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Optimization Problems under (max; min) - Linear Constraints and Some Related Topics

Department of Probability and Mathematical Statistics

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Prague, March 2014 *Mahmoud Gad*

To

The spirit of my father

My mother

My wife

My brothers

Amr and Ahmed

Prague, March 2014 Mahmoud Gad

Declaration

I declare that I carried out this doctoral thesis independently, and only with the cited sources, literature and other professional sources.

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Mahmoud Gad

Název práce: Optimalizační problémy při (max,min)-lineárních omezeních a některé související úlohy.
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Abstrakt: Úlohy na algebraických strukturách, v nichž dvojice operací (max, +) nebo (max, min) nahrazují operace sčítání a násobení v klasické lineární algebře se objevují v literatuře přibližně od šedesátých let minulého století. První výsledky s využitím těchto struktur publikovali A. Shimbel v práci [37] s aplikacemi v komunikačních sítích, a dále R. A. Cunnighame-Green [12,13], N. Vorobjov [40] a B. Giffler [18] s aplikacemi na rozvrhování práce strojů a v teorii spolehlivosti. Ucelená systematická teorie takových algebraických struktur byla publikována pravděpodobně poprvé v práci [14]. V nedávno publikované knize [4] lze nalézt nejnovější stav výzkumu teorie a algoritmů ve struktuře s operacemi (max,+). Protože operace maxima, která v uvedených strukturách nahrazuje operaci sčítání, není grupovou, ale pouze pologrupovou operací, je podstatný rozdíl mezi řešením soustav s proměnnými pouze na jedné straně rovnic resp. nerovností a soustav, v nichž se proměnné nacházejí na obou stranách těchto vztahů. Soustavy s proměnnými na jedné straně se nazývají jednostranné a soustavy, v nichž se proměnné vyskytují na obou stranách rovnic resp. nerovností nazveme dvoustranné. Cílem předkládané dizertace je poskytnout jednotící teoretický rámec pro prezentaci autorem dosažených výsledků v oblasti výzkumu soustav (max, min)-lineárních rovnic a nerovností a některých typů optimalizačních problémů s omezeními ve tvaru (max,min)-lineárních rovnic a nerovností. Kromě toho jsou navržena některá zobecnění na nelineární soustavy, které sjednocují (max,+)- a (max, min)-lineární úlohy a rozšiřují získané výsledky za rámec (max,+)- a (max, min)-lineárních struktur. V další části práce jsou studovány tzv. nekorektně formulované úlohy a uvádějí se efektivní postupy nalezení vhodného řešení těchto úloh pro soustavy (max,min)-lineárních rovnic. V práci jsou uvedeny i některé motivační příklady z oblasti operačního výzkumu a menší ilustrativní numerické příklady.

Prvním oktuhem problémů, jimiž se předložená práce zabývá jsou vlastnosti soustav (max,min)-lineárních rovnic a nerovností. Druhým tétatem je řešení optimalizačních úloh s omezeními ve tvaru soustav (max,min)-lineárních rovnic a nerovností. Třetí skupinu problémů tvoří nekorektně formulované úlohy a přístup k jejich řešení pro případ (max,min)-lineárních rovnic. Posledním okruhem problémů, jimž se předložená práce věnuje je zobecnění získaných výsledků na širší třídu nelineárních tzv. max-separabilních problémů.

Klíčová slova: Optimalizační problémy, (max, min)-lineární omezení.

Title: Optimization Problems under (max, min)-Linear Constraints and Some Related Topics
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Abstract: Problems on algebraic structures, in which pairs of operations such as (max, +) or (max, min) replace addition and multiplication of the classical linear algebra have appeared in the literature approximately since the sixties of the last century. The first publications on these algebraic structures appeared by Shimbel [37] who applied these ideas to communications networks, Cuninghame - Green [12, 13], Vorobjov [40] and Gidffer [18] applied these algebraic structures to problems of machine time - scheduling. A systematic theory of such algebraic structures was published probably for the first time in [14]. In recently appeared book [4] the readers can find latest results concerning theory and algorithms for (max,+)−linear systems of equations and inequalities. Since operation max replacing addition is no more a group, but only a semigroup operation, it is a substantial difference between solving systems with variables on one side and systems with variables occuring on both sides of the equations. The former systems will be called ”one-sided” and the latter systems ”two-sided”. The aim of this thesis is to provide a unifying survey of some recent authar's results concerning to the investigation of (max,min)− linear equations and inequality systems and some optimization problems under (max,min)−linear constraints. Besides, we propose some generalizations to non-linear systems, which unify in one model the (max,+)− and (max,min)− linear problems and extend the results beyond the (max,+)− and (max,min)− linear structures. Further a special problem called ”incorrectly posed problem” is introduced and effective methods for its solutions are proposed for (max,min)− linear and non-linear equation systems are considered. We bring also some motivating examples from the area of operations research as well as illustrating numerical examples.

The first subject of this thesis is the investigation of properties of systems of (max,min)−linear equations or inequalities. The second subject of this thesis is solving optimization problems subject to (max,min)−linear equation and inequality constraints. The third subject of research of the present thesis is the investigation of so called incorrectly posed one-sided (max,min)−linear systems of equations. The fourth part of the thesis is devoted to a generalization of the research of (max,min)−linear problems on some max-separable nonlinear problems.

Keywords: Optimization Problems, (max, min)- Linear Constraints.

Contents

1

Introduction

Approximately since the sixties of the last century appeared a number of different problems of interest to the operational researcher and the mathematical economist - for example, certain problems of optimization on graphs and networks, of machine - scheduling, of convex analysis and of approximation theory, which appeared in the mathematical literature under various name and can be formulated in a convenient way using special algebraic structures $(E, \oplus, \otimes)$. The algebraic structures are represented by a set E, $E \subset R = (-\infty, \infty)$, with two operations denoted $\oplus$ and $\otimes$, which play in the algebraic structures similar role and behave similarly like the addition and multiplication in the classic linear algebra. The operation $\oplus$ is a commutative semigroup operation with the neutral element $0_\oplus$ and the operation $\otimes$ is either a commutative group or a commutative semigroup operation with a neutral element $1_\otimes$. The set E and operations $\oplus$ and $\otimes$ are chosen in such a way that the distributive law holds. The operations $\oplus$ and $\otimes$ can be extended to E^n and to matrices over E with appropriate size similarly like in the classical linear algebra the addition and multiplication are extended to R^n and to real matrices. This make possible to define $(\oplus, \otimes)-$linear function on E^n and investigate systems of $(\oplus, \otimes)-$linear equations and inequalities as well as some optimization problems under $(\oplus, \otimes)-$linear equations and inequalities constraints.

As triplets appropriate both from the theoretical and from the application point of view are usually used triplets $(R, \max, +)$ or $(R, \max, \min)$ but we can encounter also triples $(R_+, \max, \cdot)$, $(R_+, \min, \cdot)$, where we set $R =$

$(-\infty, \infty)$, $R_+ = \{\alpha \in R \mid \alpha > 0\}$, $(Z, \max, +)$, and $(Z, \max, \min)$, where $Z = \{\alpha \in R \mid \alpha \quad \text{is integer}\}$. $([0,1], \max, \min)$ (so called fuzzy algebra), $([0,1], \max, \cdot)$ (so called boolean algebra) and other.

The first publications on these algebraic structures appeared by Shimbel [34] who applied these ideas to communications networks, Cuninghame-Green [10, 11] and Giffler [15] applied these algebraic structures to problems of machine time - scheduling. The algebraic structures $(R, \max, +)$, $(R, \max, \min)$, $(R, \max, \cdot)$, were studied by several authors Carre [6, 7], Cuninghame-Green [12] and Vorobjov [37]. The authors solve systems of $(\oplus, \otimes)-$linear equations and inequalities in which variables appear on one-side of equations or inequalities, while on the other side is a constant. Butkovič and Hegedüs [3] introduce an elimination method for finding all solutions of the system of linear equations over an extremal algebra. Tharwat and Zimmermann [35] introduce the method, which find the optimal choice of parameters in machine time scheduling Problems, Also Tharwat and Zimmermann [36] study separable optimization problems and introduce some application. Zimmermann [38] introduces a general separation theorem in extremal algebras, Zimmermann [39] studies disjunctive optimization problems, max-separable problems and extremal algebras and Zimmermann [40] discusses class of optimization problems with alternative constraints and its application.

After having extended the operations to vectors and matrices, the authors [4], [8], [9] and [16] introduce the concept of $(\oplus, \otimes)-$ eigenvalues and $(\oplus, \otimes)-$ eigenvectors and propose effective numerical methods making possible to find the eigenvalues and eigenvectors.

In [6] we find problems involving the determination of routes on networks arise in many different contexts. For example network flow problems in operations research, such as transportation and assignment problems, involve the determination of a succession of shortest or least-cost paths between commodity sources and sinks. Again, critical path analysis and certain scheduling problems involve the determination of longest paths on activity networks. Pathfinding problems of different kinds also arise in the design of logic networks, and in routing messages through congested communication networks. [6] presents an algebraic structure for the formulation and solution of such problems.

After defining the algebraic structure and giving concrete examples applicable to different kinds of routing problems, [6] uses in a general analysis of a class of directed networks, in which each of them has an associated measure (representing for instance a transportation cost, an activity duration, the state - open or closed - of a switch, or the probability of a communication link being available). It is then shown that all the routing problems mentioned above can be expressed in the same algebraic form, and that they can all be solved by variants of classical methods of linear algebra, differing from these only in the significance of the additive and multiplicative operations.

Since the $\oplus-$operation (i.e. $\max -$operation) is only a semigroup operation, so that the variables can not be simply transferred from one-sided of the equations and inequalities to the other like in classic linear algebra, where the operation $+$ is a group operation. Therefore equations and inequalities with variables on both sides of the relations must be treated by special methods different from those, which were used to investigate one-sided equations and inequalities.

The first publications denoted to some special two-sided systems by Butkovič and Hevery [4] for the $(\max, +)-$ case. A detailed survey of the research development of the structure $(R, \max, +)$ and some of its modifications can be found in the book [2] published in 2010 by Butkovič. Butkovič and Zimmermann [5] proposed a strongly polynomial algorithm for solving two-sided linear systems in max-algebra, but after that Bezem et al. [1] studied exponential behaviour of the Butkovič - Zimmermann algorithm for solving two-sided linear systems in max-algebra. Gavalec and Zimmermann [17] solve systems of two-sided $(\max, \min)$-linear equations. Besides Gavalec and Zimmermann [18] study optimization problems with two-sided systems of linear equations over distributive lattices.

Further results on $(\max, +)$ and $(\max, \min)$ eigenvalues and eigenvectors can be found in Gavalec and Plávka [16] and Cechlárová [9] and others. Operations $(\oplus, \otimes)$ make possible to introduce the concept of convexity on $(E^n, \oplus, \otimes)$. This concept led to investigating some geometrical problems as e. g. properties of $(\oplus, \otimes)-$ convex sets and functions. The corresponding results can be found e.g. Nitica and Singer [26, 27], which study Max-plus convex sets and max-plus semispaces, also Nitica and Singer [28, 29], which are contributions to max-min convex geometry segments and semispaces and convex sets. Sergeev [33] introduced algorithmic complexity of a problem of idempotent convex geometry and Helbig [19] study a Caratheodory's and Krein-Milman's theorems in fully ordered groups. Nitica and Sergeev [30] Study hyperplanes and semispaces in maxmin convex geometry.

Another type of problems which may be important mainly in the applications arises from the assumption that the coefficients of the problems are not exactly given numbers, but may move within closed intervals. This assumption led to investigation equation and inequality systems and optimization problems under $(\oplus, \otimes)-$linear constraints by making use of the methods of interval mathematics (see Cechlárová [9], Myšková [25] and other).

The investigation of the structures using methods of mathematical analysis can be found in the publications by Litvinov and Maslov [22], Litvinov et al. [23] and Maslov and Samborskij [24]. Some authors proposed explicit form as for solving some $(\oplus, \otimes)-$ linear problems (e. g. Kolokoltsov and Maslov [21] and the references there in).

The present dissertation is devoted to the investigation of $(\max, \min)-$ linear equations and inequality systems and some optimization problems under $(\max, \min)-$ linear constraints, because these problems has not yet been systematically investigated in the literature. Besides, the author proposes some generalizations to non-linear systems, which unify in one model the $(\max, +)-$ and $(\max, \min)-$ linear problems and extend the results beyond the $(\max, +)-$ and $(\max, \min)-$ linear structures. Further special problems called "incorrectly posed problem" are introduced and effective methods for their solutions are proposed for $(\oplus, \otimes)-$ linear and non-linear equation systems are considered.

When we extend operations $\oplus, \otimes,$ from E to E^n, where $E^n = E \underbrace{\times \cdots \times}_{n\ times} E$, we can define in a natural way the appropriate inner product of $x,\ y \in E^n$, namely

$$x^T \otimes y = \sum_{j=1}^{n}{}^{\oplus}(x_j \otimes y_j)$$

can be introduced. The inner product makes possible extending the operations to matrices of appropriate sizes as

$$(A \otimes B)_{ij} = \sum_{k=1}^{n}{}^{\oplus}(a_{ik} \otimes b_{kj}) \;\; \forall \;\; i \in I, \; j \in J$$

where $I = \{1, \cdots, m\}, \; j \in J = \{1, \cdots, n\}$. Multiplication $A \otimes x$ can be introduced in a similar way as

$$(A \otimes x)_i = \max_{j \in J}(a_{ij} \otimes x_j) \;\; \forall \;\; i \in I.$$

Function $f : E^n \to E^1$ for $x \in E^n$ defined as

$$f(x) = \sum_{j=1}^{n}{}^{\oplus} f_j(x_j)$$

will be called $\oplus-$separable and if $f_j(x_j) = c_j \otimes x_j$, function f is called $(\oplus, \otimes)-$linear. Examples of such functions are for instance:

$$f^{(1)}(x) = \max_{j \in J}(c_j + x_j) \qquad \text{is a } (\max, +) - \text{linear function on } (R, \max, +),$$

$$f^{(2)}(x) = \min_{j \in J}(c_j \cdot x_j) \qquad \text{is a } (\min, \cdot) - \text{linear function on } (R_+, \min, \cdot),$$

$$f^{(3)}(x) = \max_{j \in J}(c_j \wedge x_j) \qquad \text{is a } (\max, \min) - \text{linear function on } (R, \max, \min).$$

The first subject of this thesis is the investigation of properties of systems of $(\max, \min)-$ linear equations and / or inequalities, i. e. equations or inequalities, in which $(\max, \min)-$ linear functions occur. The following examples show how such inequalities may look like:

$$\max_{j \in J}(\alpha_j \wedge x_j) \;\; \square^* \;\; \beta, \tag{1.1}$$

$$\max_{j\in J}(\alpha_j \wedge x_j) \quad \Box^* \quad \max_{j\in J}(\beta_j \wedge x_j), \tag{1.2}$$

where we assume $\alpha_j,\ \beta,\ \beta_j \in R \ \ \forall \ \ j \in J$ and $\Box^*$ is one of the relations $\leq,\ =, \geq$ where to simplify the expressions we set

$$(\alpha_j \wedge x_j) = \min(\alpha_j, x_j).$$

We will use this notation throughout the next chapters of the thesis. Let us point out the difference between relations (1.1) and (1.2): relations (1.1) contain variables on the left side only while in relations (1.2) variables x_j occur in both sides of the relations. Since operation $\oplus = \max$ is only a semigroup operation, the variables cannot be simply transferred from one side to the other like in classic linear algebra, where the operation $+$ is a group operation. We will call relations (1.1) one-sided equations or inequalities and relations having the form (1.2) two-sided equations or inequalities. The absence of inverse elements leads to the necessity to investigate one-sided and two-sided inequalities or equations separately using different methods.

The second subject of this thesis is solving optimization problems, the set of feasible solutions of which is described by a finite system of $(\max, \min)-$linear equations and inequalities (both one-sided and two-sided relations are considered). The objective function of the optimization problems are continuous max-separable functions of the form:

$$f(x_1, x_2, \cdots, x_n) = \max_{j\in J} f_j(x_j).$$

Examples of operations research show that $(\max, \min)-$linear optimization problems studied in this work can be applied to processing time scheduling, network capacity problems, investigating reliability of complex systems and others as well as to some problems connected with the fuzzy set theory. Numerical examples demonstrate the behaviour of the proposed algorithms.

The third subject of research of the present thesis is the investigation of so called incorrectly posed one-sided $(\max, \min)-$linear systems of equations. The concept of incorrectly posed (or improper posed problem) is used in the literature for problems, which have no solutions for given coefficients and we look for a close set (with respect to a given distance function) of coefficients generating a solvable problem (see e.g. Eremin et al. [13] and Eremin and Vatolin [14]). In the present work this concept is introduced for $(\max, \min)-$linear equations systems. Various approaches to solving such incorrectly posed problems are proposed.

