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Gutachten zur Dissertation/
Report on the scientific value of doctoral thesis
"Finite Element Approximation of Problems in Non-Newtonian Fluid Mechanics"
written by
Adrian Hirn

The main theme of the thesis is numerical analysis for generalized Stokes and Navier-Stokes equations where the generalization consists in considering the dependence of the viscosity on the shear rate and the pressure. Such a class of fluids is capable of describing two kinds of non-Newtonian phenomena, namely shear thinning/shear thickening and pressure thickening. The fact that the viscosity for incompressible fluids may depend on the pressure (mean normal stress) is possible to justify only if one goes beyond the classical approaches in continuum mechanics in which the Cauchy stress is supposed to be a functional depending *explicitly* on the velocity gradient and other relevant quantities. In fact, an elegant justification of considered class of fluids stems from a recent new approach: *implicit constitutive theory* (see Rajagopal (2003, 2005)).

The first part (Chapters 3-6) focuses on the numerical analysis for models where the viscosity depends on the shear rate only. For the generalized Stokes system, the Q_1/Q_1 finite elements pair for the velocity and the pressure is considered. Such a pair does not fulfil inf-sup stability condition and the local projection stabilization method is used to make the numerical scheme stable.

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While in stabilizing the (linear) Stokes system the extra terms introduced by LPS are to provide a weighted L^2-control over the pressure gradient fluctuations, the stabilization proposed in the thesis involves also non-linear extra terms aiming for a control in $L^{p'}(\Omega)$, which is the natural space for the pressure in the generalized Stokes system. The thesis then answers the question concerning the convergence of the stabilized discrete solutions by deriving a priori estimates on the discretization error. In the shear thinning case the estimates are optimal with respect to the available regularity and to earlier established results, as is also indicated by the presented numerical experiments. The achieved estimates improve those previously published [BN90,BL93b,BL94] and are analogous to the estimates developed recently in [BBDR10], where a inf-sup stable finite element pair is used. Chapter 4 includes not-only nonlinear p'-Laplace stabilization, but also the other stabilizations, such as classical Laplace stabilization for the pressure and the gradient of the divergence for the velocity are used. Although a detailed discussion comparing the available theoretical results with numerical experiments is given, the final recommendation what kind of stabilization is preferable is somehow missing. Also, one would like to know the motivation why *the gradient of the divergence for the velocity* stabilization helps to treat singular case ($p < 2$ with $\varepsilon = 0$). In Chapter 4, both steady and unsteady p-Stokes system are analyzed.

In Chapter 5, the convection is taken into account and the theory is extended, providing analogous error estimates for the generalized Oseen system. Towards FE approximation of complete p-Navier-Stokes problems, a posteriori error estamtions is studied. Applying dual weighted residual method, a goal-oriented a posteriori error estimation allowing addaptive mesh refinement reducing numerical costs without loosing the accuracy is investigated.

The singular power-law system is succesfully treated in Chapter 6 by a suitable (standard) regularization and stable numerical variant of Newton's method is designed, analyzed (the convergence result is proved) and tested. In this method, the regularization parameter is coupled to the mesh size. Numerical experiments document that this approach improves both the accuracy and numerical efficiency of the discrete approximation.

Finally in Chapter 7, the thesis focuses on the generalized Stokes system where the viscosity is both shear thinning and pressure thickening. The situation is also put into more realistic setting by considering mixed boundary conditions including both the Dirichlet parts and inflow/outflow parts of the boundary with given (normal part of) traction, which makes the situation physically more relevant (in contrast to the full Dirichlet setting). Building upon the well-posedness results achieved only recently for a class of such fluids, where a restricted sub-linear growth of the viscosity with pressure is allowed, dominated by Carreau-type shear thinning response, the thesis develops a priori estimates of the finite element approximation error. This result appears to be the first concerning numerical analysis of the incompressible piezoviscous fluids. Here, the inf-sup stable discretization is considered, leaving the possibility to use stabilized equal order FE for a future research. The obtained estimates are again analogous to those for Carreau-type models.

To conclude, I would like to emphasize that Adrian Hirn's thesis is a very nice piece of work with a lot of interesting and notrivial results that should be to my opinion published in top journals on numerical analysis of partial differential equations. It is eveident that Adrian Hirn mastered several very recent approaches in non-Newtonian fluid mechanics, mathematical analysis of partial differential equations and, above all, numerical analysis of incompressible fluid problems. He has also proved the ability to perform detailed test computations with software package Gascoigne - the calculations support his analysis. Although the basic plan of the thesis is simple, namely to extend the theory (i.e. numerical analysis) for the Stokes and Navier-Stokes equations to the p-Stokes and p-Navier-Stokes equations, this extension is far from being simple as one has to use different tools to cross from the linear to non-linear operator.

Without any doubts, Adrian Hirn proved that he is capable of performing high level research on its own. With pleasure I recommend his thesis for its defense suggesting as possible evaluation *sehr gut.*

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