b}\cdot\nabla)\boldsymbol{v}, \boldsymbol{w}\Big)_\Omega + \sigma(\boldsymbol{v}, \boldsymbol{w})_\Omega - (\pi, \nabla\cdot\boldsymbol{w})_\Omega + (\nabla\cdot\boldsymbol{v}, q)_\Omega \quad (5.1)$$

so that we can equivalently write the p-Oseen Problem **(P6)** as follows: For given $\boldsymbol{f} \in \boldsymbol{L}^{p'}(\Omega)$ find $\boldsymbol{u} \equiv (\boldsymbol{v}, \pi) \in \boldsymbol{\mathcal{X}}^p \times \mathcal{Q}^p$ (the continuous solution) such that

$$A(\boldsymbol{u})(\boldsymbol{\omega}) = (\boldsymbol{f}, \boldsymbol{w})_\Omega \qquad \forall \boldsymbol{\omega} \equiv (\boldsymbol{w}, q) \in \boldsymbol{\mathcal{X}}^p \times \mathcal{Q}^p. \quad (5.2)$$

Below, we always assume that the vector field $\boldsymbol{b}$ belongs to $\boldsymbol{W}^{1,\infty}(\Omega)$ and satisfies $\nabla\cdot\boldsymbol{b} = 0$ a.e.. The Galerkin discretization of **(P6)** reads: Find $\boldsymbol{u}_h \equiv (\boldsymbol{v}_h, \pi_h) \in \boldsymbol{\mathcal{X}}_h^p \times \mathcal{Q}_h^p$ such that

$$A(\boldsymbol{u}_h)(\boldsymbol{\omega}_h) = (\boldsymbol{f}, \boldsymbol{w}_h)_\Omega \qquad \forall \boldsymbol{\omega}_h \equiv (\boldsymbol{w}_h, q_h) \in \boldsymbol{\mathcal{X}}_h^p \times \mathcal{Q}_h^p. \quad (5.3)$$

As mentioned in Section 3.1, the formulation (5.3) may suffer from violation of the discrete inf-sup condition and locally dominating advection. Both instability phenomena can be handled by the local projection stabilization method. In particular, the stabilization of the pressure gradient can be carried out as described for p-Stokes systems in Section 4.1.

Following the literature [MST07] or Section 4.1, we introduce the coarse mesh $\mathbb{M}_h = \{M\}$ constructed by coarsening the basic mesh $\mathbb{T}_h$ such that each macro element $M \in \mathbb{M}_h$ with diameter h_M is the union of one or more neighboring elements $K \in \mathbb{T}_h$. We assume that the decomposition $\mathbb{M}_h$ of Ω is non-overlapping and shape-regular. The interior elements are supposed to be of similar size as the macro element, i.e., $\exists C > 0$: $h_M \le C h_K$ for all $K \in \mathbb{T}_h$ and $M \in \mathbb{M}_h$ with $K \subset M$. Since we deal with equal-order discretizations, we do not need to assign separate projection spaces for the velocity and pressure. Similarly as in Section 4.1, we introduce the space Y_h as a (possibly discontinuous) finite element space defined on the macro partition $\mathbb{M}_h$ so that the pairing X_h/Y_h satisfies the local inf-sup condition Assumption 4.1. The restriction of Y_h on a patch $M \in \mathbb{M}_h$ is denoted by $Y_h(M) := \{w_h|_M;\, w_h \in Y_h\}$. Let $P_M : L^\nu(M) \to Y_h(M)$ be a local projection. The global projection $P_h : L^\nu(\Omega) \to Y_h$ is then given by $(P_h w)|_M := P_M(w|_M)$ for all $M \in \mathbb{M}_h$. The associated fluctuation operator $\theta_h : L^\nu(\Omega) \to L^\nu(\Omega)$ is defined by $\theta_h := \mathrm{id} - P_h$. We modify the discrete problem (5.3) by adding the stabilization term

$$\begin{aligned} S_h(\boldsymbol{u}_h)(\boldsymbol{\omega}_h) :=& \sum_{M\in\mathbb{M}_h} \alpha_M \Big((\tau + |\boldsymbol{\theta}_h\nabla\pi_h|)^{p'-2}\boldsymbol{\theta}_h\nabla\pi_h, \boldsymbol{\theta}_h\nabla q_h\Big)_M \\ &+ \sum_{M\in\mathbb{M}_h} \varrho_M \Big(\boldsymbol{\theta}_h(\boldsymbol{b}\cdot\nabla)\boldsymbol{v}_h, \boldsymbol{\theta}_h(\boldsymbol{b}\cdot\nabla)\boldsymbol{w}_h\Big)_M. \end{aligned} \quad (5.4)$$

Later the patch-wise constants α_M and ϱ_M will depend on the local mesh size h_M. Their dependence on h_M will be determined by the convergence analysis of the method. The stabilized finite element system reads as follows:

(P6$_h$) *Find* $\boldsymbol{u}_h \equiv (\boldsymbol{v}_h, \pi_h) \in \boldsymbol{\mathcal{X}}_h^p \times \mathcal{Q}_h^p$ *(the discrete solution) such that*

$$A(\boldsymbol{u}_h)(\boldsymbol{\omega}_h) + S_h(\boldsymbol{u}_h)(\boldsymbol{\omega}_h) = (\boldsymbol{f}, \boldsymbol{w}_h)_\Omega \qquad \forall \boldsymbol{\omega}_h \equiv (\boldsymbol{w}_h, q_h) \in \boldsymbol{\mathcal{X}}_h^p \times \mathcal{Q}_h^p. \tag{5.5}$$

Note that in the particular case $p = 2$ the stabilization scheme (5.5), (5.4) coincides with the standard LPS scheme for Oseen systems presented in Matthies et al. [MST07]. In order to control the consistency error caused by the $\boldsymbol{\theta}_h$-dependent stabilization terms, the space Y_h has to be rich enough or, in other words, it should satisfy the following

Assumption 5.1. *Let* $\nu > 1$. *We assume that the fluctuation operator* θ_h *satisfies*

$$\|\theta_h w\|_{\nu;M} \leq C h_M^k \|\nabla^k w\|_{\nu;M} \qquad \forall w \in W^{k,\nu}(\Omega), \qquad \forall M \in \mathbb{M}_h, \qquad k \in \{0,1\},$$

where $C > 0$ *does not depend on the local mesh size.*

Below we always assume that Assumption 5.1 is satisfied.

5.2 Properties of the stabilization scheme

In this section, we summarize important properties of the proposed stabilization term (5.4) and we discuss the well-posedness of Problem **(P6$_h$)**. We proceed similarly as in Section 4.2. Let $\boldsymbol{\mathcal{G}}$ be defined by (4.6). For $\boldsymbol{u} \equiv (\boldsymbol{v}, \pi)$ and $\boldsymbol{\omega} \equiv (\boldsymbol{w}, q)$ we define the distance

$$\begin{aligned} |\boldsymbol{u} - \boldsymbol{\omega}|_{\mathrm{lps}}^2 := &\sum_{M \in \mathbb{M}_h} \varrho_M \|\boldsymbol{\theta}_h (\boldsymbol{b} \cdot \nabla)\boldsymbol{v} - \boldsymbol{\theta}_h (\boldsymbol{b} \cdot \nabla)\boldsymbol{w}\|_{2;M}^2 \\ &+ \sum_{M \in \mathbb{M}_h} \alpha_M \|\boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h \nabla \pi) - \boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h \nabla q)\|_{2;M}^2. \end{aligned} \tag{5.6}$$

This definition is justified by the following observation:

Lemma 5.1. *For* $p \in (1, \infty)$ *let* S_h *be defined by* (5.4). *There holds*

$$S_h(\boldsymbol{u})(\boldsymbol{u} - \boldsymbol{\omega}) - S_h(\boldsymbol{\omega})(\boldsymbol{u} - \boldsymbol{\omega}) \sim |\boldsymbol{u} - \boldsymbol{\omega}|_{\mathrm{lps}}^2 \qquad \forall \boldsymbol{u}, \boldsymbol{\omega} \in \boldsymbol{W}^{1,2}(\Omega) \times W^{1,p'}(\Omega).$$

Proof. The assertion follows from the vector-valued version of Lemma 2.4. □

Lemma 5.2. *Let* $p \in (1, 2]$. *For all* $(\boldsymbol{v}, \pi)$, $(\boldsymbol{w}, q) \in \boldsymbol{W}^{1,2}(\Omega) \times W^{1,p'}(\Omega)$ *there holds*

$$\begin{aligned} |(\boldsymbol{v} - \boldsymbol{w}, \pi - q)|_{\mathrm{lps}}^2 \leq &\sum_{M \in \mathbb{M}_h} \varrho_M \|(\boldsymbol{b} \cdot \nabla)(\boldsymbol{v} - \boldsymbol{w})\|_{2;M}^2 \\ &+ \|\tau + |\nabla \pi| + |\nabla q|\|_{p'}^{p'-2} \Big(\sum_{M \in \mathbb{M}_h} \alpha_M^{\frac{p'}{2}} \|\nabla(\pi - q)\|_{p';M}^{p'} \Big)^{\frac{2}{p'}}. \end{aligned}$$

Proof. We can easily derive the desired estimate using Lemma 4.1, Assumption 5.1, and Hölder's inequality with $\frac{2}{p'}+\frac{p'-2}{p'}=1$. More precisely,

$$\begin{aligned}|(\boldsymbol{v}-\boldsymbol{w},\pi-q)|_{\mathrm{lps}}^2 &\lesssim \sum_{M\in\mathbb{M}_h}\varrho_M\|\boldsymbol{\theta}_h(\boldsymbol{b}\cdot\nabla)\boldsymbol{v}-\boldsymbol{\theta}_h(\boldsymbol{b}\cdot\nabla)\boldsymbol{w}\|_{2;M}^2\\ &\quad+\sum_{M\in\mathbb{M}_h}\alpha_M\|\tau+|\boldsymbol{\theta}_h\nabla\pi|+|\boldsymbol{\theta}_h\nabla q|\|_{p';M}^{p'-2}\|\boldsymbol{\theta}_h\nabla\pi-\boldsymbol{\theta}_h\nabla q\|_{p';M}^2\\ &\lesssim \sum_{M\in\mathbb{M}_h}\varrho_M\|(\boldsymbol{b}\cdot\nabla)(\boldsymbol{v}-\boldsymbol{w})\|_{2;M}^2\\ &\quad+\Big(\sum_{M\in\mathbb{M}_h}\|\tau+|\nabla\pi|+|\nabla q|\|_{p';M}^{p'}\Big)^{\frac{p'-2}{p'}}\Big(\sum_{M\in\mathbb{M}_h}\alpha_M^{\frac{p'}{2}}\|\nabla(\pi-q)\|_{p';M}^{p'}\Big)^{\frac{2}{p'}}.\end{aligned}$$

This yields the assertion. □

Lemma 5.3. *For $p\in(1,\infty)$ let S_h be defined by (5.4). For all $\delta>0$ there exists a constant $c=c(\delta,p)$ such that for all $\boldsymbol{u},\bar{\boldsymbol{u}},\boldsymbol{\omega}\in\boldsymbol{W}^{1,2}(\Omega)\times W^{1,p'}(\Omega)$ there holds*

$$S_h(\boldsymbol{u})(\boldsymbol{u}-\boldsymbol{\omega})-S_h(\bar{\boldsymbol{u}})(\boldsymbol{u}-\boldsymbol{\omega})\le c_\delta|\boldsymbol{u}-\bar{\boldsymbol{u}}|_{\mathrm{lps}}^2+\delta|\boldsymbol{u}-\boldsymbol{\omega}|_{\mathrm{lps}}^2.$$

Proof. Let φ and φ^* be given by (2.37). Using the vector-valued version of Lemma 2.4 (with p, ε, φ replaced by p', τ, φ^*), for $\boldsymbol{u}\equiv(\boldsymbol{v},\pi)$, $\bar{\boldsymbol{u}}\equiv(\bar{\boldsymbol{v}},\bar{\pi})$, $\boldsymbol{\omega}\equiv(\boldsymbol{w},q)$ we conclude that

$$\begin{aligned}&S_h(\boldsymbol{u})(\boldsymbol{u}-\boldsymbol{\omega})-S_h(\bar{\boldsymbol{u}})(\boldsymbol{u}-\boldsymbol{\omega})\\ &\lesssim\sum_{M\in\mathbb{M}_h}\alpha_M\int_M(\varphi^*)'_{\tau+|\boldsymbol{\theta}_h\nabla\pi|}(|\boldsymbol{\theta}_h\nabla\pi-\boldsymbol{\theta}_h\nabla\bar{\pi}|)|\boldsymbol{\theta}_h\nabla\pi-\boldsymbol{\theta}_h\nabla q|\,\mathrm{d}\boldsymbol{x}\\ &+\sum_{M\in\mathbb{M}_h}\varrho_M\int_M|\boldsymbol{\theta}_h(\boldsymbol{b}\cdot\nabla)\boldsymbol{v}-\boldsymbol{\theta}_h(\boldsymbol{b}\cdot\nabla)\bar{\boldsymbol{v}}||\boldsymbol{\theta}_h(\boldsymbol{b}\cdot\nabla)\boldsymbol{v}-\boldsymbol{\theta}_h(\boldsymbol{b}\cdot\nabla)\boldsymbol{w}|\,\mathrm{d}\boldsymbol{x}\\ &=:I_1+I_2.\end{aligned}$$

Lemma 2.2, Lemma 4.2 and Young's inequality imply that for arbitrary $\delta>0$

$$\begin{aligned}I_1&\le c_\delta\sum_{M\in\mathbb{M}_h}\alpha_M\|\boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h\nabla\pi)-\boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h\nabla\bar{\pi})\|_{2;M}^2+\delta\sum_{M\in\mathbb{M}_h}\alpha_M\|\boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h\nabla\pi)-\boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h\nabla q)\|_{2;M}^2,\\ I_2&\le c_\delta\sum_{M\in\mathbb{M}_h}\varrho_M\|\boldsymbol{\theta}_h(\boldsymbol{b}\cdot\nabla)\boldsymbol{v}-\boldsymbol{\theta}_h(\boldsymbol{b}\cdot\nabla)\bar{\boldsymbol{v}}\|_{2;M}^2+\delta\sum_{M\in\mathbb{M}_h}\varrho_M\|\boldsymbol{\theta}_h(\boldsymbol{b}\cdot\nabla)\boldsymbol{v}-\boldsymbol{\theta}_h(\boldsymbol{b}\cdot\nabla)\boldsymbol{w}\|_{2;M}^2.\end{aligned}$$

Recalling (5.6), we easily complete the proof. □

Lemma 5.4. *Let $p\in(1,2]$ and $q\ge\frac{2d}{d+1}$. Then, for each $\delta>0$ there exists $c_\delta>0$ such that for all $\boldsymbol{u}\equiv(\boldsymbol{v},\pi)\in\boldsymbol{W}^{2,q}(\Omega)\times W^{k+1,p'}(\Omega)$, $k\in\{0,1\}$, $\boldsymbol{\omega}\in\boldsymbol{W}^{1,2}(\Omega)\times W^{1,p'}(\Omega)$*

$$\begin{aligned}S_h(\boldsymbol{u})(\boldsymbol{u}-\boldsymbol{\omega})&\le c_\delta\|\boldsymbol{v}\|_{2,q;\Omega}\Big(\sum_{M\in\mathbb{M}_h}\Big[\varrho_M h_M\|\boldsymbol{b}\|_{1,\infty;M}^2\Big]^q\|\boldsymbol{v}\|_{2,q;M}^q\Big)^{\frac1q}\\ &\quad+c_\delta\|\tau+|\nabla\pi|\|_{p'}^{p'-2}\Big(\sum_{M\in\mathbb{M}_h}\Big[\alpha_M h_M^{2k}\Big]^{\frac{p'}{2}}\|\nabla^{k+1}\pi\|_{p';M}^{p'}\Big)^{\frac{2}{p'}}+\delta|\boldsymbol{u}-\boldsymbol{\omega}|_{\mathrm{lps}}^2.\end{aligned}$$

Proof. Setting $\bar{\boldsymbol{u}} = \boldsymbol{0}$ in Lemma 5.3, we observe that for each $\delta > 0$ there exists $c_\delta > 0$:

$$S_h(\boldsymbol{u})(\boldsymbol{u}-\boldsymbol{\omega}) \leq c_\delta |\boldsymbol{u}|^2_{\mathrm{lps}} + \delta|\boldsymbol{u}-\boldsymbol{\omega}|^2_{\mathrm{lps}}.$$

Applying Lemma 4.1, noting the fact $p' \geq 2$, using Hölder's inequality and Assumption 5.1, for $k \in \{0,1\}$ we can estimate the term $|\boldsymbol{u}|^2_{\mathrm{lps}}$ by

$$\begin{aligned}
|\boldsymbol{u}|^2_{\mathrm{lps}} &= \sum_{M\in\mathbb{M}_h} \varrho_M \|\boldsymbol{\theta}_h(\boldsymbol{b}\cdot\nabla)\boldsymbol{v}\|^2_{2;M} + \sum_{M\in\mathbb{M}_h} \alpha_M \|\boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h\nabla\pi)\|^2_{2;M} \\
&\lesssim \sum_{M\in\mathbb{M}_h} \varrho_M \|\boldsymbol{\theta}_h(\boldsymbol{b}\cdot\nabla)\boldsymbol{v}\|_{q';M} \|\boldsymbol{\theta}_h(\boldsymbol{b}\cdot\nabla)\boldsymbol{v}\|_{q;M} \\
&\qquad + \sum_{M\in\mathbb{M}_h} \alpha_M \|\tau + |\boldsymbol{\theta}_h\nabla\pi|\|^{p'-2}_{p';M} \|\boldsymbol{\theta}_h\nabla\pi\|^2_{p';M} \\
&\lesssim \sum_{M\in\mathbb{M}_h} \varrho_M \|(\boldsymbol{b}\cdot\nabla)\boldsymbol{v}\|_{q';M} h_M \|(\boldsymbol{b}\cdot\nabla)\boldsymbol{v}\|_{1,q;M} \\
&\qquad + \sum_{M\in\mathbb{M}_h} \alpha_M \|\tau + |\nabla\pi|\|^{p'-2}_{p';M} h_M^{2k} \|\nabla^{k+1}\pi\|^2_{p';M}.
\end{aligned}$$

Using Hölder's inequality twice (with $\frac{2}{p'} + \frac{p'-2}{p'} = 1$ in the second sum), we arrive at

$$\begin{aligned}
|\boldsymbol{u}|^2_{\mathrm{lps}} \lesssim &\Big(\sum_{M\in\mathbb{M}_h} \|\nabla\boldsymbol{v}\|^{q'}_{q';M}\Big)^{\frac{1}{q'}} \Big(\sum_{M\in\mathbb{M}_h} \Big[\varrho_M h_M \|\boldsymbol{b}\|^2_{1,\infty;M}\Big]^q \|\nabla\boldsymbol{v}\|^q_{1,q;M}\Big)^{\frac{1}{q}} \\
&+ \Big(\sum_{M\in\mathbb{M}_h} \Big[\alpha_M h_M^{2k}\Big]^{\frac{p'}{2}} \|\nabla^{k+1}\pi\|^{p'}_{p';M}\Big)^{\frac{2}{p'}} \Big(\sum_{M\in\mathbb{M}_h} \|\tau + |\nabla\pi|\|^{p'}_{p';M}\Big)^{\frac{p'-2}{p'}}.
\end{aligned}$$

Since $q \geq \frac{2d}{d+1}$, the embedding $W^{2,q}(\Omega) \hookrightarrow W^{1,q'}(\Omega)$ holds. This implies the assertion. □

The following lemma represents a simple modification of Lemma 5.4:

Lemma 5.5. *Let $p \in (1,2]$ and $q \in [\frac{2d}{d+2}, 2]$. Assume that the fluctuation operator θ_h additionally satisfies the approximation property $\|\theta_h w\|_{2;M} \leq C h_M^{1-d/q+d/2} \|w\|_{1,q;M}$ for all $M \in \mathbb{M}_h$ and $w \in W^{1,q}(\Omega) \hookrightarrow L^2(\Omega)$. Then, for each $\delta > 0$ there exists $c_\delta > 0$ such that for $\boldsymbol{u} \equiv (\boldsymbol{v}, \pi) \in \boldsymbol{W}^{2,q}(\Omega) \times W^{k+1,p'}(\Omega)$, $k \in \{0,1\}$, $\boldsymbol{\omega} \in \boldsymbol{W}^{1,2}(\Omega) \times W^{1,p'}(\Omega)$ there holds*

$$\begin{aligned}
S_h(\boldsymbol{u})(\boldsymbol{u}-\boldsymbol{\omega}) \leq c_\delta &\Big(\sum_{M\in\mathbb{M}_h} \Big[\varrho_M h_M^{2-\frac{2d}{q}+d}\Big]^{\frac{q}{2}} \|(\boldsymbol{b}\cdot\nabla)\boldsymbol{v}\|^q_{1,q;M}\Big)^{\frac{2}{q}} \\
&+ c_\delta \|\tau + |\nabla\pi|\|^{p'-2}_{p'} \Big(\sum_{M\in\mathbb{M}_h} \Big[\alpha_M h_M^{2k}\Big]^{\frac{p'}{2}} \|\nabla^{k+1}\pi\|^{p'}_{p';M}\Big)^{\frac{2}{p'}} + \delta|\boldsymbol{u}-\boldsymbol{\omega}|^2_{\mathrm{lps}}.
\end{aligned}$$

Proof. Setting $\bar{\boldsymbol{u}} = \boldsymbol{0}$ in Lemma 5.3, we realize that for each $\delta > 0$ there exists $c_\delta > 0$:

$$S_h(\boldsymbol{u})(\boldsymbol{u}-\boldsymbol{\omega}) \leq c_\delta |\boldsymbol{u}|^2_{\mathrm{lps}} + \delta|\boldsymbol{u}-\boldsymbol{\omega}|^2_{\mathrm{lps}}.$$

Using Lemma 4.1, the assumption on θ_h, and Hölder's inequality with $\frac{2}{p'} + \frac{p'-2}{p'} = 1$, and noting $q \in [\frac{2d}{d+2}, 2]$, $p' \geq 2$, for $k \in \{0,1\}$ we can estimate

$$\begin{aligned}
|\boldsymbol{u}|_{\mathrm{lps}}^2 &= \sum_{M\in\mathbb{M}_h} \varrho_M \|\boldsymbol{\theta}_h(\boldsymbol{b}\cdot\nabla)\boldsymbol{v}\|_{2;M}^2 + \sum_{M\in\mathbb{M}_h} \alpha_M \|\boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h\nabla\pi)\|_{2;M}^2 \\
&\lesssim \sum_{M\in\mathbb{M}_h} \varrho_M h_M^{2-\frac{2d}{q}+d} \|(\boldsymbol{b}\cdot\nabla)\boldsymbol{v}\|_{1,q;M}^2 + \sum_{M\in\mathbb{M}_h} \alpha_M \|\tau + |\nabla\pi|\|_{p';M}^{p'-2} h_M^{2k} \|\nabla^{k+1}\pi\|_{p';M}^2 \\
&\lesssim \Big(\sum_{M\in\mathbb{M}_h} \Big[\varrho_M h_M^{2-\frac{2d}{q}+d}\Big]^{\frac{q}{2}} \|(\boldsymbol{b}\cdot\nabla)\boldsymbol{v}\|_{1,q;M}^q \Big)^{\frac{2}{q}} \\
&\quad + \Big(\sum_{M\in\mathbb{M}_h} \Big[\alpha_M h_M^{2k}\Big]^{\frac{p'}{2}} \|\nabla^{k+1}\pi\|_{p';M}^{p'} \Big)^{\frac{2}{p'}} \Big(\sum_{M\in\mathbb{M}_h} \|\tau + |\nabla\pi|\|_{p';M}^{p'} \Big)^{\frac{p'-2}{p'}} .
\end{aligned}$$

This completes the proof. □

Remark 5.1. If the fluctuation operator θ_h satisfies Assumption 5.1, then it also fulfills the assumption of Lemma 5.5, cf. the homogeneity argument in (3.18).

Using similar arguments as in the proofs of Lemmas 4.6, 4.7, and taking into account the properties of $\boldsymbol{b}$ and S_h, we can easily conclude the well-posedness of Problem $(\mathbf{P6}_h)$:

Lemma 5.6. *For $p \in (1,\infty)$ and $\varepsilon \in [0,\varepsilon_0]$ let $\boldsymbol{\mathcal{S}}$ satisfy Assumption 2.1. Let S_h be defined by* (5.4) *with $\alpha_M := \alpha_0 h_M^s$, $s \in [0,p']$, $\tau \in [0,\tau_0]$, $\varrho_M := \varrho_0 h_M$. Then for $p \geq \frac{2d}{d+1}$ there exists a unique solution $\boldsymbol{u}_h \equiv (\boldsymbol{v}_h, \pi_h) \in \boldsymbol{\mathcal{X}}_h^p \times \mathcal{Q}_h^p$ to Problem $(\mathbf{P6}_h)$ satisfying*

$$\begin{aligned}
\|\boldsymbol{v}_h\|_{1,p}^p + \sigma\|\boldsymbol{v}_h\|_2^2 + \|\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}_h)\|_{p'}^{p'} + S_h(\boldsymbol{u}_h)(\boldsymbol{u}_h) &\leq C_1(\Omega, p, \varepsilon_0, \sigma_0, \sigma_1, \boldsymbol{f}), \\
\tilde{\beta}(p)\|\pi_h\|_{p'} &\leq C_2(\Omega, p, \varepsilon_0, \sigma_0, \sigma_1, \sigma, \boldsymbol{f}, \alpha_0, \tau_0, \varrho_0),
\end{aligned} \tag{5.7}$$

where $\tilde{\beta}(p) > 0$ is the constant appearing in (4.30)*. The restriction on p solely comes from the availability of the $L^{p'}$-pressure-estimate* (5.7)$_2$*. The constants C_1 and C_2 only depend on the data quoted within the brackets. If $p \leq 2$ then the constant C_2 does not depend on τ_0, whereas if $p > 2$ the constant C_1 does not depend on ε_0.*

Proof. Due to $([\boldsymbol{b}\cdot\nabla]\boldsymbol{v}_h, \boldsymbol{v}_h)_\Omega = 0$, the well-posedness of $(\mathbf{P6}_h)$ follows along the lines of Lemmas 4.6 and 4.7. The restriction on p, $p \geq \frac{2d}{d+1}$, results from the continuous embedding $W^{1,p}(\Omega) \hookrightarrow L^{p'}(\Omega)$ which is needed for the derivation of (5.7)$_2$. More precisely, the discrete pressure π_h is estimated by (cf. the proof of Lemma 4.6)

$$\begin{aligned}
\tilde{\beta}\|\pi_h\|_{p'} &\leq \sup_{\boldsymbol{w}_h\in\boldsymbol{\mathcal{X}}_h^p} \frac{(\nabla\cdot\boldsymbol{w}_h, \pi_h)_\Omega}{\|\nabla\boldsymbol{w}_h\|_p} + \Big(\sum_{M\in\mathbb{M}_h} h_M^{p'} \|\boldsymbol{\theta}_h\nabla\pi_h\|_{p';M}^{p'} \Big)^{\frac{1}{p'}} \\
&\leq \sup_{\boldsymbol{w}_h\in\boldsymbol{\mathcal{X}}_h^p} \frac{|(\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}_h), \boldsymbol{D}\boldsymbol{w}_h)_\Omega - (\boldsymbol{f}, \boldsymbol{w}_h)_\Omega|}{\|\nabla\boldsymbol{w}_h\|_p} + \Big(\sum_{M\in\mathbb{M}_h} h_M^{p'} \|\boldsymbol{\theta}_h\nabla\pi_h\|_{p';M}^{p'} \Big)^{\frac{1}{p'}} \\
&\quad + \sup_{\boldsymbol{w}_h\in\boldsymbol{\mathcal{X}}_h^p} \frac{\Big|([\boldsymbol{b}\cdot\nabla]\boldsymbol{v}_h, \boldsymbol{w}_h)_\Omega + \sigma(\boldsymbol{v}_h, \boldsymbol{w}_h)_\Omega + S_h\Big((\boldsymbol{v}_h, 0)\Big)\Big((\boldsymbol{w}_h, 0)\Big)\Big|}{\|\nabla\boldsymbol{w}_h\|_p} .
\end{aligned}$$

The first two terms on the right-hand side also appear in the proof of Lemma 4.6 and they are estimated similarly as there. Compared to the proof of Lemma 4.6, we additionally need to control the latter term. Using $W^{1,p}(\Omega) \hookrightarrow L^{p'}(\Omega)$, for $p \geq \frac{2d}{d+1}$ we conclude that

$$\begin{aligned}([\boldsymbol{b} \cdot \nabla]\boldsymbol{v}_h, \boldsymbol{w}_h)_\Omega &\leq \|\boldsymbol{b}\|_\infty \|\nabla \boldsymbol{v}_h\|_p \|\boldsymbol{w}_h\|_{p'} \lesssim \|\boldsymbol{b}\|_\infty \|\boldsymbol{v}_h\|_{1,p} \|\boldsymbol{w}_h\|_{1,p}, \\ \sigma(\boldsymbol{v}_h, \boldsymbol{w}_h)_\Omega &\leq \sigma \|\boldsymbol{v}_h\|_2 \|\boldsymbol{w}_h\|_2 \lesssim \sigma \|\boldsymbol{v}_h\|_{1,p} \|\boldsymbol{w}_h\|_{1,p}.\end{aligned}$$

Using Assumption 5.1, the local inverse inequality (3.19), for $p \geq \frac{2d}{d+1}$ we arrive at

$$\begin{aligned} S_h\Big((\boldsymbol{v}_h, 0)\Big)\Big((\boldsymbol{w}_h, 0)\Big) &\leq \Big(\sum_{M \in \mathbb{M}_h} \varrho_M \|\boldsymbol{\theta}_h (\boldsymbol{b} \cdot \nabla) \boldsymbol{v}_h\|_{2;M}^2 \Big)^{\frac{1}{2}} \Big(\sum_{M \in \mathbb{M}_h} \varrho_M \|\boldsymbol{\theta}_h (\boldsymbol{b} \cdot \nabla) \boldsymbol{w}_h\|_{2;M}^2 \Big)^{\frac{1}{2}} \\ &\lesssim \|\boldsymbol{b}\|_\infty^2 \Big(\sum_{M \in \mathbb{M}_h} \varrho_M h_M^{-\frac{2d}{p}+d} \|\boldsymbol{v}_h\|_{1,p;M}^2 \Big)^{\frac{1}{2}} \Big(\sum_{M \in \mathbb{M}_h} \varrho_M h_M^{-\frac{2d}{p}+d} \|\boldsymbol{w}_h\|_{1,p;M}^2 \Big)^{\frac{1}{2}} \\ &\lesssim \|\boldsymbol{b}\|_\infty^2 \Big(\sum_{M \in \mathbb{M}_h} \|\boldsymbol{v}_h\|_{1,p;M}^p \Big)^{\frac{1}{p}} \Big(\sum_{M \in \mathbb{M}_h} \|\boldsymbol{w}_h\|_{1,p;M}^p \Big)^{\frac{1}{p}} \lesssim \|\boldsymbol{b}\|_\infty^2 \|\boldsymbol{v}_h\|_{1,p} \|\boldsymbol{w}_h\|_{1,p}. \end{aligned}$$

We note that $\varrho_M \sim h_M$ and $1 - \frac{2d}{p} + d \geq 0 \Leftrightarrow p \geq \frac{2d}{d+1}$. We easily complete the proof. □

5.3 Error estimates for the stabilized p-Oseen system

In this section we derive a priori error estimates which quantify the convergence of the method. We restrict ourselves to the case $p \leq 2$ since we may perform a similar analysis in the case $p \geq 2$. The following theorem extends Theorem 4.11 to p-Oseen systems. Its a priori error estimates represent one of the main results of the thesis. They provide optimal rates of convergence with respect to the supposed regularity of the solution.

Theorem 5.7. *Let $p \in (1, 2]$ and $\varepsilon \in [0, \infty)$. Let $(\boldsymbol{v}, \pi) \in \boldsymbol{\mathcal{X}}^p \times \mathcal{Q}^p$ be the unique solution of* **(P6)***, and let $(\boldsymbol{v}_h, \pi_h) \in \boldsymbol{\mathcal{X}}_h^p \times \mathcal{Q}_h^p$ be the unique solution of* **(P6$_h$)** *where the stabilization term S_h is defined by* (5.4)*. We assume that $(\boldsymbol{v}, \pi)$ satisfies the additional regularity $\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) \in W^{1,2}(\Omega)^{d \times d}$ and $\pi \in W^{k+1,p'}(\Omega)$ with $k \in \{0, 1\}$ and $1/p + 1/p' = 1$. Let the stabilization parameters ϱ_M, α_M be chosen as follows:*

$$\varrho_M := \varrho_0 \frac{h_M}{\|\boldsymbol{b}\|_{1,\infty;M}}, \quad \alpha_M := \alpha_0 h_M^2 \quad \textit{if} \quad k = 0, \quad \textit{and} \quad \alpha_M := \alpha_0 h_M^{2/p'} \quad \textit{if} \quad k = 1.$$

Then, the error of approximation can be estimated in terms of the maximum mesh size $h := \max\{h_M;\, M \in \mathbb{M}_h\}$ as follows: There exist constants $C_{\boldsymbol{v}}, C'_{\boldsymbol{v}} > 0$ such that

$$\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2 \leq C_{\boldsymbol{v}} h, \qquad \|\boldsymbol{v} - \boldsymbol{v}_h\|_{1,p} \leq C'_{\boldsymbol{v}} h. \tag{5.8}$$

Moreover, if $p \geq \frac{2d}{d+1}$, then there exists a constant $C_\pi > 0$ such that

$$\|\pi - \pi_h\|_{p'} \leq C_\pi h^{\frac{2}{p'}}. \tag{5.9}$$

The constants $C_{\boldsymbol{v}}, C'_{\boldsymbol{v}}, C_\pi > 0$ only depend on $\|\nabla \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})\|_2$, $\|\pi\|_{k+1,p'}$, p, ε_0, σ_0, σ_1, σ, Ω, $\boldsymbol{f}$, ϱ_0, α_0, τ_0, and C_π additionally depends on the constant $\tilde{\beta}(p)$ appearing in (4.30).

Assume additionally that $\mathbb{M}_h$ is quasi-uniform. Then, there exists $C'_\pi > 0$ such that

$$\|\pi - \pi_h\|_{p'} \le C'_\pi h^{\min\{2-\frac{2d}{p}+d, \frac{2}{p'}\}}. \tag{5.10}$$

The constant $C'_\pi > 0$ depends on the same quantities as C_π.

Remark 5.2. If $\mathbb{M}_h$ is quasi-uniform, then (5.10) improves (5.9) concerning the admissible range of p. In particular, it holds $\min\{2 - \frac{2d}{p} + d, \frac{2}{p'}\} = \frac{2}{p'}$ provided that $p \ge \frac{2d-2}{d}$. Hence, if $d = 2$, (5.10) yields an $\mathcal{O}(h^{2/p'})$ bound for the pressure error in $L^{p'}(\Omega)$ provided that $p > 1$, whereas (5.9) ensures the same convergence rate for $p \ge \frac{4}{3}$ only. If $d = 3$, (5.10) predicts the convergence rate $\frac{2}{p'}$ for $p \ge \frac{4}{3}$.

Proof of Theorem 5.7. Let j_h and $\boldsymbol{j}_h$ be interpolation operators as in Lemma 4.4. It is sufficient to estimate the projection error $\boldsymbol{\xi}_h := (\boldsymbol{j}_h\boldsymbol{v} - \boldsymbol{v}_h)$ and $\eta_h := (j_h\pi - \pi_h)$ with respect to the distance $\|(\boldsymbol{\xi}_h, \eta_h)\|_{\mathrm{lps}}$ defined by

$$\|(\boldsymbol{\xi}_h, \eta_h)\|^2_{\mathrm{lps}} := \|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2^2 + \sigma\|\boldsymbol{\xi}_h\|_2^2 + |(\boldsymbol{\xi}_h, \eta_h)|^2_{\mathrm{lps}}. \tag{5.11}$$

Using Lemma 2.5 and Lemma 5.1, we observe the equivalence

$$\begin{aligned}\|(\boldsymbol{\xi}_h, \eta_h)\|^2_{\mathrm{lps}} \sim (\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{v}) - \boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}_h), \boldsymbol{D}\boldsymbol{\xi}_h)_\Omega + \sigma\|\boldsymbol{\xi}_h\|_2^2 \\ + S_h\Big((\boldsymbol{j}_h\boldsymbol{v}, j_h\pi)\Big)\Big((\boldsymbol{\xi}_h, \eta_h)\Big) - S_h\Big((\boldsymbol{v}_h, \pi_h)\Big)\Big((\boldsymbol{\xi}_h, \eta_h)\Big).\end{aligned}$$

Adding the following trivial identities to the right-hand side,

$$\begin{aligned}0 &= \Big((\boldsymbol{b}\cdot\nabla)\boldsymbol{\xi}_h, \boldsymbol{\xi}_h\Big)_\Omega = \Big((\boldsymbol{b}\cdot\nabla)(\boldsymbol{v} - \boldsymbol{v}_h), \boldsymbol{\xi}_h\Big)_\Omega + \Big((\boldsymbol{b}\cdot\nabla)(\boldsymbol{j}_h\boldsymbol{v} - \boldsymbol{v}), \boldsymbol{\xi}_h\Big)_\Omega, \\ 0 &= -(\pi - \pi_h, \nabla\cdot\boldsymbol{\xi}_h)_\Omega + (\nabla\cdot(\boldsymbol{v} - \boldsymbol{v}_h), \eta_h)_\Omega - (j_h\pi - \pi, \nabla\cdot\boldsymbol{\xi}_h)_\Omega + (\nabla\cdot(\boldsymbol{j}_h\boldsymbol{v} - \boldsymbol{v}), \eta_h)_\Omega,\end{aligned}$$

and using the disturbed Galerkin orthogonality, which reads

$$A(\boldsymbol{u})(\boldsymbol{\omega}_h) - A(\boldsymbol{u}_h)(\boldsymbol{\omega}_h) = S_h(\boldsymbol{u}_h)(\boldsymbol{\omega}_h) \qquad \forall \boldsymbol{\omega}_h \in \boldsymbol{\mathcal{X}}_h^p \times \mathcal{Q}_h^p$$

for $\boldsymbol{u} \equiv (\boldsymbol{v}, \pi)$ and $\boldsymbol{u}_h \equiv (\boldsymbol{v}_h, \pi_h)$, we arrive at

$$\begin{aligned}\|(\boldsymbol{\xi}_h, \eta_h)\|^2_{\mathrm{lps}} &\sim (\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{v}) - \boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{\xi}_h)_\Omega + \sigma(\boldsymbol{j}_h\boldsymbol{v} - \boldsymbol{v}, \boldsymbol{\xi}_h) + \Big((\boldsymbol{b}\cdot\nabla)(\boldsymbol{j}_h\boldsymbol{v} - \boldsymbol{v}), \boldsymbol{\xi}_h\Big)_\Omega \\ &\quad - (j_h\pi - \pi, \nabla\cdot\boldsymbol{\xi}_h)_\Omega + (\nabla\cdot(\boldsymbol{j}_h\boldsymbol{v} - \boldsymbol{v}), \eta_h)_\Omega + S_h\Big((\boldsymbol{j}_h\boldsymbol{v}, j_h\pi)\Big)\Big((\boldsymbol{\xi}_h, \eta_h)\Big) \\ &=: I_1 + I_2 + I_3 + I_4 + I_5 + I_6.\end{aligned} \tag{5.12}$$

We estimate the terms $I_1, \dots, I_6$ in (5.12) separately. Similarly to the proof of Theorem 4.11, for arbitrary $\delta_1 > 0$ the term I_1 can be bounded by

$$\begin{aligned}I_1 &\le c_{\delta_1}\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})\|_2^2 + \delta_1\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2^2 \\ &\le c_{\delta_1} h^2\|\nabla\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})\|_2^2 + \delta_1\|(\boldsymbol{\xi}_h, \eta_h)\|_{\mathrm{lps}}.\end{aligned} \tag{5.13}$$

Lemma 4.10 implies that $\boldsymbol{v} \in \boldsymbol{W}^{2,q}(\Omega)$ with $q := 2 - \delta$ for arbitrary $\delta \in (0,1]$ in the case $d = 2$ and $q := \frac{3p}{p+1}$ in the case $d = 3$. Hence, by means of Young's inequality and the approximation property of $\boldsymbol{j}_h$, for arbitrary $\delta_2 > 0$ the term I_2 can be estimated by

$$I_2 \le c_{\delta_2}\sigma\|\boldsymbol{j}_h\boldsymbol{v} - \boldsymbol{v}\|_2^2 + \delta_2\sigma\|\boldsymbol{\xi}_h\|_2^2 \le c_{\delta_2}\sigma h^{4-\frac{2d}{q}+d}\|\boldsymbol{v}\|_{2,q}^2 + \delta_2\sigma\|(\boldsymbol{\xi}_h, \eta_h)\|_{\text{lps}}^2. \tag{5.14}$$

Using integration by parts ($\nabla \cdot \boldsymbol{b} = 0$), and the interpolation properties of $\boldsymbol{j}_h$, we conclude that for arbitrary $\delta_3 > 0$ there exists a constant $c_{\delta_3} > 0$ such that

$$\begin{aligned}
I_3 &= \Big((\boldsymbol{b}\cdot\nabla)(\boldsymbol{j}_h\boldsymbol{v} - \boldsymbol{v}), \boldsymbol{\xi}_h\Big)_\Omega = -\Big(\boldsymbol{j}_h\boldsymbol{v} - \boldsymbol{v}, \boldsymbol{\theta}_h(\boldsymbol{b}\cdot\nabla)\boldsymbol{\xi}_h\Big)_\Omega \\
&\le c_{\delta_3}\sum_{M\in\mathbb{M}_h}\varrho_M^{-1}\|\boldsymbol{j}_h - \boldsymbol{v}\|_{2;M}^2 + \delta_3\sum_{M\in\mathbb{M}_h}\varrho_M\|\boldsymbol{\theta}_h(\boldsymbol{b}\cdot\nabla)\boldsymbol{\xi}_h\|_{2;M}^2 \\
&\le c_{\delta_3}\sum_{M\in\mathbb{M}_h}\varrho_M^{-1}h_M^{4-\frac{2d}{q}+d}\|\boldsymbol{v}\|_{2,q;S_M}^2 + \delta_3\sum_{M\in\mathbb{M}_h}\varrho_M\|\boldsymbol{\theta}_h(\boldsymbol{b}\cdot\nabla)\boldsymbol{\xi}_h\|_{2;M}^2 \\
&\le c_{\delta_3}\Big(\sum_{M\in\mathbb{M}_h}\Big[\varrho_M^{-1}h_M^{4-\frac{2d}{q}+d}\Big]^{\frac{q}{2}}\|\boldsymbol{v}\|_{2,q;S_M}^q\Big)^{\frac{2}{q}} + \delta_3\|(\boldsymbol{\xi}_h, \eta_h)\|_{\text{lps}}^2.
\end{aligned} \tag{5.15}$$

Applying Lemma 2.6, recalling the $\boldsymbol{W}^{1,p}$-stability of $\boldsymbol{j}_h$, using the uniform a priori $\boldsymbol{W}^{1,p}$-bounds for $\boldsymbol{v}_h$, $\boldsymbol{v}$ as (5.7), we deduce that for each $\delta_4 > 0$ there exists $c_{\delta_4} > 0$ so that

$$\begin{aligned}
I_4 &\le \|\pi - j_h\pi\|_{p'}\|\varepsilon + |\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{v}| + |\boldsymbol{D}\boldsymbol{v}_h|\|_p^{\frac{2-p}{2}}\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2 \\
&\le c_{\delta_4}\|\varepsilon + |\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{v}| + |\boldsymbol{D}\boldsymbol{v}_h|\|_p^{2-p}\|\pi - j_h\pi\|_{p'}^2 + \delta_4\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2^2 \\
&\le c_{\delta_4}h^{2k+2}\|\pi\|_{k+1,p'}^2 + \delta_4\|(\boldsymbol{\xi}_h, \eta_h)\|_{\text{lps}}^2.
\end{aligned} \tag{5.16}$$

The constant c_{δ_4} only depends on $\Omega, p, \varepsilon_0, \sigma_0, \sigma_1, \boldsymbol{f}, \delta_4$. The term I_5 can be estimated similarly to (4.51). Using integration by parts (the discrete pressure is continuous), the orthogonality of $\boldsymbol{j}_h$ with respect to $\boldsymbol{Y}_h$, Hölder's and Young's inequality, Lemma 4.1 with $p' \ge 2$, and the approximation property of $\boldsymbol{j}_h$, for arbitrary $\delta_5 > 0$ we estimate

$$\begin{aligned}
I_5 &\le \Big|(\nabla\cdot(\boldsymbol{j}_h\boldsymbol{v} - \boldsymbol{v}), \eta_h)_\Omega\Big| = \Big|(\boldsymbol{j}_h\boldsymbol{v} - \boldsymbol{v}, \nabla\eta_h)_\Omega\Big| = \Big|(\boldsymbol{j}_h\boldsymbol{v} - \boldsymbol{v}, \boldsymbol{\theta}_h(\nabla\eta_h))_\Omega\Big| \\
&\le c_{\delta_5}\sum_{M\in\mathbb{M}_h}\alpha_M^{-(p-1)}\|\boldsymbol{j}_h\boldsymbol{v} - \boldsymbol{v}\|_{p;M}^p + \delta_5\sum_{M\in\mathbb{M}_h}\alpha_M\|\boldsymbol{\theta}_h\nabla j_h\pi - \boldsymbol{\theta}_h\nabla\pi_h\|_{p';M}^{p'} \\
&\le c_{\delta_5}\sum_{M\in\mathbb{M}_h}\alpha_M^{1-p}h_M^{2p}\|\boldsymbol{v}\|_{2,p;S_M}^p + \delta_5 c\|(\boldsymbol{\xi}_h, \eta_h)\|_{\text{lps}}^2,
\end{aligned} \tag{5.17}$$

where the constant c_{δ_5} only depends on p, α_0, δ_5. Finally, the term I_6 can be estimated by means of Lemmas 5.3, 5.4, 5.2. We need to check that the assumptions of Lemma 5.4 are satisfied: If $d = 2$, the condition $q \ge \frac{2d}{d+1}$ is clearly satisfied. If $d = 3$, the requirement

$\frac{3p}{p+1} \geq \frac{3}{2}$ is equivalent to $p \geq 1$. Hence, for each $\delta_6 > 0$ there exists $c_{\delta_6} > 0$ such that

$$\begin{aligned} I_6 &\equiv S_h\Big((\boldsymbol{v},\pi)\Big)\Big((\boldsymbol{\xi}_h,\eta_h)\Big) + \Big\{S_h\Big((\boldsymbol{j}_h\boldsymbol{v}, j_h\pi)\Big)\Big((\boldsymbol{\xi}_h,\eta_h)\Big) - S_h\Big((\boldsymbol{v},\pi)\Big)\Big((\boldsymbol{\xi}_h,\eta_h)\Big)\Big\} \\ &\leq S_h\Big((\boldsymbol{v},\pi)\Big)\Big((\boldsymbol{\xi}_h,\eta_h)\Big) + c_{\delta_6}|(\boldsymbol{j}_h\boldsymbol{v}-\boldsymbol{v}, j_h\pi-\pi)|^2_{\mathrm{lps}} + \delta_6\|(\boldsymbol{\xi}_h,\eta_h)\|^2_{\mathrm{lps}} \\ &\leq c_{\delta_6}\|\boldsymbol{v}\|_{2,q;\Omega}\Big(\sum_{M\in\mathbb{M}_h}\Big[\varrho_M h_M\|\boldsymbol{b}\|^2_{1,\infty;M}\Big]^q\|\boldsymbol{v}\|^q_{2,q;M}\Big)^{\frac{1}{q}} \\ &\quad + c_{\delta_6}\|\tau+|\nabla\pi|\|^{p'-2}_{p'}\Big(\sum_{M\in\mathbb{M}_h}\Big[\alpha_M h_M^{2k}\Big]^{\frac{p'}{2}}\|\nabla^{k+1}\pi\|^{p'}_{p';M}\Big)^{\frac{2}{p'}} \\ &\quad + c_{\delta_6}\|\tau+|\nabla j_h\pi|+|\nabla\pi|\|^{p'-2}_{p'}\Big(\sum_{M\in\mathbb{M}_h}\alpha_M^{\frac{p'}{2}}\|\nabla(j_h\pi-\pi)\|^{p'}_{p';M}\Big)^{\frac{2}{p'}} \\ &\quad + c_{\delta_6}\sum_{M\in\mathbb{M}_h}\varrho_M\|(\boldsymbol{b}\cdot\nabla)(\boldsymbol{j}_h\boldsymbol{v}-\boldsymbol{v})\|^2_{2;M} + \delta_6 2\|(\boldsymbol{\xi}_h,\eta_h)\|^2_{\mathrm{lps}}. \end{aligned}$$

Using the properties of $\boldsymbol{j}_h$ and j_h, we can hence estimate the term I_6 as follows:

$$\begin{aligned} I_6 &\leq c_{\delta_6}\|\boldsymbol{v}\|_{2,q;\Omega}\Big(\sum_{M\in\mathbb{M}_h}\Big[\varrho_M h_M\|\boldsymbol{b}\|^2_{1,\infty;M}\Big]^q\|\boldsymbol{v}\|^q_{2,q;M}\Big)^{\frac{1}{q}} \\ &\quad + c_{\delta_6}\Big[\tau|\Omega|^{\frac{1}{p'}}+\|\pi\|_{1,p'}\Big]^{p'-2}\Big(\sum_{M\in\mathbb{M}_h}\Big[\alpha_M h_M^{2k}\Big]^{\frac{p'}{2}}\|\pi\|^{p'}_{k+1,p';S_M}\Big)^{\frac{2}{p'}} \\ &\quad + c_{\delta_6}\Big(\sum_{M\in\mathbb{M}_h}\Big[\varrho_M\|\boldsymbol{b}\|^2_{\infty;M}h_M^{2-\frac{2d}{q}+d}\Big]^{\frac{q}{2}}\|\boldsymbol{v}\|^q_{2,q;S_M}\Big)^{\frac{2}{q}} + \delta_6 2\|(\boldsymbol{\xi}_h,\eta_h)\|^2_{\mathrm{lps}}. \end{aligned} \tag{5.18}$$

