Charles University in Prague

Faculty of mathematics and physics

DOCTORAL THESIS

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# Complete Boolean Algebras and Extremally Disconnected Compact Spaces

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Study branch: (M2) geometry, topology and general structures

Prague 2013

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V Praze dne 6. 12. 2013

**Název práce**: Úplné Booleovy algebry a extremálně nesouvislé kompakty

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**Abstrakt:** Zkoumáme existenci specielních bodů v nekonečných extremálně nesouvislých kompaktních topologických prostorech, které dosvědčují jejich nehomogenitu. S použitím Stoneovy duality ekvivalentně hledáme ultrafiltry na úplných Booleových algebrách s jistými kombinatorickými vlastnostmi. Zavádíme pojem koherentního ultrafiltru (koherentního $P$-bodu, koherentně selektivního ultrafiltru). Ukazujeme, že generická existence těchto ultrafiltrů na úplných Booleových ccc algebrách s váhou nepřesahující kontinuum je konzistentní s teorií množin, a že tyto ultrafiltry slouží jako svědci nehomogenity duálních Stoneových prostorů.

Studujeme vlastnosti sekvenciální topologie na $\sigma$-úplných Booleových algebrách a její vztah k otázkám spojeným s měřitelností a subměřitelností těchto algeber. Ptáme se, zda sekvenciální topologie Booleovy algebry může být kompaktní a tuto otázku částečně zodpovídáme pro speciální případ Suslinovy algebry.

**Klíčová slova:** Booleova algebra, nedotčený bod, koherentní $P$-bod, sekvenciální topologie, spojitý funkcionál

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**Title:** Complete Boolean Algebras and Extremally Disconnected Compact Spaces

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**Abstract:** We study the existence of special points in extremally disconnected compact topological spaces that witness their nonhomogeneity. Via Stone duality, we are looking for ultrafilters on complete Boolean algebras with special combinatorial properties. We introduce the notion of a coherent ultrafilter (coherent $P$-point, coherently selective). We show that generic existence of such ultrafilters on every complete ccc Boolean algebra of weight not exceeding the continuum is consistent with set theory, and that they witness the nonhomogeneity of the corresponding Stone spaces.

We study the properties of the order-sequential property on $\sigma$-complete Boolean algebras and its relation to measure-theoretic properties. We ask whether the order-sequential topology can be compact in a nontrivial case, and partially answer the question in a special case of the Suslin algebra associated with a Suslin tree.

**Keywords:** Boolean algebra, untouchable point, coherent $P$-point, sequential topology, continuous functional

# Acknowledgements

My heartfelt thanks go to Bohuslav Balcar and Petr Simon for running the Prague Set Theory seminar. Their ongoing support has taken all forms possible on the scale from hand-holding to inspiration, and their respective roles of the *advisor* and the *supervisor* could hardly be met with a superior implementation.

I am hugely indebted to B. Balcar particularly for investing so much of his time into the development of this text. Virtually every idea presented here originated in the stimulating discussions with him. It goes without saying that I incorporated all the errors effortlessly myself.

I want to thank the Prague set-theorists, topologists and bankers whom I have had the pleasure to spend mathematical time with and who have handed me their corrective lenses many times, namely Egbert Thümmel, Eva Murtinová, Tom Pazák, Jana Flašková, David Chodounský, Jonathan Verner and Michal Doucha.

Last but not least, I want to thank Vendula for everything.

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# Chapter I

# Introduction

We deal with three topics in this text.

## 1 Nonhomogeneity and coherent structures

The first topic is the question of *nonhomogeneity* of extremally disconnected compact spaces (EDC spaces). A topological space $X$ is *homogeneous* if for every pair of points $x, y \in X$ there is an autohomeomorphism $h$ such that $h(x) = y$. The cautious reader will notice right away that we are only interested in the infinite case.

The EDC spaces, i.e. the Stone spaces of complete Boolean algebras, are long known *not* to be homogeneous. However, the original elegant proof due to Frolík [Fro] suggests no *topological* reason for this. A topological way of exploiting nonhomogeneity would be to exhibit a pair of points which simply cannot be swapped by an automorphism. Those would be the *witnesses of nonhomogeneity*. In large subclasses of the EDC spaces, such witnesses have been found. This was done by isolating a topological property that is shared by some, but not all, points in the space — while an automorphisms would have to preserve the property.

A scale of such properties has been developed in set-theoretic topology. Clearly, such a property has to be strong enough so that not all points enjoy it, yet weak enough so that points having the property can actually be found. A good candidate for such a topological property is the following.

> A point $x$ of an (infinite) topological space $X$ is *discretely untouchable* if $x \notin \bar{D}$ for every countable discrete subset $D \subseteq X$ not containing $x$.

Escaping the closure of sets which are *small* in some sense is also the main idea in other properties designed to witness nonhomogeneity. What makes discrete untouchability a good candidate is that it is the weakest of the properties introduced so far, yet there have to be points without it: by the very compactness, some points have to be discretely *touched*. So finding a discretely untouchable point would indeed be a topological exhibition of nonhomogeneity.

The subclass of EDC spaces where a witness of nonhomogeneity hasn't been found yet is currently reduced to the class of *ccc* spaces of weight at most continuum. In others EDC spaces, points with properties even stronger that discrete untouchability have been found. We deal with what we call the *Simon Conjecture*:

> *Every infinite, extremally disconnected compact Hausdorff space contains a discretely untouchable point.*

By Stone duality, the topic has a Boolean translation: we are looking for discretely untouchable ultrafilters on complete ccc Boolean algebras of size (or, equivalently, algebraic density) at most continuum. It is in this form that we actually deal with the question. To this end, we introduce the notion of *coherent ultrafilters*.

> Let $\mathcal{U}$ be an ultrafilter on a complete ccc Boolean algebra $\mathcal{B}$. If for every partition $\{p_n; n \in \omega\} \subseteq \mathcal{B}$, the family $\{A \subseteq \omega; \bigvee_A p_n \in \mathcal{U}\}$ is a $P$-point on $\omega$, call $\mathcal{U}$ a *coherent $P$-point* on $\mathcal{B}$.

Analogous definitions can be introduced for other traditional properties of ultrafiters on $\omega$. We show that coherent $P$-points and coherent Ramsey ultrafilters consistently exist on every complete ccc algebra of size not exceeding the continuum.

> **Proposition:** *Let $\mathcal{B}$ be a complete ccc Boolean algebra of size at most $\mathfrak{c}$. Then every filter $\mathcal{F}$ on $\mathcal{B}$ with a base smaller than $\mathfrak{c}$ can be extended to a coherent selective ultrafilter on $\mathcal{B}$ if and only if $\mathfrak{c} = \mathrm{cov}(\mathcal{M})$.*

> **Proposition:** *Let $\mathcal{B}$ be a complete ccc Boolean algebra of size at most $\mathfrak{c}$. Every filter on $\mathcal{B}$ with a base smaller than $\mathfrak{c}$ can be extended to a coherent $P$-ultrafilter on $\mathcal{B}$ if and only if $\mathfrak{c} = \mathfrak{d}$.*

Finally, we show that they serve as witnesses of nonhomogeneity for the corresponding Stone spaces.

> **Proposition:** *Let $\mathcal{B}$ be a complete ccc algebra. Let $\mathcal{U}$ be a coherent $P$-ultrafilter on $\mathcal{B}$. Then $\mathcal{U}$ is an untouchable point in* $\mathrm{St}(\mathcal{B})$.

# 2 The order-sequential topology

Our second topic is the order-sequential topology $\tau_s$ on $\sigma$-complete Boolean algebras. This topology has been introduced and shown to be relevant to measure-theoretic properties of Boolean algebras in [M2], and systematically developed in [BGJ]. In particular, a Boolean algebra is a Maharam algebra if and only if it is Hausdorff when equipped with the order-sequential topology.

We add a characterization of Maharam algebras using the order-sequential topology, extending the list known from [BGJ].

> **Proposition:** *A $\sigma$-complete weakly distributive ccc algebra $\mathcal{B}$ is a Maharam algebra if and only if the cartesian product topology of $(\mathcal{B}, \tau_s) \times (\mathcal{B}, \tau_s)$ coincides with the order-sequential topology of the algebra $\mathcal{B} \times \mathcal{B}$.*

We study the question of compactness for this topology. It is known from [Gl] that the only compact *Hausdorff* topology on an (infinite) Boolean algebra is the case of $P(\omega)$. Dropping the condition of Hausdorfness, we ask which algebras are compact in their order-sequential topology, and pick the Suslin algebra as a natural candidate. First we discuss the relevance of KC spaces to this question.

> A topological space is a *(countably) KC* space if every (countably) compact subset of $X$ is closed in $X$.

The KC property is a natural substitute for $T_2$ in cases $T_2$ cannot be guaranteed, as the closedness of compacts, a property readily implied by $T_2$, is often precisely the desired feature. The property relevant for us is

> **Theorem** (Bella, Costantini): *Minimal KC spaces are compact.*

In a partial result, we show that the order-sequential topology of the complete algebra determined by a Suslin tree is close to compact:

> **Proposition**: *The Suslin algebra is a minimal strongly KC space.*

We employ a coloring reformulation of compactness due to E. Thümmel and show that it is satisfied by the inherent coloring of the algebra determined by a Suslin tree added by the Jech forcing as a subset of $2^{<\omega_1}$.

# 3 Measures and functionals

The third topic is the study the similarities and differences between measures and submeasures on Boolean algebras. We look at examples of how certain measure-theoretic statements and constructions can be generalized to submeasures or even more general functionals. We are using and extending some propositions form [Pa].

> **Proposition**: *Let $(X, d)$ be a separable metric space without isolated points. Let $\mu$ be a Maharam submeasure on $Borel(X)$. Then $X$ can be decomposed into a meager set $M \subseteq supp(\mu)$ and a $G_\delta$ set $N$ with $\mu(N) = 0$.*

> **Proposition**: *Every exhaustive submeasure on $CO(2^\kappa)$ extends to a continuous regular submeasure on $Baire(2^\kappa)$.*

> **Proposition**: *An ultraproduct of algebras carrying finitely additive measures carries a $\sigma$-additive measure; its Maharam type is $\mathfrak{c}$.*

> **Proposition**: *Let $\mathcal{B}$ be a Boolean algebra carrying a monotone strictly positive exhaustive functional. Then every tree in $\mathcal{B}$ is countable.*

# Chapter II

# Basic notions

In this chapter, we recall some standard terminology and notation, and introduce the notions and results that we will be using later. Our basic references are [Je] for set theory, [Ku] for forcing, [HST] for topology, and [HBA] for Boolean algebras.

## 1 Set Theory

We work in ZFC and its extensions. Our set-theoretical notation is standard. Whenever $\kappa$ appears, it denotes an infinite cardinal number, and $\alpha, \beta, \gamma, \ldots$ are ordinal numbers; $\omega = \aleph_0$ denotes the first infinite cardinal, and $\aleph_1$ is the first uncountable cardinal; $i, j, k, l, m, n$ will usually be used as natural numbers, indexes in particular. The cardinality of a set $A$ is denoted as $|A|$. The cardinal number $\mathfrak{c} = 2^\omega = |\mathbb{R}|$ is the cardinality of the continuum. As usual, CH stays for the Continuum Hypothesis, i.e. the statement $\mathfrak{c} = \aleph_1$, and AC stays for the Axiom of Choice.

Throughout the text, the gentle reader is advised to substitute *nonempty set* for *set* whenever appropriate.

### 1.1 Combinatorics

#### Almost disjoint systems

Let $A, B$ be two infinite sets. If $A \setminus B$ is finite, we say that $A$ is *almost a subset* of $B$ and write $A \subseteq^* B$. If the symmetric difference $A \triangle B$ is finite, which means that both $A \subseteq^* B$ and $B \subseteq^* A$ hold, we say that $A$ and $B$ are *almost equal* and write $A =^* B$. If $A \cap B$ is finite, we say that $A$ and $B$ are *almost disjoint*.

A family $\mathcal{A}$ of infinite subsets of an infinite set $X$ is an *almost disjoint system* (or an *AD family*) on $X$ if the sets in $\mathcal{A}$ are pairwise almost disjoint. If the family $\mathcal{A}$ is maximal (with respect to inclusion) among the AD families on $X$, it is a *maximal almost disjoint system* (or a *MAD family*) on $X$. The smallest possible cardinality of a MAD family on $\omega$ is denoted by $\mathfrak{a} = \min\{|\mathcal{A}|; \mathcal{A} \subseteq [\omega]^\omega \text{ is a MAD family}\}$.

By a standard application of the Zorn lemma, maximal almost disjoint systems do exist. For example, there is a MAD family of size $\mathfrak{c}$ on $2^{<\omega}$ (and therefore on $\omega$ too, via any bijection): for $f \in 2^\omega$, let $A_f = \{f \restriction n; n \in \omega\}$ and put $\mathcal{A} = \{A_f; f \in 2^\omega\}$. Now extend the AD family $\mathcal{A}$ to a maximal one.

### Delta systems

**1.1 Definition.** A family $\mathcal{X}$ of sets is a $\Delta$-*system* if there is a set $r$ (called the *root*) such that $x \cap y = r$ for every $x, y \in \mathcal{X}$.

**1.2 Theorem.** *Let $\kappa < \lambda$ be infinite cardinals, $\lambda$ regular, and assume that $|\alpha^{<\kappa}| < \lambda$ for every $\alpha < \lambda$. If $\mathcal{X}$ is a family of sets, $|\mathcal{X}| = \lambda$, and every $x \in \mathcal{X}$ is of size $< \kappa$, then there is a subsystem $\mathcal{X}_0 \subseteq \mathcal{X}$ of full size $\lambda$ that forms a delta system.*

Two instances of the $\Delta$-system lemma we will need are the following: an uncountable family of finite sets contains an uncountable $\Delta$-system; a family of size $\mathfrak{c}^+$ consisting of countable sets contains a $\Delta$-system of size $\mathfrak{c}^+$.

### Diamonds

**1.3 Definition.** A set $C \subseteq \omega_1$ is a *club* if it is *closed* in the order topology and *unbounded* in $\omega_1$, i.e. for every $\alpha \in \omega_1$ there is some $\beta \in C$ such that $\alpha < \beta$. A set $S \subseteq \omega_1$ is *stationary* if it intersects every club; otherwise it is *nonstationary*.

**1.4 Definition.** Let $\diamondsuit$ stay for the following statement: there exists a *diamond sequence* $(A_\alpha \subseteq \alpha; \alpha \in \omega_1)$ such that for every $A \subseteq \omega_1$, the set $\{\alpha \in \omega_1; A \cap \alpha = A_\alpha\}$ is stationary.

The diamond principle is a combinatorial statement known to be consistent with ZFC; for example, it holds in the constructible universe. It is easily seen that $\diamondsuit$ implies CH, and it is known that the existence of Suslin trees follows from $\diamondsuit$.

### Families of functions

**1.5 Definition.** For two function $f, g \in \omega^\omega$, we say that $f$ is *dominated* by $g$ and write $f \leq g$ if $f(n) \leq g(n)$ for every $n \in \omega$. We say that $f$ is *eventually dominated* by $g$, and write $f \leq^* g$, if the set $\{n \in \omega; f(n) > g(n)\}$ is finite.

The (pre)ordered set $(\omega^\omega, \leq^*)$ gives rise to two ubiquitous cardinals:

**1.6 Definition.** A family $\mathcal{F} \subseteq \omega^\omega$ is *unbounded* if it has no upper bound in $(\omega^\omega, \leq^*)$. The *bounding number* $\mathfrak{b}$ is the smallest possible cardinality of an unbounded family.

**1.7 Definition.** A family $\mathcal{F} \subseteq \omega^\omega$ is *dominating* if $(\forall f \in \omega^\omega)(\exists g \in \mathcal{F}) f \leq^* g$. The *dominating number* $\mathfrak{d}$ is the smallest possible cardinality of a dominating family.

Clearly, $\mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c}$, and a diagonalization argument shows they are uncountable. While it is known that $\mathfrak{b}$ is a regular cardinal, $\mathfrak{d}$ can be forced to attain the singular value of $\aleph_{\omega_1}$ by a famous result of Hechler [He].

A family $\{f_\alpha; \alpha < \lambda\} \subseteq \omega^\omega$ is a $\lambda$-*scale* if it is dominating and $f_\alpha <^* f_\beta$ for $\alpha < \beta < \lambda$. In other words, a scale is a well-ordered dominating family.

**1.8 Fact.** $\mathfrak{b} = \mathfrak{d}$ *if and only if there is a scale.*

## 1.2 Ideals and filters

An *ideal* on a nonempty set $X$ is a family $\mathcal{I} \subseteq P(X)$ such that for $A \subseteq B \in \mathcal{I}$ we have $A \in \mathcal{I}$, and for $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$. In other words, an ideal is a family closed on finite unions and subsets. An ideal $\mathcal{I}$ on $X$ is *proper* if $X \notin \mathcal{I}$.

A subfamily $\mathcal{B} \subseteq \mathcal{I}$ of an ideal $\mathcal{I}$ on a set $X$ is a *base* of $\mathcal{I}$ if for every $A \in \mathcal{I}$ there is some $B \in \mathcal{B}$ such that $A \subseteq B$.

Given an ideal $\mathcal{I}$ on $X$, a subset $A \subseteq X$ such that $A \notin \mathcal{I}$ is $\mathcal{I}$-*positive*; the $\mathcal{I}$-positive subsets form a *coideal* denoted by $\mathcal{I}^+$. A family $\mathcal{A} \subseteq P(X)$ is easily seen to be a coideal iff for every partition of $A \in \mathcal{A}$ into two disjoint sets $A_0, A_1$, at least one of $A_0, A_1$ is in $\mathcal{A}$.

With every ideal on a set, four cardinal characteristics are associated.

**1.9 Definition.** For $\mathcal{I} \supseteq Fin(X)$ an ideal on an infinite set $X$, call

(i) $\mathrm{add}(\mathcal{I}) = \min\{|\mathcal{A}|; \mathcal{A} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{A} \notin \mathcal{I}\}$ the *additivity* of $\mathcal{I}$.
The ideal is $\kappa$-*additive* if $add(\mathcal{I}) \geq \kappa$; an $\aleph_1$-additive ideal is a $\sigma$-*ideal*.

(ii) $\mathrm{cov}(\mathcal{I}) = \min\{|\mathcal{A}|; \mathcal{A} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{A} = X\}$ the *covering number* of $\mathcal{I}$.

(iii) $\mathrm{cof}(\mathcal{I}) = \min\{|\mathcal{A}|; \mathcal{A} \subseteq \mathcal{I} \text{ a base of } \mathcal{I}\}$ the *cofinality* of $\mathcal{I}$.

(iv) $\mathrm{non}(\mathcal{I}) = \min\{|A|; A \subseteq X \text{ and } A \notin \mathcal{I}\}$ the *uniformity* of $\mathcal{I}$.

These cardinals are mostly of interest for $\sigma$-ideals on Polish spaces. We will recall these with the Cichoń diagram in the topology section.

**1.10 Definition** ($P$-ideal)**.** In ideal $\mathcal{I}$ on $\omega$ is a *P-ideal* if for every sequence of $A_n \in \mathcal{I}$ there is some $A \in I$ such that $A_n \subseteq^* A$ for every $n$.

Dually, a *filter* on a nonempty set $X$ is a family $\mathcal{F} \subseteq P(X)$ such that $\mathcal{I} = \{A \subseteq X; X \setminus A \in \mathcal{F}\}$ is an ideal. That is, for every $A \in \mathcal{F}$ and $A \subseteq B$ we have $B \in \mathcal{F}$, and for every $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$. A filter $\mathcal{F}$ is $\kappa$-*complete* if the dual ideal is $\kappa$-additive.

For a filter $\mathcal{F}$ on $X$, call $\mathcal{B} \subseteq \mathcal{F}$ a *base* of $\mathcal{F}$ if for every $F \in \mathcal{F}$ there is some $B \in \mathcal{B}$ such that $F \supseteq B$. Call a family $\mathcal{B} \subseteq P(X)$ *centered* if $\bigcap \mathcal{A}$ is nonempty for every finite $\mathcal{A} \subseteq \mathcal{B}$. For a centered family $\mathcal{B}$, let

$$\langle \mathcal{B} \rangle = \left\{ A \subseteq X; (\exists \mathcal{A} \subseteq \mathcal{B} \text{ finite}) \bigcap \mathcal{A} \subseteq A \right\}$$

be the filter *generated* by $\mathcal{B}$, for which $\{\bigcap \mathcal{A}; \mathcal{A} \subseteq \mathcal{B} \text{ finite}\}$ is a base.

### Ultrafilters

A filter $\mathcal{F}$ on a set $X$ is an *ultrafilter* if it is maximal (with respect to inclusion) among all filters on $X$; that is to say, if $\mathcal{G} = \mathcal{F}$ for every filter $\mathcal{G} \supseteq \mathcal{F}$ on $X$. An ultrafilter of the form $\{A \subseteq X; x \in A\}$ for some $x \in X$ is a *trivial* ultrafilter; other ultrafilters are *nontrivial* or *free*. Denote the set of free ultrafilters on $\omega$ by $\omega^*$.

Without AC, it is possible that only trivial ultrafilters exist. On the other hand, the maximality principle implies there are $2^{\mathfrak{c}}$ nontrivial ultrafilters on $\omega$.

Beyond ZFC, nontrivial ultrafilters with various interesting properties can be constructed. We are interesetd in the following two, for their topological significance.

**1.11 Definition.** An ultrafilter $\mathcal{U}$ on $\omega$ is a *P-ultrafilter* if for every partition $\{A_n\}$ of $\omega$, either some $A_n \in \mathcal{U}$, or there is a set $X \in \mathcal{U}$ such that every $X \cap A_n$ is finite.

**1.12 Fact.** *For a nontrivial ultrafilter $\mathcal{U}$ on $\omega$, the following are equivalent.*

(a) *$\mathcal{U}$ is a $P$-ultrafilter.*

(b) *For every descending sequence of $A_n \in \mathcal{U}$, there is some $A \in \mathcal{U}$ such that $A \subseteq^* A_n$ for every $n$.*

(c) *$\mathcal{U}$ is a $P$-point in the topology of $\omega^*$.*

Clearly, every trivial ultrafilter is a $P$-ultrafilter. The existence of nontrivial $P$-ultrafilters is undecidable in ZFC. On one hand, it is consistent with ZFC that there are many $P$-ultrafilters.

**1.13 Theorem** ([Ke])**.** *Every filter on $\omega$ with a base smaller than $\mathfrak{c}$ can be extended to a $P$-ultrafilter iff $\mathfrak{c} = \mathfrak{d}$.*

However, it is also consistent with ZFC that there are no $P$-ultrafilters. This was originally proved by Shelah; see [Wi] for an exposition, and [Wo] for a very readable account of this result.

**1.14 Definition.** For a partition $P$ of $\omega$ into infinite sets, call $X \subseteq \omega$ a *selector* for $P$ if $A \cap X$ is a singleton for every $A \in P$. An ultrafilter $\mathcal{U}$ on $\omega$ is *selective* or *Ramsey* if for every partition $P$ of $\omega$ into infinite sets, $\mathcal{U}$ contains either some $A \in P$ or a selector for $P$.

Clearly, every selective ultrafilter is a $P$-ultrafilter.

**1.15 Theorem** ([Ca])**.** *Every filter on $\omega$ with a base smaller than $\mathfrak{c}$ can be extended to a selective ultrafilter iff $\mathfrak{c} = \mathrm{cov}(meager)$.*

# 2 Topology

All spaces are understood to be nonempty. *A priori*, we assume no separation axioms beyond $T_1$, as we will encounter spaces for which even the basic separation properties are nontrivial, or in fact the point in question. We will be explicit about any separation requirements; in particular, *compact* does not mean compact Hausdorff.

A space without isolated points is *perfect*.

A space with a clopen base is *zero-dimensional*.

## 2.1 Polish spaces

A *Polish space* is a separable, completely metrizable space.

Let us first recall the prominent *Cantor spaces* and *Baire spaces*.

**2.1 Example.** For $\kappa$ an infinite cardinal, equip $2^\kappa$ with the product topology, where $2 = \{0, 1\}$ carries the discrete topology; that is, take the family of sets of the form $\{f \in 2^\kappa; f(\alpha) = 1\}$ for a subbase. It is clear that these are in fact clopen. Being a product of compact Hausdorff spaces, it is compact Hausdorff. This is the *Cantor space of weight* $\kappa$. In particular, $2^\omega$ is simply the *Cantor space*.

**2.2 Example.** For $\kappa$ an infinite cardinal, equip $\kappa^\omega$ with the product topology where $\kappa$ carries the discrete topology; that is, take the family of $\{f \in \kappa^\omega; f(n) = \alpha\}$ for a clopen subbase. This is the *Baire space of weight* $\kappa$. In particular, $\omega^\omega$ is simply the *Baire space*.

The standard metric for the Cantor spaces and Baire spaces is $\rho(f, g) = 2^{-\delta(f,g)}$ where $\delta(f, g) = \min\{n; f(n) \neq g(n)\}$. Note however that the definition does not fix a chosen metric; in fact, some definitions call Polish spaces *Polishable*, and reserve *Polish* for a space with a fixed metric. It is well known that the Baire space is homeomorphic to the set $\mathbb{R} \setminus \mathbb{Q}$ of irrational numbers, which, as a subset of the real line, is not complete in the usual Euclidean metric.

The Cantor space and the Baire space can be uniquely characterized.

**2.3 Theorem** (Brouwer)**.** *Up to homeomorphism, the Cantor space* $2^\omega$ *is the unique perfect compact metrizable zero-dimensional space.*

**2.4 Theorem** (Alexandroff)**.** *Up to homeomorphism, the Baire space* $\omega^\omega$ *is the unique Polish zero-dimensional space in which evrey compact has an empty interior.*

The Cantor space and the Baire space are universal in their respective classes.

**2.5 Theorem.** *Every uncountable Polish space without isolated points is a continuous image of the Baire space* $\omega^\omega$ *under a one-to-one maping.*

**2.6 Theorem.** *Every uncountable Polish compact without isolated points is a continuous image of the Cantor space* $2^\omega$.

**2.7 Fact.** *Let* $X$ *be a Polish space and* $Y \subseteq X$ *a subspace. Then* $Y$ *is a Polish space itself if and only if* $Y$ *is a* $G_\delta$ *subset of* $X$.

For example, the set $[\omega]^{<\omega}$ of finite subsets of $\omega$, as a subset of $2^\omega$, is clearly an $F_\sigma$ subset. Hence the subspace $[\omega]^\omega = 2^\omega \setminus [\omega]^{<\omega}$ of infinite sets is a $G_\delta$ subset of $2^\omega$, therefore a Polish space itself.

## 2.2 Baire category

Let $X$ be a topological space. A subset $N \subseteq X$ is *nowhere dense* if the closure of $N$ has an empty interior. A subset $M \subseteq X$ is *meager* if it is a union of countably many nowhere dense sets. A complement of a meager set is a *comeager set*. The space $X$ has the *Baire category property* (BCP) if the only meager open set is the empty set.

It is easily checked that a topological space $X$ has the BCP if and only if every intersection of countably many open dense sets is a dense set.

For every topological space $X$, the family $nwd(X)$ of nowhere dense sets is an ideal, and the family $meager(X)$ of meager sets is a $\sigma$-ideal if $X$ has the BCP.

**2.8 Theorem** (Baire Category Theorem)**.** *Locally compact Hausdorff spaces and complete metric spaces do have the Baire category property.*

Recall the four cardinal characteristics of an ideal as defined in 1.9 and note that for a space $X$ having the BCP, $cov(nwd(X)) = cov(meager(X))$.

### Cichoń diagram

The cardinal characteristics of ideals have been well studied for two traditional ideals on the real line: the $\sigma$-ideal $\mathcal{N}$ of sets of Lebesgue measure zero, and the $\sigma$-ideal $\mathcal{M}$ of meager sets of reals. See [BJST] for a general introduction into the set theory of the reals.

**2.9 Theorem.** *Let $X, Y$ be two uncountable Polish space without isolated points. Let $\mathcal{I}$ and $\mathcal{J}$ be the $\sigma$-ideals of meager sets in $X$ and $Y$, respectively. Then the additivity, covering, cofinality, and uniformity characteristics of $\mathcal{I}$ and $\mathcal{J}$ and equal.*

By this theorem, we can write just $\mathrm{cov}(\mathcal{M})$ for $\mathrm{cov}(meager(\mathbb{R}))$ or $\mathrm{cov}(nwd(X))$ for any other convenient Polish space $X$.

The ZFC inequalities between the ideal characteristics can be compactly summarized in the famous *Cichoń diagram* below.

$$\begin{tikzcd}
\mathrm{cov}(\mathcal{N}) \arrow[r] & \mathrm{non}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{N}) \\
 & \mathfrak{b} \arrow[u] & \mathfrak{d} \arrow[u] & \\
\mathrm{add}(\mathcal{N}) \arrow[r] \arrow[uu] & \mathrm{add}(\mathcal{M}) \arrow[r] \arrow[u] & \mathrm{cov}(\mathcal{M}) \arrow[r] \arrow[u] & \mathrm{non}(\mathcal{N}) \arrow[uu]
\end{tikzcd}$$

Every arrow in the diagram means an inequality provable in ZFC. It is also known that $\mathrm{add}(\mathcal{M}) = \min\{\mathrm{cov}(\mathcal{M}), \mathfrak{b}\}$ and $\mathrm{cof}(\mathcal{M}) = \max\{\mathrm{non}(\mathcal{M}), \mathfrak{d}\}$. Clearly, all of these cardinals are between $\aleph_1$ and $\mathfrak{c}$; hence CH collapses the Cichoń diagram to a single point. It is also known that Martin's Axiom makes all cardinals of the diagram equal to $\mathfrak{c}$.

These inequalities are *all* that is provable in ZFC; any assignment of $\aleph_1$ and $\aleph_2$ to these cardinals, compatible with the above, is realized in a suitable model of ZFC. See [Fr1] for full proofs.