The fourth part of the thesis is devoted to a generalization of the research of $(\max, \min)-$linear problems. We consider equations and inequalities, in which the following max-separable functions occur :

$$g(x) = \max_{j \in J} (\nu_j \wedge q_j(x_j))$$

where $\nu_j \in R, \quad q_j : R \to R,$ are strictly increasing continuous functions. We consider finite systems of equations and / or inequalities of the one-sided form:

$$\max_{j \in J}(a_{ij} \wedge r_{ij}(x_j)) \quad \Box^* \quad b_i, \quad \forall \ \ i \in I \tag{1.3}$$

where $\Box^*$ is one of $\leq, =, \geq,$ as well as some special types of two-sided equations and / or inequalities systems. Properties of the systems are investigated and used to solve optimization problems with a max-separable objective function and set of feasible solutions described by the "non-linear" systems with functions (1.3). Also the results concerning incorrectly posed problems are generalized for the one-sided equation systems of the form (1.3).

Before we begin to study the main problem in this thesis, which represented in solving the optimization problems under $(\max, \min)$-linear constraints and the investigation of properties of systems of $(\max, \min)-$ linear equations or inequalities, In the next chapter we will present briefly the idea to solve a two-sided systems $(\max, +)-$ linear equations and we introduce a finite algorithm for finding the optimal solution of the optimization problems under a two-sided $(\max, +)-$linear constraints.

2

Optimization Problems under (max,+)-Linear Constraints

We consider optimization problems, the objective function of which is equal to the maximum of a finite number of continuous strictly increasing functions of one variable. The set of feasible solutions is described by a system of (max,+)-linear equations with variables on both sides. A finite algorithm for finding the optimal solution of the problem is proposed.

2.1 Notation, Problem Formulation

We will assume that $A,\ B$ are two real $(m, n)-$matrices with entries $a_{ij},\ b_{ij},\ i \in I = \{1, 2,\ \dots,\ m\},\ j \in J = \{1, 2,\ \dots,\ n\}$. Let

$$a_i(x) = \max_{j\in J}(a_{ij} + x_j),\ b_i(x) = \max_{j\in J}(b_{ij} + x_j),\ \forall i \in I.$$

We will further assume that $f_j(x_j), \forall j \in J$ are continuous strictly increasing functions and set

$$f(x) = f(x_1,\ x_2,\ \dots,\ x_n) = \max_{j\in J} f_j(x_j).$$

If $f(x) = f_p(x_p)$, then variable x_p will be called active variable of function f at point x. By analogy we will define the concept of active variables of functions

$a_i(x),\ b_i(x),\ i \in I.$
Function $\phi(x)$ of $x = (x_1,\ x_2,\ \ldots,\ x_n)$ having the form $\phi(x) = \max_{j \in J} \phi_j(x_j)$ will be called max-separable function. Functions $f(x),\ a_i(x),\ b_i(x)$ are examples of max-separable functions.
We will consider the following optimization problem:

PROBLEM I.

$$f(x) \ \longrightarrow \min \tag{2.1}$$

subject to

$$a_i(x) = b_i(x),\ \forall i \in I, \tag{2.2}$$

$$\underline{x}\ \leq\ x\ \leq\ \overline{x}, \tag{2.3}$$

where $\overline{x},\ \underline{x}$ are given finite vectors. In the sequel, the set of feasible solutions of PROBLEM I will be denoted $M(\overline{x}, \underline{x})$.

2.2 Properties of the Set of Feasible Solutions.

Definition 2.2.1 Let $L \subseteq R^n$, $\tilde{x} \in L$. Then $\tilde{x}$ is called the *maximum element* of L, if $x \leq \tilde{x}$ holds for every $x \in L$.

Theorem 2.2.1 *[5] Let $M(\overline{x}) = \{x\ ;\ a_i(x) = b_i(x)\ \forall i \in I \& x \leq \overline{x}\}$. If $M(\overline{x}) \neq \emptyset$, there exists always element $x^{\max} \in M(\overline{x})$ such that $x \leq x^{\max}\ \ \forall x \in M(\overline{x})$.*

Therefore the following theorem is true:

Theorem 2.2.2

$M(\overline{x}, \underline{x}) \neq \emptyset$ if and only if $\underline{x} \leq x^{\max}$.

Let us note that element $x^{\max}$ is called the maximum element of $M(\overline{x}, \underline{x})$. Let us remark further that if $M(\overline{x}, \underline{x}) \neq \emptyset$, then the maximum element $x^{\max}$ of $M(\overline{x})$ is at the same time the maximum element of $M(\overline{x}, \underline{x})$. The method proposed in [5] either finds after a finite number of steps element $x^{\max}$ or finds out that $M(\overline{x}) = \emptyset$. The method of [5]is in general pseudopolynomial (see [1]). Therefore we can assume for the algorithm solving PROBLEM I that the set of its feasible solutions is nonempty and we obtained using the method of [5] the maximum element $x^{\max}$.

To simplify the description of the algorithm for solving PROBLEM I, we introduce the concept of threshold values of a max-separable function. Let $\phi(x) = \max_{j \in J} \phi_j(x_j)$ be a max-separable function, and $\tilde{x}$ an arbitrary point. Let us set:

$$t_1(\tilde{x}) = \max_{j \in J} \phi_j(\tilde{x}_j),\ P_1(\tilde{x}) = \{p \in J\ ;\ \phi_p(\tilde{x}_p) = t_1(\tilde{x})\}.$$

For all $k,\ 2 \leq k \leq n$, for which $H_{k-1}(\tilde{x}) = (J \setminus \cup_{1 \leq r \leq (k-1)}(P_r(\tilde{x}))) \neq \emptyset$ we set:

$$t_k(\tilde{x}) = \max_{j \in H_{k-1}(\tilde{x})} \phi_j(\tilde{x}_j),\ P_k(\tilde{x}) = \{p \in J\ ;\ t_k(\tilde{x}) = \phi_p(\tilde{x}_p)\}.$$

The values $t_k(\tilde{x}),\ k = 1,\ \ldots,\ h$ obtained in this way are called threshold values of function ϕ at point $\tilde{x}$. Let us note that there is always $1 \leq h \leq n$, i.e. each such max-separable function has at least one and at most n different threshold values.

Example 2.2.1
Let $n = 4,\ J = \{1, 2, 3, 4\},\ \phi(x) = \max_{j \in J}(x_j),\ \tilde{x} = (1, 1, 3, 5)$. *We obtain the following theshold values of function* ϕ *at point* $\tilde{x}$*:*

$$t_1(\tilde{x}) = 5,\ t_2(\tilde{x}) = 3,\ t_3(\tilde{x}) = 1$$

We have in this case $h = 3$*,* $P_1(\tilde{x}) = \{4\},\ P_2(\tilde{x}) = \{3\},\ P_3(\tilde{x}) = \{1\},\ H_1(\tilde{x}) = \{4\},\ H_2(\tilde{x}) = \{3, 4\},\ H_3(\tilde{x}) = \{1, 2, 3, 4\}$.

2.3 Algorithm

In this section we propose a finite algorithm for finding the optimal solution of PROBLEM I.

Let $t_k^f(x)$ denote threshold values of the objective function f, and let similarly $t_r^{a_i}(x),\ t_s^{b_i}(x),\ \forall i \in I$ be threshold values of functions $a_i,\ b_i$. The corresponding sets of indices of active variables in the threshold values will be denoted $P_k^f(x),\ P_r^{a_i}(x),\ P_s^{b_i}(x)$ respectively. Note that $k,\ r,\ s$ are always smaller or equal

to n.
We will assume further that using the algorithm of [5] we found out that $M(\overline{x}, \underline{x}) \neq \emptyset$ and that we have at our disposal maximum element $x^{\max} \in M(\overline{x}, \underline{x})$. The main idea of the proposed algorithm consists in successive decreasing active variables in thresholds of the objective function f without leaving the feasible set until we cannot decrease the objective function without violating the constraint $\underline{x} \leq x$.

Algorithm 2.3.1

$\boxed{0}$ $\tilde{x} := x^{\max}$;

$\boxed{1}$ $V(\tilde{x}) := P_1^f(\tilde{x})$, *find* $P_1^{a_i}(\tilde{x}),\ P_1^{b_i}(\tilde{x})\ \forall i \in I$;

$\boxed{2}$ $I^1(\tilde{x}) := \{i \in I\ ;\ P_1^{a_i}(\tilde{x}) \not\subseteq V(\tilde{x})\ \&\ P_1^{b_i}(\tilde{x}) \subseteq V(\tilde{x})\}$;
$I^2(\tilde{x}) := \{i \in I\ ;\ P_1^{a_i}(\tilde{x}) \subseteq V(\tilde{x})\ \&\ P_1^{b_i}(\tilde{x}) \not\subseteq V(\tilde{x})\}$;
$I^3(\tilde{x}) := \{i \in I\ ;\ P_1^{a_i}(\tilde{x}) \subseteq V(\tilde{x})\ \&\ P_1^{b_i}(\tilde{x}) \subseteq V(\tilde{x})\}$;

$\boxed{3}$ $V_1(\tilde{x}) := V(\tilde{x}) \cup \bigcup_{i \in I^1(\tilde{x})} P_1^{a_i}(\tilde{x}) \cup \bigcup_{i \in I^2(\tilde{x})} P_1^{b_i}(\tilde{x})$;

$\boxed{4}$ *If* $V_1(\tilde{x}) \neq V(\tilde{x})$, *set* $V(\tilde{x}) := V_1(\tilde{x})$, *go to* $\boxed{2}$;

$\boxed{5}$ $x_j(t); = \tilde{x}_j - t\ \ \forall j \in V(\tilde{x}),\ \ x_j(t) := \tilde{x}_j$ *otherwise;*

$\boxed{6}$ $\alpha_i(\tilde{x}) := \max_{j \in (J \setminus V(\tilde{x}))}(a_{ij} + \tilde{x}_j)\ \forall i \in I^{13} \equiv I^1(\tilde{x}) \cup I^3(\tilde{x})$,
$\beta_i(\tilde{x}) := \max_{j \in (J \setminus V(\tilde{x}))}(b_{ij} + \tilde{x}_j)\ \forall i \in I^{23} \equiv I^2(\tilde{x}) \cup I^3(\tilde{x})$;
$\gamma(\tilde{x}) := \max_{j \in (J \setminus V(\tilde{x}))} f_j(\tilde{x}_j)$;

$\boxed{7}$ $\tau^{(1)} := \min_{i \in I^{13}}(a_i(\tilde{x}) - \alpha_i(\tilde{x}))$,
$\tau^{(2)} := \min_{i \in I^{23}}(b_i(\tilde{x}) - \beta_i(\tilde{x}))$,
$\tau^{(3)} := \max_{j \in P_1^f(\tilde{x})}(\tilde{x}_j - f_j^{-1}(\gamma(\tilde{x}))$,
$\tau^{(4)} := \min_{j \in V(\tilde{x})}(\tilde{x}_j - \underline{x}_j)$,
$\tau := \min_{1 \leq v \leq 4} \tau^{(v)}$;

$\boxed{8}$ *If* $f(x(\tau)) < f(\tilde{x})$, *Set* $\tilde{x} := x(\tau)$, *go to* $\boxed{1}$;

$\boxed{9}$ *Set* $x_{opt} := x(\tau)$, *STOP;*

Remark 2.3.1
Symbol t in step $\boxed{5}$ denotes a nonnegative parameter, which is increased until a new threshold value is reached. Symbols $\tau^{(1)}, \tau^{(2)}$ denote the value of t, at which the active variables of $a_i(\tilde{x}), b_i(\tilde{x})$ reach the next threshold yielded by non-decreased variables $\tilde{x}_j,\ j \in J \setminus V(\tilde{x})$. Symbols $\tau^{(3)}$ denotes the value of t, at which $f_j(x_j(t)) \leq \gamma(\tilde{x})$ for all active variables of $f(\tilde{x})$. Symbols $\tau^{(4)}$ denotes the value of t, at which for the first time some decreased variable reaches its lower bound i.e. for the first time $x_j(t) = \underline{x}_j$ for some $j \in V(\tilde{x})$

Remark 2.3.2
We used the following convention in algorithm 2.3.1: the maximum over the empty set is defined as $-\infty$ and if e.g. $\alpha(\tilde{x}) = -\infty$ because of $J \setminus V(\tilde{x}) = \emptyset$, we set $\tau^{(1)} = +\infty$. Since $\tau^{(4)}$ is always finite, the algorithm ends with a finite optimal value of the objective function.

Remark 2.3.3
The complexity of algorithm 2.3.1 depends in general on the behaviour of the objective function. If the objective function is (max,+)-linear, the complexity can be estimated as follows:
The maximal number of thresholds (threshold values) of functions $a_i(x),\ b_i(x)$, $\forall i \in I$ is $2mn$ and the evaluation of the threshold values for each $i \in I$ requires $O(mn)$ operations, which makes together compledxity $O(m^2n^2)$. The other steps have complexity $O(n)$ so that the resulting complexity of algorithm 2.3.1 is $O(m^2n^2)$.

We will illustrate the perfomance of algorithm 2.3.1 by the following small numerical example.

Example 2.3.1
Let $m = 3,\ n = 4$, matrices $A,\ B$ will be defined as follows:

$$A = \begin{pmatrix} 4 & 3 & 0 & 2 \\ 5 & -1 & 6 & 3 \\ 7 & 3 & 0 & 4 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 2 & 6 & 1 \\ 5 & 5 & 0 & 3 \\ 2 & 10 & 6 & 3 \end{pmatrix}$$

We will assume that $\underline{x} = (1,0,1,1), \overline{x} = (24,21,22,26)$ *and* $f(x) = \max(x_1, x_2, x_3, x_4)$, $(i.e. f_j(x_j) = x_j \ \forall \ j \in J = \{1,2,3,4\})$.

Iteration 1:

$\boxed{0}$ $\tilde{x} = (24,21,22,26)$;

$\boxed{1}$ $V(\tilde{x}) = \{4\}$;

$\boxed{2}$ $I^r(\tilde{x}) = \emptyset$ *for* $r = 1,2,3$;

$\boxed{3}$, $\boxed{4}$ $V_1(\tilde{x}) = \{4\}$;

$\boxed{5}$ $x(t) = (24,21,22,26-t)$;

$\boxed{6}$ $\gamma(\tilde{x}) = 24, \ \alpha(\tilde{x}) = \beta(\tilde{x}) = -\infty$;

$\boxed{7}$ $\tau^{(1)} = \tau^{(2)} = \infty, \ \tau^{(3)} = 2, \ \tau^{(4)} = 21, \ \tau = 2$;

$\boxed{8}$ $\tilde{x} = (24,21,22,24)$, *go to* $\boxed{1}$;

Iteration 2:

$\boxed{1}$ $V(\tilde{x}) = \{1,4\}$;

$\boxed{2}$ $I^1(\tilde{x}) = \emptyset, \ I^2(\tilde{x}) = \{3\}, \ I^3(\tilde{x}) = \{2\}$;

$\boxed{3}$ $V_1(\tilde{x}) = \{1,2,4\}$;

$\boxed{4}$ $V(\tilde{x}) = \{1,2,4\}$;

$\boxed{2}$ $I^1(\tilde{x}) = \emptyset, \ I^2(\tilde{x}) = \{1,3\}, \ I^3(\tilde{x}) = \{2\}$;

$\boxed{3}$ $V_1(\tilde{x}) = \{1,2,3,4\}$;

$\boxed{4}$ $V(\tilde{x}) = \{1,2,3,4\}$;

$\boxed{2}$ $I^1(\tilde{x}) = \ I^2(\tilde{x}) = \emptyset, \ I^3(\tilde{x}) = \{1,2,3\}$;

$\boxed{3}$ $V_1(\tilde{x}) = \{1,2,3,4\}$;

$\boxed{5}$ $x(t) = (24-t, 21-t, 22-t, 24-t)$;

$\boxed{6}$ $\alpha(\tilde{x}) = \beta(\tilde{x}) = \gamma(\tilde{x}) = -\infty$;

$\boxed{7}$ $\tau^{(v)} = \infty$ *for* $v = 1,2,3, \ \tau^{(4)} = 21$;

$\boxed{8}$ $\tilde{x} = (3,0,1,3)$, *go to* $\boxed{1}$;

Iteration 3:

$\boxed{1}$ $v(\tilde{x}) = \{1,4\}$;

$\vdots$

$\boxed{4}$ $V(\tilde{x}) = \{1,2,3,4\}, \tilde{x}_2 = \underline{x}_2 = 0$;

$\vdots$

$\boxed{8}$ *If* $f(x(\tau)) = f(\tilde{x}) = 3$;

$\boxed{9}$ $x^{opt} = \tilde{x} = (3, 0, 1, 3)$, *STOP.*

We obtained the optimal value $f(x^{opt}) = 3$.