Combining (5.12) – (5.18), choosing $\delta_1,\ldots,\delta_6$ sufficiently small, we easily arrive at

$$\begin{aligned} \|(\boldsymbol{\xi}_h,\eta_h)\|^2_{\mathrm{lps}} &\lesssim h^2\|\nabla\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})\|^2_2 + \sigma h^{4-\frac{2d}{q}+d}\|\boldsymbol{v}\|^2_{2,q} + \Big(\sum_{M\in\mathbb{M}_h}\Big[\varrho_M^{-1}h_M^{4-\frac{2d}{q}+d}\Big]^{\frac{q}{2}}\|\boldsymbol{v}\|^q_{2,q;S_M}\Big)^{\frac{2}{q}} \\ &\quad + h^{2k+2}\|\pi\|^2_{k+1,p'} + \sum_{M\in\mathbb{M}_h}\alpha_M^{1-p}h_M^{2p}\|\boldsymbol{v}\|^p_{2,p;S_M} \\ &\quad + \|\boldsymbol{v}\|_{2,q}\Big(\sum_{M\in\mathbb{M}_h}\Big[\varrho_M h_M\|\boldsymbol{b}\|^2_{1,\infty;M}\Big]^q\|\boldsymbol{v}\|^q_{2,q;M}\Big)^{\frac{1}{q}} \\ &\quad + \Big[\tau|\Omega|^{\frac{1}{p'}}+\|\pi\|_{1,p'}\Big]^{p'-2}\Big(\sum_{M\in\mathbb{M}_h}\Big[\alpha_M h_M^{2k}\Big]^{\frac{p'}{2}}\|\pi\|^{p'}_{k+1,p';S_M}\Big)^{\frac{2}{p'}} \\ &\quad + \Big(\sum_{M\in\mathbb{M}_h}\Big[\varrho_M\|\boldsymbol{b}\|^2_{\infty;M}h_M^{2-\frac{2d}{q}+d}\Big]^{\frac{q}{2}}\|\boldsymbol{v}\|^q_{2,q;S_M}\Big)^{\frac{2}{q}}. \end{aligned} \tag{5.19}$$

We equilibrate the terms in (5.19) involving ϱ_M and α_M through

$$\varrho_M \sim \frac{h_M}{\|\boldsymbol{b}\|_{1,\infty;M}}, \quad \alpha_M \sim h_M^2 \quad \text{if} \quad k=0, \quad \alpha_M \sim h_M^{\frac{2}{p'}} \quad \text{if} \quad k=1. \tag{5.20}$$

As a result, we easily derive the first inequality in (5.8) combining (5.19) with (5.20) and noting that $1-\frac{2d}{q}+d\geq 0$ for q as defined above. In fact, the condition $1-\frac{2d}{q}+d\geq 0$ amounts to $q\geq\frac{2d}{d+1}$ which is satisfied for q as defined above. Hence, we obtain

$$\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})-\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2\leq\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})-\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{v})\|_2+\|(\boldsymbol{\xi}_h,\eta_h)\|_{\text{lps}}\leq C_{\boldsymbol{v}}h. \tag{5.21}$$

The second inequality in (5.8) follows from (5.21) and the estimate

$$\|\boldsymbol{v}-\boldsymbol{v}_h\|_{1,p}\lesssim\|\boldsymbol{D}\boldsymbol{v}-\boldsymbol{D}\boldsymbol{v}_h\|_p\lesssim\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})-\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2, \tag{5.22}$$

which is a simple consequence of the Poincaré & Korn inequality, Lemma 2.6 (i), and the uniform $W^{1,p}$-bounds on $\boldsymbol{v}$ and $\boldsymbol{v}_h$. Next we prove the pressure-estimate (5.9). For this, it is sufficient to estimate the projection error η_h. From **(P6)** and **(P6$_h$)** we conclude that

$$\begin{aligned}(j_h\pi-\pi_h,\nabla\cdot\boldsymbol{w}_h)_\Omega&=(\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v})-\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}_h),\boldsymbol{D}\boldsymbol{w}_h)_\Omega+(j_h\pi-\pi,\nabla\cdot\boldsymbol{w}_h)_\Omega\\&\quad+\Big((\boldsymbol{b}\cdot\nabla)(\boldsymbol{v}-\boldsymbol{v}_h),\boldsymbol{w}_h\Big)_\Omega+\sigma(\boldsymbol{v}-\boldsymbol{v}_h,\boldsymbol{w}_h)_\Omega-S_h\Big((\boldsymbol{v}_h,0)\Big)\Big((\boldsymbol{w}_h,0)\Big)\end{aligned}$$

for all $\boldsymbol{w}_h\in\boldsymbol{\mathcal{X}}_h^p$. Hence, by means of Lemma 4.5 we deduce that η_h is bounded by

$$\begin{aligned}\tilde\beta\|\eta_h\|_{p'}&\leq\sup_{\boldsymbol{w}_h\in\boldsymbol{\mathcal{X}}_h^p}\frac{(\nabla\cdot\boldsymbol{w}_h,\eta_h)_\Omega}{\|\nabla\boldsymbol{w}_h\|_p}+\Big(\sum_{M\in\mathbb{M}_h}h_M^{p'}\|\boldsymbol{\theta}_h(\nabla\eta_h)\|_{p';M}^{p'}\Big)^{\frac{1}{p'}}\\&\leq\sup_{\boldsymbol{w}_h\in\boldsymbol{\mathcal{X}}_h^p}\frac{\Big|(\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v})-\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}_h),\boldsymbol{D}\boldsymbol{w}_h)_\Omega+(j_h\pi-\pi,\nabla\cdot\boldsymbol{w}_h)_\Omega\Big|}{\|\nabla\boldsymbol{w}_h\|_p}\\&\quad+\sup_{\boldsymbol{w}_h\in\boldsymbol{\mathcal{X}}_h^p}\frac{\Big|\Big((\boldsymbol{b}\cdot\nabla)(\boldsymbol{v}-\boldsymbol{v}_h),\boldsymbol{w}_h\Big)_\Omega+\sigma(\boldsymbol{v}-\boldsymbol{v}_h,\boldsymbol{w}_h)_\Omega\Big|}{\|\nabla\boldsymbol{w}_h\|_p}\\&\quad+\sup_{\boldsymbol{w}_h\in\boldsymbol{\mathcal{X}}_h^p}\frac{\Big|\sum_{M\in\mathbb{M}_h}\varrho_M(\boldsymbol{\theta}_h(\boldsymbol{b}\cdot\nabla)[\boldsymbol{v}-\boldsymbol{v}_h],\boldsymbol{\theta}_h(\boldsymbol{b}\cdot\nabla)\boldsymbol{w}_h)_M\Big|}{\|\nabla\boldsymbol{w}_h\|_p}\\&\quad+\sup_{\boldsymbol{w}_h\in\boldsymbol{\mathcal{X}}_h^p}\frac{\Big|\sum_{M\in\mathbb{M}_h}\varrho_M(\boldsymbol{\theta}_h(\boldsymbol{b}\cdot\nabla)\boldsymbol{v},\boldsymbol{\theta}_h(\boldsymbol{b}\cdot\nabla)\boldsymbol{w}_h)_M\Big|}{\|\nabla\boldsymbol{w}_h\|_p}\\&\quad+\Big(\sum_{M\in\mathbb{M}_h}h_M^{p'}\|\boldsymbol{\theta}_h(\nabla\eta_h)\|_{p';M}^{p'}\Big)^{\frac{1}{p'}}=:J_1+J_2+J_3+J_4+J_5.\end{aligned}\tag{5.23}$$

Using Hölder's inequality, Lemma 2.7 (i), and the properties of j_h, $\boldsymbol{j}_h$, we estimate J_1 by

$$J_1\lesssim\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})-\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2^{\frac{2}{p'}}+h\|\pi\|_{1,p'}. \tag{5.24}$$

For $2\geq p\geq\frac{2d}{d+1}$ there holds the continuous embedding $W^{1,p}(\Omega)\hookrightarrow L^{p'}(\Omega)\hookrightarrow L^2(\Omega)$. Therefore, by means of (5.22) we conclude that the term J_2 is bounded by

$$\begin{aligned}J_2&\leq\sup_{\boldsymbol{w}_h\in\boldsymbol{\mathcal{X}}_h^p}\frac{1}{\|\nabla\boldsymbol{w}_h\|_p}\Big\{\|\boldsymbol{b}\|_\infty\|\nabla(\boldsymbol{v}-\boldsymbol{v}_h)\|_p\|\boldsymbol{w}_h\|_{p'}+\sigma\|\boldsymbol{v}-\boldsymbol{v}_h\|_2\|\boldsymbol{w}_h\|_2\Big\}\\&\lesssim(\|\boldsymbol{b}\|_\infty+\sigma)\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})-\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2.\end{aligned}\tag{5.25}$$

Before we estimate J_3, we firstly derive an upper bound for $|(\boldsymbol{w}_h, 0)|_{\mathrm{lps}}$. Noting the fact $p \le 2$, recalling the stability of $\boldsymbol{\theta}_h$ and the local inverse inequality (3.19), we observe that

$$|(\boldsymbol{w}_h, 0)|_{\mathrm{lps}} \equiv \Big(\sum_{M \in \mathbb{M}_h} \varrho_M \|\boldsymbol{\theta}_h(\boldsymbol{b}\cdot\nabla)\boldsymbol{w}_h\|_{2;M}^2 \Big)^{\frac{1}{2}}$$
$$\lesssim \Big(\sum_{M \in \mathbb{M}_h} \varrho_M \|\boldsymbol{b}\|_{\infty;M}^2 h_M^{-\frac{2d}{p}+d} \|\boldsymbol{w}_h\|_{1,p;M}^2 \Big)^{\frac{p}{2}\frac{1}{p}} \lesssim h^{\frac{1}{2}-\frac{d}{p}+\frac{d}{2}} \|\boldsymbol{b}\|_\infty^{1/2} \|\boldsymbol{w}_h\|_{1,p} \qquad (5.26)$$

provided that $p \ge \frac{2d}{d+1}$ and $\varrho_M \sim h_M / \|\boldsymbol{b}\|_{1,\infty;M}$. Taking into account (5.26), we conclude

$$J_3 \le \sup_{\boldsymbol{w}_h \in \boldsymbol{\mathcal{X}}_h^p} \frac{\Big| \sum_{M \in \mathbb{M}_h} \varrho_M (\boldsymbol{\theta}_h(\boldsymbol{b}\cdot\nabla)\boldsymbol{\xi}_h, \boldsymbol{\theta}_h(\boldsymbol{b}\cdot\nabla)\boldsymbol{w}_h)_M \Big|}{\|\nabla \boldsymbol{w}_h\|_p}$$
$$+ \sup_{\boldsymbol{w}_h \in \boldsymbol{\mathcal{X}}_h^p} \frac{\Big| \sum_{M \in \mathbb{M}_h} \varrho_M (\boldsymbol{\theta}_h(\boldsymbol{b}\cdot\nabla)[\boldsymbol{v} - \boldsymbol{j}_h \boldsymbol{v}], \boldsymbol{\theta}_h(\boldsymbol{b}\cdot\nabla)\boldsymbol{w}_h)_M \Big|}{\|\nabla \boldsymbol{w}_h\|_p}$$
$$\le \sup_{\boldsymbol{w}_h \in \boldsymbol{\mathcal{X}}_h^p} \frac{|(\boldsymbol{\xi}_h, 0)|_{\mathrm{lps}} |(\boldsymbol{w}_h, 0)|_{\mathrm{lps}}}{\|\nabla \boldsymbol{w}_h\|_p} + \sup_{\boldsymbol{w}_h \in \boldsymbol{\mathcal{X}}_h^p} \frac{|(\boldsymbol{v} - \boldsymbol{j}_h \boldsymbol{v}, 0)|_{\mathrm{lps}} |(\boldsymbol{w}_h, 0)|_{\mathrm{lps}}}{\|\nabla \boldsymbol{w}_h\|_p}$$
$$\le c h^{\frac{1}{2}-\frac{d}{p}+\frac{d}{2}} \|\boldsymbol{b}\|_\infty^{1/2} \Bigg[|(\boldsymbol{\xi}_h, 0)|_{\mathrm{lps}} + \Bigg(\sum_{M \in \mathbb{M}_h} \varrho_M \|\boldsymbol{b}\|_{\infty;M}^2 \|\nabla(\boldsymbol{v} - \boldsymbol{j}_h \boldsymbol{v})\|_{2;M}^2 \Bigg)^{\frac{1}{2}} \Bigg].$$

Since $\boldsymbol{v} \in \boldsymbol{W}^{2,\frac{3p}{p+1}}(\Omega)$ and $\varrho_M \sim h_M / \|\boldsymbol{b}\|_{1,\infty;M}$, we arrive at

$$J_3 \le c h^{\frac{1}{2}-\frac{d}{p}+\frac{d}{2}} \|\boldsymbol{b}\|_\infty^{1/2} \Bigg[|(\boldsymbol{\xi}_h, 0)|_{\mathrm{lps}} + \Bigg(\sum_{M \in \mathbb{M}_h} h_M \|\boldsymbol{b}\|_{\infty;M} h_M^{2-\frac{2d(p+1)}{3p}+d} \|\boldsymbol{v}\|_{2,\frac{3p}{p+1};S_M}^2 \Bigg)^{\frac{1}{2}} \Bigg]$$
$$\le c h^{\frac{1}{2}-\frac{d}{p}+\frac{d}{2}} \|\boldsymbol{b}\|_\infty^{1/2} \Bigg[\|(\boldsymbol{\xi}_h, \eta_h)\|_{\mathrm{lps}} + h^{\frac{3}{2}-\frac{d(p+1)}{3p}+\frac{d}{2}} \|\boldsymbol{b}\|_\infty^{1/2} \|\boldsymbol{v}\|_{2,\frac{3p}{p+1}} \Bigg]. \qquad (5.27)$$

Similarly we may estimate the term J_4. Using Hölder's inequality with $\frac{p+1}{3p} + \frac{2p-1}{3p} = 1$, the approximation property of $\boldsymbol{\theta}_h$, and the local inverse inequality (3.19), we deduce that

$$J_4 \le \sup_{\boldsymbol{w}_h \in \boldsymbol{\mathcal{X}}_h^p} \frac{1}{\|\nabla \boldsymbol{w}_h\|_p} \sum_{M \in \mathbb{M}_h} \varrho_M \|\boldsymbol{\theta}_h(\boldsymbol{b}\cdot\nabla)\boldsymbol{v}\|_{\frac{3p}{p+1};M} \|\boldsymbol{\theta}_h(\boldsymbol{b}\cdot\nabla)\boldsymbol{w}_h\|_{\frac{3p}{2p-1};M}$$
$$\le \sup_{\boldsymbol{w}_h \in \boldsymbol{\mathcal{X}}_h^p} \frac{1}{\|\nabla \boldsymbol{w}_h\|_p} \sum_{M \in \mathbb{M}_h} \varrho_M h_M \|\boldsymbol{b}\|_{1,\infty;M}^2 \|\boldsymbol{v}\|_{2,\frac{3p}{p+1};M} h_M^{-\frac{d}{p}+\frac{d(2p-1)}{3p}} \|\boldsymbol{w}_h\|_{1,p;M}$$
$$\le c h^{\frac{1}{2}-\frac{d}{p}+\frac{d}{2}} \|\boldsymbol{b}\|_{1,\infty} h^{\frac{3}{2}-\frac{d(p+1)}{3p}+\frac{d}{2}} \|\boldsymbol{v}\|_{2,\frac{3p}{p+1}}. \qquad (5.28)$$

We observe that $\frac{3}{2} - \frac{d(p+1)}{3p} + \frac{d}{2} \ge 1$ for $p \ge 1$. Hence, the convergence order of the stabilization part $J_3 + J_4$ is restricted to $\frac{3}{2} - \frac{d}{p} + \frac{d}{2}$ by virtue of (5.27) and $\|(\boldsymbol{\xi}_h, \eta_h)\|_{\mathrm{lps}} = \mathcal{O}(h)$.

We remark that $\frac{3}{2} - \frac{d}{p} + \frac{d}{2} \geq \frac{2}{p'}$ for $p \geq 1$. Finally, Lemma 4.1 implies that

$$\begin{aligned} J_5 &\lesssim h^{1-\frac{2}{p'}} \Big(\sum_{M \in \mathbb{M}_h} \alpha_M \|\boldsymbol{\theta}_h \nabla j_h \pi - \boldsymbol{\theta}_h \nabla \pi_h\|_{p';M}^{p'} \Big)^{\frac{1}{p'}} \\ &\lesssim h^{1-\frac{2}{p'}} \Big(\sum_{M \in \mathbb{M}_h} \alpha_M \|\boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h \nabla j_h \pi) - \boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h \nabla \pi_h)\|_{2;M}^2 \Big)^{\frac{1}{p'}} \lesssim h^{1-\frac{2}{p'}} \|(\boldsymbol{\xi}_h, \eta_h)\|_{\text{lps}}^{\frac{2}{p'}}. \end{aligned} \tag{5.29}$$

Combining (5.23), (5.24), (5.25), (5.27)–(5.29), in view of (5.21) we get the desired estimate (5.9). It remains to prove (5.10). If the triangulation $\mathbb{M}_h$ is quasi-uniform, then we can make use of the global inverse inequality (3.20) in order to better estimate the term J_2. Since the case $p \geq \frac{2d}{d+1}$ is already treated above, for the remainder of the proof we may suppose that $p \leq \frac{2d}{d+1}$. We recall the embedding $W^{1,p}(\Omega) \hookrightarrow L^{q^*}(\Omega)$ with $q^* = \frac{dp}{d-p}$. Since $q^* \leq p'$ for $p \leq \frac{2d}{d+1}$, the inverse estimate (3.20) yields

$$\|\boldsymbol{w}_h\|_{p'} \lesssim h^{-\frac{d}{q^*}+\frac{d}{p'}} \|\boldsymbol{w}_h\|_{q^*} \lesssim h^{-\frac{d}{q^*}+\frac{d}{p'}} \|\boldsymbol{w}_h\|_{1,p}.$$

Using the latter estimate, Poincaré's inequality, and (5.8), we obtain the upper bound

$$J_2 \lesssim h^{-\frac{d}{q^*}+\frac{d}{p'}} \Big(\|\boldsymbol{b}\|_\infty \|\nabla(\boldsymbol{v} - \boldsymbol{v}_h)\|_p + \sigma \|\boldsymbol{v} - \boldsymbol{v}_h\|_p \Big) \lesssim (\|\boldsymbol{b}\|_\infty + \sigma) h^{1-\frac{d}{q^*}+\frac{d}{p'}}. \tag{5.30}$$

We easily compute that $1 - \frac{d}{q^*} + \frac{d}{p'} = 2 - \frac{2d}{p} + d$. The remaining terms are estimated just as above. Besides, we remark that the exponent $\frac{1}{2} - \frac{d}{p} + \frac{d}{2}$ in (5.26) becomes negative for $p < \frac{2d}{d+1}$. Combining (5.23), (5.24), (5.30), (5.27)–(5.29), and taking into account (5.21), we finally obtain the desired estimate (5.10). This completes the proof. □

Remark 5.3. We briefly discuss in which sense in the case $p = 2$ we recover the well-known results for Oseen systems. Note that the LPS method was proposed and studied for Oseen systems in Matthies et al. [MST07]. In view of Lemma 5.5, in (5.19) the term

$$\|\boldsymbol{v}\|_{2,q} \Big(\sum_{M \in \mathbb{M}_h} \Big[\varrho_M h_M \|\boldsymbol{b}\|_{1,\infty;M}^2 \Big]^q \|\boldsymbol{v}\|_{2,q;M}^q \Big)^{\frac{1}{q}}$$

can be replaced by the following one

$$\Big(\sum_{M \in \mathbb{M}_h} \Big[\varrho_M h_M^{2-\frac{2d}{q}+d} \Big]^{\frac{q}{2}} \|(\boldsymbol{b} \cdot \nabla) \boldsymbol{v}\|_{1,q;M}^q \Big)^{\frac{2}{q}}.$$

In doing so, we observe that for $p = q = 2$ the a priori error estimate (5.19) coincides with the well-known error estimate for Oseen systems presented in [MST07].

Remark 5.4. Usually the function $\boldsymbol{b}$ is given as a finite element solution to system (2.17), cf. [BL09]. Consequently $\boldsymbol{b}$ satisfies $(\nabla \cdot \boldsymbol{b}, q_h)_\Omega = 0$ for all $q_h \in \mathcal{Q}_h^p$ but it does not fulfill $\nabla \cdot \boldsymbol{b} = 0$ pointwise as required by Theorem 5.7. If we recall the proof of Theorem 5.7, we realize that the assumption $\nabla \cdot \boldsymbol{b} = 0$ a.e. is only needed for the estimation of the term

I_3. We can relax the assumption $\nabla \cdot \boldsymbol{b} = 0$ a.e. if, in (5.5), we replace the convective term $([\boldsymbol{b} \cdot \nabla]\boldsymbol{v}_h, \boldsymbol{w}_h)_\Omega$ by the skew-symmetric tri-linear form $B(\boldsymbol{b}, \boldsymbol{v}_h, \boldsymbol{w}_h)$ defined by

$$B(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) := \frac{1}{2}\Big(([\boldsymbol{u} \cdot \nabla]\boldsymbol{v}, \boldsymbol{w})_\Omega - ([\boldsymbol{u} \cdot \nabla]\boldsymbol{w}, \boldsymbol{v})_\Omega\Big) \qquad \forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{\mathcal{X}}^p, \tag{5.31}$$

cf. [BL09]. The tri-linear form $B(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$ naturally extends the term $([\boldsymbol{u} \cdot \nabla]\boldsymbol{v}, \boldsymbol{w})_\Omega$ for $\boldsymbol{u} \in \boldsymbol{\mathcal{V}}^p$ and $\boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{\mathcal{X}}^p$, cf. [PR02]. Indeed, B preserves the skew-symmetry property, i.e., $B(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{v}) = 0$. By virtue of (2.79) the definition of B is compatible in the sense that there holds $B(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = ([\boldsymbol{u} \cdot \nabla]\boldsymbol{v}, \boldsymbol{w})_\Omega$ for all $\boldsymbol{u} \in \boldsymbol{\mathcal{V}}^p$ and $\boldsymbol{v}$, $\boldsymbol{w} \in \boldsymbol{\mathcal{X}}^p$.

Corollary 5.8. *For $p \in (\frac{2d}{d+2}, 2]$ and $k = 0$ let the assumptions of Theorem 5.7 be satisfied. Assume additionally that $\varepsilon > 0$. Then there exists a constant $C > 0$ such that*

$$\|\boldsymbol{v} - \boldsymbol{v}_h\|_{1,p} + \tilde{\beta}(2)\|\pi - \pi_h\|_2 \le Ch. \tag{5.32}$$

The constant C only depends on $\|\nabla \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})\|_2$, $\|\pi\|_{1,p'}$, p, ε, σ_0, σ_1, σ, Ω, $\boldsymbol{f}$, ϱ_0, α_0, τ_0.

Proof. In view of Theorem 5.7 it is sufficient to prove the error estimate for the pressure. The starting point is estimate (5.23) with p and p' replaced by 2. Using the identity $([\boldsymbol{b} \cdot \nabla](\boldsymbol{v} - \boldsymbol{v}_h), \boldsymbol{w}_h)_\Omega = -(\boldsymbol{b} \otimes (\boldsymbol{v} - \boldsymbol{v}_h), \nabla \boldsymbol{w}_h^\top)_\Omega$, Lemma 2.7 (i), Hölder's and Poincaré's inequality, for $\varepsilon > 0$ we estimate the expression $J_1 + J_2$ as follows:

$$J_1 + J_2 \lesssim \varepsilon^{\frac{p-2}{2}}\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2 + \|j_h\pi - \pi\|_2 + (\|\boldsymbol{b}\|_\infty + \sigma)\|\boldsymbol{v} - \boldsymbol{v}_h\|_2.$$

Because of $p \ge \frac{2d}{d+2}$ it holds $W^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$. Using this and (5.22), we arrive at

$$J_1 + J_2 \lesssim \Big(\varepsilon^{\frac{p-2}{2}} + \|\boldsymbol{b}\|_\infty + \sigma\Big)\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2 + h\|\pi\|_{1,2}. \tag{5.33}$$

Along the lines of (5.27), we can estimate the term J_3 by

$$\begin{aligned} J_3 &\le ch^{\frac{1}{2}}\|\boldsymbol{b}\|_\infty^{1/2}\Bigg[|(\boldsymbol{\xi}_h, 0)|_{\mathrm{lps}} + \Big(\sum_{M \in \mathbb{M}_h} \varrho_M \|\boldsymbol{b}\|_{\infty;M}^2 \|\nabla(\boldsymbol{v} - \boldsymbol{j}_h\boldsymbol{v})\|_{2;M}^2\Big)^{\frac{1}{2}}\Bigg] \\ &\le ch^{\frac{1}{2}}\|\boldsymbol{b}\|_\infty^{1/2}\Big[|(\boldsymbol{\xi}_h, 0)|_{\mathrm{lps}} + h^{\frac{3}{2} - \frac{d(p+1)}{3p} + \frac{d}{2}}\|\boldsymbol{b}\|_\infty^{1/2}\|\boldsymbol{v}\|_{2,\frac{3p}{p+1}}\Big]. \end{aligned} \tag{5.34}$$

Note that $\frac{3}{2} - \frac{d(p+1)}{3p} + \frac{d}{2} \ge 1$ for $p \ge 1$. Similarly to (5.28), we conclude that

$$\begin{aligned} J_4 &\le \sup_{\boldsymbol{w}_h \in \boldsymbol{\mathcal{X}}_h^2} \frac{1}{\|\nabla \boldsymbol{w}_h\|_2} \sum_{M \in \mathbb{M}_h} \varrho_M \|\boldsymbol{\theta}_h(\boldsymbol{b} \cdot \nabla)\boldsymbol{v}\|_{\frac{3p}{p+1};M}\|\boldsymbol{\theta}_h(\boldsymbol{b} \cdot \nabla)\boldsymbol{w}_h\|_{\frac{3p}{2p-1};M} \\ &\le \sup_{\boldsymbol{w}_h \in \boldsymbol{\mathcal{X}}_h^2} \frac{1}{\|\nabla \boldsymbol{w}_h\|_2} \sum_{M \in \mathbb{M}_h} \varrho_M h_M \|\boldsymbol{b}\|_{1,\infty;M}^2 \|\boldsymbol{v}\|_{2,\frac{3p}{p+1};M} h_M^{-\frac{d}{2} + \frac{d(2p-1)}{3p}} \|\boldsymbol{w}_h\|_{1,2;M} \\ &\le ch^{\frac{1}{2}}\|\boldsymbol{b}\|_{1,\infty} h^{\frac{3}{2} - \frac{d(p+1)}{3p} + \frac{d}{2}}\|\boldsymbol{v}\|_{2,\frac{3p}{p+1}}. \end{aligned} \tag{5.35}$$

Using Hölder's inequality with $\frac{2}{p'} + \frac{p'-2}{p'} = 1$ and Lemma 4.1, we bound the term J_5 by

$$\begin{aligned}
J_5^2 &= \sum_{M \in \mathbb{M}_h} h_M^2 \int\limits_M |\boldsymbol{\theta}_h(\nabla \eta_h)|^2 \,\mathrm{d}\boldsymbol{x} \leq h^{2-\frac{4}{p'}} \sum_{M \in \mathbb{M}_h} h_M^{\frac{4}{p'}} \Big(\int\limits_M |\boldsymbol{\theta}_h(\nabla \eta_h)|^{p'} \,\mathrm{d}\boldsymbol{x} \Big)^{\frac{2}{p'}} |M|^{\frac{p'-2}{p'}} \\
&\leq h^{2-\frac{4}{p'}} \Big(\sum_{M \in \mathbb{M}_h} h_M^2 \|\boldsymbol{\theta}_h(\nabla \eta_h)\|_{p';M}^{p'} \Big)^{\frac{2}{p'}} \Big(\sum_{M \in \mathbb{M}_h} |M| \Big)^{\frac{p'-2}{p'}} \\
&\lesssim h^{2-\frac{4}{p'}} \Big(\sum_{M \in \mathbb{M}_h} \alpha_M \|\boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h \nabla j_h \pi) - \boldsymbol{\mathcal{G}}(\boldsymbol{\theta}_h \nabla \pi_h)\|_{2;M}^2 \Big)^{\frac{2}{p'}} |\Omega|^{\frac{p'-2}{p'}} \lesssim h^{2-\frac{4}{p'}} \|(\boldsymbol{\xi}_h, \eta_h)\|_{\mathrm{lps}}^{\frac{4}{p'}}.
\end{aligned} \tag{5.36}$$

Inserting (5.33)–(5.36) into the L^2-norm version of (5.23), and recalling the error estimate (5.21), we can easily complete the proof. □

5.4 The non-steady p-Navier-Stokes equations

We depict how the results of Section 5.3 are applied in the numerical analysis of non-steady p-Navier-Stokes systems (2.14). Although in this section we only deal with well-known results from [BL09] and [BDR09], we state them for sake of completeness in order to provide a motivation for the analysis of p-Oseen systems performed in Section 5.3. A standard numerical approach which is frequently used for the approximation of the Navier-Stokes problem can be stated as follows (cf. [BL09]): Firstly, discretize the continuous problem in time with an A-stable implicit time-step scheme and, secondly, discretize the resulting quasi-steady problem in space with finite elements. Following the literature [BDR09], we present a semi-implicit Euler scheme for the approximation of a transient flow. For $N \in \mathbb{N}$ let us introduce the time step size $k := T/N > 0$ and the corresponding net $t_n := nk$, $n = 1, \ldots, N$. System (2.14) is discretized in time as in Algorithm 5.1.

Algorithm 5.1. Semi-implicit Euler scheme

1: Set $\boldsymbol{v}^0 = \hat{\boldsymbol{v}}$.

2: For $n = 1, 2, \ldots$ determine the solution $\boldsymbol{v}^n$ to the system

$$\left.\begin{aligned} d_t \boldsymbol{v}^n - \nabla \cdot \boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}^n) + [\boldsymbol{v}^{n-1} \cdot \nabla]\boldsymbol{v}^n + \nabla \pi^n &= \boldsymbol{f}(t_n) \\ \nabla \cdot \boldsymbol{v}^n &= 0 \end{aligned}\right\} \quad \text{in } \Omega \tag{5.37}$$

endowed with homogeneous Dirichlet boundary conditions, where

$$d_t \boldsymbol{v}^n := \frac{\boldsymbol{v}^n - \boldsymbol{v}^{n-1}}{k}.$$

The weak formulation of (5.37) reads: For $n = 1, 2, \dots$ find $(\boldsymbol{v}^n, \pi^n) \in \boldsymbol{\mathcal{X}}^p \times \mathcal{Q}^p$ such that

$$\begin{aligned}(d_t \boldsymbol{v}^n, \boldsymbol{w})_\Omega + (\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}^n), \boldsymbol{D}\boldsymbol{w})_\Omega + ([\boldsymbol{v}^{n-1} \cdot \nabla]\boldsymbol{v}^n, \boldsymbol{w})_\Omega \\ - (\pi^n, \nabla \cdot \boldsymbol{w})_\Omega + (\nabla \cdot \boldsymbol{v}^n, q)_\Omega = (\boldsymbol{f}, \boldsymbol{w})_\Omega \quad \forall (\boldsymbol{w}, q) \in \boldsymbol{\mathcal{X}}^p \times \mathcal{Q}^p.\end{aligned} \tag{5.38}$$

We define $\langle \boldsymbol{F}, \boldsymbol{w} \rangle := (\boldsymbol{f}, \boldsymbol{w})_\Omega + (\sigma \boldsymbol{v}^{n-1}, \boldsymbol{w})_\Omega$ for all $\boldsymbol{w} \in \boldsymbol{\mathcal{X}}^p$. Setting $\sigma := k^{-1}$ and $\boldsymbol{b} := \boldsymbol{v}^{n-1}$, we observe that problem (5.38) is equivalent to the p-Oseen problem

$$(\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}^n), \boldsymbol{D}\boldsymbol{w})_\Omega + (\sigma \boldsymbol{v}^n, \boldsymbol{w})_\Omega + ([\boldsymbol{b} \cdot \nabla]\boldsymbol{v}^n, \boldsymbol{w})_\Omega - (\pi^n, \nabla \cdot \boldsymbol{w})_\Omega + (\nabla \cdot \boldsymbol{v}^n, q)_\Omega = \langle \boldsymbol{F}, \boldsymbol{w} \rangle$$

for all $\boldsymbol{w} \in \boldsymbol{\mathcal{X}}^p$ and $q \in \mathcal{Q}^p$. Next, we discretize this p-Oseen problem in space and we apply the LPS technique discussed in Section 5.1. In particular, the numerical analysis performed in Section 5.3 can be applied for each time step.

For the time-discretization of system (2.14), the Algorithm 5.1 was proposed and analyzed in [BDR09] in the case of space-periodic boundary conditions. It can be interpreted as a semi-implicit Euler scheme since the convective term is treated semi-implicitly while the nonlinear extra stress tensor is treated implicitly. The semi-implicit treatment of the convective term allows to prove the uniqueness of the solutions to system (5.37), see [BDR09]. For regular initial values $\hat{\boldsymbol{v}} \in \boldsymbol{W}^{2,2}(\Omega)$ it is shown in [BDR10, BDR09] that there exists a unique strong solution $\boldsymbol{v}^n$ to system (5.37) satisfying the weak formulation (5.38) and the regularity $\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}^n) \in W^{1,2}(\Omega)^{d \times d}$, see Lemma 2.28. If the initial value $\hat{\boldsymbol{v}}$ belongs to $\boldsymbol{W}^{2,2}(\Omega)$, then the semi-implicit Euler scheme 5.1 also allows for an optimal a priori error estimate with respect to the convergence order, see [BDR09] or Lemma 4.19.

5.5 Numerical experiments

In this section we present numerical experiments for p-Navier-Stokes systems. On the basis of more natural flow configurations we demonstrate the convergence of the (stabilized) FEM, see Section 5.3. As illustrative examples, we consider steady channel flows such as a planar flow in a channel with a sudden expansion. Note that the observed order of convergence may provide hints on the smoothness of the solution. In case of Dirichlet boundary conditions, the regularity of (weak) solutions up to the boundary is subject of current research (cf. [Ebm06]). Numerical experiments may support analytical studies.

Let us consider the steady p-Navier-Stokes equations (2.15). We restrict ourselves to fluid models of class (2.10). We consider planar flows driven by the difference of the pressure between inlet and outlet. We assume that Ω is a $2d$ channel and that its boundary $\partial\Omega$ consists of a solid part Γ (upper and lower edge), of an inflow boundary S_1 (left), and of a free outflow boundary S_2 (right), see e.g. Figure 5.3(a). On the solid part we prescribe homogeneous Dirichlet boundary conditions: $\boldsymbol{v} = \boldsymbol{0}$ on Γ. On the inlet and outlet we prescribe the following natural inflow and outflow boundary conditions

$$-\frac{\mu(|\boldsymbol{D}\boldsymbol{v}|^2)}{2} \partial_{\boldsymbol{n}} \boldsymbol{v} + \pi \boldsymbol{n} = b_i \boldsymbol{n} \qquad \text{on } S_i, \qquad i \in \{1, 2\}, \tag{5.39}$$

for given $b_i \in \mathbb{R}$. Here, $\boldsymbol{n}$ denotes the outer normal on $\partial\Omega$, and $\partial_{\boldsymbol{n}}\boldsymbol{v}$ is the corresponding directional derivative. We recall that $\nabla\boldsymbol{v} = (\partial_j v_i)_{i,j=1}^d$ and $\partial_{\boldsymbol{n}}\boldsymbol{v} = (\boldsymbol{n}\cdot\nabla)\boldsymbol{v} = [\nabla\boldsymbol{v}]\boldsymbol{n}$. The boundary conditions (5.39) arise from the variational formulation and they implicitly normalize the pressure which is initially determined up to a constant only, compare Remarks 5.5 and 5.9 below. Related to boundary conditions of type (5.39), extensive discussions can be found in Heywood et al. [HRT96] in the context of Navier-Stokes equations.

Remark 5.5. Alternatively, we can prescribe the boundary conditions (see [LS11b])

$$\left.\begin{aligned} -\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v})\boldsymbol{n}\cdot\boldsymbol{n} + \pi &= b_i \\ \boldsymbol{v} &= (\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{n} \end{aligned}\right\} \quad \text{on } S_i, \qquad i \in \{1,2\}. \tag{5.40}$$

In case of simple channel flows, the boundary conditions (5.39) and (5.40) lead to the same flow behavior as depicted below. In a simple channel $\Omega = (0,L)\times(0,H)$, the condition $\boldsymbol{v} = (\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{n}$ on S_i ensures that stream lines are orthogonal to the inflow and outflow boundary, i.e., that $\boldsymbol{v} = (v_1, 0)^{\mathsf{T}}$ on S_i. We note that $\boldsymbol{n}|_{S_i}$ is a constant vector and that $\boldsymbol{v} = (\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{n}$ implies $\partial_{x_1} v_1 = \nabla\cdot\boldsymbol{v} = 0$. Hence, we conclude that $v_1 = v_1(x_2)$ and, consequently, $[\nabla\boldsymbol{v}]\boldsymbol{n} = \boldsymbol{0}$ on S_i. Let $v_n = (\boldsymbol{v}\cdot\boldsymbol{n})$ be the normal component of $\boldsymbol{v}$. Let $\boldsymbol{t}$ be the tangential vector on $\partial\Omega$, and v_t the corresponding tangential component of $\boldsymbol{v}$. Using $[\nabla\boldsymbol{v}]\boldsymbol{n} = \boldsymbol{0}$ on S_i and $[\nabla\boldsymbol{v}]^{\mathsf{T}}\boldsymbol{n} = \nabla v_n$, we equivalently write the condition (5.40) as follows:

$$-\frac{\mu(|\boldsymbol{D}\boldsymbol{v}|^2)}{2}\partial_{\boldsymbol{n}} v_n + \pi = b_i \qquad \text{on } S_i.$$

Integrating this over S_i and observing $\partial_{\boldsymbol{n}} v_n = -\partial_{\boldsymbol{t}} v_t$ due to $\nabla\cdot\boldsymbol{v} = 0$, we finally arrive at

$$\int_{S_i} \pi \,\mathrm{d}o = |S_i| b_i - \frac{1}{2}\int_{S_i} \mu(|\boldsymbol{D}\boldsymbol{v}|^2)\partial_{\boldsymbol{t}} v_t \,\mathrm{d}o, \qquad i \in \{1,2\}. \tag{5.41}$$

Multiplying (5.39) by $\boldsymbol{n}$ and integrating the result over S_i, we obtain the condition (5.41) as well. As a result, both (5.39) and (5.40) lead to (5.41). If $\boldsymbol{v} = (\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{n}$ on S_i, i.e., if $v_t \equiv 0$ on S_i, then $\partial_{\boldsymbol{t}} v_t = 0$ and, hence, $\int_{S_i} \pi \,\mathrm{d}o = |S_i| b_i$. We realize that the prescribed value b_i can be interpreted as the mean-value of the pressure over S_i.

The variational formulation **(P5)** has to be adapted to the current flow configuration. Since Dirichlet boundary conditions are prescribed on Γ only, the used function spaces have to be modified. We define the velocity and pressure space as follows:

$$\boldsymbol{\mathcal{X}}_\Gamma^p := \{\boldsymbol{w} \in \boldsymbol{W}^{1,p}(\Omega);\, \mathrm{tr}\,\boldsymbol{w} = \boldsymbol{0} \text{ on } \Gamma\}, \qquad \mathcal{Q}_\Gamma^p := L^{p'}(\Omega). \tag{5.42}$$

Let the semi-linear form $A(\cdot)(\cdot)$ be defined by (3.23). Then the weak pressure-drop problem reads: Find a velocity $\boldsymbol{v}$ and pressure π, $\boldsymbol{u} \equiv (\boldsymbol{v}, \pi) \in \boldsymbol{\mathcal{X}}_\Gamma^p \times \mathcal{Q}_\Gamma^p$, that solve the system

$$A(\boldsymbol{u})(\boldsymbol{\omega}) = \sum_i \left(\frac{\mu(|\boldsymbol{D}\boldsymbol{v}|^2)}{2}[\nabla\boldsymbol{v}]^{\mathsf{T}}\boldsymbol{n} - b_i\boldsymbol{n}, \boldsymbol{w}\right)_{S_i} \qquad \forall\boldsymbol{\omega} \equiv (\boldsymbol{w}, q) \in \boldsymbol{\mathcal{X}}_\Gamma^p \times \mathcal{Q}_\Gamma^p \tag{5.43}$$

($\boldsymbol{f} \equiv \boldsymbol{0}$). The weak formulation (5.43) implicitly contains natural boundary conditions on the free inflow and outflow boundaries as depicted by Remark 5.6.

Remark 5.6. Below we derive the free inflow and outflow boundary conditions which are implicitly hidden in the weak formulation (5.43). We assume that there exists a solution $(\boldsymbol{v}, \pi)$ to problem (5.43) which is smooth enough in order to be a classical solution. Using $(\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{w})_\Omega = (\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}), \nabla \boldsymbol{w})_\Omega$ and integration by parts, from (5.43) we deduce that

$$\begin{aligned}\sum_i \Big(\frac{\mu(|\boldsymbol{D}\boldsymbol{v}|^2)}{2}[\nabla\boldsymbol{v}]^\top\boldsymbol{n} - b_i\boldsymbol{n}, \boldsymbol{w}\Big)_{S_i} &= \int_\Omega \boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}) : \nabla\boldsymbol{w} - \pi\nabla\cdot\boldsymbol{w} + ([\boldsymbol{v}\cdot\nabla]\boldsymbol{v})\cdot\boldsymbol{w}\,\mathrm{d}\boldsymbol{x} \\ &= \int_\Omega \Big(-\nabla\cdot\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}) + \nabla\pi + [\boldsymbol{v}\cdot\nabla]\boldsymbol{v}\Big)\cdot\boldsymbol{w}\,\mathrm{d}\boldsymbol{x} \\ &\quad + \int_{S_1\cup S_2} \Big(\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v})\boldsymbol{n} - \pi\boldsymbol{n}\Big)\cdot\boldsymbol{w}\,\mathrm{d}o\end{aligned}$$

since the test functions $\boldsymbol{w} \in \boldsymbol{\mathcal{X}}^p_\Gamma$ do not vanish on the boundary part $S_1 \cup S_2$ where no Dirichlet boundary condition is prescribed. Consequently, the inflow/outflow conditions

$$\sum_i \int_{S_i} \Big(\frac{\mu(|\boldsymbol{D}\boldsymbol{v}|^2)}{2}[\nabla\boldsymbol{v}]\boldsymbol{n} - \pi\boldsymbol{n} + b_i\boldsymbol{n}\Big)\cdot\boldsymbol{w}\,\mathrm{d}o = 0 \qquad \forall\boldsymbol{w}\in\boldsymbol{\mathcal{X}}^p_\Gamma \tag{5.44}$$

follow. The natural inflow/outflow conditions (5.44) lead to the boundary conditions (5.39).

Remark 5.7. Problem (5.43) is not well-posed in general since, unless the velocity $\boldsymbol{v}$ is sufficiently smooth, the boundary integral $(\mu(|\boldsymbol{D}\boldsymbol{v}|^2)[\nabla\boldsymbol{v}]^\top\boldsymbol{n}, \boldsymbol{w})_{S_i}$ is not well-defined. Up to now, for $p \neq 2$ an existence theory is only established for pressure-drop problems of the following type (cf. [LS11b]): Find $\boldsymbol{u} \equiv (\boldsymbol{v}, \pi) \in \boldsymbol{\mathcal{X}}^p_\Gamma \times \mathcal{Q}^p_\Gamma$ such that

$$A(\boldsymbol{u})(\boldsymbol{\omega}) = -\sum_i (b_i\boldsymbol{n}, \boldsymbol{w})_{S_i} \qquad \forall\boldsymbol{\omega} \equiv (\boldsymbol{w}, q) \in \boldsymbol{\mathcal{X}}^p_\Gamma \times \mathcal{Q}^p_\Gamma. \tag{5.45}$$

Similarly to Remark 5.6, from (5.45) we derive the natural inflow/outflow conditions

$$\sum_i \int_{S_i} (\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v})\boldsymbol{n} - \pi\boldsymbol{n} + b_i\boldsymbol{n})\cdot\boldsymbol{w}\,\mathrm{d}o = 0 \qquad \forall\boldsymbol{w}\in\boldsymbol{\mathcal{X}}^p_\Gamma. \tag{5.46}$$

Note that the natural inflow/outflow conditions (5.44) lead to the following boundary conditions: (2.24b) with $\boldsymbol{b} := b_1\boldsymbol{n}$ on S_1, and (2.24b) with $\boldsymbol{b} := b_2\boldsymbol{n}$ on S_2. Although the boundary condition (2.24b) is popular from analytical point of view (it allows an existence theory such as in [LS11b]), it is less suitable from practical point of view since it is not satisfied even for simple flows such as Poiseuille flows. In particular, if the boundary condition (2.24b) is required on S_i, then the stream lines of simple flows are generally curved inwards or outwards on S_i (see [HRT96]). Such flow behavior is not desirable since it does not reflect the physical objectivity. In contrast, the boundary condition (5.39) seems to be the proper choice since it well recovers simple Poiseuille flows.

In order to guarantee that stream lines are orthogonal to the boundary S_i, we can require the additional condition $\boldsymbol{v} = (\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{v}$ on S_i which we can incorporate into the weak formulation (5.45) by altering the velocity space as follows (cf. [LS11b]):

$$\overline{\boldsymbol{\mathcal{X}}}^p_\Gamma := \Big\{\boldsymbol{w}\in\boldsymbol{W}^{1,p}(\Omega);\ \mathrm{tr}\,\boldsymbol{w} = \boldsymbol{0} \text{ on } \Gamma,\ \mathrm{tr}\,\boldsymbol{w} = (\mathrm{tr}\,\boldsymbol{w}\cdot\boldsymbol{n})\boldsymbol{n} \text{ on } S_i,\ i\in\{1,2\}\Big\}. \tag{5.47}$$

For a 2d channel Ω as in Remark 5.5, functions $\boldsymbol{w}$ from $\overline{\boldsymbol{\mathcal{X}}}^p_\Gamma$ satisfy $\boldsymbol{w} = (\boldsymbol{w} \cdot \boldsymbol{n})\boldsymbol{n}$ on S_i and, hence, they take the form $\boldsymbol{w} = (w_1, 0)^\mathsf{T}$ on S_i. If the trial space $\boldsymbol{\mathcal{X}}^p_\Gamma$ in (5.45) is replaced by the modified one $\overline{\boldsymbol{\mathcal{X}}}^p_\Gamma$, then the natural inflow/outflow conditions (5.46) lead to inflow/outflow boundary conditions that are equivalent to (5.39). More details on free inflow/outflow boundary conditions can be found in [HRT96].

Problem (5.43) was discretized with equal-order d-linear $\mathbb{Q}_1/\mathbb{Q}_1$ finite elements, see Section 3.1. The used FE spaces are given by $\boldsymbol{\mathcal{X}}^p_{\Gamma;h} := \boldsymbol{X}_h \cap \boldsymbol{\mathcal{X}}^p_\Gamma$ and $\mathcal{Q}^p_{\Gamma;h} := X_h \cap \mathcal{Q}^p_\Gamma$ with $X_h := X_{h,1}$ defined in (3.2). Since this discretization is not “inf-sup” stable, the following stabilized discrete problem was solved: Find $\boldsymbol{u}_h \equiv (\boldsymbol{v}_h, \pi_h) \in \boldsymbol{\mathcal{X}}^p_{\Gamma;h} \times \mathcal{Q}^p_{\Gamma;h}$ such that

$$A(\boldsymbol{u}_h)(\boldsymbol{\omega}_h) + S_h(\boldsymbol{u}_h)(\boldsymbol{\omega}_h) = \sum_i \left(\frac{\mu(|\boldsymbol{D}\boldsymbol{v}_h|^2)}{2}[\nabla \boldsymbol{v}_h]^\mathsf{T}\boldsymbol{n} - b_i\boldsymbol{n}, \boldsymbol{w}_h \right)_{S_i} \qquad \forall \boldsymbol{\omega}_h \equiv (\boldsymbol{w}_h, q_h) \in \boldsymbol{\mathcal{X}}^p_{\Gamma;h} \times \mathcal{Q}^p_{\Gamma;h}. \tag{5.48}$$

In the subsequent simulations, we used the Carreau model (2.10) & (2.11b). The stabilization term S_h was always chosen as in (3.54). We determined the experimental order of convergence (EOC) with respect to the quantities $E^{1,p}_{\boldsymbol{v}}$, $E^p_{\boldsymbol{v}}$, $E^{p'}_\pi$, E^2_π defined in (4.104).