### Meager ideals

As $P(\omega)$ is bijective with $2^\omega$, it carries a topology of the Cantor space. Subfamilies of $P(\omega)$ such as ideals and ultrafilters become subsets of $2^\omega$ under this identification, and one might investigate their topological properties.

**2.10 Theorem** ([T1]). *For an ideal $\mathcal{I} \supseteq Fin$ on $\omega$, the following are equivalent.*

(a) *$\mathcal{I} \subseteq 2^\omega$ is a meager set.*

(b) *There is a partition of $\omega$ into intervals $I_n$ such that every union of infinitely many $I_n$'s is $\mathcal{I}$-positive.*

## 2.3 Sequential spaces

A topological space $X$ is *first-countable* iff every point has a countable local base; $X$ is *Fréchet* if for every subset $A \subseteq X$ and every $x \in \bar{A}$, there is a sequence of points $x_n \in A$ converging to $x$; $X$ is *sequential* iff every subset $A \subseteq X$ that is *sequentially closed*, i.e. closed under limits of sequences from $A$, is closed.

It is easily seen that the properties above are given in descending strength, and all follow from metrizability. Standard examples from [Fra] show that none of the implications can be reversed.

**2.11 Fact.** *A mapping $f$ from a sequential space $X$ to any topological space $Y$ is continuous if and only if it is sequentially continuous, i.e., $f(x) = \lim f(x_n)$ for every sequence $(x_n)$ in $X$ converging to $x \in X$.*

**2.12 Fact.** *Let $X$ be a sequential $T_1$ space. Then $X$ is sequentially compact if and only if it is countably compact.*

The closure operator in a sequential space $(X, \tau)$ can be described as follows. For $A \subseteq X$, let $u(A) \subseteq X$ contain those points $x \in X$ for which there is a sequence of $a_n \in A$ converging to $x$. Put $u_0(A) = A$ and $u_\alpha(A) = u(\bigcup\{u_\xi(A); \xi < \alpha\})$ for $\alpha < \omega_1$. Then $\bar{A} = \bigcup\{u_\alpha(A); \alpha < \omega_1\}$.

In particular, the space $(X, \tau)$ is Fréchet if and only if $\bar{A} = u_1(A)$ for every $A \subseteq X$, i.e. if the sequential closure stabilizes after the first step.

**2.13 Definition.** Let $(X, \tau)$ be a topological space. The *sequential modification* of $\tau$ is the finest topology $\tau_s \supseteq \tau$ on $X$ such that every $\tau$-convergent sequence in $X$ is $\tau_s$-convergent.

The standard construction of the sequential modification is to consider all topologies $\sigma \supseteq \tau$ on $X$ with the property that every $\tau$-convergent sequence is $\sigma$-convergent, and define $\tau_s$ to be the topology on $X$ generated by the union of all such $\sigma$. Obviously, $\tau_s \supseteq \tau$, and it is easy to see that $(X, \tau_s)$ is a sequential space. The spaces $(X, \tau)$ and $(X, \tau_s)$ have the same class of convergent sequences.

**2.14 Example.** Consider the set $2^\kappa$ equipped with two topologies: the Cantor topology $\tau_c$, and the sequential modification $\tau_s$. If $\kappa = \omega$, the two topologies coincide, as $(2^\omega, \tau_c)$ is a metric space, hence sequential already. On the other hand, for $\kappa > \omega$, the sequential topology $\tau_s$ is strictly finer than $\tau_c$; indeed, the subset of countably supported functions is $\tau_s$-closed but not $\tau_c$-closed. Hence $(2^\kappa, \tau_s)$ cannot be compact.

**2.15 Example.** The space $\beta\omega$ contains no convergent sequences beside the eventually constant ones. Hence the sequential modification is the discrete topology; $\beta\omega$ is said to be *sequentialy discrete*.

We will return to sequential spaces when we discuss the order-sequential topology.

## 2.4 Descriptive theory

Here we recall the hierarchies of descriptive theory: the Borel sets, the Baire sets, and the analytic sets of a given topological space.

### Borel sets

Given a topological space $X$, denote by $\Sigma_0$ the family of all open sets, and by $\Pi_0$ the family of all closed sets. For $\alpha > 0$, let $\Sigma_\alpha$ be the family of all countable unions of sets from $\bigcup_{\beta<\alpha} \Pi_\beta$, and let $\Pi_\alpha$ be the family of all countable intersections of sets from $\bigcup_{\beta<\alpha} \Sigma_\beta$.

Call the sets from $\Sigma_1$ the $F_\sigma$ subsets, and call the sets from $\Pi_1$ the $G_\delta$ subsets of $X$. Apparently, $F_\sigma$ sets are the complements of $G_\delta$ sets.

The hierarchy of $\Sigma_\alpha$ and $\Pi_\alpha$ stabilizes after $\omega_1$ steps; for a countable set $\mathcal{A} \subseteq \bigcup_{\alpha<\omega_1} \Sigma_\alpha$ there is some $\alpha < \omega_1$ such that $\mathcal{A} \subseteq \Sigma_\alpha$, as $\omega_1$ is a regular cardinal. Let

$$Borel(X) = \bigcup_{\alpha<\omega_1} \Sigma_\alpha = \bigcup_{\alpha<\omega_1} \Pi_\alpha.$$

This is the *Borel algebra* of $X$ — the smallest $\sigma$-algebra of sets containing all open sets. Being a $\sigma$-algebra makes it a natural setting for the study of measures.

The Borel algebra of a Polish space is in fact unique.

**2.16 Theorem.** *Let $X, Y$ be uncountable Polish space withut isolated points. Then the algebras $Borel(X)$ and $Borel(Y)$ are isomorphic.*

### Baire sets

For $X$ a topological space, call a subset $Z \subseteq X$ a *zero-set* if there is a continuous function $f : X \to \mathbb{R}$ such that $Z = \{x \in X; f(x) = 0\}$. Let $Baire(X)$ be the smallest $\sigma$-subalgebra of $P(X)$ containing all zero sets.

Clearly every clopen set is a zero set, and every zero set is a closed $G_\delta$ set, hence a Borel set. Thus for every topological space we have

$$CO(X) \subseteq Baire(X) \subseteq Borel(X) \subseteq P(X).$$

Note that in a *normal* space, being a zero set is equivalent to being a closed $G_\delta$, by the Tietze-Urysohn theorem. In the special case of *perfectly normal* spaces, where every closed set is $G_\delta$, we have $Baire(X) = Borel(X)$.

We are mostly interested in the Borel and Baire sets of the Cantor spaces $2^\kappa$. These are normal spaces, but perfectly normal only for $\kappa = \omega$: for $\kappa > \omega$, any singleton $\{f\} \subseteq 2^\kappa$ is a closed set that is not $G_\delta$.

By definition, $Baire(X)$ is $\sigma$-generated by the zero sets; in the case of $Baire(2^\kappa)$, it is in fact $\sigma$-generated by the clopen sets: if $Z \subseteq 2^\kappa$ is a zero set, hence closed $G_\delta$, let $Z = \bigcap U_n$ for some open $U_n$. Every $U_n$ is a sum of some basic clopen sets $B^n_\iota$, hence the compact $Z \subseteq U_n$ is covered by some finite union $V_n = B^n_{\iota_1} \cup \dots \cup B^n_{\iota_k}$. So the zero sets that $\sigma$-generate $Baire(2^\kappa)$ are themselves $\sigma$-generated by $CO(2^\kappa)$.

We describe now some topological properties of the Baire sets in $2^\kappa$.

**2.17 Definition.** For $X \subseteq 2^\kappa$ and $I \subseteq \kappa$, let $X \restriction I = \{f \restriction I; f \in X\}$. For $Y \subseteq 2^I$, let the *extension* of $Y$ be $Ext(Y) = \{f \in 2^\kappa; (\exists \varphi \in Y)(f \supseteq \varphi)\}$. Say that $X$ *only depends on* $I$ if $X = Ext(X \restriction I)$.

**2.18 Fact.** *Every $B \in Baire(2^\kappa)$ only depends on a countable set.*

*Proof.* We show that $B = Ext(B \restriction I)$ for a countable set $I \subseteq \kappa$. The inclusion $B \subseteq Ext(B \restriction I)$ is immediate. As the Baire subsets of $2^\kappa$ are $\sigma$-generated by the clopen sets, we proceed by induction on the Baire complexity of $B$.

Every clopen set clearly depends on a *finite* set only. If $B_n$ only depends on a countable $I_n$, put $I = \bigcup I_n$; then $B = \bigcup B_n = \bigcup Ext(B_n \restriction I_n) = Ext(B \restriction I)$, i.e. $B$ only depends on $I$, which is countable. For complements, the statement is immediate; hence also for countable intersections. $\square$

**2.19 Fact.** *Let $B$ be a closed Baire set in $2^\kappa$. If $I \subseteq \kappa$ is a countable set such that $B$ only depends on $I$, then $B \restriction I$ is a closed set in $2^I$.*

*Proof.* Let $\varphi \in 2^I \setminus (B \restriction I)$. Then $\varphi \neq f \restriction I$ for every $f \in B$. Hence no $g \in 2^\kappa$ with $g \supseteq \varphi$ is in $B$. As $B \subseteq 2^\kappa$ is closed, there is a neighbourhood $g \restriction K$ of $g$ that isolates $g \notin B$ from $B$. If $K \cap I$ was empty, then $g \restriction K$ would not isolate $g$ from $B$, as $B$ only depends on $I$; hence $I \cap K \neq \emptyset$. This gives a neighbourhood $g \restriction (I \cap K) = \varphi \restriction (I \cap K)$ isolating $\varphi$ from $B \restriction I$. $\square$

As the space $2^I$ is homeomorphic to $2^\omega$, the closed set $B \restriction I \subseteq 2^I$ corresponding to a closed Baire $B$ is in fact a closed $G_\delta$ in $2^I$. But then $B = Ext(B \restriction I)$ is also a closed $G_\delta$ in $2^\kappa$, being the preimage of $B \restriction I$ under the projection $\pi_I : 2^\kappa \to 2^I$.

**2.20 Corollary.** *The only compact sets in $Baire(2^\kappa)$ are the $\sigma$-generating zero sets. Dually, every open set in $Baire(2^\kappa)$ is $K_\sigma$.*

Once we introduce the order-sequential topology on Boolean algebras, we will see $Baire(2^\kappa)$ as the order-sequential closure of $CO(2^\kappa)$.

**Analytic sets**

**2.21 Definition.** A subset $A$ of a Polish space $X$ is *analytic* if there is a continuous function $f : \omega^\omega \to X$ such that $A = f[\omega^\omega]$. A complement of an analytic set is *coanalytic*.

Analytic sets can be equivalently described by the following construction.

**2.22 Definition.** For a topological space $X$, a *Suslin scheme* is any family $\S = \{A_s; s \in \omega^{<\omega}\}$ of subsets of $X$ indexed be the finite sequences on natural numbers. Let the *Suslin operation* be the function $\mathcal{A}$ which maps any such scheme $\S$ to

$$\mathcal{A}(\S) = \bigcup_{f \in \omega^\omega} \bigcap_{n \in \omega} \S(f \upharpoonright n)$$

**2.23 Theorem.** *A subset of a Polish space $X$ is analytic if and only if it is of the form $\mathcal{A}(\S)$ for some Suslin scheme $\S$ consisting of closed subsets of $X$.*

**2.24 Theorem** (Suslin). *A subset of a Polish space is Borel if and only if it is both analytic and coanalytic.*

Hence the class of analytic subsets is generally not an algebra of sets.

## 2.5 KC spaces

Here we describe a property of topological space which, while elementary in nature, is not widely known.

**2.25 Definition.** A topological space $(X, \tau)$ is *KC* (*strongly KC*) if every compact (countably compact) subset $K \subseteq X$ is closed.

It is clear that a Hausdorff space is strongly KC. Singletons are compact, hence a KC space is $T_1$. So KC can be viewed as a separation axiom between $T_2$ and $T_1$.

Standard counterexamples show that none of these implications can be reversed: the cofinite topology on an infinite set is $T_1$ but not KC, as every subset is compact, but only finite sets are closed. The cocountable topology on an uncountable set is KC, even strongly KC, as the only countably compact subsets are the finite sets, which are closed; but it is obviously not Hausdorff.

**2.26 Definition.** A topological space $(X, \tau)$ has the *unique limit property* (ULP) if every sequence in $X$ has at most one limit.

It is easily seen that every KC space has ULP, for if $(x_n)$ converges to both $x$ and $y$, then the set $\{x_n; n \in \omega\} \cup \{x\}$ is compact but not closed. Also, every ULP space is $T_1$, because if $y$ cannot be $T_1$-separated from $x$, then the constant sequence of $(x)$ converges to both $x$ and $y$.

To summarize, $T_2 \to SKC \to KC \to ULP \to T_1$.

The role of the KC property is that many folklore results about compact Hausdorff spaces continue to hold for compact KC spaces, as the closedness of compacts is often precisely the feature desired in a proof. In this sense, the KC property is a substitute for $T_2$ in situations where $T_2$ cannot be guaranteed. For example:

**2.27 Lemma.** *(i) A continuous bijection between a compact space and a KC space is a homeomorphism.*

*(ii) A compact KC topology on $X$ is minimal among all KC topologies and maximal among all compact topologies.*

*(iii) A countably compact strongly KC topology is minimal strongly KC and maximal countably compact.*

The nontrivial converse to (ii) will be relevant when studying the compactness of order-sequential topology on Boolean algebras.

**2.28 Theorem** ([BC]). *Minimal KC spaces are compact.*

## 2.6 Topological groups

A *topological group* $(G, *, \tau)$ is a group $(G, *, ^{-1}, e)$ equipped with a topology $\tau$ in such a way that the group operations $*$ and $^{-1}$ are continuous.

The group structure relevant for us is a Boolean algebra equipped with the operation of symmetric difference, i.e. the structure $(\mathcal{B}, \triangle)$. It is easily checked that this is indeed an Abelian group structure, where $0_{\mathcal{B}}$ is the neutral element and every $x \in \mathcal{B}$ is its own group inverse.

For example, $(P(\omega), \triangle)$ is a topological group, in its natural Polish topology.

It was shown in [M2] that the question of metrizability of groups of this form is relevant to measure-theoretic problems, due to a theorem of Kakutani:

**2.29 Theorem** ([Ka]). *A first-countable topological group is metrizable.*

An ideal $\mathcal{I}$ on $\omega$ is also a group when equipped with the operation of symmetric difference. Such an ideal is called *Polishable* if there is a Polish group topology with the same Borel structure as that inherited from $2^\omega$.

**2.30 Theorem** ([So]). *The following are equivalent for an ideal $\mathcal{I}$ on $\omega$.*

(a) *$\mathcal{I}$ is Polishable.*

(b) *$\mathcal{I}$ is an analytic $P$-ideal.*

## 2.7 Connectedness

A topological space is *connected* if it has no clopen subsets beside itself and the empty set. A space is *totally disconnected* if every pair of points can be separated by a clopen set.

It is easily seen that a $T_1$ space is totally disconnected if and only if it has a base consisting of clopen sets. So Cantor spaces and Baire spaces are totally disconnected.

A topological space is *extremally disconnected* if the closure of every open set is open, hence clopen. We write EDC for an extremally disconnected compact Hausdorff space.

A *regular open set* $U$ in a topological space $X$ is an open set whose closure is open, hence clopen. Note that we have $CO(X) \subseteq RO(X)$ and recall that $X$ is extremally disconnected if and only if $CO(X) = RO(X)$.

## 2.8 Homogeneity

A topological space $X$ is *homogeneous* if for any pair of points $x, y \in X$, there is an autohomeomorphism $h$ of $X$ such that $h(x) = y$.

As an example, every topological group is a homogeneous space, the witnessing automorphisms being the group translations. A $\sigma$-complete Boolean algebra equipped with the order-sequential topology (see below) is a homogeneous space.

One of our topics is the study of *non*homogeneity. See the introduction for a history of its development. The starting point for us is the famous result of Frolík.

**2.31 Theorem** ([Fro])**.** *Let $X$ be an extremally disconnected space. If $Y \subseteq X$ is an infinite compact subspace, the $Y$ is not homogeneous.*

## 2.9 Direct limits

The direct limit of a directed system is one of the standard constructions in category theory. Without employing the abstract nonsense of categories, we define just the limit of topological spaces, and later Boolean algebras.

**2.32 Definition.** Let $(D, \leq)$ be a directed set. A *directed system* of topological spaces is a family $\{X_\alpha; \alpha \in D\}$ of topological spaces together with a family of continuous mappings $\left\{f_\alpha^\beta : X_\alpha \to X_\beta; \alpha < \beta \text{ in } D\right\}$ such that $f_\beta^\gamma f_\alpha^\beta = f_\alpha^\gamma$ for every $\alpha < \beta < \gamma$ in $D$. A *direct limit* of such a directed system is a topological space $X$ together with a family of continuous mappings $f_\alpha : X_\alpha \to X$ such that

(i) $f_\beta f_\alpha^\beta = f_\alpha$ for every $\alpha < \beta$ in $D$

(ii) For every other topological space $Y$ together with a family $g_\alpha : X_\alpha \to Y$ of continuous mappings satisfying (i), there is a unique homeomorphism $h : X \to Y$ such that $g_\alpha = h f_\alpha$ for every $\alpha \in D$.

The standard construction of a direct limit is to consider the disjoint topological sum $S = \bigcup X_\alpha$ of the spaces $X_\alpha$, and take the quotient $X$ of $S$ by an equivalence $\approx$ naturally prescribed by the binding maps: let $s \approx t$ for $s, t \in S$ if $s \in X_\alpha, t \in X_\beta$ and $f_\alpha^\gamma(s) = f_\beta^\gamma(t)$ for some $\alpha, \beta < \gamma$ in $D$.

# 3 Partially ordered sets

Let $(\mathbb{P}, \leq)$ be a nonempty partially ordered set. If there is a smallest element in $\mathbb{P}$, we denote it by 0 (or $0_\mathbb{P}$ if necessary); similarly, 1 (or $1_\mathbb{P}$) denotes the largest element if there is one. For a subset $X \subseteq \mathbb{P}$, let $X^+ = X \setminus \{0\}$. For $p \in \mathbb{P}$, call $\mathbb{P} \restriction p = \{x \in \mathbb{P}; x \leq p\}$ a *factor* of $\mathbb{P}$.

Two elements $x, y \in \mathbb{P}$ are *compatible* if there is some $z \in \mathbb{P}^+$ such that $z \leq x$ and $z \leq y$; we write $x \parallel y$ in that case. A subset $X \subseteq \mathbb{P}$ in which every two elements are compatible is *linked*. A subset $X \subseteq \mathbb{P}$ is *centered* if for every finitely many $x_1, \ldots, x_n \in X$ there is a lower bound $x$ in $\mathbb{P}$; if moreover the lower bound is itself in $X$, we call $X$ *filtered.*

Elements $x, y$ which are not compatible are *disjoint*; in that case, we write $x \perp y$. For $x \in \mathbb{P}$, let $x^\perp$ denote the set of all elements disjoint with $x$. A set consisting of mutually disjoint elements is an *antichain*.

An element $a \in \mathbb{P}^+$ is an *atom* of $\mathbb{P}$ if there are no mutually disjoint elements below $p$. The poset $(\mathbb{P}, \leq)$ is *atomic* if there is an atom below every $p \in \mathbb{P}$, and is *atomless* if it has no atoms.

A subset $X \subseteq (\mathbb{P}, \leq)$ is *upper bounded* (*lower bounded*) if there is some $p \in \mathbb{P}$ such that $x \leq p$ ($p \leq x$) for every $x \in X$. A subset is *bounded* if it is both upper and lower bounded.

If every two-element subset $\{x, y\}$ of $(\mathbb{P}, \leq)$ has a supremum (denoted by $x \vee y$) and an infimum (denoted by $x \wedge y$), then $(\mathbb{P}, \leq, \vee, \wedge)$ is a *lattice*. A lattice in which every bounded subset has a supremum and an infimum is *Dedekind complete*. A lattice in which *every* subset has a supremum and an infimum is *complete*.

**3.1 Fact.** *A lattice $(L, \leq)$ is Dedekind complete iff every nonempty upper bounded set has a supremum iff every nonempty lower bounded set has an infimum.*

**3.2 Theorem.** *A lattice $(L, \leq)$ is complete if and only if every monotone mapping from $(L, \leq)$ to itself has a fixed point.*

## 3.1 Chain conditions

A subset $X$ of a poset $(\mathbb{P}, \leq)$ is a *chain* if its elements are pairwise comparable, and an *antichain* if its elements are pairwise disjoint. A *maximal antichain* is also called a *partition* of $P$.

For a poset $(P, \leq)$, the *cellularity* $c(P, \leq)$ is the supremum of all cardinalities $|X|$ for $X \subseteq P$ an antichain. For a cardinal $\kappa$, say that $(P, \leq)$ is *$\kappa$-cc* if every antichain in $P$ is smaller than $\kappa$. The poset ic *ccc* if it is *$\omega_1$-cc*. The *saturation* of $(P, \leq)$ is $sat(P) = min\,\{\kappa; P \text{ is } \kappa\text{-cc}\}$

Clearly $c(P) \leq sat(P)$, and there are two possibilities: either some antichain $X \subseteq P$ attains the supremal cardinality $c(P)$, in which case $sat(P) = (c(P))^+$, or there is no antichain with the supremal cardinality, in which case $c(P) = sat(P)$.

**3.3 Proposition** ([HBA])**.** *The saturation of a poset is a regular cardinal.*

The question whether the ccc property is *productive*, i.e. whether $P \times Q$ is ccc whenever both $P$ and $Q$ are, cannot be resolved in ZFC. Under $MA_{\omega_1}$, ccc is productive. However, it is also consistent that there exist two ccc posets whose product is not ccc; see [Ga].

## 3.2 Separative quotient

A poset $(\mathbb{P}, \leq)$ is called *separative* if it that satisfies any (all) of the following equivalent conditions.

(i) for every $p \not\leq q$ in $P$, there is some $z \in P^+$ such that $z \leq p$ and $z \perp q$

(ii) $p \leq q$ if and only if $(\forall x)(x \parallel p) \to (x \parallel q)$

(iii) $p \leq q$ if and only if $p^\perp \supseteq q^\perp$

For a poset that is not separative, as standard construction exists to arrive at a *separative quotient*. For $(\mathbb{P}, \leq)$, let $p \preceq q$ if $p^\perp \supseteq q^\perp$. Then

(a) $(\mathbb{P}, \preceq)$ is separative.

(b) $(\mathbb{P}, \preceq)$ preserves the order relation of $(P, \leq)$

(c) $(\mathbb{P}, \preceq)$ preserves the disjointness relation of $(P, \leq)$

The relation $p \approx q$ defined on $\mathbb{P}$ by $(p \preceq q)\&(q \preceq p)$ is an equivalence. The quotient $\mathbb{P}/\approx$, with order induced by $\preceq$, is the *separative quotient* of $(\mathbb{P}, \leq)$.

As a standard example, consider the poset $([\omega]^\omega, \subseteq)$. This is not separative, and the separative quotient is easily seen to be $([\omega]^\omega, \subseteq^*)$.

## 3.3 Density

A subset $D \subseteq P$ of a poset $(P, \leq)$ is *dense* if for every $p \in P^+$ there is some $d \in D^+$ such that $d \leq p$. A dense subset is *open dense* if it is *downward closed*, i.e. $(\forall d \in D)(\forall p \in P)((p \leq d) \to (p \in D))$. The *density* of $(P, \leq)$ is $\pi(P) = min\,\{|D|; D \subseteq P \text{ is dense}\}$.

A dense subset $D$ of $(P, \leq)$ can be characterized by the distinguishing property that for every $p \in P^+$ there is some $X \subseteq D$ which is a maximal antichain in $\{q \in P; q \leq p\}$, that is, a partition of $p$.

## 3.4 Embeddings

For two posets $(P, \leq)$ and $(Q, \leq)$, a one-to-one mapping $e : P \to Q$ is an *embedding* if for every $x, y \in P$ we have $x \leq y$ in $P$ if and only if $e(x) \leq e(y)$ in $Q$. The embedding is *dense* if $e[P]$ is dense in $Q$, and is *regular* if every maximal antichain $X \subseteq P$ maps to a maximal antichain $e[X] \subseteq Q$.

In particular, a subposet $P$ of $Q$ is *regular* if the inclusion is regular. It is easily checked that every dense subset of $P$ is regular in $P$, hence every dense embedding is a regular embedding. The notion of dense and regular embedding is of particular interest when the posets are viewed as forcing notions, which we will recall later.

As an example, let $P_\alpha$ be arbitrary posets, each having a largest element $1_\alpha$. Then every $P_\alpha$ is regularly embedded into the product $\Pi P_\alpha$ by the natural mapping $e_\alpha$ that sends $p \in P_\alpha$ to the function $e_\alpha(p) \in \Pi P_\alpha$ which attains $p$ at $\alpha$ and $1_\beta$ at every other $\beta \neq \alpha$.

## 3.5 Trees and linear orders

**3.4 Definition.** A partially ordered set $(T, \leq)$ is a *tree* if for every *node* $t \in T$, the set $(\leftarrow, t) = \{s \in T; s < t\}$ of its *predecessors* is well-ordered. For $t \in T$, let the *height of $t$ in $T$* be the ordinal type of $(\leftarrow, t)$, denoted by $h_T(t)$. Let $T_\alpha = \{t \in T; h_T(t) = \alpha\}$ be the $\alpha$-th *level* of $T$ and call the tree *rooted* if $T_0$ consists of a single point, the *root*. Let $h(T) = \min\{\alpha; T_\alpha = \emptyset\}$ be the *height of $T$*.

Familiar examples of trees are the *Cantor tree* $2^{<\omega}$ and the *Baire tree* $\omega^{<\omega}$, both ordered by inclusion. More generally, for an ordinal $\gamma$, both $2^{<\gamma}$ and $\omega^{<\gamma}$ are trees of height $\gamma$.

**3.5 Definition.** A maximal chain in $T$ is a *branch* of $T$. A branch is *cofinal* if its order type (its *length*) is equal to $h(T)$. A subset $X \subseteq T$ is an *antichain* if the members of $X$ are mutually incomparable in $(T, \leq)$.

It is easily seen that $x, y \in T$ are incomparable in $(T, \leq)$ precisely when they are disjoint in $(T, \geq)$; hence the term for antichain. Every level $T_\alpha \subseteq T$ is an antichain in $T$, but there might be other antichains too.

**3.6 Definition.** For an infinite kardinal $\kappa$, a tree $T$ of height $\kappa$ such that every level of $T$ is of size $< \kappa$ is called a *$\kappa$-tree*.

As an easy example, $2^{<\omega}$ is an $\omega$-tree but $\omega^{<\omega}$ is not. They both have cofinal branches. We are mostly interested in $\omega_1$-trees without cofinal branches.

**3.7 Definition.** An $\omega_1$-tree without cofinal branches is an *Aronszajn tree*. An Aronszajn tree where every antichain is countable is a *Suslin tree*.

The existence of an Aronszajn tree can be proved in ZFC. The existence of a Suslin tree follows from $\diamondsuit$ and hence is consistent with ZFC. On the other hand, MA implies that there are no Suslin trees. The existence of Suslin trees is independent of ZFC. We will get back to Suslin trees in the section on forcing, and later when studying the corresponding Suslin algebras.

**3.8 Definition.** A subset $D$ of linearly ordered set $(L, \leq)$ is *dense* if $D$ intersects every nonempty open interval of $(L, \leq)$, i.e. if $D$ is topologically dense in the order topology. The linear order $L$ is *ccc* if every family of nonempty, mutually disjoint open intervals is at most countable. A linear order which is Dedekind complete, ccc, but not separable is a *Suslin line*.

It can be shown in ZFC that a Suslin line exists if and only if a Suslin tree exists.

# 4 Boolean algebras

We recall the algebraic, order-theoretic, and combinatorial properties needed later.

## 4.1 Fields of sets

A subalgebra of a powerset $P(X)$ is a *field of sets*; that is, the elements are subsets of $X$, and the Boolean operations are the set-theoretical operations. Similarly, a $\sigma$-subalgebra of $P(X)$, i.e. a subalgebra that also contains unions and intersection of countably many members, is a *$\sigma$-field of sets*.

Every Boolean algebra is isomorphic to a field of sets, as a consequence of the Stone duality described in 4.7.

Let us recall the fields of sets appearing naturally in topology. For a topological space $X$, denote the field of clopen sets by $CO(X)$; denote the smallest $\sigma$-field containing all zero sets by $Baire(X)$, the $\sigma$-algebra of *Baire sets*; denote the smallest $\sigma$-field containing all open sets by $Borel(X)$, the $\sigma$-algebra of *Borel sets*.

It is clear that $CO(X) \subseteq Baire(X) \subseteq Borel(X) \subseteq P(X)$ for any topological space. We will mostly be interested in these fields for the Cantor spaces $2^\kappa$. In the case $\kappa = \omega$, the $\sigma$-algebras $Baire(2^\omega)$ and $Borel(2^\omega)$ coincide: as $2^\omega$ is a metric space, every closed subset is closed $G_\delta$, hence a zero set. In a general Cantor space $2^\kappa$, the zero sets are precisely the $G_\delta$ compacts, and every compact Baire set is $G_\delta$.

The prominence of $Borel(2^\omega)$ is described in the following theorem.

**4.1 Theorem.** *Let $X$, $Y$ be two Polish spaces. Then $Borel(X)$ and $Borel(Y)$ are isomorphic $\sigma$-fields of sets, via a $\sigma$-complete isomorphism.*

**4.2 Theorem** (Loomis-Sikorski)**.** *Every $\sigma$-complete Boolean algebra is isomorphic to a quotient of a $\sigma$-field of sets by a $\sigma$-ideal. If moreover $\mathcal{B}$ is $\sigma$-generated by a countable subset, then the ideal can be chosen as a $\sigma$-ideal on $Borel(X)$ for a suitable Polish space $X$.*

## 4.2 Complete algebras

A Boolean algebra $\mathcal{B}$ is *complete*, resp. *$\kappa$-complete* if the lattice $(\mathcal{B}, \leq)$ is complete, resp. $\kappa$-complete; an $\aleph_1$-*complete* algebra is *$\sigma$-complete*.