We can easily verify that $x^{\max}$ *is a feasible solution:* $a_1(x^{\max}) = b_1(x^{\max}) = 7$, $a_2(x^{\max}) = b_2(x^{\max}) = 8$, $a_3(x^{\max}) = b_3(x^{\max}) = 10$, *and inequality* $x^{\max} \geq \underline{x}$ *is fulfilled.*

The following example shows one possible application of the the optimization problem considered in this section.

Example 2.3.2

Let one group of passangers be transported from places $P_j,\ j \in J$ *to places* $R_i,\ i \in I$;

Let another group be transported from places $Q_k,\ k \in K$ *to the same places* $R_i,\ i \in I$;

Let a_{ij} *be traveling times from* P_j *to* R_i, *and let* b_{ik} *be travelling times from* Q_k *to* R_i. *We require to determine departure times* $x_j,\ j \in J$, $y_k\ , k \in J$ *such that the last passangers of both groups meet in* R_i *at the same time. It means*

$$\max_{j \in J}(a_{ij} + x_j) = \max_{k \in J}(b_{ik} + y_k) \quad \forall i \in I.$$

We require additionally that $\underline{x} \leq x \leq \overline{x},\ \underline{y} \leq y \leq \overline{y}$. *By introducing new variables* $z = (x, y)$ *and appropriate sufficiently small coefficients* a_{ij} *for* $j > n$ *and sufficiently small coefficients* b_ik *for* $k < n$ *we obtain the system*

$$\max_{j \in K}(a_{ij} + z_j) = \max_{j \in K}(b_{ij} + z_j) \quad \forall i \in I,\ \underline{z} \leq z \leq \overline{z},$$

where $K = \{1,\ 2,\ \ldots,\ 2n\}$. *This system has the same form as the system, which describes the set of feasible solutions of PROBLEM I. We assume that there is given a panalty function* $f_j(z_j)$ *for each time* $z_j,\ j \in K$ *and require that the maximum of the penalties* $f_j(z_j)$ *is minimized. Such problem can be solved by the algorithm described above.*

3

Optimization Problems under One-Sided $(\max, \min)$-Linear Inequality Constraints

In this chapter, we will begin our studies by studying one-sided $(\max, \min)$-linear systems of inequalities where the unknowns appear in the left side only of inequalities and on the right side of these systems of inequalities we have constant variables only. Here we will provide an algorithm, which determines whether the set of all feasible solutions is empty or not, and if the set of feasible solutions is not empty this algorithm finds the maximum element of the set of all feasible solutions. Also, we will extend our studies to study one-sided $(\max, \min)$-linear systems of inequalities if there are another boundary conditions on the variables in the left side of the system of inequalities and we will modify algorithm for approval the existence boundary conditions as we will see in detail in the following. Also in this chapter we study an optimization problems under one-sided $(\max, \min)$-linear inequality constraints and we introduce an algorithm, which finds an optimal solution of these optimization problems under one-sided $(\max, \min)$-linear inequality constraints under the assumption that the set of all feasible solutions of one-sided $(\max, \min)$-linear systems of inequalities is not empty. We bring also some motivating examples from the area of operations research as well as examples illustrating the numerical performance of these algorithms.

3.1 One-Sided $(\max, \min)$-Linear Systems of Inequalities

Let us introduce the following notations:

$J = \{1, \ldots, n\}$, $I^{(1)} = \{1, \ldots, m\}$, $I^{(2)} = \{m+1, \ldots, k\}$, where n, m and k are integer numbers, $R = (-\infty, \infty)$, $\overline{R} = R \cup \{-\infty, \infty\}$,

$R^n = R \times \cdots \times R$ (n-times), similarly $\overline{R}^n = \overline{R} \times \cdots \times \overline{R}$, $x = (x_1, \ldots, x_n) \in \overline{R}^n$, $\alpha \wedge \beta = \min\{\alpha, \beta\}$, $\alpha \vee \beta = \max\{\alpha, \beta\}$ for any $\alpha, \beta \in \overline{R}$, we set per definition $-\infty \wedge \infty = -\infty$, $-\infty \vee \infty = \infty$,

$a_{ij} \in R$, $b_i \in R$, $\forall\, i \in I$, $j \in J$ are given finite numbers,
In what follows we will consider the following system of inequalities:

$$\max_{j \in J}(a_{ij} \wedge x_j) \geq b_i,\ i \in I^{(1)}, \tag{3.1}$$

$$\max_{j \in J}(a_{ij} \wedge x_j) \leq b_i,\ i \in I^{(2)}, \tag{3.2}$$

where $I^{(1)} \cup I^{(2)} = I$.

The set of all solutions of system (3.1) and (3.2), will be denoted M. Before investigating properties of set M, we will bring an example, which shows one possible application, which leads to solving the system given above.

Example 3.1.1

Let us assume that m places $i \in I^{(1)} \equiv \{1,\ 2,\ \ldots,\ m\}$ are connected with n places $j \in J \equiv \{1,\ 2,\ \ldots,\ n\}$ by roads with given capacities. The capacity of the road connecting place i with place j is equal to $a_{ij} \in R$. We have to extend for all $i \in I$, $j \in J$ the road between i and j by a road connecting j with a terminal place T and choose an appropriate capacity x_j for this road. If a capacity x_j is chosen, then the capacity of the road from i to T via j is equal to $a_{ij} \wedge x_j = \min(a_{ij}, x_j)$. We require that the connection between places i and T is for at least one j greater or equal to a given number $b_i \in R$ and the chosen *capacity* x_j *lies in a given finite interval i.e.* $x_j \in [\underline{x}_j, \overline{x}_j]$, *where* $\underline{x}_j, \overline{x}_j \in R$ *are given finite numbers. Therefore feasible vectors of capacities* $x = (x_1,\ x_2,\ \ldots,\ x_n)$ *(i.e. the vectors, the components of which are capacities* x_j *having the required properties) must satisfy system (3.1).*

In what follows, we will investigate some properties of set M described by system (3.1), (3.2). Also, to simplify the formulas in what follows we will set

$$a_i(x) \ \equiv \ \max_{j \in J}(a_{ij} \wedge x_j) \quad \text{for all } \ i \in I\,,$$

Let us note that for any fixed $i \in I^{(2)}$ the inequality

$$a_i(x) = \max_{j \in J}(a_{ij} \wedge x_j) \leq b_i$$

implies $a_{ij} \wedge x_j \leq b_i \ \ \forall j \in J$.

Lemma 3.1.1 *Let us set for all* $i \in I^{(2)}$

$$V_{ij} = \{x_j\ ;\ (a_{ij} \wedge x_j) \leq b_i \ \&\ x_j \in R = (-\infty, \infty)\}$$

For any fixed $i \in I^{(2)}$ *and* $j \in J$, *the following statements hold:*
$V_{ij} = (-\infty, \infty)$ *if* $a_{ij} \leq b_i$*;*
$V_{ij} = (-\infty, b_i]$ *if* $a_{ij} > b_i$*;*

Proof:
If $a_{ij} \leq b_i$, then $a_{ij} \wedge x_j = a_{ij} \leq b_i$ for arbitrary x_j.
If $a_{ij} > b_i$, then $a_{ij} \wedge x_j \leq b_i$ if and only if $x_j \leq b_i$.

□

It follows from Lemma 3.1.1 that

$$x \in M \ \Rightarrow \ x_j \in \bigcap_{i \in I^{(2)}} V_{ij} \ \ \forall j \in J.$$

Therefore if for any $i \in I^{(2)},\ j \in J$ set V_{ij} is not empty, then $M \neq \emptyset$. In other words, from the above lemma we can find that $x \in M$ is bounded from above by $\overline{x}$, which specifies from the previous discussion.
Then the set of all solutions of system of inequalities (3.1) and (3.2), M can be described as follows:

$$a_i(x) = \max_{j \in J}(a_{ij} \wedge x_j) \geq b_i,\ i \in I^{(1)}, \tag{3.3}$$

$$x \leq \overline{x} \tag{3.4}$$

we can specify $\overline{x}$ from the previous discussion as follows:

$$\overline{x}_j = min_{i \in I_j^{(3)}} b_i$$

where $I_j^{(3)} = \left\{i; i \in I^{(2)}\ \& a_{ij} > b_i\right\}$ for all $j \in J$ and we will set the minimum equal to ∞ if $I_j^{(3)} = \emptyset$.
Now it is appropriate to define

$$M(\overline{x}) = \{x\ ;\ x \in M \& x \leq \overline{x}\}$$

Let $\overline{x}_{ij}$ denote the upper bound of any nonempty set V_{ij}, i.e. $\overline{x}_{ij} = -\infty$ if $a_{ij} \leq b_i$ and $\overline{x}_{ij} = b_i$ if $a_{ij} > b_i$, then $x_j \leq \min_{i \in I^{(2)}} \overline{x}_{ij}$ for all $j \in J$. So that we replace system (3.2) by introducing new upper bounds $x_j^{\max} \equiv \min_{i \in I^{(2)}} \overline{x}_{ij}$ for all $j \in J$. Let $M(\overline{x})$ is nonempty set and $\overline{x}$ be defined as above, the element $\overline{x}$ will be the maximum element of $M(\overline{x})$; we denote further this element as $x^{\max}$, which satisfy relations (3.3), (3.4), i.e. if x is any element satisfying (3.3), (3.4), then $x \leq x^{\max}$. In what follows, we will solve system (3.3) and (3.4) taking into account that if $x \nleq \overline{x}$, then some inequalities of (3.2) are not satisfied.

Lemma 3.1.2 *Let us define sets $T_{ij},\ i \in I^{(1)},\ j \in J$ as follows:*

$$T_{ij} \equiv \{x_j\ ;\ a_{ij} \wedge x_j \geq b_i\}.$$

For any fixed i, j the following equalities hold:

$T_{ij} = [b_i, \overline{x}_j]$ *if* $a_{ij} \geq b_i$ & $b_i \leq \overline{x}_j$*;*
$T_{ij} = \emptyset$ *otherwise, i.e. if either* $a_{ij} < b_i$ *or* $b_i > \overline{x}_j$*.*

Proof:
Let $a_{ij} \geq b_i$ and $b_i \leq \overline{x}_j$. Then $[b_i, \overline{x}_j] \neq \emptyset$ and for any $x_j \in [b_i, \overline{x}_j]$ we have $a_{ij} \wedge x_j \geq x_j \geq b_i$, which proves that $T_{ij} = [b_i, \overline{x}_j]$.
Let us assume now that either $a_{ij} < b_i$ or $b_i > \overline{x}_j$. Then we have either $a_{ij} \wedge x_j < b_i$ or $a_{ij} \wedge x_j \geq b_i > \overline{x}_j$ so that set T_{ij} must be empty.

□

Lemma 3.1.3 *For any pair of indices* $i_1, i_2 \in I^{(1)},\ i_1 \neq i_2$ *and arbitrary* $j \in J$ *either* $T_{i_1 j} \subseteq T_{i_2 j}$ *or* $T_{i_2 j} \subseteq T_{i_1 j}$ *holds .*

Proof:
If one of the sets $T_{i_1 j},\ T_{i_2 j}$ is empty, the assertion is evident. Let us assume that both sets are nonempty so that we have according to Lemma 3.1.2 $T_{i_r j} = [b_{i_r}, \overline{x}_j]$ for $r = 1, 2$. We can assume w.l.o.g. that $(b_{i_1} \geq (b_{i_2})$. Then $T_{i_1 j} \subseteq T_{i_2 j}$.

□

As a consequence of Lemma 3.1.3 we obtain that for any fixed $j \in J$ there exists a permutation of indices $\{i_1, \ \ldots, \ i_m\}$ of set $I^{(1)}$ such that the inclusions $T_{i_1 j} \subseteq T_{i_2 j} \ldots, \subseteq T_{i_m, j}$ hold. In the sequel, we will call this property of sets T_{ij} "chain property".

Lemma 3.1.4

$$M(\overline{x}) \neq \emptyset \quad \Leftrightarrow \quad \forall i \in I^{(1)} \ \exists \ \ j(i) \in J, \quad \textit{such that} \ \ T_{ij(i)} \neq \emptyset.$$

Proof:
Let $M(\overline{x}) \neq \emptyset$ and x be an arbitrary element of $M(\overline{x})$. Let $i \in I$ be arbitrarily chosen. Then we have:

$$\max_{j \in J}(a_{ij} \wedge x_j) = a_{ij(i)} \wedge x_{j(i)} \geq b_i$$

so that $x_{j(i)} \in T_{ij(i)}$ and therefore $T_{ij(i)} \neq \emptyset$.
Let now $\forall \ i \in I^{(1)} \ \exists \ j(i) \in J$ such that $T_{ij(i)} \neq \emptyset$ and let us consider the element $\overline{x}$. Let $i \in I$ be arbitrary, then $\overline{x}_j \in T_{ij}$ for any nonempty set T_{ij}. Therefore we obtain according to the definition of $T_{ij(i)}$ that

$$\max_{j \in J}(a_{ij} \wedge \overline{x}_j) \geq a_{ij(i)} \wedge \overline{x}_{j(i)} \geq b_i.$$

Therefore $\overline{x} \in M(\overline{x})$ and thus $M(\overline{x}) \neq \emptyset$. Let $T_k = \{i \in I \mid j(i) = k\} \ \forall \ k \in J$, choose $\tilde{x}_k \in T_k$ if $T_k \neq \emptyset$, $\tilde{x}_k \leq \overline{x}_k$ otherwise and prove that $\tilde{x} \in M(\overline{x})$ so that $M(\overline{x}) \neq \emptyset$.

□

As a consequence of Lemma 3.1.4 we obtain:

$$M(\overline{x}) = \emptyset \ \Leftrightarrow \ \exists \ i \in I^{(1)} \ \forall \ j \in J \ \ T_{ij} = \emptyset.$$

It follows further that if $M(\overline{x}) \neq \emptyset$, then $\overline{x} = x^{\max}$ is the maximum element of set $M(\overline{x})$.

Algorithm 3.1.1 *We will provide algorithm, which summarizes the above discussion and determines whether $M(\overline{x}) = \emptyset$ or finds the maximum element of set $M(\overline{x}) \neq \emptyset$*

[0] *Input $I^{(1)}$, $I^{(2)}$, J, a_{ij} and b_i for all $i \in I^{(1)} \cup I^{(2)}$ and $j \in J$*

[1] *For all $j \in J$ set $I_j^{(3)} = \left\{i; i \in I^{(2)} \ \& a_{ij} > b_i\right\}$*

[2] *$\overline{x}_j = min_{i \in I_j^{(3)}} b_i$ if $I_j^{(3)} \neq \emptyset$ or $\overline{x}_j = \infty$ if $I_j^{(3)} = \emptyset$;*

[3] *For all $i \in I^{(1)}$ and $j \in J$ set:*
$T_{ij} = [b_i, \overline{x}_j]$ if $a_{ij} \geq b_i$ & $b_i \leq \overline{x}_j$;
$T_{ij} = \emptyset$ otherwise, i.e. if either $a_{ij} < b_i$ or $b_i > \overline{x}_j$

[4] *If there exists $i \in I^{(1)}$ such that $T_{ij} = \emptyset$ for all $j \in J$, then $M(\overline{x}) = \emptyset$, STOP;*
Otherwise $\overline{x}$ is the maximum element of set $M(\overline{x})$ STOP;

We will illustrate the performance of the Algorithm 3.1.1 by the following small numerical example.

Example 3.1.2
Let $J = \{1, 2, 3\}$, $I^{(1)} = \{1, 2, 3\}$, $I^{(2)} = \{4, 5, 6, 7\}$, and consider the following system of inequalities:

$$\max(3 \wedge x_1, 2 \wedge x_2, 1 \wedge x_3) \geq 1,$$

$$\max(2 \wedge x_1, 5 \wedge x_2, 4 \wedge x_3) \geq 3,$$

$$\max(5 \wedge x_1, 1 \wedge x_2, 3 \wedge x_3) \geq 2,$$

$$\max(6 \wedge x_1, 5 \wedge x_2, 1 \wedge x_3) \leq 5,$$

$$\max(3 \wedge x_1, 9 \wedge x_2, 7 \wedge x_3) \leq 6,$$

$$\max(9 \wedge x_1, 10 \wedge x_2, 3 \wedge x_3) \leq 8,$$

$$\max(8 \wedge x_1, 2 \wedge x_2, 11 \wedge x_3) \leq 7,$$

$V_{41} = (-\infty, 5]$, $V_{42} = (-\infty, \infty)$, $V_{43} = (-\infty, \infty)$;
$V_{51} = (-\infty, \infty)$, $V_{52} = (-\infty, 6]$, $V_{53} = (-\infty, 6]$;
$V_{61} = (-\infty, 8]$, $V_{62} = (-\infty, 8]$, $V_{63} = (-\infty, \infty)$;
$V_{71} = (-\infty, 7]$, $V_{72} = (-\infty, \infty)$, $V_{73} = (-\infty, 7]$;
$I_1^{(3)} = \{4, 6, 7\}$ $I_2^{(3)} = \{5, 6\}$ $I_3^{(3)} = \{5, 7\}$
Then we can find that $\overline{x} = (5, 6, 6)$
$T_{11} = [1, 5]$, $T_{12} = [1, 6]$, $T_{13} = [1, 6]$;
$T_{21} = \emptyset$, $T_{22} = [3, 6]$, $T_{23} = [3, 6]$;
$T_{31} = [2, 5]$, $T_{32} = [2, 6]$, $T_{33} = \emptyset$.