Table 5.1. Numerical verification of Theorem 5.7

(a) $p = 1.3$

#cells	$E^{\boldsymbol{\mathcal{F}}}_{\boldsymbol{v}}$	$E^{1,p}_{\boldsymbol{v}}$	$E^p_{\boldsymbol{v}}$	$E^{\boldsymbol{\mathcal{S}}}_{\boldsymbol{v}}$
512	0.99	0.99	1.98	1.00
2048	1.00	1.00	2.00	1.00
8192	1.00	1.00	2.00	1.00
32768	1.00	1.00	2.00	1.00
131072	1.00	1.00	2.00	1.00
expected	1	1	2	

(b) $p = 1.5$

#cells	$E^{\boldsymbol{\mathcal{F}}}_{\boldsymbol{v}}$	$E^{1,p}_{\boldsymbol{v}}$	$E^p_{\boldsymbol{v}}$	$E^{\boldsymbol{\mathcal{S}}}_{\boldsymbol{v}}$
512	1.00	1.00	2.00	1.00
2048	1.00	1.00	2.00	1.00
8192	1.00	1.00	2.00	1.00
32768	1.00	1.00	2.00	1.00
131072	1.00	1.00	2.00	1.00
expected	1	1	2	

Example 1: First of all, let $\Omega := (0, L) \times (0, H)$ be a simple channel. Below, we set $b_1 = L/2$ and $b_2 = 0$. In the case under consideration, it can easily be verified that for $\varepsilon = 0$ the unique (strong) solution $(\boldsymbol{v}, \pi)$ to (5.43) is given by

$$v_1(\boldsymbol{x}) = c_p\left(\left(\frac{1}{2}\right)^{\frac{p}{p-1}} - \left|\frac{x_2 - H/2}{H}\right|^{\frac{p}{p-1}}\right), \qquad v_2(\boldsymbol{x}) \equiv 0, \qquad \pi(\boldsymbol{x}) = -\frac{1}{2}x_1 + \frac{1}{2}L, \tag{5.49}$$

where $c_p = \mu_0^{-\frac{1}{p-1}} \sqrt{2}^{\frac{p-2}{p-1}} \frac{p-1}{p} H^{\frac{p}{p-1}}$. We briefly motivate why we consider this particular simple pressure-drop example here: The data such as $\boldsymbol{f}$ are independent of p. The function $(\boldsymbol{v}, \pi)$ defined in (5.49) captures the typical flow behavior of a shear thinning fluid. For $1 < p \ll 2$ sharp boundary layers occur near Γ, and the measure of the critical set $\Omega_c := \{\boldsymbol{x} \in \Omega;\ \nabla \boldsymbol{v}(\boldsymbol{x}) \approx \boldsymbol{0}\}$ becomes large. In this example we set $L = 1.64$, $H = 0.41$,

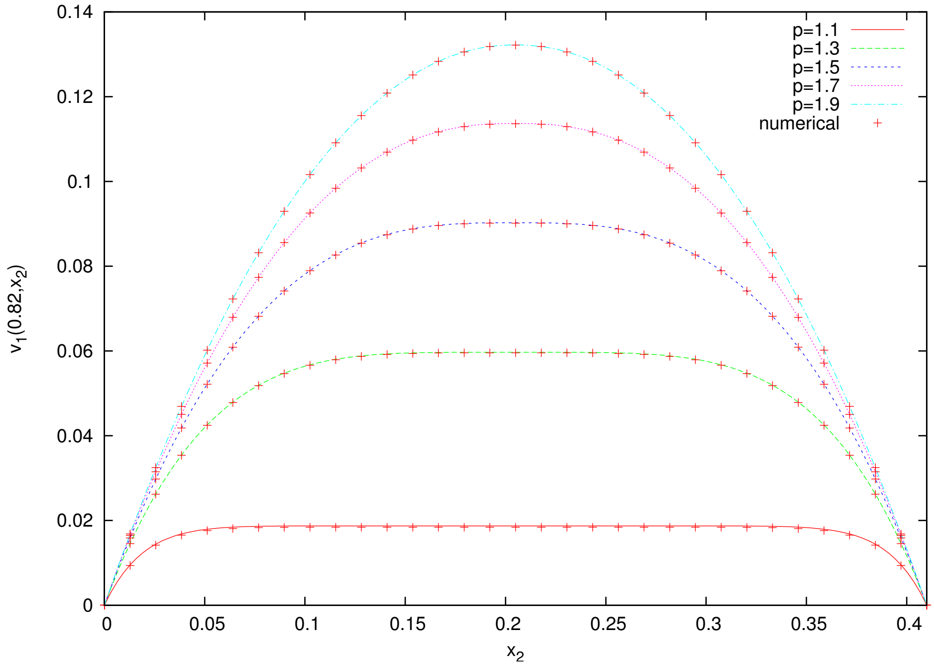


Figure 5.1. Comparison of analytical and numerical velocity profiles

$b_1 = 0.82$, $b_2 = 0$, $\mu_0 = 0.15$. Due to Theorem 5.7 we expect linear convergence for the quantities $E_{\boldsymbol{v}}^{1,p}$ and $E_{\boldsymbol{v}}^{\mathcal{F}}$. The observed convergence rates are presented in Table 5.1. We realize that the experimental order of convergence agrees with the theoretical one very well. Obviously, convergence rates for the pressure are not presented. This is due to the fact that the pressure π belongs to the finite element space $\mathcal{Q}^p_{\Gamma;h}$ and, hence, π was resolved exactly up to machine accuracy. It is worth mentioning that we were not able to determine the solutions to (5.48) numerically for a smaller range of p using Newton's method (Algorithm 3.1). For instance, if $p = 1.2$, then the Newton iteration did not reach the prescribed tolerance $TOL = 10^{-11}$ for the nonlinear residual on the grid with 32768 elements. The reason is simple: Since $|\Omega_c|$ is large for $1 < p \ll 2$, in view of (3.53) the Newton matrices arising in Algorithm 3.1 become singular. This usually causes numerical instabilities, and Newton's method does not converge. As a result, system (5.48) can generally be solved for $\varepsilon > 0$ only. Then the parameter $\varepsilon > 0$ plays the role of a regularization parameter. The question arises how solutions to (5.43) can properly be approximated in the case $p < 2$ and $\varepsilon = 0$. This will be the topic of Chapter 6. Figure 5.1 depicts the analytical velocity profiles (5.49) and it shows the solutions to (5.48) for small $\varepsilon > 0$ which apparently represent good approximations to the exact solution. In Chapter 6 we will analytically show that for $\varepsilon \searrow 0$ the (discrete) solutions to (5.48) indeed approximate the solution to (5.43) with $\varepsilon = 0$.

In Table 5.2(a) we solved (5.48) for $p = 1.1$ and $\varepsilon = 10^{-7}$. Clearly, the overall approximation error $(\boldsymbol{v} - \boldsymbol{v}_h)$ can be splitted into two contributions: The first one results from discretization, while the second one is caused by regularization of (5.48) with $\varepsilon > 0$. Here, the regularization parameter ε was chosen sufficiently small so that the discretization error dominates the regularization error on the considered meshes by several orders of magnitude.

For instance, on the grid with 128 elements we obtained a $W^{1,p}$-error of about 4.55e-02 for $\varepsilon = 0$, which agrees with the corresponding value for $E_{\boldsymbol{v}}^{1,p}$ in Table 5.2(a). Considering Table 5.2(a), we realize that $E_{\boldsymbol{v}}^{1,p}$ behaves as $\mathcal{O}(h)$. This agrees with Theorem 5.7.

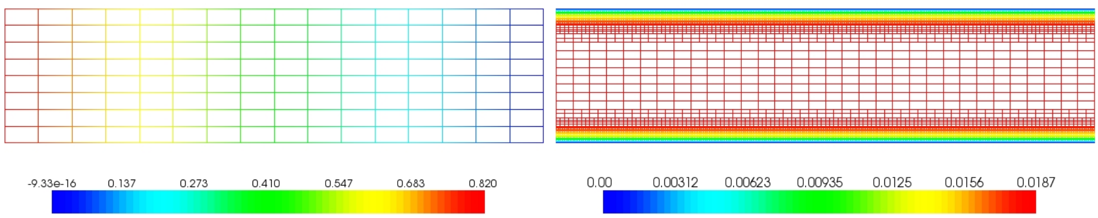
Table 5.2. Global vs. local mesh refinement: Case $p = 1.1$

(a) Global mesh refinement

	$E_{\boldsymbol{v}}^{1,p}$	
#cells	error	conv.
128	4.55e-02	–
512	2.50e-02	0.87
2048	1.28e-02	0.97
8192	6.44e-03	0.99
32768	3.22e-03	1.00
131072	1.61e-03	1.00

(b) Local mesh refinement

	$E_{\boldsymbol{v}}^{1,p}$	
#cells	error	conv.
128	4.55e-02	–
320	2.50e-02	0.87
704	1.33e-02	0.91
2816	6.73e-03	0.99
9728	3.42e-03	0.98
35840	1.80e-03	0.93



(a) Coarse grid (128 elements)   (b) Locally refined grid (9728 elements)

Figure 5.2. Pressure (left) and velocity (right) profile on different meshes

Local mesh refinement: Considering (5.49), for diminishing values of p we observe sharp boundary layers near the Dirichlet boundary Γ and we realize that the measure of Ω_c increases. In Ω_c the velocity is almost constant. Hence, a good refinement strategy consists in the following one which, in each refinement cycle, refines elements along the Dirichlet boundary Γ only: After covering Ω with a coarse mesh, we refine all elements that are located in a neighborhood of Γ. Since the velocity is almost constant in Ω_c, we do not refine the mesh within Ω_c. As is common practice, the local mesh refinement is performed by bisection of edges (cf. Algorithm 5.2 below): Each quadrilateral is subdivided into four subelements. Here, the neighborhood of Γ, in which elements are refined, was chosen as the set $\{\boldsymbol{x} \in \Omega;\, \mathrm{dist}(\boldsymbol{x}, \Gamma) < 0.05\}$. In Table 5.2 we compared the global refinement strategy with the described local one for $p = 1.1$, while in Table 5.3 we applied the same refinement strategies for $p = 1.03$. Table 5.2 indicates that local mesh refinement is more efficient than global mesh refinement: On globally refined meshes the approximation error $E_{\boldsymbol{v}}^{1,p}$ behaves as $\mathcal{O}(h)$. Similar values of $E_{\boldsymbol{v}}^{1,p}$ were obtained if the mesh is refined along Γ only (see Figure

5.2). In view of Table 5.3 we realize even better agreement of the approximation errors for the two different refinement strategies. The reason is that, compared to the case $p = 1.1$, sharper boundary layers occur. Hence, we could further reduce the complexity by choosing a smaller refinement area without losing accuracy. For such simple flows, the smaller the power-law exponent p is, the more efficient local mesh refinement works. To sum up, we benefit from local mesh refinement since we obtain the same accuracy as in the case of global mesh refinement while saving random access memory. Since the numerical solution of power-law flows becomes more complex and more cost-intensive for diminishing p, we should make use of local mesh refinement in order to counter the increasing complexity and to efficiently approximate power-law flows.

Table 5.3. Global vs. local mesh refinement: Case $p = 1.03$

(a) Global mesh refinement

	$E_{\boldsymbol{v}}^{\mathcal{F}}$		$E_{\boldsymbol{v}}^{1,p}$	
#cells	error	conv.	error	conv.
128	6.41e-02	–	9.48e-03	–
512	4.24e-02	0.60	7.82e-03	0.28
2048	2.35e-02	0.85	4.86e-03	0.69
8192	1.21e-02	0.96	2.59e-03	0.91
32768	6.10e-03	0.99	1.32e-03	0.98
131072	3.06e-03	1.00	6.62e-04	0.99

(b) Local mesh refinement

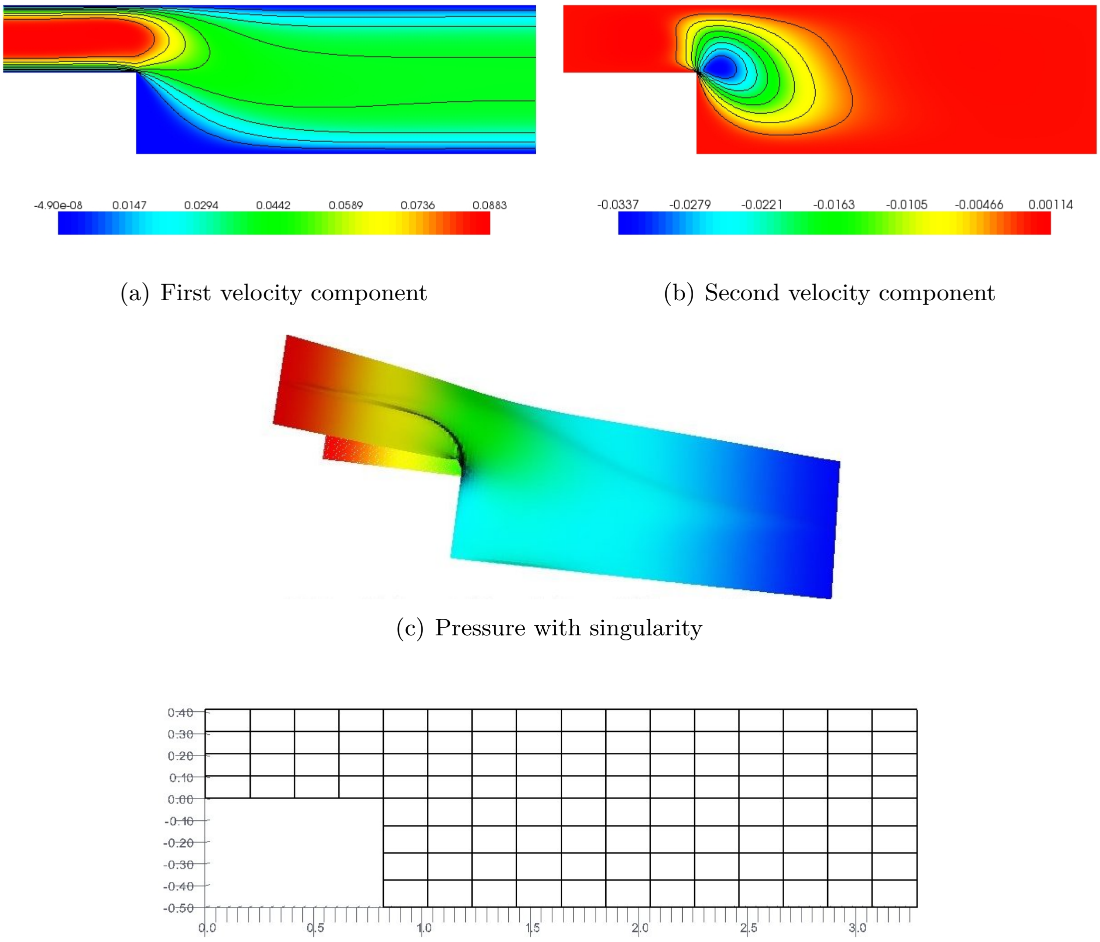
	$E_{\boldsymbol{v}}^{\mathcal{F}}$		$E_{\boldsymbol{v}}^{1,p}$	
#cells	error	conv.	error	conv.
128	6.41e-02	–	9.48e-03	–
320	4.24e-02	0.60	7.82e-03	0.28
704	2.35e-02	0.85	4.86e-03	0.69
2816	1.21e-02	0.96	2.59e-03	0.91
9728	6.10e-03	0.99	1.32e-03	0.98
35840	3.06e-03	1.00	6.62e-04	0.99
137216	1.53e-03	1.00	3.31e-04	1.00

Example 2: We consider a steady flow in a channel with sudden expansion driven by the difference of the pressure between inlet (left) and outlet (right), see Figure 5.3(d). For $p = 1.3$, $\varepsilon = 10^{-7}$, $\mu_0 = 0.15$, $b_1 = 1$ the observed convergence rates are presented in Table 5.4. Note that the exact solution is unknown. As the reference solution we took an accurate FE solution that was determined on a fine grid with 7340032 elements (cf. the subsequent discussion). In Table 5.4 we discover almost linear convergence for $E_{\boldsymbol{v}}^{1,p}$. Apparently, the experimental convergence rate for $E_{\pi}^{p'}$ is less than the one stated in Theorem 5.7. To sum up, the observed convergence does not agree with the a priori error estimates of Theorem 5.7. As a result we conclude that the regularity of the solution is not sufficient to ensure the optimal order of convergence. Note that the pressure and the velocity gradient exhibit a singular behavior in the corner, see Figure 5.3.

Reference solution: For general flow configurations, the exact solution $(\boldsymbol{v}, \pi)$ to (5.43) is generally not available. Hence, a finite element solution $(\boldsymbol{v}_H, \pi_H)$, that is computed on a very fine grid with mesh size H, is employed as the reference solution $(\boldsymbol{v}_H, \pi_H) \approx (\boldsymbol{v}, \pi)$. Let us reconsider Example 2. In Table 5.5 we compare the approximation errors and convergence rates obtained for the reference solution $(\boldsymbol{v}_H, \pi_H)$ with those obtained for the "better" reference solution $(\boldsymbol{v}_{H/2}, \pi_{H/2})$. Here, H corresponds to the grid with 1835008 elements, which results from the finest grid in Table 5.5(a) by double refinement. In view

Table 5.4. Experimental order of convergence: Case $p = 1.3$

#cells	$E_{\boldsymbol{v}}^{1,p}$	$E_{\boldsymbol{v}}^{p}$	$E_{\pi}^{p'}$	E_{π}^{2}
448	0.89	1.66	0.27	1.11
1792	0.92	1.56	0.24	0.88
7168	0.92	1.43	0.24	0.83
28672	0.91	1.31	0.24	0.80
114688	0.91	1.27	0.26	0.79
458752	0.92	1.31	0.38	0.81



(a) First velocity component

(b) Second velocity component

(c) Pressure with singularity

(d) Coarse mesh (112 elements)

Figure 5.3. Steady flow in a channel with sudden expansion

of Table 5.5 the approximation errors for $(\boldsymbol{v}_H, \pi_H)$ differ from the ones for $(\boldsymbol{v}_{H/2}, \pi_{H/2})$ on the finer grids. As a result, the observed convergence rates are reliable only in their first significant digit. Throughout the thesis, we use the following convention: If the exact solution $(\boldsymbol{v}, \pi)$ is unknown, then the finite element solution $(\boldsymbol{v}_H, \pi_H)$ is set as the reference solution and the fine grid with mesh size H is chosen as the grid that is obtained after second refinement of the finest grid stated in the table.

Table 5.5. Comparison of different reference solutions

(a) Reference solution $(\boldsymbol{v}_H, \pi_H)$

	$\lVert\boldsymbol{v}_H - \boldsymbol{v}_h\rVert_{1,p}$		$\lVert\pi_H - \pi_h\rVert_2$	
#cells	error	conv.	error	conv.
112	1.75e-01	–	2.34e-02	–
448	9.48e-02	0.89	1.09e-02	1.11
1792	5.00e-02	0.92	5.89e-03	0.88
7168	2.65e-02	0.92	3.32e-03	0.83
28672	1.40e-02	0.92	1.89e-03	0.81
114688	7.33e-03	0.93	1.07e-03	0.82

(b) Reference solution $(\boldsymbol{v}_{H/2}, \pi_{H/2})$

	$\lVert\boldsymbol{v}_{H/2} - \boldsymbol{v}_h\rVert_{1,p}$		$\lVert\pi_{H/2} - \pi_h\rVert_2$	
#cells	error	conv.	error	conv.
112	1.75e-01	–	2.34e-02	–
448	9.45e-02	0.89	1.08e-02	1.11
1792	5.01e-02	0.92	5.88e-03	0.88
7168	2.66e-02	0.92	3.32e-03	0.83
28672	1.41e-02	0.91	1.90e-03	0.80
114688	7.48e-03	0.91	1.10e-03	0.79
458752	3.95e-03	0.92	6.25e-04	0.81

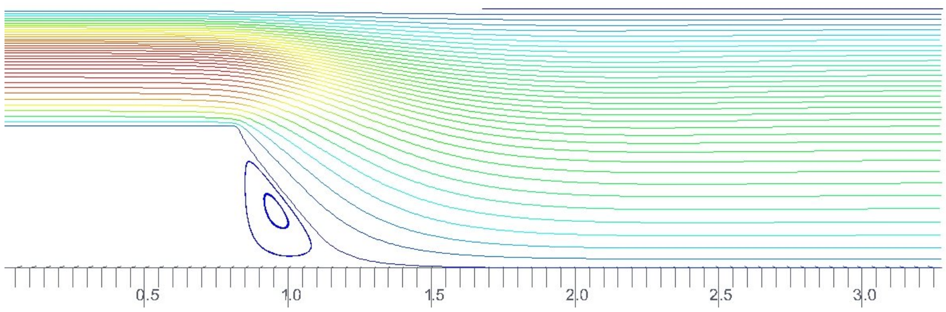


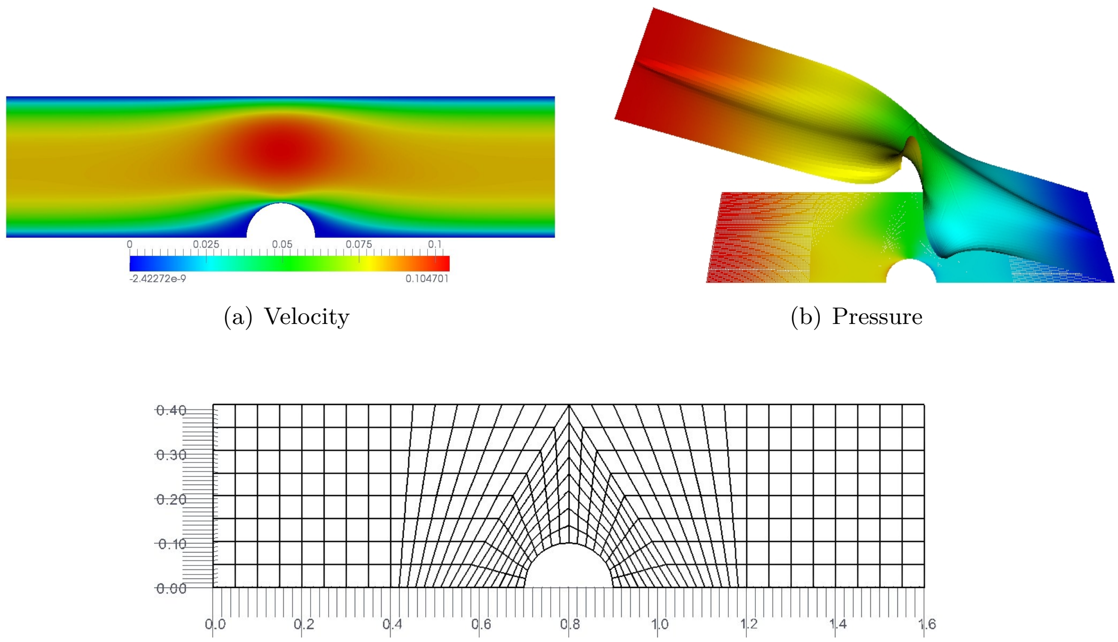
Figure 5.4. Stream lines in a channel with expansion: case $p = 1.5$, $b_1 = 2.5$

Example 3: We consider a steady flow in a channel with stenosis driven by pressure drop, see Figure 5.5. The parameters were given as in Example 2. The experimental convergence rates are depicted in Table 5.6. We realize that $E_{\boldsymbol{v}}^{1,p}$ behaves as $\mathcal{O}(h)$. The apparent convergence is in agreement with Theorem 5.7. For $E_\pi^{p'}$ the observed rates of convergence are better than expected from Theorem 5.7. We believe that the improved order of convergence for the pressure can be explained by super approximation (cf. Example 7 in Section 4.8). The velocity converges in $L^p(\Omega)$ quadratically. As a result, we conclude that the solution is smooth and that, hence, a duality argument (similar to the one described in [BS94]) seems to be applicable here.

Conclusion: We extended the LPS-based approach of Chapter 4 to p-Oseen equations in order to cope with dominating advection. For p-Oseen systems we established optimal a priori error estimates that quantify the convergence of the method, see Theorem 5.7. At least for $1 < p \ll 2$ a convergence analysis of stabilized p-Navier-Stokes systems remain an open problem. If p is sufficiently large and the Reynolds number is small enough, we can easily generalize Theorem 5.7 to p-Navier-Stokes systems using the skew-symmetric tri-linear form (5.31) for approximation of the convective term. We performed several numerical experiments on p-Navier-Stokes systems. They indicate that the proposed stabilization leads to a stable discretization. Furthermore, the experimental order of convergence agrees with the expected one for p-Oseen systems (Theorem 5.7). Hence we conjecture that for the considered experiments Theorem 5.7 remains valid in the context of p-Navier-Stokes systems.

Table 5.6. Numerical verification of Theorem 5.7: Case $p = 1.3$

#cells	$E_{\boldsymbol{v}}^{1,p}$	$E_{\boldsymbol{v}}^{p}$	$E_{\pi}^{p'}$	E_{π}^{2}
384	0.87	1.48	0.48	1.04
1536	0.96	1.71	0.65	1.18
6144	1.00	1.87	1.57	1.63
24576	1.01	1.97	1.66	1.80
98304	1.01	1.95	1.53	1.80



(a) Velocity

(b) Pressure

(c) Coarse mesh (384 elements)

Figure 5.5. Steady flow in channel with stenosis

5.6 A posteriori error estimation and adaptive mesh refinement

This section deals with a posteriori error estimation and adaptive mesh refinement. The adaptive finite element method (AFEM) consists of a loop (see [DK08]): First of all the FE problem is solved on the current mesh, then the a posteriori error estimator is evaluated, and finally elements are marked for refinement with the help of the estimator:

$$\text{SOLVE} \quad \rightarrow \quad \text{ESTIMATE} \quad \rightarrow \quad \text{MARK} \quad \rightarrow \quad \text{REFINE.}$$

In this section we introduce the well-known dual weighted residual (DWR) method following the literature [BR03] and [Sch10]. Later we will apply the DWR method to the steady p-Navier-Stokes equations. If $\boldsymbol{u}$ represents the unknown exact solution, $\boldsymbol{u}_h$ the corresponding approximation and J an output functional such as the drag force, then the goal-oriented DWR estimator aims at assessing the error between $J(\boldsymbol{u})$ and $J(\boldsymbol{u}_h)$. The DWR method yields weighted a posteriori error bounds such as

$$|J(\boldsymbol{u}) - J(\boldsymbol{u}_h)| \leq \sum_{K \in \mathbb{T}_h} \varrho_K \omega_K$$

where the weights ω_K are determined by means of approximative solutions to a linearized dual problem and the quantities ϱ_K represent computable residuals. We briefly recall the DWR method within an abstract framework. For details on the DWR method we refer to [BR03] and [Sch10]. Let $\boldsymbol{Y}$ be a function space. In the context of p-Navier-Stokes systems, the space $\boldsymbol{Y}$ will be chosen as a subspace of $\boldsymbol{W}^{1,p}(\Omega) \times L^{p'}(\Omega)$. For given $F \in \boldsymbol{Y}^*$ we seek a solution $\boldsymbol{u} \in \boldsymbol{u}_0 + \boldsymbol{Y}$ to the abstract variational problem

$$B(\boldsymbol{u})(\boldsymbol{\omega}) = F(\boldsymbol{\omega}) \qquad \forall \boldsymbol{\omega} \in \boldsymbol{Y}. \tag{5.50}$$

Here, $\boldsymbol{u}_0$ stands for non-homogeneous Dirichlet data, and the semi-linear form B is supposed to be three-times differentiable on $\boldsymbol{Y} \times \boldsymbol{Y}$. The problem (5.50) is approximated by conforming finite elements, i.e., the finite dimensional spaces $\boldsymbol{Y}_h$ satisfy $\boldsymbol{Y}_h \subset \boldsymbol{Y}$. The discrete problem reads: Find $\boldsymbol{u}_h \in \boldsymbol{u}_{0,h} + \boldsymbol{Y}_h$ such that

$$\tilde{B}(\boldsymbol{u}_h)(\boldsymbol{\omega}_h) = F(\boldsymbol{\omega}_h) \qquad \forall \boldsymbol{\omega}_h \in \boldsymbol{Y}_h. \tag{5.51}$$

Here, $\boldsymbol{u}_{0,h}$ denotes an approximation of $\boldsymbol{u}_0$ and the semi-linear form $\tilde{B}$ stands for an approximation of B. In the context of p-Navier-Stokes systems, the discrete approximation $\tilde{B}$ will include stabilization terms such as (5.4). We introduce the Lagrangians

$$L(\boldsymbol{u}; \boldsymbol{z}) := J(\boldsymbol{u}) + F(\boldsymbol{z}) - B(\boldsymbol{u})(\boldsymbol{z}) \tag{5.52}$$

$$\tilde{L}(\boldsymbol{u}_h; \boldsymbol{z}_h) := J(\boldsymbol{u}_h) + F(\boldsymbol{z}_h) - \tilde{B}(\boldsymbol{u}_h)(\boldsymbol{z}_h). \tag{5.53}$$

Let $\boldsymbol{u}$ and $\boldsymbol{u}_h$ be the solutions to (5.50) and (5.51) (the primal solutions). Hence,

$$J(\boldsymbol{u}) = L(\boldsymbol{u}; \boldsymbol{z}), \qquad J(\boldsymbol{u}_h) = \tilde{L}(\boldsymbol{u}_h, \boldsymbol{z}_h). \tag{5.54}$$

Below we proceed as in [Sch10]. We observe that the solutions $\boldsymbol{u}$ and $\boldsymbol{u}_h$ can be interpreted as the first component of the stationary points of the corresponding Lagrangians:

$$L'_{\boldsymbol{z}}(\boldsymbol{u};\boldsymbol{z})(\boldsymbol{\omega}) = F(\boldsymbol{\omega}) - B(\boldsymbol{u})(\boldsymbol{\omega}) = 0 \qquad \forall \boldsymbol{\omega} \in \boldsymbol{Y} \tag{5.55}$$

$$\tilde{L}'_{\boldsymbol{z}}(\boldsymbol{u}_h;\boldsymbol{z}_h)(\boldsymbol{\omega}_h) = F(\boldsymbol{\omega}_h) - \tilde{B}(\boldsymbol{u}_h)(\boldsymbol{\omega}_h) = 0 \qquad \forall \boldsymbol{\omega}_h \in \boldsymbol{Y}_h. \tag{5.56}$$

Let $\boldsymbol{z} \in \boldsymbol{Y}$ be the solution of the so-called dual problem:

$$B'_{\boldsymbol{u}}(\boldsymbol{u})(\boldsymbol{\omega},\boldsymbol{z}) = J'_{\boldsymbol{u}}(\boldsymbol{u})(\boldsymbol{\omega}) \qquad \forall \boldsymbol{\omega} \in \boldsymbol{Y} \qquad \Leftrightarrow \qquad L'_{\boldsymbol{u}}(\boldsymbol{u};\boldsymbol{z})(\boldsymbol{\omega}) = 0 \qquad \forall \boldsymbol{\omega} \in \boldsymbol{Y}. \tag{5.57}$$

Its Galerkin approximation reads: Find $\boldsymbol{z}_h \in \boldsymbol{Y}_h$ such that

$$\begin{aligned} & \tilde{B}'_{\boldsymbol{u}}(\boldsymbol{u}_h)(\boldsymbol{\omega}_h,\boldsymbol{z}_h) = J'_{\boldsymbol{u}}(\boldsymbol{u}_h)(\boldsymbol{\omega}_h) && \forall \boldsymbol{\omega}_h \in \boldsymbol{Y}_h \\ \Leftrightarrow \qquad & \tilde{L}'_{\boldsymbol{u}}(\boldsymbol{u}_h;\boldsymbol{z}_h)(\boldsymbol{\omega}_h) = 0 && \forall \boldsymbol{\omega}_h \in \boldsymbol{Y}_h. \end{aligned} \tag{5.58}$$

We define $\boldsymbol{e}_h^{\boldsymbol{u}} := \boldsymbol{u} - \boldsymbol{u}_h$ and $\boldsymbol{e}_h^{\boldsymbol{z}} := \boldsymbol{z} - \boldsymbol{z}_h$. The main theorem of calculus implies that

$$\begin{aligned} L(\boldsymbol{u};\boldsymbol{z}) - \tilde{L}(\boldsymbol{u}_h;\boldsymbol{z}_h) &= L(\boldsymbol{u};\boldsymbol{z}) - L(\boldsymbol{u}_h;\boldsymbol{z}_h) + (L-\tilde{L})(\boldsymbol{u}_h;\boldsymbol{z}_h) \\ &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} L(\boldsymbol{u}_h + s\boldsymbol{e}_h^{\boldsymbol{u}}; \boldsymbol{z}_h + s\boldsymbol{e}_h^{\boldsymbol{z}})\,\mathrm{d}s + (L-\tilde{L})(\boldsymbol{u}_h;\boldsymbol{z}_h) \\ &= \int_0^1 L'_{\boldsymbol{u}}(\boldsymbol{u}_h + s\boldsymbol{e}_h^{\boldsymbol{u}}; \boldsymbol{z}_h + s\boldsymbol{e}_h^{\boldsymbol{z}})(\boldsymbol{e}_h^{\boldsymbol{u}})\,\mathrm{d}s \\ &\quad + \int_0^1 L'_{\boldsymbol{z}}(\boldsymbol{u}_h + s\boldsymbol{e}_h^{\boldsymbol{u}}; \boldsymbol{z}_h + s\boldsymbol{e}_h^{\boldsymbol{z}})(\boldsymbol{e}_h^{\boldsymbol{z}})\,\mathrm{d}s + (L-\tilde{L})(\boldsymbol{u}_h;\boldsymbol{z}_h). \end{aligned}$$

For approximation of the integrals we use the trapezoidal rule,

$$\int_0^1 f(s)\,\mathrm{d}s = \frac{1}{2}f(0) + \frac{1}{2}f(1) + \frac{1}{2}\int_0^1 f''(s)s(s-1)\,\mathrm{d}s,$$

so that we arrive at

$$\begin{aligned} L(\boldsymbol{u};\boldsymbol{z}) - \tilde{L}(\boldsymbol{u}_h;\boldsymbol{z}_h) &= \frac{1}{2}L'_{\boldsymbol{u}}(\boldsymbol{u}_h;\boldsymbol{z}_h)(\boldsymbol{e}_h^{\boldsymbol{u}}) + \frac{1}{2}L'_{\boldsymbol{z}}(\boldsymbol{u}_h;\boldsymbol{z}_h)(\boldsymbol{e}_h^{\boldsymbol{z}}) \\ &\quad + \frac{1}{2}\underbrace{L'_{\boldsymbol{u}}(\boldsymbol{u};\boldsymbol{z})(\boldsymbol{e}_h^{\boldsymbol{u}})}_{=0} + \frac{1}{2}\underbrace{L'_{\boldsymbol{z}}(\boldsymbol{u};\boldsymbol{z})(\boldsymbol{e}_h^{\boldsymbol{z}})}_{=0} + (L-\tilde{L})(\boldsymbol{u}_h;\boldsymbol{z}_h) + R \\ &= \frac{1}{2}L'_{\boldsymbol{u}}(\boldsymbol{u}_h;\boldsymbol{z}_h)(\boldsymbol{e}_h^{\boldsymbol{u}}) + \frac{1}{2}L'_{\boldsymbol{z}}(\boldsymbol{u}_h;\boldsymbol{z}_h)(\boldsymbol{e}_h^{\boldsymbol{z}}) + (L-\tilde{L})(\boldsymbol{u}_h;\boldsymbol{z}_h) + R \end{aligned}$$

where for $\boldsymbol{e}_h := (\boldsymbol{e}_h^{\boldsymbol{u}}, \boldsymbol{e}_h^{\boldsymbol{z}})$ the remainder term R is given by

$$R := \frac{1}{2}\int_0^1 L'''(\boldsymbol{u}_h + s\boldsymbol{e}_h^{\boldsymbol{u}}; \boldsymbol{z}_h + s\boldsymbol{e}_h^{\boldsymbol{z}})(\boldsymbol{e}_h, \boldsymbol{e}_h, \boldsymbol{e}_h)s(s-1)\,\mathrm{d}s. \tag{5.59}$$

This simple argument needs the assumption $\boldsymbol{e}_h^{\boldsymbol{u}} \in \boldsymbol{Y}$ which requires exact representation of boundary data, i.e., $\boldsymbol{u}_0 = \boldsymbol{u}_{0,h}$. Let $\tilde{\boldsymbol{u}}_h \in \boldsymbol{Y}_h$ be arbitrary. Using (5.58), we conclude

$$\begin{aligned} L'_{\boldsymbol{u}}(\boldsymbol{u}_h;\boldsymbol{z}_h)(\boldsymbol{e}_h^{\boldsymbol{u}}) &= L'_{\boldsymbol{u}}(\boldsymbol{u}_h;\boldsymbol{z}_h)(\boldsymbol{u}-\tilde{\boldsymbol{u}}_h) + L'_{\boldsymbol{u}}(\boldsymbol{u}_h;\boldsymbol{z}_h)(\tilde{\boldsymbol{u}}_h-\boldsymbol{u}_h) \\ &= L'_{\boldsymbol{u}}(\boldsymbol{u}_h;\boldsymbol{z}_h)(\boldsymbol{u}-\tilde{\boldsymbol{u}}_h) + (L-\tilde{L})'_{\boldsymbol{u}}(\boldsymbol{u}_h;\boldsymbol{z}_h)(\tilde{\boldsymbol{u}}_h-\boldsymbol{u}_h). \end{aligned}$$

Let $\tilde{\boldsymbol{z}}_h$ be an arbitrary element of $\boldsymbol{Y}_h$. Similarly as above, in view of (5.56) we deduce

$$L'_{\boldsymbol{z}}(\boldsymbol{u}_h;\boldsymbol{z}_h)(\boldsymbol{e}_h^{\boldsymbol{z}}) = L'_{\boldsymbol{z}}(\boldsymbol{u}_h;\boldsymbol{z}_h)(\boldsymbol{z}-\tilde{\boldsymbol{z}}_h) + (L-\tilde{L})'_{\boldsymbol{z}}(\boldsymbol{u}_h;\boldsymbol{z}_h)(\tilde{\boldsymbol{z}}_h-\boldsymbol{z}_h).$$

Collecting all results above, we finally get the following error representation

$$\begin{aligned} J(\boldsymbol{u}) - J(\boldsymbol{u}_h) &= \frac{1}{2}L'_{\boldsymbol{u}}(\boldsymbol{u}_h;\boldsymbol{z}_h)(\boldsymbol{u}-\tilde{\boldsymbol{u}}_h) + \frac{1}{2}L'_{\boldsymbol{z}}(\boldsymbol{u}_h;\boldsymbol{z}_h)(\boldsymbol{z}-\tilde{\boldsymbol{z}}_h) \\ &\quad + \frac{1}{2}(L-\tilde{L})'_{\boldsymbol{u}}(\boldsymbol{u}_h;\boldsymbol{z}_h)(\tilde{\boldsymbol{u}}_h-\boldsymbol{u}_h) + \frac{1}{2}(L-\tilde{L})'_{\boldsymbol{z}}(\boldsymbol{u}_h;\boldsymbol{z}_h)(\tilde{\boldsymbol{z}}_h-\boldsymbol{z}_h) \\ &\quad + (L-\tilde{L})(\boldsymbol{u}_h;\boldsymbol{z}_h) + R. \end{aligned} \tag{5.60}$$

Introducing the primal and dual residual,

$$\varrho(\boldsymbol{u})(\boldsymbol{\omega}) := L'_{\boldsymbol{z}}(\boldsymbol{u};\boldsymbol{z})(\boldsymbol{\omega}), \qquad \varrho^*(\boldsymbol{u};\boldsymbol{z})(\boldsymbol{\omega}) := L'_{\boldsymbol{u}}(\boldsymbol{u};\boldsymbol{z})(\boldsymbol{\omega}),$$

the above identity may be rewritten as

$$J(\boldsymbol{u}) - J(\boldsymbol{u}_h) \approx \eta_h := \frac{1}{2}\varrho(\boldsymbol{u}_h)(\boldsymbol{z}-\tilde{\boldsymbol{z}}_h) + \frac{1}{2}\varrho^*(\boldsymbol{u}_h;\boldsymbol{z}_h)(\boldsymbol{u}-\tilde{\boldsymbol{u}}_h) \tag{5.61}$$

for all $\tilde{\boldsymbol{u}}_h$, $\tilde{\boldsymbol{z}}_h \in \boldsymbol{Y}_h$ provided that the remainder term R and the additional terms involving $(L-\tilde{L})$ can be neglected. Note that in many cases the remainder term R is of higher order in the errors $\boldsymbol{u}-\boldsymbol{u}_h$ and $\boldsymbol{z}-\boldsymbol{z}_h$. If a priori information on $\boldsymbol{u}-\boldsymbol{u}_h$ and $\boldsymbol{z}-\boldsymbol{z}_h$ is known, then the neglect of R can often be justified. Concerning the practical realization of the DWR method, we study Algorithm 5.2 and the subsequent discussion. For details we refer Bangerth/Rannacher [BR03] and the literature cited therein.

Practical aspects and adaptive mesh refinement: Below we deal with the practical evaluation of η_h in (5.61) and we discuss adaptive mesh refinement within the DWR framework. Since $\tilde{\boldsymbol{u}}_h$ and $\tilde{\boldsymbol{z}}_h$ are arbitrary, the weights $\boldsymbol{u}-\tilde{\boldsymbol{u}}_h$ and $\boldsymbol{z}-\tilde{\boldsymbol{z}}_h$ appearing in (5.61) basically represent interpolation errors. Because the weights $\boldsymbol{u}-\tilde{\boldsymbol{u}}_h$ and $\boldsymbol{z}-\tilde{\boldsymbol{z}}_h$ depend on the unknown primal and dual solution, they cannot be evaluated numerically and in case of $\mathbb{Q}_1/\mathbb{Q}_1$ elements they are replaced by $\boldsymbol{i}_{2h}^{(2)}\boldsymbol{u}_h - \boldsymbol{u}_h$ and $\boldsymbol{i}_{2h}^{(2)}\boldsymbol{z}_h - \boldsymbol{z}_h$. Here, the operator $i_{2h}^{(2)} : X_{h,1} \to X_{2h,2}$ denotes the nodal interpolant into the space of d-quadratic finite elements. (It can be constructed easily since the underlying mesh exhibits patch structure.) Details on this approach can be found in Becker/Rannacher [BR01]. The presented error estimator η_h also enables adaptive mesh refinement which plays an important role when the accuracy should be improved efficiently. The aim of adaptivity is to compute the functional value $J(\boldsymbol{u})$ up to a prescribed accuracy $TOL > 0$ and to refine the meshes only locally in order to get along with the available random access memory. In order to achieve an adaptive method, the information of η_h is localized to element-wise contributions via the representation $\eta_h = \sum_{K\in\mathbb{T}_h} \eta_K$. The quantities η_K are called local error indicators. Note that different representations of η_K are possible: For instance, the error estimator η_h can simply be splitted into its cell-wise contributions, or cell-wise integration by parts can be applied to the cell-wise contributions so that the resulting local error indicators η_K involve strong cell-wise residuals of the equation and jumps of the discrete solution over faces of elements (see [BR03, Ran09]). For an extensive discussion on localization we refer to [Sch10]. Adaptive mesh refinement can be carried out on the basis of standard strategies such as: successive “error balancing” or “fixed fraction” strategy. In case of the first strategy we balance the values of η_K until we achieve $\eta_K \approx TOL/(\#\mathbb{T}_h)$, while in case of the second strategy we refine a certain fraction, say 20 – 30 %, of the elements with the largest value of η_K. An alternative approach, which was used for the subsequent computations, is described in Richter [Ric05].

Algorithm 5.2. Adaptive finite element method (AFEM)

1: Choose a tolerance $TOL > 0$ and an initial discretization $\mathbb{T}_{h_0}$.
2: Set $L := 0$.
3: On the grid $\mathbb{T}_{h_L}$ determine the solution $\boldsymbol{u}_{h_L}$ to the discrete primal problem (5.51) and compute the solution $\boldsymbol{z}_{h_L}$ to the discrete dual problem (5.58).
4: Compute the local error indicators $\eta_K = \eta_K(\boldsymbol{u}_{h_L}, \boldsymbol{z}_{h_L})$ for all $K \in \mathbb{T}_{h_L}$ on the basis of (5.61) and take into account the following conventions (see [Ran09]):

   - Neglect the higher-order remainder term R and all terms involving $(L - \tilde{L})$ in (5.60).
   - Approximate the weights in (5.61) by higher-order interpolation

     $$(\boldsymbol{z} - \tilde{\boldsymbol{z}}_{h_L})\Big|_K \approx \Big(\boldsymbol{i}^{(2)}_{2h_L}\boldsymbol{z}_{h_L} - \boldsymbol{z}_{h_L}\Big)\Big|_K,$$

     where $\boldsymbol{i}^{(2)}_{2h_L}\boldsymbol{z}_{h_L}$ denotes the d-quadratic nodal interpolant on patches of the current mesh $\mathbb{T}_{h_L}$ applied to the computed d-linear approximation $\boldsymbol{z}_{h_L}$. Use a similar replacement for the weights involving $\boldsymbol{u}$.

5: If $\eta_h = \sum_{K\in\mathbb{T}_{h_L}} \eta_K \leq TOL$ then STOP.
6: For mesh adaptation choose a subset of $\mathbb{T}_{h_L}$ on the basis of the local error indicators η_K by an appropriate strategy such as successive “error balancing” or “fixed fraction” strategy. Mark the elements for refinement.
7: Perform a refinement of $\mathbb{T}_{h_L}$ using bisection of edges. Each marked element K is subdivided in 2^d subelements. Since quadrilateral meshes are involved, the refinement process leads to hanging nodes. The degrees of freedom, that correspond to hanging nodes, are eliminated using interpolation between neighboring degrees of freedom (see, e.g., [AO00]).
8: Increment L and go to step 3.

Numerical example: As a simple test we considered the p-Laplace system (3.44) provided with $\Omega := (0,1)^2$, $\boldsymbol{f} \equiv (2,0)^\mathsf{T}$, $\boldsymbol{v}_D \equiv \boldsymbol{0}$, $\varepsilon = 10^{-5}$, and $p = 1.1$. We applied the dual weighted residual (DWR) method, which serves for two purposes: the quantitative assessment of the discretization error and the adaptive refinement of the underlying meshes. The quality of the a posteriori error estimation was measured by the effectivity index I_{eff},

$$I_{\text{eff}} := \frac{J(\boldsymbol{u}) - J(\boldsymbol{u}_h)}{\eta_h},$$

where $J(\boldsymbol{u}) - J(\boldsymbol{u}_h)$ is the true error and η_h is the estimated error. In Table 5.7 we chose the output functional $J(\boldsymbol{u}) = 1/p' \int_\Omega |\nabla \boldsymbol{u}|^p \, \mathrm{d}\boldsymbol{x}$. We used the reference value $J(\boldsymbol{u}) = 0.082979144$ which was obtained by Richardson extrapolation of approximations $\{J(\tilde{\boldsymbol{u}}_{2^j H})\}_{j=0}^4$. Here the functions $\tilde{\boldsymbol{u}}_h$ denote the d-quadratic FE solutions computed on uniformly refined meshes and H corresponds to the mesh with 1048576 elements. In view of Table 5.7(a) we observe that the estimated errors agree with the actual errors very well and, in particular, $I_{\text{eff}} \approx 1$ on finer grids. Apparently local mesh refinement is more efficient than uniform mesh refinement, even though only marginally for this example. More significant differences of the two refinement strategies usually occur when the involved quantities are not smooth as exemplarily depicted by the subsequent experiment.

Table 5.7. A posteriori estimation of the energy

(a) Local mesh refinement

#cells	η_h	$J(\boldsymbol{u}) - J(\boldsymbol{u}_h)$	I_{eff}
64	2.15e-02	4.79e-02	2.23
160	1.41e-02	2.53e-02	1.80
640	5.70e-03	7.21e-03	1.26
1600	1.78e-03	1.98e-03	1.11
4480	5.63e-04	5.91e-04	1.05
14656	1.65e-04	1.68e-04	1.02
52672	4.57e-05	4.64e-05	1.02
200464	1.21e-05	1.22e-05	1.01

(b) Uniform mesh refinement

#cells	η_h	$J(\boldsymbol{u}) - J(\boldsymbol{u}_h)$	I_{eff}
64	2.15e-02	4.79e-02	2.23
256	1.11e-02	1.59e-02	1.44
1024	3.79e-03	4.36e-03	1.15
4096	1.07e-03	1.12e-03	1.05
16384	2.78e-04	2.83e-04	1.01
65536	7.05e-05	7.08e-05	1.00
262144	1.77e-05	1.77e-05	1.00

Remark 5.8. For $\varepsilon = 0$ let $\boldsymbol{u}$ be the primal weak solution to (3.44), and let $J(\boldsymbol{u}) := \frac{1}{p'} \int_\Omega |\nabla \boldsymbol{u}|^p \, \mathrm{d}\boldsymbol{x}$. If $\varepsilon = 0$, then obviously $\boldsymbol{z} = \boldsymbol{u}$ is a solution to the dual problem (5.57). In the case under consideration, the dual equation (5.57) formally takes the form

$$\int_\Omega |\nabla \boldsymbol{u}|^{p-2} \nabla \boldsymbol{\omega} : \nabla \boldsymbol{z} \, \mathrm{d}\boldsymbol{x} + (p-2) \int_\Omega |\nabla \boldsymbol{u}|^{p-4} (\nabla \boldsymbol{u} : \nabla \boldsymbol{\omega})(\nabla \boldsymbol{u} : \nabla \boldsymbol{z}) \, \mathrm{d}\boldsymbol{x}$$
$$\equiv B'_{\boldsymbol{u}}(\boldsymbol{u})(\boldsymbol{\omega}, \boldsymbol{z}) = J'_{\boldsymbol{u}}(\boldsymbol{u})(\boldsymbol{\omega}) \equiv (p-1) \int_\Omega |\nabla \boldsymbol{u}|^{p-2} \nabla \boldsymbol{u} : \nabla \boldsymbol{\omega} \, \mathrm{d}\boldsymbol{x} \qquad \forall \boldsymbol{\omega} \in \boldsymbol{Y} \equiv \boldsymbol{W}_0^{1,p}(\Omega).$$

Below we show that for $\varepsilon = 0$ the solution $\boldsymbol{z} = \boldsymbol{u}$ is uniquely determined within the class $\{\boldsymbol{v} \in \boldsymbol{Y};\, B'_{\boldsymbol{u}}(\boldsymbol{u})(\boldsymbol{\omega}, \boldsymbol{v}) < \infty \text{ for all } \boldsymbol{\omega} \in \boldsymbol{Y}\}$. To this end, we suppose that there is a further solution $\tilde{\boldsymbol{z}}$ with $\tilde{\boldsymbol{z}} \neq \boldsymbol{z}$. This implies that $B'_{\boldsymbol{u}}(\boldsymbol{u})(\boldsymbol{\omega}, \boldsymbol{z} - \tilde{\boldsymbol{z}}) = 0$ for all $\boldsymbol{\omega} \in \boldsymbol{Y}$. Using the latter identity with $\boldsymbol{\omega} := \boldsymbol{z} - \tilde{\boldsymbol{z}}$, Lemma 2.4 with $\varepsilon = 0$, for $p < 2$ we conclude that

$$\begin{aligned}
0 &= B'_{\boldsymbol{u}}(\boldsymbol{u})(\boldsymbol{u} - \tilde{\boldsymbol{z}}, \boldsymbol{u} - \tilde{\boldsymbol{z}}) \\
&\equiv \int_\Omega |\nabla \boldsymbol{u}|^{p-2} |\nabla(\boldsymbol{u} - \tilde{\boldsymbol{z}})|^2 \,\mathrm{d}\boldsymbol{x} + (p-2) \int_\Omega |\nabla \boldsymbol{u}|^{p-4} (\nabla \boldsymbol{u} : \nabla(\boldsymbol{u} - \tilde{\boldsymbol{z}}))(\nabla \boldsymbol{u} : \nabla(\boldsymbol{u} - \tilde{\boldsymbol{z}})) \,\mathrm{d}\boldsymbol{x} \\
&\geq (p-1) \int_\Omega |\nabla \boldsymbol{u}|^{p-2} |\nabla(\boldsymbol{u} - \tilde{\boldsymbol{z}})|^2 \,\mathrm{d}\boldsymbol{x} \geq (p-1) \int_\Omega (|\nabla \boldsymbol{u}| + |\nabla \tilde{\boldsymbol{z}}|)^{p-2} |\nabla(\boldsymbol{u} - \tilde{\boldsymbol{z}})|^2 \,\mathrm{d}\boldsymbol{x} \\
&\sim (p-1) \int_\Omega (\boldsymbol{\mathcal{S}}(\nabla \boldsymbol{u}) - \boldsymbol{\mathcal{S}}(\nabla \tilde{\boldsymbol{z}})) : (\nabla \boldsymbol{u} - \nabla \tilde{\boldsymbol{z}}) \,\mathrm{d}\boldsymbol{x}.
\end{aligned}$$

Since $\boldsymbol{\mathcal{S}}$ is monotone and $\tilde{\boldsymbol{z}} \neq \boldsymbol{u}$, the last term is strictly positive. This yields the desired contradiction. As a result, the solution $\boldsymbol{z} = \boldsymbol{u}$ is uniquely determined.