Recall that the algebra $RO(X)$ of regular open sets of a topological space $X$ is a complete algebra. Another rich source of complete Boolean algebras is the process of *completion* of an arbitrary poset.

**4.3 Fact.** *Let $(\mathbb{P}, \leq)$ be a partially ordered set. Then there is a complete Boolean algebra $\mathcal{B}(\mathbb{P})$, unique up to isomorphism, and a mapping $e : \mathbb{P} \to \mathcal{B}(\mathbb{P})$ with the following properties:*

(i) *$e$ preserves the ordering $\leq$ and disjointness $\perp$ of $\mathbb{P}$, i.e. $e(p) \leq e(q)$ in $\mathcal{B}(\mathbb{P})$ if $p \leq q$ in $\mathbb{P}$ and $e(p) \perp e(q)$ in $\mathcal{B}(\mathbb{P})$ if $p \perp q$ in $\mathbb{P}$*

(ii) *the image $e[P]$ is a dense subset of $\mathcal{B}(\mathbb{P})$*

(ii) *$e$ is an isomorphic embedding iff $(\mathbb{P}, \leq)$ is separative*

In particular, for a Boolean algebra $\mathcal{A}$ which is not complete, there is a complete Boolean algebra $\mathcal{B}$ in which $\mathcal{A}$ is a dense subalgebra.

For instance, the algebra $Baire(2^\omega) = Borel(2^\omega)$ is $\sigma$-complete but not complete. It contains all the (closed) singletons as atoms, hence the completion can only be the whole powerset $P(2^\omega)$. We will return to $Borel(2^\kappa)$ for $\kappa > \omega$ later.

**Solovay embeddings**

**4.4 Definition.** Let $\mathcal{B}$ be a Boolean algebra. A subset $X \subseteq \mathcal{B}$ *generates* ($\sigma$-*generates*, *completely generates*) $\mathcal{B}$ if the smallest subalgebra ($\sigma$-subalgebra, complete subalgebra) of $\mathcal{B}$ that contains $X$ is $\mathcal{B}$ itself. In that case, $X$ is a *set of generators* ($\sigma$-*generators*, *complete generators*).

The smallest possible cardinality of a subset $X \subseteq \mathcal{B}$ that generates ($\sigma$-generates, completely generates) $\mathcal{B}$ will be denoted by $g(\mathcal{B})$ (by $g_\sigma(\mathcal{B})$, by $g_c(\mathcal{B})$).

It is well known that there are complete Boolean algebras of arbitrarily large cardinalities that are completely generated by a countable subset. Consider the poset $Fn(\omega, \kappa)$ of partial functions $p : F \to \kappa$ where $F \subseteq \omega$ is finite, ordered by reverse inclusion. The complete Boolean algebra determined by this poset is isomorphic to $RO(\kappa^\omega)$. This is the *collapsing algebra*, denoted by $C(\omega, \kappa)$.

**4.5 Theorem** (Solovay)**.** *Every ccc algebra of size at most $\mathfrak{c}$ can be embedded into a complete ccc algebra of size at most $\mathfrak{c}$ with a countable set of complete generators.*

## 4.3 Chain conditions

Viewing a Boolean algebra $\mathcal{B}$ as the partially ordered set $(\mathcal{B}, \leq)$, the notions of cellularity and saturatedness apply as introduced in 3.1

**4.6 Definition.** A Boolean algebra is

(i) $\sigma$-*centered* if $\mathcal{B}^+ = \bigcup X_n$ with each $X_n$ centered;

(ii) $\sigma$-*linked* if $\mathcal{B}^+ = \bigcup X_n$ with each $X_n$ linked;

(iii) $\sigma$-*bounded-cc* if $\mathcal{B}^+ = \bigcup X_n$ with each $X_n$ being $(n+1)$-cc;

(iv) $\sigma$-*finite-cc* if $\mathcal{B}^+ = \bigcup X_n$ with each $X_n$ being $\omega$-cc;

Clearly every $\sigma$-bounded-cc algebra is $\sigma$-finite-cc. It was an open problem since 1948 (see [HT]) whether the opposite implication also holds. This has recently been solved by E. Thümmel ([Th]) by exhibiting a $\sigma$-finite-cc poset which is not $\sigma$-bounded-cc.

A $\sigma$-linked algebra $\mathcal{B}$ cannot be larger than the continuum: if $\mathcal{B} = \bigcup X_n$ witnesses the $\sigma$-linkedness, let $f(x) = \{n; x \in X_n\}$. It is easily verified that $f$ is an injective mapping into $P(\omega)$.

## 4.4 Distributivity properties

Let $\kappa, \lambda$ be cardinal numbers. A Boolean algebra $\mathcal{B}$ is $(\kappa, \lambda)$-*distributive* if it satisfies

$$\prod_{\alpha<\kappa} \sum_{\beta<\lambda} a_{\alpha,\beta} = \sum_{f\in\kappa^\lambda} \prod_{\alpha<\kappa} a_{\alpha,f(\alpha)}$$

for any family $\{a_{\alpha,\beta}; \alpha < \kappa, \beta < \lambda\} \subseteq \mathcal{B}$. The algebra is $\kappa, \infty$-*distributive* (or simply $\kappa$-*distributive)* if it is $(\kappa, \lambda)$-distributive for every cardinal $\lambda$.

Of course, the algebra needs to be suitably complete for the formula to even make sense; we will mainly deal with distributivity properties of complete algebras.

If $\kappa, \lambda$ are finite, the above equality holds in every Boolean algebra. The first nontrivial case is the question of $(\omega, 2)$-distributivity.

The distributivity property can be reformulated in the language of order, and thus generalizes to partially ordered sets.

**4.7 Fact.** *A complete Boolean algebra $\mathcal{B}$ is $(\kappa, \lambda)$-distributive if and only if for every family $\{X_\alpha; \alpha < \kappa\}$ of partitions of $\mathcal{B}$, with each $|X_\alpha| \leq \lambda$, has a common refinement.*

**4.8 Definition.** A partially ordered set $(\mathbb{P}, \leq)$ is $\kappa$-*closed* if for every $\alpha \leq \kappa$ and every descending sequence $(p_\xi; \xi < \alpha)$ there is a lower bound of $(p_\xi)$, i.e. $p \in \mathbb{P}$ such that $p \leq p_\xi$ for all $\xi < \alpha$.

It is eaily verified that a partially ordered set which is $\kappa$-closed is $\kappa$-distributive and that a Boolean algebra with a $\kappa$-closed dense subset is $\kappa$-distributive.

**4.9 Definition.** An complete atomless ccc algebra which is $\omega$-distributive and has density $\kappa$ is called a $\kappa$-*Suslin algebra*. The case $\kappa = \omega_1$ is simply the *Suslin algebra*.

The existence of a Suslin algebra is undecidable in set theory. In fact, a Suslin algebra can be obtained by completing a Suslin tree, and conversely a Suslin tree can be found as a dense subset in a Suslin algebra.

### Weak distributivity

The notion of distributivity has a natural generalization, relevant in forcing and measure-theoretic considerations.

**4.10 Definition.** A Boolean algebra is *weakly distributive* if for any countable family $\{X_n; n \in \omega\}$ of partitions, there is a partition $X$ such that every $x \in X$ only intersects finitely many members of every $X_n$.

Clearly, every $\omega$-distributive algebra is weakly distributive, and an atomless measure algebra is an example of a complete algebra which is weakly distributive, but not $(\omega, 2)$-distributive.

The Cohen algebra $\mathcal{C}$ is an example where even weak distributivity *fails everywhere*, i.e. no factor $\mathcal{C} \restriction x$ is weakly distributive.

The forcing significance of weak distributivity of an algebra $\mathcal{B}$ is that forcing with $\mathcal{B}$ is $\omega^\omega$-bounding, i.e. every function $f \in \omega^\omega$ in the extension is bounded by a function from the ground model.

## 4.5 Countable separation

**4.11 Definition.** A Boolean algebra $\mathcal{B}$ has the *countable separation property* (CSP for short) if for every two countable subsets $X, Y \subseteq \mathcal{B}$ such that every $x \in X$ and every $y \in Y$) are mutually disjoint, there is a *separating* $b \in \mathcal{B}$ such that

$$(\forall x \in X)(x \leq b) \text{ and } (\forall y \in Y)(b \perp y).$$

It is immediate that every $\sigma$-complete algebra has the CSP. Is is also easily checked that a quotient of a CSP algebra is CSP again. In particular, $P(\omega)$ and $P(\omega)/fin$ have the CSP.

**4.12 Fact.** *Every ccc algebra with CSP is complete.*

*Proof.* Let $A = \{a_n; n \in \omega\}$ be a countable subset of $\mathcal{B}$. Put $x_n = a_n - \bigvee_{m<n} a_m$ and let $X = \{x_n; n \in \omega\}$. Clearly $X$ is an antichain in $\mathcal{B}$; extend $X$ by $Y$ so that $X \cup Y \subseteq \mathcal{B}$ is a maximal antichain. Now $X$ and $Y$ are two countable subsets by ccc, consisting of disjoint elements, so by CSP there is a separating $a \in \mathcal{B}$. It is clear that $a$ is the supremum of $X$, hence also a supremum of $A$. $\square$

**4.13 Proposition** (Smith-Tarski)**.** *Let $\mathcal{B}$ be a Boolean algebra with CSP, let $I$ be an ideal on $\mathcal{B}$ such that $\mathcal{B}/I$ is ccc. Then $\mathcal{B}/I$ is complete.*

## 4.6 Direct limits

**4.14 Definition.** Let $(D, \leq)$ be a directed set. A *directed system* of Boolean algebras is a family $\{\mathcal{B}_\alpha; \alpha \in D\}$ of Boolean algebras together with a family of homomorphisms $\left\{f_\alpha^\beta : \mathcal{B}_\alpha \to \mathcal{B}_\beta; \alpha < \beta \text{ in } D\right\}$ such that $f_\beta^\gamma f_\alpha^\beta = f_\alpha^\gamma$ for every $\alpha < \beta < \gamma$ in $D$. A *direct limit* of such a directed system is a Boolean algebra $\mathcal{B}$ together with a family of homomorphisms $f_\alpha : \mathcal{B}_\alpha \to \mathcal{B}$ such that

(i) $f_\beta f_\alpha^\beta = f_\alpha$ for every $\alpha < \beta$ in $D$

(ii) For every other Boolean algebra $\mathcal{C}$ together with a family $g_\alpha : \mathcal{B}_\alpha \to \mathcal{C}$ of homomorphisms satisfying (i), there is a unique isomorphism $h : \mathcal{B} \to \mathcal{C}$ such that $g_\alpha = h f_\alpha$ for every $\alpha \in D$.

To obtain a direct limit of a given directed system, consider the disjoint union $\mathcal{B} = \bigcup \mathcal{B}_\alpha$ of the algebras $\mathcal{B}_\alpha$, and identify $x \approx y$ for $x, y \in \mathcal{B}$ if $x \in \mathcal{B}_\alpha, y \in \mathcal{B}_\beta$ and $f_\alpha^\gamma(x) = f_\beta^\gamma(y)$ for some $\alpha, \beta < \gamma$ in $D$. This imposes an algebraic structure on $\mathcal{B}/\approx$ in a natural way.

## 4.7 Stone duality

We recall the Stone duality between Boolean algebras and compact Hausdorff totally disconnected spaces, called *Boolean spaces*.

For a Boolean algebra $\mathcal{B}$, let $\mathrm{St}(\mathcal{B})$ be the space of all ultrafilters on $\mathcal{B}$, equipped with the *Stone topology* whose base consists of the sets $s(b) = \{\mathcal{U} \in \mathrm{St}(\mathcal{B}); b \in \mathcal{U}\}$, for $b \in \mathcal{B}$. This is the *Stone space* of $\mathcal{B}$.

**4.15 Theorem** (M. H. Stone)**.** *For a Boolean algebra $\mathcal{B}$, the space* $\mathrm{St}(\mathcal{B})$ *is a compact Hausdorff totally disconnected space. The clopen algebra $CO(\mathrm{St}(\mathcal{B}))$ is isomorphic to $\mathcal{B}$. The properties of $\mathcal{B}$ and $\mathrm{St}(\mathcal{B})$ are in the following correspondence.*

(a) *$\mathcal{B}$ is $\sigma$-centered iff* $\mathrm{St}(\mathcal{B})$ *is separable.*

(b) *$\mathcal{B}$ is weakly distributive iff the nowhere dense sets of* $\mathrm{St}(\mathcal{B})$ *form a $\sigma$-ideal.*

(c) *$\mathcal{B}$ is complete iff* $\mathrm{St}(\mathcal{B})$ *is extremally disconnected.*

When looking for special points in extremally disconnected compacts spaces, we are in fact looking for special types of ultrafilters under the Stone duality disguise.

**4.16 Corollary.** *Every Boolean algebra is isomorphic to a field of sets.*

Given a Boolean algebra $\mathcal{B}$, it is a natural question what is the least possible cardinality $|X|$ of a set $X$ such that $\mathcal{B}$ is a subfield of $P(X)$. Using Stone duality, it can be shown that this is exactly the topological density of $\mathrm{St}(\mathcal{B})$.

# 5 Forcing and Generic Extensions

Here we recall extensions of models of ZFC, the basic concepts of forcing, and describe some standard forcing notions. Our basic reference is [Ku]. We will be translating freely between the language of partial orders and Boolean algebras.

**5.1 Definition.** (i) Let $\mathcal{V}$ and $\mathcal{W}$ be transitive models of ZFC. Say that $\mathcal{W}$ is an *extension* of $\mathcal{V}$ if $\mathcal{V} \subseteq \mathcal{W}$ and both have the same ordinals.

(ii) Let $(\mathbb{P}, \leq)$ be a poset in a model $\mathcal{V}$ of ZFC. A filter $G$ on $\mathbb{P}$ is *generic over* $\mathcal{V}$ if for every dense subset $D \subseteq \mathbb{P}$ such that $D \in \mathcal{V}$, the intersection $D \cap G$ is nonempty.

(iii) An extension $\mathcal{W}$ of $\mathcal{V}$ is a *generic extension* if there is a poset $(\mathbb{P}, \leq) \in \mathcal{V}$ and a generic filter $G \in \mathcal{W}$ on $\mathbb{P}$ such that $\mathcal{W}$ is the smallest model of ZFC extending $\mathcal{V}$ and containing $G$ as a set. Denote this smallest extension as $V[G]$.

In this context, $\mathcal{V}$ is called the *ground model*, $\mathbb{P} \in \mathcal{V}$ is the *forcing notion*, and the elements of $\mathbb{P}$ are the *forcing conditions*. We follow the "western" notation where $p \leq q$ means that $p$ is a *stronger* forcing condition. Note that if the forcing notion $\mathbb{P} \in \mathcal{V}$ is atomless, then a generic filter $G \subseteq \mathbb{P} \in \mathcal{V}$ cannot be a set in $\mathcal{V}$.

By 4.3, every separative poset $\mathbb{P}$ in $\mathcal{V}$ uniquely determines a complete Boolean algebra $\mathcal{B}$ in $\mathcal{V}$, together with an embedding of $\mathbb{P}$ onto a dense subset of $\mathcal{B}$. Every generic filter $G$ on $\mathbb{P}$ then determines a *generic ultrafilter* $\bar{G} = \{b \in \mathcal{B}; (\exists p \in G) p \leq b\}$ on $\mathcal{B}$, satisfying

(i) $G \subseteq \mathcal{B}^+$;

(ii) if $b \in G$ and $b \leq c$, then $c \in \mathcal{B}^+$;

(iii) either $b \in G$ or $-b \in G$, for every $b \in B$;

(iv) $\bigwedge X \in G$ for every $X \subseteq G$ such that $X \in \mathcal{V}$.

In the other direction, a generic ultrafilter $\bar{G} \subseteq \mathcal{B}$ determines a generic filter $G = \bar{G} \cap \mathbb{P}$ on (the embedded copy of) $\mathbb{P}$. The generic extensions determined by $G$ and $\bar{G}$ are indentical.

We will follow the usual practice of using either the poset $\mathbb{P}$ or the corresponding complete algebra $\mathcal{B}$, as convenient. Two forcing notions will be called *forcing equivalent* if their corresponding complete algberas are isomorphic.

## 5.1 New reals

We are interested in generic extensions that *add new reals*, i.e. such that there are $r \subseteq \omega$ in $\mathcal{W}$ which are not in $\mathcal{V}$. In a forcing context, adding a new "real" means adding a new member of $2^\omega$ or $\omega^\omega$; this is justified by the fact that a new member of $2^\omega$ codes a new path through the halved subintervals of $[0, 1]$, and thus a new real number in their intersection.

For a set $a \in \mathcal{V}$, the mappings $f : a \to \mathcal{B}$ in $\mathcal{V}$ are *Boolean names* for new subsets of $a$: given a generic filter $G$ on $\mathcal{B}$, the name $f$ is *evaluated* in $\mathcal{V}[G]$ as

$$f_G = \{x \in a; f(x) \in G\} \subseteq a.$$

In particular, every $f : \omega \to \mathcal{B}$ in the ground model is a Boolean name for a new real. $P(\omega)$ in the extension $\mathcal{V}[G]$ consists precisely of the sets $f_G \subseteq \omega$ for $f \in \mathcal{B}^\omega$ in the ground model. Note that the set $\mathcal{B}^\omega$ of all names for new reals is itself a complete Boolean algebra in $\mathcal{V}$.

If a new real is added in a generic extension, various combinatorial properties with respect to the ground model reals are of interest.

**5.2 Definition.** Let $\mathcal{V}[G]$ be a generic extension of $\mathcal{V}$.

(i) A new real $b : \omega \to \omega$ in $\mathcal{V}[G]$ is an *unbounded real* if $b \not\leq^* f$ for every $f : \omega \to \omega$ in the ground model.

(ii) A new real $d : \omega \to \omega$ in $\mathcal{V}[G]$ is a *dominating real* if $f \leq^* d$ for every $f : \omega \to \omega$ in the ground model.

(iii) A new real $r \subseteq \omega$ in $\mathcal{V}[G]$ is a *splitting real*, also called an *independent real*, if for every $x \subseteq \omega$ in the ground model both $x \cap r$ and $x \setminus r$ are infinite sets.

(iv) We say that a forcing notion $\mathcal{B} \in \mathcal{V}$ *adds an ubounded (dominating, independent) real* if there is a Boolean name $f : \omega \to \mathcal{B}$ in $\mathcal{V}$ such that for every generic $G \subseteq \mathcal{B}$, the added real $f_G$ is unbounded (dominating, independent).

**5.3 Fact.** *A forcing that adds a dominating real adds an independent real as well.*

A combinatorial property of Boolean algebras characterizing the adding of independent reals has been isolated in [Pa].

**5.4 Definition.** Let $\mathcal{B}$ be a complete Boolean algebra. Say that $\mathcal{B}$ *almost regularly embedds* the Cantor algebra $\mathcal{A} = CO(2^\omega)$ if there are $x_n \in \mathcal{B}$ such that

$$\bigvee\{x_n; n \in A\} = 1_\mathcal{B} \text{ and } \bigwedge\{x_n; n \in A\} = 0_\mathcal{B}$$

for every infinite set $A \subseteq \omega$.

Clearly, this condition is weaker than embedding the Cantor algebra regularly.

**5.5 Proposition.** *A complete Boolean algebra $\mathcal{B}$ adds an independent real if and only if $\mathcal{B}$ almost regularly embedds the Cantor algebra.*

**5.6 Question.** If $\mathcal{B}$ adds an independent real, does $\mathcal{B} * \mathcal{B}$ add a Cohen real? Equivalently, if $\mathcal{B}$ embeds $\mathcal{A}$ almost regularly, does $\mathcal{B} * \mathcal{B}$ embed $\mathcal{A}$ regularly?

## 5.2 Some standard forcings

Here we describe some of the traditional forcing notions.

### Cohen forcing

Let $\mathbb{P} = \{p : \omega \to 2; \mathrm{dom}(p) \text{ is finite}\}$ be ordered by reverse inclusion; this is the *Cohen forcing*. The Boolean completion of $\mathbb{P}$ is the *Cohen algebra* $\mathcal{C} = RO(2^\omega)$. For a generic $G \subseteq \mathcal{C}$, the generically added real $r = \bigcup G \in 2^\omega$ is a *Cohen real*.

Equivalently, we can consider the set of finite functions $p : \omega \to \omega$ ordered by reverse inclusion, which is forcing equivalent to the $\mathbb{P}$ above. The added Cohen real can be viewed then as a new function $r : \omega \to \omega$.

Cohen forcing is known to add a Suslin tree by suitably modifying an Aronszajn tree from the ground model. In $\mathcal{V}$, let $\{f_\alpha; \alpha \to \omega_1\}$ be a *coherent system of functions*, i.e. a family such that every $f_\alpha$ is one-to-one and $f_\alpha \subseteq^* f_\beta$ for $\alpha < \beta$; this means that $f_\beta \restriction \alpha$ is a finite modification of $f_\alpha$. A trivial coherent system can be obtained by taking $f_\alpha = f \restriction \alpha$ for any one-to-one function $f : \omega_1 \to \omega_1$, but a nontrivial coherent system also exists. See [Ku] for the construction.

If $r : \omega \to \omega$ is the Cohen real, put $S = \{rf; \mathrm{dom}(f) = \alpha \text{ and } f =^* f_\alpha\}$. Then $(S, \subseteq)$ is a Suslin tree in $\mathcal{V}[G]$. A detailed proof can be found in [St].

### Jech forcing

This partial order adds a Suslin tree with countable forcing conditions. These are *normal binary $\alpha$-trees* for $\alpha < \omega_1$, i.e. trees $T \subseteq 2^{<\alpha}$ such that

(i) every $s \in T$ with $h(s) < \alpha$ splits into $s^\frown 0, s^\frown 1 \in T$

(ii) for every $s \in T$ and every $\xi < \alpha$ there is some $t \in T_\xi$ with $s \subseteq t$ or $t \subseteq s$.

(iii) for every $\xi < \alpha$, the level $T \cap 2^\xi$ is countable.

The ordering is the *end extension*, i.e. $T_1 \leq T_2$ if $T_2 = T_1 \restriction \alpha$ for some $\alpha$.

For a generic filter $G$, let $S = \bigcup\{T; T \in G\}$. Then $S$ is a normal Suslin tree in the generic extension $\mathcal{V}[G]$. It is straightforward that the forcing is $\sigma$-closed, hence $\mathcal{V}[G]$ contains no new countable subsets. In particular, $\omega_1^{\mathcal{V}} = \omega_1^{\mathcal{V}[G]}$ and there are no new reals. A density argument shows that $S$ is a normal binary $\omega_1$-tree. It can be shown that every antichain in $S$ is countable; see [Je] for full proofs.

The construction can be generalized to regular $\kappa \geq \omega_1$. Note that Jech forcing adds a Suslin tree with countable conditions; independently, Tennenbaum [Te] forces a Suslin tree with finite conditions.

### Forcing with the Baire sets

The $\sigma$-algebra $Baire(2^\kappa)$ for $\kappa > \omega$ was introduced in the previous sections. Being $\sigma$-complete but not complete, we ask from the forcing point of view what the Boolean completion is. It is in fact a standard forcing in disguise.

For a topological space $(X, \tau)$, let the $G_\delta$-*modification* of $\tau$, denoted by $\tau_\delta$, be the topology on $X$ whose base sets are the countable intersections of $\tau$-open sets. It is immediate that $\tau_\delta \supseteq \tau$ and that$(X, \tau_\delta)$ is discrete if $(X, \tau)$ is first-countable. On the other hand, the $G_\delta$-modification of $\beta\omega$ for instance remains unchanged.

For $(2^\kappa, \tau_c)$, the sets of the form $\{f \in 2^\kappa; f \supseteq \varphi\}$, where $\varphi : I \to 2$ for some countable $I \subseteq \kappa$, form a base of the $G_\delta$-modification. Hence $(2^\kappa, \tau_\delta)$ is precisely the $\sigma$-*box topology* on $2^\kappa$, for which $\{f \in 2^\kappa; f \supseteq \varphi\}$ form a clopen base.

**5.7 Proposition.** *$Baire(2^\kappa, \tau_c)$ for $\kappa > \omega$ is isomorphic to $CO(2^\kappa, \tau_\delta)$. Hence its Boolean completion is isomorphic to $RO(2^\kappa, \tau_\delta)$.*

*Proof.* First we show that every $B \in Baire(2^\kappa, \tau_c)$ is an open set in $(2^\kappa, \tau_\delta)$. By 2.18, $B = Ext(B \restriction I)$ for some countable $I \subseteq \kappa$, so given $f \in B$, the set $Ext(f \restriction I)$, which is a $\tau_\delta$-neighbourhood of $f$, is a subset of $B$. As $2^\kappa \setminus B$ is $\tau_\delta$-open by the same argument, we see that the Baire sets of $(2^\kappa, \tau_c)$ are clopen in $(2^\kappa, \tau_\delta)$; in fact, they form a clopen base. Now recall that for any totally disconnected space $X$, the completion of $CO(X)$ is $RO(X)$. □

With this description in hand, we can describe the properties of $Baire(2^\kappa)$ interesting from the forcing point of view. Firstly, $Baire(2^\kappa)$ is apparently $\sigma$-closed, hence $(\omega, 2)$-distributive, and it does not add any new reals. As for cellularity,

**5.8 Fact.** *The algebra $Baire(2^\kappa, \tau_c)$ is $(2^\omega)^+$-cc.*

*Proof.* Let $B_\alpha, \alpha < \mathfrak{c}^+$ be an antichain in $Baire(2^\kappa)$. We can assume that the $B_\alpha$ are of the form $\{f \in 2^\kappa; f \supseteq \varphi_\alpha\}$ for some $\varphi_\alpha : \mathrm{dom}(\varphi_\alpha) \to 2$ with $\mathrm{dom}(\varphi_\alpha)$ countable. The system $\{\mathrm{dom}(\varphi_\alpha; \alpha < \mathfrak{c}^+\}$ contains a delta system of full size, with (countable) root $D \subseteq \bigcap \varphi_\alpha$. But then $\varphi_\alpha \restriction D$ is $\mathfrak{c}^+$ many distinct functions in $2^D$, a contradiction. □

Hence the cellularity of $RO(2^\kappa, \tau_\delta)$ is also $\mathfrak{c}$; forcing with $Baire(2^\kappa)$ preserves all cardinalities above the continuum.

**5.9 Fact.** *$Baire(2^\kappa)$ is not $(\omega_1, 2)$-distributive.*

*Proof.* Let $\kappa \geq \omega_1$. For any countable $I \subseteq \kappa$, the family $\mathcal{A}_I = \{Ext(\varphi); \varphi \in 2^I\}$ consists of Baire sets and is an antichain in $Baire(2^\kappa)$. In fact, this antichain is maximal: for $B \in Baire(2^\kappa, \tau_c)$, choose a basis clopen set $C \subseteq B$ in $CO(2^\kappa, \tau_\delta)$, of the form $C = \{f; f \supseteq \psi\}$ for some countable $\psi : \kappa \to 2$. The function $\psi \restriction I$ is one of the $\varphi \in 2^I$, hence $C \subseteq B$ is compatible with the corresponding $Ext(\varphi)$.

A generic filter $G$ on $Baire(2^\kappa)$ chooses exactly one $Ext(\varphi)$ from every $\mathcal{A}_I$. This corresponds to choosing one $\varphi_I$ from every $2^I$. These $\varphi_I$ have to be mutually consistent, as $G$ is a filter. Hence $f = \bigcup \{\varphi; Ext(\varphi) \in G\} \in 2^\kappa$.

Every restriction $f \restriction I$ for $I \subseteq \kappa$ countable is a set from the ground model, as the forcing is $\sigma$-closed, but $f$ is a new subset of $\kappa$, by a standard density argument: for every $g \in 2^\kappa$, the set $D_g = \{\varphi : \mathrm{dom}(\varphi) \to 2; \varphi \nsubseteq g\}$ is dense, i.e. the set of the corresponding $Ext(\varphi)$ is dense in $Baire(2^\kappa)$. $\square$

For the case $\kappa = \omega_1$, the poset $Baire(2^\kappa, \supseteq)$ is forcing equivalent to

(i) $Fn(\omega_1, 2, \omega_1)$, adding a new subset of $\omega_1$

(ii) $Fn(\omega_1, \mathbb{R}, \omega_1)$, forcing CH

(iii) Jech's forcing for adding a Suslin tree

In fact, it is shown in [BDH] that these forcing have isomorphic *base trees*.

# 6 Order-sequential topology

We recall now an important class of sequential spaces: Boolean algebras equipped with the order-sequential topology. This structure is relevant to measure-theoretic questions we will be studying later. We recall the basic properties of this topology as developed mostly in [BGJ] and add a few examples.

### Algebraic convergence

**6.1 Definition.** For a sequence $(a_n)$ in a $\sigma$-complete Boolean algebra, put

$$\limsup a_n = \bigwedge_{m \in \omega} \bigvee_{n \geq m} a_n$$

$$\liminf a_n = \bigvee_{m \in \omega} \bigwedge_{n \geq m} a_n$$

and say that $(a_n)$ *converges algebraically* to $a \in \mathcal{B}$ if $\limsup a_n = a = \liminf a_n$; write $a_n \to a$ if this is the case.

**6.2 Example.** Let $(A_n)$ be a sequence of subsets of $X$. Then $x \in \limsup A_n$ iff $x \in A_n$ for infinitely many $n$ while $x \in \liminf A_n$ iff $x \in A_n$ for almost all $n$. Hence the sequence converges algebraically iff every $x \in X$ that belongs to infinitely many $A_n$ belongs to almost all $A_n$. In particular, the sequence algebraically converges to $\emptyset$ if and only if the family $\{A_n; n \in \omega\}$ is point-finite.