We find $\cup_{j\in J} T_{ij} \neq \emptyset$ *for all* $i \in I^{(1)}$. *Then* $\overline{x} = (5, 6, 6)$ *is the maximum element of set* $M(\overline{x})$.

Now we will consider the system of inequalities (3.1) and (3.2) with the upper and lower bounds $\overline{x}$ and $\underline{x}$ respectively. Let $M(\underline{x}, \overline{x})$ be the set of all solutions of system (3.1) and (3.2) with the upper and lower bounds $\overline{x}$ and $\underline{x}$ respectively. In what follows, we will investigate some properties of set $M(\underline{x}, \overline{x})$.

Remark 3.1.1
In fact the inequalities $\underline{x} \leq x \leq \overline{x}$ *could be included in the original system by introducing new inequalities with appropriately chosen coefficients of the left-hand sides and appropriately chosen additional* $b_i's$ *in the right-hand sides. i.e. instead of requiring* $x_1 \geq \underline{x}_1$ *we can include in the system additional inequality:*

$$a_{m+1}(x) \geq b_{m+1}$$

where $a_{m+1}(x) \equiv \max_{j\in J}(a_{m+1j} \wedge x_j)$ *and* $a_{m+11} = \infty$, $a_{m+1j} = -\infty, j = 2, ..., n$ *and* $b_{m+1} = \underline{x}_1$, *so that this inequality is equivalent to* $x_1 \geq \underline{x}_1$.
Similarly we can proceed with other $\geq$, $\leq$*-inequalities. Such a procedure requires including additionally* m $\geq$*-inequalities and* m $\leq$*-inequalities, which is a disadvantage of such procedure. Therefore we prefer to take into account inequalities* $\underline{x} \leq x \leq \overline{x}$ *explicitly in the process of finding the maximum element* x^{max}. *Such approach simplifies (or shortens) the necessary computations and memory requirements. In what follows we will introduce the procedures to find the maximum element* x^{max} *of the system of inequalities (3.1) and (3.2) with the lower and upper bounds* $\underline{x}$ *and* $\overline{x}$ *respectively.*

Lemma 3.1.5 *Let us set for all* $i \in I^{(2)}$

$$V_{ij} = \{x_j \ ; \ (a_{ij} \wedge x_j) \leq b_i \ \& \ \underline{x}_j \leq x_j \leq \overline{x}_j\}$$

For any fixed i, j *the following equalities hold:*
$V_{ij} = [\underline{x}_j, \overline{x}_j]$ *if* $a_{ij} \leq b_i$;
$V_{ij} = [\underline{x}_j, \overline{x}_j \wedge b_i]$ *if* $a_{ij} > b_i \ \& \ b_i \geq \underline{x}_j$;
$V_{ij} = \emptyset$ *if* $a_{ij} > b_i \ \& \ b_i < \underline{x}_j$.

Proof:
If $a_{ij} \le b_i$, then $a_{ij} \wedge x_j = a_{ij} \le b_i$ for arbitrary x_j.
If $a_{ij} > b_i$ & $b_i \ge \underline{x}_j$, then $a_{ij} \wedge x_j \le b_i$ if and only if $x_j \le b_i$ & $x_j \le \overline{x}_j$.
If $a_{ij} > b_i$ & $b_i < \underline{x}_j$, we obtain: $a_{ij} \wedge x_j \le b_i$ if and only if $x_j \le b_i < \underline{x}_j$ so that in this case $x_j \notin [\underline{x}_j, \overline{x}_j]$ and therefore $V_{ij} = \emptyset$.

□

It follows that from Lemma 3.1.5 that

$$x \in M(\underline{x}, \overline{x}) \;\Rightarrow\; x_j \in \bigcap_{i \in I^{(2)}} V_{ij} \;\; \forall j \in J.$$

Therefore if for any $i \in I^{(2)},\ j \in J$ set V_{ij} is empty, then $M(\underline{x}, \overline{x}) = \emptyset$. Or in other words, if there exist indices $i \in I^{(2)},\ j \in J$ such that $a_{ij} > b_i$ & $b_i < \underline{x}_j$, then $M(\underline{x}, \overline{x}) = \emptyset$.
Let $\overline{x}_{ij}$ denote the upper bound of any nonempty set V_{ij}, i.e. $\overline{x}_{ij} = \overline{x}_j$ if $a_{ij} \le b_i$ and $\overline{x}_{ij} = \overline{x}_j \wedge b_i$ if $a_{ij} > b_i$ & $b_i \ge \underline{x}_j$, then $x_j \le \min_{i \in I^{(2)}} \overline{x}_{ij}$ for all $j \in J$. It follows that we can replace system (3.2) by introducing new upper bounds $x_j^{\max} \equiv \min_{i \in I^{(2)}} \overline{x}_{ij}$ for all $j \in J$.
Element $x^{\max}$ is the maximum element satisfying relations (3.1), with the upper and lower bounds $\overline{x}$ and $\underline{x}$ respectively, i.e. if x is any element satisfying (3.1), with the upper and lower bounds $\overline{x}$ and $\underline{x}$ respectively, then $x \le x^{\max}$. We will redefine therefore the upper bound setting $\overline{x} = x^{\max}$ and consider only the subsystem (3.1), with the upper and lower bounds $\overline{x}$ and $\underline{x}$ Respectively, with this new upper bound taking into account that if $x \not\le x^{\max}$, relations (3.2) do not hold.

Lemma 3.1.6 *Let us define sets $T_{ij},\ i \in I^{(1)},\ j \in J$ as follows:*

$$T_{ij} \equiv \{x_j \;;\; a_{ij} \wedge x_j \ge b_i \;\&\; \underline{x}_j \le x_j \le \overline{x}_j\}.$$

For any fixed i, j the following equalities hold:
$T_{ij} = [b_i \vee \underline{x}_j, \overline{x}_j]$ if $a_{ij} \ge b_i$ & $b_i \le \overline{x}_j$;
$T_{ij} = \emptyset$ otherwise, i.e. if either $a_{ij} < b_i$ or $b_i > \overline{x}_j$.

Proof:
Let $a_{ij} \geq b_i$ and $b_i \leq \overline{x}_j$. Then $[b_i \vee \underline{x}_j, \overline{x}_j] \neq \emptyset$ and for any $x_j \in [b_i \vee \underline{x}_j, \overline{x}_j]$ we have $a_{ij} \wedge x_j \geq (b_i \vee \underline{x}_j) \geq b_i$, which proves that $T_{ij} = [b_i \vee \underline{x}_j, \overline{x}_j]$.
Let us assume now that either $a_{ij} < b_i$ or $b_i > \overline{x}_j$. Then we have either $a_{ij} \wedge x_j < b_i$ or $a_{ij} \wedge x_j \geq b_i > \overline{x}_j$ so that set T_{ij} must be empty.

□

As a consequence of Lemma 3.1.3 we obtain that for any fixed $j \in J$ there exists a permutation of indices $\{i_1, \ \ldots, \ i_m\}$ of set $I^{(1)}$ such that the inclusions $T_{i_1 j} \subseteq T_{i_2 j} \ldots, \ \subseteq T_{i_m,j}$ hold. In the sequel, we will call this property of sets T_{ij} "chain property".
In the same way as a consequence of Lemma 3.1.4 we can introduce the next lemma.

Lemma 3.1.7

$$M(\underline{x}, \overline{x}) \neq \emptyset \quad \Leftrightarrow \quad \forall \ \ i \in I^{(1)} \ \ \exists \ \ j(i) \in J \ \ \textit{such that} \ \ T_{ij} \neq \emptyset.$$

Proof:
Let $M(\underline{x}, \overline{x}) \neq \emptyset$ and assume that $\forall \ \ i \in I^{(1)} \ \ \exists \ \ j(i) \in J$ such that $T_{ij} \neq \emptyset$ is not fulfilled, i.e. $\exists \ \ i_0 \in I$ such that $T_{i_0 j} = \emptyset \ \ \forall \ \ j \in J$. Then for any $x \in R^n$, for each $j \in J$ either $x_j > \overline{x}_j$ or $a_{i_0 j} \wedge x_j < b_{i_0}$. If $x_j > \overline{x}_j$, then at least one inequality of (3.4) is not satisfied and therefore $x \notin M(\underline{x}, \overline{x})$. If $x_j \leq \overline{x}_j$, for all $j \in J$ and $a_{i_0 j} \wedge x_j < b_{i_0}, \ \ \forall \ \ j \in J$, then $max_{j \in J}(a_{i_0 j} \wedge x_j) < b_{i_0}$ so that the i_0-th inequality of (3.3) is not satisfied so that again $x \notin M(\underline{x}, \overline{x})$. It follows that $M(\underline{x}, \overline{x}) = \emptyset$. We proved therefore non $\left[\forall \ \ i \in I^{(1)} \ \ \exists \ \ j(i) \in J \ \text{ such that } \ T_{ij} \neq \emptyset\right] \ \Rightarrow \ M(\underline{x}, \overline{x}) = \emptyset$ or in other words $M(\underline{x}, \overline{x}) \neq \emptyset \ \ \Rightarrow \left[\forall \ \ i \in I^{(1)} \ \ \exists \ \ j(i) \in J \ \text{ such that } \ T_{ij} \neq \emptyset\right]$ must be satisfied.
It is remains to prove the implication:

$$\left[\forall \ \ i \in I^{(1)} \ \ \exists \ \ j(i) \in J \ \text{ such that } \ T_{ij} \neq \emptyset\right] \ \Rightarrow \ M(\underline{x}, \overline{x}) \neq \emptyset.$$

We will show that $\overline{x} \in M(\underline{x}, \overline{x})$ if $[\ \forall \ \ i \in I^{(1)} \ \ \exists \ \ j(i) \in J$ such that $T_{ij} \neq \emptyset\ ]$ holds. Let $i_0 \in I$ be arbitrarily chosen so that according to implication

$\left[\forall \;\; i \in I^{(1)} \;\; \exists \;\; j(i) \in J\right.$ such that $\left.T_{ij} \neq \emptyset\right]$, we have $T_{i_0 j(i_0)} \neq \emptyset$. Note that it must be $\overline{x}_{j(i_0)} \in T_{i_0 j(i_0)}$ (otherwise it would be $T_{i_0 j(i_0)} = \emptyset$, since if $\overline{x}_{j(i_0)} \notin T_{i_0 j(i_0)}$, it must be $a_{i_0 j(i_0)} \wedge \overline{x}_{j(i_0)} < b_{i_0}$, and thus $a_{i_0 j(i_0)} \wedge x_{j(i_0)} < b_{i_0}$, for any $x_j \leq \overline{x}_j$). Therefore we obtain:

$$\max_{j \in J}(a_{i_0 j} \wedge \overline{x}_j) \geq a_{i_0 j(i_0)} \wedge \overline{x}_{j(i_0)} \geq b_{i_0}$$

since $i_0 \in I$ was arbitrarily chosen, we obtain that $\overline{x} \in M(\underline{x}, \overline{x})$ and therefore $M(\underline{x}, \overline{x}) \neq \emptyset$, which completes the proof.

□

It follows further that if $M(\underline{x}, \overline{x}) \neq \emptyset$, then $\overline{x} = x^{\max}$ is the maximum element of set $M(\underline{x}, \overline{x})$.

Algorithm 3.1.2 *We will provide algorithm, which summarizes the above discussion and determines whether $M(\underline{x}, \overline{x}) = \emptyset$ or finds the maximum element of set $M(\underline{x}, \overline{x}) \neq \emptyset$*

[0] *Input $I^{(1)}$, $I^{(2)}$, J, $\underline{x}, \overline{x}$, a_{ij} and b_i for all $i \in I^{(1)} \cup I^{(2)}$ and $j \in J$;*

[1] *For all $i \in I^{(2)}$ and $j \in J$ set*
$V_{ij} = [\underline{x}_j, \overline{x}_j]$ if $a_{ij} \leq b_i$;
$V_{ij} = [\underline{x}_j, \overline{x}_j \wedge b_i]$ if $a_{ij} > b_i$ & $b_i \geq \underline{x}_j$;
$V_{ij} = \emptyset$ if $a_{ij} > b_i$ & $b_i < \underline{x}_j$.

[2] *If $V_{ij} = \emptyset$ for any $i \in I^{(2)}$ and $j \in J$*
Then $M(\underline{x}, \overline{x})$ is empty set, STOP.

[3] *For all $j \in J$ and $i \in I^{(2)}$ set*
$\overline{x}_{ij} = \overline{x}_j$ if $a_{ij} \leq b_i$
$\overline{x}_{ij} = \overline{x}_j \wedge b_i$ if $a_{ij} > b_i$ & $b_i \geq \underline{x}_j$

[4] *$x_j^{max} = min_{i \in I_j^{(2)}} \overline{x}_{ij}$ For all $j \in J$;*

[5] *$\overline{x} = x^{max}$;*

$\boxed{6}$ *For all* $i \in I^{(1)}$ *and* $j \in J$ *set:*
$T_{ij} = \left[b_i \vee \underline{x}_j, \overline{x}_j\right]$ *if* $a_{ij} \geq b_i$ & $b_i \leq \overline{x}_j$*;*
$T_{ij} = \emptyset$ *otherwise, i.e. if either* $a_{ij} < b_i$ *or* $b_i > \overline{x}_j$

$\boxed{7}$ *If there exists* $i \in I^{(1)}$ *such that* $T_{ij} = \emptyset$ *for all* $j \in J$*, then* $M(\overline{x}) = \emptyset$*, STOP;*
Otherwise $\overline{x}$ *is the maximum element of set* $M(\overline{x})$ *STOP;*

We will illustrate the performance of the Algorithm 3.1.2 by the following small numerical example.

Example 3.1.3
Let $J = \{1, 2, 3\}$*,* $I^{(1)} = \{1, 2, 3\}$*,* $I^{(2)} = \{4, 5, 6\}$*,* $\underline{x} = (0, 0, 0)$ *and* $\overline{x} = (7, 7, 7)$ *and consider the following system of inequalities:*

$$\max(4 \wedge x_1, 3 \wedge x_2, 2 \wedge x_3) \geq 2,$$

$$\max(3 \wedge x_1, 5 \wedge x_2, 4 \wedge x_3) \geq 4,$$

$$\max(5 \wedge x_1, 1 \wedge x_2, 3 \wedge x_3) \geq 1,$$

$$\max(6 \wedge x_1, 5 \wedge x_2, 1 \wedge x_3) \leq 5,$$

$$\max(3 \wedge x_1, 5 \wedge x_2, 8 \wedge x_3) \leq 4,$$

$$\max(2 \wedge x_1, 4 \wedge x_2, 1 \wedge x_3) \leq 3,$$

$V_{41} = [0, 5]$, $V_{42} = [0, 7]$, $V_{43} = [0, 7]$*;*
$V_{51} = [0, 7]$, $V_{52} = [0, 4]$, $V_{53} = [0, 4]$
$V_{61} = [0, 7]$, $V_{62} = [0, 3]$, $V_{63} = [0, 7]$
Then we can find that $\overline{x} = (5, 3, 4)$
$T_{11} = [2, 5]$, $T_{12} = [2, 3]$, $T_{13} = [2, 4]$*;*

$T_{21} = \emptyset, \quad T_{22} = \emptyset, \quad T_{23} = \{4\};$
$T_{31} = [1,5], \quad T_{32} = [1,3], \quad T_{33} = [1,4].$
We find $\cup_{j \in J} T_{ij} \neq \emptyset$ *for all* $i \in I^{(1)}$. *Then* $\overline{x} = (5,3,4)$ *is the maximum element of set* $M(\underline{x}, \overline{x})$.