In Table 5.8 we estimated the error at some point $\boldsymbol{x}_0 \in \Omega$. Since the point functional $J(\boldsymbol{u}) := u_1(\boldsymbol{x}_0)$ is not well-defined on the trial space $\boldsymbol{Y} := \boldsymbol{W}_0^{1,p}(\Omega)$, for $B_\delta := \{\boldsymbol{x} \in \mathbb{R}^2;\, |\boldsymbol{x} - \boldsymbol{x}_0| < \delta\}$ the regularized functional $J_\delta(\boldsymbol{u}) := |B_\delta|^{-1} \int_{B_\delta} u_1 \,\mathrm{d}\boldsymbol{x}$ is employed within the DWR method. It is well-known that for small δ it holds $J_\delta(\boldsymbol{u}) = u_1(\boldsymbol{x}_0) + \mathcal{O}(\delta^2)$. Here, for $\boldsymbol{x}_0 := (0.8, 0.8)$ the reference value $J(\boldsymbol{u}) = 0.082979144$ was used. It was obtained analogously to the previous experiment by extrapolation of approximations computed on uniformly refined meshes. Considering Table 5.8, we observe that the estimated errors agree with the actual errors very well. In particular we realize that $I_{\text{eff}} \approx 1$ on finer grids.

Table 5.8. A posteriori estimation of a point value

(a) Local mesh refinement

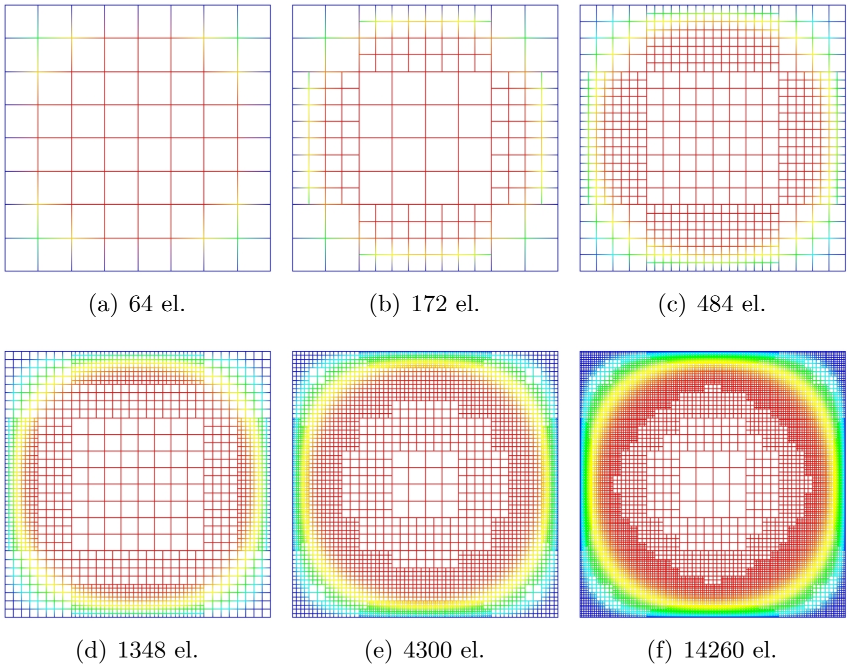
#cells	η_h	$J(\boldsymbol{u}) - J(\boldsymbol{u}_h)$	I_{eff}
64	1.58e-02	4.20e-02	2.66
172	9.38e-03	1.96e-02	2.09
484	4.95e-03	5.80e-03	1.17
1348	1.66e-03	1.79e-03	1.08
4300	4.91e-04	5.00e-04	1.02
14260	1.40e-04	1.47e-04	1.05
50788	3.94e-05	3.95e-05	1.00

(b) Uniform mesh refinement

#cells	η_h	$J(\boldsymbol{u}) - J(\boldsymbol{u}_h)$	I_{eff}
64	1.58e-02	4.20e-02	2.66
256	8.45e-03	1.70e-02	2.01
1024	4.22e-03	4.78e-03	1.13
4096	1.22e-03	1.25e-03	1.03
16384	3.12e-04	3.02e-04	0.97
65536	7.64e-05	7.90e-05	1.03

Lemma 5.9. *For $p \in (1, 2]$ let us consider the p-Laplace problem (3.44), i.e., let $B = \tilde{B}$ be given by $B(\boldsymbol{u})(\boldsymbol{\omega}) := \int_\Omega (\varepsilon^2 + |\nabla \boldsymbol{u}|^2)^{\frac{p-2}{2}} \nabla \boldsymbol{u} : \nabla \boldsymbol{\omega} \,\mathrm{d}\boldsymbol{x}$. If for some constant $C > 0$ the functional J satisfies $J'_{\boldsymbol{u}}(\boldsymbol{u}_h)(\boldsymbol{\omega}_h) \leq C \|\boldsymbol{\omega}_h\|_{1,p}$ for all $\boldsymbol{\omega}_h \in \boldsymbol{\mathcal{X}}_h^p$, then for each $h > 0$ there exists a unique solution $\boldsymbol{z}_h \in \boldsymbol{Y}_h \equiv \boldsymbol{\mathcal{X}}_h^p$ to problem (5.58) satisfying*

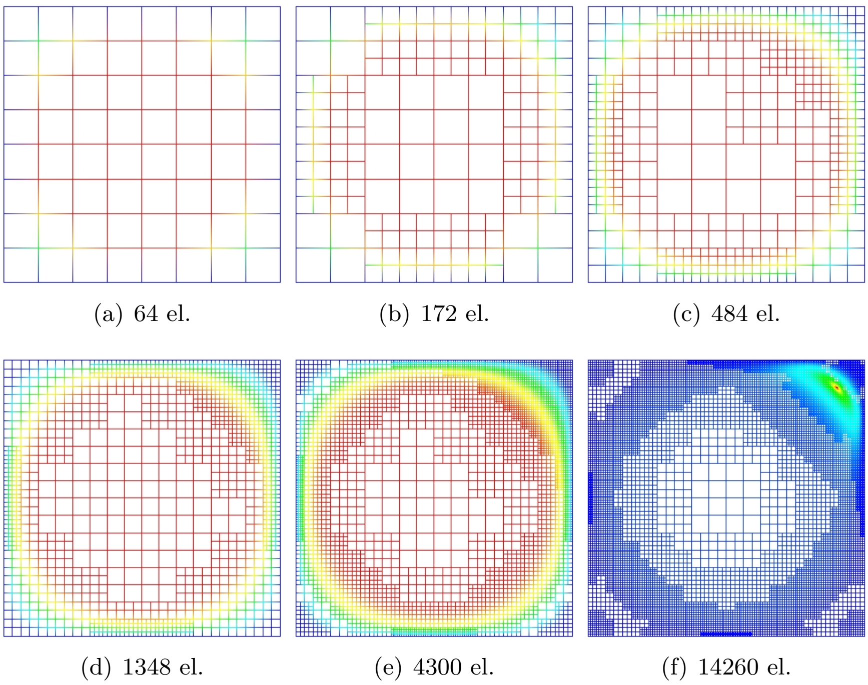
$$\|\boldsymbol{z}_h\|_{1,p} \leq C = C(p, \varepsilon_0, \Omega, \boldsymbol{f}). \tag{5.62}$$



(a) 64 el. (b) 172 el. (c) 484 el.

(d) 1348 el. (e) 4300 el. (f) 14260 el.

Figure 5.6. Adaptively refined meshes for the computation of $J(\boldsymbol{u}) = 1/p' \int_\Omega |\nabla \boldsymbol{u}|^p \,\mathrm{d}\boldsymbol{x}$: primal solutions



(a) 64 el. (b) 172 el. (c) 484 el.

(d) 1348 el. (e) 4300 el. (f) 14260 el.

Figure 5.7. Adaptively refined meshes for the computation of $\boldsymbol{u}(0.8, 0.8)$: primal solutions (a)–(e) and dual solution (f)

Proof. First of all we prove that if there exists a solution $\boldsymbol{z}_h$ to problem (5.58), then $\boldsymbol{z}_h$ is determined uniquely. For this we assume that $\boldsymbol{z}_h^1$ and $\boldsymbol{z}_h^2$ are two functions satisfying (5.58). Setting $\boldsymbol{\xi}_h := \boldsymbol{z}_h^1 - \boldsymbol{z}_h^2$, we observe that

$$B'_{\boldsymbol{u}}(\boldsymbol{u}_h)(\boldsymbol{\omega}_h, \boldsymbol{\xi}_h) = 0 \qquad \forall \boldsymbol{\omega}_h \in \boldsymbol{\mathcal{X}}_h^p.$$

Using Hölder's inequality with $\frac{2-p}{2} + \frac{p}{2} = 1$ and taking into account $p < 2$, we conclude

$$\begin{aligned}
(p-1)\|\nabla \boldsymbol{w}_h\|_p^2 &= (p-1)\Big(\int_\Omega \Big(\varepsilon^2 + |\nabla \boldsymbol{u}_h|^2\Big)^{\frac{(2-p)}{4}p}\Big(\varepsilon^2 + |\nabla \boldsymbol{u}_h|^2\Big)^{\frac{(p-2)}{4}p}|\nabla \boldsymbol{w}_h|^p \,\mathrm{d}\boldsymbol{x}\Big)^{\frac{2}{p}} \\
&\le (p-1)\|\varepsilon + |\nabla \boldsymbol{u}_h|\|_p^{2-p} \int_\Omega \Big(\varepsilon^2 + |\nabla \boldsymbol{u}_h|^2\Big)^{\frac{p-2}{2}} |\nabla \boldsymbol{w}_h|^2 \,\mathrm{d}\boldsymbol{x} \\
&\le \|\varepsilon + |\nabla \boldsymbol{u}_h|\|_p^{2-p} \Big[\int_\Omega \Big(\varepsilon^2 + |\nabla \boldsymbol{u}_h|^2\Big)^{\frac{p-2}{2}} |\nabla \boldsymbol{w}_h|^2 \,\mathrm{d}\boldsymbol{x} \\
&\qquad + (p-2)\int_\Omega \Big(\varepsilon^2 + |\nabla \boldsymbol{u}_h|^2\Big)^{\frac{p-4}{2}} |\nabla \boldsymbol{u}_h|^2 |\nabla \boldsymbol{w}_h|^2 \,\mathrm{d}\boldsymbol{x}\Big] \\
&\le \|\varepsilon + |\nabla \boldsymbol{u}_h|\|_p^{2-p} B'_{\boldsymbol{u}}(\boldsymbol{u}_h)(\boldsymbol{w}_h, \boldsymbol{w}_h)
\end{aligned} \tag{5.63}$$

for all $\boldsymbol{w}_h \in \boldsymbol{\mathcal{X}}_h^p$. We recall that $\boldsymbol{u}_h$ is uniformly bounded in $\boldsymbol{W}^{1,p}(\Omega)$ by a constant only depending on the data. Hence, we deduce that there exists a constant $c = c(p, \varepsilon_0, \Omega, \boldsymbol{f})$:

$$0 = B'_{\boldsymbol{u}}(\boldsymbol{u}_h)(\boldsymbol{\xi}_h, \boldsymbol{\xi}_h) \ge c\|\nabla \boldsymbol{\xi}_h\|_p^2. \tag{5.64}$$

We infer that $\boldsymbol{\xi}_h \equiv \boldsymbol{0}$ and, hence, $\boldsymbol{z}_h^1 = \boldsymbol{z}_h^2$. Since system (5.58) is linear and the space $\boldsymbol{\mathcal{X}}_h^p$ is finite dimensional, we can conclude that there exists a solution $\boldsymbol{z}_h$ to system (5.58). In order to show (5.62), we test (5.58) with $\boldsymbol{\omega}_h := \boldsymbol{z}_h$ and we apply (5.63) so that

$$C\|\boldsymbol{z}_h\|_{1,p} \ge J'_{\boldsymbol{u}}(\boldsymbol{u}_h)(\boldsymbol{z}_h) = B'_{\boldsymbol{u}}(\boldsymbol{u}_h)(\boldsymbol{z}_h, \boldsymbol{z}_h) \ge (p-1)\|\varepsilon + |\nabla \boldsymbol{u}_h|\|_p^{p-2}\|\nabla \boldsymbol{z}_h\|_p^2.$$

This completes the proof. □

For instance, the assumptions of Lemma 5.9 are satisfied for the functional $J(\boldsymbol{u}) = 1/p' \int_\Omega |\nabla \boldsymbol{u}|^p \,\mathrm{d}\boldsymbol{x}$ since the primal solution $\boldsymbol{u}_h$ of the discrete p-Laplace problem is uniformly bounded in $\boldsymbol{W}^{1,p}(\Omega)$ by a constant only depending on the data. Discrete dual solutions $\boldsymbol{z}_h$ usually exist. In contrast the well-posedness of the continuous dual problem (5.57) cannot be established in general, even for the simple p-Laplace equation.

5.7 Application to the p-Navier-Stokes equations

In this section we apply the well-known DWR method to the p-Navier-Stokes equations (2.15). As an illustrative example, we compute the drag coefficient of an obstacle immersed into a fluid of class (2.10). We consider the planar flow around an obstacle between two steady parallel plates driven by an inflow profile for the velocity. We assume that Ω is a simple channel whose boundary consists of a solid part Γ_s (upper and lower edge), of an inflow boundary Γ_i (left), and of a free outflow boundary S (right), see Figure 5.8(b). In addition we suppose that an obstacle with surface Γ_o is immersed into the fluid. On $\Gamma := \Gamma_s \cup \Gamma_o \cup \Gamma_i$ we prescribe boundary conditions of Dirichlet type: $\boldsymbol{v}|_{\Gamma_s \cup \Gamma_o} = 0$ and $\boldsymbol{v}|_{\Gamma_i} = \boldsymbol{v}_D$. Here, $\boldsymbol{v}_D$ is given by the trace of a globally defined function $\boldsymbol{v}_0 \in \boldsymbol{W}^{1,p}(\Omega)$. On S we prescribe the natural outflow boundary condition

$$-\frac{\mu(|\boldsymbol{D}\boldsymbol{v}|^2)}{2}\partial_{\boldsymbol{n}}\boldsymbol{v} + \pi\boldsymbol{n} = \boldsymbol{0} \qquad \text{on } S. \tag{5.65}$$

As above, $\boldsymbol{n}$ denotes the outer normal on $\partial\Omega$, and $\partial_{\boldsymbol{n}}\boldsymbol{v}$ is the corresponding directional derivative. Note that $\partial_{\boldsymbol{n}}\boldsymbol{v} = (\boldsymbol{n}\cdot\nabla)\boldsymbol{v} = [\nabla\boldsymbol{v}]\boldsymbol{n}$. If the boundary condition (5.65) is prescribed, then the pressure is uniquely determined without an additional constraint on the pressure mean value. More details can be found in [HRT96], Remarks 5.6 and 5.9.

Remark 5.9. Let $\boldsymbol{t}$ be the tangential vector on $\partial\Omega$ and let $v_t = \boldsymbol{v}\cdot\boldsymbol{t}$ be the corresponding tangential component of $\boldsymbol{v}$. Multiplying (5.65) by $\boldsymbol{n}$, integrating the result over S, and observing $\partial_{\boldsymbol{n}} v_n = -\partial_{\boldsymbol{t}} v_t$ due to $\nabla\cdot\boldsymbol{v} = 0$, we obtain the condition

$$\int_S \pi \,\mathrm{d}o = -\frac{1}{2}\int_S \mu(|\boldsymbol{D}\boldsymbol{v}|^2)\partial_{\boldsymbol{t}} v_t \,\mathrm{d}o. \tag{5.66}$$

For instance, for unidirectional flows (Poiseuille flows) the stream lines are orthogonal to the outflow boundary S. In this case, the tangential component v_t is identically zero. As a result, we conclude that $\partial_{\boldsymbol{t}} v_t = 0$ and, hence, $\int_S \pi \,\mathrm{d}o = 0$.

The variational formulation **(P5)** has to be adjusted to the current flow configuration. Let the semi-linear form $A(\cdot)(\cdot)$ be defined by (3.23). As above, $\boldsymbol{\mathcal{X}}^p_\Gamma$ and $\mathcal{Q}^p_\Gamma$ denote the velocity and pressure space, and they are defined as in (5.42). Then the weak formulation reads: Find a velocity $\boldsymbol{v}$ and pressure π, $\boldsymbol{u} \equiv (\boldsymbol{v}, \pi) \in (\boldsymbol{v}_0 + \boldsymbol{\mathcal{X}}^p_\Gamma) \times \mathcal{Q}^p_\Gamma$, such that

$$A(\boldsymbol{u})(\boldsymbol{\omega}) = (\boldsymbol{f}, \boldsymbol{w})_\Omega + \left(\frac{\mu(|\boldsymbol{D}\boldsymbol{v}|^2)}{2}[\nabla\boldsymbol{v}]^\top\boldsymbol{n}, \boldsymbol{w}\right)_S \qquad \forall\boldsymbol{\omega} \equiv (\boldsymbol{w}, q) \in \boldsymbol{\mathcal{X}}^p_\Gamma \times \mathcal{Q}^p_\Gamma. \tag{5.67}$$

Similarly to Remark 5.6, we can derive the so-called “do-nothing” boundary condition (5.65) by applying integration by parts to (5.67).

Drag computation: Below we introduce the drag and lift force following the literature Giles et al. [GLLS97]. For $\boldsymbol{u} = (\boldsymbol{v}, \pi)$ the weighted boundary flux $J_\psi(\boldsymbol{u})$ is defined by

$$J_\psi(\boldsymbol{u}) := \sum_{i,j=1}^{d}\int_{\partial\Omega} n_i(\mathcal{S}_{ij}(\boldsymbol{D}\boldsymbol{v}) - \pi\delta_{ij})\psi_j \,\mathrm{d}o. \tag{5.68}$$

If $\boldsymbol{\psi}$ is a unit vector parallel to the direction of the flow, then $J_{\mathrm{drag}} := J_\psi$ is called the drag on $\partial\Omega$. If $\boldsymbol{\psi}$ is a unit vector perpendicular to the direction of the flow, then $J_{\mathrm{lift}} := J_\psi$ is referred to as the lift on $\partial\Omega$. If only a part Γ_o of the boundary $\partial\Omega$ is of concern, then $\boldsymbol{\psi}$ can be taken to have its support in Γ_o. As above, Γ_o is a closed surface which represents the boundary of an object immersed into the fluid. The space of all $\boldsymbol{w} \in \boldsymbol{W}^{1,p}(\Omega)$, which satisfy the Dirichlet boundary condition $\boldsymbol{w}|_{\partial\Omega} = \boldsymbol{\psi}$, is denoted by $\boldsymbol{W}^{1,p}_{\boldsymbol{\psi}}(\Omega)$. Let $\boldsymbol{u} = (\boldsymbol{v}, \pi)$ be a weak solution to (2.15) that is smooth enough in order to be a classical solution. Then it follows from integration by parts that for any $\boldsymbol{\omega} = (\boldsymbol{w}, q) \in \boldsymbol{W}^{1,p}_{\boldsymbol{\psi}}(\Omega) \times L^{p'}(\Omega)$

$$A(\boldsymbol{u})(\boldsymbol{\omega}) - (\boldsymbol{f}, \boldsymbol{w})_\Omega = \sum_{j=1}^{d} \int_\Omega \underbrace{\Big(-\sum_{i=1}^{d} \partial_i \mathcal{S}_{ij}(\boldsymbol{D}\boldsymbol{v}) + \partial_j \pi + \sum_{i=1}^{d} v_i \partial_i v_j - f_j \Big)}_{=0} w_j \,\mathrm{d}\boldsymbol{x}$$
$$+ \sum_{j=1}^{d} \int_{\partial\Omega} \sum_{i=1}^{d} n_i \mathcal{S}_{ij}(\boldsymbol{D}\boldsymbol{v}) w_j - n_j \pi w_j \,\mathrm{d}o = J_{\boldsymbol{\psi}}(\boldsymbol{u}).$$

Clearly, the left-hand side is independent of the choice of $\boldsymbol{\omega} \in \boldsymbol{W}^{1,p}_{\boldsymbol{\psi}}(\Omega) \times L^{p'}(\Omega)$. Here, we choose the particular test functions $\boldsymbol{\omega}_{\text{drag}} = (\boldsymbol{w}_{\text{drag}}, 0)$ and $\boldsymbol{\omega}_{\text{lift}} = (\boldsymbol{w}_{\text{lift}}, 0)$ which fulfill $\boldsymbol{w}_{\text{drag}}|_{\partial\Omega} = \boldsymbol{\psi}_{\text{drag}}$ and $\boldsymbol{w}_{\text{lift}}|_{\partial\Omega} = \boldsymbol{\psi}_{\text{lift}}$, where $\boldsymbol{\psi}_{\text{drag}}|_{\Gamma_o} = (1,0)^\mathsf{T}$, $\boldsymbol{\psi}_{\text{drag}}|_{\partial\Omega\setminus\Gamma_o} = 0$, and $\boldsymbol{\psi}_{\text{lift}}|_{\Gamma_o} = (0,1)^\mathsf{T}$, $\boldsymbol{\psi}_{\text{lift}}|_{\partial\Omega\setminus\Gamma_o} = 0$. As a result, we obtain the identities

$$J_{\text{drag}}(\boldsymbol{u}) = A(\boldsymbol{u})(\boldsymbol{\omega}_{\text{drag}}) - (\boldsymbol{f}, \boldsymbol{w}_{\text{drag}})_\Omega, \qquad J_{\text{lift}}(\boldsymbol{u}) = A(\boldsymbol{u})(\boldsymbol{\omega}_{\text{lift}}) - (\boldsymbol{f}, \boldsymbol{w}_{\text{lift}})_\Omega. \tag{5.69}$$

However, such identities are not true on the discrete level. The finite element space $\boldsymbol{X}_{h,\boldsymbol{\psi}}$ consists of all $\boldsymbol{w}_h \in \boldsymbol{X}_h$ with $\boldsymbol{w}_h|_{\partial\Omega} = \boldsymbol{\psi}$ where $\boldsymbol{\psi}$ is given by $\boldsymbol{\psi} = \boldsymbol{g}_h|_{\partial\Omega}$ for some $\boldsymbol{g}_h \in \boldsymbol{X}_h$. Motivated by (5.69), we define approximations $J^h_{\text{drag}}(\boldsymbol{u}_h)$ to $J_{\text{drag}}(\boldsymbol{u})$ by

$$J^h_{\text{drag}}(\boldsymbol{u}_h) := A\Big(\boldsymbol{u}_h\Big)\Big((\boldsymbol{w}_{\text{drag},h}, 0)\Big) - (\boldsymbol{f}, \boldsymbol{w}_{\text{drag},h})_\Omega, \qquad \boldsymbol{w}_{\text{drag},h} \in \boldsymbol{X}_{h,\boldsymbol{\psi}_{\text{drag}}}.$$

As usual, $\boldsymbol{u}_h = (\boldsymbol{v}_h, \pi_h)$ denotes the finite element solution. If the boundary is sufficiently smooth and if the used FE spaces are based on d-simplices and are inf-sup stable, then Giles et al. showed in [GLLS97] in the case $p = 2$ that the order of convergence for $J^h_{\text{drag}}(\boldsymbol{u}_h) - J_{\text{drag}}(\boldsymbol{u})$ amounts to[1] $2r$. However, for the direct approximation $J_{\text{drag}}(\boldsymbol{u}_h)$ the order of convergence is typically only r (see [GLLS97]). To sum up, for $p = 2$ there hold

$$|J_{\text{drag}}(\boldsymbol{u}_h) - J_{\text{drag}}(\boldsymbol{u})| = \mathcal{O}(h^r), \qquad |J^h_{\text{drag}}(\boldsymbol{u}_h) - J_{\text{drag}}(\boldsymbol{u})| = \mathcal{O}(h^{2r}). \tag{5.70}$$

Similar results hold true for J_{lift} and J^h_{lift}.

The DWR method applied to p-Navier-Stokes systems: As above, let $A(\cdot)(\cdot)$ be defined by (3.23). The p-Navier-Stokes problem (5.67) can be expressed equivalently by (5.50) if the product space $\boldsymbol{Y} := \boldsymbol{\mathcal{X}}^p_\Gamma \times \mathcal{Q}^p_\Gamma$ is used, and for $\boldsymbol{u} \equiv (\boldsymbol{v}, \pi)$, $\boldsymbol{\omega} \equiv (\boldsymbol{w}, q) \in \boldsymbol{Y}$ the right-hand side is given by $F(\boldsymbol{\omega}) := (\boldsymbol{f}, \boldsymbol{w})_\Omega$ and the semi-linear form B is defined by

$$B(\boldsymbol{u})(\boldsymbol{\omega}) := A(\boldsymbol{u})(\boldsymbol{\omega}) - \left(\frac{\mu(|\boldsymbol{D}\boldsymbol{v}|^2)}{2}[\nabla\boldsymbol{v}]^\mathsf{T}\boldsymbol{n}, \boldsymbol{w}\right)_S. \tag{5.71}$$

[1] Here, the variable r denotes the underlying polynomial degree of the velocity space.

We discretize the p-Navier-Stokes system (5.67) with equal-order $\mathbb{Q}_1/\mathbb{Q}_1$ finite elements. This discretization requires stabilization of the finite element equations. The stabilized discrete problem reads: Find $\boldsymbol{u}_h \equiv (\boldsymbol{v}_h, \pi_h) \in (\boldsymbol{v}_{0,h} + \boldsymbol{\mathcal{X}}^p_{\Gamma;h}) \times \mathcal{Q}^p_{\Gamma;h}$ such that

$$B(\boldsymbol{u}_h)(\boldsymbol{\omega}_h) + S_h(\boldsymbol{u}_h)(\boldsymbol{\omega}_h) = (\boldsymbol{f}, \boldsymbol{w}_h)_\Omega \qquad \forall \boldsymbol{\omega}_h \equiv (\boldsymbol{w}_h, q_h) \in \boldsymbol{\mathcal{X}}^p_{\Gamma;h} \times \mathcal{Q}^p_{\Gamma;h}. \tag{5.72}$$

The used stabilization term S_h is chosen as in (3.54). The semi-linear form $\tilde{B}$, that was introduced in (5.51), is given by the left-hand side of (5.72). We only consider the popular Carreau-type model (2.10) & (2.11b). The directional derivative $A'(\boldsymbol{u})(\boldsymbol{\omega}, \boldsymbol{z})$ is then formally given by (3.53). As already mentioned, the directional derivative (3.53) is not well-defined for $\varepsilon = 0$ and $p < 2$ since the functions $\boldsymbol{u}$, $\boldsymbol{\omega}$, $\boldsymbol{z}$ naturally belong to the space $\boldsymbol{W}^{1,p}(\Omega) \times L^{p'}$ only. If additional regularity on the primal solution $\boldsymbol{u}$ such as $\boldsymbol{u} \in \boldsymbol{W}^{1,\infty}(\Omega)$ is not available, then the existence of a unique solution $\boldsymbol{z}$ to the dual problem (5.57) is not ensured. The additional terms in (5.60) caused by stabilization are given by

$$(L - \tilde{L})(\boldsymbol{u}_h; \boldsymbol{z}_h) = S_h(\boldsymbol{u}_h)(\boldsymbol{z}_h), \quad \frac{1}{2}(L - \tilde{L})'_{\boldsymbol{z}}(\boldsymbol{u}_h; \boldsymbol{z}_h)(\tilde{\boldsymbol{z}}_h - \boldsymbol{z}_h) = \frac{1}{2} S'_h(\boldsymbol{u}_h)(\tilde{\boldsymbol{z}}_h - \boldsymbol{z}_h), \quad \dots .$$

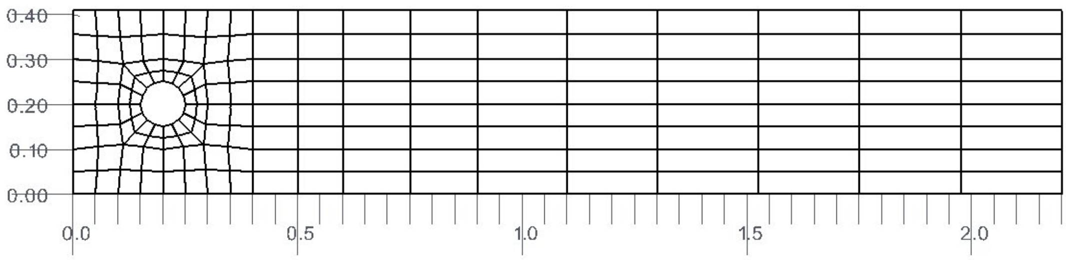
They are supposed to be neglectable. This may be justified by the fact that they include stabilization parameters which vanish for diminishing mesh size (see [Sch10]).

Example 1: For the Carreau model (2.10) & (2.11b) we reconsider the benchmark problem 2D-1 in Schäfer/Turek [TS96]. As in [TS96], the parabolic inflow profile $\boldsymbol{v}_D(x, y) = (4v_m y(H - y)/H^2, 0)^\mathsf{T}$ was prescribed on Γ_i. Here, the variable H denotes the height of the channel and it was given by $H = 0.41$. The parameters were set to $p = 1.2$, $\varepsilon = 10^{-3}$, $\mu_0 = 0.15$, $v_m = 0.3$. For $\mathbb{Q}_2/\mathbb{Q}_2$ elements on uniformly refined meshes the computation of the drag-coefficient yielded the values listed in Table 5.9.

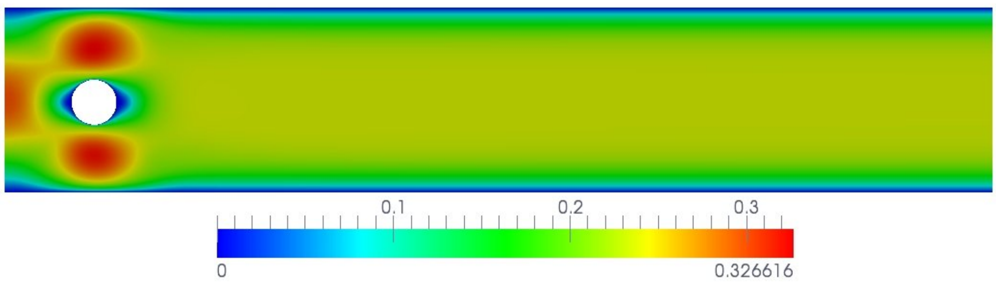
Table 5.9. Drag-coefficient ($\mathbb{Q}_2/\mathbb{Q}_2$ elements, uniform refinement): Case $p = 1.2$

#cells	$J^h_{\text{drag}}(\boldsymbol{u}_h)$
10240	0.1655069473366795
40960	0.1650880440829428
163840	0.1650477143604138
655360	0.1650447725137632
extrapolated	0.1650445410360044

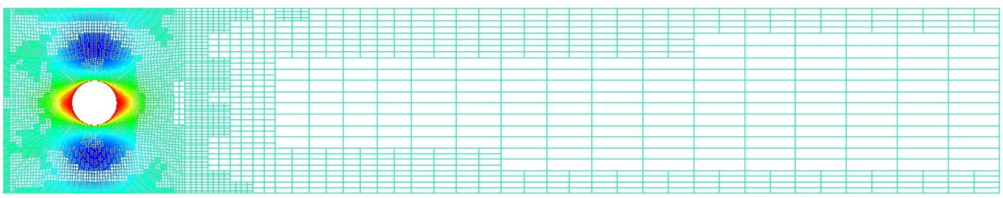
Since the exact drag coefficient $J_{\text{drag}}(\boldsymbol{u})$ is unknown, the extrapolated value was used as the reference value $J_{\text{drag}}(\boldsymbol{u}) = 0.16504454$. We applied the DWR method which enables the quantitative assessment of the discretization error and the adaptive refinement of the underlying meshes. As above, $J_{\text{drag}}(\boldsymbol{u}) - J^h_{\text{drag}}(\boldsymbol{u}_h)$ represents the actual error and η_h denotes the estimated error. The quality of the error estimation is measured by the effectivity index $I_{\text{eff}} := (J_{\text{drag}}(\boldsymbol{u}) - J^h_{\text{drag}}(\boldsymbol{u}_h))/\eta_h$. Using adaptively refined meshes, we obtained the results shown in Table 5.10. Particularly on finer grids we observe a good agreement of the estimated and the actual errors. This is illustrated by $I_{\text{eff}} \approx 1$. The



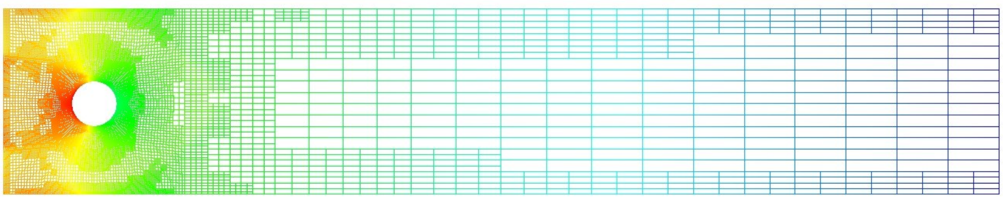
(a) Coarse mesh (160 elements)



(b) Primal solution: velocity

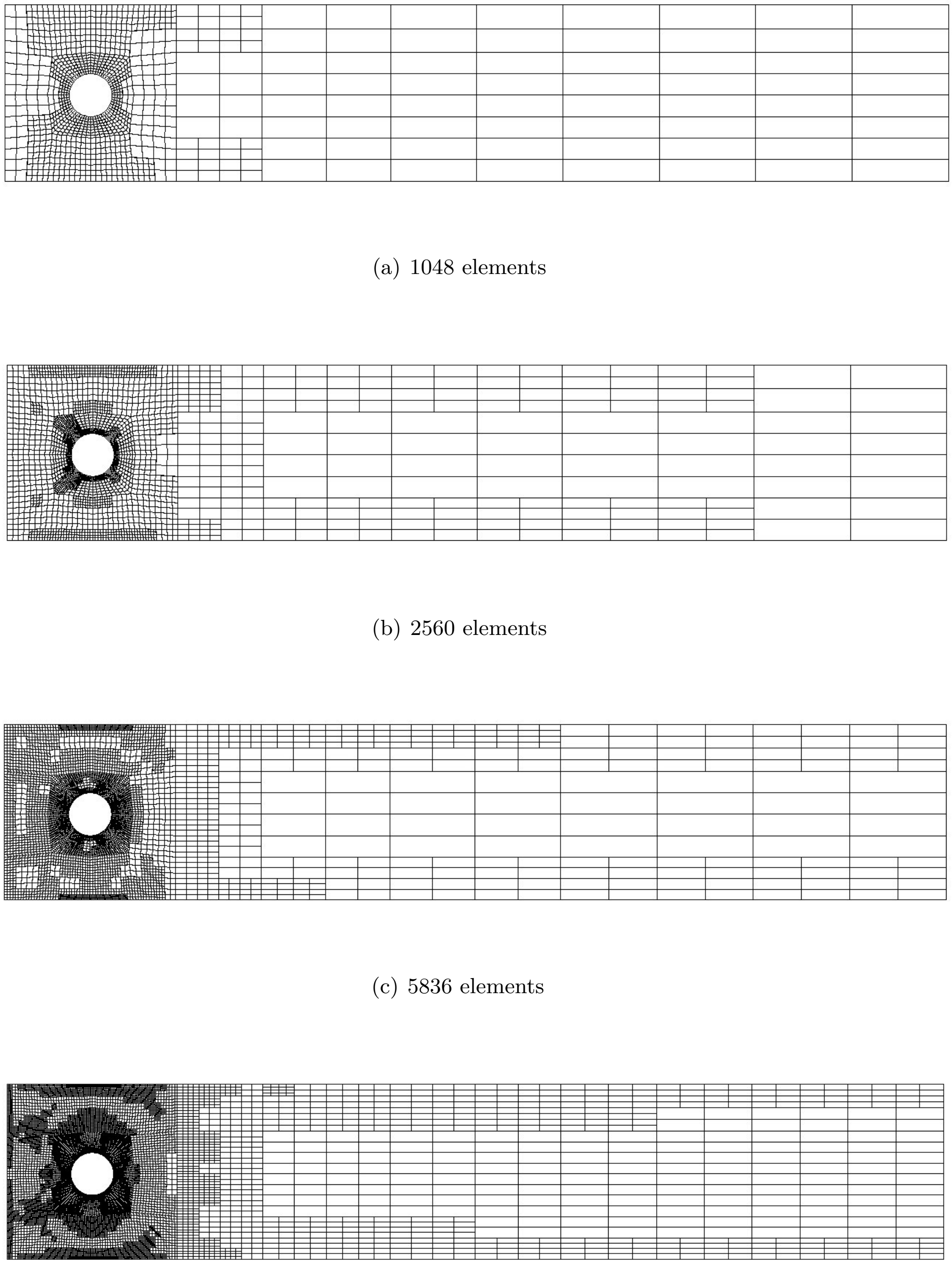


(c) Dual solution: velocity



(d) Primal solution: pressure

Figure 5.8. FE solution on an adaptively refined mesh: case $p = 1.2$



(a) 1048 elements

(b) 2560 elements

(c) 5836 elements

(d) 17308 elements

Figure 5.9. Adaptively refined meshes in case of Example 1

number of elements, which is needed to reach a relative error of one percent, is indicated by bold face. Figure 5.10 depicts the behavior of the discretization error for different refinement strategies. As expected, adaptive mesh refinement is more efficient than uniform mesh refinement.

Table 5.10. Drag-coefficient: case $p = 1.2$, $\varepsilon = 10^{-3}$

#cells	η_h	$J_{\text{drag}}(\boldsymbol{u}) - J^h_{\text{drag}}(\boldsymbol{u}_h)$	I_{eff}
160	-1.35e-02	-3.08e-02	2.28
412	-4.85e-03	-1.15e-02	2.36
1048	-2.03e-03	-3.89e-03	1.91
2560	-9.13e-04	-1.22e-03	1.34
5836	-3.90e-04	-4.14e-04	1.06
17308	-1.34e-04	-1.33e-04	0.99
54760	-4.17e-05	-4.14e-05	0.99

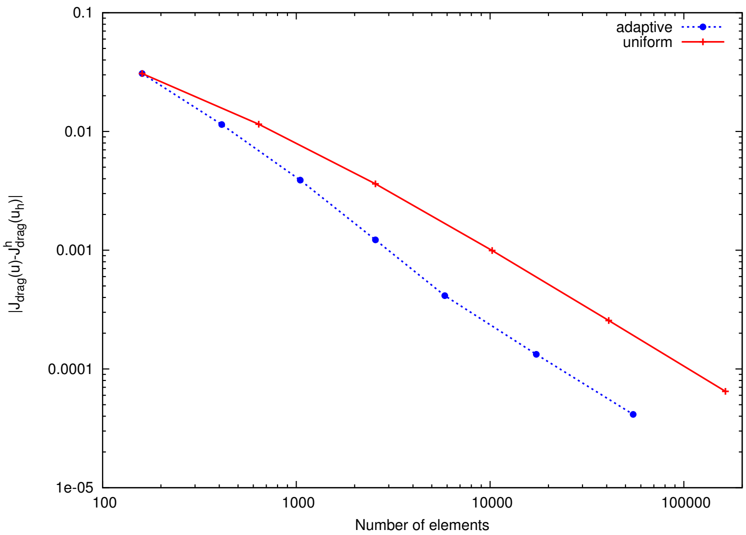


Figure 5.10. $|J_{\text{drag}}(\boldsymbol{u}) - J^h_{\text{drag}}(\boldsymbol{u}_h)|$ for different refinement strategies: $p = 1.2$

Example 2: We consider the laminar flow around an obstacle with square cross-section, see Figure 5.12. Here the parameters were set to $p = 1.5$, $\varepsilon = 10^{-3}$, $\mu_0 = 0.15$. Considering Figure 5.12, we observe singularities of the pressure and velocity-gradient caused by the obstacle. The lack of regularity leads to a reduction of the convergence-rate for $J^h_{\text{drag}}(\boldsymbol{u}_h)$ with respect to uniform refinement. As a reference value for the drag-coefficient, we used the estimated value $J_{\text{drag}}(\boldsymbol{u}) \approx 0.31244827$. We established this value by comparing the approximations obtained for bi-linear and bi-quadratic finite elements by means of adaptive and uniform refinement. Considering Table 5.11, we discover over-estimation which is indicated by $I_{\text{eff}} > 1$. The measured effectivity indices I_{eff} are worse than those in Example 1. The reason is that neither the primal nor the dual solution is regular. In view of Figure 5.11 adaptive mesh refinement is more efficient than uniform refinement. Compared to Example 1, the efficiency of adaptivity is greater.

Table 5.11. Drag-coefficient: case $p = 1.5$, $\varepsilon = 10^{-3}$

#cells	η_h	$J_{\text{drag}}(\boldsymbol{u}) - J^h_{\text{drag}}(\boldsymbol{u}_h)$	I_{eff}
160	-2.37e-02	-3.93e-02	1.66
412	-7.26e-03	-1.54e-02	2.13
868	-4.13e-03	-8.16e-03	1.97
1828	-2.22e-03	-4.15e-03	1.87
3976	-1.12e-03	-2.08e-03	1.86
6244	-7.65e-04	-1.25e-03	1.64
13840	-3.78e-04	-6.23e-04	1.64
23644	-2.20e-04	-3.39e-04	1.54
42736	-1.27e-04	-1.88e-04	1.48
79192	-7.02e-05	-9.95e-05	1.42
142204	-3.92e-05	-5.38e-05	1.37

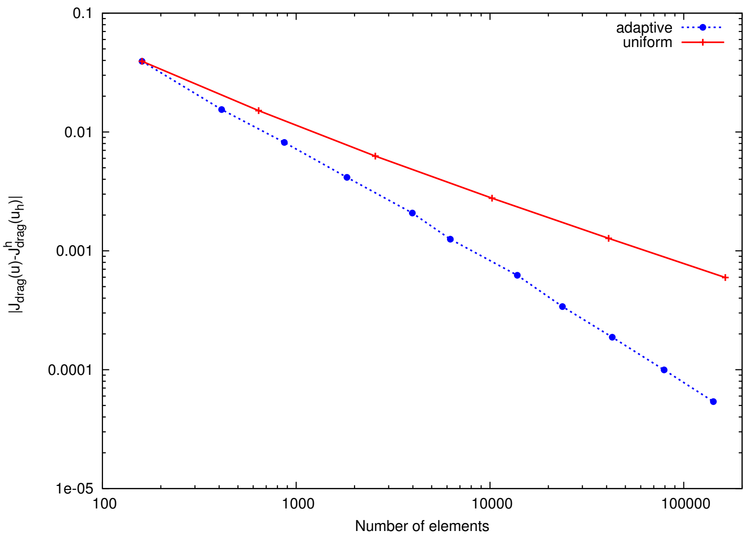
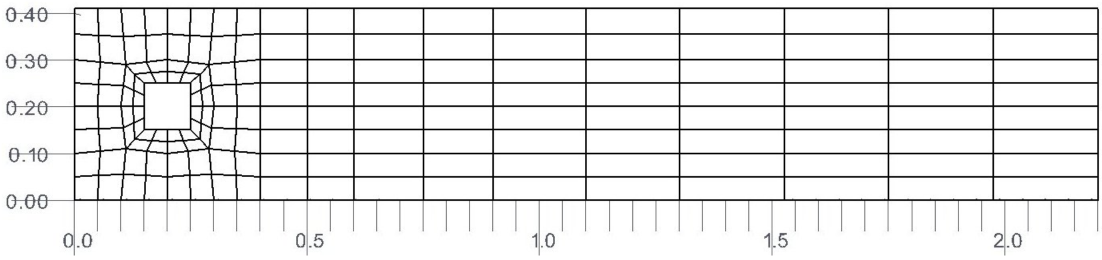
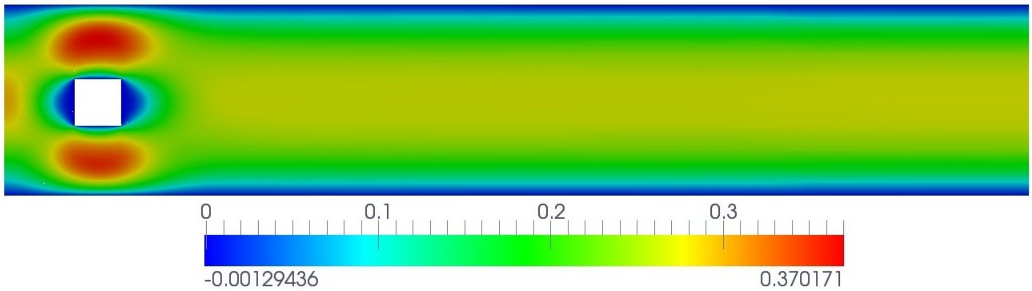


Figure 5.11. $|J_{\text{drag}}(\boldsymbol{u}) - J^h_{\text{drag}}(\boldsymbol{u}_h)|$ for different refinement strategies: $p = 1.5$

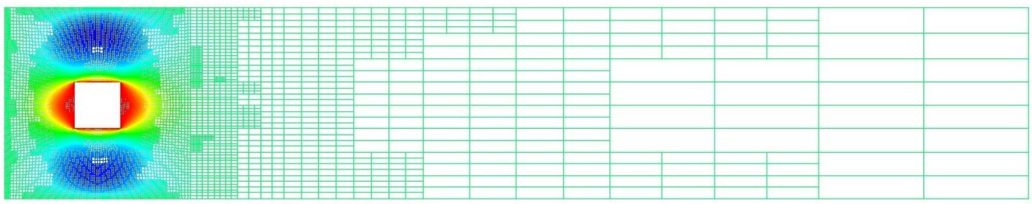
Conclusion: For a posteriori error estimation we applied the DWR method to p-Navier-Stokes systems. Our numerical experiments demonstrate that the DWR method works well in the context of p-Navier-Stokes systems: It quantitatively assesses the discretization error and it enables efficient adaptive mesh refinement. Despite its practical success, the DWR method offers many open questions when it is applied to p-Navier-Stokes systems. In fact, from theoretical point of view it has not yet been understood at all: For $p < 2$ the dual problem (5.57) is not well-posed in general even in the case $\varepsilon > 0$. Moreover the analysis performed in Section 5.6 does not include the limiting case $\varepsilon \searrow 0$. The remainder term (5.59) generally does not remain bounded as $\varepsilon \searrow 0$ for fixed $h > 0$. At least numerical experiments indicate that the DWR method works reasonably in the case $0 < \varepsilon \ll 1$.



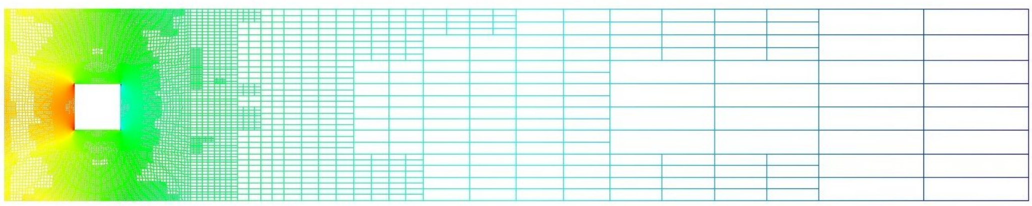
(a) Coarse mesh (160 elements)



(b) Primal solution: velocity

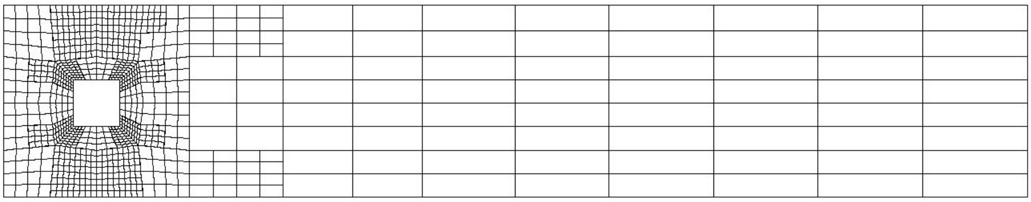


(c) Dual solution: velocity

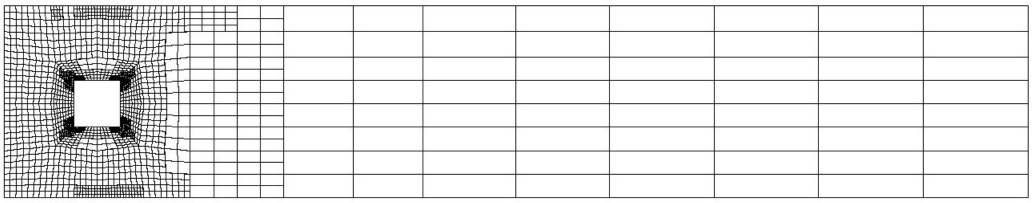


(d) Primal solution: pressure

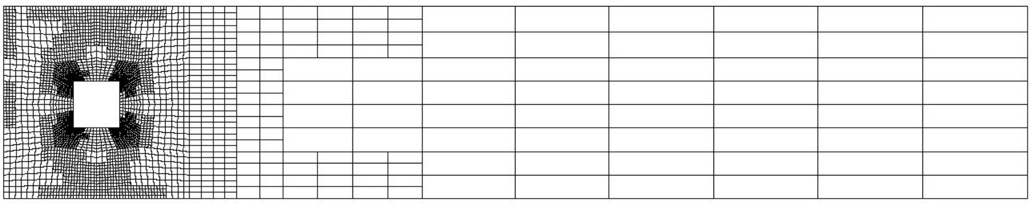
Figure 5.12. FE solution on an adaptively refined mesh: case $p = 1.5$



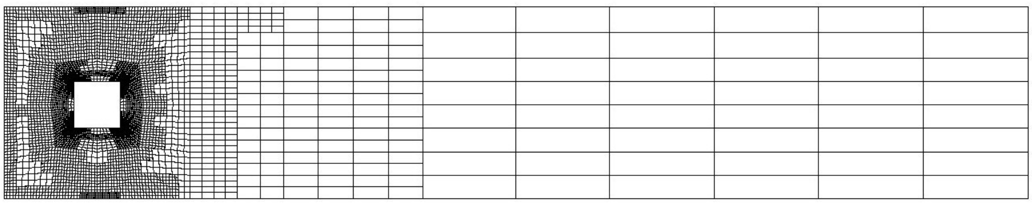
(a) 868 elements



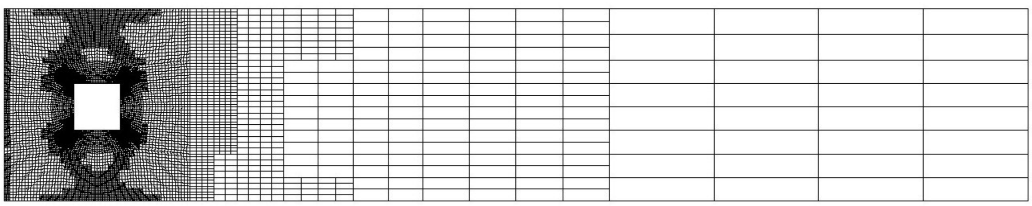
(b) 1828 elements



(c) 3976 elements



(d) 6244 elements



(e) 13840 elements

Figure 5.13. Adaptively refined meshes in case of Example 2

6 Finite Element Approximation of Singular Power-Law Systems

Non-Newtonian fluid motions are often modeled by a power-law ansatz. In this chapter, we consider the power-law model (2.10) & (2.11a) with $p < 2$ which features an unbounded viscosity in the limit of zero shear rate, and we study the finite element (FE) discretization of the corresponding equations of motion (the *singular* power-law systems). In the case under consideration, numerical instabilities usually arise when the finite element equations are solved via Newton's method. In this chapter, we aim at developing a numerical method that enables the stable approximation of singular power-law systems. First of all we identify the arising difficulties connected with the numerical solution. Then we propose an approximation method for singular power-law systems that is based on a simple regularization of the power-law model (2.11a). Our proposed method generates a sequence of discrete functions that is computable in practice via Newton's method and that converges to the power-law solution for diminishing mesh size. We derive a priori error estimates that quantify the convergence of our method, see Corollary 6.4. Furthermore, we demonstrate numerically that our regularized approximation method surpasses the non-regularized one regarding accuracy and numerical efficiency.