These are the elementary properties of algebraic convergence:

**6.3 Fact** ([BGJ]). *Let $\mathcal{B}$ be a $\sigma$-complete Boolean algebra.*

(a) *Every constant sequence converges algebraically.*

(b) *The limit of an algebraically converging sequence is unique.*

(c) *Every monotone sequence converges to its supremum/infimum.*

(d) *If $\lim a_n = a$ then also $\lim a_{\pi(n)} = a$ for any permutation $\pi$ of $\omega$.*

(e) *A sequence $(a_n)$ converges to $a$ if and only if $a$ is simultaneously a supremum of an increasing sequence $(x_n)$ and an infimum of a decreasing sequence $(y_n)$ such that $x_n \leq a_n \leq y_n$ for every $n$.*

(f) *Algebraic convergence is preserved by Boolean operations.*

(g) *If $a_n \in \mathcal{B}$ are pairwise disjoint, then $\lim a_n = 0$.*

(h) *If $(a_n)$ and $(b_n)$ are decreasing sequences, than $\bigwedge a_n \vee \bigwedge b_n = \bigwedge a_n \vee b_n$*

**6.4 Observation.** *A sequence $(x_n)$ in $\mathcal{B}$ converges to $x$ if and only if there is a partition $P$ of $\mathcal{B}$ refining $\{x, -x\} \setminus \{0\}$ such that every $p \in P, p \leq x$ is compatible with almost every $x_n$, while every $q \in P, q \perp x$ is disjoint with almost every $x_n$.*

*Proof.* If $x_n \to x$, then $\bigvee_{n \geq m} x_n$ decreases to $x$; put $q_m = \bigvee_{n \geq m} x_n - \bigvee_{n \geq m+1} x_n$; then $q_m \perp x$. Similarly, $\bigwedge_{n \geq m} x_n$ increases to $x$; put $p_m = \bigwedge_{n \geq m+1} x_n - \bigwedge_{n \geq m} x_n$; then $p_m \leq x$. All the $p_m, q_m$ are mutually disjoint, so $P = (\{p_m, q_m; m \in \omega\} \cup \{\bigwedge_n x_n, -\bigvee_n x_n\})$ is a partition. This partition works: every $p_m$ is compatible with all $x_n$ except possibly $\{x_0, \dots, x_m\}$ while every $q_m$ can only be compatible with the elements of $\{x_0, \dots, x_m\}$.

Conversely, if $P$ is a suitable partition, then $\bigwedge_m \bigvee_{n \geq m} x_n$ must be disjoint with every $q \in P, q \perp x$, as otherwise $q$ would be compatible with infinitely many $x_n$. Hence $\limsup x_n \leq x$. Similarly, $\liminf x_n \geq x$. Hence $x_n \to x$. □

The algebraic convergence of sequences in a $\sigma$-complete Boolean algebra is not necessarily a topological convergence – the operation of taking a closure with respect to algebraic limits is not necessarily idempotent. There is however a natural topology determined by the algebraic convergence.

**6.5 Definition.** Let $\mathcal{B}$ be a $\sigma$-complete Boolean algebra. The *order-sequential topology* $\tau_s$ on $\mathcal{B}$ is the finest topology for which every algebraically convergent sequence in $\mathcal{B}$ is topologically convergent in $(\mathcal{B}, \tau_s)$.

Note that that there is indeed such a finest topology on $\mathcal{B}$: simply take the topology generated by the union of all such topologies as a subbase.

**6.6 Remark.** For a subset $A \subseteq \mathcal{B}$, let $u(A) = \{a \in \mathcal{B}; (\exists(a_n) \subseteq A)(a_n \to a)\}$ be the *algebraic closure* of $A$ in $\mathcal{B}$. The algebraic closure is not necessarily necessarily idempotent, and needs to be iterated to stabilize. For a subset $A$ of an algebra $\mathcal{B}$, put $u_0(A) = A$ and define

$$u_{\alpha+1}(A) = u(u_\alpha(A))$$

at successors and

$$u_\lambda(A) = \bigcup_{\alpha<\lambda} u_\alpha(A)$$

for $\lambda$ a limit. Then $\bar{A} = \bigcup_{\alpha<\omega_1} u_\alpha(A)$ is a topological closure operator, and it can be easily checked that $\bar{A}$ is precisely the closure of $A$ in $(\mathcal{B}, \tau_s)$.

**6.7 Example.** Consider the order-sequential topology $\tau_s$ on $2^\kappa$, viewing $2^\kappa$ as the complete algebra $P(\kappa)$. The pointwise convergence in $2^\kappa$ is exactly the algebraic convergence in $P(\kappa)$, hence $\tau_c \subseteq \tau_s$. In fact, we know from example 2.14 that $(2^\kappa, \tau_s)$ is the sequential modification of $(2^\kappa, \tau_c)$. Hence $(2^\kappa, \tau_s)$ is noncompact for $\kappa > \omega$. The class of convergent sequences is the same in both topologies.

**6.8 Example.** Consider $CO(2^\kappa)$ as a subset in the $\sigma$-algebra $Baire(2^\kappa)$. We show that this is a $\tau_s$-dense subset, i.e., $Baire(2^\kappa)$ is the $\tau_s$-closure of $CO(2^\kappa)$.

To see this, note first that the zero sets of $2^\kappa$, i.e. the $\sigma$-generators of $Baire(2^\kappa)$, belong to the sequential closure of $CO(2^\kappa)$. Indeed, let $Z = \bigcap U_n$ be a closed $G_\delta$. Each $U_n$, being open, is a union of some basic $B^n_\alpha \in CO(2^\kappa)$. By compactness, we have $Z \subseteq B^n_{\alpha_1} \cup \cdots \cup B^n_{\alpha_k} = V_n \subseteq U_n$ for some finitely many. Hence $Z = \bigcap V_n$ is in fact an intersection of a decreasing sequence of $V_n \in CO(2^\kappa)$, hence a $\tau_s$-limit of clopen sets. Note that the $\sigma$-generators of $Baire(2^\kappa)$ belong into the first iteration of the sequential closure.

It remains to show that every Baire set $B \subseteq 2^\kappa$ belongs to the $\tau_s$-closure of $CO(2^\kappa)$. This is proved by induction on the Baire complexity of $B$. For example, let $B = \bigcup Z_k$ be a countable union of zero sets, with $Z_k = V^k_n$ a countable intersection of clopen sets. Let $U_1$ be the union of the increasing sequence $V^0_1, V^0_1 \cup V^1_2, \ldots, V^k_{k+1}$ of clopen sets; generally, let $U_l = \bigcup V^k_{k+l}$. Clearly every $U_l$ is in the $\tau_s$-closure of $CO(2^\kappa)$, and $B = \bigcup Z_k = \bigcap U_l$ is as well.

Proceeding by induction, we see that the Baire complexity of $B$ corresponds to the iteration of the algebraic closure as described in 6.6 above.

In fact, we see that for a subalgebra $\mathcal{A}$ of a $\sigma$-algebra of sets $\mathcal{B}$, the subalgebra $\sigma$-generated by $\mathcal{A}$ is precisely the closure of $\mathcal{A}$ in $(\mathcal{B}, \tau_s)$, by the same argument.

These are the basic topological properties of $(\mathcal{B}, \tau_s)$:

**6.9 Proposition** ([BGJ])**.** *Let $\mathcal{B}$ be a $\sigma$-complete Boolean algebra.*

(a) *A sequence converges to $x \in \mathcal{B}$ topologically iff every subsequence has a subsequence which converges to $x$ algebraically.*

(b) *The space* $(\mathcal{B}, \tau_s)$ *is sequential.*

(c) *The space* $(\mathcal{B}, \tau_s)$ *has the ULP (and hence is* $T_1$*).*

(d) *The space* $(\mathcal{B}, \tau_s)$ *is homogeneous.*

(e) *The space* $(\mathcal{B}, \tau_s)$ *has no isolated points unless* $\mathcal{B}$ *is finite.*

(f) *The topology* $\tau_s$ *is determined by the neighbourhood filter of zero.*

(g) *The space* $(\mathcal{B}, \tau_s)$ *is connected for a complete atomless* $\mathcal{B}$*.*

It is worth noting at this point that the space $(\mathcal{B}, \tau_s)$ is not necessarilly Hausdorff; indeed, the Hausdorffness of this space is a nontrivial property of the algebra.

**6.10 Lemma.** *Let* $B$ *be a* $\sigma$*-complete algebra, let* $X$ *be any topological space. Then a mapping* $f : (\mathcal{B}, \tau_s) \to X$ *is continuous if and only if* $f$ *preserves the algebraic convergence of sequences in* $\mathcal{B}$*.*

*Proof.* The space $(\mathcal{B}, \tau_s)$ is sequential, hence it is enough to show that $f$ is sequentially continuous. So let $x_n$ be a sequence in $(\mathcal{B}, \tau_s)$ that topologically converges to $x$, and imagine that $f(x_n)$ does not converge to $f(x)$ in $X$. Then there is a neighbourhood $V$ of $f(x)$ such that infinitely many $f(x_n)$ are missing from $V$; this yields a subsequence $(x_{n_k})$ of $(x_n)$. This subsequence has a subsequence $(x_{n_{k_l}})$ that converges to $x$ algebraically, so $f(x_{n_{k_l}})$ converges to $f(x)$ algebraically, hence topologically as well — a contradiction. $\square$

All algebraically convergent sequences of $\mathcal{B}$ are topologically convergent in $(\mathcal{B}, \tau_s)$ by definition. It is natural to ask whether those are precisely the sequences which converge in $(\mathcal{B}, \tau_s)$, or if new convergent sequences can emerge.

**6.11 Proposition** ([BGJ])**.** *For a* $\sigma$*-complete algebra* $\mathcal{B}$ *the following are equivalent:*

(a) *The topologically convergent sequences of* $(\mathcal{B}, \tau_s)$ *are precisely the algebraically convergent sequences of* $\mathcal{B}$

(b) $\mathcal{B}$ *does not add new reals*

(c) $\mathcal{B}$ *is* $(\omega, 2)$*-distributive*

**6.12 Example.** Enumerate $CO(2^\omega)$ as $\{a_n; n \in \omega\}$ and consider $(a_n)$ as a sequence in the Cohen algebra $RO(2^\omega)$. It is easily seen that the sequence converges topologically to zero, while $\limsup a_n = 1$ and $\liminf a_n = 0$, so the sequence does not converge algebraically. Indeed the Cohen algebra does add new reals of course.

By the Borel isomorphism theorem 2.16, the algebras $Borel(X)$ are all isomorphic for all uncountable Polish spaces $X$. We pick $Borel(2^\omega)$ for the following proposition.

**6.13 Proposition.** $(Borel(2^\omega), \tau_s)$ *is a separable, Hausdorff, non-regular space.*

*Proof.* The separability of $(Borel(2^\omega), \tau_s)$ is a special case of 6.8, as $CO(2^\omega)$ is countable and $Borel(2^\omega) = Baire(2^\omega)$.

For Hausdorffness, take two distinct Borel sets $A$ and $B$, pick some $x \in A \setminus B$, and consider $U = \{X \in \mathcal{B}; x \in X\}$. The set $U \subseteq \mathcal{B}$ is both sequentially open and sequentially closed, hence clopen in $(\mathcal{B}, \tau_s)$, and clearly separates $A$ and $B$.

To show non-regularity, consider the filter $\mathcal{F} \subseteq \mathcal{B}$ of comeager sets. Being a $\sigma$-complete filter, $\mathcal{F}$ is a closed subset of $(\mathcal{B}, \tau_s)$, and clearly $0 \notin \mathcal{F}$. We will show that the closure of every open neighbourhood $V \subseteq \mathcal{B} \setminus \mathcal{F}$ of $0_\mathcal{B}$ intersects $\mathcal{F}$; consequently, the point $0 \in \mathcal{B}$ and the closed set $\mathcal{F}$ cannot be separated by open neighbourhoods.

Fix an enumeration $\{U_k; k \in \omega\}$ of $CO(2^\omega)$. Let $\{V_0^n; n \in \omega\}$ be a maximal antichain in $CO(2^\omega)$. Put $G_0^n = \bigcup_{m \geq n} V_0^m$ and note that the sets $G_0^n$ form a decreasing sequence converging to $0 \in V$. Hence there is some $n_0 \in \omega$ such that $G_0 = G_0^{n_0} \in V$. As a next step, choose an infinite partition $\{V_1^n; n \in \omega\} \subseteq CO(2^\omega)$ of $V_0^n \cup \cdots \cup V_0^{n_0 - 1} \in CO(2^\omega)$ such that every $V_0^n$ with $n < n_0$ is infinitely partitioned. If $U_0$ misses $G_0$, make sure that $U_0$ is infinitely partitioned too. The sets $G_1^n = G_0 \cup \bigcup_{m \geq n} V_1^m$ form a decreasing sequence converging to $G_0 \in V$. Hence there is some $n_1 \in \omega$ such that $G_1 = G_1^{n_1} \in V$. See if $U_1$ meets $G_1$ and take a suitable $\{V_2^n; n \in \omega\}$.

Continue inductively in this fashion, arriving at an increasing sequence of open sets $G_k \in V$. Then the open set $G = \bigcup G_k$ is in $cl(V)$, as the $G_k \in V$ converge to $G$. In fact, $G$ is open dense: a basic clopen set $U_k$ meets $G_{k+1}$ at the latest. Hence $G$ is comeager, and $G \in \mathcal{F} \cap cl(V) \neq \emptyset$. □

# 7 Measures and submeasures

Here we recall the fundamental properties of measures and submeasures on Boolean algebras, viewed as an the interaction of Boolean algebras and real numbers.

**7.1 Definition.** Let $\mathcal{B}$ be a Boolean algebra, and let $\mu : \mathcal{B} \to \mathbb{R}^+$ be a monotone mapping such that $\mu(0) = 0$. Call $\mu$

(a) *strictly positive* if $\mu(x) > 0$ for $x > 0_\mathcal{B}$;

(b) a *measure* if $\mu(x + y) = \mu(x) + \mu(y)$ for every two disjoint $x, y \in \mathcal{B}$;

(c) a *submeasure* if $\mu(x + y) \leq \mu(x) + \mu(y)$ for every $x, y \in \mathcal{B}$;

(d) a *supermeasure* if $\mu(x + y) \geq \mu(x) + \mu(y)$ for every $x, y \in \mathcal{B}$.

Call $\mu$ *normalized* if $\mu(1_\mathcal{B}) = 1$.
Denote by $Null(\mu)$ the set $\{x \in \mathcal{B}; \mu(x) = 0\}$ of *null elements*.

If moreover the algebra $\mathcal{B}$ is $\sigma$-complete, call $\mu$

(e) a *$\sigma$-additive measure* if $\mu(\bigvee X) = \sum_{x \in X} \mu(x)$ for countable disjoint $X \subseteq \mathcal{B}$;

(f) a *probability measure* if $\mu$ is $\sigma$-additive and normalized;

(e) a *$\sigma$-subadditive submeasure* if $\mu(\bigvee X) \leq \sum_{x \in X} \mu(x)$ for countable $X \subseteq \mathcal{B}$;

(f) *continuous* if $\lim \mu(a_n) = 0$ for every decreasing sequence $(a_n)$ with $\bigwedge a_n = 0_{\mathcal{B}}$.

(g) *exhaustive* if $\lim \mu(a_n) = 0$ for every disjoint sequence $(a_n)$.

It is clear that every measure is a submeasure. For $\mu$ a submeasure, $Null(\mu)$ is an ideal, which determines the quotient algebra $\mathcal{B}/Null(\mu)$. On this algebra, a natural *quotient submeasure* $\bar{\mu}$ can be introduced by putting $\bar{\mu}([x]) = \mu(x)$.

Note that the continuity defined above is precisely the continuity of $\mu$ as a real function on the sequential space $(\mathcal{B}, \tau_s)$, as introduced in 6.5. In particular, for $\mu$ a measure, continuity is equivalent to $\sigma$-additivity.

### Asymptotic density and Solecki's Theorem

As an illustration of the notions just introduced, we mention the known results concerning submeasures on the fundamental complete Boolean algebra $P(\mathbb{N})$ of subsets of natural numbers. Here we write $\mathbb{N} = \omega \setminus \{0\}$ for the natural numbers to avoid possible divisions by zero.

**7.2 Definition.** For a subset $X \subseteq \mathbb{N}$, call

(i) $d_*(X) = \liminf_n |X \cap n|/n$ the *lower asymptotic density* of $X$,

(ii) $d^*(X) = \limsup_n |X \cap n|/n$ the *upper asymptotic density* and

(iii) $d(X) = \lim_n |X \cap n|/n$ the *asymptotic density* of $X$ if the limit exists.

It is easily seen that $d^*$ is a normalized submeasure on $P(\mathbb{N})$ and $d_*$ is a normalized supermeasure on $P(\mathbb{N})$, both extending $d$. Note that the family of sets having an asymptotic density does not form a Boolean algebra, as it is not closed on intersections.

**7.3 Definition.** Every measure $\mu$ extending the asymptotic density $d$ to the complete algebra $P(\mathbb{N})$ is a *density* on $\mathbb{N}$.

Every nontrivial ultrafilter $\mathcal{U}$ on $\omega$ determines such an extension of asymptotic density: for $A \subseteq \mathbb{N}$, put $\mu(X) = \mathcal{U} - \lim_n |X \cap n|/n$.

**7.4 Definition.** The *density ideal* is the family $\mathcal{Z} = \{X \subseteq \mathbb{N}; d(X) = 0\}$ sets having zero asymptotic density. Clearly, $\mathcal{Z} = \{X \subseteq \mathbb{N}; d^*(X) = 0\}$.

Note that the family $\{X \subseteq \mathbb{N}; d_*(X) = 0\}$ of sets having zero *lower* density does not form an ideal. In fact, $\mathbb{N}$ can be partitioned into two sets of lower density zero.

**7.5 Lemma.** *For $k \geq 2$, partition $\mathbb{N}$ into intervals $I_n = [k^n, k^{n+1})$. Then*

(a) $d(X) = \lim \frac{|X \cap I_n|}{|I_n|}$ *for every $X$ having asymptotic density;*

(b) $X \in \mathcal{Z}$ *iff* $\lim \frac{|X \cap I_n|}{|I_n|} = 0$.

*Proof.* Let $X \subseteq \mathbb{N}$ be a set with density $d(X) = \lim |X \cap n|/n = \gamma$. Then $\lim |X \cap k^{n+1}|/k^{n+1} = \gamma$ as well. Fix $\varepsilon > 0$; then for $n$ large enough we have $|X \cap k^{n+1}|/k^{n+1} < \gamma + \varepsilon$ and so $|X \cap k^{n+1}|/k^n < k\gamma + k\varepsilon$. At the same time, for $n$ large enough we have $|X \cap k^n|/k^n > \gamma - \varepsilon$. Putting these two inequalities together we get

$$\frac{|X \cap k^{n+1}|}{k^n} - \frac{|X \cap k^n|}{k^n} \leq (k\gamma + k\varepsilon) - (\gamma - \varepsilon) = (k-1)\gamma + (k+1)\varepsilon$$

and so

$$\frac{|X \cap [k^n, k^{n+1})|}{k^n(k-1)} = \frac{|X \cap I_n|}{|I_n|} \leq \gamma + \frac{k+1}{k-1}\varepsilon.$$

Similarly, we get $|X \cap I_n|/|I_n| \geq \gamma - ((k+1)/(k-1))\varepsilon$ for $n$ large enough.

□

Note that the other implication in (a) above does not hold; that is, the existence of $\lim |X \cap I_n|/|I_n|$ for $X \subseteq \mathbb{N}$ does not even imply that $X$ has a density. Simply put $X = \bigcup \{[2^n, 2^n + 2^{n-1}); n \in \omega\}$ so that $X$ consists of the first halves of intervals $I_n = [2^n, 2^{n+1})$. Then clearly $\lim |X \cap I_n|/|I_n| = 1/2$ while

$$\limsup |X \cap n|/n = 1 \neq 0 = \liminf |X \cap n|/n.$$

We will show now that $\mathcal{Z}$ is an analytic $P$-ideal, using Solecki's characterization via semicontinuous functions.

**7.6 Definition.** A real-valued function $f : X \to \mathbb{R}$, defined on an arbitrary topological space $X$, is *lower (upper)* semicontinuous if for every $r \in \mathbb{R}$, the set $\{x \in X; f(x) \leq r\}$ is closed (the set $\{x \in X; f(x) \geq r\}$ is open).

**7.7 Fact.** *A submeasure $\mu$ on $P(\mathbb{N})$ is lower semicontinuous if*

$$\mu(A) = \lim_n \mu(A \cap n)$$

*for every $A \subseteq \mathbb{N}$.*

As a corollary we see that a lower semicontinuous submeasure is uniquely determined by its values on finite sets. So a finite subadditive function on $[\mathbb{N}]^{<\omega}$ uniquely extends to a lower semicontinuous submeasure on $P(\mathbb{N})$.

In the present context, we are viewing $(P(\mathbb{N}), \tau_s)$ as the metric space $(2^{\mathbb{N}}, \tau_c)$.

**7.8 Definition.** For a lower semicontinuous submeasure $\mu$ on $P(\mathbb{N})$ with finite values on singletons, let $Exh(\mu) = \{A \subseteq \omega; \lim \mu(A \setminus n) = 0\}$ be the *exhaustive ideal* of $\mu$.

It is easily seen that $Exh(\mu)$ is indeed an ideal. In fact, it is a $P$-ideal on $\mathbb{N}$, and cannot be of complexity higher than $F_{\sigma\delta}$.

For let $\mu$ be a lower semicontinuous submeasure. For every $n, k \in \omega$, the set $\varphi^{-1}[0, 1/n] \cap P(\omega \setminus k) \subseteq 2^\omega$ is closed by the lower semicontinuity of $\varphi$. Hence

$\bigcap_n \bigcup_k \varphi^{-1}[0, 1/n] \cap P(\omega \setminus k)$ is an $F_{\sigma\delta}$ subset, and it is not hard to verify that it contains precisely the members of $Exh(\varphi)$.

If $A_n \in Exh(\varphi)$ are disjoint sets, choose for $n \in \omega$ a number $k_n \in \omega$ large enough so that $\varphi(A_n \setminus k_n) < 2^{-n}$. Put $A = \bigcup_n (A_n \cap k_n)$; then $A_n \subseteq^* A$ for every $n$, and $A \in Exh(\varphi)$ by definition.

**7.9 Theorem** ([So]). *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. Then $\mathcal{I}$ is an analytic $P$-ideal iff $\mathcal{I} = Exh(\varphi)$ for some finite, lower semicontinuous submeasure $\varphi$ on $P(\mathbb{N})$.*

As an application of Solecki's theorem we show a well-known property of the zero density ideal $\mathcal{Z}$ on $\mathbb{N}$.

**7.10 Proposition.** *$\mathcal{Z}$ is a $F_{\sigma\delta}$ $P$-ideal.*

*Proof.* We will describe a suitable submeasure $\varphi$ with $\mathcal{Z} = Exh(\varphi)$. Let $\mathbb{N} = \bigcup I_n$ be a decomposition of $\mathbb{N}$ into intervals $I_n = [2^n, 2^{n+1})$ and let $\mu_n$ be the counting measure on $I_n$. For a finite $A \subseteq \mathbb{N}$, put

$$\varphi(A) = \max \{\mu_n(A \cap I_n); A \cap I_n \neq \emptyset\}$$

It is easy to see that $\varphi$ is subadditive and strictly positive. Extend $\varphi$ to infinite sets $A \subseteq \mathbb{N}$ as is necessary: by putting $\varphi(A) = \lim_n \varphi(A \cap n)$. This makes $\varphi$ a finite lower semicontinuous submeasure on $P(\mathbb{N})$ with $||\varphi|| = 1$. Now see that $Exh(\varphi) = \mathcal{Z}$. □

**7.11 Example.** Let $I_k = [2^k, 2^{k+1})$ as in the previous proof. Consider a function $f : \mathbb{N} \to \mathbb{R}$ defined as $f(n) = \mu_k(\{n\})$, where $k$ is the unique index such that $n \in I_k$.

For $X \in [\mathbb{N}]^{<\omega}$ put $\nu(X) = \sum \{f(n); n \in X\}$. Then $\nu$ is subadditive on $[\mathbb{N}]^{<\omega}$ and so extends to a lower semicontinuous submeasure $\nu : P(\mathbb{N}) \to \mathbb{R} \cup \{\infty\}$. Now $Exh(\nu) = \left\{A \subseteq \mathbb{N}; \sum_{n \in A} f(n) \text{ converges}\right\}$ is an example of a summable ideal on $\mathbb{N}$, hence an $F_\sigma$ $P$-ideal.

Note that if we put $\bar{\nu}(A) = \nu(A)/(1 + \nu(A))$ and $\infty/\infty = 1$, then $\bar{\nu}$ is a finite normalized submeasure and $Exh(\bar{\nu}) = Exh(\nu)$.

**7.12 Proposition.** *The algebra $P(\omega)/\mathcal{Z}$*

*(a) is atomless,*

*(b) is algebraically homogeneous,*

*(c) contains a regularly embedded copy of $P(\omega)/fin$,*

*(d) is a complete metric space with respect to $\mu_{d^*}$.*

*Proof.* (a) Every set $A \subseteq \mathbb{N}$ of nonzero upper density can be split into two disjoint sets, both of nonzero upper density again. Hence $[A]$ is never an atom of $P(\mathbb{N})/\mathcal{Z}$.

(b) We need to find an isomorphism between $P(\mathbb{N})/\mathcal{Z}$ and a factor given by any $[A] \neq 0$. So let $A \subseteq \mathbb{N}$ be a set of nonzero upper density. Surely $A$ is infinite, so let $e_A : \mathbb{N} \to A$ be the enumeration of $A$. This bijection naturally lifts to an isomorphism of $P(\mathbb{N})/\mathcal{Z}$ and $P(A)/Z$, and it is not hard to see that $P(A)/\mathcal{Z} \simeq ((P(\mathbb{N})/\mathcal{Z}) \upharpoonright [A])$.

(c) Put $I_n = [2^n, 2^{n+1})$, and for $A \subseteq \omega$ let $h([A]_{fin}) = [\bigcup\{I_n; n \in A\}]_{\mathcal{Z}}$ (we drop the subscripts of the equivalence classes of the respective quotients in what follows).

This is a well defined mapping from $P(\omega)/fin$ to $P(\omega)/\mathcal{Z}$, because if $A =^* B$, then $h([A])$ differs from $h([B])$ only by a finite union of the $I_n$'s, which has zero density. It is easy to check that $h$ is a Boolean morphism. Also, $h$ is one-to-one: for a nonzero $[A]$ in $P(\omega)/fin$, hence an infinite $A \subseteq \omega$, the set $\bigcup_{n\in A} I_n$ has upper density at least $1/2$, so $h([A]) \neq 0$.

For regularity, let $\mathcal{A}$ be a maximal antichain in $P(\omega)/fin$; this means that $\mathcal{A}$ is a MAD family on $\omega$. Clearly $h[\mathcal{A}]$ is an antichain in $P(\omega)/\mathcal{Z}$ under the embedding. If $h[\mathcal{A}]$ fails to be maximal, then there is a nonzero element $[X]$ in $P(\omega)/\mathcal{Z}$ disjoint to every $h(A), A \in \mathcal{A}$. So $X \subseteq \omega$ has nonzero upper density, but every $h(A) \cap X$ has zero upper density, and by 7.5

$$\lim_k \frac{|X \cap \bigcup_A I_n \cap I_k|}{|I_k|} = 0$$

for every $A \in \mathcal{A}$. But then $X$ only meets every $A \in \mathcal{A}$ in a finite set: if some $X \cap A$ was infinite, then for the infinitely many indexes $k \in X \cap A$ we would have $|X \cap \bigcup_A I_n \cap I_k|/|I_k| = 1$. This is a contradiction as $\mathcal{A}$ is a MAD family.

(d) Let $(X_n)$ be a sequence in $P(\mathbb{N})/\mathcal{Z}$ such that $\rho([X_n],[X_{n+1}]) < 2^{-n}$. We can assume that the representatives $X_n$ are such that $d^*(X_n \triangle X_i) < 2^{-i+1}$ for $n \geq i$. Choose a sequence $(k_n)$ in $\mathbb{N}$ increasing fast enough so that $k_{n+1} \geq 2k_n$ and

$$1/m < |X_n \triangle X_i \cap m| < 1/2^{i-2}$$

for $n \geq i$ and $m \geq k_n$. Put $X = \bigcup_n (X_n \cap (k_{n+1} - k_n))$. Then $([X_n])$ converges to $[X]$ in $(P(\mathbb{N})/\mathcal{Z}, \rho)$. □

## 7.1 Measure algebras

Here we recall Maharam's algebraic characterization of measure algebras, and the structural lemmas used in its proof.

**7.13 Definition.** A *measure algebra* is a complete ccc Boolean algebra that carries a $\sigma$-additive, strictly positive measure. If $\mu$ is a measure on $\mathcal{A}$ and $\nu$ is a measure on $\mathcal{B}$, then $(A, \mu)$ and $(\mathcal{B}, \nu)$ are *measure-isomorphic* if there is a Boolean isomorphism $f : \mathcal{A} \to \mathcal{B}$ such that $\mu(a) = \nu(f(a))$ for every $a \in \mathcal{A}$.

**7.14 Theorem** (Kelley)**.** *A complete Boolean algebra is a measure algebra if and only if it is weakly distributive and carries a strictly positive, finitely additive measure.*

**7.15 Definition.** An infinite complete algebra $\mathcal{A}$ is of *Maharam type* $\kappa$ if $g_c(\mathcal{A}) = \kappa$. $\mathcal{A}$ is *homogeneous in type* if moreover $g_c(\mathcal{A} \restriction a) = \kappa$ holds for every $a \in \mathcal{A}^+$.

**7.16 Theorem** ([M1])**.** *Let $(\mathcal{A}, \mu)$ and $(\mathcal{B}, \nu)$ be two measure algebras, both homogeneous in type $\kappa$, such that $\mu(1) = \nu(1)$. Then they are measure-isomorphic.*

Hence the the normalized measure algebra of $2^\kappa$ is the unique measure algebra homogeneous in type $\kappa$.