Example 3.1.4
Let $J = \{1,2,3\}$, $I^{(1)} = \{1,2,3,4,5\}$, $I^{(2)} = \{6,7,8,9\}$, $\underline{x} = (0,0,0)$ *and* $\overline{x} = (20,20,20)$ *and consider the following system of inequalities:*

$$\max(21 \wedge x_1, 16 \wedge x_2, 12 \wedge x_3) \geq 16,$$

$$\max(15 \wedge x_1, 16 \wedge x_2, 19 \wedge x_3) \geq 13,$$

$$\max(14 \wedge x_1, 13 \wedge x_2, 22 \wedge x_3) \geq 14,$$

$$\max(15 \wedge x_1, 15 \wedge x_2, 19 \wedge x_3) \geq 17,$$

$$\max(15 \wedge x_1, 15 \wedge x_2, 16 \wedge x_3) \geq 13,$$

$$\max(15 \wedge x_1, 26 \wedge x_2, 19 \wedge x_3) \leq 16,$$

$$\max(13 \wedge x_1, 15 \wedge x_2, 18 \wedge x_3) \leq 14,$$

$$\max(25 \wedge x_1, 17 \wedge x_2, 21 \wedge x_3) \leq 17,$$

$$\max(19 \wedge x_1, 18 \wedge x_2, 23 \wedge x_3) \leq 18,$$

$V_{61} = [0,20], \quad V_{62} = [0,16], \quad V_{63} = [0,16]$
$V_{71} = [0,20], \quad V_{72} = [0,14]\ , \quad V_{73} = [0,14];$
$V_{81} = [0,17], \quad V_{82} = [0,20], \quad V_{83} = [0,17]$

$V_{91} = [0, 18]$, $V_{92} = [0, 20]$, $V_{93} = [0, 18]$
Then we can find that $\overline{x} = (17, 14, 14)$
$T_{11} = [16, 17]$, $T_{12} = \emptyset$, $T_{13} = \emptyset$;
$T_{21} = [13, 17]$, $T_{22} = [13, 14]$, $T_{23} = [13, 14]$;
$T_{31} = [14, 17]$, $T_{32} = \emptyset$, $T_{33} = [14, 14]$.
$T_{41} = \emptyset$, $T_{42} = \emptyset$, $T_{43} = \emptyset$,.
$T_{51} = [13, 17]$, $T_{52} = [13, 14]$, $T_{53} = [13, 14]$;
We find $\cup_{j\in J} T_{4j} = \emptyset$ *which means that the inequality 4 does not satisfy, because* $a_{41} = 15 < b_4 = 17$, $a_{42} = 15 < b_4 = 17$ *and* $a_{43} = 19 > b_4 = 17$ *but* $b_4 = 17 > \overline{x}_3 = 14$. *Then the set* $M(\underline{x}, \overline{x})$ *is empty set.*

3.2 Solving Optimization Problems under One-Sided (max, min)-Linear Inequality Constraints

In this section we will solve the following optimization problem:

$$f(x) \equiv \max_{j\in J} f_j(x_j) \quad \longrightarrow \quad \min \tag{3.5}$$

subject to

$$\max_{j\in J}(a_{ij} \wedge x_j) \ \geq \ b_i, \ \forall i \in I, \tag{3.6}$$

$$\underline{x} \leq x \leq \overline{x}, \tag{3.7}$$

where $\underline{x}, \ \overline{x} \in R^n$, $I \equiv \{1, \ \dots, \ m\}$, $J \equiv \{1, \ \dots, \ ...n\}$, $a_{ij}, b_i \in R \ \forall i \in I, j \in J$ are given. We assume further that $f_j : R \to R$ are continuous functions, $M(\underline{x}, \overline{x})$ denotes the set of feasible solutions of the problem and $M(\underline{x}, \overline{x}) \neq \emptyset$ (note that the emptiness of set $M(\underline{x}, \overline{x})$ can be verified using the considerations of the preceding section). Let us note further that the formulation of the optimization problem (3.5), (3.6), (3.7) includes also the case of one-sided $\leq$-inequality constraints,which can be included by adjusting the upper bounds $\overline{x}_j, \ j \in J$ like in the preceding section and since each equality can be replaced by two inequalities $\leq, \ \geq$, the formulation (3.5), (3.6), (3.7) includes also (max, min)-linear equality constraints.

Let us define for all $i \in I,\ j \in J$ sets T_{ij} as follows:

$$T_{ij} \equiv \{x_j\ ;\ a_{ij} \wedge x_j \geq b_i\ \&\ x_j \in [\underline{x}_j, \overline{x}_j]\}.$$

Then we have similarly like in the preceding section:
$x \in M(\underline{x}, \overline{x})$ if and only if for each $i \in I$ there exists at least one $j(i) \in J$ such that $x_{j(i)} \in T_{ij(i)}$, or in other words x is a feasible solution of problem (3.5), (3.6), (3.7) if and only if for each fixed $i \in I$ either $x_1 \in T_{i1}$ or $x_2 \in T_{i2}$ or, ... or $x_n \in T_{in}$. Let us set for $i \in I,\ j \in J$

$$\tilde{T}_{ij} \equiv \{x = (x_1, \ldots, x_n)\ ;\ x_j \in T_{ij}\}.$$

Then we can replace problem (3.5), (3.6), (3.7) by the optimization problem

$$f(x) \equiv \max_{j \in J} f_j(x_j) \quad \longrightarrow \quad \min \tag{3.8}$$

subject to

$$x \in M(\underline{x}, \overline{x}) = \bigcap_{i \in I} \bigcup_{j \in J} \tilde{T}_{ij} \tag{3.9}$$

Let us introduce the following notations:

$$f_j(x_j^{(i)}) = \min_{x_j \in T_{ij}} (f_j(x_j)),$$

where we set $f_j(x_j^{(i)}) = \infty$ if $T_{ij} = \emptyset$;

$$f_{p(i)}(x_{p(i)}^{(i)}) = \min_{j \in J}(f_j(x_j^{(i)}));$$

$$R_k \equiv \{i \in I\ ;\ p(i) = k\} \quad \forall k \in J;$$

$$T_k \equiv \bigcap_{i \in R_k} T_{ik} \quad \forall k \in J;$$

Let $\hat{x}$ be defined as follows:

$$f_k(\hat{x}_k) \equiv \min_{x_k \in T_k} (f_k(x_k)),$$

for all $k \in J$ such that $R_k \neq \emptyset$;

$$f_k(\hat{x}_k) \equiv \min_{x_k \in [\underline{x}_k, \overline{x}_k]} f_k(x_k),$$

if $R_k = \emptyset$.

Lemma 3.2.1 *$\hat{x}$ is a feasible solution of problem (3.5), (3.6), (3.7).*

Proof:
Let us note that since sets T_{ij} have the chain property (see Lemma 3.1.3), there exists for any nonempty set $R_k,\ k \in J$ an index $i(k) \in R_k$ such that $T_k = T_{i(k)k}$ and therefore $T_{i(k)k} \ \subseteq T_{ik} \ \ \forall i \in R_k$. Since we assumed that $M(\underline{x}, \overline{x}) \neq \emptyset$, there exists according to Lemma 3.1.4 for each $i \in I$ at least one index $j(i) \in J$ such that T_{ij} is nonempty. Therefore there exists for each $i \in I$ an index $k(i) \in J$ such that $i \in R_{k(i)}$ or in other words and $\hat{x}_{k(i)} \in T_{k(i)} \subseteq T_{ik(i)}$.

$$a_i(\hat{x}) = \max_{j \in J}(a_{ij} \wedge \hat{x}_j) \geq a_{ik(i)} \wedge \hat{x}_{k(i)} \geq b_i.$$

Since $i \in I$ was arbitrarily chosen, we obtain that $\hat{x} \in M(\underline{x}, \overline{x})$, i.e. $\hat{x}$ is a feasible solution of problem (3.5), (3.6), (3.7).

□

Theorem 3.2.1 *$\hat{x}$ is the optimal solution of problem (3.5), (3.6), (3.7).*

Proof:
Let us note that $\hat{x}$ satisfies (3.6), (3.7) according to Lemma 3.2.1 so that it is a feasible solution of the optimization problem in question. It remains to prove its optimality. We have to prove that $f(x) \geq f(\hat{x})$ for all $x \in M(\underline{x}, \overline{x})$. Let us assume on the contrary that there exists a feasible solution $\tilde{x}$ such that $f(\tilde{x}) < f(\hat{x})$. Let us assume that $f(\hat{x}) = f_p(\hat{x}_p)$. Since $f(\tilde{x}) < f(\hat{x})$, it must be $f_p(\tilde{x}_p) < f_p(\hat{x}_p)$ so that $\tilde{x}_p \notin T_p$. Let $i(p) \in I$ is such that $T_p = T_{i(p)p}$ so that $\tilde{x}_p \notin T_{i(p)p}$. Therefore it must exist an index $r \in J$ such that $\tilde{x}_r \in T_{i(p)r}$ (otherwise it would be $\tilde{x}_j \notin T_{i(p)j} \ \ \forall j \in J$ and thus $a_{i(p)}(\tilde{x}) = \max_{j \in J}(a_{ij} \wedge \tilde{x}_j) < b_{i(p)}$ and $\tilde{x}$ would be infeasible). But if $\tilde{x}_r \in T_{i(p)r}$, we obtain:

$$f(\tilde{x}) \geq f_r(\tilde{x}_r) \geq \min_{x_r \in T_{i(p)r}} (f_r(x_r)) \geq \min_{x_p \in T_{i(p)p}} (f_p(x_p)) = f_p(\hat{x}_p) = f(\hat{x}),$$

which is a contradiction with the assumed inequality $f(\tilde{x}) < f(\hat{x})$. This contradiction proves the optimality of $\hat{x}$.

□

Let us note that the complexity of finding the optimal solution of (3.5), (3.6), (3.7) depends on the complexity of finding the minimum of $f_j(x_j)$ on a closed interval. Such minimum can be easily found if function f_j is for instance increasing, decreasing, convex, concave or unimodal.

Algorithm 3.2.1 *We will provide algorithm, which summarizes the above discussion and under the assumption that the set $M(\underline{x}, \overline{x})$, which is the set of all a feasible solutions of system (3.5), (3.6), (3.7) is not empty. This algorithm finds the optimal solution of problem (3.5), (3.6), (3.7) if set $M(\underline{x}, \overline{x}) \neq \emptyset$*

[0] *Input I, J, $\underline{x}, \overline{x}$, a_{ij} and b_i for all $i \in I$ and $j \in J$;*

[1] *For all $i \in I$ and $j \in J$ set:*
$T_{ij} = \left[b_i \vee \underline{x}_j, \overline{x}_j\right]$ *if* $a_{ij} \geq b_i$ & $b_i \leq \overline{x}_j$;
$T_{ij} = \emptyset$ *otherwise, i.e. if either* $a_{ij} < b_i$ *or* $b_i > \overline{x}_j$;

[2] *Find* $x_j^{(i)}$ $\forall i \in I$ *and* $j \in J$ *with* $T_{ij} \neq \emptyset$;

[3] *Find* $f_{p(i)}(x_{p(i)}^{(i)}) = \min_{j \in J}(f_j(x_j^{(i)}))$ *for all* $i \in I$;

[4] *Find* $R_k = \{i \in I \ ; \ p(i) = k\}$ $\forall k \in J$;

[5] *Find* $T_k = \bigcap_{i \in R_k} T_{ik}$ $\forall k \in J$;

[5] *Find* $\hat{x}_k$ *where* $f_k(\hat{x}_k) = \min_{x_k \in T_k}(f_k(x_k))$, *for all* $k \in J$ & $R_k \neq \emptyset$, $f_k(\hat{x}_k) = \min_{x_k \in [\underline{x}_k, \overline{x}_k]} f_k(x_k)$, *if* $R_k = \emptyset$;

[6] *Set* $x^{opt} = \hat{x}$, x^{opt} *is the optimal solution, STOP.*

We will illustrate the performance of the Algorithm 3.2.1 by the following small numerical example.

Example 3.2.1
Let $J = \{1, 2, 3\}$, $I = \{1, 2, 3\}$, $\underline{x} = (0, 0, 0)$ *and* $\overline{x} = (10, 10, 10)$ *and consider the following system of inequalities:*

$$\max(7 \wedge x_1, 5 \wedge x_2, 6 \wedge x_3) \geq 6,$$

$$\max(6 \wedge x_1, 7 \wedge x_2, 8 \wedge x_3) \geq 8,$$

$$\max(8 \wedge x_1, 5 \wedge x_2, 4 \wedge x_3) \geq 4,$$

Consider the objective function in the form:

$$f(x_1, x_2, x_3) = max\,(f_1(x_1), f_2(x_2), f_3(x_3))$$

where $f_j(x_j) = c_j x_j + d_j$, *where* $c = (0.5, 0.8, 0.7)$ *and* $d = (1.4, 5.2, 3.1)$.
By using Algorithm 3.2.1 we find:
$T_{11} = [6, 10]$, $T_{12} = \emptyset$, $T_{13} = [6, 10]$;
$T_{21} = \emptyset$, $T_{22} = \emptyset$, $T_{23} = [8, 10]$;
$T_{31} = [4, 10]$, $T_{32} = [4, 10]$, $T_{33} = [4, 10]$.
$x_1^{(1)} = 6$, $x_3^{(1)} = 6$, $x_3^{(2)} = 8$ $x_1^{(3)} = 4$, $x_2^{(3)} = 4$, *and* $x_3^{(3)} = 4$.
$R_1 = \{1, 3\}$, $R_2 = \emptyset$, *and* $R_3 = \{2\}$.
$T_1 = [6, 10]$, $T_2 = [0, 10]$, $T_3 = [8, 10]$.
Then $x^{opt} = (6, 0, 8)$ *is the optimal solution of the set* $M(\underline{x}, \overline{x})$ *and* $f\,(x^{opt}) = \max\,(4.4, 5.2, 8.7)$, *then the objective function is equal to* 8.7.

Example 3.2.2
Let $J = \{1, 2, 3\}$, $I^{(1)} = \{1, 2, 3, 4, 5\}$, $I^{(2)} = \{6, 7, 8, 9\}$, $\underline{x} = (0, 0, 0)$ *and* $\overline{x} = (10, 10, 10)$ *and consider the following system of inequalities:*

$$\max(11 \wedge x_1, 3 \wedge x_2, 3 \wedge x_3) \geq 4,$$

$$\max(5 \wedge x_1, 6 \wedge x_2, 9 \wedge x_3) \geq 5,$$

$$\max(5 \wedge x_1, 3 \wedge x_2, 12 \wedge x_3) \geq 4,$$

$$\max(3 \wedge x_1, 5 \wedge x_2, 9 \wedge x_3) \geq 2,$$

$$\max(5 \wedge x_1, 5 \wedge x_2, 6 \wedge x_3) \geq 5,$$

$$\max(5 \wedge x_1, 16 \wedge x_2, 9 \wedge x_3) \leq 6,$$

$$\max(3 \wedge x_1, 5 \wedge x_2, 8 \wedge x_3) \leq 5$$

$$\max(15 \wedge x_1, 7 \wedge x_2, 11 \wedge x_3) \leq 7,$$

$$\max(9 \wedge x_1, 8 \wedge x_2, 13 \wedge x_3) \leq 8,$$

Consider the objective function in the form:

$$f(x_1, x_2, x_3) = max\,(f_1(x_1), f_2(x_2), f_3(x_3))$$

where $f_j(x_j) = |c_j x_j - d_j|\,,$ *where* $c = (1, 0.3, 1.5)$ *and* $d = (7.2, 1.9, 3.2)$.
In this example we will use in the beginning Algorithm 3.1.2 to find the new upper bounds of set $M(\underline{x}, \overline{x})$*, which is equivalent to solving the system of* $\leq$*-inequalities.*

By using Algorithm 3.1.2 we find:
$V_{61} = [0, 10]$, $V_{62} = [0, 6]$, $V_{63} = [0, 6]$
$V_{71} = [0, 10]$, $V_{72} = [0, 10]$, $V_{73} = [0, 5]$*;*
$V_{81} = [0, 7]$, $V_{82} = [0, 10]$, $V_{83} = [0, 7]$
$V_{91} = [0, 8]$, $V_{92} = [0, 10]$, $V_{93} = [0, 8]$
Then we find that the new upper bounds of set $M(\underline{x}, \overline{x})$ *is* $\overline{x} = (7, 6, 5)$
$T_{11} = [4, 7]$, $T_{12} = \emptyset$, $T_{13} = \emptyset$*;*

$T_{21} = [5, 7], \quad T_{22} = [5, 6], \quad T_{23} = [5, 5];$
$T_{31} = [4, 7], \quad T_{32} = \emptyset, \quad T_{33} = [4, 5].$
$T_{41} = [2, 7], \quad T_{42} = [2, 6], \quad T_{43} = [2, 5],.$
$T_{51} = [5, 7], \quad T_{52} = [5, 6], \quad T_{53} = [5, 5];$
By using Algorithm 3.2.1 we find:
$x_1^{(1)} = 7, \quad x_1^{(2)} = 7, \quad x_2^{(2)} = 6, \quad x_3^{(2)} = 5 \quad x_1^{(3)} = 7, \quad x_3^{(3)} = 4, \quad x_1^{(4)} = 7, \quad x_2^{(4)} = 6, \quad x_3^{(4)} = 2.12, \quad x_1^{(5)} = 7, \quad x_2^{(5)} = 6, \quad x_3^{(5)} = 5,.$
$R_1 = \{1, 3\}, \quad R_2 = \{2, 5\},$ *and* $\quad R_3 = \{4\}.$
$T_1 = [4, 7], \quad T_2 = [5, 6], \quad T_3 = [2, 5].$
Then $x^{opt} = (7, 6, 2.12)$ *is the optimal solution of the set* $M(\underline{x}, \overline{x})$ *and* $f\,(x^{opt}) = \max(0.2, 0.1, 0.02)$*, then the objective function is equal to* 0.2.