In Section 6.1, we recall the weak formulation and we introduce its discretization. For ease of presentation, we restrict ourselves to stable discretizations that satisfy the inf-sup stability condition **(IS)**. Section 6.2 deals with Newton's method and its stability. For the regularized model we show the stability of Newton's method in the sense that we derive an upper bound for the condition number of the Newton matrix. In Section 6.3, we present our regularized approximation scheme and we derive a priori error estimates for it. In Section 6.4, we illustrate the a priori error estimates by numerical experiments.

6.1 Problem formulation

For ease of presentation, we only study power-law/Carreau-type models (2.10)–(2.11b). Such models are derived from a potential. For $p \in (1, \infty)$ and $\varepsilon \geq 0$, we define the extra stress tensor $\boldsymbol{\mathcal{S}}_\varepsilon$ by means of a convex function $\varPhi_\varepsilon : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ as follows:

$$\boldsymbol{\mathcal{S}}_\varepsilon(\boldsymbol{Q}) := \varPhi'_\varepsilon(|\boldsymbol{Q}|)\frac{\boldsymbol{Q}}{|\boldsymbol{Q}|} \qquad \forall \boldsymbol{Q} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}, \qquad \varPhi_\varepsilon(t) := \int_0^t (\varepsilon^2 + s^2)^{\frac{p-2}{2}} s \,\mathrm{d}s. \tag{6.1}$$

The subscript $_\varepsilon$ highlights the dependence on ε which will be of relevance below.

Weak formulation: The weak formulation of system (2.16) & (6.1) reads:

$(\mathbf{P}^\varepsilon)$ *For given* $\boldsymbol{f} \in (\boldsymbol{\mathcal{X}}^p)^*$ *find* $(\boldsymbol{v}^\varepsilon, \pi^\varepsilon) \in \boldsymbol{\mathcal{X}}^p \times \mathcal{Q}^p$ *such that*

$$(\boldsymbol{\mathcal{S}}_\varepsilon(\boldsymbol{D}\boldsymbol{v}^\varepsilon), \boldsymbol{D}\boldsymbol{w})_\Omega - (\pi^\varepsilon, \nabla \cdot \boldsymbol{w})_\Omega = \langle \boldsymbol{f}, \boldsymbol{w} \rangle \qquad \forall \boldsymbol{w} \in \boldsymbol{\mathcal{X}}^p \tag{6.2}$$

$$(\nabla \cdot \boldsymbol{v}^\varepsilon, q)_\Omega = 0 \qquad \forall q \in \mathcal{Q}^p. \tag{6.3}$$

The well-posedness of Problem $(\mathbf{P}^\varepsilon)$ has been established in Section 2.5: There exists a unique solution $(\boldsymbol{v}^\varepsilon, \pi^\varepsilon) \in \boldsymbol{\mathcal{X}}^p \times \mathcal{Q}^p$ to Problem $(\mathbf{P}^\varepsilon)$ that satisfies the a priori estimate

$$\|\boldsymbol{v}^\varepsilon\|_{1,p} \leq c\Big(\|\boldsymbol{f}\|_{-1,p'}^{\frac{1}{p-1}} + \varepsilon\Big) \tag{6.4}$$

where $c > 0$ only depends on Ω, p (see Lemma 2.19). Since $\boldsymbol{\mathcal{S}}_\varepsilon$ is derived from the potential Φ_ε, we can introduce the functional $\mathcal{J}_\varepsilon : \boldsymbol{\mathcal{X}}^p \to \mathbb{R}$ associated with Φ_ε:

$$\mathcal{J}_\varepsilon(\boldsymbol{u}) := \int_\Omega \Phi_\varepsilon(|\boldsymbol{D}\boldsymbol{u}|)\,\mathrm{d}\boldsymbol{x} - \langle \boldsymbol{f}, \boldsymbol{u} \rangle \qquad \forall \boldsymbol{u} \in \boldsymbol{\mathcal{X}}^p. \tag{6.5}$$

In Section 2.5 we have shown that Problem $(\mathbf{P}^\varepsilon)$ is equivalent to the minimization problem

$(\mathbf{M}^\varepsilon)$ *For given* $\boldsymbol{f} \in (\boldsymbol{\mathcal{X}}^p)^*$ *find* $\boldsymbol{v}^\varepsilon \in \boldsymbol{\mathcal{V}}^p$ *such that*

$$\mathcal{J}_\varepsilon(\boldsymbol{v}^\varepsilon) \leq \mathcal{J}_\varepsilon(\boldsymbol{w}) \qquad \forall \boldsymbol{w} \in \boldsymbol{\mathcal{V}}^p. \tag{6.6}$$

Finite element discretization: Let $\boldsymbol{\mathcal{X}}_h^p$ and $\mathcal{Q}_h^p$ be two appropriate finite element spaces as in (3.21). Their precise definition is not important for the purpose of the following sections. The Galerkin approximation of $(\mathbf{P}^\varepsilon)$ reads as follows:

$(\mathbf{P}_h^\varepsilon)$ *Find* $(\boldsymbol{v}_h^\varepsilon, \pi_h^\varepsilon) \in \boldsymbol{\mathcal{X}}_h^p \times \mathcal{Q}_h^p$ *such that*

$$(\boldsymbol{\mathcal{S}}_\varepsilon(\boldsymbol{D}\boldsymbol{v}_h^\varepsilon), \boldsymbol{D}\boldsymbol{w}_h)_\Omega - (\pi_h^\varepsilon, \nabla \cdot \boldsymbol{w}_h)_\Omega = \langle \boldsymbol{f}, \boldsymbol{w}_h \rangle \qquad \forall \boldsymbol{w}_h \in \boldsymbol{\mathcal{X}}_h^p \tag{6.7}$$

$$(\nabla \cdot \boldsymbol{v}_h^\varepsilon, q_h)_\Omega = 0 \qquad \forall q_h \in \mathcal{Q}_h^p. \tag{6.8}$$

For ease of presentation, throughout the chapter we require that the discrete inf-sup condition **(IS)** is satisfied. We may easily verify the well-posedness of the discrete Problem $(\mathbf{P}_h^\varepsilon)$ using the same arguments as in the continuous case (see Lemma 2.19).

Lemma 6.1. *Let* **(IS)** *be satisfied. Then, there exists a unique solution* $(\boldsymbol{v}_h^\varepsilon, \pi_h^\varepsilon) \in \boldsymbol{\mathcal{X}}_h^p \times \mathcal{Q}_h^p$ *to Problem* $(\mathbf{P}_h^\varepsilon)$ *that satisfies the a priori bound* (6.4) *with* $\boldsymbol{v}^\varepsilon$ *replaced by* $\boldsymbol{v}_h^\varepsilon$.

Remark 6.1. It is well-known that equal-order finite elements (such as $\mathbb{Q}_1/\mathbb{Q}_1$) lead to an unstable discretization, i.e., they do not fulfill the inf-sup stability condition **(IS)**. If the pairing $\boldsymbol{\mathcal{X}}_h^p \times \mathcal{Q}_h^p$ does not satisfy **(IS)**, we need to stabilize the Galerkin discretization $(\mathbf{P}_h^\varepsilon)$. Stabilization methods, that are frequently used, are the local projection stabilization (LPS) and the pressure-stabilization Petrov-Galerkin (PSPG) method (see Section 3.2). If we discretize $(\mathbf{P}^\varepsilon)$ with the unstable $\mathbb{Q}_1/\mathbb{Q}_1$ elements, we can apply the LPS-based stabilization method proposed in Section 4.1: One adds an appropriate stabilization term $s_h(\pi_h)(q_h)$ to (6.8) which gives a weighted $L^{p'}$-control over the fluctuations of the pressure-gradient.

In order to ensure approximation properties, one clearly needs to specify the choice of the discrete spaces. Since approximation properties are not important for the purpose of the forthcoming section, we will discuss particular choices of the discrete spaces later on.

6.2 Stability of Newton's method

Nonlinear FE systems are frequently solved via Newton's method, see Algorithm 3.1. This section is dedicated to computational aspects: We discuss Newton's method and its stability in the context of power-law/Carreau-type models. In particular, we derive an upper bound for the condition number of the matrix resulting from linearization of the viscous part.

Solution of the discrete problems: Below we investigate the numerical scheme that solves the FE systems (6.7) & (6.8). Due to their nonlinear nature, the discrete equations need to be linearized. For linearization we apply Newton's method, see Algorithm 3.1. For ease of presentation, we introduce a semi-linear form $a_\varepsilon(\cdot)(\cdot)$ associated with $\boldsymbol{\mathcal{S}}_\varepsilon$:

$$a_\varepsilon(\boldsymbol{v})(\boldsymbol{w}) := (\boldsymbol{\mathcal{S}}_\varepsilon(\boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{w})_\Omega \qquad \forall \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{\mathcal{X}}^p.$$

Formally, we may compute the Gâteaux-derivative of $a_\varepsilon(\cdot)(\cdot)$:

$$\begin{aligned} a'_\varepsilon(\boldsymbol{v})(\boldsymbol{\xi}, \boldsymbol{w}) = &\int_\Omega \left(\varepsilon^2 + |\boldsymbol{D}\boldsymbol{v}|^2\right)^{\frac{p-2}{2}} \boldsymbol{D}\boldsymbol{\xi} : \boldsymbol{D}\boldsymbol{w} \,\mathrm{d}\boldsymbol{x} \\ &+ (p-2) \int_\Omega \left(\varepsilon^2 + |\boldsymbol{D}\boldsymbol{v}|^2\right)^{\frac{p-4}{2}} (\boldsymbol{D}\boldsymbol{v} : \boldsymbol{D}\boldsymbol{\xi})(\boldsymbol{D}\boldsymbol{v} : \boldsymbol{D}\boldsymbol{w}) \,\mathrm{d}\boldsymbol{x} \end{aligned} \tag{6.9}$$

for $\boldsymbol{v}$, $\boldsymbol{\xi}$, $\boldsymbol{w} \in \boldsymbol{\mathcal{X}}^p$. We recall Newton's method applied to (6.7) & (6.8), see Algorithm 3.1: Choose an initial guess $(\boldsymbol{v}_h^0, \pi_h^0)$. For $k = 0, 1, 2, \ldots$ compute $(\boldsymbol{\xi}_h^k, \eta_h^k) \in \boldsymbol{\mathcal{X}}_h^p \times \mathcal{Q}_h^p$ from

$$\begin{aligned} a'_\varepsilon(\boldsymbol{v}_h^k)(\boldsymbol{\xi}_h^k, \boldsymbol{w}_h) - (\eta_h^k, \nabla \cdot \boldsymbol{w}_h)_\Omega + (\nabla \cdot \boldsymbol{\xi}_h^k, q_h)_\Omega = -a_\varepsilon(\boldsymbol{v}_h^k)(\boldsymbol{w}_h) \\ + (\pi_h^k, \nabla \cdot \boldsymbol{w}_h)_\Omega - (\nabla \cdot \boldsymbol{v}_h^k, q_h)_\Omega + \langle \boldsymbol{f}, \boldsymbol{w}_h \rangle \quad \forall (\boldsymbol{w}_h, q_h) \in \boldsymbol{\mathcal{X}}_h^p \times \mathcal{Q}_h^p \end{aligned} \tag{6.10}$$

and set $(\boldsymbol{v}_h^{k+1}, \pi_h^{k+1}) := (\boldsymbol{v}_h^k, \pi_h^k) + (\boldsymbol{\xi}_h^k, \eta_h^k)$. For $p < 2$ and $\varepsilon = 0$ the Gâteaux-derivative $a'_\varepsilon(\boldsymbol{v}_h^k)(\boldsymbol{\xi}_h^k, \boldsymbol{w}_h)$ does not exist in general when the critical set $\Omega_c \equiv \{\boldsymbol{x} \in \Omega;\, \nabla \boldsymbol{v}_h^k(\boldsymbol{x}) \approx \boldsymbol{0}\}$ is not empty. Since Newton's method requires the existence of first derivatives, for $\varepsilon = 0$ its convergence is not ensured in the case $\Omega_c \neq \emptyset$. Hence, for $\varepsilon = 0$ the solution to Problem $(\mathbf{P}_h^\varepsilon)$ cannot (approximatively) be determined by means of Newton's method in general. However, if $\varepsilon > 0$, it can easily be shown that a_ε is Gâteaux differentiable on $\boldsymbol{\mathcal{X}}_h^p \times \boldsymbol{\mathcal{X}}_h^p$.

Stability of Newton's method: First of all we discuss the algebraic structure of Newton's algorithm. For simplicity, we assume that Problem $(\mathbf{P}^\varepsilon)$ is discretized with inf-sup stable finite elements. In this case, the Galerkin discretization (6.7), (6.8) does not need to be modified by additional stabilization terms. If equal-order discretizations are considered (cf. Remark 6.1), then the forthcoming investigations can easily be generalized. Let $\{\boldsymbol{\psi}_j,\, j = 1, \ldots, N := \dim(\boldsymbol{\mathcal{X}}_h^p)\}$ and $\{\chi_j,\, j = 1, \ldots, M := \dim(\mathcal{Q}_h^p)\}$ be the nodal basis of the finite element space $\boldsymbol{\mathcal{X}}_h^p$ and $\mathcal{Q}_h^p$, respectively. In view of the representations

$$\boldsymbol{\xi}_h^k = \sum_{j=1}^{N} \alpha_j^k \boldsymbol{\psi}_j, \qquad \eta_h^k = \sum_{j=1}^{M} \beta_j^k \chi_j, \tag{6.11}$$

Newton's system (6.10) is equivalent to the linear system of equations

$$\begin{pmatrix} \boldsymbol{A}^k & \boldsymbol{B} \\ -\boldsymbol{B}^\top & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha}^k \\ \boldsymbol{\beta}^k \end{pmatrix} = \begin{pmatrix} \boldsymbol{b}^k \\ \boldsymbol{c}^k \end{pmatrix} \tag{6.12}$$

for the unknowns $\boldsymbol{\alpha}^k \in \mathbb{R}^N$ and $\boldsymbol{\beta}^k \in \mathbb{R}^M$ where

$$\begin{aligned} \boldsymbol{A}^k &:= \Big(a_\varepsilon'(\boldsymbol{v}_h^k)(\boldsymbol{\psi}_j, \boldsymbol{\psi}_i)\Big)_{i,j=1}^{N}, \qquad \boldsymbol{B} := -\Big((\chi_j, \nabla\cdot\boldsymbol{\psi}_i)_\Omega\Big)_{i,j=1}^{N,M}, \\ \boldsymbol{b}^k &:= \Big(-a_\varepsilon(\boldsymbol{v}_h^k)(\boldsymbol{\psi}_i) + (\pi_h^k, \nabla\cdot\boldsymbol{\psi}_i)_\Omega + \langle \boldsymbol{f}, \boldsymbol{\psi}_i\rangle\Big)_{i=1}^{N}, \qquad \boldsymbol{c}^k := \Big(-(\nabla\cdot\boldsymbol{v}_h^k, \chi_i)_\Omega\Big)_{i=1}^{M}. \end{aligned} \tag{6.13}$$

The following theorem provides an upper bound for the condition number of $\boldsymbol{A}^k$.

Theorem 6.2. *Let $p \in (1, 2]$ and $\varepsilon \in (0, \infty)$. Then, the matrix $\boldsymbol{A}^k$ defined in* (6.13) *is symmetric and positive definite for all $k \in \mathbb{N}$. Consequently, $\boldsymbol{A}^k$ is regular for all $k \in \mathbb{N}$. Furthermore, the condition number* $\mathrm{cond}_2(\boldsymbol{A}^k)$ *of the matrix $\boldsymbol{A}^k$ can be estimated by*

$$\mathrm{cond}_2(\boldsymbol{A}^k) := \frac{\lambda_{\max}(\boldsymbol{A}^k)}{\lambda_{\min}(\boldsymbol{A}^k)} \le c(p-1)^{-1}\Big(\varepsilon + \|\boldsymbol{D}\boldsymbol{v}_h^k\|_\infty\Big)^{2-p} \varepsilon^{p-2} h^{-2}. \tag{6.14}$$

Here, $\lambda_{\max}(\boldsymbol{A}^k)$ and $\lambda_{\min}(\boldsymbol{A}^k)$ denotes the largest and smallest eigenvalue of the matrix $\boldsymbol{A}^k$. The constant c only depends on Ω and on the shape-regularity of the grid $\mathbb{T}_h$.

Proof. In the context of the regularized p-Laplace equation, an estimate similar to (6.14) has been proven by Hirn [Hir08]. Here, we similarly derive estimate (6.14) following the arguments in [Hir08]. Clearly, the matrix $\boldsymbol{A}^k$ defined in (6.13) is symmetric. Let $\boldsymbol{w}_h \in \boldsymbol{\mathcal{X}}_h^p$ be an arbitrary finite element function with corresponding nodal vector $\boldsymbol{\zeta} = (\zeta_i)_{i=1}^N \in \mathbb{R}^N$, i.e., $\boldsymbol{w}_h = \sum_{i=1}^N \zeta_i \boldsymbol{\psi}_i$. Then, there holds true the following identity:

$$a_\varepsilon'(\boldsymbol{v}_h^k)(\boldsymbol{w}_h, \boldsymbol{w}_h) = \sum_{i,j=1}^{N} \zeta_i a_\varepsilon'(\boldsymbol{v}_h^k)(\boldsymbol{\psi}_i, \boldsymbol{\psi}_j)\zeta_j = \sum_{i,j=1}^{N} \zeta_i A_{ij}^k \zeta_j.$$

In view of (6.9), we observe that

$$\begin{aligned} a_\varepsilon'(\boldsymbol{v}_h^k)(\boldsymbol{w}_h, \boldsymbol{w}_h) &= \int_\Omega \Big(\varepsilon^2 + |\boldsymbol{D}\boldsymbol{v}_h^k|^2\Big)^{\frac{p-2}{2}} |\boldsymbol{D}\boldsymbol{w}_h|^2 \,\mathrm{d}\boldsymbol{x} \\ &\quad + (p-2)\int_\Omega \Big(\varepsilon^2 + |\boldsymbol{D}\boldsymbol{v}_h^k|^2\Big)^{\frac{p-4}{2}} |\boldsymbol{D}\boldsymbol{v}_h^k : \boldsymbol{D}\boldsymbol{w}_h|^2 \,\mathrm{d}\boldsymbol{x}. \end{aligned} \tag{6.15}$$

Taking into account $p \le 2$, applying the Cauchy-Schwarz inequality, we deduce from (6.15):

$$a'_\varepsilon(\boldsymbol{v}_h^k)(\boldsymbol{w}_h,\boldsymbol{w}_h) \ge (p-1)\Big(\varepsilon + \|\boldsymbol{D}\boldsymbol{v}_h^k\|_\infty\Big)^{p-2}\|\boldsymbol{D}\boldsymbol{w}_h\|_2^2. \tag{6.16}$$

Using Korn's and Poincaré's inequality, for some $c = c(\Omega) > 0$ we arrive at the estimate

$$\sum_{i,j=1}^{N} \zeta_i A_{ij}^k \zeta_j \ge c(p-1)\Big(\varepsilon + \|\boldsymbol{D}\boldsymbol{v}_h^k\|_\infty\Big)^{p-2}\|\boldsymbol{w}_h\|_{1,2}^2.$$

As a result, the matrix $\boldsymbol{A}^k$ is positive definite. Let $\boldsymbol{M}$ be the mass matrix associated with $\boldsymbol{\mathcal{X}}_h^p$, i.e., let $\boldsymbol{M} \in \mathbb{R}^{N\times N}$ be defined by $M_{ij} := (\boldsymbol{\psi}_i, \boldsymbol{\psi}_j)_\Omega$. Then, it is well-known that

$$\|\boldsymbol{w}_h\|_2^2 = \sum_{i,j=1}^{N} \zeta_i M_{ij}\zeta_j, \qquad \operatorname{cond}_2(\boldsymbol{M}) = \mathcal{O}(1). \tag{6.17}$$

The smallest eigenvalue $\lambda_{\min}(\boldsymbol{A}^k)$ is bounded from below by

$$\begin{aligned}\lambda_{\min}(\boldsymbol{A}^k) &= \min_{\boldsymbol{\zeta}\in\mathbb{R}^N} \frac{\sum_{i,j=1}^N \zeta_i A_{ij}^k\zeta_j}{|\boldsymbol{\zeta}|^2}\\ &\ge \min_{\boldsymbol{\zeta}\in\mathbb{R}^N} \frac{\sum_{i,j=1}^N \zeta_i A_{ij}^k\zeta_j}{\sum_{i,j=1}^N \zeta_i M_{ij}\zeta_j} \min_{\boldsymbol{\zeta}\in\mathbb{R}^N} \frac{\sum_{i,j=1}^N \zeta_i M_{ij}\zeta_j}{|\boldsymbol{\zeta}|^2} = \min_{\boldsymbol{\zeta}\in\mathbb{R}^N} \frac{\sum_{i,j=1}^N \zeta_i A_{ij}^k\zeta_j}{\sum_{i,j=1}^N \zeta_i M_{ij}\zeta_j}\lambda_{\min}(\boldsymbol{M}).\end{aligned}$$

Using (6.16), Poincaré's and Korn's inequality, we conclude that for some $c = c(\Omega) > 0$:

$$\lambda_{\min}(\boldsymbol{A}^k) \ge \min_{\boldsymbol{w}_h\in\boldsymbol{\mathcal{X}}_h^p} \frac{a'_\varepsilon(\boldsymbol{v}_h^k)(\boldsymbol{w}_h,\boldsymbol{w}_h)}{\|\boldsymbol{w}_h\|_2^2}\lambda_{\min}(\boldsymbol{M}) \ge c(p-1)\Big(\varepsilon + \|\boldsymbol{D}\boldsymbol{v}_h^k\|_\infty\Big)^{p-2}\lambda_{\min}(\boldsymbol{M}).$$

Similarly, we get an upper bound for the largest eigenvalue $\lambda_{\max}(\boldsymbol{A}^k)$:

$$\lambda_{\max}(\boldsymbol{A}^k) = \max_{\boldsymbol{\zeta}\in\mathbb{R}^N} \frac{\sum_{i,j=1}^N \zeta_i A_{ij}^k\zeta_j}{|\boldsymbol{\zeta}|^2} \le \max_{\boldsymbol{\zeta}\in\mathbb{R}^N} \frac{\sum_{i,j=1}^N \zeta_i A_{ij}^k\zeta_j}{\sum_{i,j=1}^N \zeta_i M_{ij}\zeta_j}\lambda_{\max}(\boldsymbol{M}).$$

In view of $p < 2$, it easily follows from (6.15) that

$$a'_\varepsilon(\boldsymbol{v}_h^k)(\boldsymbol{w}_h,\boldsymbol{w}_h) \le \int_\Omega \Big(\varepsilon^2 + |\boldsymbol{D}\boldsymbol{v}_h^k|^2\Big)^{\frac{p-2}{2}} |\boldsymbol{D}\boldsymbol{w}_h|^2 \,\mathrm{d}\boldsymbol{x} \le \varepsilon^{p-2}\|\boldsymbol{D}\boldsymbol{w}_h\|_2^2.$$

Using the global inverse estimate (3.20), we can estimate the largest eigenvalue of $\boldsymbol{A}^k$ by

$$\begin{aligned}\lambda_{\max}(\boldsymbol{A}^k) &\le \max_{\boldsymbol{w}_h\in\boldsymbol{\mathcal{X}}_h^p} \frac{a'_\varepsilon(\boldsymbol{v}_h^k)(\boldsymbol{w}_h,\boldsymbol{w}_h)}{\|\boldsymbol{w}_h\|_2^2}\lambda_{\max}(\boldsymbol{M})\\ &\le \frac{\varepsilon^{p-2}\|\boldsymbol{D}\boldsymbol{w}_h\|_2^2}{\|\boldsymbol{w}_h\|_2^2}\lambda_{\max}(\boldsymbol{M}) \le C\varepsilon^{p-2}h^{-2}\lambda_{\max}(\boldsymbol{M}),\end{aligned}$$

where $C > 0$ only depends on the shape-regularity of $\mathbb{T}_h$. To sum up, we have proven that

$$c(p-1)\Big(\varepsilon + \|\boldsymbol{D}\boldsymbol{v}_h^k\|_\infty\Big)^{p-2}\lambda_{\min}(\boldsymbol{M}) \le \lambda_{\min}(\boldsymbol{A}^k) \le \lambda_{\max}(\boldsymbol{A}^k) \le C\varepsilon^{p-2}h^{-2}\lambda_{\max}(\boldsymbol{M}).$$

Using (6.17), we easily complete the proof. □

Remark 6.2. Finally, we comment on Theorem 6.2:

- For $p = 2$, the result (6.14) is well-known in the context of Stokes systems. If the elasticity problem is studied, i.e., if the pressure term and the constraint $\nabla \cdot \boldsymbol{v} = 0$ are omitted, then the Newton matrix stated in (6.12) only consists of the bloc $\boldsymbol{A}^k$.

- Theorem 6.2 only yields an upper bound for the condition number of the matrix $\boldsymbol{A}^k$. However, numerical experiments indicate that, indeed, the condition number of $\boldsymbol{A}^k$ can behave as the expression on the right-hand side of (6.14). This means that for diminishing $\varepsilon > 0$ and $p > 1$ the condition number of $\boldsymbol{A}^k$ increases.

- Since $\boldsymbol{A}^k$ is regular for all $k \in \mathbb{N}$, one can eliminate the variable $\boldsymbol{\alpha}^k$ in the system of equations (6.12) so that $\boldsymbol{\beta}^k$ and $\boldsymbol{\alpha}^k$ can be determined from

$$\boldsymbol{B}^\top (\boldsymbol{A}^k)^{-1} \boldsymbol{B} \boldsymbol{\beta}^k = \boldsymbol{c}^k + \boldsymbol{B}^\top (\boldsymbol{A}^k)^{-1} \boldsymbol{b}^k, \qquad \boldsymbol{\alpha}^k = (\boldsymbol{A}^k)^{-1} \Big(\boldsymbol{b}^k - \boldsymbol{B} \boldsymbol{\beta}^k \Big). \tag{6.18}$$

  The matrix $\boldsymbol{B}^\top (\boldsymbol{A}^k)^{-1} \boldsymbol{B}$ is referred to as the "Schur complement". For its computation, one needs to determine the inverse matrix of $\boldsymbol{A}^k$. According to (6.14), the condition number of $\boldsymbol{A}^k$ can be large for $0 < \varepsilon \ll 1$ and $p < 2$. Hence, one has to construct appropriate preconditioning methods in order to solve (6.18) numerically.

6.3 Approximation of singular power-law systems

This section is dedicated to the finite element approximation of power-law solutions ($\varepsilon = 0$). As already mentioned, the nonlinear operator related to $\boldsymbol{\mathcal{S}}_\varepsilon$ is not differentiable for $\varepsilon = 0$ in the shear thinning case. The lack of differentiability may cause numerical instabilities when the nonlinear discrete systems are solved via Newton's method. In this section, we propose a numerical method that enables the stable approximation of singular power-law systems. The proposed method generates a sequence of discrete functions which is computable in practice via Newton's method and which converges to the exact solution of the power-law system. It is based on a simple regularization of the power-law model. Clearly, the Carreau model (6.1) with $\varepsilon > 0$ can be interpreted as a regularized power-law model. Let the quantities $\boldsymbol{\mathcal{S}}_\varepsilon$, Φ_ε, $\mathcal{J}_\varepsilon$ be defined as in Section 6.1. In order to highlight the dependence on ε, we also relabel $\boldsymbol{\mathcal{F}}$ introduced in (2.39) as

$$\boldsymbol{\mathcal{F}}_\varepsilon(\boldsymbol{Q}) := \Big(\varepsilon^2 + |\boldsymbol{Q}|^2 \Big)^{\frac{p-2}{4}} \boldsymbol{Q} \qquad \forall \boldsymbol{Q} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}. \tag{6.19}$$

Let us set $\boldsymbol{\mathcal{S}} := \boldsymbol{\mathcal{S}}_0$, $\Phi := \Phi_0$, $\boldsymbol{\mathcal{F}} := \boldsymbol{\mathcal{F}}_0$, and $\mathcal{J} := \mathcal{J}_0$. As depicted by the following theorem, the solutions to the Carreau systems ($\varepsilon > 0$) approximate the solution to the power-law system ($\varepsilon = 0$) for diminishing $\varepsilon \searrow 0$.

Theorem 6.3. *For $p \in (1, 2)$ and $\varepsilon \in [0, \varepsilon_0]$ let the extra stress tensor $\boldsymbol{\mathcal{S}}_\varepsilon$ be given by* (6.1) *and let $\boldsymbol{\mathcal{F}}_\varepsilon$ be defined by* (6.19)*. For each $\varepsilon \in [0, \varepsilon_0]$ let $(\boldsymbol{v}^\varepsilon, \pi^\varepsilon) \in \boldsymbol{\mathcal{X}}^p \times \mathcal{Q}^p$ be the unique* *solution to* $(\mathbf{P}^\varepsilon)$*. Let us define* $(\boldsymbol{v},\pi) := (\boldsymbol{v}^0,\pi^0)$*. Then, there hold the a priori estimates*

$$\|\boldsymbol{\mathcal{F}}_\varepsilon(\boldsymbol{D}\boldsymbol{v}^\varepsilon) - \boldsymbol{\mathcal{F}}_\varepsilon(\boldsymbol{D}\boldsymbol{v})\|_2 \le c(p,\Omega)\varepsilon^{p/2} \tag{6.20}$$

$$\|\boldsymbol{D}\boldsymbol{v}^\varepsilon - \boldsymbol{D}\boldsymbol{v}\|_p \le c(p,\varepsilon_0,\Omega,\boldsymbol{f})\varepsilon^{p/2} \tag{6.21}$$

$$\|\pi^\varepsilon - \pi\|_{p'} \le c(\beta(p),p,\Omega)\varepsilon^{p-1}, \tag{6.22}$$

where the constants only depend on the quantities quoted within the brackets. In particular, $(\boldsymbol{v}^\varepsilon,\pi^\varepsilon)$ *converge to the power-law solution* $(\boldsymbol{v},\pi)$ *in* $\boldsymbol{\mathcal{X}}^p \times \mathcal{Q}^p$ *strongly for* $\varepsilon \to 0$.

Proof. Since $\boldsymbol{v}^\varepsilon$ is the unique solution to $(\mathbf{P}^\varepsilon)$ for $\varepsilon \ge 0$, it can be characterized as the unique minimizer of the functional $\mathcal{J}_\varepsilon$ in $\boldsymbol{\mathcal{V}}^p$, i.e., it satisfies

$$\mathcal{J}_\varepsilon(\boldsymbol{v}^\varepsilon) \le \mathcal{J}_\varepsilon(\boldsymbol{u}) \qquad \forall \boldsymbol{u} \in \boldsymbol{\mathcal{V}}^p$$

for each $\varepsilon \ge 0$. Using the trivial inequality

$$\Phi_\varepsilon(t) = \frac{1}{p}\left[\left(\varepsilon^2 + t^2\right)^{p/2} - \varepsilon^p\right] \le \frac{1}{p}t^p = \Phi(t) \qquad \forall t \in \mathbb{R}_0^+,$$

we conclude that $\mathcal{J}_\varepsilon(\boldsymbol{u}) \le \mathcal{J}(\boldsymbol{u})$ for all $\boldsymbol{u} \in \boldsymbol{\mathcal{V}}^p$. We recall that $\boldsymbol{v} \in \boldsymbol{\mathcal{V}}^p$ is the unique minimizer of $\mathcal{J}$. Consequently, we arrive at the inequalities

$$\mathcal{J}_\varepsilon(\boldsymbol{v}) \le \mathcal{J}(\boldsymbol{v}) \le \mathcal{J}(\boldsymbol{v}^\varepsilon). \tag{6.23}$$

From the main theorem of calculus we deduce that

$$\begin{aligned}
\mathcal{J}_\varepsilon(\boldsymbol{v}) - \mathcal{J}_\varepsilon(\boldsymbol{v}^\varepsilon) &= \int_0^1 \mathcal{J}_\varepsilon'(\boldsymbol{v}^\varepsilon + s(\boldsymbol{v}-\boldsymbol{v}^\varepsilon))(\boldsymbol{v}-\boldsymbol{v}^\varepsilon)\,\mathrm{d}s \\
&= \int_0^1 \Big[\mathcal{J}_\varepsilon'(\boldsymbol{v}^\varepsilon + s(\boldsymbol{v}-\boldsymbol{v}^\varepsilon))([\boldsymbol{v}^\varepsilon + s(\boldsymbol{v}-\boldsymbol{v}^\varepsilon)] - \boldsymbol{v}^\varepsilon) \\
&\qquad - \mathcal{J}_\varepsilon'(\boldsymbol{v}^\varepsilon)([\boldsymbol{v}^\varepsilon + s(\boldsymbol{v}-\boldsymbol{v}^\varepsilon)] - \boldsymbol{v}^\varepsilon)\Big]\frac{\mathrm{d}s}{s} + \mathcal{J}_\varepsilon'(\boldsymbol{v}^\varepsilon)(\boldsymbol{v}-\boldsymbol{v}^\varepsilon) \\
&=: I + \mathcal{J}_\varepsilon'(\boldsymbol{v}^\varepsilon)(\boldsymbol{v}-\boldsymbol{v}^\varepsilon).
\end{aligned}$$

Since $\boldsymbol{v}^\varepsilon$ is the minimizer of $\mathcal{J}_\varepsilon$, the last term equals zero: $\mathcal{J}_\varepsilon'(\boldsymbol{v}^\varepsilon)(\boldsymbol{v}-\boldsymbol{v}^\varepsilon) = 0$. Let us estimate the term I. On the one hand, inequality (6.23) implies that

$$\begin{aligned}
I &= \mathcal{J}_\varepsilon(\boldsymbol{v}) - \mathcal{J}_\varepsilon(\boldsymbol{v}^\varepsilon) \le \mathcal{J}(\boldsymbol{v}^\varepsilon) - \mathcal{J}_\varepsilon(\boldsymbol{v}^\varepsilon) \\
&= \int_\Omega \frac{1}{p}|\boldsymbol{D}\boldsymbol{v}^\varepsilon|^p\,\mathrm{d}\boldsymbol{x} - \int_\Omega \frac{1}{p}\left[\left(\varepsilon^2 + |\boldsymbol{D}\boldsymbol{v}^\varepsilon|^2\right)^{p/2} - \varepsilon^p\right]\mathrm{d}\boldsymbol{x} \\
&= \frac{1}{p}\int_\Omega \left[|\boldsymbol{D}\boldsymbol{v}^\varepsilon|^p - \left(\varepsilon^2 + |\boldsymbol{D}\boldsymbol{v}^\varepsilon|^2\right)^{p/2}\right]\mathrm{d}\boldsymbol{x} + \frac{|\Omega|}{p}\varepsilon^p \le \frac{|\Omega|}{p}\varepsilon^p.
\end{aligned} \tag{6.24}$$

On the other hand, Lemma 2.4 and (2.46) imply that

$$
\begin{aligned}
I &= \int_0^1 \Big(\boldsymbol{\mathcal{S}}_\varepsilon(\boldsymbol{D}\boldsymbol{v}^\varepsilon + s(\boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{v}^\varepsilon)) - \boldsymbol{\mathcal{S}}_\varepsilon(\boldsymbol{D}\boldsymbol{v}^\varepsilon), [\boldsymbol{D}\boldsymbol{v}^\varepsilon + s(\boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{v}^\varepsilon)] - \boldsymbol{D}\boldsymbol{v}^\varepsilon\Big)_\Omega \frac{\mathrm{d}s}{s} \\
&\sim \int_0^1 \int_\Omega \Big(\varepsilon + |\boldsymbol{D}\boldsymbol{v}^\varepsilon + s\boldsymbol{D}(\boldsymbol{v} - \boldsymbol{v}^\varepsilon)| + |\boldsymbol{D}\boldsymbol{v}^\varepsilon|\Big)^{p-2} s|\boldsymbol{D}(\boldsymbol{v} - \boldsymbol{v}^\varepsilon)|^2 \,\mathrm{d}\boldsymbol{x}\mathrm{d}s \\
&\sim \int_0^1 \int_\Omega \Big(\varepsilon + |\boldsymbol{D}\boldsymbol{v}^\varepsilon| + s|\boldsymbol{D}(\boldsymbol{v} - \boldsymbol{v}^\varepsilon)|\Big)^{p-2} s|\boldsymbol{D}(\boldsymbol{v} - \boldsymbol{v}^\varepsilon)|^2 \,\mathrm{d}\boldsymbol{x}\mathrm{d}s \\
&\sim \int_\Omega \Big(\varepsilon + |\boldsymbol{D}\boldsymbol{v}^\varepsilon| + |\boldsymbol{D}(\boldsymbol{v} - \boldsymbol{v}^\varepsilon)|\Big)^{p-2} |\boldsymbol{D}(\boldsymbol{v} - \boldsymbol{v}^\varepsilon)|^2 \,\mathrm{d}\boldsymbol{x} \sim \|\boldsymbol{\mathcal{F}}_\varepsilon(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{F}}_\varepsilon(\boldsymbol{D}\boldsymbol{v}^\varepsilon)\|_2^2,
\end{aligned}
$$

where the constants only depend on p. This inequality together with the upper bound (6.24) yields the desired estimate (6.20). Using (2.43) and (6.20), we conclude that

$$
\|\boldsymbol{D}\boldsymbol{v}^\varepsilon - \boldsymbol{D}\boldsymbol{v}\|_p \le c\Big(\varepsilon_0|\Omega|^{\frac{1}{p}} + \|\boldsymbol{D}\boldsymbol{v}\|_p + \|\boldsymbol{D}\boldsymbol{v}^\varepsilon\|_p\Big)^{\frac{2-p}{2}} \varepsilon^{p/2}
$$

for some $c = c(p) > 0$. Due to the a priori bound (6.4), the expression within the brackets is uniformly bounded by a constant only depending on p, ε_0, Ω and $\boldsymbol{f}$. This proves (6.21).

In order to show (6.22), we recall that the functions $(\boldsymbol{v}, \pi)$ and $(\boldsymbol{v}^\varepsilon, \pi^\varepsilon)$ satisfy

$$
\begin{aligned}
(\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{w})_\Omega - (\pi, \nabla\cdot\boldsymbol{w})_\Omega &= \langle \boldsymbol{f}, \boldsymbol{w}\rangle \\
(\boldsymbol{\mathcal{S}}_\varepsilon(\boldsymbol{D}\boldsymbol{v}^\varepsilon), \boldsymbol{D}\boldsymbol{w})_\Omega - (\pi^\varepsilon, \nabla\cdot\boldsymbol{w})_\Omega &= \langle \boldsymbol{f}, \boldsymbol{w}\rangle
\end{aligned}
$$

for all $\boldsymbol{w} \in \boldsymbol{\mathcal{X}}^p$. Subtracting these equations, we immediately conclude that

$$
(\pi - \pi^\varepsilon, \nabla\cdot\boldsymbol{w})_\Omega = (\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{S}}_\varepsilon(\boldsymbol{D}\boldsymbol{v}^\varepsilon), \boldsymbol{D}\boldsymbol{w})_\Omega \qquad \forall \boldsymbol{w} \in \boldsymbol{\mathcal{X}}^p. \tag{6.25}
$$

Using the inf-sup inequality (2.68) for $\boldsymbol{\mathcal{X}}^p \times \mathcal{Q}^p$ and the identity (6.25), we deduce that

$$
\begin{aligned}
\beta\|\pi - \pi^\varepsilon\|_{p'} &\le \sup_{\boldsymbol{w}\in\boldsymbol{\mathcal{X}}^p} \frac{(\pi - \pi^\varepsilon, \nabla\cdot\boldsymbol{w})_\Omega}{\|\boldsymbol{w}\|_{1,p}} = \sup_{\boldsymbol{w}\in\boldsymbol{\mathcal{X}}^p} \frac{(\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{S}}_\varepsilon(\boldsymbol{D}\boldsymbol{v}^\varepsilon), \boldsymbol{D}\boldsymbol{w})_\Omega}{\|\boldsymbol{w}\|_{1,p}} \\
&\le \|\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{S}}_\varepsilon(\boldsymbol{D}\boldsymbol{v})\|_{p'} + \|\boldsymbol{\mathcal{S}}_\varepsilon(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{S}}_\varepsilon(\boldsymbol{D}\boldsymbol{v}^\varepsilon)\|_{p'} =: J_1 + J_2.
\end{aligned} \tag{6.26}
$$

In order to estimate J_1, we have to control $|\Phi'(t) - \Phi'_\varepsilon(t)|$. We recall that there holds

$$
\Big||a|^q - |b|^q\Big| \sim (|a| + |b|)^{q-1}|a - b| \qquad \forall a, b \in \mathbb{R} \tag{6.27}
$$

for each $q > 0$ (cf. Lemma 2.4). Applying (6.27) with $q := (2-p)/2$, we conclude that

$$
\begin{aligned}
|\Phi'_\varepsilon(t) - \Phi'(t)| &= \Big(\varepsilon^2 + t^2\Big)^{\frac{p-2}{2}} t \left|1 - \left(\frac{\varepsilon^2 + t^2}{t^2}\right)^{\frac{2-p}{2}}\right| \sim \Big(\varepsilon^2 + t^2\Big)^{\frac{p-2}{2}} t \left(1 + \frac{\varepsilon^2 + t^2}{t^2}\right)^{-\frac{p}{2}} \frac{\varepsilon^2}{t^2} \\
&\sim \Big(\varepsilon^2 + t^2\Big)^{\frac{p-2}{2}} \Big(\varepsilon^2 + 2t^2\Big)^{-\frac{p}{2}} t^{p-1}\varepsilon^2 \sim t^{p-1}\frac{\varepsilon^2}{\varepsilon^2 + t^2}
\end{aligned} \tag{6.28}
$$

uniformly in $t \in \mathbb{R}_0^+$ where we have used $\varepsilon^2 + t^2 \sim \varepsilon^2 + 2t^2$. In particular, (6.28) implies that $|\Phi'_\varepsilon(t) - \Phi'(t)| \lesssim (\varepsilon^2 + t^2)^{\frac{p-3}{2}}\varepsilon^2 \lesssim \varepsilon^{p-1}$ uniformly in $t \in \mathbb{R}_0^+$. Consequently, we obtain

$$J_1 = \|\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{S}}_\varepsilon(\boldsymbol{D}\boldsymbol{v})\|_{p'} \leq \Big(\int\limits_\Omega |\Phi'(|\boldsymbol{D}\boldsymbol{v}|) - \Phi'_\varepsilon(|\boldsymbol{D}\boldsymbol{v}|)|^{p'}\,\mathrm{d}\boldsymbol{x}\Big)^{\frac{1}{p'}} \leq c|\Omega|^{\frac{1}{p'}}\varepsilon^{p-1} \tag{6.29}$$

where the constant c only depends on p. It remains to estimate the term J_2. Using Lemma 2.7 and inequality (6.20), we deduce that for some $c = c(p, \Omega) > 0$:

$$J_2 \leq c\|\boldsymbol{\mathcal{F}}_\varepsilon(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{F}}_\varepsilon(\boldsymbol{D}\boldsymbol{v}^\varepsilon)\|_2^{\frac{2}{p'}} \leq c\varepsilon^{\frac{p2}{2p'}} = c\varepsilon^{p-1}. \tag{6.30}$$

Combining (6.26), (6.29), and (6.30), we arrive at the desired result (6.22). □

Remark 6.3. One can also derive similar error estimates as stated in Theorem 6.3 without using the minimization property of the energy functional. Since $\boldsymbol{v}^\varepsilon$ are the solutions of $(\mathbf{P}^\varepsilon)$ for $\varepsilon \geq 0$, they satisfy $(\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{w})_\Omega = \langle \boldsymbol{f}, \boldsymbol{w}\rangle$ for all $\boldsymbol{w} \in \boldsymbol{\mathcal{V}}^p$ and $(\boldsymbol{\mathcal{S}}_\varepsilon(\boldsymbol{D}\boldsymbol{v}^\varepsilon), \boldsymbol{D}\boldsymbol{w})_\Omega = \langle \boldsymbol{f}, \boldsymbol{w}\rangle$ for all $\boldsymbol{w} \in \boldsymbol{\mathcal{V}}^p$. Subtracting the latter equations, we immediately conclude that $(\boldsymbol{\mathcal{S}}_\varepsilon(\boldsymbol{D}\boldsymbol{v}^\varepsilon) - \boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{w})_\Omega = 0$ for all $\boldsymbol{w} \in \boldsymbol{\mathcal{V}}^p$. By means of (6.28) we deduce that

$$\begin{aligned}
\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}^\varepsilon) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})\|_2^2 &\sim (\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}^\varepsilon) - \boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{v}^\varepsilon - \boldsymbol{D}\boldsymbol{v})_\Omega \\
&= (\boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}^\varepsilon) - \boldsymbol{\mathcal{S}}_\varepsilon(\boldsymbol{D}\boldsymbol{v}^\varepsilon), \boldsymbol{D}\boldsymbol{v}^\varepsilon - \boldsymbol{D}\boldsymbol{v})_\Omega \\
&= \int\limits_\Omega \Big[\Phi'(|\boldsymbol{D}\boldsymbol{v}^\varepsilon|) - \Phi'_\varepsilon(|\boldsymbol{D}\boldsymbol{v}^\varepsilon|)\Big]\frac{\boldsymbol{D}\boldsymbol{v}^\varepsilon}{|\boldsymbol{D}\boldsymbol{v}^\varepsilon|} : (\boldsymbol{D}\boldsymbol{v}^\varepsilon - \boldsymbol{D}\boldsymbol{v})\,\mathrm{d}\boldsymbol{x} \\
&\sim \varepsilon^{p-1}\int\limits_\Omega \frac{\varepsilon^{3-p}|\boldsymbol{D}\boldsymbol{v}^\varepsilon|^{p-1}}{\varepsilon^2 + |\boldsymbol{D}\boldsymbol{v}^\varepsilon|^2}\frac{\boldsymbol{D}\boldsymbol{v}^\varepsilon}{|\boldsymbol{D}\boldsymbol{v}^\varepsilon|} : (\boldsymbol{D}\boldsymbol{v}^\varepsilon - \boldsymbol{D}\boldsymbol{v})\,\mathrm{d}\boldsymbol{x}.
\end{aligned} \tag{6.31}$$

The real function $g(t) := \frac{\varepsilon^{3-p}t^{p-1}}{\varepsilon^2+t^2}$, $t \in \mathbb{R}_0^+$, is bounded by one:

$$g(t) \leq \begin{cases} \frac{\varepsilon^2}{\varepsilon^2+t^2} & \text{if } t \leq \varepsilon \\ \frac{t^2}{\varepsilon^2+t^2} & \text{if } t > \varepsilon \end{cases} \qquad \Rightarrow \qquad g(t) \leq 1 \ \forall t \in \mathbb{R}_0^+.$$

Therefore, the integral in (6.31) is well-defined. Using the continuous embedding $L^p(\Omega) \hookrightarrow L^1(\Omega)$, Lemma 2.6, and the uniform a priori bound (6.4) for $\boldsymbol{v}^\varepsilon$, we finally arrive at

$$\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}^\varepsilon) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})\|_2 \leq c\varepsilon^{p-1}$$

where the constant c only depends on p, ε_0, Ω, $\boldsymbol{f}$.

Application to the FE approximation: On the basis of Theorem 6.3, we now construct our approximation scheme for singular power-law equations. For it, we require certain approximation properties of the finite element method. For particular finite elements, the following result is known (cf. Theorem 4.11): For $p \in (1, 2]$ and $\varepsilon \in [0, \varepsilon_0]$ let $(\boldsymbol{v}^\varepsilon, \pi^\varepsilon)$ be the solution to $(\mathbf{P}^\varepsilon)$ and let $(\boldsymbol{v}_h^\varepsilon, \pi_h^\varepsilon)$ be the solution to $(\mathbf{P}_h^\varepsilon)$. Assume that the solution

$(\boldsymbol{v}^\varepsilon, \pi^\varepsilon)$ satisfies $\boldsymbol{\mathcal{F}}_\varepsilon(\boldsymbol{D}\boldsymbol{v}^\varepsilon) \in W^{1,2}(\Omega)^{d\times d}(\Omega)$ and $\pi^\varepsilon \in W^{1,p'}(\Omega)$. Then, the discretization error can be estimated in terms of the maximum mesh size h as follows:

$$\|\boldsymbol{\mathcal{F}}_\varepsilon(\boldsymbol{D}\boldsymbol{v}^\varepsilon) - \boldsymbol{\mathcal{F}}_\varepsilon(\boldsymbol{D}\boldsymbol{v}_h^\varepsilon)\|_2 \le C_1 h, \qquad \tilde{\beta}\|\pi^\varepsilon - \pi_h^\varepsilon\|_{p'} \le C_2 h^{\frac{2}{p'}}. \tag{6.32}$$

The constants C_1 and C_2 depend on $\|\nabla\boldsymbol{\mathcal{F}}_\varepsilon(\boldsymbol{D}\boldsymbol{v}^\varepsilon)\|_2$, $\|\pi^\varepsilon\|_{1,p'}$, p, ε_0, Ω, and $\boldsymbol{f}$. In particular, they do not depend on ε explicitely. The error estimates (6.32) remain valid for $\varepsilon = 0$.

Remark 6.4. The error estimates (6.32) have been proven for various finite elements by [BBDR10], [Hir10] or Thm. 6.3. (Note that Thm. 6.3 has recently been published in [Hir10].) In [BBDR10], they were derived for inf-sup stable finite elements based on d-simplices such as: $\mathbb{P}_2/\mathbb{P}_0$, Crouzeix-Raviart ($\mathbb{P}_2$ plus bubble / discontinuous $\mathbb{P}_1$), MINI-element ($\mathbb{P}_1$ plus bubble / $\mathbb{P}_1$). In [Hir10] or Thm. 6.3, the error estimates (6.32) were proven for equal-order d-linear finite elements ($\mathbb{Q}_1/\mathbb{Q}_1$) based on quadrilateral meshes provided that the LPS-based stabilization proposed in Section 4.1 is used. Note that (6.32) provides optimal convergence rates with respect to the supposed regularity of the solution.

The following corollary is a simple consequence of Theorem 6.3. It yields the desired approximation scheme for singular power-law systems. Our approximation method generates a sequence of discrete functions $\boldsymbol{v}_h^\varepsilon$ which is computable in practice via Newton's method and which converges to the power-law solution.