**7.17 Lemma** (Vladimirov). *Let $\mathcal{A}$ be a complete Boolean algebra, let $\mathcal{B}$ be a complete subalgebra of $\mathcal{A}$ such that $(\forall a \in \mathcal{A}^+)(\exists a' \in \mathcal{A}^+)(\forall b \in \mathcal{B}^+) b \not\leq a'$; i.e., $\mathcal{B}$ is not dense below any $a \in \mathcal{A}^+$. Let $x \in \mathcal{A}$ be arbitrary. Then there is an element $r \in \mathcal{A}$ such that, simultaneously, $r$ is independent with respect to $\mathcal{B}$, and $x \in \mathcal{B}[r]$.*

**7.18 Lemma** (Fremlin). *Let $\mu$ be a $\sigma$-additive measure on a complete ccc Boolean algebra $\mathcal{A}$. Let $\mathcal{B}$ be a complete subalgebra of $\mathcal{A}$ with $\{a \wedge b; b \in \mathcal{B}\} \neq \mathcal{A} \restriction a$ for every $a \in \mathcal{A}$, and let $\nu \leq \mu \restriction \mathcal{B}$ be a finitely additive measure on $\mathcal{B}$. Then there is some $a \in \mathcal{A}$ such that $\nu(x) = \mu(x \wedge a)$ for every $x \in \mathcal{B}$.*

Fremlin's lemma can be seen as an abstract version of the Radon-Nikodým Theorem. Note that to satisfy the assumptions, the algebra $\mathcal{A}$ cannot have atoms. Also, the measure $\nu$ is itself $\sigma$-additive.

Note the two apparently distinct conditions for a complete subalgebra $\mathcal{B}$ of a complete algebra $\mathcal{A}$ in Fremlin's and Vladimirov's lemma:

(F) $(\forall a \in \mathcal{A}^+)(\exists a' \in (\mathcal{A} \restriction a)^+)(\forall b \in \mathcal{B}^+)(b \wedge a \neq a')$

(V) $(\forall a \in \mathcal{A}^+)(\exists a' \in (\mathcal{A} \restriction a)^+)(\forall b \in \mathcal{B}^+)(b \not\leq a')$

It is clear that (F) implies (V): if (V) fails and $\mathcal{B}$ is dense below some $a \in \mathcal{A}^+$, then $a \in \mathcal{B}$, hence (F) fails. The following example due to E. Thümmel shows that the reverse implication does not hold.

**7.19 Example.** Let $\mathcal{A}$ be the completion of a free product of $P(\omega)$ with a complete atomless algebra $\mathcal{B}$. Then $\mathcal{A}$ itself is a complete atomless algebra, and the product $P(\omega) \times \mathcal{B}$ is dense in $\mathcal{A}$. Both $P(\omega)$ and $\mathcal{B}$ are regular subalgebras of $\mathcal{A}$, and we identify them with their embedded copies via $b \mapsto (1, b)$ for $b \in \mathcal{B}$, resp. $x \mapsto (x, 1)$ for $x \subseteq \omega$. The regular subalgebra $\mathcal{B}$ of $\mathcal{A}$ satisfies (V): below every $a \in \mathcal{A}^+$ there is some nonzero $(x, b') \in P(\omega) \times \mathcal{B}$, and for any $n \in x$, the element $(\{n\}, b') \leq (x, b')$ has no $(1, b) \in \mathcal{B}$ bellow it. On the other hand, the (F) condition fails: the only members of $\mathcal{A}$ below $a = (\{n\}, 1)$ are of the form $a \wedge b = (\{n\}, 1) \wedge (1, b) = (\{n\}, b)$ for some $b \in \mathcal{B}$.

## 7.2 Maharam algebras

The class of Maharam algebras was introduced in [M2] in a measure-theoretic context: the motivation was to describe the *algebraic* properties of an algebra $\mathcal{B}$ necessary and sufficient for $\mathcal{B}$ to carry a measure. Various algebraic (and not so algebraic) equivalents have been found. See [Ve] for an exhaustive survey.

**7.20 Definition.** A submeasure $\mu$ on a $\sigma$-complete Boolean algebra $\mathcal{B}$ is a *continuous submeasure* or a *Maharam submeasure* if for every decreasing sequence $(a_n)$ in $\mathcal{B}$ such that $\bigwedge a_n = 0$ we have $\lim \mu(a_n) = 0$. An algebra that carries a strictly positive Maharam submeasure is a *Maharam algebra*.

A Maharam algebra is necessarily ccc and weakly distributive.

A number of equivalent reformulations of being a Maharam algebra has been developed in [BGJ].

**7.21 Theorem.** *Let $\mathcal{B}$ be a $\sigma$-complete Boolean algebra. Then the following conditions are equivalent.*

(i) *$\mathcal{B}$ is a Maharam algebra.*

(ii) *The sequential space $(\mathcal{B}, \tau_s)$ is metrizable.*

(iii) *$\mathcal{B}$ is ccc and the space $(\mathcal{B}, \tau_s)$ is Hausdorff.*

(iv) *The space $(\mathcal{B}, \tau_s)$ is regular.*

(v) *$\mathcal{B}$ is ccc and $(\mathcal{B}, \triangle, \tau_s)$ is a topological group.*

**7.22 Example.** The completeness of the algebra cannot be omitted in the characterization theorem above – the algebra $\mathcal{B} \leq P(\omega_1)$ consisting of countable sets and their complements is $\sigma$-complete, not complete, and indeed the theorem fails: $(\mathcal{B}, \triangle, \tau_s)$ is a topological group, yet is not a Maharam algebra.

To show this, we need the following fact: while the order-sequential topology $\tau_s$ of $2^{\omega_1}$ is strictly finer than the Cantor topology $\tau_c$ on $2^{\omega_1}$, these two topologies coincide on the subspace $[\omega_1]^{\leq\omega}$; hence also on the dual filter, as complementation is an automorphism.

So the zero element 0 cannot be a countable intersection of neighbourhoods, and $\mathcal{B}$ is not a Maharam algebra. On the other hand, the group $(\mathcal{B}, \triangle, \tau_s)$ is topological.

## 7.3 Metric from a submeasure

A Boolean algebra carrying a submeasure carries the structure of a (pseudo)metric space. We recall some basic results from [Fr1].

**7.23 Definition.** For $\mu$ a submeasure on a complete ccc Boolean algebra $\mathcal{B}$, put

$$\rho_\mu(x, y) = \mu(x \triangle y)$$

for every $x, y \in \mathcal{B}$.

It is easily verified that $\rho_\mu$ is a pseudometric on $\mathcal{B}$; in case $\mu$ is strictly positive, $\rho_\mu$ is a metric. The Boolean operations $\wedge, -, \vee$ are uniformly continuous functions with respect to this metric.

**7.24 Theorem.** *Let $\mathcal{B}$ be a $\sigma$-complete Boolean algebra and let $\mu$ be a strictly positive submeasure on $\mathcal{B}$.*

*(i) If $\mu$ is a Maharam submeasure, then the metric space $(\mathcal{B}, \rho_\mu)$ is complete.*

*(ii) If $\mu$ is exhaustive, then the metric completion $\bar{\mathcal{B}}$ of $(\mathcal{B}, \rho)$ carries a structure of a complete Boolean algebra that makes $\mathcal{B}$ a subalgebra.*

The reason for (ii) is that the Boolean operations, being uniformly continuous, have a unique continuous extension to $\bar{\mathcal{B}}$, and the validity of Boolean-algebraic axioms is preserved. Clearly $\mathcal{B}$ is a dense subset of $\bar{\mathcal{B}}$ metrically, but not necessarily a dense subset in the order-theoretic sense.

As a consequence, we get the following

**7.25 Proposition.** *Let $(\mathcal{B}, \mu)$ be an algebra with a strictly positive exhaustive submeasure. Then the Boolean completion of $\mathcal{B}$ also carries a strictly positive exhaustive submeasure.*

*Proof.* Consider the metric completion $\bar{\mathcal{B}}$ of $(\mathcal{B}, \rho_\mu)$ as above. By [Fr2], the original $\mu$ can be extended to a strictly positive exhaustive submeasure $\bar{\mu}$ on $\bar{\mathcal{B}}$. Being a complete algebra, $\bar{\mathcal{B}}$ is injective by Sikorski's theorem. Hence the Boolean completion of $\mathcal{B}$ also embeds into $\bar{\mathcal{B}}$ and inherits $\bar{\mu}$. □

The following theorem describes the correspondence between the measure-theoretic properties of an algebra equipped with a measure and the properties of the corresponding metric space.

**7.26 Theorem.** *Let $(\mathcal{B}, \mu)$ be an infinite measure algebra and let $\kappa \geq \omega$. Then*

*(a) $(\mathcal{B}, \mu)$ has Maharam type $\kappa$ iff the metric space $(\mathcal{B}, \rho_\mu)$ has density $\kappa$.*

*(b) $(\mathcal{B}, \mu)$ is homogeneous in Maharam type $\kappa$ iff the metric space $(\mathcal{B}, \rho_\mu)$ has hereditary density $\kappa$.*

# Chapter III

# Coherent Structures and Nonhomogeneity

Motivated by the search for untouchable points, we introduce in this chapter the notion of a coherent structure on a Boolean algebra.

We give a short overview of the nonhomogeneity problem. We describe the lattice of partitions on a complete ccc Boolean algebra and the structure of subalgebras and quotients it induces. We look at how an ultrafilter on the given algebra reflects in the partition structure and we introduce *coherent ultrafilters*, specifically the *coherent $P$-points* and *coherently Ramsey* ultrafilters. We show that these consistently exist on every complete ccc algebra, and use them as witnesses of nonhomogeneity for the corresponding Stone spaces.

## 1 The nonhomogeneity problem

A Stone space of an infinite *complete* Boolean algebra is never homogeneous. That is to say, there are pairs of points that cannot be swapped by an autohomeomorphism. This was proved by Z. Frolík for a wider class of $F$-spaces, using a cardinality argument.

It is a natural quest to provide a transparent topological property of ultrafilters on complete Boolean algebras that would – as points in the corresponding Stone spaces – visibly violate the homogeneity of the extremally disconnected compact. A candidate for such a topological property has been formulated as "not to be a cluster point of a countable discrete subset".

**1.1 Definition.** Let $X$ be a topological space. A point $x \in X$ is *discretely untouchable* if $x \notin cl(D \setminus \{x\})$ for every countable discrete set $D \subseteq X$.

A homeomorphism obviously preserves the property of being discretely untouchable. In a compact Hausdorff space, every infinite subset must also *have* a cluster point – which cannot be swapped with a discretely untouchable point then. So finding a discretely untouchable point is a straight way to show the nonhomogeneity of a compact Hausdorff space.

Note that trying to find points that are not cluster points of *any* countable set (not necessarilly discrete) would be too much to ask: there are separable EDC spaces – via Stone's duality, these are Stone spaces of $\sigma$-centered complete algebras. In these, such points cannot exist. That's why we need to specify which class of countable sets our points are escaping.

**1.2 Conjecture** (Simon)**.** In every extremally disconnected ccc compact space of weight $\mathfrak{c}$ without isolated points, there is a discretely untouchable point. Equivalently, every complete atomless ccc Boolean algebra of size $\mathfrak{c}$ carries a discretely untouchable ultrafilter.

Consistently, the conjecture holds. It is also known that points with properties even stringer than discrete untouchability do exist in EDCs with weight $> \mathfrak{c}$ (van Mill), with uncountable cellularity (van Douwen) and with $cf(g(RO(X))) > \omega$ (Simon). Hence the spaces where a discretely untouchable point is yet to be found are the Stone spaces of complete, atomless, ccc algebras of size (or, equivaently, algebraic density) equal to the continuum.

# 2 The lattice of partitions

Recall that a *partition* of a Boolean agebra is a maximal antichain. We will denote the set of all partitions of an algebra $\mathcal{B}$ by $Part(\mathcal{B})$, while $Part_{fin}(\mathcal{B})$ and $Part_\infty(\mathcal{B})$ will stand for the set of finite and infinite partitions, respectively.

**2.1 Definition.** Let $\mathcal{B}$ be a Boolean algebra. For two partitions $P, Q$ of $\mathcal{B}$, we say that $P$ *refines* $Q$ and write $P \preceq Q$ if for each $p \in P$ there is exactly one $q \in Q$ such that $p \leq q$. We say that $P$ *finitely (infinitely) refines* $Q$ if $P \preceq Q$ and for each $q \in Q$, the set $\{p \in P; p \leq q\}$ is finite (infinite). We call $P \wedge Q = \{p \wedge q; p \in P, q \in Q\} \setminus \{0\}$ the *common refinement* of $P$ and $Q$.

The relation $P \preceq Q$ is easily seen to be a partial order on $Part(\mathcal{B})$. We note that $P \wedge Q$ is indeed a partition of $\mathcal{B}$ that refines both $P$ and $Q$. In fact, it is the infimum of $\{P, Q\}$ in $(Part(\mathcal{B}), \preceq)$, and makes $(Part(\mathcal{B}), \wedge, \{1_\mathcal{B}\}, \preceq)$ a semilattice with unit.

**2.2 Observation.** *For a complete Boolean algebra $\mathcal{B}$, the order $(Part(\mathcal{B}), \preceq)$ is a lattice. This lattice is complete iff $\mathcal{B}$ is atomic.*

*Proof.* We show that, in fact, every system $\{P_\alpha; \alpha \in \kappa\} \subseteq Part(\mathcal{B})$ has a supremum. Fix any $P$ from the system. For $p \in P$, put $p_0 = p$ and inductively define

$$q_n = \bigvee \left\{q \in \bigcup P_\alpha; q \parallel p_n\right\}$$

$$p_{n+1} = \bigvee \{p \in P; p \parallel q_n\}.$$

It is clear that $p \leq p_n \leq q_n \leq p_{n+1} \leq q_{n+1}$ for each $n \in \omega$. Put $u(p) = \bigvee \{p_n; n \in \omega\} = \bigvee \{q_n; n \in \omega\}$. It is easily checked that the set $\bigvee P_\alpha = \{u_p; p \in P\}$

does not depend on the choice of the starting partition $P$. Clearly, $\bigvee P_\alpha$ is a partition refined by every $P_\alpha$; we show that it is the finest among such partitions.

Let $P_\alpha \preceq R$ for every $\alpha \in \kappa$. It suffices to see that whenever $p \leq r$ for some $p \in P_\alpha$ and $r \in R$, we also have $u(p) \leq r$; this can be shown by induction for every $p_n, q_n$ as defined above. Let $p \in P$ and let $r$ be the only member of $R$ such that $p \leq r$. Every $q \in \bigcup P_\alpha$ is below exactly one $r' \in R$, and if $r \neq r'$, then $q \perp p$; hence $q_0 = \bigvee \{q \in \bigcup P_\alpha; q \parallel p\} \leq r$. Similarly, $p_1 = \bigvee \{p \in P; p \parallel q_0\} \leq r$, and it follows by induction that every $p_n \leq q_n \leq r$. Hence $u(p) \leq r$ and $\bigvee P_\alpha \preceq R$.

For completeness, let $\{P_\alpha; \alpha \in \kappa\}$ be a system of partitions. A supremum $\bigvee P_\alpha$ exists in $Part(\mathcal{B})$ by the above. A complete atomic algebra is a powerset algebra, which is completely distributive. The partition $P = \left\{\bigwedge_{\alpha\in\kappa} f(\alpha); f \in \Pi P_\alpha\right\} \setminus \{0\}$ is easily seen to be the infimum of the $P_\alpha$. In particular, the set of all atoms is the finest partition of $\mathcal{B}$, i.e., the smallest element of $Part(\mathcal{B})$. In the other direction, if $(Part(\mathcal{B}), \preceq)$ is complete, it must have a smallest element, which clearly needs to be a partition consisting exclusively of atoms of $\mathcal{B}$. □

Note that for an atomless algebra $\mathcal{B}$, completeness is actually necessary in the previous observation: we will show that in an atomless algebra that is not $\sigma$-complete, two partitions can always be found that do not have a supremum.

**2.3 Example.** Let $A = \{a_n; n \in \omega\} \subseteq \mathcal{B}$ be a countable subset without a supremum in $\mathcal{B}$; without loss of generality, $A$ is an antichain. Let $\mathcal{C}$ be the completion of $\mathcal{B}$, and consider $c = \bigvee^{\mathcal{C}} A \in \mathcal{C} \setminus \mathcal{B}$. The element $-c \in \mathcal{C}$ can be partitioned into some $\{x_\alpha; \alpha \in \kappa\} = X \subseteq \mathcal{B}$, as $\mathcal{B}$ is dense in $\mathcal{C}$.

Split every $a_n \in A$ into $a_n^0 \vee a_n^1$, put $b_0 = a_0^0$, $b_{n+1} = a_n^1 \vee a_{n+1}^0$ and $B = \{b_n; n \in \omega\}$. Then clearly $\bigvee^{\mathcal{C}} B = \bigvee^{\mathcal{C}} A = c$. Put $P = A \cup X, Q = B \cup X$. Now $P, Q$ are partitions of $\mathcal{B}$, and we show that $\{P, Q\}$ has no supremum in $Part(\mathcal{B})$.

Let $R \in Part(\mathcal{B})$ satisfy $P, Q \preceq R$. Then there must be some $r \in R$ such that $r \geq a_n, b_n$ for all $n$; but $r \in \mathcal{B}$ cannot be a supremum of $a_n$, hence $r$ meets some $x \in X$. In fact, we have $x \leq r$, as $X \subseteq P \cap Q$ and $P, Q \preceq R$. Then the partition $R_0$ which contains $r - x, x \in R_0$ instead of $r \in R$ satisfies $P, Q \preceq R_0 \prec R$. Hence $R$ is not a supremum.

**2.4 Definition.** Partitions $P, Q \in Part(\mathcal{B})$ are *independent* if $p \wedge q \neq 0$ for every $p \in P, q \in Q$. More generally, $\{P_i; i \in I\}$ is an *independent system of partitions* if for every finite $K \subseteq I$ and every $f \in \Pi \{P_i; i \in K\}$, the intersection $\bigwedge \{f(i); i \in K\}$ is nonzero.

Note that if $P, Q$ are independent, then $P \vee Q = \{1_B\}$ in $Part(\mathcal{B})$.

**2.5 Definition.** A *partition filter* is a filter in $(Part(\mathcal{B}), \preceq)$; i.e., a subset $F \subseteq Part(\mathcal{B})$ that contains with every $P \in F$ all $Q \in Part(\mathcal{B})$ such that $P \preceq Q$, and contains $P \wedge Q$ with every $P, Q \in F$. Clearly, $Part_{fin}(\mathcal{B})$ is a partition filter.

# 3 The structure induced by partitions

Let $\mathcal{B}$ be a complete ccc Boolean algebra. For $P \in Part(\mathcal{B})$, let $\mathcal{B}_P$ be the subalgebra completely generated by $P \subseteq \mathcal{B}$. Denote the inclusion as $e_P : \mathcal{B}_P \subseteq \mathcal{B}$. If $P \preceq Q$, let $e_P^Q$ be the inclusion of $\mathcal{B}_Q$ in $\mathcal{B}_P$. The family $\{\mathcal{B}_P; P \in Part(\mathcal{B})\}$ together with the mappings $e_Q^P$ forms a directed system of complete Boolean algebras indexed by the directed set $(Part(\mathcal{B}), \succeq)$.

The restriction to complete ccc algebras is not strictly necessary in the above definition; a more general situation could be described, minding the possible size of partitions and suitable $\kappa$-completeness of the algebra. It is however the complete ccc case which we are mostly interested in.

**3.1 Fact.** *Let $\mathcal{B}$ be a complete ccc Boolean algebra.*

(a) *For each $P \in Part_\infty(\mathcal{B})$, the algebra $\mathcal{B}_P$ is isomorphic to $P(\omega)$.*

(b) *$\mathcal{B}_{P\wedge Q}$ is completely generated by $\mathcal{B}_P \cup \mathcal{B}_Q$, and $\mathcal{B}_{P\vee Q} = \mathcal{B}_P \cap \mathcal{B}_Q$.*

(c) *$\mathcal{B}_P \cap \mathcal{B}_Q = \{0_\mathcal{B}, 1_\mathcal{B}\}$ iff $P \vee Q = \{1_\mathcal{B}\}$.*

(d) *For $P \preceq Q \in Part(\mathcal{B})$, the inclusion $e_P^Q : \mathcal{B}_Q \subseteq \mathcal{B}_P$ is a regular embedding.*

(e) *For each $P \in Part(\mathcal{B})$, the inclusion $e_P : \mathcal{B}_P \subseteq \mathcal{B}$ is a regular embedding.*

**3.2 Lemma.** *The algebra $\mathcal{B}$, together with the regular embeddings $e_P : \mathcal{B}_P \to \mathcal{B}$, is a direct limit of the directed system of algebras $\mathcal{B}_P$ and mappings $e_P^Q$. In fact, $\mathcal{B}$ is a limit of every subsystem consisting of $\mathcal{B}_P$ and $e_P^Q$ for $P, Q \notin F$, where $F \subseteq Part(\mathcal{B})$ is a partition filter.*

*Proof.* Every triangle commutes, i.e. $e_P \circ e_P^Q = e_Q$ whenever $P \preceq Q$. The algebra $\mathcal{B}$ is easily seen to be isomorphic to the direct limit as described in Chapter I. Put $\varphi(x) = [x]_\approx$ for $x \in \mathcal{B}$. Then $\varphi : \mathcal{B} \to (\bigsqcup \mathcal{B}_P / \approx)$ is well defined; in fact, the equivalence relation $x \approx y$ iff $x \in \mathcal{B}_P, y \in \mathcal{B}_Q, e^P_{P\wedge Q}(x) = e^Q_{P\wedge Q}(y)$ reduces to $x = y$ in $\mathcal{B}$, and merely factorizes out the formal distinction between multiple copies of $x \in \mathcal{B}$ coming from different components $\mathcal{B}_P$ of the disjoint union; hence $\varphi$ is one-to-one, too. Clearly, $\varphi$ is onto, and it is easily checked to be homomorphic. The second part follows from the fact that for every partition filter $F$, the system formed by $\mathcal{B}_P$ and $e_P^Q$ for $P, Q \notin F$ is a cofinal directed subsystem. $\square$

For $P \in Part(\mathcal{B})$, let $\mathcal{J}_P$ be the ideal on $\mathcal{B}$ generated by $P \subseteq \mathcal{B}$. Note that $\mathcal{J}_{P\wedge Q} = \mathcal{J}_P \cap \mathcal{J}_Q$ and $\mathcal{J}_P \subseteq \mathcal{J}_Q$ for $P \preceq Q$. Write $\mathcal{B}/P$ for $\mathcal{B}/\mathcal{J}_P$ and $\mathcal{B}_P/P$ for $\mathcal{B}_P/\mathcal{J}_P$. Whenever $P \preceq Q \in Part(\mathcal{B})$, we have $\mathcal{J}_P \subseteq \mathcal{J}_Q$, hence the algebra $\mathcal{B}/Q$ is a quotient of $\mathcal{B}/P$; denote the quotient mapping by $f_P^Q : \mathcal{B}/P \to \mathcal{B}/Q$. The family of algebras $\mathcal{B}/P$ and mappings $f_P^Q$ for $P, Q \in Part(\mathcal{B})$ forms an inverse system indexed by $(Part(\mathcal{B}), \succeq)$.

**3.3 Observation.** *Let $\mathcal{B}$ be a complete ccc Boolean algebra. Then*

(a) *For each $P \in Part_\infty(\mathcal{B})$, the quotient $\mathcal{B}_P/P$ is isomorphic to $P(\omega)/fin$.*

(b) *The inclusion $\mathcal{B}_P/P \subseteq \mathcal{B}/P$ is a regular embedding.*

**3.4 Lemma.** *The algebra $\mathcal{B}$, together with the quotient mappings $f_P : \mathcal{B} \to \mathcal{B}/P$, is an inverse limit of the inverse system $\{\mathcal{B}/P, f_P^Q\}$.*

Employing the Stone duality, we see that

**3.5 Corollary.** (a) *Every infinite complete ccc algebra is a limit of a directed system of copies of $P(\omega)$. Dually, every infinite ccc EDC space is an inverse limit of a directed system of copies of $\beta\omega$.*

(b) *Every infinite complete ccc Boolean algebra is an inverse limit of an inverse system of copies of $P(\omega)/fin$. Dually, every infinite ccc EDC space is a direct limit of directed system of copies of $\omega^*$.*

# 4 Ultrafilters and the partition structure

In this section, we fix an ultrafilter $\mathcal{U}$ on a complete ccc algebra $\mathcal{B}$ and look at how $\mathcal{U}$ reflects in the partition structure described above.

Let $\mathcal{B}$ be a complete, ccc algebra, $\mathcal{U}$ an ultrafilter on $\mathcal{B}$, and $P$ a partition of $\mathcal{B}$. Put $\mathcal{U}_P = \mathcal{U} \cap \mathcal{B}_P$, which is clearly an ultrafilter on $\mathcal{B}_P$; as $\mathcal{B}_P$ is isomorphic to $P(\omega)$, the ultrafilter $\mathcal{U}_P$ can be viewed as an ultrafilter on $\omega$.

**4.1 Observation.** *Let $\mathcal{B}$ be a complete atomless ccc algebra, let $P, Q$ be partitions of $\mathcal{B}$, and let $\mathcal{U}$ be an ultrafilter on $\mathcal{B}$. Then*

(a) $P \cap \mathcal{U} \neq \emptyset$ *if and only if* $\mathcal{U}_P$ *is trivial.*

(b) $\{P \in Part(\mathcal{B}); \mathcal{U} \cap P \neq \emptyset\}$ *is a proper partition filter if* $\mathcal{U}$ *is nontrivial.*

(c) $\{P \in Part(\mathcal{B}); \mathcal{U} \cap P = \emptyset\}$ *is an open dense subset of* $(Part(\mathcal{B}), \preceq)$.

(d) $\mathcal{U}_Q = \mathcal{U}_P \cap \mathcal{B}_Q$ *for* $P \preceq Q$.

(e) $\mathcal{B} = \bigcup \{\mathcal{B}_P; P \cap \mathcal{U} = \emptyset\}$

# 5 Coherent families

**5.1 Definition.** Let $\mathcal{B}$ be a complete, atomless, ccc algebra. For a property $\varphi$ of families of subsets of $\omega$, we say that a subset $X \subseteq \mathcal{B}$ is a *coherent $\varphi$-family* on $\mathcal{B}$ if for every partition $P = \{p_n; n \in \omega\}$ of $\mathcal{B}$, the family $\{A \subseteq \omega; \bigvee \{p_n; n \in A\} \in X\}$ of subsets of $\omega$ satisfies $\varphi$.

For some properties $\varphi$, the *coherent* $\varphi$ is actually no stronger than $\varphi$ itself. As an easy example, any antichain in $\mathcal{B}$ is a coherent antichain; and any filter $\mathcal{F}$ on $\mathcal{B}$ is a coherent filter, as for every partition $P$ of $\mathcal{B}$, the family $\{A \subseteq \omega; \bigvee \{p_n; n \in A\} \in \mathcal{F}\}$ is a filter on $\omega$. Similarly, every ultrafilter on $\mathcal{B}$ is a coherent ultrafilter, and an ultrafilter that is coherently trivial is a generic ultrafilter on $\mathcal{B}$. We will be interested in ultrafilters with special properties, where the coherent version becomes nontrivial.

It can be seen from the very definition that the ZFC implications between various classes of ultrafilters on $\omega$ continue to hold for the corresponding classes of coherent ultrafilters on $\mathcal{B}$. For instance, every coherent selective ultrafilter on $\mathcal{B}$ is a coherent $P$-ultrafilter on $\mathcal{B}$, as every selective ultrafilter on $\omega$ is a $P$-ultrafilter on $\omega$.

# 6 Coherent P-ultrafilters

**6.1 Definition.** An ultrafilter $\mathcal{U}$ on a complete ccc algebra $\mathcal{B}$ is a *coherent $P$-ultrafilter* if for every partition $P$ of $\mathcal{B}$, the family $\{A \subseteq \omega; \bigvee \{p_n; n \in A\} \in \mathcal{U}\}$ is a $P$-ultrafilter on $\omega$

Seeing that the subalgebra $\mathcal{B}_P$ is a copy of $P(\omega)$, we can equivalently characterize coherent $P$-ultrafilters as follows.

**6.2 Observation.** *Let $\mathcal{B}$ be a complete ccc algebra. An ultrafilter $\mathcal{U}$ on $\mathcal{B}$ is a coherent $P$-ultrafilter iff for every pair of partitions $P$ and $Q$ of $\mathcal{B}$ such that $P \preceq Q$, either $\mathcal{U} \cap Q \neq \emptyset$, or there is a set $X \subseteq P$ such that $\bigvee X \in \mathcal{U}$ and for every $q \in Q$, the set $\{p \in X; p \wedge q \neq 0\}$ is finite.*

It should probably be noted explicitly that as the notion of a coherent $P$-ultrafilter on $\mathcal{B}$ only depends on countable partitions $P$ of $\mathcal{B}$, and the $P$-point condition is only evaluated in the corresponding subalgebras $\mathcal{B}_P$, a coherent $P$-ultrafilter on $\mathcal{B}$ is in no way a $P$-point in the Stone space of $\mathcal{B}$ — unless $\mathcal{B}$ happens to be $P(\omega)$ itself.

We show now that coherent $P$-points consistently exist. The proof is an iteration of the Ketonen argument ([Ke]) for the existence of $P$-points on $\omega$.

**6.3 Proposition.** *Let $\mathcal{B}$ be a complete ccc Boolean algebra of size at most $\mathfrak{c}$. Every filter on $\mathcal{B}$ with a base smaller than $\mathfrak{c}$ can be extended to a coherent $P$-ultrafilter on $\mathcal{B}$ if and only if $\mathfrak{c} = \mathfrak{d}$.*

*Proof.* Assume $\mathfrak{c} = \mathfrak{d}$ and let $\mathcal{F} \subseteq \mathcal{B}$ be a filter with a base smaller than $\mathfrak{c}$. We will construct an increasing chain of filters $\mathcal{F}_\alpha$ extending $\mathcal{F}$, eventually arriving at a filter $\bigcup \mathcal{F}_\alpha$, where each $\mathcal{F}_\alpha$ takes care of a pair of partitions, as requested by 6.2.