4

Optimization Problems under One-Sided $(\max, \min)$-Linear Equality Constraints

In this chapter we will consider the optimization problems under one-sided (max, min)-linear equality constraints. It is evident that problems with one-sided equality constraints of the form:

$$\max_{j \in J}(a_{ij} \wedge x_j) = b_i,\ i \in I,$$

can be solved by the appropriate formulation of equivalent one-sided inequality constraints and using the methods presented in the preceding chapter. Namely, we can consider inequality systems of the form:

$$\max_{j \in J}(a_{ij} \wedge x_j) \leq b_i,\ i \in I,$$

$$\max_{j \in J}(a_{ij} \wedge x_j) \geq b_i,\ i \in I.$$

Such systems have $2m$ inequalities (if $|I| = m$), which have to be taken into account. It arises an idea, whether it is not more effective to solve the problems with equality constraints directly without replacing the equality constraints with the double numbers of inequalities. In this article, we are going to propose such an approach to the equality constraints. First we will study the structure of the set of all solutions of the given system of equations with finite entries a_{ij} & b_i, for all $i \in I$ &$j \in J$. Using some of the theorems characterizing the structures of the solution set of such systems, we will propose an algorithm, which finds an optimal solution of minimization problems with objective functions of the form:

$$f(x) \equiv \max_{j \in J} f_j(x_j),$$

where $f_j,\ j \in J$ are continuous functions. Complexity of the proposed method of monotone or unimodal functions $f_j,\ j \in J$ will be studied, possible generalizations and extensions of the results will be discussed.

4.1 One-Sided (max, min)-Linear Systems of Equations

In this chapter we will consider the following system of equations:

$$\max_{j \in J}(a_{ij} \wedge x_j) = b_i,\ i \in I, \tag{4.1}$$

$$\underline{x} \leq x \leq \overline{x}. \tag{4.2}$$

The set of all solutions of the system (4.1), will be denoted $M^=$. Before investigating properties of the set $M^=$, we will bring an example, which shows one possible application, which leads to solving this system.

Example 4.1.1

The practical problem, which be described by system (4.1) and (4.2) may be as in Example3.1.1 in the previous chapter with a simple change so that, we will require further to find such capacities $x_j,\ j \in J$ that for each $i \in I$ the maximum capacity of the roads connecting i to T via j over all $j \in J$ is exactly equal to a given positive value b_i. Therefore feasible vectors of capacities $x = (x_1,\ x_2,\ \ldots,\ x_n)$ (i.e. the vectors, the components of which are capacities x_j having the required properties) must satisfy the system (4.1) and (4.2).

Remark 4.1.1 *Since each equality can be replaced by two inequalities $\leq$, $\geq$, so that the system of equations in formulation (4.1), can be solved by the algorithm 3.1.2 given in chapter 3, but the disadvantage of the this technique is coming from the fact that we need to solve double the number of inequalities, which need more time and memory. In this chapter we will introduce a new technique to solve the system of equations in formulation (4.1) and (4.2), without transforming each equality to two inequalities.*

In what follows, we will investigate some properties of the set $M^{=}$ described by the system (4.1). Also, to simplify the formulas in what follows we will set

$$a_i(x) \equiv \max_{j \in J}(a_{ij} \wedge x_j) \quad \text{for all } \; i \in I \, ,$$

Let us define for any fixed $j \in J$ the set I_j as follows:

$$I_j = \{i \in I \; \& \; a_{ij} \geq b_i\} \, ,$$

and define the set $M^{=}(\overline{x})$ as follows:

$$M^{=}(\overline{x}) = \{x \in M^{=} \; \& \; x \leq \overline{x}\} \, ,$$

also we define the set $S_j(x_j)$ as follows:

$$S_j(x_j) = \{k \in I \; \& \; a_{kj} \wedge x_j = b_k\} \, , \quad \forall j \in J.$$

Lemma 4.1.1 *Let us set for all $i \in I$ and $j \in J$*

$$T_{ij}^{=} = \{x_j \; ; \; (a_{ij} \wedge x_j) = b_i \; \& \; x_j \leq \overline{x}_j\}$$

Then for any fixed i, j the following equalities hold:

(i) $T_{ij}^{=} = \{b_i\}$ *if* $a_{ij} > b_i$ & $b_i \leq \overline{x}_j$;

(ii) $T_{ij}^{=} = [b_i, \overline{x}_j]$ *if* $a_{ij} = b_i$ & $b_i \leq \overline{x}_j$;

(iii) $T_{ij}^{=} = \emptyset$ *if either* $a_{ij} < b_i$ *or* $b_i > \overline{x}_j$.

Proof:

(i) If $a_{ij} > b_i$, then $a_{ij} \wedge x_j > b_i$ for any $x_j > b_i$, also $a_{ij} \wedge x_j < b_i$ for any $x_j < b_i$, so that the only solution for equation $a_{ij} \wedge x_j = b_i$ is $x_j = b_i \leq \overline{x}_j$.

(ii) If $a_{ij} = b_i$ & $b_i \leq \overline{x}_j$, then $a_{ij} \wedge x_j < b_i$ for arbitrary $x_j < b_i$, but $a_{ij} \wedge x_j = b_i$ for arbitrary $b_i \leq x_j \leq \overline{x}_j$.

(iii) If $a_{ij} < b_i$, then either $a_{ij} \wedge x_j = a_{ij} < b_i$ for arbitrary $x_j \geq a_{ij}$, or $a_{ij} \wedge x_j = x_j < b_i$ for arbitrary $x_j < a_{ij}$. Therefore there is no solution for equation $a_{ij} \wedge x_j = b_i$, which means $T_{ij}^{=} = \emptyset$.

Also if $b_i > \overline{x}_j$, then either $x_j \geq b_i > \overline{x}_j$, so that $T_{ij}^{=} = \emptyset$, or $x_j < b_i$ there are two cases, the first is $x_j \leq a_{ij}$ and $a_{ij} \wedge x_j = x_j < b_i$ and the second case is $x_j > a_{ij}$ and $a_{ij} \wedge x_j = a_{ij} < x_j < b_i$. therefore there is no solution for equation $a_{ij} \wedge x_j = b_i$, which means $T_{ij}^{=} = \emptyset$.

□

Lemma 4.1.2 *Let us set for all $i \in I$ and $j \in J$*

$$x_j^{(i)} = \begin{cases} b_i & \text{if} \quad a_{ij} > b_i \ \& \ b_i \leq \overline{x}_j, \\ \overline{x}_j & \text{if} \quad a_{ij} = b_i \ \& \ b_i \leq \overline{x}_j \quad \text{or} \quad T_{ij}^{=} = \emptyset, \end{cases}$$

and let

$$\hat{x}_j = \begin{cases} \min_{k \in I_j} x_j^{(k)} & \text{if} \quad I_j \neq \emptyset, \\ \overline{x}_j & \text{if} \quad I_j = \emptyset. \end{cases}$$

Let

$$S_j(\hat{x}_j) = \left\{ k \in I \ ; \ x_j^{(k)} = \hat{x}_j \right\}, \forall j \in J,$$

and the following statements hold:

(i) $\hat{x} \in M^{=}(\overline{x}) \Leftrightarrow \bigcup_{j \in J} S_j(\hat{x}_j) = I$

(ii) Let $M^{=}(\overline{x}) \neq \emptyset$, then $\hat{x} \in M^{=}(\overline{x})$ and for any $x \in M^{=}(\overline{x}) \Rightarrow x \leq \hat{x}$, i.e. $\hat{x}$ is the maximum element of $M^{=}(\overline{x})$.

Proof:

(i) To prove the necessary condition we suppose $\hat{x} \in M^{=}(\overline{x})$, then $\max_{j\in J}(a_{ij} \wedge \hat{x}_j) = b_i$, for all $i \in I$. So that for all $i \in I$, there exists at least one $j(i) \in J$ such that $a_{ij(i)} \wedge \hat{x}_{j(i)} = \max_{j\in J}(a_{ij} \wedge \hat{x}_j) = b_i$, then either $\hat{x}_{j(i)} = b_i$, if $a_{ij(i)} > b_i$ & $b_i \leq \overline{x}_{j(i)}$, or $\hat{x}_{j(i)} > b_i$, if $a_{ij(i)} = b_i$ & $b_i \leq \overline{x}_{j(i)}$, so that $i \in I_{j(i)} \Rightarrow \hat{x}_{j(i)} = x^{(i)}_{j(i)}$. Otherwise if $a_{ik} \wedge \hat{x}_k < b_i$, for all $k \neq j(i)$ & $k \in J$, then $\hat{x}_k < b_i$, and $a_{ik} < b_i, \Rightarrow I_k = \emptyset$, therefore we can choose $\hat{x}_k = \overline{x}_k$. Hence for all $i \in I$ there exists at least one $j(i) \in J$ such that $S_{j(i)}(\hat{x}_{j(i)}) \neq \emptyset$ and $i \in S_{j(i)}(\hat{x}_{j(i)}) \Rightarrow \bigcup_{j\in J} S_j(\hat{x}_j) = I$.
To prove the sufficient condition, let $\bigcup_{j\in J} S_j(\hat{x}_j) = I$, then for all $i \in I$ there exists at least one $j(i) \in J$ such that $S_{j(i)}(\hat{x}_{j(i)}) \neq \emptyset$ and $i \in S_{j(i)}(\hat{x}_{j(i)})$. Therefore $\hat{x}_{j(i)} = b_i$ if $a_{ij(i)} > b_i$ & $b_i \leq \overline{x}_{j(i)}$, then $a_{ij(i)} \wedge \hat{x}_{j(i)} = \hat{x}_{j(i)} = b_i$. Otherwise $\hat{x}_{j(i)} = \overline{x}_{j(i)}$ if either $a_{ij(i)} = b_i$ & $b_i \leq \overline{x}_{j(i)}$, then $a_{ij(i)} \wedge \hat{x}_{j(i)} = a_{ij(i)} = b_i$ or $T^{=}_{ij(i)} = \emptyset$. Then for all $i \in I$ there exists at least one $j(i) \in J$ such that $\max_{j\in J}(a_{ij} \wedge \hat{x}_j) = a_{ij(i)} \wedge \hat{x}_{j(i)} = b_i$. Then $\hat{x} \in M^{=}(\overline{x})$.

(ii) Let $M^{=}(\overline{x}) \neq \emptyset$, then for each $i \in I$, there exists at least one $j(i) \in J$ such that $T^{=}_{ij(i)} \neq \emptyset$, and $b_i \leq \overline{x}_{j(i)}$ & $a_{ij(i)} \geq b_i$. Therefore there exists at least one $j(i) \in J$ such that either $a_{ij(i)} > b_i$ & $b_i \leq \overline{x}_{j(i)}$, then $x^{(i)}_{j(i)} = b_i$. Or $a_{ij(i)} = b_i$ & $b_i \leq \overline{x}_{j(i)}$, then $x^{(i)}_{j(i)} = \overline{x}_{j(i)}$ so that $\hat{x}_{j(i)} = b_i$ if $i \in I_{j(i)}$ and $a_{ij(i)} \wedge \hat{x}_{j(i)} = b_i$ is satisfied. Otherwise if $I_j = \emptyset$, we set $\hat{x}_j = \overline{x}_j$. Then $\hat{x} \in M^{=}(\overline{x})$ and for any $x \in M^{=}(\overline{x})$ we have $x \leq \hat{x}$, i.e. $\hat{x}$ is the maximum element of $M^{=}(\overline{x})$.

□

Another proof for lemma 4.1.2:
It follows from the definition of $x^{(i)}_j$ and $\hat{x}_j$ for $i \in I,\ j \in J$ that

$$\max_{j\in J}(a_{ij} \wedge \hat{x}_j) \leq b_i,\ i \in I, \qquad (*)$$

$$\text{If } x \in M^{=}(\overline{x}), \text{ then } x \leq \hat{x} \qquad (**)$$

$$\text{Further} \begin{cases} a_{ij} \wedge \hat{x}_j = b_i & \text{if } \hat{x}_j = x^{(k)}_j \,, \\ a_{ij} \wedge \hat{x}_j < b_i & \text{otherwise.} \end{cases} \qquad (***)$$

Let $\bigcup_{j\in J} S_j(\hat{x}_j) \subset I$ and $i_0 \in I \setminus \bigcup_{j\in J} S_j(\hat{x}_j)$. Then it follows from (*) - (***) that $a_{i_0 j} \wedge x_j < b_{i_0} \;\; \forall \;\; j \in J$ and therefore $\max_{j\in J}(a_{i_0 j} \wedge x_j) \leq b_{i_0}$, so that $M^{=}(\overline{x}) = \emptyset$.
Let now $\bigcup_{j\in J} S_j(\hat{x}_j) = I$ and $k \in I$ be arbitrary. Then there exists $j(k) \in J$ such that $k \in S_{j(k)}(\hat{x}_j)$ and we have $a_{kj(k)} \wedge \hat{x}_{j(k)} = a_{kj(k)} \wedge x_{j(k)}^{(k)} = b_k$ so that $\max_{j\in J}(a_{ij} \wedge \hat{x}_j) = a_{kj(k)} \wedge \hat{x}_{j(k)} = b_k$. Since $k \in I$ was arbitrarily chosen it follows that $\hat{x} \in M^{=}(\overline{x})$ and therefore $M^{=}(\overline{x}) \neq \emptyset$.
Note that if $x \leq \hat{x}$ and $S_j(x_j) = \left\{k \in I \;;\; x_j^{(k)} = x_j\right\}$, then we have $x \leq \hat{x}$ and $x \in M^{=}(\overline{x}) \;\; \Leftrightarrow \;\; \bigcup_{j\in J} S_j(x_j) = I$. It follows that $M^{=}(\overline{x}) \neq \emptyset$, then $\hat{x} \in M^{=}(\overline{x})$ and $\hat{x}$ is the maximum element of $M^{=}(\overline{x})$.

□

It is appropriate now to define $M^{=}(\underline{x}, \overline{x})$, which is the set of all solutions of the system describe by (4.1) and (4.2) as follows:

$$M^{=}(\underline{x}, \overline{x}) = \{x \in M^{=}(\overline{x}) \;\&\; x \geq \underline{x}\},$$

Theorem 4.1.1 *Let $\hat{x}$ and $S_j(\hat{x}_j)$ be defined as in Lemma 4.1.2 then:*

(i) $M^{=}(\underline{x}, \overline{x}) \neq \emptyset$ *if and only if* $\hat{x} \in M^{=}(\overline{x})$ & $\underline{x} \leq \hat{x}$,

(ii) If $M^{=}(\underline{x}, \overline{x}) \neq \emptyset$, *then* $\hat{x}$ *is the maximum element of* $M^{=}(\underline{x}, \overline{x})$,

(iii) Let $M^{=}(\underline{x}, \overline{x}) \neq \emptyset$ *and* $\tilde{J} \subseteq J$. *Let us set*

$$\tilde{x}_j = \begin{cases} \hat{x}_j & \text{if} \quad j \in \tilde{J}, \\ \underline{x}_j & \text{otherwise}, \end{cases}$$

then $\tilde{x} \in M^{=}(\underline{x}, \overline{x}) \Leftrightarrow \bigcup_{j\in \tilde{J}} S_j(\tilde{x}_j) = I$

Proof:

(i) If $\hat{x} \in M^{=}(\overline{x})$ & $\underline{x} \leq \hat{x}$, from definition $M^{=}(\underline{x}, \overline{x})$ we have $\hat{x} \in M^{=}(\underline{x}, \overline{x})$, then $M^{=}(\underline{x}, \overline{x}) \neq \emptyset$. If $M^{=}(\underline{x}, \overline{x}) \neq \emptyset$ so that $M^{=}(\overline{x}) \neq \emptyset$ and from lemma 4.1.2, it is verified that $\hat{x} \in M^{=}(\overline{x})$ and $\hat{x}$ is the maximum element of $M^{=}(\overline{x})$, therefore $\hat{x} \geq \underline{x}$.

(ii) If $M^{=}(\underline{x}, \overline{x}) \neq \emptyset$, so that $M^{=}(\overline{x}) \neq \emptyset$ and $M^{=}(\underline{x}, \overline{x}) \subset M^{=}(\overline{x})$, and since $\hat{x}$ is the maximum element of $M^{=}(\overline{x})$, then $\hat{x}$ is the maximum element of $M^{=}(\underline{x}, \overline{x})$.