Corollary 6.4. *Let $p \in (1,2)$ and $\varepsilon \in [0, \varepsilon_0]$. For each ε let $(\boldsymbol{v}^\varepsilon, \pi^\varepsilon) \in \boldsymbol{\mathcal{X}}^p \times \mathcal{Q}^p$ be the unique solution to $(\mathbf{P}^\varepsilon)$, and let $(\boldsymbol{v}_h^\varepsilon, \pi_h^\varepsilon) \in \boldsymbol{\mathcal{X}}_h^p \times \mathcal{Q}_h^p$ be the unique solution to $(\mathbf{P}_h^\varepsilon)$. Let us define $(\boldsymbol{v}, \pi) := (\boldsymbol{v}^0, \pi^0)$ and $(\boldsymbol{v}_h, \pi_h) := (\boldsymbol{v}_h^0, \pi_h^0)$. We assume that* **(IS)** *is satisfied. Furthermore, we suppose that $\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) \in W^{1,2}(\Omega)^{d\times d}$ and $\pi \in W^{1,p'}(\Omega)$, and that* (6.32) *holds true for $\varepsilon = 0$. Then, the solution $(\boldsymbol{v}, \pi)$ to the power-law problem can be approximated by the discrete functions $(\boldsymbol{v}_h^\varepsilon, \pi_h^\varepsilon)$ for $\varepsilon, h \searrow 0$ in the following sense:*

$$\|\boldsymbol{v} - \boldsymbol{v}_h^\varepsilon\|_{1,p} \le C_1\Big(\varepsilon^{p/2} + h\Big), \qquad \|\pi - \pi_h^\varepsilon\|_{p'} \le C_2\Big(\varepsilon^{p-1} + h^{2/p'}\Big). \tag{6.33}$$

The constants C_1, C_2 only depend on $\|\nabla\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})\|_2$, $\|\pi\|_{1,p'}$, p, ε_0, Ω, $\boldsymbol{f}$, and C_2 additionally depends on $\tilde{\beta}(p)$. As a result, for $\varepsilon := h^{2/p}$ it follows from (6.33) *that*

$$\|\boldsymbol{v} - \boldsymbol{v}_h^\varepsilon\|_{1,p} \le 2C_1 h, \qquad \|\pi - \pi_h^\varepsilon\|_{p'} \le 2C_2 h^{2/p'}.$$

Proof. Since **(IS)** is fulfilled, the discrete mixed formulation $(\mathbf{P}_h^\varepsilon)$ is equivalent to a discrete version of $(\mathbf{M}^\varepsilon)$. In particular, for each $\varepsilon \ge 0$ the discrete solution $\boldsymbol{v}_h^\varepsilon$ can be characterized as the unique minimizer of the functional $\mathcal{J}_\varepsilon$ in $\boldsymbol{\mathcal{V}}_h^p$, i.e., $\boldsymbol{v}_h^\varepsilon$ satisfies

$$\mathcal{J}_\varepsilon(\boldsymbol{v}_h^\varepsilon) = \inf_{\boldsymbol{w}_h \in \boldsymbol{\mathcal{V}}_h^p} \mathcal{J}_\varepsilon(\boldsymbol{w}_h), \qquad \boldsymbol{\mathcal{V}}_h^p := \{\boldsymbol{w}_h \in \boldsymbol{\mathcal{X}}_h^p;\ (\nabla\cdot\boldsymbol{w}_h, q_h)_\Omega = 0\,\forall q_h \in \mathcal{Q}_h^p\}.$$

Hence, we can adjust the proof of Theorem 6.3 to the discrete setting. We conclude that

$$\|\boldsymbol{D}\boldsymbol{v}_h - \boldsymbol{D}\boldsymbol{v}_h^\varepsilon\|_p \le c_1\varepsilon^{p/2}, \qquad \|\pi_h - \pi_h^\varepsilon\|_{p'} \le c_2\varepsilon^{p-1}$$

where $c_1 = c_1(p, \varepsilon_0, \Omega, \boldsymbol{f})$ and $c_2 = c_2(p, \tilde{\beta}(p), \Omega)$. Using the latter inequalities, (6.32) with $\varepsilon = 0$, Poincaré's and Korn's inequality, we easily deduce the desired estimates (6.33). □

Remark 6.5. When we choose finite element pairings $\boldsymbol{\mathcal{X}}_h^p \times \mathcal{Q}_h^p$ which do not satisfy the discrete inf-sup condition **(IS)**, then we need to stabilize the Galerkin discretization $(\mathbf{P}_h^\varepsilon)$. If standard stabilization methods such as LPS or PSPG are applied (cf. [BBJL07]), then the discrete velocity $\boldsymbol{v}_h^\varepsilon$ cannot be interpreted as the minimizer of $\mathcal{J}_\varepsilon$ in $\boldsymbol{\mathcal{V}}_h^p$ any more. Hence, we cannot apply Theorem 6.3 to the discrete setting as carried out in Corollary 6.4. However, Theorem 6.3 yields an upper bound of $(\boldsymbol{v}-\boldsymbol{v}^\varepsilon)$ in terms of ε so that

$$\|\boldsymbol{v}-\boldsymbol{v}_h^\varepsilon\|_{1,p} \le \|\boldsymbol{v}-\boldsymbol{v}^\varepsilon\|_{1,p} + \|\boldsymbol{v}^\varepsilon-\boldsymbol{v}_h^\varepsilon\|_{1,p} \le c\varepsilon^{\frac{p}{2}} + \|\boldsymbol{v}^\varepsilon-\boldsymbol{v}_h^\varepsilon\|_{1,p}. \tag{6.34}$$

In order to derive an estimate similar to (6.33), the discretization error $(\boldsymbol{v}^\varepsilon-\boldsymbol{v}_h^\varepsilon)$ needs to be estimated. For this, the error estimate (6.32) is available. Note that the constant in (6.32) depends on $\|\nabla\boldsymbol{\mathcal{F}}_\varepsilon(\boldsymbol{D}\boldsymbol{v}^\varepsilon)\|_2$ and $\|\pi^\varepsilon\|_{1,p'}$. In order to be able to deduce (6.33) from (6.34), we have to assume that there exist constants C, $C'>0$ independent of $\varepsilon\in[0,\varepsilon_0]$:

$$\|\nabla\boldsymbol{\mathcal{F}}_\varepsilon(\boldsymbol{D}\boldsymbol{v}^\varepsilon)\|_2 \le C, \qquad \|\nabla\pi^\varepsilon\|_{p'} \le C'. \tag{6.35}$$

The question is whether it is allowed to assume (6.35). In fact, the assumption $(6.35)_1$ is satisfied at least in the case of space-periodic boundary conditions, see Lemma 2.28. However assumption $(6.35)_2$ seems to be rather sophisticated. In particular, Lemma 2.28 does not enable us to make any statement: According to Lemma 2.28, the pressure-gradient $\nabla\pi^\varepsilon$ is only bounded in $L^2(\Omega)$ by a constant which might explode as $\varepsilon\searrow 0$. Alternatively, in Corollary 6.4 we can avoid assumption **(IS)** if we employ (6.31) on the discrete level instead of using a discrete version of Theorem 6.3. But then the order of the resulting error estimates is less than the one of (6.33).

6.4 Numerical experiments

For $p<2$ and $\varepsilon\ge 0$ let the generalized viscosity μ be given by (2.11b) and let the extra stress tensor $\boldsymbol{\mathcal{S}}_\varepsilon$ be given by $\boldsymbol{\mathcal{S}}_\varepsilon(\boldsymbol{D}\boldsymbol{v}) \equiv \mu(|\boldsymbol{D}\boldsymbol{v}|^2)\boldsymbol{D}\boldsymbol{v}$. From mathematical point of view, the singular power-law model ($\varepsilon=0$) is more interesting and more challenging than its regularized counterpart ($\varepsilon>0$). In particular, the discrete power-law systems cannot numerically be solved in general without regularization. In this section, we numerically justify the regularized approximation method proposed by Corollary 6.4.

We reconsider the pressure-drop problem described in Section 5.5: Find a velocity field $\boldsymbol{v}^\varepsilon \in \boldsymbol{\mathcal{X}}_\Gamma^p \equiv \{\boldsymbol{w}\in\boldsymbol{W}^{1,p}(\Omega);\, \boldsymbol{w}|_\Gamma=\boldsymbol{0}\}$ and a pressure $\pi^\varepsilon\in\mathcal{Q}_\Gamma^p\equiv L^{p'}(\Omega)$ such that

$$\begin{aligned}(\boldsymbol{\mathcal{S}}_\varepsilon(\boldsymbol{D}\boldsymbol{v}^\varepsilon),\boldsymbol{D}\boldsymbol{w})_\Omega &- (\pi^\varepsilon,\nabla\cdot\boldsymbol{w})_\Omega + (\nabla\cdot\boldsymbol{v}^\varepsilon,q)_\Omega \\ &= \sum_i \left(\frac{\mu(|\boldsymbol{D}\boldsymbol{v}^\varepsilon|^2)}{2}[\nabla\boldsymbol{v}^\varepsilon]^{\mathsf{T}}\boldsymbol{n} - b_i\boldsymbol{n},\boldsymbol{w}\right)_{S_i} \qquad \forall(\boldsymbol{w},q)\in\boldsymbol{\mathcal{X}}_\Gamma^p\times\mathcal{Q}_\Gamma^p.\end{aligned} \tag{6.36}$$

Let Ω be a rectangular channel with length L and height H, and let $b_1 := L/2$ and $b_2 := 0$. This simple pressure-drop problem seems to be a proper example due to the following two reasons: Firstly, the data such as $\boldsymbol{f}$ are independent of p, and for $\varepsilon=0$ the exact solution $(\boldsymbol{v},\pi)$ is known and it is given by (5.49). Secondly, the solution $(\boldsymbol{v},\pi)$

captures the typical flow behavior of a shear thinning fluid: For $1 < p \ll 2$ sharp boundary layers occur along the Dirichlet boundary Γ, and the measure of the critical set $\Omega_c \equiv \{\boldsymbol{x} \in \Omega;\, \nabla \boldsymbol{v}(\boldsymbol{x}) \approx \boldsymbol{0}\}$ becomes large, see Figure 5.1. The nonlinear operator associated with $\boldsymbol{\mathcal{S}}_0$ is not differentiable on Ω_c so that the convergence of Newton's method is not ensured in general. Hence, numerical problems related to the stability of the solver may be expected when the algebraic equations arising from the FE discretization of (6.36) with $\varepsilon = 0$ are solved directly by means of Newton's method. For $1 < p \ll 2$ our numerical simulations will indicate that the solution $(\boldsymbol{v}, \pi)$ cannot numerically be approximated via the direct application of the FEM-Newton algorithm but it can be approximated with help of the method proposed by Corollary 6.4.

Remark 6.6. Below we highlight the structure of the functions that solve the pressure-drop problem under consideration of $\varepsilon \geq 0$. Here, we assume that $\mu_0 = 1$. Let Φ_ε be defined in (6.1). We introduce a function $\tilde{\boldsymbol{v}}^\varepsilon : (0, L) \times (-H/2, +H/2) \to \mathbb{R}^2$ by

$$\tilde{v}_1^\varepsilon(\boldsymbol{x}) := \sqrt{2}H[\Phi_\varepsilon^*(0.5) - \Phi_\varepsilon^*(|x_2|/H)], \qquad \Phi_\varepsilon^*(t) := \int_0^t (\Phi_\varepsilon')^{-1}(s)\,\mathrm{d}s, \qquad \tilde{v}_2^\varepsilon \equiv 0.$$

If $\varepsilon = 0$, then $\Phi_0^*(t) = \frac{1}{p'} t^{p'}$ and, hence, $\tilde{\boldsymbol{v}}^0$ coincides with $\boldsymbol{v}$ given by (5.49) up to scaling. Note that $(\Phi_\varepsilon^*)'(t) = (\Phi_\varepsilon')^{-1}(t)$ for $t > 0$. Consequently, the derivative of $\tilde{v}_1^\varepsilon$ equals

$$(\tilde{v}_1^\varepsilon)'(x_2) = -\sqrt{2}H(\Phi_\varepsilon^*)'(|x_2|/H)\frac{x_2}{H|x_2|} = -\sqrt{2}(\Phi_\varepsilon')^{-1}(|x_2|/H)\frac{x_2}{|x_2|}.$$

As a result, the symmetric part of the velocity gradient takes the form

$$\boldsymbol{D}\tilde{\boldsymbol{v}}^\varepsilon(\boldsymbol{x}) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -(\Phi_\varepsilon')^{-1}(|x_2|/H)\frac{x_2}{|x_2|} \\ -(\Phi_\varepsilon')^{-1}(|x_2|/H)\frac{x_2}{|x_2|} & 0 \end{pmatrix}.$$

Since $|\boldsymbol{D}\boldsymbol{v}| := \sqrt{\boldsymbol{D}\boldsymbol{v} : \boldsymbol{D}\boldsymbol{v}}$, we conclude that $|\boldsymbol{D}\tilde{\boldsymbol{v}}^\varepsilon| = (\Phi_\varepsilon')^{-1}(|x_2|/H)$. Hence, we obtain

$$\boldsymbol{\mathcal{S}}_\varepsilon(\boldsymbol{D}\tilde{\boldsymbol{v}}^\varepsilon) = \Phi_\varepsilon'(|\boldsymbol{D}\tilde{\boldsymbol{v}}^\varepsilon|)\frac{\boldsymbol{D}\tilde{\boldsymbol{v}}^\varepsilon}{|\boldsymbol{D}\tilde{\boldsymbol{v}}^\varepsilon|} = \frac{-|x_2|}{\sqrt{2}H}\begin{pmatrix} 0 & \frac{x_2}{|x_2|} \\ \frac{x_2}{|x_2|} & 0 \end{pmatrix} = \frac{-1}{\sqrt{2}H}\begin{pmatrix} 0 & x_2 \\ x_2 & 0 \end{pmatrix}.$$

Clearly, $\nabla \cdot \boldsymbol{\mathcal{S}}_\varepsilon(\boldsymbol{D}\tilde{\boldsymbol{v}}^\varepsilon)$ is constant. If $\tilde{\boldsymbol{v}}^\varepsilon$ represents the velocity field solving the momentum equations $-\nabla \cdot \boldsymbol{\mathcal{S}}_\varepsilon(\boldsymbol{D}\tilde{\boldsymbol{v}}^\varepsilon) + \nabla\pi^\varepsilon = \boldsymbol{0}$, then the pressure $\tilde{\pi}^\varepsilon$ necessarily needs to be a linear function satisfying $\partial_{x_2}\tilde{\pi}^\varepsilon \equiv 0$.

Problem (6.36) was discretized with equal-order $\mathbb{Q}_1/\mathbb{Q}_1$ finite elements, i.e., both the velocity and pressure were discretized with bilinear finite elements based on quadrilateral meshes. Since this discretization is not "inf-sup" stable, the stabilized discrete system

$$\begin{aligned}(\boldsymbol{\mathcal{S}}_\varepsilon(\boldsymbol{D}\boldsymbol{v}_h^\varepsilon), \boldsymbol{D}\boldsymbol{w}_h)_\Omega - (\pi_h^\varepsilon, \nabla \cdot \boldsymbol{w}_h)_\Omega + s_h(\pi_h^\varepsilon)(q_h) + (\nabla \cdot \boldsymbol{v}_h^\varepsilon, q_h)_\Omega \\ = \sum_i \left(\frac{\mu(|\boldsymbol{D}\boldsymbol{v}_h^\varepsilon|^2)}{2}[\nabla \boldsymbol{v}_h^\varepsilon]^\mathsf{T}\boldsymbol{n} - b_i\boldsymbol{n}, \boldsymbol{w}_h\right)_{S_i} \qquad \forall(\boldsymbol{w}_h, q_h) \in \boldsymbol{\mathcal{X}}_{\Gamma;h}^p \times \mathcal{Q}_{\Gamma;h}^p\end{aligned} \tag{6.37}$$

was solved. The stabilization term $s_h(\pi_h^\varepsilon)(q_h)$ was chosen as in (3.54) with $\varrho_M \equiv 0$. The algebraic equations were solved by Newton's method, the linear subproblems by the GMRES method (see Section 3.4). The subsequent computations were performed with the following parameters: $L = 1.64$, $H = 0.41$, $b_1 = 0.82$, $b_2 = 0$, $\mu_0 = 0.15$. In our numerical experiments, we measured the approximation errors $\|\nabla(\boldsymbol{v} - \boldsymbol{v}_h^\varepsilon)\|_p$ and $\|\pi - \pi_h^\varepsilon\|_{p'}$ and corresponding convergence rates under global mesh refinement.

Table 6.1. Development of $\|\nabla(\boldsymbol{v} - \boldsymbol{v}_h^\varepsilon)\|_p$: Case $\varepsilon = 0$

	$p = 1.1$		$p = 1.2$		$p = 1.3$		$p = 1.5$	
#cells	error	conv.	error	conv.	error	conv.	error	conv.
256	4.55e-02	–	6.29e-02	–	6.96e-02	–	7.48e-02	–
1024	–	–	3.22e-02	0.97	3.51e-02	0.99	3.75e-02	1.00
4096	–	–	1.62e-02	0.99	1.76e-02	1.00	1.88e-02	1.00
16384	–	–	8.10e-03	1.00	8.80e-03	1.00	9.38e-03	1.00
65536	–	–	–	–	4.40e-03	1.00	4.69e-03	1.00
262144	–	–	–	–	2.20e-03	1.00	2.34e-03	1.00
expected		1.00		1.00		1.00		1.00

Example 1: In this example, we did not regularize the singular power-law model and we directly solved the discrete system (6.37) with $\varepsilon = 0$ applying Newton's method. Table 6.1 depicts the discretization errors $(\boldsymbol{v} - \boldsymbol{v}_h)$ with respect to the $\boldsymbol{W}^{1,p}(\Omega)$-norm and corresponding convergence rates for different values of p. Note that the pressure π belongs to the finite element space $\mathcal{Q}^p_{\Gamma;h}$ and, hence, π was resolved exactly up to machine accuracy. Thus convergence rates for the pressure are not presented. For $p \geq 1.3$ we observe that the discretization error behaves as $\mathcal{O}(h)$. This agrees with Theorem 4.11. For $p < 1.3$ we were not able to determine $\boldsymbol{v}_h$ numerically using Newton's method. For instance, if $p = 1.2$, then the Newton iteration did not reach the prescribed tolerance $TOL = 10^{-11}$ for the residual in case of the mesh with 65536 elements.

Table 6.2. Development of $\|\nabla(\boldsymbol{v} - \boldsymbol{v}_h^\varepsilon)\|_p$ for $\varepsilon = \varepsilon_0 h^{\frac{2}{p}}$: Case $p = 1.1$

	$\varepsilon_0 = 0$		$\varepsilon_0 = 1$		$\varepsilon_0 = 10^2$	
#cells	error	conv.	error	conv.	error	conv.
256	4.55e-02	–	4.64e-02	–	6.06e-02	–
1024	–	–	2.52e-02	0.88	2.92e-02	1.05
4096	–	–	1.29e-02	0.97	1.39e-02	1.07
16384	–	–	6.46e-03	0.99	6.73e-03	1.05
65536	–	–	3.23e-03	1.00	3.30e-03	1.03
262144	–	–	1.61e-03	1.00	1.63e-03	1.02
expected		1.00		1.00		1.00

Below we intend to illustrate the approximation scheme proposed by Corollary 6.4: Instead of solving system (6.37) with $\varepsilon = 0$, we determine the solution $\boldsymbol{v}_h^\varepsilon$ to system (6.37) for small $\varepsilon > 0$. For diminishing mesh size $h \searrow 0$, the error caused by regularization of the power-law model with $\varepsilon > 0$ dominates the discretization error. In order to obtain a convergent method for $h \searrow 0$, we couple the parameter ε with the mesh size h so that we preserve the convergence rate of the discretization error. The choice $\varepsilon = \varepsilon_0 h^{2/p}$ implies that the regularization error is of same order as the discretization error at least.

Table 6.3. Development of $\|\nabla(\boldsymbol{v} - \boldsymbol{v}_h^\varepsilon)\|_p$ for $\varepsilon = \varepsilon_0 h^{\frac{2}{p}}$: Case $p = 1.2$

	$\varepsilon_0 = 0$		$\varepsilon_0 = 1$		$\varepsilon_0 = 10^2$	
#cells	error	conv.	error	conv.	error	conv.
256	6.29e-02	–	6.33e-02	–	7.61e-02	–
1024	3.22e-02	0.97	3.23e-02	0.97	3.59e-02	1.09
4096	1.62e-02	0.99	1.62e-02	0.99	1.71e-02	1.07
16384	8.10e-03	1.00	8.11e-03	1.00	8.34e-03	1.04
65536	–	–	4.06e-03	1.00	4.11e-03	1.02
262144	–	–	2.03e-03	1.00	2.04e-03	1.01
expected		1.00		1.00		1.00

Example 2: We solved the regularized discrete system (6.37) with $\varepsilon = \varepsilon_0 h^{2/p}$. It is easy to see that π^ε coincides with π from (5.49) for all $\varepsilon \geq 0$. Indeed, π^ε is a linear function that satisfies $\partial_{x_2}\pi^\varepsilon = 0$, cf. Remark 6.6. Hence, the condition $\int_{S_i} \pi \, \mathrm{d}o = |S_i| b_i$ actually fixes the absolute value of the pressure on the inlet and outlet: $\pi^\varepsilon(\boldsymbol{x}) = b_i$ for $\boldsymbol{x} \in S_i$. Consequently, there holds $\pi^\varepsilon = \pi$ for all $\varepsilon \geq 0$. Since π is linear and, hence, $\pi \in \mathcal{Q}^p_{\Gamma;h}$ holds true, the pressure was resolved exactly up to machine accuracy: $\pi_h^\varepsilon = \pi$. Tables 6.2 – 6.3 depict the errors of approximation $(\boldsymbol{v} - \boldsymbol{v}_h^\varepsilon)$ in $\boldsymbol{W}^{1,p}(\Omega)$ with $\varepsilon = \varepsilon_0 h^{2/p}$. Since the pressure was resolved exactly, only velocity errors are presented. Independently of the value of ε_0, we expect that the error $(\boldsymbol{v} - \boldsymbol{v}_h^\varepsilon)$ in $\boldsymbol{W}^{1,p}(\Omega)$ behaves as $\mathcal{O}(h)$ due to Corollary 6.4. Considering Tables 6.2 – 6.3, we realize that the numerical results agree with the theoretical ones very well. In case of $\varepsilon_0 = 0$, the numerical results coincide with those from Example 1. The missing numbers indicate that Newton's method did not converge. More precisely, the Newton iteration did not reach the prescribed tolerance $TOL = 10^{-11}$ for the residual. Comparing the absolute errors for $\varepsilon_0 = 0$ with those for $\varepsilon_0 \neq 0$ depicted in Tables 6.2 – 6.3, we finally observe that despite the additional regularization errors the proposed approximation scheme leads to better approximation results and higher accuracy compared to the non-regularized FE approximation of singular power-law systems.

Example 3: In this example, we considered another flow configuration which is less realistic from a physical point of view but which exhibits a non-smooth analytical solution. Here, we chose the computational domain $\Omega := (-0.5, 0.5)^2$ and we defined

$$\boldsymbol{v}(\boldsymbol{x}) := |\boldsymbol{x}|^7 \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} \qquad \text{and} \qquad \pi(\boldsymbol{x}) := |\boldsymbol{x}|^b x_1 x_2.$$

Table 6.4. Development of $\|\nabla(\boldsymbol{v} - \boldsymbol{v}_h^\varepsilon)\|_p$ for $\varepsilon = \varepsilon_0 h^{\frac{2}{p}}$: Case $p = 1.3$

	$\varepsilon_0 = 0$		$\varepsilon_0 = 1$		$\varepsilon_0 = 20$	
#cells	error	conv.	error	conv.	error	conv.
256	1.12e-02	–	1.47e-02	–	3.52e-02	–
1024	5.54e-03	1.02	6.47e-03	1.18	1.37e-02	1.37
4096	2.76e-03	1.01	2.99e-03	1.11	5.05e-03	1.44
16384	1.37e-03	1.00	1.43e-03	1.06	1.95e-03	1.37
65536	6.87e-04	1.00	7.00e-04	1.03	8.27e-04	1.24
262144	3.43e-04	1.00	3.46e-04	1.02	3.76e-04	1.14
expected		1.00		1.00		1.00

The right-hand side $\boldsymbol{f}$ was chosen accordingly as $\boldsymbol{f} := -\nabla \cdot \boldsymbol{\mathcal{S}}(\boldsymbol{D}\boldsymbol{v}) + \nabla\pi$, and system (2.16) was complemented with non-homogeneous Dirichlet boundary conditions: On $\partial\Omega$ the boundary values $\boldsymbol{v}|_{\partial\Omega}$ were prescribed. The parameter b was chosen so that $\pi \in W^{1,p'}(\Omega)$. This condition is ensured for $b > -\frac{2}{p'} - 1$. We approximatively solved the corresponding weak boundary value problem for the following parameters: $p = 1.3$, $\mu_0 = 1$, and $b = -1.45$. Table 6.4 depicts the approximation errors $(\boldsymbol{v} - \boldsymbol{v}_h^\varepsilon)$ and corresponding convergence rates for $\varepsilon = \varepsilon_0 h^{2/p}$. We realize that $\boldsymbol{v}_h^\varepsilon$ converges to $\boldsymbol{v}$ in $\boldsymbol{W}^{1,p}(\Omega)$ with order one at least. The numerical results for the pressure are presented in Table 6.5. We observe that the error $(\pi - \pi_h^\varepsilon)$ measured in $L^{p'}(\Omega)$ behaves as $\mathcal{O}(h)$. In view of Tables 6.4 and 6.5, the numerical results agree with the theoretical ones stated in Corollary 6.4.

Table 6.5. Development of $\|\pi - \pi_h^\varepsilon\|_{p'}$ for $\varepsilon = \varepsilon_0 h^{\frac{2}{p}}$: Case $p = 1.3$

	$\varepsilon_0 = 0$		$\varepsilon_0 = 1$		$\varepsilon_0 = 20$	
#cells	error	conv.	error	conv.	error	conv.
256	2.84e-02	–	2.66e-02	–	3.63e-02	–
1024	1.28e-02	1.45	1.14e-02	1.22	1.17e-02	1.63
4096	6.37e-03	1.01	5.47e-03	1.06	5.19e-03	1.17
16384	3.22e-03	0.98	2.67e-03	1.03	2.53e-03	1.04
65536	1.60e-03	1.01	1.31e-03	1.02	1.24e-03	1.03
262144	7.96e-04	1.01	6.46e-04	1.02	6.08e-04	1.03
expected		0.46		0.46		0.46

Numerical complexity: Finally, for Examples 1–3 we compare the proposed regularized approximation scheme with the non-regularized one regarding numerical complexity. The numerical costs were measured by the number of iterations carried out by Newton's algorithm, see Algorithm 3.1. In Table 6.6 we depict the number of Newton iterations that were performed in order to reduce the residual up to the prescribed tolerance $TOL = 10^{-11}$ for each refinement level l. Here, $l = 1$ corresponds to the mesh with 256 cells. The number within the brackets represents the total number of iterations performed by the step-size control and it equals the number l^* that appears in Algorithm 3.1 with $\lambda = 3/4$. As initial guess for Newton's method on level l, we chose the FE solution corresponding to level $l-1$. In particular, as initial guess for Newton's method on level $l = 1$, we took the discrete solution on the mesh with 64 cells. Comparing the number of iterations for $\varepsilon_0 = 0$ with those for $\varepsilon_0 \neq 0$ depicted in Table 6.6, we observe that the solution of the non-regularized systems ($\varepsilon_0 = 0$) requires more iterations of the Newton algorithm and step-size control than the solution of the regularized systems ($\varepsilon_0 \neq 0$). We recall that in view of Tables 6.2 – 6.3 we achieved higher accuracy for reasonable values of $\varepsilon_0 > 0$. Hence, we realize that the regularized approximation method proposed by Corollary 6.4 is more efficient than the non-regularized FE approximation of singular power-law equations. To sum up, we conclude that the regularized FE approximation surpasses the non-regularized one regarding accuracy and numerical efficiency.

Table 6.6. Number of Newton iterations ($TOL = 10^{-11}$) w.r.t. refinement level

(a) Ex. with $p = 1.1$

l	$\varepsilon_0 = 0$	$\varepsilon_0 = 1$	$\varepsilon_0 = 10^2$
1	8 (11)	6 (2)	5 (3)
2	–	5 (2)	6 (2)
3	–	5 (2)	5 (2)
4	–	5 (2)	5 (2)
5	–	5 (2)	5 (2)
6	–	6 (2)	5 (2)

(b) Ex. with $p = 1.2$

l	$\varepsilon_0 = 0$	$\varepsilon_0 = 1$	$\varepsilon_0 = 10^2$
1	6 (4)	5 (2)	5 (1)
2	6 (4)	5 (1)	5 (1)
3	6 (4)	5 (1)	5 (1)
4	7 (4)	5 (1)	5 (1)
5	–	5 (1)	5 (1)
6	–	4 (1)	4 (1)

(c) Ex. with $p = 1.3$

l	$\varepsilon_0 = 0$	$\varepsilon_0 = 1$	$\varepsilon_0 = 20$
1	12 (6)	7 (0)	6 (0)
2	17 (11)	5 (0)	7 (0)
3	16 (13)	5 (0)	6 (0)
4	19 (16)	5 (0)	5 (0)
5	21 (18)	5 (0)	5 (0)
6	16 (17)	5 (0)	5 (0)

Conclusion: In this chapter we studied singular power-law systems and their numerical approximation. The application of Newton's method usually suffers from instabilities. We proposed an approximation scheme that is based on a regularization of the singular power-law model and that enables the stable approximation of singular power-law systems via Newton's method. In Corollary 6.4 we derived a priori error estimates that quantify the convergence of the proposed method. We practically validated them by numerical experiments. The numerical examples indicate that our regularized approximation scheme surpasses the non-regularized one regarding accuracy and numerical efficiency.

7 Fluids with Shear-Rate- and Pressure-Dependent Viscosity

In this chapter, we extend the finite element analysis performed so far to a wider class of fluid models and more general boundary conditions such as (2.24). We consider a class of incompressible viscous fluids whose viscosity depends on the shear rate and pressure. We restrict ourselves to shear thinning fluid models that are similar to the Carreau model, but we allow a restricted sub-linear dependence of the viscosity on the pressure (see Assumption 2.2). The fluid models under consideration appear in many practical problems, for instance, in elasto-hydrodynamic lubrication where very high pressures occur. We deal with the isothermal steady flow under various boundary conditions. First of all, we analyze the Galerkin discretization of the governing equations: We discuss the existence and uniqueness of discrete solutions, and their convergence to the solution of the original problem. Note that the mathematical theory concerned with the self-consistency of the governing equations has emerged recently, see [MNR02, FMR05, BMR07, Lan09]. We adopt the established theory in the context of discrete approximations. As before, our aim is to quantify the convergence if a finite element (FE) discretization is applied. Since the considered equations come up with additional difficulties due to the complicated structure of the viscosity, only inf-sup stable elements are considered so that no additional pressure-stabilization is needed. We derive a priori error estimates similar to (4.1), which provide optimal rates of convergence with respect to the supposed regularity of the solution, see Corollary 7.13. Finally, we demonstrate the established error estimates by numerical experiments. To the best of my knowledge, there is no further literature presenting a rigorous FE analysis for fluids with pressure-dependent viscosity. The derived error estimates coincide with the optimal error estimates for Carreau-type models established in Theorem 4.11, which are covered as a special case. The results of this chapter have already been published in Hirn et al. [Hir10].

The chapter is organized as follows: Section 7.1 deals with the weak formulation. In Section 7.2 we introduce the Galerkin discretization and we discuss its well-posedness, while in Section 7.3 we show that the discrete solutions converge to a weak solution. Many estimates of Sections 7.3 are also employed in Section 7.4, in which a priori error estimates are derived in the form of best approximation results. In Section 7.5 we apply the abstract error estimates of Section 7.4 to finite element discretizations. Section 7.6 is dedicated to numerical experiments. Finally, in Section 7.7 we verify Assumption 2.2 for a particular fluid model.

7.1 Galerkin formulation

Throughout the chapter, we assume that for $p \in (1,\infty)$ and $\varepsilon \in (0,\varepsilon_0]$ the extra stress $\boldsymbol{\mathcal{S}}$ satisfies Assumption 2.2. We consider system (2.16) complemented with mixed boundary conditions (2.24). The natural spaces for the velocity and pressure are given by

$$\begin{aligned}\boldsymbol{\mathcal{X}}^p_\Gamma &:= \{\boldsymbol{w} \in \boldsymbol{W}^{1,p}(\Omega);\ \operatorname{tr}\boldsymbol{w} = \boldsymbol{0} \text{ on } \Gamma\},\\ \mathcal{Q}^p_\Gamma &:= \{q \in L^{p'}(\Omega);\ \text{ if } |S| = 0 \text{ then } \textstyle\int_\Omega q\,\mathrm{d}\boldsymbol{x} = 0\}.\end{aligned}$$

As usual, $p' := p/(p-1)$. The following Korn inequality holds in $\boldsymbol{\mathcal{X}}^p_\Gamma$ as long as $|\Gamma| > 0$:

Lemma 7.1 (Korn's inequality). *Let $\nu \in (1,\infty)$, let $\Omega \subset \mathbb{R}^d$ be a bounded domain with $\partial\Omega, \Gamma \in C^{0,1}$, where $\Gamma \subset \partial\Omega$ has nonzero $(d-1)$-dimensional measure. Then there exists a constant $c_K := c_K(\Omega,\Gamma,\nu) > 0$ such that*

$$c_K\|\boldsymbol{w}\|_{1,\nu} \le \|\boldsymbol{D}\boldsymbol{w}\|_\nu \qquad \forall \boldsymbol{w} \in \boldsymbol{\mathcal{X}}^\nu_\Gamma.$$

Proof. The result can be found e.g. in [MNRR96, Theorem 1.10 on p. 196]; although it is formulated for $\Gamma = \partial\Omega$ there, its proof covers the case $|\Gamma| > 0$. □

Let us summarize the general assumptions that will be used in the following sections.

Assumption 7.1. *We suppose that*

- *$\Omega \subset \mathbb{R}^d$, $d \ge 2$, is a bounded domain, $\partial\Omega = \Gamma \cup S$ and $\partial\Omega, \Gamma, S \in C^{0,1}$, $|\Gamma| > 0$;*
- *Let $\varepsilon_0 > 0$ be arbitrary. The extra stress tensor $\boldsymbol{\mathcal{S}}$ belongs to the class* (2.12) *and for $p \in (1,2)$, $\varepsilon \in (0,\varepsilon_0]$, $\gamma_0 \in (0,\infty)$ it satisfies* **(A1)** – **(A2)***, see Assumption 2.2.*
- *The following data are given:*

$$\begin{aligned}&\boldsymbol{v}_0 \in \boldsymbol{W}^{1,p}(\Omega), \quad \nabla\cdot\boldsymbol{v}_0 = 0 \text{ a.e. in } \Omega, \quad \boldsymbol{v}_0 = \boldsymbol{v}_D \text{ on } \Gamma,\\ &\boldsymbol{f} \in \boldsymbol{L}^{p'}(\Omega) \quad \text{and} \quad \boldsymbol{b} \in \boldsymbol{L}^{(p^\#)'}(S), \quad \text{with } (p^\#)' := \tfrac{(d-1)p}{d(p-1)}.\end{aligned}$$

Remark 7.1. In Assumption 7.1, $p^\#$ is given by $p^\# := \frac{(d-1)p}{d-p}$ so that $\operatorname{tr}(W^{1,p}(\Omega)) \hookrightarrow L^{p^\#}(\partial\Omega)$. Indeed, it holds $\operatorname{tr}(W^{1,p}(\Omega)) \hookrightarrow W^{1-\frac{1}{p},p}(\partial\Omega)$ due to the trace theorem, and $W^{1-\frac{1}{p},p}(\partial\Omega) \hookrightarrow L^{p^\#}(\partial\Omega)$ for $\frac{p-1}{p} - \frac{d-1}{p} = -\frac{d-1}{p^\#}$ due to Sobolev's embedding theorem. The condition on $p^\#$ is equivalent to $\frac{p-d}{p} = -\frac{d-1}{p^\#}$ and, hence, $p^\# = \frac{(d-1)p}{d-p}$.

The weak formulation of system (2.16), (2.24), (2.25) reads:

(P8) *Find $(\boldsymbol{v},\pi) \in (\boldsymbol{v}_0 + \boldsymbol{\mathcal{X}}^p_\Gamma) \times \mathcal{Q}^p_\Gamma$ (the weak solution) such that*

$$(\boldsymbol{\mathcal{S}}(\pi,\boldsymbol{D}\boldsymbol{v}),\boldsymbol{D}\boldsymbol{w})_\Omega - (\pi,\nabla\cdot\boldsymbol{w})_\Omega = (\boldsymbol{f},\boldsymbol{w})_\Omega - (\boldsymbol{b},\boldsymbol{w})_S \qquad \forall \boldsymbol{w} \in \boldsymbol{\mathcal{X}}^p_\Gamma, \tag{7.1}$$

$$(\nabla\cdot\boldsymbol{v},q)_\Omega = 0 \qquad \forall q \in \mathcal{Q}^p_\Gamma. \tag{7.2}$$

The following observation plays an essential role in the analysis of **(P8)**.

Lemma 7.2. *Let Assumption 7.1 be satisfied. For any $\nu \in (1,\infty)$ there exists a constant $\beta(\nu)$ (depending on ν, Ω and Γ_P) such that*

$$0 < \beta(\nu) \le \inf_{q \in \mathcal{Q}^\nu_\Gamma} \sup_{\boldsymbol{w} \in \boldsymbol{\mathcal{X}}^\nu_\Gamma} \frac{(q, \nabla\cdot\boldsymbol{w})_\Omega}{\|q\|_{\nu'}\|\boldsymbol{w}\|_{1,\nu}}. \tag{7.3}$$

In particular, there exists a constant $\beta_0(\nu)$ depending on ν and Ω such that

$$0 < \beta_0(\nu) \le \inf_{q \in L^{\nu'}_0(\Omega)} \sup_{\boldsymbol{w} \in \boldsymbol{W}^{1,\nu}_0(\Omega)} \frac{(q, \nabla\cdot\boldsymbol{w})_\Omega}{\|q\|_{\nu'}\|\boldsymbol{w}\|_{1,\nu}}. \tag{7.4}$$

Proof. We refer to Haslinger/Stebel [HS11] and Hirn et al. [HLS10]. □

Remark 7.2 (See Remark 2.3 in [HLS10]). Lemma 7.2 reveals, in terms of the spaces $\boldsymbol{\mathcal{X}}^p_\Gamma$, $\mathcal{Q}^p_\Gamma$, why the additional constraint (2.25) is requisite to fix the level of pressure if $\partial\Omega = \Gamma$. Note that $(1, \nabla\cdot\boldsymbol{w})_\Omega = 0$ for all $\boldsymbol{w} \in \boldsymbol{W}^{1,\nu}_0(\Omega)$ and, consequently,

$$\inf_{q \in L^{\nu'}(\Omega)} \sup_{\boldsymbol{w} \in \boldsymbol{W}^{1,\nu}_0(\Omega)} \frac{(q, \nabla\cdot\boldsymbol{w})_\Omega}{\|q\|_{\nu'}\|\boldsymbol{w}\|_{1,\nu}} = 0.$$

7.2 Galerkin discretization and its well-posedness

For given $h > 0$, let $\boldsymbol{X}_h$, Q_h be finite-dimensional spaces and

$$\boldsymbol{\mathcal{X}}^p_{\Gamma;h} := \boldsymbol{X}_h \cap \boldsymbol{\mathcal{X}}^p_\Gamma, \qquad \mathcal{Q}^p_{\Gamma;h} := Q_h \cap \mathcal{Q}^p_\Gamma,$$
$$\boldsymbol{\mathcal{V}}^p_{\Gamma;h} := \left\{\boldsymbol{w}_h \in \boldsymbol{\mathcal{X}}^p_{\Gamma;h};\ (\nabla\cdot\boldsymbol{w}_h, q_h)_\Omega = 0 \text{ for all } q_h \in \mathcal{Q}^p_{\Gamma;h}\right\}.$$

We will specify the spaces $\boldsymbol{X}_h$ and Q_h in the context of finite elements in Section 7.5. As before, the symbol h will then stand for the mesh parameter. At this stage, we only require that $\boldsymbol{\mathcal{X}}^p_{\Gamma;h}$ and $\mathcal{Q}^p_{\Gamma;h}$ approximate $\boldsymbol{\mathcal{X}}^p_\Gamma$ and $\mathcal{Q}^p_\Gamma$ in the following sense

$$\lim_{h\searrow 0} \inf_{\boldsymbol{w}_h \in \boldsymbol{\mathcal{X}}^p_{\Gamma;h}} \|\boldsymbol{w} - \boldsymbol{w}_h\|_{1,p} = \lim_{h\searrow 0} \inf_{q_h \in \mathcal{Q}^p_{\Gamma;h}} \|q - q_h\|_{p'} = 0 \qquad \forall \boldsymbol{w} \in \boldsymbol{\mathcal{X}}^p_\Gamma,\ \forall q \in \mathcal{Q}^p_\Gamma. \tag{7.5}$$

The pure Galerkin approximation of Problem **(P8)** consists in replacing the Banach spaces $\boldsymbol{\mathcal{X}}^p_\Gamma$ and $\mathcal{Q}^p_\Gamma$ by their finite dimensional subspaces $\boldsymbol{\mathcal{X}}^p_{\Gamma;h}$ and $\mathcal{Q}^p_{\Gamma;h}$:

(P8$_h$) *Find $(\boldsymbol{v}_h, \pi_h) \in (\boldsymbol{v}_{0,h} + \boldsymbol{\mathcal{X}}^p_{\Gamma;h}) \times \mathcal{Q}^p_{\Gamma;h}$ (the discrete solution) such that*

$$(\boldsymbol{S}(\pi_h, \boldsymbol{D}\boldsymbol{v}_h), \boldsymbol{D}\boldsymbol{w}_h)_\Omega - (\pi_h, \nabla\cdot\boldsymbol{w}_h)_\Omega = (\boldsymbol{f}, \boldsymbol{w}_h)_\Omega - (\boldsymbol{b}, \boldsymbol{w}_h)_S \qquad \forall \boldsymbol{w}_h \in \boldsymbol{\mathcal{X}}^p_{\Gamma;h}, \tag{7.6}$$
$$(\nabla\cdot\boldsymbol{v}_h, q_h)_\Omega = 0 \qquad \forall q_h \in \mathcal{Q}^p_{\Gamma;h}. \tag{7.7}$$

Here, $\boldsymbol{v}_{0,h}$ is any[1] *appropriate approximation of the Dirichlet data which satisfies*

$$(\nabla\cdot\boldsymbol{v}_{0,h}, q_h)_\Omega = 0 \quad \forall q_h \in \mathcal{Q}^p_{\Gamma;h} \qquad \text{and} \qquad \lim_{h\searrow 0}\|\boldsymbol{v}_0 - \boldsymbol{v}_{0,h}\|_{1,p} = 0. \tag{7.8}$$

[1] In the context of finite elements $\boldsymbol{v}_{0,h}$ typically belongs to $\boldsymbol{X}_h$.

Since we have to cope with additional difficulties due to the complex structure of the viscosity, here we restrict ourselves to inf-sup stable discrete spaces $\boldsymbol{X}_h$, Q_h so that we avoid any further term in (7.7), which would be necessary for pressure stabilization. For shear rate dependent viscosities, local projection stabilization has been analyzed in Chapter 4. Below, we require that for $\nu \in (1,\infty)$ the pair $\boldsymbol{\mathcal{X}}^\nu_{\Gamma;h}$, $\mathcal{Q}^\nu_{\Gamma;h}$ satisfies the inf–sup condition:

($\mathbf{IS}^\nu_\Gamma$) *For given $\nu \in (1,\infty)$, there exists a constant $\tilde\beta(\nu)$ independent of h such that*

$$0 < \tilde\beta(\nu) \le \inf_{q\in\mathcal{Q}^\nu_{\Gamma;h}} \sup_{\boldsymbol{w}\in\boldsymbol{\mathcal{X}}^\nu_{\Gamma;h}} \frac{(q,\nabla\cdot\boldsymbol{w})_\Omega}{\|q\|_{\nu'}\|\boldsymbol{w}\|_{1,\nu}}.$$

The availability of $(\mathbf{IS}^\nu_\Gamma)$ and the value of $\tilde\beta(\nu)$ depend on the choice of the spaces $\boldsymbol{X}_h$ and Q_h. For the purposes of Thm. 7.5, we also require the following modification of $(\mathbf{IS}^\nu_\Gamma)$.

($\mathbf{IS}^\nu_0$) *There exists a constant $\tilde\beta_0(\nu)$, independent of h, such that*

$$0 < \tilde\beta_0(\nu) \le \inf_{q\in Q_h\cap L^{\nu'}_0(\Omega)} \sup_{\boldsymbol{w}\in\boldsymbol{X}_h\cap\boldsymbol{W}^{1,\nu}_0(\Omega)} \frac{(q,\nabla\cdot\boldsymbol{w})_\Omega}{\|q\|_{\nu'}\|\boldsymbol{w}\|_{1,\nu}}.$$

Below we will use $(\mathbf{IS}^2_0)$ in conjunction with the following observation:

Remark 7.3. Let $(\mathbf{IS}^2_0)$ hold, let $|S|>0$ and $p\in(1,2)$. For arbitrary $q\in\mathcal{Q}^p_{\Gamma;h}$, we write $q = q_0 + \fint_\Omega q\,\mathrm{d}\boldsymbol{x}$, where[2] $q_0 \in Q_h\cap L^2_0(\Omega)$. Since $\|q\|_2 \le \|q_0\|_2 + |\Omega|^{1/2}|\fint_\Omega q\,\mathrm{d}\boldsymbol{x}|$, we obtain

$$\tilde\beta_0(2)\left(\|q\|_2 - |\Omega|^{1/2}|\fint_\Omega q\,\mathrm{d}\boldsymbol{x}|\right) \le \sup_{\boldsymbol{w}\in\boldsymbol{\mathcal{X}}^2_{\Gamma;h}} \frac{(q,\nabla\cdot\boldsymbol{w})_\Omega}{\|\boldsymbol{w}\|_{1,2}}, \qquad \forall q\in\mathcal{Q}^p_{\Gamma;h}. \tag{7.9}$$

Below, in Thm. 7.3 we show the existence of solutions to $(\mathbf{P8}_h)$, and in Thm. 7.4 we discuss the conditions that guarantee the uniqueness of solutions to both $(\mathbf{P8}_h)$ and $(\mathbf{P8})$.

Theorem 7.3 (Existence of discrete solutions). *Let Assumption 7.1 hold. Let $\boldsymbol{\mathcal{X}}^p_{\Gamma;h}$ and $\mathcal{Q}^p_{\Gamma;h}$ fulfill $(\mathbf{IS}^p_\Gamma)$ with $\tilde\beta(p)>0$ arbitrary. Then there exists a solution to $(\mathbf{P8}_h)$. Moreover, any such solution $(\boldsymbol{v}_h,\pi_h)$ satisfies the a priori estimate*

$$\|\boldsymbol{v}_h\|_{1,p} + \|\boldsymbol{\mathcal{S}}(\pi_h,\boldsymbol{D}\boldsymbol{v}_h)\|_{p'} + \tilde\beta(p)\|\pi_h\|_{p'} \le K. \tag{7.10}$$

The constant K only depends on $\Omega,\Gamma,p,\varepsilon_0,\sigma_0,\sigma_1,\|\boldsymbol{f}\|_{p'},\|\boldsymbol{b}\|_{(p^\#)';S}$ and $\|\boldsymbol{v}_{0,h}\|_{1,p}$.

Proof. The proof is similar to the proof of Lemma 4.6. For any $\delta>0$ (small), we consider the quasi-compressible problem $(\mathbf{P8}^\delta_h)$: find $(\boldsymbol{v}^\delta_h,\pi^\delta_h)\in(\boldsymbol{v}_{0,h}+\boldsymbol{\mathcal{X}}^p_{\Gamma;h})\times\mathcal{Q}^p_{\Gamma;h}$ such that

$$(\boldsymbol{\mathcal{S}}(\pi^\delta_h,\boldsymbol{D}\boldsymbol{v}^\delta_h),\boldsymbol{D}\boldsymbol{w}_h)_\Omega - (\pi^\delta_h,\nabla\cdot\boldsymbol{w}_h)_\Omega = (\boldsymbol{f},\boldsymbol{w}_h)_\Omega - (\boldsymbol{b},\boldsymbol{w}_h)_S \qquad \forall\boldsymbol{w}_h\in\boldsymbol{\mathcal{X}}^p_{\Gamma;h}, \tag{7.11}$$

$$\delta(\pi^\delta_h,q_h)_\Omega + (q_h,\nabla\cdot\boldsymbol{v}^\delta_h)_\Omega = 0 \qquad \forall q_h\in\mathcal{Q}^p_{\Gamma;h}. \tag{7.12}$$

[2] Here we assume that constants belong to Q_h.