Start with $\mathcal{F}_0 = \mathcal{F}$ and enumerate all partition pairs $P \preceq Q$ as $(P_\alpha, Q_\alpha)$, where $\alpha < \mathfrak{d}$ runs through all isolated ordinals. If an increasing chain $(\mathcal{F}_\beta; \beta < \alpha)$ of filters has already been found such that every $\mathcal{F}_\beta$ has a base smaller than $\mathfrak{c}$ and has the $P$-ultrafilter property 6.2 with respect to the partition pairs $P_\gamma \preceq Q_\gamma$ for $\gamma < \beta$, proceed as follows.

If $\alpha$ is a limit, take for $\mathcal{F}_\alpha$ the filter generated by $\bigcup \{\mathcal{F}_\beta; \beta < \alpha\}$; then $\mathcal{F}_\alpha$ still has a base smaller than $\mathfrak{c} = \mathfrak{d}$. We didn't miss a partition pair here.

If $\alpha = \beta + 1$ is a successor, consider the partition pair $P_\beta \preceq Q_\beta$. If some $q \in Q_\beta$ is compatible with $\mathcal{F}_\beta$, let $\mathcal{F}_\alpha = \mathcal{F}_{\beta+1}$ be the filter generated by $\mathcal{F}_\beta \cup \{q\}$ and be done with $(P_\beta, Q_\beta)$. If there is no such $q$ in $Q_\beta$, enumerate $Q_\beta$ as $\{q_n; n \in \omega\}$ and consider the refinement $P_\beta$ of $Q_\beta$. Without loss of generality, every $q_n \in Q_\beta$ is partitioned into infinitely many $p \in P_\beta$; enumerate $\{p \in P; p < q_n\}$ as $\{p_n^m; m \in \omega\}$. Let $\{a_\xi; \xi < \kappa\}$ be the base of $\mathcal{F}_\beta$, for some $\kappa < \mathfrak{c}$.

Now perform the Ketonen construction for this step: for each $\xi < \kappa$, put $f_\xi(n) = \min\{m; a_\xi \wedge p_n^m \neq 0\}$ if there is such an $m$. The value of $f_\xi(n)$ is defined for infinitely many $n$, corresponding to those $q_n$ which $a_\xi$ meets. In the missing places, fill the value of $f_\xi(n)$ with the *next* defined value (there must be some). This yields a family $\{f_\xi : \omega \to \omega; \xi < \kappa\}$ of functions — which cannot be dominating, as $\kappa < \mathfrak{c} = \mathfrak{d}$. Therefore, there is a function $f : \omega \to \omega$ which is not dominated by any $f_\xi$; that is, for each $\xi$, we have $f(n) > f_\xi(n)$ for infinitely many $n$. We can assume that $f$ is strictly increasing.

Put $a = \bigvee\{p_n^m; n \in \omega, m \leq f(n)\}$. The element $a$ is compatible with $\mathcal{F}_\beta$, because it meets every $a_\xi$, as witnessed by $f \not\leq f_\xi$. Let $\mathcal{F}_\alpha$ be the filter generated by $\mathcal{F}_\beta \cup \{a\}$. This filter obviously extends $\mathcal{F}_\beta$, is generated by fewer than $\mathfrak{c}$ elements, and has the $P$-ultrafilter property with respect to $(P_\beta, Q_\beta)$.

Now every ultrafilter extending $\bigcup\{\mathcal{F}_\alpha; \alpha < \mathfrak{c}\}$ is a coherent $P$-ultrafilter on $\mathcal{B}$ that extends $\mathcal{F}$, because we have taken care of all possible partition pairs $P \preceq Q$, as requested by 6.2.

The other direction follows from [Ke] immediately. Being able to extend every small filter $\mathcal{F} \subseteq \mathcal{B}$ into a coherent $P$-ultrafilter is apparently stronger than being able to extend every small filter $\mathcal{F}$ on $\omega$ to a $P$-point, which itself implies $\mathfrak{c} = \mathfrak{d}$. □

For completeness, we translate the Ketonen argument for the opposite direction into the algebra $\mathcal{B}$, showing how $\mathfrak{d} < \mathfrak{c}$ can break the coherence *anywhere*.

Assume $\mathfrak{d} < \mathfrak{c}$ and let $\{f_\alpha; \alpha < \mathfrak{d}\}$ be a dominating family of functions. Choose any two countable partitions $P \preceq Q$ of $\mathcal{B}$ such that every $q_n \in Q$ is partitioned into countably many $p_n^m \in P$. For each $\alpha < \mathfrak{d}$, put $a_\alpha = \bigcup\{p_n^m; m > f_\alpha(n)\}$. The family $\{a_\alpha; \alpha < \mathfrak{d}\} \cup \{-q_n; n \in \omega\} \subseteq \mathcal{B}$ is centered, and the filter $\mathcal{F}$ that it generates has $\mathfrak{d} < \mathfrak{c}$ generators. No ultrafilter on $\mathcal{B}$ that extends $\mathcal{F}$ can be a coherent $P$-ultrafilter, as witnessed by $P \preceq Q$.

We have shown that coherent $P$-ultrafilters consistently exist on complete ccc algebras of size $\leq \mathfrak{c}$. On the other hand, there consistently is no coherent $P$-ultrafilter on any complete ccc algebra, as even the classical $P$-points need not exist. Hence the existence of coherent $P$-ultrafilters is undecidable in ZFC.

**6.4 Question.** The consistency we have shown is what [Ca] calls "generic existence" — under our assumption, coherent $P$-ultrafilters not only exist, but every small filter can be enlarged into one. Questions arise:

(a) If $\mathcal{B}$ is a complete ccc algebra of size $> \mathfrak{c}$, is it consistent that there is a coherent $P$-ultrafilter on $\mathcal{B}$?

(b) Is it consistent that $P$-ultrafilters exist in $\omega$, but there are no coherent $P$-ultrafilters on any complete atomless ccc algebra?

(c) Is it consistent that a coherent $P$-ultrafilters exists on a complete atomles ccc algebra $\mathcal{B}$, but does not exist on another?

(d) Is there a single "testing" algebra $\mathcal{B}$ with the property that if there is a coherent $P$-ultrafilter on $\mathcal{B}$, then necessarily $\mathfrak{c} = \mathfrak{d}$, and hence $P$-ultrafilters exist generically?

**An application to nonhomogeneity**

Now we show the relevance of coherent $P$-ultrafilters to the Simon Conjecture: they provide a consistent positive answer.

**6.5 Proposition.** *Let $\mathcal{B}$ be a complete ccc algebra. Let $\mathcal{U}$ be a coherent $P$-ultrafilter on $\mathcal{B}$. Then $\mathcal{U}$ is an untouchable point in* $\mathrm{St}(\mathcal{B})$.

*Proof.* We assume that $\mathcal{U}$ is not an atom, otherwise there is nothing to prove. Let $R = \{\mathcal{F}_n; n \in \omega\}$ be a countable nowhere dense set in $\mathrm{St}(\mathcal{B})$ such that $\mathcal{F}_n \neq \mathcal{U}$ for all $n$. Choose some $a_0 \in \mathcal{F}_0$ with $-a_0 \in \mathcal{U}$ and put $R_0 = \{\mathcal{F} \in R; a_0 \in \mathcal{F}\} \subseteq R$. Generally, if $a_i \in \mathcal{B}^+$ for $i < k$ are disjoint elements such that $\bigvee_{i<k} a_i \notin \mathcal{U}$ and $R_i = \{\mathcal{F} \in R; a_i \in \mathcal{F}\}$, consider $\bigcup_{i<k} R_i \subseteq R$. If $\bigcup_{i<k} R_i = R$, we are done, as $-\bigvee_{i<k} a_i \in \mathcal{U}$ guarantees $\mathcal{U} \notin cl(R)$. Otherwise, let $n_k$ be the first index such that $\mathcal{F}_{n_k} \notin \bigcup_{i<k} R_i$ and choose some $a_k$ disjoint with $\bigvee_{i<k} a_i$ such that $a_k \in \mathcal{F}_{n_k}$ and $a_k \notin \mathcal{U}$.

This construction either stops at some $k$ and we are done, or we arrive at an infinite disjoint system $Q = \{a_i; i \in \omega\} \subseteq \mathcal{B}^+$. Again, if $\bigvee Q \notin \mathcal{U}$, we have $\mathcal{U} \notin cl(R)$. Otherwise, we can assume that $\bigvee Q = 1$, so $Q$ is a partition of $\mathcal{B}$. For each $a_i \in Q$, choose an infinite partition $P_i$ of $a_i$ such that $P_i \cap \bigcup R_i = \emptyset$ – this is possible, because $R_i \subseteq R$ is nowhere dense. Now $P = \bigcup P_i \preceq Q$ is a partition pair in $\mathcal{B}$.

As $\mathcal{U}$ is a coherent $P$-ultrafilter and misses $Q$, there is some $X \subseteq P$ with $u = \bigvee X \in \mathcal{U}$ such that for every $i$, the set $\{p \in X; p \leq a_i\}$ is finite. This means that $u \notin \mathcal{F}_n$ for all $n$: every $\mathcal{F}_n$ is in one particular $a_i$, so $u \in \mathcal{F}_n$ would mean that $\mathcal{F}_n$ contains one of the finitely many $\{p \leq u; p \leq a_i\}$. But this is in contradiction with $P_i \cap \bigcup R_i = \emptyset$. So $u \in \mathcal{U}$ isolates $\mathcal{U}$ from $cl(R)$. □

In fact, we have proven something slightly stronger: $\mathcal{U}$ escapes the closure of any nowhere dense set that can be covered by countably many disjoint open sets.

# 7 Coherent selective ultrafilters

Similarly to coherent $P$-ultrafilters, we start with the following characterization of coherent selective ultrafilters via partitions.

**7.1 Observation.** *Let $\mathcal{B}$ be a complete ccc algebra. An ultrafilter $\mathcal{U}$ on $\mathcal{B}$ is a coherent selective ultrafilter iff for every pair of partitions $P$ and $Q$ of $\mathcal{B}$ such that $P \preceq Q$, either $\mathcal{U} \cap Q \neq \emptyset$, or there is a set $X \subseteq P$ such that $\bigvee X \in \mathcal{U}$ and for every $q \in Q$, the set $\{p \in X; p \wedge q \neq 0\}$ is at most a singleton.*

The following proposition generalizes the arguments from [Ke] and [Ca] on existence of selective ultrafilters on $\omega$ to coherent selective ultrafilters on complete ccc algebras.

**7.2 Proposition.** *Let $\mathcal{B}$ be a complete ccc Boolean algebra of size at most $\mathfrak{c}$. Then every filter $\mathcal{F}$ on $\mathcal{B}$ with a base smaller than $\mathfrak{c}$ can be extended to a coherent selective ultrafilter on $\mathcal{B}$ if and only if $\mathfrak{c} = \mathrm{cov}(\mathcal{M})$.*

*Proof.* Assume $\mathfrak{c} = \mathrm{cov}(\mathcal{M})$ and let $\mathcal{F}$ be a filter with a base smaller than $\mathfrak{c}$. We will construct an increasing chain of filters extending $\mathcal{F}$. Put $\mathcal{F}_0 = \mathcal{F}$ and enumerate all partition pairs $P \preceq Q$ as $\{(P_\alpha, Q_\alpha); \alpha < \mathrm{cov}(\mathcal{M}) \text{ isolated }\}$.

If an increasing chain $(\mathcal{F}_\beta; \beta < \alpha)$ of filters has been found such that every $\mathcal{F}_\beta$ has a base smaller than $\mathfrak{c}$ and has the selective property with respect to all $\{(P_\gamma, Q_\gamma); \gamma < \beta\}$, proceed as follows.

If $\alpha$ is a limit, take for $\mathcal{F}_\alpha$ the filter generated by $\bigcup\{\mathcal{F}_\beta; \beta < \alpha\}$; then $\mathcal{F}_\alpha$ still has a base smaller than $\mathfrak{c}$.

If $\alpha = \beta+1$ is a successor, consider $(P, Q) = (P_\beta, Q_\beta)$. Without loss of generality, both partitions are infinite, and every $q_n \in Q$ is infinitely partitioned into $p_n^m \in P$.

If there is some $q \in Q$ compatible with $\mathcal{F}_\beta$, let $\mathcal{F}_\alpha$ be the filter generated by $\mathcal{F}_\beta \cup \{q\}$. If there is no such $q \in Q$, consider some base $\{a_\xi; \xi < \kappa\}$ of $\mathcal{F}_\beta$, where $\kappa < \mathfrak{c}$. Every $a_\xi$ intersects infinitely many $q \in Q$: if $a_\xi$ only meets $q_1, \dots, q_n \in Q$, choose $a_\xi^i$ disjoint with $q_i$, respectively; then $a_\xi \leq \bigvee q_i$ is disjoint with $\bigwedge a_\xi^i$ — a contradiction.

Consider the set $T = \Pi_{n\in\omega}\{p_n^m; m \in \omega\}$; the functions $\varphi \in T$ are the selectors for $Q$. View $T$ as a copy of the Baire space $\omega^\omega$. If no selector for $Q$ is compatible with $\mathcal{F}_\beta$, put $T_\xi = \{\varphi \in T; \bigvee rng(\varphi) \perp a_\xi\}$; then we have $T = \bigcup_{\xi<\kappa} T_\xi$. But the sets $T_\xi$ cannot cover $T$, as $\kappa < \mathrm{cov}(\mathcal{M})$ and every $T_\xi$ is a nowhere dense subset of $T$, which is seen as follows.

For a basic clopen subset $[s]$ of $T$, there is some $n > |s|$ such that $a_\xi$ meets $q_n \in Q$, because $a_\xi$ meets infinitely many $q_n$. Hence some $p_n^m$ meets $a_\xi$. Extend $s$ into $t$ so that $t(n) = m$. Then $[t] \subseteq [s]$ is disjoint with $T_\xi$.

Thus there must be a selector $\varphi \in T$ with $b = \bigvee rng(\varphi)$ compatible with every $a_\xi$. Let $\mathcal{F}_{\beta+1}$ be the filter generated by $\mathcal{F}_\beta \cup \{b\}$. Iterating this process, we obtain a monotone sequence of filters $(\mathcal{F}_\alpha; \alpha \in \mathfrak{c})$ extending $\mathcal{F} = \mathcal{F}_0$. Now every ultrafilter extending $\bigcup \mathcal{F}_\alpha$ is a coherent selective ultrafilter on $\mathcal{B}$ by 7.1. $\square$

We note in closing that this contribution to the nonhomogeneity question in not a ZFC solution, and consistent solution have been found before under additive set-theoretic assumptions.

**7.3 Question.** Does the Simon conjecture hold in ZFC?

# Chapter IV

# The order-sequential topology

## 1 Downward closed neighbourhoods

For a $\sigma$-complete Boolean algebra equipped with the order-sequential topology, the space $(\mathcal{B}, \tau_s)$ is sequential by definition. It is natural to ask whether it has some stronger sequential property, such as being Fréchet or metrizable.

Metrizability of the sequential space $(\mathcal{B}, \tau_s)$ is equivalent to $\mathcal{B}$ being Maharam. An algebra that is not Maharam, hence not metrizable, can still be Frchet, as is the case with the Suslin algebra. An algebra that adds a Cohen real cannot be metrizable, and cannot even be Fréchet, as the Cohen algebra is not weakly distributive.

We show here another property of Fréchet algebras that gets violated in every algebra $\mathcal{B}$ which regularly embedds the Cantor algebra.

**1.1 Proposition** ([BGJ]). *Let $\mathcal{B}$ be a $\sigma$-complete Boolean algebra.*

*(i) The space $(\mathcal{B}, \tau_s)$ is Fréchet if and only if $\mathcal{B}$ is weakly distributive and $\mathfrak{b}$-cc.*

*(ii) If $(\mathcal{B}, \tau_s)$ is a Fréchet space, then every neighbourhood $V$ of $0_{\mathcal{B}}$ contains an neighbourhood $U \subseteq V$ which is downward closed.*

**1.2 Proposition** ([BGJ]). *If $(\mathcal{B}, \tau_s)$ is a Fréchet space, then every neighbourhood $V$ of $0_{\mathcal{B}}$ contains an neighbourhood $U \subseteq V$ which is downward closed.*

**1.3 Lemma.** *In a $\sigma$-complete algebra $\mathcal{B}$, let $y_m \to 0$, $x_{m,n} \to y_m$ and $z_m^n \leq x_m^n$. If $0 \notin cl(\{z_m^n; m, n \in \omega\})$, then $U = \mathcal{B} \setminus cl(\{z_m^n; m, n \in \omega\})$ is an open neighbourhood of zero which does not contain any downward closed neighbourhood of zero.*

*Proof.* Let $0 \in V \subseteq U$ be a downward closed neighbourhood. Then some $y_m \in V$, hence some $x_m^n \in V$. But then also $z_m^n \leq x_m^n$ is in $V \subseteq U$, a contradiction. $\square$

**1.4 Proposition.** *If a complete Boolean algebra regularly embedds the Cantor algebra, and hence adds a Cohen real, then $0 \in (\mathcal{B}, \tau_s)$ cannot have a local base consisting of downward closed neighbourhoods.*

*Proof.* We will construct the situation described in the above lemma inside the Cohen algebra. Let $\mathcal{I} \subseteq \mathcal{A}$ be an independent system in the Cantor algebra.

Decompose $\mathcal{I}$ into infinite $\{a_k; k \in \omega\}$, $\{b_l^m; i, n \in \omega\}$, $\{c_n; n \in \omega\}$ and put

$$y_m = \bigwedge_{k \leq m} a_k$$

$$x_m^n = y_m \vee \bigwedge_{l \leq n} b_l^m$$

$$z_m^n = (y_m \wedge c_n) \vee \bigwedge_{l \leq n} b_l^m.$$

It is clear that $y_m \to 0$, $x_m^n \to y_m$ and $z_m^n \leq x_m^n$. Also, every $z_m^n > 0$ by independence.

We claim now that the set $Z = \{z_m^n; m, n \in \omega\}$ is closed in $(\mathcal{B}, \tau_s)$; in particular, $0 \notin \bar{Z}$. Suppose not. As the space $(\mathcal{B}, \tau_s)$ is sequential, there is a sequence $(z_{m_i}^{n_i})$ in $Z$ converging to some $b \notin Z$, so that $Z$ is not sequentially closed.

If the set of $\{m_i\}$ is infinite, then $\limsup z_{m_i}^{n_i} = 1$, while $\liminf z_{m_i}^{n_i} \leq a_0 < 1$ — a contradiction. If the set of $\{m_i\}$ is finite, then without loss of generality the sequence $(z_{m_i}^{n_i})$ is in fact of the form $(z_m^{n_i})$ for a fixed $m$. Then $\limsup z_m^{n_i} > 0$, while $\liminf z_m^{n_i} = 0$ — a contradiction. □

# 2 Characterizing Maharam with $\tau_s \times \tau_s$

In this section, we formulate one more equivalence of $\mathcal{B}$ being a Maharam algebra, using properties of the order-sequential topology.

The idea concerns the product topology $\tau_s \times \tau_s$. Given a $\sigma$-complete algebra $\mathcal{B}$, there are two natural topologies on the cartesian product algebra $\mathcal{B} \times \mathcal{B}$: the product topology of $(\mathcal{B}, \tau_s) \times (\mathcal{B}, \tau_s)$, and the order-sequential topology of $\mathcal{B} \times \mathcal{B}$, which is a $\sigma$-complete algebra itself. It is a natural question whether these two topologies coincide.

**2.1 Lemma.** *The order-sequential topology of the cartesian product algebra $\mathcal{B} \times \mathcal{B}$ is the sequential modification of the topological product $(\mathcal{B}, \tau_s) \times (B, \tau_s)$.*

**2.2 Proposition.** *Let $\mathcal{B}$ be a $\sigma$-complete weakly distributive ccc algebra. Then $\mathcal{B}$ is a Maharam algebra if and only if the topological product $(\mathcal{B}, \tau_s) \times (\mathcal{B}, \tau_s)$ is homeomorphic to the order-sequential topology of the cartesian product algebra $\mathcal{B} \times \mathcal{B}$.*

*Proof.* We need to show that the operation $x \triangle y$ is continuous, as a function from $\mathcal{B} \times \mathcal{B}$ to $\mathcal{B}$. Assuming that the product topology is the order-sequential topology, it suffices to verify that $\triangle$ preserves the limits of *algebraically* convergent sequences in the product algebra $\mathcal{B} \times \mathcal{B}$.

Conversely, if $\mathcal{B}$ is Maharam, the space $(\mathcal{B}, \tau_s)$ is metrizable. So $(\mathcal{B}, \tau_s) \times (\mathcal{B}, \tau_s)$ is a metrizable space as well, hence sequential, and the product topology coincides with its sequential modification. By the previous lemma, this is precisely the order-sequential topology on the $\sigma$-complete algebra $\mathcal{B} \times \mathcal{B}$. □

Note that if the two topologies coincide, the topology of the product becomes simultaneously extreme in two opposing ways: being the product topology, it is the coarsest that makes the projections continuous; being order-sequential, it is the finest that lets algebraic sequences converge.

If a ccc algebra $\mathcal{B}$ fails to be Maharam, we know that $(\mathcal{B}, \tau_s)$ is not Hausdorff. In that case, we can easily exhibit the subset of $\mathcal{B} \times \mathcal{B}$ that exploits the difference of the two topologies considered in the previous proposition.

Recall the elementary lemma which says that a topological space $(X, \tau)$ is Hausdorff if and only if the diagonal $\{(x, x); x \in X\}$ is closed in $X \times X$. So if $(\mathcal{B}, \tau_s)$ is not Hausdorff, the diagonal $\{(x, x); x \in \mathcal{B}\}$ is not closed in $(\mathcal{B}, \tau_s) \times (\mathcal{B}, \tau_s)$. However, it is easily seen to be sequentially closed, hence closed in $(\mathcal{B} \times \mathcal{B}, \tau_s)$.

# 3 Compactness of the order-sequential topology

The general problem we deal with in this section is to describe the complete Boolean algebras which are compact in their order-sequential topology. We remind the gentle reader that by *compact* we do not mean compact Hausdorff.

Obviously, we only consider infinite algebras. Some further restrictions need to be made at the outset.

**3.1 Observation.** *Let $\mathcal{B}$ be a complete Boolean algebra. If the order-sequential topology $\tau_s$ of $\mathcal{B}$ is compact, then $\mathcal{B}$ is ccc.*

*Proof.* If not, let $X$ be an antichain of size $\aleph_1$. By completeness, we can assume that $X$ is a maximal antichain (add $-\bigvee X$ to the antichain otherwise). This generates a copy of $P(\omega_1)$ as a complete subalgebra in $\mathcal{B}$; this copy is a sequentially closed, hence closed subspace of $(\mathcal{B}, \tau_s)$. So if $(\mathcal{B}, \tau_s)$ is compact, then $(2^{\omega_1}, \tau_s)$ is a compact Hausdorff space, with a topology strictly finer than $(2^{\omega_1}, \tau_c)$ — a contradiction. $\square$

**3.2 Proposition** ([BJP])**.** *For a complete Boolean algebra $\mathcal{B}$, the space $(\mathcal{B}, \tau_s)$ is countably compact if and only if $\mathcal{B}$, as a forcing, does not add an independent real.*

**3.3 Proposition** (Głowczyński)**.** *Let $\mathcal{B}$ be an infinite complete Boolean algebra such that $(\mathcal{B}, \tau_s)$ is a compact Hausdorff space. Then $\mathcal{B}$ is isomorphic to $P(\omega)$.*

The previous proposition was originally proved in [Gl]. It follows from the results of [BJP] as well. Being compact, the algebra must be ccc by the above observation. For $(\mathcal{B}, \tau_s)$ ccc and Hausdorff, the algebra $\mathcal{B}$ is Maharam, and an atomless Maharam algebra adds an independent real. Hence to not add an independent real, a Hausdorff algebra has to be atomic. Hence $\mathcal{B}$ is isomorphic to the only complete atomic ccc algebra, namely $P(\omega)$.

Thus in the search for compact order-sequential topology, we restrict ourselves to complete infinite ccc algebras which are not Hausdorff, hence not Maharam, and do not add independent reals, hence do not almost regularly embedd the Cantor algebra. It is a natural question now where to get a supply of such algebras. We don't know of any within ZFC. (The candidates described in [BJP] require either a measurable cardinal or constructibility features to work.)

Note that if a non-compact algebra $\mathcal{B}$ embeds regularly into a complete algebra $\mathcal{C}$, then $\mathcal{C}$ cannot be compact either, as the embedded copy of $\mathcal{B}$ is a sequentially closed subspace.

### Suslin: a compactness candidate

We restrict ourselves to a special case which is a natural candidate, namely the complete Boolean algebra $\mathcal{B} = \mathcal{B}(T)$ which is a completion of a Suslin tree $T$. Note that for an Aronszajn tree $T$ which is not Suslin, the algebra $\mathcal{B}(T)$ cannot be compact, as it is not ccc. Note also that our candidate is out of ZFC.

Let $(T, \leq)$ be a normal Suslin tree, and let $\mathcal{B} = \mathcal{B}(T)$ be the complete algebra determined by $(T, \leq)$. We recall that this is a complete, atomless, ccc and $\sigma$-closed, hence $\omega$-distributive Boolean algebra of density $\omega_1$. It is known that these properties characterize the Suslin algebra.

Let $T_\alpha$ for $\alpha < \omega_1$ be the countable levels of $T$. These are partitions of $\mathcal{B}$ and $T_\beta$ is a refinement of $T_\alpha$ for $\alpha < \beta < \omega_1$. Let $\mathcal{B}_\alpha$ be the subalgebra completely generated by $T_\alpha \subseteq \mathcal{B}$. It is clear that $\mathcal{B}_\alpha$ is a copy of $P(\omega)$ and $\mathcal{B}_\alpha$ is a regular subalgebra of $\mathcal{B}_\beta$ for $\alpha < \beta < \omega_1$. Hence we have a chain of regularly embedded copies of $P(\omega)$ – a special case of a directed system of complete algebras.

**3.4 Fact.** *The Suslin algebra $\mathcal{B}$ is a direct limit of the chain of $\mathcal{B}_\alpha$.*

*Proof.* As $\mathcal{B}$ is ccc, we have $\mathcal{B} = \bigcup \mathcal{B}_\alpha$: every $x \in \mathcal{B}$ is a join of some countably many $x_n \in T$, and every $x_n \in T_{\alpha_n}$ for some $\alpha_n < \omega_1$. Hence $x \in \mathcal{B}_\alpha$ for any $\alpha \in \omega_1$ such that $\alpha \geq \sup \{\alpha_n; n \in \omega\}$. Commutativity of the regular inclusions is clear. □

As $\mathcal{B}$ is atomless, the space $(\mathcal{B}, \tau_s)$ is connected. The algebra $\mathcal{B}$ is $\omega$-distributive, hence topological convergence in $(\mathcal{B}, \tau_s)$ coincides with algebraic convergence in $\mathcal{B}$. Distributivity and ccc implies that the space $(\mathcal{B}, \tau_s)$ is Fréchet.

It is well known that the Suslin algebra is not a Maharam algebra. In fact, this is the original example from [M2]. Hence $(\mathcal{B}, \tau_s)$ is not Hausdorff. Being $\omega$-distributive, it adds no new reals, in particular does not add independent reals, and is therefore sequentially compact; equivalently, it is countably compact, being a $T_1$ space.

**3.5 Fact.** *The space $(\mathcal{B}, \tau_s)$ is a direct limit of the spaces $(\mathcal{B}_\alpha, \tau_s)$.*

*Proof.* The subalgebras $\mathcal{B}_\alpha$, in their order-sequential topology, are copies of the compact Cantor space $(2^\omega, \tau_c)$. It is easy to see that this coincides with the subspace topology imposed by $(\mathcal{B}, \tau_s)$. Being sequentially closed, they are closed compact subspaces of $(\mathcal{B}, \tau_s)$.

To show that $\tau_s$ is the topology of a direct limit, consider a set $A \subseteq \mathcal{B}$ such that every $A \cap \mathcal{B}_\alpha$ is closed in $\mathcal{B}_\alpha$; we will show that $A$ is sequentially closed in $\mathcal{B}$, hence $\tau_s$-closed. So let $(x_n; n \in \omega)$ be a sequence in $A$, converging to $x \in \mathcal{B}$. Choose $\alpha < \omega_1$ large enough so that $\mathcal{B}_\alpha$ contains the countable set $\{x_n; n \in \omega\} \cup \{x\}$. Then

$(x_n; n \in \omega)$ is a sequence in $A \cap \mathcal{B}_\alpha \subseteq \mathcal{B}_\alpha$, which we assumed to be a closed subset of $\mathcal{B}_\alpha$. So $x \in A \cap \mathcal{B}_\alpha \subseteq A$, and $A$ is sequentially closed. $\square$

We assume the Suslin tree $T$ to be normal, in particular $T$ splits everywhere. Hence every $x \in T_\alpha$ is partitioned into infinitely many members of $T_{\alpha+1}$. As a consequence, every $\mathcal{B}_\alpha$ is a closed nowhere dense subset of $\mathcal{B}_{\alpha+1}$.

Next we show that the Suslin tree $T$ itself, as a subset of $(\mathcal{B}(T), \tau_s)$, can be assumed to be closed and compact.

**3.6 Lemma.** *Let $T$ be a Suslin tree. Then there is a Suslin tree $S$ in $\mathcal{B}(T)$ such that $\mathcal{B}(S) = \mathcal{B}(T)$ and $S \cup \{0\}$ is a closed compact subset of $(\mathcal{B}(T), \tau_s)$.*

*Proof.* $S$ can be constructed from $T$ inductively by refining the levels. Put $S_0 = T_0$. On isolated levels $\alpha = \beta + 1$, let $S_\alpha$ be the common refinement of $S_\beta$ and $T_\beta$. For $\alpha$ limit, consider all branches $f$ in $\bigcup_{\beta<\alpha} S_\beta$ of length $\alpha$ such that $\bigwedge f \neq 0$, and let $S_\alpha$ consist of all the $\bigwedge f$.