(iii) Let $\tilde{x} \in M^{=}(\underline{x}, \overline{x}) \Rightarrow \tilde{x} \in M^{=}(\overline{x})$, also from definition $\tilde{x}$ we have $\tilde{x}_j = \hat{x}_j$ for all $j \in \tilde{J}$, so that $S_j(\tilde{x}_j) \neq \emptyset \;\; \forall \;\; j \in \tilde{J}$. And $\tilde{x}_j < \hat{x}_j$ for all $j \in J \setminus \tilde{J}$, so that $S_j(\tilde{x}_j) = \emptyset \;\; \forall \;\; j \in J \setminus \tilde{J}$. Hence $\tilde{x} \in M^{=}(\overline{x})$, by lemma 4.1.2, we have $\bigcup_{j \in \tilde{J}} S_j(\tilde{x}_j) = I$.
Let $\bigcup_{j \in \tilde{J}} S_j(\tilde{x}_j) = I$, since $\tilde{J} \subseteq J$ we have $\bigcup_{j \in J} S_j(\tilde{x}_j) = I$. By lemma 4.1.2, $\tilde{x} \in M^{=}(\overline{x})$ also we have $\tilde{x} \geq \underline{x}$ therefore $\tilde{x} \in M^{=}(\underline{x}, \overline{x})$.

□

4.2 Solving Optimization Problems under One-Sided $(\max, \min)$-Linear Equality Constraints

In this section we will solve the following optimization problem:

$$f(x) \equiv \max_{j \in J} f_j(x_j) \quad \longrightarrow \quad \min \tag{4.3}$$

subject to

$$x \in M^{=}(\underline{x}, \overline{x}) \tag{4.4}$$

We assume further that $f_j : R \to R$ are continuous and monotone functions (i.e. increasing or decreasing), $M^{=}(\underline{x}, \overline{x})$ denotes the set of all feasible solutions of the system described by (4.1) and (4.2) and assuming that $M^{=}(\underline{x}, \overline{x}) \neq \emptyset$ (note that the emptiness of the set $M^{=}(\underline{x}, \overline{x})$ can be verified using the considerations of the preceding section).

Let $J^* \equiv \{j \mid f_j \;\; decreasing \;\; function\}$ so that

$$\min_{x_j \in [\underline{x}_j, \hat{x}_j]} f_j(x_j) = f_j(\hat{x}_j), \quad \forall j \in J^*.$$

Then we can propose an algorithm for solving problem (4.3) and (4.4) under the assumption that $M^{=}(\underline{x}, \overline{x}) \neq \emptyset$ which means $\bigcup_{j \in J} S_j(\hat{x}_j) = I$, i.e. for finding an optimal solution x^{opt} of problem (4.3) and (4.4).

Algorithm 4.2.1 *We will provide algorithm, which summarizes the above discussion and finds an optimal solution x^{opt} of problem (4.3) and (4.4), where $f_j(x_j)$ are continuous and monotone functions.*

$\boxed{0}$ *Input* $I, J, \underline{x}, \overline{x}, a_{ij}$ *and* b_i *for all* $i \in I$ *and* $j \in J$.

$\boxed{1}$ *Find* $\hat{x}$, *and set* $\tilde{x} = \hat{x}$.

$\boxed{2}$ *Find* $J^* \equiv \{j \mid f_j \;\; decreasing \;\; function\}$.

$\boxed{3}$ $F = \{p \mid \max_{j \in J} f_j(\tilde{x}_j) = f_p(\tilde{x}_p)\}$.

$\boxed{4}$ *If* $F \cap J^* \neq \emptyset$, *then* $x^{opt} = \tilde{x}$, *Stop.*

$\boxed{5}$ *Set* $y_p = \underline{x}_p \;\; \forall \;\; p \in F$, & $y_j = \tilde{x}_j$, *otherwise.*

$\boxed{6}$ *If* $\bigcup_{j \in J} S_j(y_j) = I$, *set* $\tilde{x} = y$ *go to* $\boxed{3}$.

$\boxed{8}$ $x^{opt} = \tilde{x}$, *Stop.*

We will illustrate the performance of this algorithm by the following numerical example.

Example 4.2.1 *Consider the following optimization problem:*
Minimize $f(x) \equiv \max_{j \in J}(f_j(x_j))$
where $f_j(x_j) \;\; \forall \;\; j \in J$ *are continuous and monotone functions in the form* $f_j(x_j) \equiv c_j \times x_j + d_j$,

$$C = \begin{bmatrix} -0.2057 & 4.8742 & 2.8848 & 0.9861 & 1.7238 & 1.1737 & -3.3199 \end{bmatrix}$$

and
$D = \begin{bmatrix} 1.4510 & 1.5346 & -3.6121 & -0.9143 & -2.0145 & 1.9373 & -4.8467 \end{bmatrix}$

subject to

$$x \in M^{=}(\underline{x}, \overline{x})$$

where the set $M^{=}(\underline{x}, \overline{x})$ *is given by the system (4.1) and (4.2) where* $J = \{1, 2, \ldots, 7\}$, $I = \{1, 2, \ldots, 6\}$, $\underline{x}_j = 0 \ \ \forall \ \ j \in J$ *and* $\overline{x}_j = 10 \ \ \forall \ \ j \in J$ *and consider the system (4.1) of equations where* a_{ij} & $b_i \ \ \forall \ \ i \in I$ *and* $j \in J$ *are given by the matrix* A *and vector* B *as follows:*

$$A = \begin{pmatrix} 6.1221 & 9.0983 & 9.5032 & 6.0123 & 6.1112 & 4.1221 & 5.5776 \\ 8.2984 & 3.3920 & 2.5185 & 1.1925 & 8.9742 & 6.7594 & 8.6777 \\ 2.0115 & 6.3539 & 4.4317 & 7.7452 & 0.6465 & 9.4098 & 1.3576 \\ 6.4355 & 1.6404 & 3.1850 & 3.7361 & 7.2605 & 3.0201 & 5.3808 \\ 8.5668 & 5.8310 & 2.5146 & 8.7804 & 3.7709 & 4.4770 & 2.3007 \\ 5.2690 & 9.6900 & 5.1598 & 9.2889 & 6.1585 & 1.0786 & 7.0121 \end{pmatrix}$$

and
$B^T = \begin{bmatrix} 6.1221 & 7.0955 & 6.3539 & 6.4355 & 6.5712 & 7.0121 \end{bmatrix}$

By the method in section 2 we get $\hat{x}$, *which is the maximum element of* $M^{=}(\underline{x}, \overline{x})$, *as follows:*
$\hat{x} = (6.5712,\ 6.1221,\ 6.1221,\ 6.3539,\ 6.4355,\ 6.3539,\ 7.0955)$
By using algorithm 4.2.1 we find:
Iteration 1*:*
$\boxed{1}$ $\tilde{x} = (6.5712,\ 6.1221,\ 6.1221,\ 6.3539,\ 6.4355,\ 6.3539,\ 7.0955)$;
$\boxed{2}$ $J^* = \{1, 7\}$;
$\boxed{3}$ $F = \{2\}$;
$f(\tilde{x}) = 31.3750$;
$\boxed{5}$ $y = (6.5712,\ 0,\ 6.1221,\ 6.3539,\ 6.4355,\ 6.3539,\ 7.0955)$;
$\boxed{6}$ $\bigcup_{j \in J} S_j(y_j) = I$;
$\tilde{x} = (6.5712,\ 0,\ 6.1221,\ 6.3539,\ 6.4355,\ 6.3539,\ 7.0955)$.
Iteration 2:

$\boxed{3}$ $F = \{3\}$;
$f(\tilde{x}) = 14.0490$;
$\boxed{5}$ $y = (6.5712,\ 0,\ 0,\ 6.3539,\ 6.4355,\ 6.3539,\ 7.0955)$;
$\boxed{6}$ $\bigcup_{j\in J} S_j(y_j) = I$;
$\tilde{x} = (6.5712,\ 0,\ 0,\ 6.3539,\ 6.4355,\ 6.3539,\ 7.0955)$.

Iteration 3:
$\boxed{3}$ $F = \{6\}$;
$f(\tilde{x}) = 9.3946$;
$\boxed{5}$ $y = (6.5712,\ 0,\ 0,\ 6.3539,\ 6.4355,\ 0,\ 7.0955)$;
$\boxed{6}$ $\bigcup_{j\in J} S_j(y_j) = I$;
$\tilde{x} = (6.5712,\ 0,\ 0,\ 6.3539,\ 6.4355,\ 0,\ 7.0955)$.

Iteration 4:
$\boxed{3}$ $F = \{5\}$;
$f(\tilde{x}) = 9.0789$;
$\boxed{5}$ $y = (6.5712,\ 0,\ 0,\ 6.3539,\ 0,\ 0,\ 7.0955)$;
$\boxed{6}$ $\bigcup_{j\in J} S_j(y_j) = I$;
$\tilde{x} = (6.5712,\ 0,\ 0,\ 6.3539,\ 0,\ 0,\ 7.0955)$.

Iteration 5:
$\boxed{3}$ $F = \{4\}$;
$f(\tilde{x}) = 5.3510$;
$\boxed{5}$ $y = (6.5712,\ 0,\ 0,\ 0,\ 0,\ 0,\ 7.0955)$;
$\boxed{6}$ $\bigcup_{j\in J} S_j(y_j) \neq I$;
$\boxed{7}$ $x^{opt} = \tilde{x} = (6.5712,\ 0,\ 0,\ 6.3539,\ 0,\ 0,\ 7.0955)$, *STOP.*

In iteration 5 we find that if we set $x_4 = 0$ *the third equation of the system (4.1) is given as follows:*

$$\begin{aligned} a_3 &= max\,(2.0115 \wedge 6.5712, 6.3539 \wedge 0, 4.4317 \wedge 0, 7.7452 \wedge 0, 0.6465 \wedge 0, \\ &\quad 9.4098 \wedge 0, 1.3576 \wedge 7.0955) = 2.0115 \neq b_3 = 6.3539. \end{aligned}$$

Therefore Algorithm 4.2.1 go to step $\boxed{8}$ *and take* $x^{opt} = \tilde{x} = (6.5712,\ 0,\ 0,\ 6.3539,\ 0,\ 0,\ 7.0955)$ *and stop. We obtained the optimal value for the objective function* $f(x^{opt}) = 5.3510$*. We can easily verify that* x^{opt} *is a feasible solution as follows:*

$a_1(x^{opt}) = b_1 = 6.1221, \quad a_2(x^{opt}) = b_2 = 7.0955, \quad a_3(x^{opt}) = b_3 = 6.3539,$
$a_4(x^{opt}) = b_4 = 6.4355, \quad a_5(x^{opt}) = b_5 = 6.5712, \quad a_6(x^{opt}) = b_6 = 7.0121,$
and inequality $\underline{x} \leq x^{opt} \leq \overline{x}$ *is fulfilled.*

Remark 4.2.1 *By reference to the lemma 4.1.2 and theorem 4.1.1 it is not difficult to note that the maximum number of arithmetic or logic operations in any step to get* $\hat{x}$ *can not exceed* $O(nm)$ *operations. This will happen when we calculate* $x_j^{(i)}, \ \forall \ i \in I \ \& \ j \in J$. *Also from the above example we can remark that the maximum number of operations in each step in any iterations from algorithm 4.2.1 is less than or equal to the number of variables* n *and the maximum number of iterations from step* $\boxed{3}$ *to step* $\boxed{6}$ *of this algorithm can not exceed* $O(n)$. *Therefore the computational complexity of the algorithm 4.2.1 is* $O(max(n^2, \ nm))$.

In what follows let us modify algorithm 4.2.1, which find an optimal solution for continuous and monotone functions, to be suitable to find the optimal solution for any general continuous functions $f_j(x_j)$ as follows:

Algorithm 4.2.2 *We will provide algorithm, which summarizes the above discussion and finds an optimal solution* x^{opt} *of problem (4.3) and (4.4), where* $f_j(x_j)$ *are general continuous functions.*

$\boxed{0}$ *Input* $I, \ J, \ \underline{x}, \overline{x}, \ a_{ij}$ *and* b_i *for all* $i \in I$ *and* $j \in J$.

$\boxed{1}$ *Find* $\hat{x}$, *and set* $\tilde{x} = \hat{x}$.

$\boxed{2}$ *Find* $\min_{x_j \in [\underline{x}_j, \hat{x}_j]} f_j(x_j) = f_j(x_j^*), \ \forall \ j \in J$.

$\boxed{3}$ *Set* $J^* \equiv \{j \ | \ f_j(\hat{x}_j) = f_j(x_j^*)\}$.

$\boxed{4}$ $F = \{p \ | \ \max_{j \in J} f_j(\tilde{x}_j) = f_p(\tilde{x}_p)\}$.

$\boxed{5}$ *If* $F \cap J^* \neq \emptyset$, *then* $x^{opt} = \tilde{x}$, *Stop.*

$\boxed{6}$ *Set* $y_p = x_p^* \ \forall \ p \in F, \ \& \ y_j = \tilde{x}_j$, *otherwise.*

$\boxed{7}$ *If* $\bigcup_{j \in J} S_j(y_j) = I$, *set* $\tilde{x} = y$ *go to* $\boxed{3}$.

$\boxed{8}$ $x^{opt} = \tilde{x}$, *Stop.*

We will illustrate the performance of this algorithm by the following numerical examples.

Example 4.2.2 *Consider the following optimization problem:*
Minimize $f(x) \equiv \max_{j\in J}(f_j(x_j))$
where $f_j(x_j) \quad \forall \quad j \in J$ *are continuous functions given in the following form* $f_j(x_j) \equiv (x_j - \xi_j)^2$,
$\xi = (3.3529,\ 1.4656,\ 5.6084,\ 5.6532,\ 6.1536,\ 6.5893)$
subject to

$$x \in M^{=}(\underline{x}, \overline{x})$$

where the set $M^{=}(\underline{x}, \overline{x})$ *is given by the system (4.1) and (4.2) where* $J = \{1, 2, \ldots, 6\}$, $I = \{1, 2, \ldots, 6\}$, $\underline{x}_j = 0 \ \forall \ j \in J$ *and* $\overline{x}_j = 10 \ \forall \ j \in J$ *and consider the system (4.1) of equations where* a_{ij} & $b_i \ \forall \ i \in I$ *and* $j \in J$ *are given by the matrix* A *and vector* B *as follows:*

$$A = \begin{pmatrix} 3.6940 & 0.8740 & 0.5518 & 4.6963 & 2.1230 & 1.4673 \\ 1.9585 & 8.3470 & 5.8150 & 8.5545 & 8.9532 & 8.7031 \\ 1.3207 & 8.9610 & 1.5718 & 3.7155 & 0.1555 & 4.3611 \\ 8.4664 & 9.1324 & 6.6594 & 2.5637 & 6.0204 & 6.0846 \\ 2.4219 & 9.6081 & 1.9312 & 2.5218 & 1.3976 & 4.1969 \\ 1.1172 & 3.6992 & 7.5108 & 4.7686 & 4.4845 & 4.3301 \end{pmatrix}$$

and
$B^T = \begin{bmatrix} 4.0195 & 7.2296 & 4.2766 & 6.6594 & 4.1969 & 6.9874 \end{bmatrix}$

By the method in section 2 we get $\hat{x}$, *which is the maximum element of* $M^{=}(\underline{x}, \overline{x})$, *as follows:*
$\hat{x} = (6.6594,\ 4.1969,\ 6.9874,\ 4.0195,\ 7.2296,\ 4.2766)$.
By using algorithm 4.2.2 we find:
Iteration 1:
1 $\tilde{x} = (6.6594,\ 4.1969,\ 6.9874,\ 4.0195,\ 7.2296,\ 4.2766)$;
2 $x^* = 3.3297,\ 1.4689,\ 5.5899,\ 4.0195,\ 6.1452,\ 4.2766$;
3 $J^* = \{4, 6\}$;
4 $F = \{1\}$;