The inserted term $\delta(\pi_h^\delta, q_h)_\Omega$ ensures the coercivity of the equations with respect to the pressure and allows to use the Brouwer fixed-point theorem to establish the solution to $(\mathbf{P8}_h^\delta)$. Indeed, setting $\boldsymbol{w}_h := \boldsymbol{v}_h^\delta - \boldsymbol{v}_{0,h}$ and $q_h := \pi_h^\delta$, summing the equations and using Hölder's and Korn's inequality, $(7.8)_1$, the embedding $\operatorname{tr}(\boldsymbol{W}^{1,p}(\Omega)) \hookrightarrow \boldsymbol{L}^{p^\#}(\partial\Omega)$, the following estimate (which can be derived from (2.52) and Hölder's inequality)

$$(\boldsymbol{\mathcal{S}}(\pi_h^\delta, \boldsymbol{D}\boldsymbol{v}_h^\delta), \boldsymbol{D}\boldsymbol{v}_h^\delta - \boldsymbol{D}\boldsymbol{v}_{0,h})_\Omega \geq \frac{\sigma_0}{2p}\|\boldsymbol{D}\boldsymbol{v}_h^\delta\|_p^p - \frac{\sigma_1}{p-1}\|\boldsymbol{D}\boldsymbol{v}_h^\delta\|_p^{p-1}\|\boldsymbol{D}\boldsymbol{v}_{0,h}\|_p - \frac{\sigma_0}{2p}|\Omega|\varepsilon^p,$$

and Young's inequality, we obtain the a priori bound

$$\delta\|\pi_h^\delta\|_2^2 + \|\boldsymbol{v}_h^\delta\|_{1,p}^p + \|\boldsymbol{\mathcal{S}}(\pi_h^\delta, \boldsymbol{D}\boldsymbol{v}_h^\delta)\|_{p'}^{p'} \leq C,$$

where $C > 0$ depends on $\Omega, \Gamma, p, \varepsilon_0, \sigma_0, \sigma_1, \|\boldsymbol{f}\|_{p'}, \|\boldsymbol{b}\|_{(p^\#)';S}$ and $\|\boldsymbol{v}_{0,h}\|_{1,p}$. In particular, C is independent of δ and h. Therefore, using $(\mathbf{IS}_\Gamma^p)$ and (7.11), we observe that

$$\tilde{\beta}(p)\|\pi_h^\delta\|_{p'} \leq \sup_{\boldsymbol{w}_h \in \boldsymbol{\mathcal{X}}_{\Gamma;h}^p} \frac{(\pi_h^\delta, \nabla\cdot\boldsymbol{w}_h)_\Omega}{\|\boldsymbol{w}_h\|_{1,p}} \leq C,$$

with $C > 0$ and $\tilde{\beta}(p) > 0$ independent of δ and h. The same arguments applied to $(\mathbf{P8}_h)$ prove (7.10). Since $\boldsymbol{\mathcal{X}}_{\Gamma;h}^p$ and $\mathcal{Q}_{\Gamma;h}^p$ are of finite dimension, the uniform bounds above imply that there exists $(\boldsymbol{v}_h, \pi_h) \in (\boldsymbol{v}_{0,h} + \boldsymbol{\mathcal{X}}_{\Gamma;h}^p) \times \mathcal{Q}_{\Gamma;h}^p$ such that (for some sequence $\delta_n \searrow 0$)

$$\begin{aligned} \boldsymbol{v}_h^{\delta_n} &\to \boldsymbol{v}_h && \text{in } \boldsymbol{W}^{1,p}(\Omega),\\ \pi_h^{\delta_n} &\to \pi_h && \text{in } L^{p'}(\Omega),\\ \boldsymbol{\mathcal{S}}(\pi_h^{\delta_n}, \boldsymbol{D}\boldsymbol{v}_h^{\delta_n}) &\to \boldsymbol{\mathcal{S}}(\pi_h, \boldsymbol{D}\boldsymbol{v}_h) && \text{in } L^{p'}(\Omega)^{d\times d}. \end{aligned}$$

Consequently, $(\boldsymbol{v}_h, \pi_h)$ is a solution to $(\mathbf{P8}_h)$. □

The constant K in (7.10) does not depend on h since $\|\boldsymbol{v}_{0,h}\|_{1,p} \leq 2\|\boldsymbol{v}_0\|_{1,p}$ for $h \leq h_0$. For the subsequent analysis we recall the natural distance $d(\cdot,\cdot)$ defined in (2.53). According to Thm. 7.3, discrete solutions exist regardless of **(A2)**. However, uniqueness of a solution can only be shown by means of **(A2)** under a smallness assumption on γ_0 as depicted by

Theorem 7.4 (Uniqueness). *Provided that* $(\mathbf{IS}_\Gamma^2)$ *is satisfied and*

$$\gamma_0 < \tilde{\beta}(2)\varepsilon^{\frac{2-p}{2}}\frac{\sigma_0}{\sigma_0+\sigma_1}, \tag{7.13}$$

the solution to $(\mathbf{P8}_h)$ *in Theorem 7.3 is uniquely determined.*

Similarly, there is at most one solution to **(P8)** *if Assumption 7.1 is satisfied and*

$$\gamma_0 < \beta(2)\varepsilon^{\frac{2-p}{2}}\frac{\sigma_0}{\sigma_0+\sigma_1}.$$

Proof. We prove the uniqueness of a solution to $(\mathbf{P8}_h)$. The second statement can be proven analogously. Let $(\boldsymbol{v}_h^i, \pi_h^i)$, $i = 1, 2$, be two solutions to $(\mathbf{P8}_h)$. Then we realize that

$$(\boldsymbol{\mathcal{S}}(\pi_h^1, \boldsymbol{D}\boldsymbol{v}_h^1) - \boldsymbol{\mathcal{S}}(\pi_h^2, \boldsymbol{D}\boldsymbol{v}_h^2), \boldsymbol{D}\boldsymbol{w}_h)_\Omega = (\pi_h^1 - \pi_h^2, \nabla \cdot \boldsymbol{w}_h)_\Omega \quad \forall \boldsymbol{w}_h \in \boldsymbol{\mathcal{X}}^p_{\Gamma;h}.$$

In particular, choosing $\boldsymbol{w}_h := \boldsymbol{v}_h^1 - \boldsymbol{v}_h^2$ we observe

$$(\boldsymbol{\mathcal{S}}(\pi_h^1, \boldsymbol{D}\boldsymbol{v}_h^1) - \boldsymbol{\mathcal{S}}(\pi_h^2, \boldsymbol{D}\boldsymbol{v}_h^2), \boldsymbol{D}\boldsymbol{v}_h^1 - \boldsymbol{D}\boldsymbol{v}_h^2)_\Omega = 0$$

and we thus obtain from (2.54) that

$$d(\boldsymbol{v}_h^1, \boldsymbol{v}_h^2)^2 \leq \frac{\gamma_0^2}{\sigma_0^2} \|\pi_h^1 - \pi_h^2\|_2^2. \tag{7.14}$$

Hence, $(\mathbf{IS}^2_\Gamma)$ and (2.56) yield the following estimate

$$\begin{aligned}
\tilde{\beta}(2)\|\pi_h^1 - \pi_h^2\|_2 &\leq \sup_{\boldsymbol{w}_h \in \boldsymbol{\mathcal{X}}^2_{\Gamma;h}} \frac{(\pi_h^1 - \pi_h^2, \nabla \cdot \boldsymbol{w}_h)_\Omega}{\|\boldsymbol{w}_h\|_{1,2}} \\
&\leq \|\boldsymbol{\mathcal{S}}(\pi_h^1, \boldsymbol{D}\boldsymbol{v}_h^1) - \boldsymbol{\mathcal{S}}(\pi_h^2, \boldsymbol{D}\boldsymbol{v}_h^2)\|_2 \\
&\leq \sigma_1 \varepsilon^{\frac{p-2}{2}} d(\boldsymbol{v}_h^1, \boldsymbol{v}_h^2) + \gamma_0 \varepsilon^{\frac{p-2}{2}} \|\pi_h^1 - \pi_h^2\|_2,
\end{aligned} \tag{7.15}$$

which together with (7.14) and (7.13) leads to $\pi_h^1 = \pi_h^2$ a.e. in Ω and to $d(\boldsymbol{v}_h^1, \boldsymbol{v}_h^2) = 0$. But this completes the proof, because (2.60), (2.43) and the a priori bound (7.10) ensure that $\|\boldsymbol{D}\boldsymbol{v}_h^1 - \boldsymbol{D}\boldsymbol{v}_h^2\|_p^2 \leq C\, d(\boldsymbol{v}_h^1, \boldsymbol{v}_h^2)^2 = 0$. Since $|\Gamma| > 0$, Lemma 7.1 yields $\boldsymbol{v}_h^1 = \boldsymbol{v}_h^2$ a.e. in Ω.□

7.3 Convergence of the discrete solutions

In this section, we show that the discrete solutions generated by $(\mathbf{P8}_h)$ converge to a weak solution solving the original problem $(\mathbf{P8})$. In particular, we establish the existence of a solution to $(\mathbf{P8})$ as the limit of the discrete solutions. Note that the well-posedness of $(\mathbf{P8})$ has already been resolved: For $\Gamma = \partial\Omega$ this was published in [FMR05, Lan09], while the case $|S| > 0$ was conducted in [LS11a]. In these works, the proof was carried out in a different way than here: First a quasi-compressible approximation to $(\mathbf{P8})$ was established (by the Galerkin method), and later it was shown that this approximation converges (on the continuous level) to the "incompressible" solution to $(\mathbf{P8})$. Here, since our concern lies with the finite element discretization, the weak solution is established directly as a limit of discrete solutions satisfying the incompressibility constraint (7.7).

Theorem 7.5 (Convergence of discrete solutions). *Let the assumptions of Theorem 7.3 hold, let the discrete spaces* $\{(\boldsymbol{\mathcal{X}}^p_{\Gamma;h}, \mathcal{Q}^p_{\Gamma;h})\}_{h>0}$ *satisfy* (7.5)*, and let* $\{\boldsymbol{v}_{0,h}\}_{h>0}$ *satisfy* (7.8)*. In addition, let* $(\mathbf{IS}^2_0)$ *hold and let* γ_0 *fulfill*

$$\gamma_0 < \tilde{\beta}_0(2)\varepsilon^{\frac{2-p}{2}} \frac{\sigma_0}{\sigma_0 + \sigma_1}. \tag{7.16}$$

Then, the solutions to $(\mathbf{P8}_h)$ *converge to a solution to* $(\mathbf{P8})$ *as follows,*

$$(\boldsymbol{v}_{h_n}, \pi_{h_n}) \to (\boldsymbol{v}, \pi) \quad \textit{strongly in } \boldsymbol{W}^{1,p}(\Omega) \times L^{p'}(\Omega), \quad \textit{for some } h_n \searrow 0. \tag{7.17}$$

If the solution to $(\mathbf{P8})$ *is unique, then the whole sequence* $\{(\boldsymbol{v}_h, \pi_h)\}_{h>0}$ *tends to* $(\boldsymbol{v}, \pi)$.

Remark 7.4. Note that $\tilde{\beta}_0(2)$ appears in (7.16) even in the case $|S| > 0$.

Proof of Theorem 7.5. Theorem 7.3 ensures that solutions $(\boldsymbol{v}_h, \pi_h) \in (\boldsymbol{v}_{0,h} + \boldsymbol{\mathcal{X}}^p_{\Gamma;h}) \times \mathcal{Q}^p_{\Gamma;h}$ to $(\mathbf{P8}_h)$ exist and satisfy the a priori estimate (7.10). Hence, there exist $(\boldsymbol{v}, \pi) \in (\boldsymbol{v}_0 + \boldsymbol{\mathcal{X}}^p_\Gamma) \times \mathcal{Q}^p_\Gamma$ and $\overline{\boldsymbol{\mathcal{S}}} \in L^{p'}(\Omega)^{d\times d}$ such that for a sequence $h_n \searrow 0$ there hold

$$\boldsymbol{v}_{h_n} \rightharpoonup \boldsymbol{v} \qquad \text{weakly in } \boldsymbol{W}^{1,p}(\Omega), \tag{7.18}$$

$$\pi_{h_n} \rightharpoonup \pi \qquad \text{weakly in } L^{p'}(\Omega), \tag{7.19}$$

$$\boldsymbol{\mathcal{S}}(\pi_{h_n}, \boldsymbol{D}\boldsymbol{v}_{h_n}) \rightharpoonup \overline{\boldsymbol{\mathcal{S}}} \qquad \text{weakly in } L^{p'}(\Omega)^{d\times d}. \tag{7.20}$$

Obviously, the weak limits satisfy equation (7.2) and

$$(\overline{\boldsymbol{\mathcal{S}}}, \boldsymbol{D}\boldsymbol{w})_\Omega - (\pi, \nabla\cdot\boldsymbol{w})_\Omega = (\boldsymbol{f}, \boldsymbol{w})_\Omega - (\boldsymbol{b}, \boldsymbol{w})_S \qquad \forall \boldsymbol{w} \in \boldsymbol{\mathcal{X}}^p_\Gamma. \tag{7.21}$$

Here, we have used the density (7.5). Subtracting (7.21) and (7.6), we observe

$$(\boldsymbol{\mathcal{S}}(\pi_{h_n}, \boldsymbol{D}\boldsymbol{v}_{h_n}) - \overline{\boldsymbol{\mathcal{S}}}, \boldsymbol{D}\boldsymbol{w}_{h_n})_\Omega = (\pi_{h_n} - \pi, \nabla\cdot\boldsymbol{w}_{h_n})_\Omega \qquad \forall \boldsymbol{w}_{h_n} \in \boldsymbol{\mathcal{X}}^p_{\Gamma;h_n}. \tag{7.22}$$

Then, (7.22) with $\boldsymbol{w}_h := \boldsymbol{v}_{h_n} - \boldsymbol{v}_{0,h_n}$ implies

$$\begin{aligned}
&(\boldsymbol{\mathcal{S}}(\pi_{h_n}, \boldsymbol{D}\boldsymbol{v}_{h_n}) - \boldsymbol{\mathcal{S}}(\pi, \boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{v}_{h_n} - \boldsymbol{D}\boldsymbol{v})_\Omega = (\pi_{h_n} - \pi, \nabla\cdot(\boldsymbol{v}_{h_n} - \boldsymbol{v}_{0,h_n}))_\Omega \\
&\quad + (\overline{\boldsymbol{\mathcal{S}}}, \boldsymbol{D}\boldsymbol{v}_{h_n} - \boldsymbol{D}\boldsymbol{v}_{0,h_n})_\Omega + (\boldsymbol{\mathcal{S}}(\pi_{h_n}, \boldsymbol{D}\boldsymbol{v}_{h_n}), \boldsymbol{D}\boldsymbol{v}_{0,h_n} - \boldsymbol{D}\boldsymbol{v})_\Omega - (\boldsymbol{\mathcal{S}}(\pi, \boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{v}_{h_n} - \boldsymbol{D}\boldsymbol{v})_\Omega.
\end{aligned}$$

Using (7.8), (7.7), and (7.2), we realize that

$$\begin{aligned}
(\boldsymbol{\mathcal{S}}(\pi_{h_n}, \boldsymbol{D}\boldsymbol{v}_{h_n}) - \boldsymbol{\mathcal{S}}(\pi, \boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{v}_{h_n} - \boldsymbol{D}\boldsymbol{v})_\Omega &= (\pi, \nabla\cdot(\boldsymbol{v} - \boldsymbol{v}_{h_n}))_\Omega + (\pi, \nabla\cdot(\boldsymbol{v}_{0,h_n} - \boldsymbol{v}_0))_\Omega \\
&\quad + (\overline{\boldsymbol{\mathcal{S}}}, \boldsymbol{D}\boldsymbol{v}_{h_n} - \boldsymbol{D}\boldsymbol{v})_\Omega + (\overline{\boldsymbol{\mathcal{S}}} - \boldsymbol{\mathcal{S}}(\pi_{h_n}, \boldsymbol{D}\boldsymbol{v}_{h_n}), \boldsymbol{D}\boldsymbol{v})_\Omega \\
&\quad + (\boldsymbol{\mathcal{S}}(\pi_{h_n}, \boldsymbol{D}\boldsymbol{v}_{h_n}) - \overline{\boldsymbol{\mathcal{S}}}, \boldsymbol{D}\boldsymbol{v}_{0,h_n})_\Omega - (\boldsymbol{\mathcal{S}}(\pi, \boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{v}_{h_n} - \boldsymbol{D}\boldsymbol{v})_\Omega.
\end{aligned}$$

Recalling (7.18)–(7.20) and using (7.8), we conclude that

$$(\boldsymbol{\mathcal{S}}(\pi_{h_n}, \boldsymbol{D}\boldsymbol{v}_{h_n}) - \boldsymbol{\mathcal{S}}(\pi, \boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{v}_{h_n} - \boldsymbol{D}\boldsymbol{v})_\Omega = \mathrm{o}(1), \qquad h_n \searrow 0, \tag{7.23}$$

where $\mathrm{o}(1)$ denotes an arbitrary sequence that tends to zero for $h_n \searrow 0$. Furthermore, from (2.43), (7.10), (2.54), and (7.23) we deduce (cf. (7.14))

$$C\,\|\boldsymbol{D}\boldsymbol{v}_{h_n} - \boldsymbol{D}\boldsymbol{v}\|_p^2 \le d(\boldsymbol{v}_{h_n}, \boldsymbol{v})^2 \le \frac{\gamma_0^2}{\sigma_0^2}\|\pi_{h_n} - \pi\|_2^2 + o(1) \tag{7.24}$$

for some $C > 0$ independent of h_n. We suppose for a while that

$$\tilde{\beta}_0(2)\|\pi_{h_n} - \pi\|_2 \le \|\boldsymbol{\mathcal{S}}(\pi_{h_n}, \boldsymbol{D}\boldsymbol{v}_{h_n}) - \boldsymbol{\mathcal{S}}(\pi, \boldsymbol{D}\boldsymbol{v})\|_2 + o(1). \tag{7.25}$$

Then, combining (7.25) and (2.56), we arrive at

$$\tilde{\beta}_0(2)\|\pi_{h_n} - \pi\|_2 \le \sigma_1 \varepsilon^{\frac{p-2}{2}} d(\boldsymbol{v}_{h_n}, \boldsymbol{v}) + \gamma_0 \varepsilon^{\frac{p-2}{2}} \|\pi_{h_n} - \pi\|_2 + o(1), \qquad h_n \searrow 0.$$

Using (7.24) and assumption (7.16), we conclude that $\|\pi_{h_n} - \pi\|_2 \le o(1)$. Consequently, (7.24) also yields $\|\boldsymbol{D}\boldsymbol{v}_{h_n} - \boldsymbol{D}\boldsymbol{v}\|_p \le o(1)$, which finally implies that

$$\pi_{h_n} \to \pi \text{ a.e. in } \Omega \quad \text{and} \quad \boldsymbol{D}\boldsymbol{v}_{h_n} \to \boldsymbol{D}\boldsymbol{v} \text{ a.e. in } \Omega.$$

This allows us to apply Vitali's lemma and to identify $\overline{\boldsymbol{\mathcal{S}}}$,

$$\int_\Omega \boldsymbol{\mathcal{S}}(\pi_{h_n}, \boldsymbol{D}\boldsymbol{v}_{h_n}) : \boldsymbol{D}\boldsymbol{w}\, \mathrm{d}\boldsymbol{x} \to \int_\Omega \boldsymbol{\mathcal{S}}(\pi, \boldsymbol{D}\boldsymbol{v}) : \boldsymbol{D}\boldsymbol{w}\, \mathrm{d}\boldsymbol{x} = \int_\Omega \overline{\boldsymbol{\mathcal{S}}} : \boldsymbol{D}\boldsymbol{w}\, \mathrm{d}\boldsymbol{x} \qquad \forall \boldsymbol{w} \in \boldsymbol{\mathcal{X}}^p_\Gamma.$$

Therefore, it only remains to show (7.25). Define $\tilde{\boldsymbol{w}}_{h_n} \in \boldsymbol{\mathcal{X}}^2_{\Gamma;h_n}$, $\|\tilde{\boldsymbol{w}}_{h_n}\|_{1,2} = 1$, such that

$$\sup_{\boldsymbol{w}_{h_n} \in \boldsymbol{\mathcal{X}}^2_{\Gamma;h_n}} \frac{(\pi_{h_n} - \pi, \nabla \cdot \boldsymbol{w}_{h_n})_\Omega}{\|\boldsymbol{w}_{h_n}\|_{1,2}} = (\pi_{h_n} - \pi, \nabla \cdot \tilde{\boldsymbol{w}}_{h_n})_\Omega.$$

Then, there exists $\tilde{\boldsymbol{w}} \in \boldsymbol{\mathcal{X}}^2_\Gamma$ such that (for a not-relabelled subsequence) $\tilde{\boldsymbol{w}}_{h_n} - \tilde{\boldsymbol{w}} \rightharpoonup 0$ weakly in $\boldsymbol{\mathcal{X}}^2_\Gamma$ and[3] $\|\tilde{\boldsymbol{w}}_{h_n} - \tilde{\boldsymbol{w}}\|_{1,2} \le 1$. Hence, using (7.22) and (7.20), we obtain:

$$\begin{aligned}(\pi_{h_n} - \pi, \nabla \cdot \tilde{\boldsymbol{w}}_{h_n})_\Omega &= (\boldsymbol{\mathcal{S}}(\pi_{h_n}, \boldsymbol{D}\boldsymbol{v}_{h_n}) - \overline{\boldsymbol{\mathcal{S}}}, \boldsymbol{D}\tilde{\boldsymbol{w}}_{h_n} - \boldsymbol{D}\tilde{\boldsymbol{w}})_\Omega + o(1)\\ &= (\boldsymbol{\mathcal{S}}(\pi_{h_n}, \boldsymbol{D}\boldsymbol{v}_{h_n}) - \boldsymbol{\mathcal{S}}(\pi, \boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\tilde{\boldsymbol{w}}_{h_n} - \boldsymbol{D}\tilde{\boldsymbol{w}})_\Omega + o(1)\\ &\le \|\boldsymbol{\mathcal{S}}(\pi_{h_n}, \boldsymbol{D}\boldsymbol{v}_{h_n}) - \boldsymbol{\mathcal{S}}(\pi, \boldsymbol{D}\boldsymbol{v})\|_2 + o(1), \qquad h_n \searrow 0.\end{aligned}$$

Recalling (7.9) and using that $\fint_\Omega \pi_{h_n} - \pi \, \mathrm{d}\boldsymbol{x} \to 0$, we deduce that for any $q_{h_n} \in \mathcal{Q}^p_{\Gamma;h_n}$:

$$\begin{aligned}\tilde{\beta}_0(2)\|\pi_{h_n} - q_{h_n}\|_2 &\le \sup_{\boldsymbol{w}_{h_n} \in \boldsymbol{\mathcal{X}}^2_{\Gamma;h_n}} \frac{(\pi_{h_n} - q_{h_n}, \nabla \cdot \boldsymbol{w}_{h_n})_\Omega}{\|\boldsymbol{w}_{h_n}\|_{1,2}} + \tilde{\beta}_0(2)|\Omega|^{1/2} \left| \fint_\Omega \pi_{h_n} - q_{h_n} \, \mathrm{d}\boldsymbol{x} \right|\\ &\le \sup_{\boldsymbol{w}_{h_n} \in \boldsymbol{\mathcal{X}}^2_{\Gamma;h_n}} \frac{(\pi_{h_n} - \pi, \nabla \cdot \boldsymbol{w}_{h_n})_\Omega}{\|\boldsymbol{w}_{h_n}\|_{1,2}} + \|\pi - q_{h_n}\|_2 + C \left| \fint_\Omega \pi_{h_n} - q_{h_n} \, \mathrm{d}\boldsymbol{x} \right|\\ &\le \|\boldsymbol{\mathcal{S}}(\pi_{h_n}, \boldsymbol{D}\boldsymbol{v}_{h_n}) - \boldsymbol{\mathcal{S}}(\pi, \boldsymbol{D}\boldsymbol{v})\|_2 + C\,\|\pi - q_{h_n}\|_2 + o(1), \qquad h_n \searrow 0,\end{aligned}$$

with $C > 0$ independent of h_n. Using the density of $\{\mathcal{Q}^p_{\Gamma;h_n}\}$ in $\mathcal{Q}^p_\Gamma$, we finally assert (7.25):

$$\begin{aligned}\tilde{\beta}_0(2)\|\pi_{h_n} - \pi\|_2 &\le \tilde{\beta}_0(2) \inf_{q_{h_n} \in \mathcal{Q}^p_{\Gamma;h_n}} \{\|\pi_{h_n} - q_{h_n}\|_2 + \|q_{h_n} - \pi\|_2\}\\ &\le \|\boldsymbol{\mathcal{S}}(\pi_{h_n}, \boldsymbol{D}\boldsymbol{v}_{h_n}) - \boldsymbol{\mathcal{S}}(\pi, \boldsymbol{D}\boldsymbol{v})\|_2 + o(1), \qquad h_n \searrow 0.\end{aligned}$$

This completes the proof. □

Theorem 7.5 guarantees the existence of a solution to **(P8)** provided that there is a suitable family of discrete spaces $\{\boldsymbol{\mathcal{X}}^p_{\Gamma;h}, \mathcal{Q}^p_{\Gamma;h}\}_{h>0}$. The proper existence result is formulated in Corollary 7.6. For its proof one constructs an appropriate family of discrete spaces that approximates the Banach spaces $\{\boldsymbol{\mathcal{X}}_\Gamma, \mathcal{Q}_\Gamma\}$ and that satisfies the inf-sup condition $(\mathbf{IS}^2_0)$ with a constant $\tilde{\beta}_0(2)$ which is almost equal to $\beta_0(2)$. To any discrete pressure space one assigns a rich enough discrete velocity space. The construction of such spaces, which is carried out in [Hir10], is based on the fact that the used Banach spaces are separable.

[3]Indeed, $\|\tilde{\boldsymbol{w}}\|^2_{1,2} \le 2(\tilde{\boldsymbol{w}}_{h_n}, \tilde{\boldsymbol{w}})_{1,2;\Omega}$ for n large enough, which implies $\|\tilde{\boldsymbol{w}}_{h_n} - \tilde{\boldsymbol{w}}\|^2_{1,2} \le \|\tilde{\boldsymbol{w}}_{h_n}\|^2_{1,2}$ $(= 1)$.

Corollary 7.6 (Existence of solutions). *Let Assumption 7.1 hold and*

$$\gamma_0 < \beta_0(2)\varepsilon^{\frac{2-p}{2}} \frac{\sigma_0}{\sigma_0 + \sigma_1}. \tag{7.26}$$

Then there exists a weak solution to **(P8)**. *Any solution to* **(P8)** *fulfills the a priori estimate*

$$\|\boldsymbol{v}\|_{1,p} + \|\boldsymbol{\mathcal{S}}(\pi, \boldsymbol{D}\boldsymbol{v})\|_{p'} + \beta(p)\|\pi\|_{p'} \leq K. \tag{7.27}$$

The constant K only depends on $\Omega, \Gamma, p, \varepsilon_0, \sigma_0, \sigma_1, \|\boldsymbol{f}\|_{p'}, \|\boldsymbol{b}\|_{(p^\#)';S}$ and $\|\boldsymbol{v}_0\|_{1,p}$.

Proof. The proof follows from Theorems 7.3 and 7.5 if the family of discrete spaces $\{\boldsymbol{\mathcal{X}}^p_{\Gamma;h}, \mathcal{Q}^p_{\Gamma;h}\}_{h>0}$ is chosen appropriately. Details can be found in [Hir10]. □

7.4 A priori error estimates

In this section we aim to derive a priori estimates for the error of approximation $\boldsymbol{v} - \boldsymbol{v}_h$ and $\pi - \pi_h$. For the remainder of this chapter, let us use the convention that $(\boldsymbol{v}, \pi)$ and $(\boldsymbol{v}_h, \pi_h)$ denote the solutions to **(P8)** and **(P8$_h$)**, respectively. Their existence and uniqueness was shown in Sections 7.2 and 7.3. The main results are given by Corollaries 7.9 and 7.10 which state a priori error estimates in the form of a best approximation result.

Lemma 7.7. *Let Assumption 7.1 hold, let $d(\cdot,\cdot)$ be defined by* (2.53). *For each $\delta > 0$ there exists a constant $c_\delta > 0$ such that for all $\boldsymbol{u}_h \in (\boldsymbol{v}_{0,h} + \boldsymbol{\mathcal{V}}^p_{\Gamma;h})$, $r_h \in \mathcal{Q}^p_{\Gamma;h}$ there holds*

$$d(\boldsymbol{v}, \boldsymbol{v}_h) \leq c_\delta\Big(d(\boldsymbol{v}, \boldsymbol{u}_h) + \|\boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{u}_h\|_p + \|\pi - r_h\|_{p'}\Big) + \Big(\frac{1}{\sigma_0} + \delta\Big)\gamma_0\|\pi - \pi_h\|_2,$$

where the constant c_δ also depends on p, ε_0, σ_0, σ_1, Γ, Ω, $\|\boldsymbol{f}\|_{p'}$, $\|\boldsymbol{b}\|_{(p^\#)';S}$ and $\|\boldsymbol{v}_0\|_{1,p}$.

Proof. Let $(\boldsymbol{u}_h, r_h) \in (\boldsymbol{v}_{0,h} + \boldsymbol{\mathcal{V}}^p_{\Gamma;h}) \times \mathcal{Q}^p_{\Gamma;h}$ be arbitrary. From **(P8)**, **(P8$_h$)** it follows that

$$(\boldsymbol{\mathcal{S}}(\pi, \boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{S}}(\pi_h, \boldsymbol{D}\boldsymbol{v}_h), \boldsymbol{D}\boldsymbol{w}_h)_\Omega = (\pi - \pi_h, \nabla\cdot\boldsymbol{w}_h)_\Omega = (\pi - r_h, \nabla\cdot\boldsymbol{w}_h)_\Omega$$

for all $\boldsymbol{w}_h \in \boldsymbol{\mathcal{V}}^p_{\Gamma;h}$. This, with $\boldsymbol{w}_h := (\boldsymbol{u}_h - \boldsymbol{v}_h) \in \boldsymbol{\mathcal{V}}^p_{\Gamma;h}$, implies

$$\begin{aligned}(\boldsymbol{\mathcal{S}}(\pi, \boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{S}}(\pi_h, \boldsymbol{D}\boldsymbol{v}_h), \boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{v}_h)_\Omega &= (\boldsymbol{\mathcal{S}}(\pi, \boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{S}}(\pi_h, \boldsymbol{D}\boldsymbol{v}_h), \boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{u}_h)_\Omega \\ &\quad + (\pi - r_h, \nabla\cdot(\boldsymbol{u}_h - \boldsymbol{v}_h))_\Omega =: I_1 + I_2.\end{aligned}$$

Applying (2.54), we conclude that

$$\frac{\sigma_0}{2} d(\boldsymbol{v}, \boldsymbol{v}_h)^2 \leq I_1 + I_2 + \frac{\gamma_0^2}{2\sigma_0}\|\pi - \pi_h\|_2^2. \tag{7.28}$$

It remains to estimate I_1 and I_2. First of all, we split the term I_1 in the following way,

$$\begin{aligned}I_1 &= (\boldsymbol{\mathcal{S}}(\pi, \boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{S}}(\pi_h, \boldsymbol{D}\boldsymbol{u}_h), \boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{u}_h)_\Omega \\ &\quad + (\boldsymbol{\mathcal{S}}(\pi_h, \boldsymbol{D}\boldsymbol{u}_h) - \boldsymbol{\mathcal{S}}(\pi_h, \boldsymbol{D}\boldsymbol{v}_h), \boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{u}_h)_\Omega =: I_3 + I_4.\end{aligned}$$

Due to (2.55), for each $\delta_1 > 0$ there exists $c_{\delta_1} > 0$ such that

$$I_3 \le c_{\delta_1} d(\boldsymbol{v}, \boldsymbol{u}_h)^2 + \delta_1 \gamma_0^2 \|\pi - \pi_h\|_2^2.$$

Let φ be defined by (2.37) and let φ_a be given as in Definition 2.4. In order to get an upper bound of I_4, we apply Lemma 2.8 and Young's inequality (2.36) with $\psi_a := \varphi_a$ taking into account that the Δ_2-constants of φ_a, $(\varphi_a)^*$ only depend on p and do not depend on the shift-parameter $a \ge 0$. Hence, for each $\delta_2 > 0$ there is a constant $c_{\delta_2} > 0$ so that

$$\begin{aligned} I_4 &\le c \int_\Omega \varphi'_{\varepsilon+|\boldsymbol{D}\boldsymbol{u}_h|}(|\boldsymbol{D}\boldsymbol{u}_h - \boldsymbol{D}\boldsymbol{v}_h|)|\boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{u}_h| \,\mathrm{d}\boldsymbol{x} \\ &\le \delta_2 \int_\Omega \varphi_{\varepsilon+|\boldsymbol{D}\boldsymbol{u}_h|}(|\boldsymbol{D}\boldsymbol{u}_h - \boldsymbol{D}\boldsymbol{v}_h|) \,\mathrm{d}\boldsymbol{x} + c_{\delta_2} \int_\Omega \varphi_{\varepsilon+|\boldsymbol{D}\boldsymbol{u}_h|}(|\boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{u}_h|) \,\mathrm{d}\boldsymbol{x} \\ &\sim \delta_2 \|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{u}_h) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2^2 + c_{\delta_2} \|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{u}_h)\|_2^2 \\ &\le \delta_2 c d(\boldsymbol{v}, \boldsymbol{v}_h)^2 + c_{\delta_2} d(\boldsymbol{v}, \boldsymbol{u}_h)^2. \end{aligned}$$

Here, we have also used Lemma 2.11. Collecting the estimates above, we arrive at

$$I_1 \le c_{\delta_1,\delta_2} d(\boldsymbol{v}, \boldsymbol{u}_h)^2 + \delta_2 c d(\boldsymbol{v}, \boldsymbol{v}_h)^2 + \delta_1 \gamma_0^2 \|\pi - \pi_h\|_2^2. \tag{7.29}$$

Next, we estimate the term I_2. Using Korn's and Young's inequality, applying Lemma 2.6 (i) with $\nu = p$ and Lemma 2.11, and recalling the uniform a priori bounds (7.10) and (7.27), we deduce that for each $\delta_3 > 0$ there exists a constant $c_{\delta_3} > 0$ such that

$$\begin{aligned} I_2 &\le \Big|(\pi - r_h, \nabla \cdot (\boldsymbol{u}_h - \boldsymbol{v}_h))_\Omega\Big| \le c\|\pi - r_h\|_{p'} \|\boldsymbol{D}\boldsymbol{u}_h - \boldsymbol{D}\boldsymbol{v}_h\|_p \\ &\le \delta_3 \Big(\|\boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{u}_h\|_p^2 + \|\boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{v}_h\|_p^2\Big) + c_{\delta_3} \|\pi - r_h\|_{p'}^2 \\ &\le \delta_3 \|\boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{u}_h\|_p^2 + \delta_3 c d(\boldsymbol{v}, \boldsymbol{v}_h)^2 \|\varepsilon + |\boldsymbol{D}\boldsymbol{v}| + |\boldsymbol{D}\boldsymbol{v}_h|\|_p^{2-p} + c_{\delta_3} \|\pi - r_h\|_{p'}^2 \\ &\le \delta_3 \|\boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{u}_h\|_p^2 + \delta_3 c d(\boldsymbol{v}, \boldsymbol{v}_h)^2 + c_{\delta_3} \|\pi - r_h\|_{p'}^2. \end{aligned} \tag{7.30}$$

Combining the estimates (7.28), (7.29) and (7.30), we conclude that

$$\begin{aligned} \frac{\sigma_0}{2} d(\boldsymbol{v}, \boldsymbol{v}_h)^2 &\le \delta_2 c d(\boldsymbol{v}, \boldsymbol{v}_h)^2 + \delta_3 c d(\boldsymbol{v}, \boldsymbol{v}_h)^2 + c_{\delta_1,\delta_2} d(\boldsymbol{v}, \boldsymbol{u}_h)^2 + \delta_3 \|\boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{u}_h\|_p^2 \\ &\quad + c_{\delta_3} \|\pi - r_h\|_{p'}^2 + \Big(\frac{1}{2\sigma_0} + \delta_1\Big) \gamma_0^2 \|\pi - \pi_h\|_2^2. \end{aligned}$$

Multiplying this with $2/\sigma_0$, taking the square root, we easily complete the proof. □

Lemma 7.7 enables us to estimate the pressure error in the L^2-norm.

Theorem 7.8. *Let Assumption 7.1 hold. Let the discrete spaces fulfill* $(\mathbf{IS}^2_\Gamma)$ *and let the parameters meet the condition (7.13):* $\gamma_0 < \tilde{\beta}(2) \varepsilon^{\frac{2-p}{2}} \frac{\sigma_0}{\sigma_0 + \sigma_1}$. *Then, there exists a constant* $c > 0$, *which only depends on* $p, \varepsilon, \gamma_0, \sigma_0, \sigma_1, \tilde{\beta}(2), \Gamma, \Omega, \|\boldsymbol{f}\|_{p'}, \|\boldsymbol{b}\|_{(p^\#)';S}, \|\boldsymbol{v}_0\|_{1,p}$ *and which may explode as* $\varepsilon \searrow 0$, *such that the pressure error is bounded in* $L^2(\Omega)$ *by*

$$\|\pi - \pi_h\|_2 \le c \inf_{\boldsymbol{u}_h \in \boldsymbol{v}_{0,h} + \boldsymbol{\mathcal{V}}^p_{\Gamma;h}} \Big(\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{u}_h)\|_2 + \|\boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{u}_h\|_p\Big) + c \inf_{r_h \in \mathcal{Q}^p_{\Gamma;h}} \|\pi - r_h\|_{p'}.$$

Proof. Let $(\boldsymbol{u}_h, r_h) \in (\boldsymbol{v}_{0,h} + \boldsymbol{\mathcal{V}}^p_{\Gamma;h}) \times \mathcal{Q}^p_{\Gamma;h}$ be arbitrary. Then, **(P8)** and **(P8$_h$)** imply

$$(r_h - \pi_h, \nabla\cdot \boldsymbol{w}_h)_\Omega = (\boldsymbol{\mathcal{S}}(\pi, \boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{S}}(\pi_h, \boldsymbol{D}\boldsymbol{v}_h), \boldsymbol{D}\boldsymbol{w}_h)_\Omega + (r_h - \pi, \nabla\cdot \boldsymbol{w}_h)_\Omega \tag{7.31}$$

for all $\boldsymbol{w}_h \in \boldsymbol{\mathcal{X}}^p_{\Gamma;h}$. Using **(IS$^2_\Gamma$)** and (7.31), we deduce, cf. (7.15),

$$\tilde{\beta}(2)\|r_h - \pi_h\|_2 \le \sup_{\boldsymbol{w}_h \in \boldsymbol{\mathcal{X}}^2_{\Gamma;h}} \frac{(r_h - \pi_h, \nabla\cdot \boldsymbol{w}_h)_\Omega}{\|\boldsymbol{w}_h\|_{1,2}} \le \|\boldsymbol{\mathcal{S}}(\pi, \boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{S}}(\pi_h, \boldsymbol{D}\boldsymbol{v}_h)\|_2 + \|r_h - \pi\|_2.$$

Applying (2.56) & Lemma 7.7, we conclude that for each $\delta > 0$ there exists $c_\delta > 0$ so that

$$\begin{aligned}\tilde{\beta}(2)\|r_h - \pi_h\|_2 &\le \sigma_1 \varepsilon^{\frac{p-2}{2}} d(\boldsymbol{v}, \boldsymbol{v}_h) + \gamma_0 \varepsilon^{\frac{p-2}{2}} \|\pi - \pi_h\|_2 + \|r_h - \pi\|_2 \\ &\le \sigma_1 \varepsilon^{\frac{p-2}{2}} c_\delta \Big(d(\boldsymbol{v}, \boldsymbol{u}_h) + \|\boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{u}_h\|_p + \|\pi - r_h\|_{p'} \Big) \\ &\quad + \sigma_1 \varepsilon^{\frac{p-2}{2}} \Big(\frac{1}{\sigma_0} + \delta \Big) \gamma_0 \|\pi - \pi_h\|_2 + \gamma_0 \varepsilon^{\frac{p-2}{2}} \|\pi - \pi_h\|_2 + \|r_h - \pi\|_2 .\end{aligned}$$

Using Minkowski's inequality, Lemma 2.11, and $L^{p'}(\Omega) \hookrightarrow L^2(\Omega)$ for $p \le 2$, we arrive at

$$\begin{aligned}\|\pi - \pi_h\|_2 &\le c_\delta \Big(\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{u}_h)\|_2 + \|\boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{u}_h\|_p + \|\pi - r_h\|_{p'} \Big) \\ &\quad + \tilde{\beta}(2)^{-1} \sigma_1 \varepsilon^{\frac{p-2}{2}} \Big(\frac{1}{\sigma_0} + \delta \Big) \gamma_0 \|\pi - \pi_h\|_2 + \tilde{\beta}(2)^{-1} \gamma_0 \varepsilon^{\frac{p-2}{2}} \|\pi - \pi_h\|_2 .\end{aligned}$$

Recalling (7.13), and choosing $\delta > 0$ sufficiently small, we can absorb all terms, which include the pressure error, into the left-hand side. Hence, we get the desired result. □

Corollary 7.9. *Let the assumptions of Theorem 7.8 be satisfied. Then, the approximation error of the velocity field is bounded with respect to the natural distance as follows*

$$\begin{aligned}\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2 &\le c \inf_{\boldsymbol{u}_h \in (\boldsymbol{v}_{0,h} + \boldsymbol{\mathcal{V}}^p_{\Gamma;h})} \Big(\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{u}_h)\|_2 + \|\boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{u}_h\|_p \Big) \\ &\quad + c \inf_{r_h \in \mathcal{Q}^p_{\Gamma;h}} \|\pi - r_h\|_{p'} .\end{aligned} \tag{7.32}$$

Proof. The estimate follows from Lemma 2.11, Lemma 7.7, and Theorem 7.8. □

Corollary 7.10. *Let the assumptions of Thm. 7.8 hold. In addition, let* **(IS$^p_\Gamma$)** *hold and*

$$\gamma_0 < \tilde{\beta}(p) \varepsilon^{\frac{2-p}{2}} . \tag{7.33}$$

Then, the approximation error of the pressure field is bounded in $L^{p'}(\Omega)$ by

$$\|\pi - \pi_h\|_{p'} \le c\, \|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2^{\frac{2}{p'}} + c \inf_{r_h \in \mathcal{Q}^p_{\Gamma;h}} \|r_h - \pi\|_{p'} . \tag{7.34}$$

Proof. The estimate is again based on the inf–sup inequality $(\mathbf{IS}^p_\Gamma)$. Using $(\mathbf{IS}^p_\Gamma)$, Hölder's inequality, (7.31), (2.57) and (2.60), for arbitrary $r_h \in \mathcal{Q}^p_{\Gamma;h}$ we obtain the estimate

$$\begin{aligned}\tilde{\beta}(p)\|r_h - \pi_h\|_{p'} &\le \sup_{\boldsymbol{w}_h \in \boldsymbol{\mathcal{X}}^p_{\Gamma;h}} \frac{(r_h - \pi_h, \nabla \cdot \boldsymbol{w}_h)_\Omega}{\|\boldsymbol{w}_h\|_{1,p}} \\ &\le \|\boldsymbol{\mathcal{S}}(\pi, \boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{S}}(\pi_h, \boldsymbol{D}\boldsymbol{v}_h)\|_{p'} + \|r_h - \pi\|_{p'} \\ &\le c\,\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2^{\frac{2}{p'}} + \gamma_0 \varepsilon^{\frac{p-2}{2}} \|\pi - \pi_h\|_{p'} + \|r_h - \pi\|_{p'}.\end{aligned}$$

Due to assumption (7.33), this completes the proof. □

In practice, one never obtains the solution $(\boldsymbol{v}_h, \pi_h)$ to Problem $(\mathbf{P8}_h)$ exactly. Instead, one obtains its approximation $(\tilde{\boldsymbol{v}}_h, \tilde{\pi}_h) \in (\boldsymbol{v}_{0,h} + \boldsymbol{\mathcal{V}}^p_{\Gamma;h}) \times \mathcal{Q}^p_{\Gamma;h}$, satisfying

$$\begin{aligned}(\boldsymbol{\mathcal{S}}(\tilde{\pi}_h, \boldsymbol{D}\tilde{\boldsymbol{v}}_h), \boldsymbol{D}\boldsymbol{w}_h)_\Omega - (\tilde{\pi}_h, \nabla \cdot \boldsymbol{w}_h)_\Omega &= (\boldsymbol{f}, \boldsymbol{w}_h)_\Omega - (\boldsymbol{b}, \boldsymbol{w}_h)_S + \langle \boldsymbol{e}, \boldsymbol{w}_h \rangle \qquad \forall \boldsymbol{w}_h \in \boldsymbol{\mathcal{X}}^p_{\Gamma;h}, \\ (\nabla \cdot \tilde{\boldsymbol{v}}_h, q_h)_\Omega &= \langle g, q_h \rangle \qquad \forall q_h \in \mathcal{Q}^p_{\Gamma;h},\end{aligned}$$

where $\boldsymbol{e} \in (\boldsymbol{\mathcal{X}}^p_{\Gamma;h})^*$, $g \in (\mathcal{Q}^p_{\Gamma;h})^*$, and the brackets $\langle \cdot, \cdot \rangle$ denote the corresponding duality pairings. Here, $\boldsymbol{e} = \boldsymbol{e}(\tilde{\boldsymbol{v}}_h, \tilde{\pi}_h)$ and $g = g(\tilde{\boldsymbol{v}}_h, \tilde{\pi}_h)$ represent some additional error which includes, e.g., the residual associated with the approximative solution of the nonlinear algebraic problem, or the error due to numerical integration. However, provided that one is able to estimate $\boldsymbol{e}$ and g, then one can derive estimates for $\boldsymbol{v} - \tilde{\boldsymbol{v}}_h$ and $\pi - \tilde{\pi}_h$ similar to those derived above by following the same procedure. For instance, assuming that $|\langle \boldsymbol{e}, \boldsymbol{w}_h \rangle| \le E\,\|\boldsymbol{w}_h\|_{1,p}$ and $|\langle g, q_h \rangle| \le G\,\|q_h\|_2$ for E, G independent of h (say, $E, G \le 1$, such that $\|\boldsymbol{D}\tilde{\boldsymbol{v}}_h\|_p$ remains reasonably bounded), one can show (cf. (7.32), (7.34)):

$$\begin{aligned}\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\tilde{\boldsymbol{v}}_h)\|_2 &\le c \inf_{\boldsymbol{u}_h \in (\boldsymbol{v}_{0,h} + \boldsymbol{\mathcal{V}}^p_{\Gamma;h})} \Big(\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{u}_h)\|_2 + \|\boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{u}_h\|_p \Big) \\ &\quad + c \inf_{r_h \in \mathcal{Q}^p_{\Gamma;h}} \|\pi - r_h\|_{p'} + c\,(E + G) \\ \|\pi - \tilde{\pi}_h\|_{p'} &\le c\,\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\tilde{\boldsymbol{v}}_h)\|_2^{\frac{2}{p'}} + c \inf_{r_h \in \mathcal{Q}^p_{\Gamma;h}} \|r_h - \pi\|_{p'} + c\,E.\end{aligned}$$

7.5 Finite element approximation

In this section, we consider some finite element approximations of **(P8)** that satisfy the abstract theory of the previous sections. We assume that, for ease of exposition, Ω is a polygonal/polyhedral domain and that $\mathbb{T}_h$ is a shape-regular decomposition of Ω into quadrilaterals/hexahedra (or d-dimensional simplices) so that $\overline{\Omega} = \bigcup_{K \in \mathbb{T}_h} \overline{K}$, see Section 3.1. As usual, the symbol h_K denotes the diameter of an element $K \in \mathbb{T}_h$. The mesh parameter h represents the maximum diameter of the elements, i.e., $h := \max\{h_K;\, K \in \mathbb{T}_h\}$. As mentioned in Section 3.1, the neighborhood S_K of $K \in \mathbb{T}_h$, which denotes the union of all elements in $\mathbb{T}_h$ touching K, fulfills $|K| \sim |S_K|$ with constants independent of h. Furthermore, the number of elements in S_K is uniformly bounded with respect to $K \in \mathbb{T}_h$.

Let X_h, Q_h be appropriate finite element spaces defined on $\mathbb{T}_h$ that satisfy $X_h \subset W^{1,\infty}(\Omega)$, $Q_h \subset L^\infty(\Omega)$. We recall that the FE spaces for the velocity and pressure are given by $\boldsymbol{\mathcal{X}}^p_{\Gamma;h} := \boldsymbol{X}_h \cap \boldsymbol{\mathcal{X}}^p_\Gamma$, $\boldsymbol{X}_h = [X_h]^d$, and $\mathcal{Q}^p_{\Gamma;h} := Q_h \cap \mathcal{Q}^p_\Gamma$. In order to ensure approximation properties and the discrete inf-sup conditions, we need to specify the choice of spaces:

Assumption 7.2 (Approximation property of X_h and Q_h). *We assume that X_h contains the set of linear polynomials on Ω. Moreover, we suppose that there exist a linear projection $\boldsymbol{j}_h : \boldsymbol{W}^{1,1}(\Omega) \to \boldsymbol{X}_h$ and an interpolation operator $i_h : W^{1,1}(\Omega) \to Q_h$ so that*

(1) $\boldsymbol{j}_h$ preserves zero boundary values on Γ, such that $\boldsymbol{j}_h(\boldsymbol{\mathcal{X}}^p_\Gamma) \subset \boldsymbol{\mathcal{X}}^p_{\Gamma;h}$.

(2) $\boldsymbol{j}_h$ is locally $W^{1,1}$-stable in the sense that there exists $c > 0$ independent of h:

$$\fint_K |\boldsymbol{j}_h \boldsymbol{w}| \,\mathrm{d}\boldsymbol{x} \le c \fint_{S_K} |\boldsymbol{w}| \,\mathrm{d}\boldsymbol{x} + c \fint_{S_K} h_K |\nabla \boldsymbol{w}| \,\mathrm{d}\boldsymbol{x} \qquad \forall \boldsymbol{w} \in \boldsymbol{W}^{1,1}(\Omega),\ \forall K \in \mathbb{T}_h, \tag{7.35}$$

where S_K denotes a local neighborhood of K (as defined above).

(3) $\boldsymbol{j}_h$ preserves divergence[4] in the Q_h^*-sense, i.e.,

$$(\nabla \cdot \boldsymbol{w}, q_h)_\Omega = (\nabla \cdot \boldsymbol{j}_h \boldsymbol{w}, q_h)_\Omega \qquad \forall \boldsymbol{w} \in \boldsymbol{W}^{1,1}(\Omega),\ \forall q_h \in Q_h. \tag{7.36}$$

(4) i_h preserves mean values, i.e., $i_h(\mathcal{Q}^p_\Gamma) \subset \mathcal{Q}^p_{\Gamma;h}$, and, for any $\nu \ge 1$, i_h satisfies

$$\|q - i_h q\|_\nu \le ch \|q\|_{1,\nu} \qquad \forall q \in W^{1,\nu}(\Omega). \tag{7.37}$$

Later we will suppose that functions in X_h satisfy the following global inverse inequality:

Assumption 7.3 (Inverse property of X_h). *For $\nu, \mu \in [1, \infty]$ and $0 \le m \le l$ it holds*

$$\|w_h\|_{l,\nu} \le C h^{m-l+\min(0,\frac{d}{\nu}-\frac{d}{\mu})} \|w_h\|_{m,\mu} \qquad \forall w_h \in X_h. \tag{7.38}$$

Assumption 7.3 usually requires that the mesh is quasi-uniform, see (3.6). Assumption 7.2 is similar to Assumption 2.21 in [BBDR10]. Clearly, the existence of $\boldsymbol{j}_h$ and i_h as in Assumption 7.2 depends on the choice of the finite element pairing X_h/Q_h:

- The construction of an operator $\boldsymbol{j}_h$, that satisfies Assumptions 7.2 (1) – (3), is well-known for some particular finite elements, including the Crouzeix-Raviart and MINI element, see [BBDR10]. If $\Gamma \ne \partial\Omega$, Assumption 7.2 (1) requires that the triangulation matches Γ appropriately, cf. [SZ90].

[4] In case of $|S| > 0$ this implies $\int_S \boldsymbol{w} \cdot \boldsymbol{n} \,\mathrm{d}\boldsymbol{x} = \int_S (\boldsymbol{j}_h \boldsymbol{w}) \cdot \boldsymbol{n} \,\mathrm{d}\boldsymbol{x}$ which requires that $\mathbb{T}_h$ matches S appropriately.

- Assumption 7.2 (2) is standard in the context of interpolation in Orlicz-Sobolev spaces, see Section 3.3 or [DR07]. For standard finite elements, it is well-known that the Scott-Zhang interpolation operator satisfies (7.35), see [SZ90]. It is crucial that from (7.35) one can derive the local stability result (see Lemma 3.1 or [DR07])

$$\fint_K \psi(|\nabla \boldsymbol{j}_h \boldsymbol{w}|)\,\mathrm{d}\boldsymbol{x} \leq c \fint_{S_K} \psi(|\nabla \boldsymbol{w}|)\,\mathrm{d}\boldsymbol{x} \qquad \forall \boldsymbol{w} \in \boldsymbol{W}^{1,\psi}(\Omega) \qquad \forall K \in \mathbb{T}_h, \tag{7.39}$$

  which is valid for arbitrary N-functions ψ with $\Delta_2(\psi) < \infty$. Here, $\boldsymbol{W}^{1,\psi}(\Omega)$ is the classical Orlicz-Sobolev space and the constant c only depends on $\Delta_2(\psi)$.