Then $S \cup \{0\}$ is sequentially closed: a sequence $(x_n)$ either contains infinitely many parwise disjoint nodes and converges to 0, or contains infinitely many members of a branch in $S$, which converges either to 0 or to a member of $S$.

For compactness, let $\mathcal{U}$ be an open cover of $S \cup \{0\}$. Some $U \in \mathcal{U}$ is a neighbourhood of zero, and we can assume that $U$ is a downward closed algebraically dense subset of $\mathcal{B}$, as $(\mathcal{B}, \tau_s)$ is Fréchet.

Hence there is a countable maximal antichain $X \subseteq S \cap U$; by possibly further refining $X$, we can assume that $X = S_\alpha$ for some $\alpha < \omega_1$. Then $\bigcup\{S_\xi; \xi > \alpha\} \subseteq U$ and $\bigcup\{S_\xi; \xi \leq \alpha\}$ is compact, being a closed subset of the compact space $\mathcal{B}_\alpha$. $\square$

### Towards a minimal KC topology

We recall a general topological theorem from [BC] mentioned in the introduction and show a path towards its application.

**3.7 Theorem** (Bella, Costantini)**.** *Minimal KC spaces are compact.*

**3.8 Proposition.** *$(\mathcal{B}, \tau_s)$ is a strongly KC space.*

*Proof.* Let $K$ be a countably compact subset of $(\mathcal{B}, \tau_s)$. We need to show that $K$ is sequentially closed. So let $(x_n; n \in \omega)$ be a sequence in $K$, converging to $x \in \mathcal{B}$. Choose $\alpha < \omega_1$ so large that $\mathcal{B}_\alpha$ contains the countable set $\{x_n; n \in \omega\} \cup \{x\}$. Then $(x_n; n \in \omega)$ is a sequence in $K \cap \mathcal{B}_\alpha \subseteq \mathcal{B}_\alpha$. It suffices to show now that $K \cap \mathcal{B}_\alpha$ is a closed subset in $\mathcal{B}_\alpha$. But $\mathcal{B}_\alpha$ is a compact metric space: a copy of $2^\omega$.

Hence if $K \cap \mathcal{B}_\alpha$ is not a closed subset, it fails to be countably compact. So $K \cap \mathcal{B}_\alpha$ has a countable bad covering $\mathcal{U}$ – but then $\mathcal{U} \cup (\mathcal{B} \setminus \mathcal{B}_\alpha)$ is a countable bad covering of $K$. Hence $K \cap \mathcal{B}_\alpha$ is in fact a closed subset of $\mathcal{B}_\alpha$ and $x \in K \cap \mathcal{B}_\alpha \subseteq K$; so $K$ is sequentially closed. $\square$

**3.9 Proposition.** *$(\mathcal{B}, \tau_s)$ is a minimal strongly KC space.*

*Proof.* Let $\tau$ be a strongly KC topology on $\mathcal{B}$ that is strictly coarser than $\tau_s$. Then the identity mapping from $(\mathcal{B}, \tau_s)$ to $(\mathcal{B}, \tau)$ is a continuous bijection. But a continuous bijection from a countably compact space to a strongly KC space must be a homeomorphism — a contradiction. □

So the order-sequential topology on $\mathcal{B}$ is minimal among all strongly KC topologies. That by itself does not imply it is also minimal among all the KC topologies, which leaves a step to be made to an application of the Bella-Costantini theorem. Hence we ask:

**3.10 Question.** Is there a KC topology on $\mathcal{B}(T)$ strictly weaker than $\tau_s$?

We close with a description of the peculiar properties of such a topology.

**3.11 Proposition.** *Let $\tau$ be a KC topology on $\mathcal{B}(T)$ strictly coarser than $\tau_s$, let $A$ be a subset that is $\tau_s$-closed but not $\tau$-closed.*

(i) *The subspaces $(\mathcal{B}_\alpha, \tau)$ are copies of $2^\omega$*

(ii) *The intersections $A \cap \mathcal{B}_\alpha$ are $\tau$-closed.*

(iii) $|A \setminus \mathcal{B}_\alpha| \geq \omega_1$ *for every $\alpha < \omega_1$*

(iv) $(\mathcal{B}, \tau_s)$ *is the sequential modification of $(\mathcal{B}, \tau)$. Hence $(\mathcal{B}, \tau)$ is not sequential.*

*Proof.* (i) Every $\mathcal{B}_\alpha$ is $\tau_s$-compact, hence $\tau$-compact, and must be $\tau$-closed by KC. Being a $\tau$-closed subspace of a KC space, every $\mathcal{B}_\alpha$ is a $\tau$-compact KC space itself. So $(\mathcal{B}_\alpha, \tau)$ cannot be a strictly weaker topology than $(\mathcal{B}_\alpha, \tau_s)$.

(ii) As the order-sequential topology $\tau_s$ is the topology of the direct limit $\mathcal{B} = \bigcup \mathcal{B}_\alpha$, every $A \cap \mathcal{B}_\alpha$ is $\tau_s$-closed in $\mathcal{B}_\alpha$, hence $\tau_s$-compact. Thus $A \cap \mathcal{B}_\alpha$ is also $\tau$-compact, and must be $\tau$-closed because $\tau$ is KC.

(iii) Clearly $A \not\subseteq \mathcal{B}_\alpha$ for every $\alpha < \omega_1$, because if $A \subseteq \mathcal{B}_\alpha$, then $A$, being $\tau_s$-closed, is a compact subset of $\mathcal{B}_\alpha$; so $A$ is $\tau$-compact as well, hence $\tau$-closed, as $\tau$ is KC — a contradiction. If some $A \setminus \mathcal{B}_\alpha$ was only countable, we would have $A \subseteq \mathcal{B}_\beta$ for some suitably larger $\beta < \omega_1$.

(iv) We show that the class of convergent sequences is the same. As $\mathcal{B}$ is $\sigma$-distributive, the $\tau_s$-convergent sequences are precisely the algebraically convergent sequences. Hence a $\tau$-convergent sequence $x_n \to x$ that is not $\tau_s$-convergent means that $x$ is not the algebraic limit of $(x_n)$. But this can never happen: the space $(\mathcal{B}, \tau_s)$ is sequentially compact, so $(x_n)$ has a $\tau_s$-convergent subsequence $(x_{n_k}) \to y$, where $y$ is the algebraic limit; hence $y \neq x$. But then also $(x_{n_k}) \to y$ in $\tau$, which violates the ULP of $(\mathcal{B}, \tau)$. So both $\tau_s$ and $\tau$ have the same class of convergent sequences, namely, the algebraically convergent sequences of $\mathcal{B}$. □

### Coloring a Suslin tree

For a Suslin tree $T$, the topological density of $(\mathcal{B}(T), \tau_s)$ is $\omega_1$. Hence to show compactness, it suffices to find an $\omega_1$-accumulation point for every subset of size $\omega_1$. We use in this section a coloring reformulation of the accumulation property, due to E. Thümmel, and show that the Suslin tree added with Jech's forcing as a subset of $2^{<\omega_1}$ satisfies Thümmel's condition with respect to the inherent coloring.

**3.12 Definition.** Let $T$ be a Suslin tree. For a coloring $\chi : T \to 2$ and $\alpha < \beta < \omega_1$, say that $\beta$ *returns to* $\alpha$ if there is an increasing sequence of ordinals $\alpha_n < \beta$ such that $\alpha_0 = \alpha$, $\sup \alpha_n = \beta$, and for every node $x \in T_\alpha$ there is a fixed color $k(x) \in 2$ with the property that for every $y \in T_\beta$ with $y > x$, the set $\{n \in \omega; \chi(y \restriction \alpha_n) \neq k(x)\}$ is finite. The coloring *accummulates* if for some $\alpha < \omega_1$, there are unboundedly many $\beta > \alpha$ that return to $\alpha$.

Note that the "right" color $k(x)$ is in no relation to the color $\chi(x)$ of the node $x$ itself, or the color $\chi(y)$ of $y \in T_\beta$.

The following fact is a reformulation of the accumulating property in terms of convergence in $(\mathcal{B}(T), \tau_s)$, which is precisely the algebraic convergence.

**3.13 Fact.** *In the above notation, $\beta$ returns to $\alpha$ if and only if the sequence of points $x_n = \bigvee \{p \in T_{\alpha_n}; \chi(p) = 1\} \in \mathcal{B}_{\alpha_n} \subseteq \mathcal{B}$ converges to $x = \bigvee \{p \in T_\alpha; k(p) = 1\} \in \mathcal{B}_\alpha$.*

With this reformulation, compactness can be described as follows.

**3.14 Proposition** (Thümmel). *For a Suslin tree $T$, the following are equivalent.*

1. *The space $(\mathcal{B}(T), \tau_s)$ is compact.*
2. *For every subtree $S \subseteq T$ of the form $S = \bigcup_{\alpha \in M} T_\alpha$, where $M \in [\omega_1]^{\omega_1}$, every coloring $\chi : S \to 2$ accummulates.*
3. *For every subtree $S \subseteq T$ of the form $S = \bigcup_{\alpha \in C} T_\alpha$, where $C \subseteq \omega_1$ is a club, every coloring $\chi : S \to 2$ accummulates.*

*Proof.* Let $(\mathcal{B}(T), \tau_s)$ be compact, let $S = \bigcup_{\alpha \in M} T_\alpha$ be a subtree. Note that $\mathcal{B}(S) = \mathcal{B}(T)$, as $S$ is algebraically dense in $T$. Let $\chi : S \to 2$ be a coloring. For every $\alpha < \omega_1$, put $x_\alpha = \bigvee \{p \in S_\alpha; \chi(p) = 1\}$. Without loss of generality, this is $\aleph_1$ many distinct points. As $(\mathcal{B}, \tau_s)$ is compact, the set $\{x_\alpha; \alpha < \omega_1\}$ has a complete accumulation point; that is some $x$ in some $\mathcal{B}_\alpha$. Hence for every $\xi < \omega_1$, the point $x$ is in the closure of $\{x_\alpha; \alpha > \xi\}$. Recall that the space $(\mathcal{B}(T), \tau_s)$ is Fréchet; so being in the closure means there is a sequence in $\{x_\alpha; \alpha > \xi\}$ which converges to $x$. Also, the topological convergence in $(\mathcal{B}(T), \tau_s)$ is precisely the algebraic convergence. So by the lemma, unboundedly many $\beta$ return to $\alpha$.

Conversely, let $F_\alpha \subseteq \mathcal{B}(T), \alpha < \omega_1$ be a descending chain of nonempty closed sets. For every $\alpha < \omega_1$, the intersection $\bigcap_{\xi < \alpha} F_\xi$ is nonempty, as $(\mathcal{B}(T), \tau_s)$ is countably compact; pick some $x_\alpha$ from this intersection. If unboundedly many of these $x_\alpha$ happen to be in some fixed $\mathcal{B}_\beta$, which is a copy of $2^\omega$, then there is a complete accumulation point $x$ in $\mathcal{B}_\beta$; then $x \in F_\alpha$ for unboundedly many (hence all) of the $F_\alpha$ and we are done.

So without loss of generality, assume that $x_\alpha \notin \bigcup \{\mathcal{B}_{h(\alpha_\xi)}; \xi < \alpha\}$. Every $x_\alpha$, being a member of $\mathcal{B}_{h(x_\alpha)}$, is a join of some members $p$ of $T_{h(x_\alpha)}$; put $\chi(p) = 1$ iff $p \leq x_\alpha$. This defines a coloring of the (Suslin) subtree $S = \bigcup \{T_{h(x_\alpha)}; \alpha < \omega_1\}$.

Renumerate the levels of $S$ to ease notation and pretend that $x_\alpha \in \mathcal{B}_\alpha$. By assumption, there is unboundedly many $\beta < \omega_1$ returning to some fixed $\alpha_0 < \beta$ (each via some sequence $\alpha_n \to \beta$, possibly different). Put $x = \bigvee \{p \in T_{\alpha_0}; k(p) = 1\}$. Then by the previous lemma, for unboundedly many $F_\xi$ we have a sequence in $F_\xi$ converging to $x$. Thus $x$ is in every $\bar{F_\alpha} = F_\alpha$, and $(\mathcal{B}(T), \tau_s)$ is compact. □

We show now that the generic Suslin tree added with Jech's forcing as a new subset of $2^{<\omega_1}$ satisfies the coloring condition with respect the inherited $\{0,1\}$-coloring.

**3.15 Lemma.** *Consider the Jech forcing $\mathbb{P}$ consisting of normal binary $\gamma$-trees, for $\gamma < \omega_1$, ordered by the relation of end extension. For every $\xi < \omega_1$, the set*

$$D_\xi = \{T \in \mathbb{P}; (\exists \alpha, \beta \in h(T)) \xi < \alpha < \beta \text{ and } \beta \text{ returns to } \alpha\}$$

*is a dense subset of $\mathbb{P}$.*

*Proof.* Given any Jech tree $S \in \mathbb{P}$, we will find an end extension $T$ of $S$ which belongs to $D_\xi$. So let $S \in \mathbb{P}$. We can assume without loss of generality that the height of $S$ is already larger than $\xi$. This is a property of the Jech forcing which guarantees that the generic tree is indeed of height $\omega_1$.

Put $\alpha = h(S)$ and extend $S$ with $\omega + 1$ many more levels $T_{\alpha_n}$, $n \in \omega$ as follows. Let $\alpha_0 = \alpha$ and make $T_{\alpha_0}$ any legal extension of $S$; now $T_{\alpha_0}$ is the last level so far of the tree we are building.

Put $\alpha_n = \alpha + n$ and make $\bigcup_n T_{\alpha_n}$ form a copy of the full binary tree of height $\omega$ attached to every $x \in T_\alpha$. Let $\beta = \alpha + \omega$ and put a successor $y$ to $T_\beta$ for every branch of $\bigcup_n T_{\alpha_n}$ which is eventually constant. There is countably many such branches, so $T_\beta$ is countable. Hence $T = S \cup \bigcup_{n \leq \omega} T_{\alpha_n}$ is a condition in $\mathbb{P}$ which extends $S$.

It is clear now that $T \in D_\xi$: we have $\xi < \alpha < \beta$ and $\beta$ returns to $\alpha$ via $\alpha_n$: the "right" color $k(x)$ for $x \in T_\alpha$ is the value of the eventual constant. □

**3.16 Corollary.** *The Suslin tree $T \subseteq 2^{<\omega_1}$ generically added with Jech forcing satisfies the compactness condition with respect to the inherent coloring.*

**3.17 Question.** Let $\mathcal{V}$ be a model without Suslin trees, and let $T$ be an Aronszajn tree in $\mathcal{V}$ formed by a coherent system of one-to-one functions $\{f_\alpha : \alpha \to \omega_1; \alpha < \omega_1\}$. Consider the Suslin tree $S = \{rf; \operatorname{dom} f = \alpha \text{ and } f =^* f_\alpha\}$ in a generic extension obtained by adding a Cohen real $r$, as described in the introduction. Color with 0 the nodes "glued" together with $r$, color the others with 1. Does T satisfy the compactness condition with respect to this coloring?

**3.18 Question.** The notion of a *Suslin algebra* is more general: it is defined to be a complete ccc distributive algebra, without necessarily being a Boolean completion of a Suslin tree – which is a property we have relied upon. It is consistent that there are algebras exploiting the difference. Can these be compact?

# Chapter V

# Measures and functionals

In this chapter, we present miscelaneous measure-theoretic constructions and generalize some of them to submeasures and possibly other functionals. We give examples meant to illustrate the similarity between measures and submeasures. We are using and extending some results from chapters III and IV from T. Pazák's dissertation *Exhaustive structures on Boolean algebras* ([Pa]).

### Measures versus submeasures

Clearly, every measure is a submeasure, and every $\sigma$-additive measure is a Maharam submeasure; hence every measure algebra is a Maharam algebra. D. Maharam conjectured in [M2] the existence of an algebra carrying a strictly positive continuous submeasure which does not carry a measure.

It was only proved in 2006 by M. Talagrand [T2] that such an algebra indeed exists. While solving the original problem, the solution raised more questions on the relation between measures and submeasures.

## 1 Submeasure and category

We start with the following generalization of a theorem on paradoxial decomposition, usually stated for measures (see [Ox]).

For a submeasure $\mu$ on $Borel(X)$, let the *support* of $\mu$, denoted by $supp(\mu)$, be the largest closed subset of $X$ such that every neighbourhood of every $x \in supp(\mu)$ has a positive submeasure.

**1.1 Proposition.** *Let $(X, d)$ be a separable metric space without isolated points. Let $\mu$ be a Maharam submeasure on $Borel(X)$. Then $X$ can be decomposed into a meager set $M \subseteq supp(\mu)$ and a $G_\delta$ set $N$ with $\mu(N) = 0$.*

*Proof.* First, let $\mathcal{O}$ be the system of all open $U \subseteq X$ with $\mu(U) = 0$. Then $O = \bigcup \mathcal{O}$ is clearly open, and $\mu(O) = 0$; being a union of open null sets, it is a union of countably many, due to separability. Put $F = X \setminus O$.

Let $S \subseteq F$ be the set of all singletons with nonzero submeasure. $S$ is at most countable, by exhaustivity of $\mu$. Let $H$ be a countable dense subset of $F \setminus S$;

this exists by second-countability. Let $K_n$ be a maximal subset of $H$ satisfying $(\forall x, y \in K_n) d(x,y) > \frac{1}{n}$. Then $K = \bigcup K_n$ is again a dense subset of $F \setminus S$.

Clearly $\mu(K) = 0$, as $K \subseteq H$ is countable. Hence $K$ is contained in a $G_\delta$ set $G$ with $\mu(G) = 0$ by the claim below. Put $N = O \cup G$ and $M = S \cup (F \setminus G)$. $N$ is clearly a null $G_\delta$ set. $S$ is countable and $F \setminus G$ is meager in the closed $F$, as it misses the dense $G_\delta$ set $G$. Hence $F \setminus G$, and $M$ as well, is meager in the whole space. $\square$

**1.2 Lemma.** *Every countable subset $K$ of $(X, d)$ with $\mu(K) = 0$ is contained in a $G_\delta$ subset $G$ with $\mu(G) = 0$.*

*Proof.* Enumerate $K$ as $\{x_n; n \in \omega\}$. Choose a sequence of $\varepsilon_k > 0$ decreasing to 0, and for every $k$ choose a family of balls $B^k_n \ni x_n$ such that the sum of their diameters is less then $\varepsilon_k$. Put $G_k = \bigcup \left\{B^k_n; n \in \omega\right\}$. For every $k$, this is an open set containing $K$. Hence $K \subseteq G = \bigcap G_k$, which is a $G_\delta$ set with $\mu(G) < \varepsilon_k$ for every $k$, i.e. $\mu(G) = 0$. $\square$

# 2 Baire extensions

As an example of another property shared by measures and submeasures, we prove the following generalization of an extension theorem in [Pa] (section V.5) from the countable case to an arbitrary Cantor space.

**2.1 Definition.** Let $X$ be a topological space and let $\mathcal{B}$ be a $\sigma$-field of subsets of $X$. Call a submeasure $\mu$ on $\mathcal{B}$ *regular* ir *regular from below* if $\mu(B) = \sup\{\mu(C); C \in \mathcal{B} \text{ compact}\}$ for every $B \in \mathcal{B}$.

**2.2 Theorem.** *Every exhaustive submeasure on $CO(2^\kappa)$ extends to a continuous regular submeasure on $Baire(2^\kappa)$.*

Being a continuous function on $(Baire(2^\kappa), \tau_s)$, this extension is necesarily unique, as $CO(2^\kappa)$ is a $\tau_s$-dense subspace. So in fact, there is a correspondence between exhaustive submeasures on $CO(2^\kappa)$ and continuous submeasures on $Baire(2^\kappa)$. In the well known case of $\kappa = \omega$ we already have $Baire(2^\omega) = Borel(2^\omega)$.

*Proof.* Let $\mu$ be an exhaustive submeasure on $CO(2^\kappa)$. Firstly, we will extend $\mu$ to the closed sets in $Baire(2^\kappa)$. These are $G_\delta$ compacts of $2^\kappa$; in fact, every such $F$ is of the form $\bigcap U_n$ for some clopen $U_n$: $F$ is a countable intersection of some open $V_n$; every $V_n$ is a union of basic clopen $B^n_\alpha$, hence for every $n$, $F$ is covered by some finite clopen union $U_n$ of the $B^n_\alpha$, which is a clopen set.

Put $\mu(F) = \inf\{\mu(A); F \subseteq A \in CO(2^\kappa)\}$; equivalently, $\mu(F) = \lim_{n\in\omega} \mu(U_n)$ for any sequence of clopen $U_n$ such that $F = \bigcap U_n$. It follows from the compactness of $2^\kappa$ that the value of $\lim_{n\in\omega} \mu(U_n)$ does not depend on the particular choice of the $U_n$; in particular, the $U_n$ can be made decreasing.

Similarly, an open set $U \in Baire(2^\kappa)$ is of the form $\bigcup U_n$ for some clopen $U_n$, and for $\mu(U) = \sup\{\mu(A); U \supseteq A \in CO(2^\kappa)\}$ we equivalently have $\mu(U) = \lim_{n\in\omega} \mu(U_n)$, independent of the particular choice of the $U_n$.

We have extended $\mu$ to closed sets and open sets in $Baire(2^\kappa)$. We show now some basic properties of this extension, needed in what follows.

(a) It is clear that $\mu$ so extended is a monotone function. Also, $\mu$ is subadditive on the open sets in $Baire(2^\kappa)$: if $U = \bigcup U_n, V = \bigcup V_n$, then for $U \cup V = \bigcup (U_n \cup V_n)$ we have $\mu(U \cup V) = \lim \mu(U_n \cup V_n) \leq \lim \mu(U_n) + \lim \mu(V_n) \leq \mu(U) + \mu(V)$ by subadditivity of $\mu$ on $CO(2^\kappa)$.

(b) For a closed Baire $F = \bigcap V_n$ and an open Baire $U = \bigcup U_n$ with $F \subseteq U$, we have $U_n \setminus V_n \subseteq U \setminus F$ for every $n$, so $\mu(U) - \mu(F) = \lim \mu(U_n) - \lim \mu(V_n) \leq \lim \mu(U_n \setminus V_n) \leq \mu(U \setminus F)$ by subadditivity of $\mu$ on $CO(2^\kappa)$. Hence $\mu(U) - \mu(F) \leq \mu(U \setminus F)$.

(c) For an open Baire set $U$, and a given $\varepsilon > 0$, there is a clopen $A \subseteq U$ satisfying $\mu(U \setminus A) < \varepsilon$ by monotonicity and exhaustivity of $\mu$ on $CO(2^\kappa)$; similarly for a closed Baire $F$ there is a clopen set above $F$ with an arbitrarily small difference in submeasure.

Now we extend $\mu$ further. Let $\mathcal{B}$ be the family of those $B \subseteq 2^\kappa$ such that for every $\varepsilon > 0$, there are a closed (compact) $F \subseteq B$ and an open $U \supseteq B$ in $Baire(2^\kappa)$ with $\mu(U \setminus F) < \varepsilon$. By the above, $\mathcal{B}$ contains all closed Baire sets (the $\sigma$-generators of $Baire(2^\kappa)$) and it is easy to see that with every $B \in \mathcal{B}$, we also have $-B \in \mathcal{B}$.

In fact, $\mathcal{B}$ can be verified to be a $\sigma$-field of sets: let $B_n \in \mathcal{B}$ and $\varepsilon > 0$; fix some closed $F_n$ and open $U_n$ in $Baire(2^\kappa)$ such that $F_n \subseteq B_n \subseteq U_n$ and $\mu(U_n \setminus F_n) < \varepsilon/2^{n+1}$. Then $U = \bigcup U_n$ is again an open Baire set, hence there is some clopen $A \subseteq U$ with $\mu(U \setminus A) < \varepsilon/2^n$. By compactness, $A \subseteq \bigcup_{n<n_0} U_n$ for some $n_0 \in \omega$. Thus $\mu(U \setminus \bigcup_{n<n_0} F_n) \leq \mu(U \setminus A) + \sum_0^{n_0} \mu(U_i \setminus F_i) \leq \varepsilon + \varepsilon$, while $\bigcup_{n<n_0} F_n \subseteq \bigcup A_n \subseteq U$. Hence $\mathcal{B}$ is closed under countable unions, is a $\sigma$-field of sets, and so in particular contains $Baire(2^\kappa)$.

For $B \in \mathcal{B}$, put $\mu(B) = \inf \{\mu(U); B \subseteq U \text{ open Baire}\}$. By (b) above, we can equivalently put $\mu(B) = \sup \{\mu(F); B \supseteq F \text{ closed Baire}\}$. This extends $\mu$ to $\mathcal{B} \supseteq Baire(2^\kappa)$ and the extension is clearly regular. We need to verify that $\mu$ so extended is indeed a continuous submeasure.

For subadditivity, given $B_1, B_2 \in \mathcal{B}$, chose for $\varepsilon > 0$ some open Baire $U_1, U_2$ such that $\mu(B_i) \leq \mu(U_i) + \varepsilon$. Using (a) above, we have $\mu(B_1 \cup B_2) \leq \mu(U_1 \cup U_2) \leq \mu(U_1) + \mu(U_2) \leq \mu(B_1) + \mu(B_2) + 2\varepsilon$.

For continuity, let $B_n$ form a sequence in $\mathcal{B}$ decreasing to $\emptyset$ and suppose that the nonincreasing sequence of $\mu(B_n)$ has $\lim \mu(B_n) = \varepsilon > 0$. Choose a sequence of closed Baire $F_n \subseteq B_n$ with $\mu(B_n \setminus F_n) < \varepsilon/2^{n+1}$. For every $n$ we have $B_n \subseteq \bigcup_{k<n} (B_k \setminus F_k) \cup \bigcap_{k<n} F_k$, so $\mu(\bigcap_{k<n} F_k) > \varepsilon/2$ and $\bigcap_{k<n} F_k$ is nonempty. By compactness, also $\bigcap_{n\in\omega} F_n \subseteq \bigcap_{n\in\omega} B_n = \emptyset$ is nonempty — a contradiction. □

Note that in the special case when the starting $\mu$ in $CO(2^\kappa)$ is a measure, the extended $\mu$ on $Baire(2^\kappa)$ constructed in the previous theorem is again a measure.

# 3 An ultraproduct of measures

In this section, we describe a way of arriving at measure algebras using an ultraproduct construction.

Let $(\mathcal{B}_n, \mu_n)$ be a sequence of $\sigma$-complete Boolean algebras carrying finitely additive, normalized measures. The cartesian product $\mathcal{B} = \Pi\mathcal{B}_n$ is $\sigma$-complete as well.

Choose a free ultrafilter $\mathcal{U}$ on $\omega$, and for $b = (b_n) \in \mathcal{B}$ put $\mu_{\mathcal{U}}(b) = \mathcal{U} - \lim \mu_n(b_n)$. It follows easily from the properties of the $\mathcal{U} - \lim$ operator that $\mu_{\mathcal{U}}$ is a finitely additive measure on $\mathcal{B}$.

Denote by $Null(\mu_{\mathcal{U}})$ the null ideal of $\mu_{\mathcal{U}}$ and consider the quotient algebra $\mathcal{B}_{\mathcal{U}} = \mathcal{B}/Null(\mu_{\mathcal{U}})$. Notice that $Null(\mu_{\mathcal{U}})$ extends the ideal $\{b \in \mathcal{B}; b =_{\mathcal{U}} 0\}$ on $\mathcal{B}$. Hence $\mathcal{B}_{\mathcal{U}}$ is a quotient of the ultraproduct algebra $\Pi\mathcal{B}_n/\mathcal{U}$.

**3.1 Proposition.** *The algebra $\mathcal{B}_{\mathcal{U}}$ carries a $\sigma$-additive, strictly positive measure, and is therefore a measure algebra.*

*Proof.* Consider the quotient measure $m_{\mathcal{U}}$ on $\mathcal{B}_{\mathcal{U}}$ defined by $m_{\mathcal{U}}([b]) = \mu_{\mathcal{U}}(b)$. This is a strictly positive, finitely additive measure on $\mathcal{B}_{\mathcal{U}}$. Hence the algebra $\mathcal{B}_{\mathcal{U}}$ is ccc.

The cartesian product $\mathcal{B} = \Pi\mathcal{B}_n$ is $\sigma$-complete, and therefore has the countable separation property. Thus $\mathcal{B}_{\mathcal{U}}$ also has the countable separation property, being a quotient of $\mathcal{B} = \Pi\mathcal{B}_n$. Being ccc as well, $\mathcal{B}_{\mathcal{U}}$ is complete by the Smith-Tarski theorem.

We will show that $m_{\mathcal{U}}$ is in fact a $\sigma$-additive measure on $\mathcal{B}_{\mathcal{U}}$, by showing its continuity. Let $[b^k]$ form a sequence in $\mathcal{B}_{\mathcal{U}}$ with $\bigwedge [b^k] = 0$. We assume the sequence is decreasing, and we can also assume without loss of generality that the representatives $b^k$ themselves form a decreasing sequence in $\mathcal{B}$; if not, remove from $b^1$ the element $(b^1_n - b^0_n)$ of $\mathcal{B}$ which has zero measure as $[b^1] \leq [b^0]$, thus obtaining a better representative for $[b^1]$, lying below $b^0$; proceed inductively, removing a finite union of zero-measure elements in each step.

Now $(\mu_{\mathcal{U}}(b^k))_{k\in\omega}$ is a non-increasing sequence of positive real numbers, which is convergent. Aiming for contradiction, assume that $\lim \mu_{\mathcal{U}}(b^k) = \varepsilon > 0$. Let $U_k = \left\{n; \mu_n(b^k_n) > \varepsilon/2\right\} \setminus k$ for $k \in \omega$ be the ultrafilter sets witnessing this. So we have decreasing $U_k \in \mathcal{U}$ for $k \in \omega$ such that $\bigcap U_k = \emptyset$.