- For standard finite elements, i_h may be chosen as the L^2-projection onto Q_h:

$$(i_h q, q_h)_\Omega = (q, q_h)_\Omega \qquad \forall q_h \in Q_h \qquad \forall q \in L^1(\Omega). \tag{7.40}$$

  Indeed, it is shown in Crouzeix/Thomée [CT87] that the L^2-projection is L^ν-stable and even $W^{1,\nu}$-stable for any $\nu \in [1,\infty]$ and that, consequently, it fulfills (7.37). The results of [CT87] are derived for finite element spaces Q_h based on simplices, $Q_h := \{w \in C(\overline{\Omega});\, w|_K \in \mathbb{P}_r(K) \text{ for all } K \in \mathbb{T}_h\}$, where $\mathbb{P}_r(K)$ denotes the space of polynomials on K of degree less than or equal to r. Moreover, setting $q_h = 1$ in (7.40), we deduce that i_h preserves mean values. Hence, $i_h(\mathcal{Q}^p_\Gamma) \subset \mathcal{Q}^p_{\Gamma;h}$.

Next, we depict important consequences of Assumption 7.2:

Lemma 7.11. *Let there exist a linear projection $\boldsymbol{j}_h$ that satisfies Assumption 7.2 (2). Then, for all $K \in \mathbb{T}_h$ and $\boldsymbol{w} \in \boldsymbol{W}^{1,p}(\Omega)$ there holds*

$$\fint_K |\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{w}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{w})|^2\,\mathrm{d}\boldsymbol{x} \leq c h_K^2 \fint_{S_K} |\nabla \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{w})|^2\,\mathrm{d}\boldsymbol{x} \tag{7.41}$$

provided that $\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{w}) \in W^{1,2}(\Omega)^{d\times d}$. The constant c only depends on p.

Proof. The proof is based on the Orlicz-stability (7.39) and it is identical to the proof of Lemma 4.4 (iii). Note that in Lemma 4.4 the interpolation estimate (7.41) was especially proven for the isoparametric d-linear $\mathbb{Q}_1$ finite elements. Under Assumption 7.2, estimate (7.41) follows analogously. We also refer to [BBDR10, Hir10]. □

Moreover, the assumptions on $\boldsymbol{j}_h$ imply the discrete version of the inf-sup inequality:

Lemma 7.12. *Let there exist a linear projection $\boldsymbol{j}_h$ that satisfies Assumption 7.2 (1)–(3). Then, for $\nu \in (1,\infty)$ the discrete inf-sup inequality* $(\mathbf{IS}^\nu_\Gamma)$ *is satisfied.*

Proof. Since $\mathbb{T}_h$ is nondegenerate, the local stability result (7.39) (with $\psi(t) := t^\nu$) leads to the global $W^{1,\nu}$-stability inequality, $\|\boldsymbol{j}_h\boldsymbol{w}\|_{1,\nu} \leq C_s \|\boldsymbol{w}\|_{1,\nu}$ for all $\boldsymbol{w} \in \boldsymbol{\mathcal{X}}^\nu_\Gamma$, where

$\nu \in (1,\infty)$ and the stability constant C_s does not depend on h. Thus, the continuous inf-sup inequality (7.3) and Assumption 7.2 imply that for arbitrary $q_h \in \mathcal{Q}^\nu_{\Gamma;h} \subset \mathcal{Q}^\nu_\Gamma$

$$\begin{aligned}\|q_h\|_{\nu'} &\le \beta(\nu)^{-1} \sup_{\boldsymbol{w}\in\boldsymbol{\mathcal{X}}^\nu_\Gamma} \frac{(q_h, \nabla\cdot\boldsymbol{w})_\Omega}{\|\boldsymbol{w}\|_{1,\nu}} = \beta(\nu)^{-1} \sup_{\boldsymbol{w}\in\boldsymbol{\mathcal{X}}^\nu_\Gamma} \frac{(q_h, \nabla\cdot\boldsymbol{j}_h\boldsymbol{w})_\Omega}{\|\boldsymbol{w}\|_{1,\nu}} \\ &\le \beta(\nu)^{-1} C_s \sup_{\boldsymbol{w}\in\boldsymbol{\mathcal{X}}^\nu_\Gamma} \frac{(q_h, \nabla\cdot\boldsymbol{j}_h\boldsymbol{w})_\Omega}{\|\boldsymbol{j}_h\boldsymbol{w}\|_{1,\nu}} \le \tilde\beta(\nu)^{-1} \sup_{\boldsymbol{w}_h\in\boldsymbol{\mathcal{X}}^\nu_{\Gamma;h}} \frac{(q_h, \nabla\cdot\boldsymbol{w}_h)_\Omega}{\|\boldsymbol{w}_h\|_{1,\nu}},\end{aligned}$$

where $\tilde\beta(\nu) := \beta(\nu)/C_s$ is independent of h. □

Remark 7.5. Let us briefly discuss the case of unstable discretizations. For instance, one can consider the equal-order d-linear $\mathbb{Q}_1/\mathbb{Q}_1$ element, which uses continuous isoparametric d-linear shape functions for both the velocity and pressure approximation, see Section 3.1. In this case, the discrete inf-sup condition is violated. For p-Stokes systems (2.10) & (2.11), a stabilization technique based on the local projection stabilization (LPS) method was proposed in Chapter 4, that leads to optimal convergence results. Whether LPS can be applied to the equal-order discretization of **(P8)**, is subject of current research.

Below we state our a priori error estimates that quantify the convergence of the finite element method. For this, the regularity $\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) \in W^{1,2}(\Omega)^{d\times d}$ of the solution $\boldsymbol{v}$ is required. According to Lemma 2.27, this condition is equivalent to $\mathcal{I}(\boldsymbol{v}) < \infty$, where the quantity $\mathcal{I}(\boldsymbol{v})$ is defined in (2.82). We mention that the regularity $\mathcal{I}(\boldsymbol{v}) < \infty$ is available for sufficiently smooth data at least in the setting of space-periodic boundary conditions in two space dimensions, see Bulíček/Kaplický [BK08].

Corollary 7.13. *Let the assumptions of Theorem 7.8 hold. We suppose that there exist operators $\boldsymbol{j}_h$ and i_h satisfying Assumption 7.2. Moreover, we assume the regularity*

$$\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) \in W^{1,2}(\Omega)^{d\times d} \qquad \text{and} \qquad \pi \in W^{1,p'}(\Omega)$$

and we set $\boldsymbol{v}_{0,h} := \boldsymbol{j}_h\boldsymbol{v}_0$. Then, the error of approximation is bounded in terms of the maximum mesh size h as follows:

$$\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2 \le C_{\boldsymbol{v}} h, \qquad \|\pi - \pi_h\|_2 \le C_\pi h. \tag{7.42}$$

If additionally $\gamma_0 < \tilde\beta(p)\varepsilon^{\frac{2-p}{2}}$, then the pressure error in $L^{p'}(\Omega)$ is bounded by

$$\|\pi - \pi_h\|_{p'} \le C'_\pi h^{\frac{2}{p'}}. \tag{7.43}$$

The constants $C_{\boldsymbol{v}}$, C_π, $C'_\pi > 0$ only depend on p, ε, γ_0, σ_0, σ_1, $\tilde\beta(2)$, Γ, Ω, $\|\boldsymbol{f}\|_{p'}$, $\|\boldsymbol{b}\|_{(p^\#)';S}$, $\|\boldsymbol{v}_0\|_{1,p}$, $\|\nabla\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})\|_2$, $\|\pi\|_{1,p'}$, and C'_π additionally depends on $\tilde\beta(p)$.

Proof. According to Lemma 7.12, the discrete inf-sup inequalities $(\mathbf{IS}^2_\Gamma)$, $(\mathbf{IS}^p_\Gamma)$ hold true. Hence, the desired error estimates follow from Theorem 7.8, Corollaries 7.9 and 7.10, and the interpolation properties of $\boldsymbol{j}_h$ and i_h. More precisely, the velocity is given by $\boldsymbol{v} = \boldsymbol{v}_0 + \hat{\boldsymbol{v}}$ for some $\hat{\boldsymbol{v}} \in \boldsymbol{\mathcal{X}}^p_\Gamma$. Since $\hat{\boldsymbol{v}}$ is divergence-free, the interpolant $\boldsymbol{j}_h\hat{\boldsymbol{v}}$ fulfills $(\nabla\cdot\boldsymbol{j}_h\hat{\boldsymbol{v}}, q_h)_\Omega = 0$

for all $q_h \in \mathcal{Q}^p_{\Gamma;h}$. Hence, $\boldsymbol{j}_h\hat{\boldsymbol{v}} \in \boldsymbol{\mathcal{V}}^p_{\Gamma;h}$ and $\boldsymbol{j}_h\boldsymbol{v} = \boldsymbol{j}_h\boldsymbol{v}_0 + \boldsymbol{j}_h\hat{\boldsymbol{v}} \in (\boldsymbol{v}_{0,h} + \boldsymbol{\mathcal{V}}^p_{\Gamma;h})$. Consequently, we can set $\boldsymbol{u}_h := \boldsymbol{j}_h\boldsymbol{v}$ and $r_h := i_h\pi$ in Theorem 7.8 and Corollary 7.9. Using Lemma 2.6 (i), the global $W^{1,p}$-stability of $\boldsymbol{j}_h$ (which follows from (7.39) with $\psi(t) = t^p$ and the non-degeneracy of $\mathbb{T}_h$), the a priori bound (7.27), the interpolation properties (7.41), (7.37), we easily conclude (7.42). Finally, (7.43) follows from Corollary 7.10 and (7.42). □

Remark 7.6. Using (2.43), (7.10), and (7.27), we deduce from Corollary 7.13 that

$$\|\boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{v}_h\|_p \leq c\,\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2 \leq ch. \tag{7.44}$$

Hence, we also obtain an a priori error estimate in $\boldsymbol{W}^{1,p}(\Omega)$.

If $d = 2$, then the $W^{1,p'}$-regularity assumption for the pressure π can be avoided and confined to $\pi \in W^{1,2}(\Omega)$ provided that the velocity $\boldsymbol{v}$ additionally satisfies $\boldsymbol{v} \in \boldsymbol{W}^{1,\infty}(\Omega)$. Note that from analytical point of view we are not able to show the regularity $\pi \in W^{1,p'}(\Omega)$ but we can expect the regularity $\pi \in W^{1,2}(\Omega)$, see Bulíček/Kaplický [BK08]. Moreover note that, in case of space-periodic boundary conditions, global $C^{1,\alpha}$-regularity of $\boldsymbol{v}$ is well-established, see [BK08]. The following Corollary represents a variant of Corollary 7.13 that is motivated by our subsequent numerical experiments.

Corollary 7.14. *Let $d = 2$. Let the hypothesis of Theorem 7.8 hold true and let Assumption 7.3 be satisfied. We suppose that there exist operators $\boldsymbol{j}_h$ and i_h as in Assumption 7.2. Moreover, we assume that the solution $(\boldsymbol{v}, \pi)$ satisfies the additional regularity*

$$\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) \in W^{1,2}(\Omega)^{d\times d}, \qquad \boldsymbol{v} \in \boldsymbol{W}^{1,\infty}(\Omega), \qquad \textit{and} \qquad \pi \in W^{1,2}(\Omega).$$

We set $\boldsymbol{v}_{0,h} := \boldsymbol{j}_h\boldsymbol{v}_0$. Then, the error of approximation is bounded as follows:

$$\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2 \leq C_{\boldsymbol{v}}h, \qquad \|\pi - \pi_h\|_2 \leq C_\pi h. \tag{7.45}$$

Assume additionally (7.33) *and the $W^{1,2}$-stability of i_h. Then, there holds*

$$\|\pi - \pi_h\|_{p'} \leq C'_\pi h^{\frac{2}{p'}}. \tag{7.46}$$

The constants $C_{\boldsymbol{v}}$, C_π, $C'_\pi > 0$ only depend on p, ε, γ_0, σ_0, σ_1, $\tilde{\beta}(2)$, Γ, Ω, $\|\nabla\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})\|_2$, $\|\pi\|_{1,2}$, $\|\boldsymbol{v}\|_{1,\infty}$, and C'_π additionally depends on $\tilde{\beta}(p)$.

Proof. Under the supposed regularity, (7.45) and (7.46) are not surprising: due to $\boldsymbol{v} \in \boldsymbol{W}^{1,\infty}(\Omega)$ and $\varepsilon > 0$ the generalized viscosity μ remains bounded from below and above so that system (2.16) can basically be interpreted as a Stokes system. We only need to show that $\boldsymbol{v}_h$ is uniformly bounded in $\boldsymbol{W}^{1,\infty}(\Omega)$: Similarly to [SZ90] it can be shown that $\boldsymbol{j}_h$ is locally $W^{1,1}$-stable, i.e., there holds $\|\boldsymbol{j}_h\boldsymbol{w}\|_{1,1;K} \lesssim \|\boldsymbol{w}\|_{1,1;S_K}$ for all $\boldsymbol{w} \in \boldsymbol{W}^{1,1}(\Omega)$ and $K \in \mathbb{T}_h$. As in the proof of Corollary 4.14, we then conclude that $W^{1,1}$-stability implies $W^{1,\infty}$-stability, i.e., that $\boldsymbol{j}_h$ actually satisfies (4.67). Using the inverse inequality (3.20) with $d = 2$, the $W^{1,\infty}$-stability of $\boldsymbol{j}_h$ – (4.67), Korn's Lemma 7.1, and Lemma 2.6 (i) with $\nu = 2$, exactly as in (4.69) we can estimate the $W^{1,\infty}$-norm of $\boldsymbol{v}_h$ by

$$\|\boldsymbol{v}_h\|_{1,\infty} \leq c\Big[h^{-1}\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{v})\|_2\Big(\varepsilon_0 + \|\nabla\boldsymbol{v}_h\|_\infty + \|\nabla\boldsymbol{v}\|_\infty\Big)^{\frac{2-p}{2}} + \|\boldsymbol{v}\|_{1,\infty}\Big]. \tag{7.47}$$

Similarly to the derivation of (7.32), via Lemma 2.6 with $\nu = 2$ we can infer the estimate

$$\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2 + \|\pi - \pi_h\|_2 \lesssim \|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{v})\|_2 + \Big(\varepsilon_0 + \|\nabla \boldsymbol{j}_h\boldsymbol{v}\|_\infty + \|\nabla \boldsymbol{v}_h\|_\infty\Big)^{\frac{2-p}{2}} \|\pi - i_h\pi\|_2.$$

Using the properties of $\boldsymbol{j}_h$ and i_h, we consequently arrive at (w.l.o.g. $\varepsilon_0 \geq 1$)

$$\|\boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v}_h)\|_2 + \|\pi - \pi_h\|_2 \leq Ch\Big(\varepsilon_0 + \|\nabla \boldsymbol{v}\|_\infty + \|\nabla \boldsymbol{v}_h\|_\infty\Big)^{\frac{2-p}{2}} \tag{7.48}$$

where C depends on $\|\nabla \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})\|_2$ and $\|\pi\|_{1,2}$. Combining (7.47) and (7.48), we conclude

$$\|\boldsymbol{v}_h\|_{1,\infty} \leq C = C\Big(\|\nabla \boldsymbol{\mathcal{F}}(\boldsymbol{D}\boldsymbol{v})\|_2, \|\pi\|_{1,2}, \|\boldsymbol{v}\|_{1,\infty}\Big).$$

The constant C also depends on p, ε, ε_0, γ_0, σ_0, σ_1, $\tilde{\beta}(2)$, Ω, but it is independent of h. Thus, (7.48) yields the desired error estimates (7.45). It remains to prove the pressure estimate in $L^{p'}(\Omega)$. Interpolating $L^{p'}(\Omega)$ between $L^2(\Omega)$ and $W^{1,2}(\Omega)$, using (7.37) and the $W^{1,2}$-stability of i_h, for $p > \frac{2d}{d+2}$ and $\lambda := \frac{d}{2} - \frac{d}{p'}$ we obtain the estimate

$$\|\pi - i_h\pi\|_{p'} \leq c\,\|\pi - i_h\pi\|_{1,2}^{\lambda}\|\pi - i_h\pi\|_2^{1-\lambda} \leq ch^{1+\frac{d}{p'}-\frac{d}{2}}\|\pi\|_{1,2}. \tag{7.49}$$

For $d = 2$ the estimate (7.46) follows from the combination of (7.34), (7.45), (7.49). □

7.6 Numerical experiments

In this section we present some numerical examples, which illustrate the a priori error estimates of Corollary 7.13. Here we use the following model that goes back to [MNR02]:

$$\mu(\pi, |\boldsymbol{D}\boldsymbol{v}|^2) := \mu_0 \left(\delta_1 + \delta_2\left(\delta_3 + \exp(\alpha\pi)\right)^{-s} + \delta_4\,|\boldsymbol{D}\boldsymbol{v}|^2\right)^{\frac{p-2}{2}}, \tag{7.50}$$

where $s, \alpha, \delta_1, \ldots, \delta_4 \geq 0$.

Remark 7.7. Similarly to (e.g.) [MNR02], it can be shown that model (7.50) satisfies Assumptions **(A1)**–**(A2)**, e.g., with $\varepsilon^2 := \delta_1/\delta_4$, $\sigma_0 := \mu_0\delta_4^{(p-2)/2}(p-1)(1+\delta_2\delta_3^{-s}/\delta_1)^{(p-2)/2}$, $\sigma_1 := \mu_0\delta_4^{(p-2)/2}$, and $\gamma_0 := \mu_0\delta_4^{(p-4)/4}s\alpha\frac{2-p}{2}\delta_2^{p/4}\delta_3^{-sp/4}$, see Section 7.7.

Problem **(P8)** was discretized with bilinear $\mathbb{Q}_1/\mathbb{Q}_1$ finite elements based on quadrilateral meshes. Since the considered discretization is not inf-sup stable, we used the LPS-type stabilization (3.54). Note that in all examples the stabilization method was less sensitive with respect to the stabilization parameter. As described in Section 3.4, the algebraic equations were solved by Newton's method, the linear subproblems by the GMRES method. In the following numerical examples we depict the experimental order of convergence (EOC) with respect to the quantities $E_{\boldsymbol{v}}^{\boldsymbol{\mathcal{F}}}$, $E_{\boldsymbol{v}}^{1,\nu}$, $E_{\boldsymbol{v}}^{\nu}$, E_{π}^{ν} defined in (4.104).

Table 7.1. Numerical verification of Corollary 7.13

(a) $p = 1.7$

#cells	$E_{\boldsymbol{v}}^{\mathcal{F}}$	$E_{\boldsymbol{v}}^{p}$	E_{π}^{2}	$E_{\pi}^{p'}$
1024	1.00	2.17	1.00	0.83
4096	1.00	2.17	1.00	0.83
16384	1.00	2.17	1.00	0.82
65536	1.00	2.16	1.00	0.83
262144	1.00	2.16	1.00	0.83
expected	1		1	0.82

(b) $p = 1.5$

#cells	$E_{\boldsymbol{v}}^{\mathcal{F}}$	$E_{\boldsymbol{v}}^{p}$	E_{π}^{2}	$E_{\pi}^{p'}$
1024	1.01	2.33	1.01	0.68
4096	1.01	2.33	1.01	0.67
16384	1.00	2.32	1.01	0.67
65536	1.00	2.31	1.01	0.67
262144	1.00	2.29	1.01	0.67
expected	1		1	0.67

(c) $p = 1.3$

#cells	$E_{\boldsymbol{v}}^{\mathcal{F}}$	$E_{\boldsymbol{v}}^{p}$	E_{π}^{2}	$E_{\pi}^{p'}$
1024	0.99	2.49	1.00	0.46
4096	0.99	2.48	1.00	0.46
16384	0.99	2.45	1.00	0.46
65536	1.00	2.41	1.00	0.47
262144	1.00	2.36	1.00	0.47
expected	1		1	0.46

(d) $p = 1.1$

#cells	$E_{\boldsymbol{v}}^{\mathcal{F}}$	$E_{\boldsymbol{v}}^{p}$	E_{π}^{2}	$E_{\pi}^{p'}$
1024	0.99	2.70	0.99	0.19
4096	0.99	2.66	1.00	0.19
16384	0.99	2.56	1.00	0.19
65536	1.00	2.44	1.00	0.19
262144	1.00	2.30	1.01	0.19
expected	1		1	0.18

Example 1: In a square domain $\Omega := (-0.5, 0.5) \times (-0.5, 0.5)$, the exact solution to **(P8)** was given by $\boldsymbol{v}(\boldsymbol{x}) := |\boldsymbol{x}|^{a-1}(x_2, -x_1)^{\mathsf{T}}$ and $\pi(\boldsymbol{x}) := |\boldsymbol{x}|^b x_1 x_2$ for $a, b \in \mathbb{R}$. Here, the case $S = \emptyset$ was considered. Problem $(\mathbf{P8}_h)$ was then solved for the data $\boldsymbol{f} := -\nabla\cdot\boldsymbol{\mathcal{S}}(\pi, \boldsymbol{Dv}) + \nabla\pi$ and $\boldsymbol{v}_0 := \boldsymbol{v}$. The parameters a and b were chosen so that $\boldsymbol{\mathcal{F}}(\boldsymbol{Dv}) \in W^{1,2}(\Omega)^{d\times d}$ and $\pi \in W^{1,2}(\Omega)$. This requirement amounts to the conditions $a > 1$ and $b > -2$. Since $\|\nabla\boldsymbol{v}\|_\infty$ is bounded for $a > 1$, according to Corollary 7.14 the requirement $\pi \in W^{1,2}(\Omega)$ is sufficient to ensure the optimal rate of convergence (note that Corollary 7.13 would require $\pi \in W^{1,p'}(\Omega)$ with $p' > 2$). We set $a = 1.01$ and $b = -1.99$. Hence, as soon as (7.13) is satisfied, we expect $E_{\boldsymbol{v}}^{\mathcal{F}} = \mathrm{O}(h)$, $E_\pi^2 = \mathrm{O}(h)$, and $E_\pi^{p'} = \mathrm{O}(h^{2/p'})$, for finite elements satisfying Assumption 7.2. Note that our considered $\mathbb{Q}_1/\mathbb{Q}_1$ discretization does not fulfill Assumption 7.2. By virtue of Chapter 4 we however believe that Corollaries 7.13 and 7.14 can be extended to $\mathbb{Q}_1/\mathbb{Q}_1$ finite elements if the Galerkin system (7.6)–(7.7) is stabilized by (4.4) or (4.63). The parameters of the model (7.50) were set to $\delta_1 := 10^{-8}$, $s := 2/(2-p)$ and $\mu_0 = \delta_2 = \delta_3 = \delta_4 := 1$. Then, Remark 7.7 implies $\gamma_0 = \alpha$ and, hence, (7.13) is ensured at least for $\alpha < \tilde{\beta}(2)\delta_1^{(2-p)/4}\frac{(p-1)(1+1/\delta_1)^{(p-2)/2}}{(p-1)(1+1/\delta_1)^{(p-2)/2}+1}$, i.e., by virtue of $\delta_1 \ll 1$, (7.13) is satisfied for $\alpha \ll 1$. In this particular example, for the stated parameters we have numerically observed the expected convergence rates for $\alpha \in [0, 8]$ approximately. For greater α, Newton's method did not converge any more. One may ask, whether the assumption (7.13) could be relaxed[5]. In particular, one may ask whether the estimates (7.43) and (7.44) remain valid in the degenerate case $\varepsilon \searrow 0$. Note that in case of Carreau-

[5]However, the above observations do not allow us to *claim* that assumption (7.13) could be relaxed. We note that, in this example, the solution $(\boldsymbol{v}, \pi)$ always exists, whatever the values of α and γ_0 are.

type models (i.e., $\gamma_0 \equiv 0$), error estimates similar to (7.43) and (7.44) actually hold true and are numerically validated also for $\varepsilon = 0$, see Chapter 4. For fluids with pressure dependent viscosity, though, the behavior for $\varepsilon \searrow 0$ remains an open question. In what follows, we set $\alpha := 1$. In Tables 7.1(a)–7.1(d), we present the observed convergence rates for different values of $p \in (1, 2)$. We realize that the numerical results agree with the theoretical ones very well. In particular, the example reflects that the rate of convergence for $E_\pi^{p'}$ depends on the choice of p as predicted by the estimate (7.46). Apart from that, we observed that the experimental order of convergence declines as soon as $a < 1$ or $b < -2$. This indicates that the derived a priori error estimates are optimal with respect to the regularity of the solution. We also observe that the error $E_{\boldsymbol{v}}^p$ behaves like $\mathrm{O}(h^2)$. This observation raises hope that a duality argument (see [BS94]) may be applicable here.

Table 7.2. EOC: Pressure drop problem for the model (7.50)

(a) $p = 1.5$ and $\delta_4 = 10^{-5}$

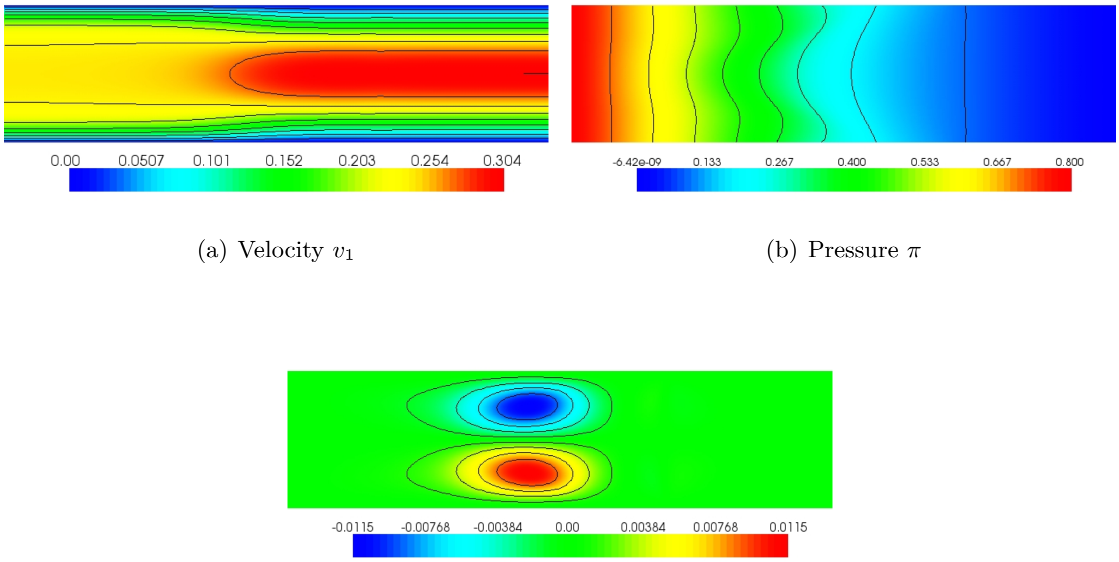
#cells	$E_{\boldsymbol{v}}^{1,p}$	$E_{\boldsymbol{v}}^p$	E_π^2
1024	1.00	1.97	1.94
4096	1.00	2.00	2.04
16384	1.01	2.01	1.98
65536	1.02	2.06	1.89
expected	1		1

(b) $p = 1.2$ and $\delta_4 = 10^{-3}$

#cells	$E_{\boldsymbol{v}}^{1,p}$	$E_{\boldsymbol{v}}^p$	E_π^2
1024	0.99	1.96	1.97
4096	1.00	1.98	1.98
16384	1.00	1.99	1.92
65536	1.01	2.03	1.90
expected	1		1

Example 2: Pressure drop problem. In order to confirm the results in a realistic flow configuration, we consider a planar flow between two steady parallel plates, driven by the difference of pressure between inlet and outlet, cf. Section 5.5. Here, $\Omega = (0, 1.64) \times (0, 0.41)$. We prescribe homogeneous Dirichlet boundary conditions on the upper and lower edge, while we set $\boldsymbol{b} := 0.8\,\boldsymbol{n}$ on the inflow (left) boundary, and $\boldsymbol{b} := \mathbf{0}$ on the outflow (right) boundary. Moreover, we additionally require[6] there that $\boldsymbol{v} = (\boldsymbol{v} \cdot \boldsymbol{n})\boldsymbol{n}$, i.e., the stream lines are orthogonal to the inflow and outflow boundary. Note that if the viscosity did not vary with the pressure, this setting would lead to a unidirectional flow (Poiseuille flow) of the form $\boldsymbol{v} = (v_1(x_2), 0)^\mathsf{T}$ and $\pi = \pi(x_1)$. Since the viscosity depends on the pressure, however, this needs not be the case; e.g., there is no such unidirectional solution for the Barus model $\mu = \mu_0 \exp(\alpha\pi)$, as was shown in [HMR01]. Here we consider the model (7.50) provided with $\mu_0 := 0.01$, $s := \frac{2}{2-p}$, $\delta_1 := 5 * 10^{-6}$, $\delta_2 = \delta_3 := 1$ and $\alpha := 10$. The resulting velocity and pressure fields are shown in Figure 7.1. For moderate and low pressures (in the middle-length and the right-hand part of the domain) this model approximates the Barus model, while for higher pressures (in the domain left-hand part) the behavior is that of the Carreau model. In Table 7.2, we present the observed convergence rates for different values of p. Since the exact solution is unknown, we have used the finite

[6] This requirement is achieved by altering the definition of the space $\boldsymbol{\mathcal{X}}_\Gamma^p$, see, e.g., [LS11b].



(a) Velocity v_1

(b) Pressure π

(c) Velocity v_2

Figure 7.1. Pressure drop problem for the model (7.50): Case $p = 1.2$

element approximation computed on a grid with 4^{10} cells as the reference solution (cf. Example 4). In view of Table 7.2, the velocity error $E_{\boldsymbol{v}}^{1,p}$ behaves as $\mathcal{O}(h)$. For the velocity the experimental convergence rate agrees with the theoretical one. However we observe that E_π^2 behaves almost as $\mathcal{O}(h^2)$ and, hence, it converges better than expected. In [HLS10] we discretized the considered problem with $p = 1.5$ not only with $\mathbb{Q}_1/\mathbb{Q}_1$ elements, but also with the inf-sup stable $\mathbb{Q}_2/\mathbb{Q}_1$ and $\mathbb{Q}_2/\mathbb{Q}_0$ elements. While for the $\mathbb{Q}_2/\mathbb{Q}_1$ discretization we discovered $E_\pi^2 \approx \mathcal{O}(h^2)$, for the $\mathbb{Q}_2/\mathbb{Q}_0$ elements we observed $E_\pi^2 \approx \mathcal{O}(h)$ which agrees with the derived a priori error estimate. Hence we believe that for the $\mathbb{Q}_1/\mathbb{Q}_1$ elements the improved convergence rates are caused by super-approximation effects.

Table 7.3. EOC: Pressure drop problem for the Barus model

#cells	$E_{\boldsymbol{v}}^{1,2}$	E_π^2
1024	1.16	1.21
4096	1.06	1.07
16384	1.03	0.88
65536	1.04	0.89

Example 3: We considered the Barus model $\mu := \mu_0 \exp(\alpha\pi)$ that corresponds to model (7.50) in the limiting case $\delta_1 = \delta_3 = \delta_4 = 0$. As in Example 2, we set $\mu_0 := 0.01$ and

$\alpha := 10$. Once again, a FE solution on a fine grid was employed as the reference solution. Table 7.3 depicts the observed convergence rates. We realize almost linear convergence for E_π^2. In contrast to Example 2, super-approximation for E_π^2 does apparently not occur.

Table 7.4. Discretization errors for different reference solutions: $p = 1.2$

(a) analytical reference solution

	$E_v^{1,2}$		E_π^2	
#cells	error	conv.	error	conv.
256	1.69e-2	0.98	5.56e-2	1.01
1024	8.38e-3	1.01	2.76e-2	1.01
4096	4.17e-3	1.01	1.38e-2	1.01
16384	2.08e-3	1.01	6.84e-3	1.01
65536	1.03e-3	1.01	3.41e-3	1.01
262144	5.15e-4	1.01	1.70e-3	1.01
1048576	2.56e-4	1.01	8.43e-4	1.01

(b) approximative reference solution

	$E_v^{1,2}$		E_π^2	
#cells	error	conv.	error	conv.
256	1.69e-2	0.98	5.55e-2	1.01
1024	8.39e-3	1.01	2.76e-2	1.01
4096	4.18e-3	1.01	1.37e-2	1.01
16384	2.09e-3	1.00	6.79e-3	1.01
65536	1.04e-3	1.00	3.32e-3	1.03

Example 4: Finally, we numerically confirm that, even if the exact solution $(\boldsymbol{v}, \pi)$ is unknown, we are able to determine reliable convergence rates using an accurate finite element element solution $(\boldsymbol{v}_H, \pi_H)$ as the reference solution $(\boldsymbol{v}, \pi)$. In Table 7.4 we depict the discretization errors that are obtained (a) if the analytical solution is known and (b) if the analytical solution is not known. In this example, the model (7.50) was used and its parameters were chosen as in Example 1. The (non-smooth) analytical solution was given by $\boldsymbol{v}(\boldsymbol{x}) := |\boldsymbol{x}|^{a-1}(x_2, -x_1)^\top$ and $\pi(\boldsymbol{x}) := |\boldsymbol{x}|^b x_1 x_2$ with $a = 1.01$ and $b = -1.99$ so that the assumptions of Corollary 7.14 are satisfied. In Table 7.4(b) we employed the finite element approximation $(\boldsymbol{v}_H, \pi_H)$ as the reference solution $(\boldsymbol{v}, \pi) \approx (\boldsymbol{v}_H, \pi_H)$, where H corresponds to the grid with 1048576 cells. Comparing Tables 7.4(a) and 7.4(b), we observe that the discretization errors and rates of convergence agree reasonable well.

Conclusion: We have shown the convergence of the finite element method in the context of fluids with shear rate and pressure dependent viscosity. The convergence of the method has been quantified by the a priori error estimates of Corollary 7.13. These error estimates have been demonstrated practically by numerical experiments. The numerical examples indicate that the problems are well posed for a wider class of models than required by the assumptions. This is encouraging for further investigation, since the assumptions are rather restrictive from the point of view of practical applications. To my best knowledge, the error estimates of Corollary 7.13 are the first of their kind for fluids with pressure dependent viscosity. All results of this chapter also cover the case of Carreau-type models. In this case, the error estimates of Corollary 7.13 coincide with the optimal error estimates for Carreau-type models which have been established in Theorem 4.11.

7.7 Verification of (A1)–(A2) for particular models

As stated in Remark 7.7, the model (7.50) satisfies Assumptions **(A1)**–**(A2)**, e.g., with $\varepsilon^2 := \delta_1/\delta_4$, $\sigma_0 := \mu_0\delta_4^{(p-2)/2}(p-1)(1+\delta_2\delta_3^{-s}/\delta_1)^{(p-2)/2}$, $\sigma_1 := \mu_0\delta_4^{(p-2)/2}$, and $\gamma_0 := \mu_0\delta_4^{(p-4)/4}s\alpha\frac{2-p}{2}\delta_2^{p/4}\delta_3^{-sp/4}$. Below we prove Remark 7.7 following Málek et al. [MNR02]. Note that in [MNR02] Remark 7.7 was shown for $\mu_0 = \delta_2 = \delta_3 = \delta_4 = 1$. Setting $\gamma(q) := \frac{\delta_2}{\delta_4}(\delta_3 + \exp(\alpha q))^{-s}$ and $\varepsilon^2 := \frac{\delta_1}{\delta_4}$, for all $\boldsymbol{P}, \boldsymbol{Q} \in \mathbb{R}^{d\times d}_{\text{sym}}$ we observe that

$$\begin{aligned}
\frac{\partial \boldsymbol{\mathcal{S}}(q,\boldsymbol{P})}{\partial \boldsymbol{P}} : (\boldsymbol{Q}\otimes\boldsymbol{Q}) &= \sum_{i,j,k,l} \frac{\partial \boldsymbol{\mathcal{S}}_{kl}(q,\boldsymbol{P})}{\partial P_{ij}} Q_{ij}Q_{kl} \\
&= \sum_{i,j,k,l} (p-2)\mu_0\delta_4^{\frac{p-2}{2}}\left(\frac{\delta_1}{\delta_4} + \gamma(q) + |\boldsymbol{P}|^2\right)^{\frac{p-4}{2}} P_{ij}Q_{ij}P_{kl}Q_{kl} \\
&\quad + \sum_{i,j,k,l} \mu_0\delta_4^{\frac{p-2}{2}}\left(\frac{\delta_1}{\delta_4} + \gamma(q) + |\boldsymbol{P}|^2\right)^{\frac{p-2}{2}} \delta_{ij,kl}Q_{ij}Q_{kl} \\
&= (p-2)\mu_0\delta_4^{\frac{p-2}{2}}\left(\varepsilon^2 + \gamma(q) + |\boldsymbol{P}|^2\right)^{\frac{p-4}{2}} |\boldsymbol{P}:\boldsymbol{Q}|^2 \\
&\quad + \mu_0\delta_4^{\frac{p-2}{2}}\left(\varepsilon^2 + \gamma(q) + |\boldsymbol{P}|^2\right)^{\frac{p-2}{2}} |\boldsymbol{Q}|^2.
\end{aligned}$$

Since $p < 2$ and $0 < \gamma(q) \le \frac{\delta_2\delta_3^{-s}}{\delta_4} = \varepsilon^2\frac{\delta_2\delta_3^{-s}}{\delta_1}$, on the one hand we get the lower bound

$$\begin{aligned}
\frac{\partial \boldsymbol{\mathcal{S}}(q,\boldsymbol{P})}{\partial \boldsymbol{P}} : (\boldsymbol{Q}\otimes\boldsymbol{Q}) &\ge (p-1)\mu_0\delta_4^{\frac{p-2}{2}}\left(\varepsilon^2 + \gamma(q) + |\boldsymbol{P}|^2\right)^{\frac{p-2}{2}} |\boldsymbol{Q}|^2 \\
&\ge \mu_0\delta_4^{\frac{p-2}{2}}(p-1)\Big(\varepsilon^2(1+\delta_2\delta_3^{-s}/\delta_1) + |\boldsymbol{P}|^2\Big)^{\frac{p-2}{2}} |\boldsymbol{Q}|^2 \\
&\ge \mu_0\delta_4^{\frac{p-2}{2}}(p-1)\Big((1+\delta_2\delta_3^{-s}/\delta_1)\Big(\varepsilon^2 + |\boldsymbol{P}|^2\Big)\Big)^{\frac{p-2}{2}} |\boldsymbol{Q}|^2 \\
&= \mu_0\delta_4^{\frac{p-2}{2}}(p-1)(1+\delta_2\delta_3^{-s}/\delta_1)^{\frac{p-2}{2}}\Big(\varepsilon^2 + |\boldsymbol{P}|^2\Big)^{\frac{p-2}{2}} |\boldsymbol{Q}|^2.
\end{aligned}$$

On the other hand, we easily obtain the upper bound

$$\frac{\partial \boldsymbol{\mathcal{S}}(q,\boldsymbol{P})}{\partial \boldsymbol{P}} : (\boldsymbol{Q}\otimes\boldsymbol{Q}) \le \mu_0\delta_4^{\frac{p-2}{2}}\Big(\varepsilon^2 + \gamma(q) + |\boldsymbol{P}|^2\Big)^{\frac{p-2}{2}}|\boldsymbol{Q}|^2 \le \mu_0\delta_4^{\frac{p-2}{2}}\Big(\varepsilon^2 + |\boldsymbol{P}|^2\Big)^{\frac{p-2}{2}}|\boldsymbol{Q}|^2.$$

Hence, the model (7.50) satisfies Assumption **(A1)** with $\sigma_1 = \mu_0\delta_4^{(p-2)/2}$ and $\sigma_0 = \mu_0\delta_4^{(p-2)/2}(p-1)(1+\delta_2\delta_3^{-s}/\delta_1)^{(p-2)/2}$. Moreover, there holds

$$\gamma'(q) = -s\frac{\delta_2}{\delta_4}(\delta_3 + \exp(\alpha q))^{-s-1}\alpha\exp(\alpha q)$$

and, hence, $|\gamma'(q)| \le s\alpha\gamma(q)$. As a result, we conclude that

$$\begin{aligned}
\left|\frac{\partial \boldsymbol{S}(q,\boldsymbol{P})}{\partial q}\right| &= \mu_0 \delta_4^{\frac{p-2}{2}} \frac{2-p}{2}\Big(\varepsilon^2+\gamma(q)+|\boldsymbol{P}|^2\Big)^{\frac{p-4}{2}} |\boldsymbol{P}||\gamma'(q)| \\
&\le \mu_0 \delta_4^{\frac{p-2}{2}} \frac{2-p}{2}\gamma(q)^{\frac{p-4}{2}}\Big(1+\gamma(q)^{-1}\Big(\varepsilon^2+|\boldsymbol{P}|^2\Big)\Big)^{\frac{p-4}{2}} |\boldsymbol{P}|\alpha s\gamma(q) \\
&= \mu_0 \delta_4^{\frac{p-2}{2}} \alpha s \frac{2-p}{2}\gamma(q)^{\frac{p-1}{2}}\Big(1+\gamma(q)^{-1}\Big(\varepsilon^2+|\boldsymbol{P}|^2\Big)\Big)^{\frac{p-4}{2}} \Big[\gamma(q)^{-1}|\boldsymbol{P}|^2\Big]^{\frac{1}{2}} \\
&\le \mu_0 \delta_4^{\frac{p-2}{2}} \alpha s \frac{2-p}{2}\gamma(q)^{\frac{p-1}{2}}\Big(1+\gamma(q)^{-1}\Big(\varepsilon^2+|\boldsymbol{P}|^2\Big)\Big)^{\frac{p-3}{2}}.
\end{aligned}$$

Since $\frac{3-p}{2} > \frac{2-p}{4}$, we finally arrive at

$$\begin{aligned}
\left|\frac{\partial \boldsymbol{S}(q,\boldsymbol{P})}{\partial q}\right| &\le \mu_0 \delta_4^{\frac{p-2}{2}} \alpha s \frac{2-p}{2}\gamma(q)^{\frac{p-1}{2}}\Big(1+\gamma(q)^{-1}\Big(\varepsilon^2+|\boldsymbol{P}|^2\Big)\Big)^{\frac{p-2}{4}} \\
&\le \mu_0 \delta_4^{\frac{p-2}{2}} \alpha s \frac{2-p}{2}\gamma(q)^{\frac{p}{4}}\Big(\varepsilon^2+|\boldsymbol{P}|^2\Big)^{\frac{p-2}{4}}.
\end{aligned}$$

To sum up, due to $\gamma(q) \le \frac{\delta_2\delta_3^{-s}}{\delta_4}$ the model (7.50) satisfies Assumption **(A2)** with

$$\gamma_0 = \mu_0 \delta_4^{\frac{p-4}{4}} \alpha s \frac{2-p}{2} \delta_2^{\frac{p}{4}} \delta_3^{-\frac{sp}{4}}. \tag{7.51}$$

In particular, γ_0 is independent of δ_1.

8 Conclusion and Outlook

In this thesis, we analyzed the finite element (FE) approximation of nonlinear equations describing the steady motion of incompressible non-Newtonian fluids whose viscosity depends on the shear rate and pressure through a general power-law with exponent $p \in (1, \infty)$, see Assumption 2.2. The studied models include the popular power-law and Carreau model. To a certain degree, the thesis closes the gap between the widely developed mathematical theory concerned with the self-consistency of the governing equations and engineering simulations performed in industrial applications.

In Chapter 4, we considered viscosities that solely depend on the shear rate. We discretized the p-Stokes equations with equal-order d-linear $\mathbb{Q}_1/\mathbb{Q}_1$ elements which uses continuous isoparametric d-linear shape functions for both the velocity and pressure approximation. Since this discretization fails to satisfy the inf-sup stability condition, we proposed a stabilization method for the pressure gradient that is based on the well-known local projection stabilization (LPS) method introduced for Stokes systems in Becker/Braack [BB01]. Our proposed stabilization scheme is adjusted to the p-structure of the problem, and it coincides with the classical LPS scheme of [BB01] in the particular case $p = 2$. We established the well-posedness of the stabilized discrete problems, and we derived a priori error estimates which quantify the convergence of the method (see Theorems 4.11 & 4.12). Our a priori error estimates improve the ones derived in the literature so far regarding the order of convergence or the assumed regularity of the solution (cf. [BN90, BL93b, BL94]). Numerical experiments indicate that, at least in the shear thinning case, our derived a priori error estimates provide optimal rates of convergence with respect to the supposed regularity. In the shear thickening case, the derived error estimates may be suboptimal. A priori error estimates were also derived for both the steady and non-steady p-Stokes equations if the classical LPS method of [BB01] is applied. They provide rates of convergence depending on the space dimension d. In contrast, our modified stabilization with p-structure enabled us to establish error estimates that do not depend on d, see Theorem 4.11.

In Chapter 5 we studied the FE approximation of p-Oseen systems which may suffer from numerical instabilities resulting from lacking inf-sup stability and locally dominating advection. We extended the LPS approach of Chapter 4 to p-Oseen systems in order to cope with both instability phenomena (see Theorem 5.7). Note that the derived a priori error estimates remain valid for the classical power-law model which, in the case $p < 2$, features an unbounded viscosity in the limit of zero shear rate.

Chapter 6 deals with singular power-law models ($p < 2$). We identified the numerical difficulties which usually arise when the algebraic systems are solved via Newton's method. By means of Corollary 6.4 we suggested a numerical method that is based on a simple regularization of the power-law model and that enables the stable approximation of singular power-law systems. We demonstrated numerically that our regularized approximation method surpasses the non-regularized one regarding accuracy and numerical efficiency.

In Chapter 7 we considered viscosities which do not only depend on the shear rate but also on the pressure. The proposed structure of the viscosity allows a restricted sublinear dependence on the pressure measured by the parameter γ_0, see Assumption 2.2. Since the equations of motion come up with additional difficulties due to the complicated structure of the viscosity, we restricted ourselves to inf-sup stable discretizations so that we avoided stabilization of the pressure. We analyzed the Galerkin discretization of the governing equations and we showed that the discrete solutions converge to the solution of the original problem provided that γ_0 is small enough. Then we established a priori error estimates (see Corollary 7.13) which provide optimal rates of convergence with respect to the expected regularity. Note that Carreau-type models are covered as a special case. For such models, the derived error estimates agree with those established in Theorem 4.11.

Regarding the achieved results of the present thesis, the following topics represent possible extensions and they can be considered as encouraging future work:

Improved interpolation estimates

In order to be able to derive optimal a priori error estimates for Taylor-Hood elements, we need to generalize the interpolation inequalities expressed in quasi-norms to quadratic elements. From analytical point of view it still is not known if interpolation inequalities such as (3.47) hold for higher polynomial degree $r > 1$.

Optimal version of Theorem 4.12

The a priori error estimates of Theorem 4.12 quantify the convergence of the proposed stabilized finite element method in the shear thickening case. However, numerical experiments indicated that they are possibly suboptimal.

Optimal error estimates for the space-time discretization

Theorem 4.23 provides a priori error estimates for the non-steady p-Stokes problem if the temporal discretization is performed before the spatial one. But the restricted regularity of the time-discretized pressure has led to a suboptimal order of convergence in space. A future project can consist in deriving error estimates if the non-steady p-Stokes system is firstly discretized in space and afterwards an A-stable time-step method is applied.

Numerical solution on anisotropic meshes

Since typical velocity profiles of shear thinning fluids often exhibit sharp boundary layers, anisotropic meshes can be used in order to efficiently resolve sharp velocity gradients perpendicular to the boundary. The numerical solution on anisotropic meshes becomes important for an efficient solution of problems with boundary layers. For linear Oseen systems, M. Braack studied the LPS method on anisotropic meshes in [Bra08]. One may ask whether the LPS analysis of Chapter 4 can be extended to anisotropic meshes.

Refinement of the results of Chapter 7

In Chapter 7 we considered viscosities μ which depend on both the shear rate and pressure. Assumption 2.2 allows a restricted sublinear dependence on the pressure and, hence, it is rather restrictive since the relation between viscosity and pressure is usually considered as $\mu \sim \exp(\alpha\pi)$. Concerning a super-linear dependence on the pressure, the well-posedness of the governing equations and the convergence of discrete solutions are however open problems. Similarly, the singular case $\varepsilon = 0$ is not included in our analysis and the behavior for $\varepsilon \searrow 0$ remains an open question. Our numerical experiments indicated that assumption (7.13) relating ε to γ_0 can possibly be relaxed. In further studies one could investigate whether the results of Chapter 7 can be extended to a wider class of models.

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List of Abbreviations

AFEM	adaptive finite element method
DWR	dual weighted residual
EOC	experimental order of convergence
FE	finite element
FEM	finite element method
GMRES	generalized minimal residual method
LPS	local projection stabilization
PDE	partial differential equation
PSPG	pressure stabilization Petrov-Galerkin
SUPG	streamline upwind Petrov-Galerkin
$(\mathbf{IS})$	discrete inf-sup stability condition
$(\mathbf{IS}_0^\nu)$	discrete inf-sup condition (L^ν-norm version, case $\Gamma = \partial\Omega$)
$(\mathbf{IS}_\Gamma^\nu)$	discrete inf-sup condition (L^ν-norm version, case $\Gamma \neq \partial\Omega$)
$(\mathbf{P1})$	mixed weak formulation of the p-Stokes system
$(\mathbf{P2})$	direct weak formulation of the p-Stokes system
$(\mathbf{P3})$	minimization problem associated with $(\mathbf{P2})$
$(\mathbf{P4})$	weak formulation of the non-steady p-Stokes system
$(\mathbf{P5})$	weak formulation of the p-Navier-Stokes system
$(\mathbf{P6})$	weak formulation of the p-Oseen system
$(\mathbf{P7})$	weak formulation of the non-steady p-Navier-Stokes system
$(\mathbf{P8})$	extension of $(\mathbf{P1})$ to viscosities depending on the shear rate and pressure
$(\mathbf{P}^\varepsilon)$	problem $(\mathbf{P1})$ for the regularized power-law model with parameter ε
$(\mathbf{M}^\varepsilon)$	minimization problem associated with $(\mathbf{P}^\varepsilon)$
$(\mathbf{P}_h^\varepsilon)$	Galerkin discretization of $(\mathbf{P}^\varepsilon)$
$(\mathbf{P1}_h)$	equal-order discretization of $(\mathbf{P1})$, stabilized
$(\mathbf{P4}^k)$	temporal discretization of $(\mathbf{P4})$ with the implicit Euler method
$(\mathbf{P4}_h^k)$	spatial discretization of $(\mathbf{P4}^k)$ with equal-order elements, stabilized
$(\mathbf{P6}_h)$	equal-order discretization of $(\mathbf{P6})$, stabilized
$(\mathbf{P8}_h)$	Galerkin discretization of $(\mathbf{P8})$

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