We will find a nonzero lower bound for $([b_k])_{k\in\omega}$ by diagonalizing the sequence. Let $b = (b_n) \in \mathcal{B}$ where $b_n = b^k_n$ for $n \in U_k \setminus U_{k+1}$, and let $b_n \in \mathcal{B}_n$ be arbitrary on the non-ultrafilter set $\omega \setminus U_0 \notin \mathcal{U}$. Then $b - b^k \in \mathcal{B}$ has measure zero for every $k \in \omega$, hence $[b] \leq [b^k]$ in $\mathcal{B}_{\mathcal{U}}$. At the same time, the measure of $b$ itself is nonzero, as $\mu_n(b_n) > \varepsilon/2$ on an ultrafilter set. This makes $[b]$ a nonzero element of $\mathcal{B}_{\mathcal{U}}$ below all $[b^k]$ — a contradiction. □

**3.2 Example.** (a) As an easy example of the above construction, let all $\mathcal{B}_n$ be copies of the finite algebra $2^2 = \{0, a, -a, 1\}$, equipped with $\mu_n(a) = p_n, \mu(-a) = 1 - p_n$. Then $\mathcal{B}_{\mathcal{U}}$ is again a copy of $2^2$, with some atomic measure. Similarly for other $2^m$.

(b) Let $\mathcal{B}_n$ be copies of $P(\mathbb{N})$, equipped with the atomic measure assigning $\mu(A) = \sum \{2^{-m}; m \in A\}$ to $A \subseteq \mathbb{N}$. Then $\mathcal{B}_{\mathcal{U}}$ is a copy of $P(\mathbb{N})$, with some atomic measure.

(c) Let $I_n \subseteq \mathbb{N}$ form a decomposition of $\mathbb{N}$ into intervals of increasing length; put $\mathcal{B}_n = P(I_n)$ and let $\mu_n$ be the counting measure on $\mathcal{B}_n$. Then $\mu_{\mathcal{U}}$ is a density, i.e. a finitely additive measure on $P(\mathbb{N})$ extending the asymptotic density.

(d) Consider the Cantor algebra $\mathcal{A}$ equipped with the usual measure. This measure extends to the Cohen algebra $\mathcal{C}$ by 7.25. Let $\mathcal{B}_n$ be a copy of the Cohen algebra with the extended measure. Then $\mathcal{B}_{\mathcal{U}}$ is the measure algebra $\mathcal{B}(\mathfrak{c})$ of length continuum.

We will show this using the metric reformulation: the hereditary density of the corresponding metric space $(\mathcal{B}_{\mathcal{U}}, \rho)$ cannot be smaller than $\mathfrak{c}$. To see this, consider the countably many elements $x_k$ of the Cantor algebra which form a measure-independent system; that is, the measure of every $x_k$ is $1/2$ and the measure of every $x_k \wedge x_l$ is $1/4$. Having countably many such $x_k$ in every copy $\mathcal{B}_n$ of $\mathcal{A}$, we have a system of size $\mathfrak{c}$ in $\Pi\mathcal{B}_n$ consisting of mutually $\mathcal{U}$-different functions attaining a value of some $x_k$ in every $\mathcal{B}_n$. This yields a system of $\mathfrak{c}$ mutually different elements $b_\alpha = (b_\alpha^n)$ in $\mathcal{B}_{\mathcal{U}}$ such that $\rho(b_\alpha, b_\beta) = \mathcal{U} - \lim \mu_{\mathcal{U}}(b_\alpha^n, b_\beta^n) = \mathcal{U} - \lim 1/2 = 1/2$ for every $\alpha \neq \beta$. Hence every dense set in $(\mathcal{B}_{\mathcal{U}}, \rho)$ must be of size at least $\mathfrak{c}$. The same argument applies to every nonempty base set of $(\mathcal{B}_{\mathcal{U}}, \rho)$, as every clopen base subset of $2^\omega$ is homeomorphic to $2^\omega$ itself.

# 4 Functionals

The notion of a functional is a wide generalization of the notion of a measure.

### Basic classification

**4.1 Definition.** A mapping $f : \mathcal{B} \to \mathbb{R}$ on an algebra $\mathcal{B}$ is a *functional* if $f(0) = 0$. Denote by $Fn(\mathcal{B})$ the set of all functionals on $\mathcal{B}$. For functionals $f$ and $g$, let $f \leq g$ if $(\forall x \in \mathcal{B}) f(x) \leq g(x)$.

Occasionally, we deal with functionals that also admit an infinite value, i.e. mappings $f : \mathcal{B} \to [0, \infty]$. In that case, we will speak of *extended functionals*. To retain a notion of ordering for extended functionals, we accept that $\infty \leq \infty$; to retain a notion of additivity, we accept that $\infty + \infty = \infty$.

**4.2 Definition.** An extended functional $f$ on $\mathcal{B}$ is

(i) *nonnegative* if $f(b) \geq 0$ for every $b \in \mathcal{B}$;

(ii) *strictly positive* if $f(b) > 0$ for $b > 0_{\mathcal{B}}$;

(iii) *monotone* if $f(a) \leq f(b)$ for $a \leq b$;

(iv) *bounded* if $(\exists r \in \mathbb{R})(\forall b \in \mathcal{B})|f(b)| \leq r$.

For $f \in Fn(\mathcal{B})$, let $||f|| = \sup\{|f(b)|; b \in \mathcal{B}\}$ denote the *norm* of $f$.
Let $Null(f) = \{b \in \mathcal{B}; f(b) = 0\}$ denote the set of *null elements* of $f$.

Denote by $Mon(\mathcal{B})$ the set of monotone functionals. Clearly every monotone functional is nonnegative and bounded, with norm equal to $f(1_{\mathcal{B}})$.

We recall now the property of functionals concerning the relation of disjointness.

**4.3 Definition.** A functional $f$ on an algebra $\mathcal{B}$ is

(i) *exhaustive* if for every disjoint sequence $(a_n)$ in $\mathcal{B}$ we have $\lim |f(a_n)| = 0$.

(ii) *uniformly exhaustive* if for every $\varepsilon > 0$ there is some $k \in \omega$ such that for every disjoint sequence $(a_n)$ in $\mathcal{B}$ we have $|\{n; |f(a_n)| \geq \varepsilon\}| \leq k$.

**4.4 Fact.** *Let $f, g$ be functionals on a Boolean algebra $\mathcal{B}$.*

*(i) $f$ is exhaustive iff $|f|$ is exhaustive iff $\min(1, |f|)$ is exhaustive.*

*(ii) If $f$ is monotone and $D \subseteq \mathcal{B}$ is a dense subset, then $f$ is exhaustive iff the restriction $f \restriction D$ is an exhaustive mapping.*

*(iii) If $|f| \leq |g|$ and $g$ is exhaustive, then $f$ is exhaustive.*

**4.5 Lemma.** *Let $f$ be an exhaustive functional on $\mathcal{B}$. Then $g = \min(1, |f|)$ is a bounded functional and there is a smallest monotone functional $h \geq g$. This functional $h$ is exhaustive as well.*

The preceding lemma and fact also remain valid with exhaustivity replaced by uniform exhaustivity in all statements.

For monotone functionals, exhaustivity can be characterized as follows.

**4.6 Lemma.** *Let $f$ be a monotone functional on $\mathcal{B}$. Then $f$ is exhaustive iff for any sequence $(a_n)$ in $\mathcal{B}$ and any given $\varepsilon > 0$, there is some $k \in \omega$ such that*

$$(\forall l > k) f(\bigvee_{n<l} a_n - \bigvee_{n<k} a_n) < \varepsilon.$$

The following properties are motivated by additivity. Note that *signed measure* is synonymous with *finitely additive functional*, and by *measure* we understand a finitely additive finite measure.

**4.7 Definition.** A functional $\mu$ on Boolean algebra $\mathcal{B}$ is

(i) *finitely additive*, also called a *signed measure*, if $\mu(a \vee b) = \mu(a) + \mu(b)$ for every two disjoint $a, b \in \mathcal{B}$

(ii) 2-*additive* if $\mu(a \vee b) = \mu(1)$ for every partition $\{a, b\}$ of $1_{\mathcal{B}}$

(iii) 3-*additive* if $\mu(a \vee b \vee c) = \mu(1)$ for every partition $\{a, b, c\}$ of $1_{\mathcal{B}}$

Clearly, every 3-additive functional is 2-additive. It is easy to see that 3-additivity is, in fact, equivalent to finite additivity.

**4.8 Definition.** Let $\mathcal{B}$ be a Boolean algebra. A non-negative functional $\mu$ on $\mathcal{B}$ is

(i) a *(finitely additive) measure* on $\mathcal{B}$ if $\mu(a \vee b) = \mu(a) + \mu(b)$ for disjoint $a, b \in \mathcal{B}$. Denote by $Meas(\mathcal{B})$ the set of all measures on $\mathcal{B}$.

(ii) a *submeasure* on $\mathcal{B}$ if it is monotone and *subadditive*, i.e. $\mu(a \vee b) \leq \mu(a) + \mu(b)$ for every disjoint $a, b \in \mathcal{B}$. Denote by $Sub(\mathcal{B})$ the set of all submeasures on $\mathcal{B}$.

(iii) a *supermeasure* on $\mathcal{B}$ if it is *superadditive*, i.e. $\mu(a \vee b) \geq \mu(a) + \mu(b)$ for every disjoint $a, b \in \mathcal{B}$. Denote by $Sup(\mathcal{B})$ the set of all supermeasures on $\mathcal{B}$.

Every measure is simultaneously a submeasure and a supermeasure. Every supermeasure is uniformly exhaustive and monotone; for submeasures, we require monotonicity explicitly, as it does not follow from subadditivity.

As an easy example, the functional mapping every $x < 1_{\mathcal{B}}$ to 0 is a supermeasure, and the functional mapping every $x \in B^+$ to 1 is a (non-exhaustive) submeasure.

### Variation

Here we introduce the variation of an arbitrary functional, generalizing the classical notion of variation for measures (see e.g. [DS]).

**4.9 Definition.** For $f \in Fn(\mathcal{B})$ and $x \in \mathcal{B}$, the *variation of $f$ on $x$* is

$$v_f(x) = \sup \left\{ \sum_1^n |f(x_i)|; x_1, \ldots, x_n \text{ a finite antichain bellow } x \right\}$$

The functional $v_f$ mapping $x$ to $v_f(x)$ is the *variation* of $f$.

It will always be clear whether by "variation" we mean the value or the mapping. Note that $v_f(1_{\mathcal{B}})$ is finite iff $v_f$ is a bounded functional with $||v_f|| = v_f(1_{\mathcal{B}})$.

**4.10 Fact.** *For any functional $f \in Fn(\mathcal{B})$*

*(i) $v_f$ is a (extended) supermeasure on $\mathcal{B}$;*

*(ii) $v_f$ is the smallest supermeasure above $|f|$;*

*(iii) $v_f = f$ iff $f$ is a supermeasure.*

*(iv) For $f$ a signed measure, $v_f$ is a (extended) submeasure.*

*Proof.* (i) For finite antichains $(x_i)$ and $(y_j)$ bellow disjoint elements $x$ and $y$ respectively, $(x_i) \cup (y_j)$ is a finite antichain bellow $x \vee y$. Hence $v_f(x \vee y) \geq v_f(x) + v_f(y)$.

(ii) $v_f$ is a supermeasure by (i), and clearly $v_f(x) \geq |f(x)|$ for every $x \in \mathcal{B}$. If $\nu \geq |f|$ is a supermeasure, then for every $x \in \mathcal{B}$ and every finite antichain $(x_i)$ below $x$ we have $\nu(x) \geq \sum \nu(x_i) \geq \sum |f(x_i)|$, hence also $\nu(x) \geq v_f(x)$.

(iii) A supermeasure $f$ is monotone, and $\sum |f(x_i)| = \sum f(x_i) \leq f(x)$ for every finite antichain $(x_i)$ bellow $x$; hence also $v_f(x) \leq f(x)$. The other inequality holds by (ii), so we have $v_f = f$. The converse is trivial. □

**4.11 Example.** We describe a uniformly exhaustive submeasure on $P(\mathbb{N})$ without a bounded variation. This indicates that having a bounded variation is a strong property that even "nice" submeasures can fail to have.

For nonempty $A \subseteq \mathbb{N}$, put $f(A) = 1/\min(A)$; in particular, $f(\{n\}) = 1/n$ for singletons. This is a submeasure on $P(\mathbb{N})$: monotonicity is clear, and subadditivity follows from $1/\min(A) + 1/\min(B) \geq 1/\min(A \cup B)$. Uniform exhaustivity is easily verified: for a given $\varepsilon > 0$, at most $1/\varepsilon$ of the disjoint members of an antichain $\{A_n\}$ can gave $\min A < 1/\varepsilon$, i.e. $f(A_n) = 1/\min A_n > \varepsilon$.

If $P, Q$ are finite partitions of $A \subseteq N$, with $P$ finer than $Q$, then $\sum_{X\in P} f(X)$ is a larger contribution to $v_f(A)$ than $\sum_{X\in Q} f(X)$, by the same inequality as above. Hence the supremal value of $v_f(A)$ is the supremum of $\sum_{n\in P} 1/n$ for $P$ a finite subset of $A$. It follows that $v_f(A) < \infty$ if and only if $A$ belongs to the summable ideal.

We note that the example can obviously be modified by taking $2^{-\min A}$ or some other function decreasing fast enough (as opposed to $1/n$) so that the variation becomes finite for all sets.

### Lattices of functionals

In this section, we describe the properties of the natural ordering $f \leq g$ of functionals on a given algebra $\mathcal{B}$.

Firstly, note that this is a lattice. For functionals $f, g \in Fn(\mathcal{B})$, put $(f \wedge g)(x) = \min\{f(x), g(x)\}$ and $(f \vee g)(x) = \max\{f(x), g(x)\}$. These are easily seen to be the supremum and infimum of $\{f, g\}$ in $Fn(\mathcal{B})$.

**4.12 Proposition.** *For any Boolean algebra $\mathcal{B}$, the sets $(Fn(\mathcal{B}), \leq)$, $(Mon(\mathcal{B}), \leq)$, $(Sub(\mathcal{B}), \leq)$, $(Sup(\mathcal{B}), \leq)$, $(Meas(\mathcal{B}), \leq)$ are Dedekind complete lattices.*

This is not to say that the infinite suprema necessarily coincide: a sublattice can well be Dedekind complete in its own right without inheriting the existing suprema.

*Proof.* (a) The constant zero function is the smallest element in each of the classes.

(b) $Fn(\mathcal{B})$ itself is a Dedekind complete lattice: if $\mathcal{F} \subseteq Fn(\mathcal{B})$ is a bounded subset, then $\sup\{f(x); f \in \mathcal{F}\}$ is the supremum of $\mathcal{F}$.

(c) $Mon(\mathcal{B})$ is a complete sublattice of $Fn(\mathcal{B})$, as the supremum from (b) is a monotone functional if all the $f \in \mathcal{F}$ are monotone.

(d) For a bounded family $S \subseteq Sub(\mathcal{B})$, put $\mu(x) = \sup\{\nu(x); \nu \in S\}$. We have $\mu(0) = 0$ immediately, and $\nu(x + y) \leq \nu(x) + \nu(y)$ for every $\nu \in S$, hence $\mu(x + y) \leq \mu(x) + \mu(y)$ as well. So $\mu$ is a submeasure on $\mathcal{B}$, and it is clear that $\mu$ is the supremum of $S$.

(e) Let $\mathcal{M} \subseteq Meas(\mathcal{B})$ be a family of measures on $\mathcal{B}$, bounded by $m$. We know that there is a *submeasure* $\mu$ on $\mathcal{B}$ which is the supremum of $\mathcal{M}$ in $Sub(\mathcal{B})$, but not necessarily a measure. For $x \in \mathcal{B}$, the variation $v_\mu(x)$ is finite, and $v_\mu$ is a supermeasure below $m$. Check that $v_\mu$ is in fact a measure, and is the supremum of $\mathcal{M}$ in $Meas(\mathcal{B})$. □

As an example of an unbounded family of measures, consider for any ultrafilter $\mathcal{U}$ on an algebra $\mathcal{B}$ the corresponding 2-valued measure $m_\mathcal{U}$ on $\mathcal{B}$. The family of all $m_\mathcal{U}$ is not bounded in $Meas(\mathcal{B})$ unless $\mathcal{B}$ is finite.

### Extremal submeasures and supermeasures

**4.13 Proposition** ([Pa]). *For any Boolean algebra $\mathcal{B}$,*

*(i) a submeasure $\mu$ on $\mathcal{B}$ is 2-additive if and only if it is minimal in $(Sub(\mathcal{B}), \leq)$ among submeasures with the same norm $\mu(1)$.*

*(ii) A supermeasure $\nu$ on $\mathcal{B}$ is 2-additive if and only if it is maximal in $(Sup(\mathcal{B}), \leq)$ among supermeasure with the same norm $\nu(1)$.*

Given the above proposition, we will also call a 2-additive submeasure (resp. supermeasure) a *minimal submeasure* (resp. a *maximal supermeasure*). It is clear that a measure is simultanelously a minimal submeasure and a maximal supermeasure.

**4.14 Proposition.** *For any Boolean algebra $\mathcal{B}$,*

*(i) for every $\mu \in Sub(\mathcal{B})$, there is a minimal submeasure $\bar{\mu} \leq \mu$ with $||\bar{\mu}|| = ||\mu||$.*

*(ii) for every $\nu \in Sup(\mathcal{B})$, there is a maximal supermeasure $\bar{\nu} \geq \nu$ with $||\bar{\nu}|| = ||\nu||$.*

*Proof.* One way to prove this is to call forth the Zorn minimality (maximality) principle. We give an explicit description instead.

(i) For $x \in \mathcal{B}$ such that $\mu(x) \leq \mu(1)/2$ (which implies $\mu(-x) > \mu(1)/2$ by subadditivity), put $\bar{\mu}(x) = \mu(x)$ and $\bar{\mu}(-x) = \mu(1) - \mu(x)$; for $x \in \mathcal{B}$ with both $\mu(x) > \mu(1)/2$ and $\mu(-x) > \mu(1)/2$, put $\bar{\mu}(x) = \bar{\mu}(-x) = \mu(1)/2$. Clearly $\bar{\mu}$ is a functional with $\bar{\mu} \leq \mu$, is 2-additive by definition, and $\bar{\mu}(1) = \mu(1)$. The subadditivity of $\bar{\mu}$ follows from the subadditivity of $\mu$.

(ii) is completely dual. Note that for a supermeasure $\nu$ with $\nu(1) > 0$, the resulting $\bar{\nu}$ is strictly positive. $\square$

### Algebraic restrictions

**4.15 Lemma.** *Let $\mathcal{B}$ be a Boolean algebra carrying a strictly monotone functional. Then $\mathcal{B}$ is ccc.*

*Proof.* Let $X \subseteq \mathcal{B}$ be an uncountable antichain in $\mathcal{B}$. For some $\varepsilon > 0$, there must be uncountably many $x \in X$ with $f(x) > \varepsilon$. Fix infinitely many such $x_n$. Then $\lim_n \sum_{i \leq n} f(x_i) = \infty$, a contradiction. $\square$

For a Boolean algebra $\mathcal{B}$, call $T \subseteq \mathcal{B}$ a *tree in* $\mathcal{B}$ if $(T, \geq)$ with the inherited (reverse) ordering is a tree in the usual set-theoretic sense, and for every $x, y \in T$, if $x, y$ are disjoint in $T$, then $x, y$ are disjoint on $\mathcal{B}$. In other words, $T$ inherits both the order relation and the disjointness relation.

**4.16 Proposition.** *Let $\mathcal{B}$ be a Boolean algebra carrying a monotone strictly positive exhaustive functional. Then every tree in $\mathcal{B}$ is countable.*

*Proof.* Working towards a contradiction, suppose $T$ is an uncountable tree in $\mathcal{B}$. Every level and every branch of $T$ must be at most countable, as $\mathcal{B}$ is ccc. Hence there must be uncountably many $\alpha \in \omega_1$ such that the $\alpha$th level $T_\alpha$ of $T$ is nonempty.

For every such $\alpha$, let $r_\alpha = \max\{f(x); x \in T_\alpha\}$. The maximum exists: by exhaustivity, only finitely many $x \in T_\alpha$ can have $f(x)$ larger than some chosen $\varepsilon > 0$. Clearly, the sequence $(r_\alpha)_{\alpha\in\omega_1}$ is not increasing.

In fact, the value $r_\alpha$ strictly decreases uncountably many times, which is a contradiction. Indeed, the sequence can only *not* decrease for a countable number of times: if for some $\alpha \in \omega_1$ the set $\{\beta \in \omega_1; \alpha < \beta \text{ and } r_\alpha = r_\beta\}$ is uncountable, look at the witnessing $x_\beta \in T_\beta$ with $f(x_\beta) = r_\alpha$. Colour the pairs of such $x_\beta, x_\gamma$ with two colors, based on whether $x_\beta$ and $x_\gamma$ are comparable or disjoint in $T$ (which means disjoint in $\mathcal{B}$).

Only countably many $x_\beta$ can be comparable, otherwise we would have an uncountable branch. Hence by the Erdös-Duschnik theorem, there must be infinitely many disjoint $x_\beta$ with $f(x_\beta) = r_\alpha$. This contradicts exhaustivity. □

The above proposition can be restated as saying that the Suslin algebra cannot carry a strictly positive, monotone exhaustive functional. For Maharam submeasures, this is known from [M2].

# Chapter VI

# Lines

In this appendix, we consider the maximal linear subposets of Boolean algebras.

## 1 General properties

**1.1 Definition.** Let $(\mathcal{B}, \leq)$ be the canonical ordering of a Boolean algebra. A subset $L \subseteq \mathcal{B}$ is a *line* in $\mathcal{B}$ if it is a maximal linear subordering of $(\mathcal{B}, \leq)$.

Clearly, every line $L \subseteq \mathcal{B}$ contains both 0 and 1 as its smallest and greatest element. Being linearly ordered, the line $(L, \leq)$ carries the order topology structure. For example, in $CO(2^\omega)$, every line is homeomorphic to the rationals with ends.

**1.2 Fact.** *Let $\mathcal{B}$ be a complete Boolean algebra, let $L \subseteq \mathcal{B}$ a line.*

*(i) The linearly ordered topological space $(L, \leq)$ is a compact Hausdorff space.*

*(ii) If $\mathcal{B}$ is atomless and ccc then the linearly ordered space $(L, \leq)$ is connected.*

*Proof.* (i) It is well known that a linearly ordered topological space is Hausdorff in its order topology; in fact, it is collectionwise normal. Compactness of a linearly ordered space is equivalent to the completeness of the linear order; so let $X \subseteq L$. The supremum $\bigvee X$ exists in $\mathcal{B}$. Every $y \in L$ is either below some $x \in X$, hence below $\bigvee X$, or above every $x \in X$, hence above $\bigvee X$. Thus $\bigvee X \in L$ by maximality of $L$, and $(L, \leq)$ is complete.

(ii) If $(L, \leq)$ is not connected, let $X$ and $Y$ be two nonempty clopen subsets of $L$. Assume $0 \in X$ and $y_0 \in Y$. Let $x = \sup\{a \in X; a < y_0\}$, where the supremum is taken in $L$; we have $x \in X$. Let $y = \inf\{b \in Y; x < b\}$; we have $y \in Y$. Clearly $x < y$, and as $\mathcal{B}$ is atomless, there is some $z \in \mathcal{B}$ such that $x < z < y$. By maximality of $L$, we have $z \in L$. But then $X$ and $Y$ do not form a decomposition of $L$ — a contradiction. $\square$

It is a natural question whether the linear order topology of $(L, \leq)$ is the same as the order-sequential topology $(L, \tau_s)$ which $L$ inherits as a subspace of $(\mathcal{B}, \tau_s)$.

**1.3 Proposition.** *Let $\mathcal{B}$ be a complete ccc algebra. Then for every line $L \subseteq \mathcal{B}$, the topologies $(L, \leq)$ and $(L, \tau_s)$ coincide. Moreover, the subspace $(L, \tau_s)$ is Fréchet. The subset $L \subseteq (\mathcal{B}, \tau_s)$ is closed.*

**1.4 Proposition** (Thümmel)**.** *Let $\mathcal{B}$ be a complete atomless algebra such that the free product $\mathcal{B} * \mathcal{B}$ is ccc. Then every line $L \subseteq \mathcal{B}$ is homeomorphic to the closed interval* $[0, 1]$.

*Proof.* $\mathcal{B}$ is a regular subalgebra of the free product $\mathcal{B} * \mathcal{B}$ which is ccc, so $\mathcal{B}$ is ccc itself. Therefore, the linearly ordered compact Hausdorff space $(L, \leq)$ is connected. Hence it suffices to show the separability of $(L, \leq)$. Suppose that $(L, \leq)$ is not separable. Then we can find by induction a family of triples $\{a_\alpha < x_\alpha < b_\alpha; \alpha \in \omega_1\}$ in $L$ such that $\{x_\alpha; \alpha < \beta\} \cap (a_\beta, b_\beta) = \emptyset$. But then $\{(x_\alpha - a_\alpha, b_\alpha - x_\alpha); \alpha \in \omega_1\}$ is an antichain of size $\omega_1$ in $\mathcal{B} * \mathcal{B}$: assume $\alpha < \beta < \omega_1$; if $x_\alpha < x_\beta$, then $x_\alpha - a_\alpha \perp x_\beta - a_\beta$; and if $x_\alpha > x_\beta$, then $b_\alpha - x_\alpha \perp b_\beta - x_\beta$. Either way, the two pairs are disjoint in the free product. $\square$

**1.5 Example.** We show that the converse to the above theorem does not hold. It is relatively consistent with ZFC that CH and SH hold simultanelously (hint: start in $L$ and kill all trees; see[DJ]). We work in a model of ZFC + CH + SH.

From CH it follows that there exists an algebra $\mathcal{B}$ which is complete atomless and ccc, but $\mathcal{B} * \mathcal{B}$ is not ccc (see [Ga] or excercise VIII.C8 in [Ku]).

Now every line $L \subseteq \mathcal{B}$ is a complete ccc order without jumps, hence the linearly ordered space $(L, \leq)$ is a compact connected ccc Hausdorff space. The linear order $(L, \leq)$ is in fact separable: if not, it is a Suslin line, which violates SH. So every line $L \subseteq \mathcal{B}$ is homeomorphic to the unit interval, yet $\mathcal{B} * \mathcal{B}$ is not ccc.

# 2 Examples

**2.1 Definition.** Let $(L, \leq)$ be a linear order. A *jump* in $(L, \leq)$ is a pair $x < y$ in $L$ such that there is no $z \in L$ with $x < z < y$. A complete linear order $(L, \leq)$ is *boolean* if it has a dense set of jumps, i.e., for every nonempty $(a, b) \subseteq L$ there is a jump $x < y$ in $L$ such that $a \leq x < y \leq b$.

It is folklore knowledge that a linearly ordered topological space $(L, \leq)$ is Boolean if and only if the order $(L, \leq)$ is itself Boolean.

**2.2 Example.** We show that the lines in $P(\omega)$ are precisely the boolean linear orders with a countable dense set of jumps.

Indeed, let $(L, \leq)$ be such an order, and let $D$ be the countable dense set. Consider $P(D)$ as a copy of $P(\omega)$. For every cut $(X, Y)$ of $(D, \leq)$, let $u_X = \bigvee X$. Obviously $L' = \{u_X; (X, Y) \text{ a cut}\}$ is a linear order in $\mathcal{B}$ which is easily seen to be isomorpic to $(L, \leq)$. It is not hard to see that $L'$ is maximal, and therefore a line[1].

[1] Note in this context how $\mathbb{R}$ is a chain, but not a line in $P(\omega)$: indeed, $\mathbb{R}$ has no jumps. The countable dense set of jumps that correspond to the rational cuts is what's missing in maximality.

The other implication is easy: a line $L \subseteq P(\omega)$ is a complete linear order in itself, hence a compact Hausdorff space. It must have a dense set of jumps by maximality, which makes it a Boolean order. At the same time, the set of jumps cannot be uncountable.

We have, in particular, that $\omega + 1$ is a line (which can be shown directly). This line is minimal in the sense that a copy of $\omega + 1$ can be found in every other line. More generally, every $\alpha + 1$ for $\alpha < \omega_1$ can be found in $P(\omega)$ as a line. It follows that there are $\mathfrak{c}$ many nonisomorphic types of lines: take the countable set of jumps in a countable ordinal $\alpha + 1$. For a subset $A$ of the set of jumps, replace every jump in $A$ with the Cantor discontinuum. Now different $A$ yield different lines in $P(\omega)$.

**2.3 Example.** Let $T$ be a Suslin tree, let $\mathcal{B} = \mathcal{B}(T)$ be the corresponding Suslin algebra. We show that every line $(L, \leq) \subseteq \mathcal{B}$ and, in fact, any closed interval $[a, b] \subseteq L$ is a Suslin line (with ends).

The compact Hausdorff space $(L, \leq)$ is ccc and connected. The interval $[a, b]$ is itself a complete dense linear order which is ccc and connected. We show that it is not separative.

Let $D \subseteq (a, b)$ be a countable dense subset. Let $G$ be a generic filter on $\mathcal{B}$ that contains $b - a$; so $b \in G$ but $a \notin G$. The filter $G$ determines a cut of the linear order $(D, \leq)$: put $X = (a, b) \setminus G$ and $Y = (a, b) \cap G$. Then $X, Y$ are two countable subsets of $D$, and hence are sets in the ground model, as $\mathcal{B}$ is a $(\omega, \infty)$-distributive algebra. By genericity of $G$, we have $x = \bigvee X \notin G$ and $y = \bigwedge Y \in G$; hence $x < y$. But then $y - x$ must be an atom — a contradiction.

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