$f(\tilde{x}) = 10.9328$;
$\boxed{6}$ $y = (3.3297,\ 4.1969,\ 6.9874,\ 4.0195,\ 7.2296,\ 4.2766)$;
$\boxed{7}$ $\bigcup_{j\in J} S_j(y_j) = I$;
$\tilde{x} = (3.3297,\ 4.1969,\ 6.9874,\ 4.0195,\ 7.2296,\ 4.2766)$.
Iteration 2:
$\boxed{3}$ $J^* = \{1, 4, 6\}$;
$\boxed{4}$ $F = \{2\}$;
$f(\tilde{x}) = 7.4600$;
$\boxed{6}$ $y = (3.3297,\ 1.4689,\ 6.9874,\ 4.0195,\ 7.2296,\ 4.2766)$;
$\boxed{7}$ $\bigcup_{j\in J} S_j(y_j) = I$;
$\tilde{x} = (3.3297,\ 1.4689,\ 6.9874,\ 4.0195,\ 7.2296,\ 4.2766)$.
Iteration 3:
$\boxed{3}$ $J^* = \{1, 2, 4, 6\}$;
$\boxed{4}$ $F = \{6\}$;
$f(\tilde{x}) = 5.3488$;
$\boxed{5}$ $F \cap J^* \neq \emptyset$, *then* $x^{opt} = \tilde{x}$;
$x^{opt} = (3.3297,\ 1.4689,\ 6.9874,\ 4.0195,\ 7.2296,\ 4.2766)$, *STOP.*
Here we find the algorithm 4.2.2 stop in step $\boxed{5}$ *since the active variable in iteration 3 is* x_6*, and at the same time the objective function has the minimum value in this value of* x_6*, so that the algorithm 4.2.2 stop. Then we obtained the optimal value of the objective function,* $f(x^{opt}) = 5.3488$*. It is easy to verify that* x^{opt} *is a feasible solution:*
$a_1(x^{opt}) = b_1 = 4.0195$, $a_2(x^{opt}) = b_2 = 7.2296$, $a_3(x^{opt}) = b_3 = 4.2766$,
$a_4(x^{opt}) = b_4 = 6.6594$, $a_5(x^{opt}) = b_5 = 4.1969$, $a_6(x^{opt}) = b_6 = 6.9874$,
and inequality $\underline{x} \leq x^{opt} \leq \overline{x}$ *is fulfilled.*

Example 4.2.3 *Consider the following optimization problem:*
Minimize $f(x) \equiv \max_{j\in J}(f_j(x_j))$
where $f_j(x_j)$ $\forall$ $j \in J$ *are continuous functions given in the following form* $f_j(x_j) \equiv |(x_j - \xi_j)(x_j - \hbar_j)|$,
where
$\xi = (3.3529,\ 1.4656,\ 5.6084,\ 5.6532,\ 6.1536,\ 6.5893)$
and
$\hbar = (0.7399,\ -0.1385,\ -4.1585,\ 1.1625,\ -2.1088,\ 1.2852)$

subject to

$$x \in M^{=}(\underline{x}, \overline{x})$$

where the set $M^{=}(\underline{x}, \overline{x})$ *is given in the same way as in example 4.2.2.*
By the method in section 2 we get $\hat{x}$*, which is the maximum element of* $M^{=}(\underline{x}, \overline{x})$*, as follows:*
$\hat{x} = (6.6594,\ 4.1969,\ 6.9874,\ 4.0195,\ 7.2296,\ 4.2766)$
By using algorithm 4.2.2 we find:
Iteration 1:
1 $\tilde{x} = (6.6594,\ 4.1969,\ 6.9874,\ 4.0195,\ 7.2296,\ 4.2766)$;
2 $x^* = 0.7325,\ 1.4689,\ 5.5899,\ 1.1656,\ 6.1452,\ 1.2830$;
3 $J^* = \emptyset$;
4 $F = \{1\}$;
$f(\tilde{x}) = 19.5728$;
6 $y = (0.7325,\ 4.1969,\ 6.9874,\ 4.0195,\ 7.2296,\ 4.2766)$;
7 $\bigcup_{j \in J} S_j(y_j) = I$;
$\tilde{x} = (0.7325,\ 4.1969,\ 6.9874,\ 4.0195,\ 7.2296,\ 4.2766)$.
Iteration 2:
3 $J^* = \{1\}$;
4 $F = \{3\}$;
$f(\tilde{x}) = 15.3692$;
6 $y = (0.7325,\ 4.1969,\ 5.5899,\ 4.0195,\ 7.2296,\ 4.2766)$;
7 $\bigcup_{j \in J} S_j(y_j) = \{1,\ 2,\ 3,\ 5\} \neq I$;
8 $x^{opt} = (0.7325,\ 4.1969,\ 6.9874,\ 4.0195,\ 7.2296,\ 4.2766)$, *STOP.*
In Iteration 2 algorithm 4.2.2 stops since the fourth and sixth equations in the system of equations (4.1) can not be verify if we change the value of the variable y_3 *to be* $y_3 = (5.5899)$*. If it is necessary to change this value of the variable* y_3 *to minimize the objective function so that our exhortation to decision-maker, it must be changed both the capacities of the ways* a_{45} *and* a_{65} *to be* $a_{45} = 6.6594$ *and* $a_{65} = 6.9874$ *in order to maintain the verification of the system of equations (4.1). In this case we can complete the operation to minimize the objective function as follows:*

Iteration 2:

$\boxed{3}$ $J^* = \{1\}$;
$\boxed{4}$ $F = \{3\}$;
$f(\tilde{x}) = 15.3692$;
$\boxed{6}$ $y = (0.7325,\ 4.1969,\ 5.5899,\ 4.0195,\ 7.2296,\ 4.2766)$;
$\boxed{7}$ $\bigcup_{j \in J} S_j(y_j) = I$;
$\tilde{x} = (0.7325,\ 4.1969,\ 5.5899,\ 4.0195,\ 7.2296,\ 4.2766)$;
Iteration 3:
$\boxed{3}$ $J^* = \{1,\ 3\}$;
$\boxed{4}$ $F = \{2\}$;
$f(\tilde{x}) = 11.8413$;
$\boxed{6}$ $y = (0.7325,\ 1.4689,\ 5.5899,\ 4.0195,\ 6.1452,\ 4.2766)$;
$\boxed{7}$ $\bigcup_{j \in J} S_j(y_j) = I$;
$\tilde{x} = (0.7325,\ 1.4689,\ 5.5899,\ 4.0195,\ 7.2296,\ 4.2766)$;
Iteration 4:
$\boxed{3}$ $J^* = \{1,\ 2,\ 3\}$;
$\boxed{4}$ $F = \{5\}$;
$f(\tilde{x}) = 10.0481$;
$\boxed{6}$ $y = (0.7325,\ 1.4689,\ 5.5899,\ 4.0195,\ 6.1452,\ 4.2766)$;
$\boxed{7}$ $\bigcup_{j \in J} S_j(y_j) = \{1,\ 3,\ 5\} \neq I$;
$x^{opt} = (0.7325,\ 1.4689,\ 5.5899,\ 4.0195,\ 7.2296,\ 4.2766)$;
We obtained the optimal value $f(x^{opt}) = 10.0481$, *We can easily verify that* x^{opt} *is a feasible solution.*

Remark 4.2.2 *Step* $\boxed{2}$ *in algorithm 4.2.2 depends on the method, which finds the minimum for each function* $f_j(x_j)$ *in the interval* $[\underline{x}_j, \hat{x}_j]$, *but this appears in the first iteration only and only once. Also from the above examples we can remark that the maximum number of operations in each step in any iterations from algorithm 4.2.2 is less than or equal to the number of variables* n *and the maximum number of iterations from step* $\boxed{3}$ *to step* $\boxed{7}$ *of this algorithm can not exceed* n. *Therefore the computational complexity of the algorithm 4.2.2 is given by* $\max\left\{O(\max(n^2,\ n \times m)), \widetilde{O}\right\}$, *where* $\widetilde{O}$ *is complexity of the method, which finds the minimum for each function* $f_j(x_j)$ *in the interval* $[\underline{x}_j, \hat{x}_j]$.

5

Optimization Problems under Two-Sided $(\max, \min)-$ Linear Equation Constraints - an Iteration Method

We consider optimization problems, the objective function of which is equal to the maximum of a finite number of continuous and unimodal functions of one variable. The set of feasible solutions in described by a system of $(\max, \min)-$linear (or using an alternative notation $(\max, \wedge)-$linear) equations with variables on both sides. Finite iteration method for solving these systems of $(\max, \wedge)-$linear equations is used to introduce an algorithm for finding the optimal solution of the problem.

Max-algebras naturally arise in many contexts, such as decision theory, discrete event dynamic systems, and operations research. Namely find the optimal solution for the $\min - \max$ objective function subject to systems of two-sided linear equation in $\max -$algebra. In section 4.4, we will shows one possible application of the the optimization problem considered in this chapter.
In recent time, the attention was devoted to systems, in which the variables occur on both sides in the equations or inequalities (see e.g. [1], [5], [2], [8], [12], [17], [18], [31], [32], [37], [41]). In this chapter we will consider optimization problems, the set of feasible solutions of which is described by systems of two-sided (max,

min)-linear separable equations. The objective functions of the problems considered are equal to the maximum of a finite number of continuous and unimodal functions $f_j(x_j)$ of one variable. The extension to systems of inequalities of the same structure is a purely technical problem, we simply need to introduce slack variables on appropriate sides and transform the inequalities to equations. Similarly, the algorithm proposed below can be extended to objective functions with convex or concave function.

5.1 Two-Sided (max, min)− Linear Systems of Equations

Let $a_{ij}, b_{ij} \in R$, $i \in I$, $j \in J$ be given numbers, let

$$a_i(x) \equiv \max_{j\in J}(a_{ij} \wedge x_j) \quad \text{for all } \; i \in I\,,$$
$$b_i(x) \equiv \max_{j\in J}(b_{ij} \wedge x_j) \quad \text{for all } \; i \in I\,,$$

We will consider the following system of (max, min)-linear equations

$$a_i(x) = b_i(x) \qquad \text{for all } i \in I. \tag{5.1}$$

The set of all solutions of system (5.1) will be denoted by M. We define further sets $M(\overline{x}),\; M_i(\overline{x})$ for any $\overline{x} \in \overline{R}^n,\; i \in I$

$$M(\overline{x}) \equiv \{x | x \in M \;\&\; x \leq \overline{x}\} \tag{5.2}$$
$$M_i(\overline{x}) \equiv \{x | a_i(x) = b_i(x) \;\&\; x \leq \overline{x}\} \tag{5.3}$$

Clearly, $M(\overline{x}),\; M_i(\overline{x})$ are always nonempty, since e.g. $x(\alpha) \equiv (\alpha,\; \ldots,\; \alpha) \in M(\overline{x})$, if $\alpha \leq \min_{(i,j)\in I\times J}(a_{ij} \wedge b_{ij} \wedge \overline{x}_j)$. Moreover, if $\overline{x} = (\infty,\; \ldots,\; \infty)$, then evidently $M(\overline{x}) = M$, and if $\underline{x} = (-\infty,\; \ldots,\; -\infty)$, then $\underline{x} \leq x$ for any $x \in M$.

Remark 5.1.1 The algorithm presented in [17] finds the maximum element of $M(\overline{x})$ for any given $\overline{x}$, i.e. such an element $x^{\max} \in M(\overline{x})$ that $x \leq x^{\max}$ for all $x \in M(\overline{x})$.

In what follows we will prepare the theoretical background for an algorithm which finds the maximum element $x^{\max}$ of $M(\overline{x})$. Using the results of [17], if $\overline{x} \in M(\overline{x}))$ then we have evidently $x^{\max} = \overline{x}$. Therefore we will assume in what follows that $\overline{x} \notin M(\overline{x})$. Further, we can assume w.l.o.g. that the notation was possibly changed in such a way that $a_i(\overline{x}) \leq b_i(\overline{x})$ for all $i \in I$. Since we assumed that $\overline{x} \notin M(\overline{x})$, the set $I^{<}(\overline{x}) \equiv \{i \in I \ ; \ a_i(\overline{x}) < b_i(\overline{x})\}$ is nonempty. Let us set further $I^{=}(\overline{x}) \equiv \{i \in I \ ; \ a_i(\overline{x}) = b_i(\overline{x})\}$.
Let us introduce the following notation for any given upper bound $\overline{x}$

$$\begin{aligned}
\alpha(\overline{x}) &\equiv \min_{i \in I^{<}(\overline{y})} a_i(\overline{x}), \\
I^{<}(\alpha(\overline{x})) &\equiv \{i \in I^{<}(\overline{x}) \ ; \ a_i(\overline{x}) = \alpha(\overline{x})\}\,, \\
I^{=}(\alpha(\overline{x})) &\equiv \{i \in I^{=}(\overline{x}) \ ; \ a_i(\overline{x}) \leq \alpha(\overline{x})\}\,, \\
J(\alpha(\overline{x})) &\equiv \{j \in J \ ; \ \exists\, i \in I^{<}(\alpha(\overline{x})) \text{ such that } b_{ij} \wedge \overline{x}_j > \alpha(\overline{x})\}\,.
\end{aligned}$$

To simplify the explanations, we will replace in what follows the notation $\alpha(\overline{x})$ with α, if it does not cause any confusion.

Theorem 5.1.1 *Let $\overline{x} \notin M(\overline{x})$, let vector $\tilde{x}$ be defined as follows*

$$\tilde{x}_j = \alpha \text{ for } j \in J(\alpha), \quad \tilde{x}_j = \overline{x}_j \text{ for } j \in J \setminus J(\alpha). \tag{5.4}$$

Then $\tilde{x}$ is the maximum element of the set of all solutions of the system

$$a_i(x) = b_i(x) \quad \text{ for all } i \in I^{<}(\alpha) \cup I^{=}(\alpha), \tag{5.5}$$

$$x \leq \overline{x} \qquad \text{ for all } j \in J. \tag{5.6}$$

Proof: Let $k \in I^{<}(\alpha)$ be arbitrarily chosen. Then $a_k(\overline{x}) = \alpha < b_k(\overline{x})$. Let us set $J_k(\alpha) \equiv \{j \in J \ ; \ b_{kj} \wedge \overline{x}_j \ > \ \alpha\}$. Then $J_k(\alpha) \neq \emptyset$, $J_k(\alpha) \ \subseteq \ J(\alpha)$. Note that for any $j \in J_k(\alpha)$ both $b_{kj} > \alpha$ and $\overline{x}_j > \alpha$. It follows immediately from the definition of $\tilde{x}$ (compare (5.4)) that $b_{kj} \wedge \tilde{x}_j \leq \alpha$ for all $j \in J$ and $b_{kj} \wedge \tilde{x}_j = \alpha$ for all $j \in J_k(\alpha)$ so that $b_k(\tilde{x}) = \alpha$. Let us remind that $a_k(\overline{x}) = \alpha$. Let p be any index of J such that $a_k(\overline{x}) = a_{kp} \wedge \overline{x}_p$ so that $a_{kp} \wedge \overline{x}_p = \alpha$ and we have according to (5.4) $a_{kp} \wedge \tilde{x}_p = a_{kp} \wedge \overline{x}_p = a_k(\overline{x}) = \alpha$ if $p \notin J(\alpha)$, and

$a_{kp} \wedge \tilde{x}_p = a_{kp} \wedge \alpha = \alpha$ if $j \in J(\alpha)$. Since $a_{kj} \wedge \tilde{x}_j \leq a_{kj} \wedge \overline{x}_j$ for all $j \in J$, we obtain that $a_k(\tilde{x}) = \alpha = b_k(\tilde{x})$.

Let us assume now that k is an arbitrary index of $I^{=}(\alpha)$ so that we have $a_k(\overline{x}) \leq \alpha$ and $a_k(\overline{x}) = b_k(\overline{x}) = \beta_k \leq \alpha$. Let $s \in J$ be an index such that $a_k(\overline{x}) = a_{ks} \wedge \overline{x}_s$. If $s \notin J(\alpha)$, then $\tilde{x}_s = \overline{x}_s$ (compare (5.4)) and thus $a_{ks} \wedge \tilde{x}_s = a_{ks} \wedge \overline{x}_s = \beta_k$.

Let us assume now that $s \in J(\alpha)$. Then there exists index $i \in I$ such that $b_{is} \wedge \overline{x}_s > \alpha$ and therefore it must be $\overline{x}_s > \alpha$. Since we assumed that $a_k(\overline{x}) = a_{ks} \wedge \overline{x}_s = \beta_k \leq \alpha$, and we have $\overline{x}_s > \alpha$, it must be $a_{ks} = \beta_k$. Since $s \in J(\alpha)$, we have $\tilde{x}_s = \alpha \geq \beta_k$. We have therefore $a_{ks} \wedge \tilde{x}_s = \beta_k$. Since otherwise for all $j \in J$ the inequality $a_{kj} \wedge \tilde{x}_j \leq a_{kj} \wedge \overline{x}_j \leq \beta_k$ holds, we obtain $a_k(\tilde{x}) = \beta_k$.

Let us derive now value $b_k(\tilde{x})$. We assumed that $b_k(\overline{x}) = \beta_k \leq \alpha$. Let us assume that $b_k(\overline{x}) = b_{ks} \wedge \overline{x}_s$. Similarly as above, we have $b_{ks} \wedge \tilde{x}_s = b_{ks} \wedge \overline{x}_s$ if $s \notin J(\alpha)$. If $s \in J(\alpha)$, then similarly as above $\overline{x}_s > \alpha \geq \beta_k$, $\tilde{x}_s = \alpha \geq \beta_k$ and therefore it must be $b_{ks} = \beta_k$, so that $b_{ks} \wedge \tilde{x}_s = b_{ks} \wedge \alpha = \beta_k$. Since otherwise $b_{kj} \wedge \tilde{x}_j \leq b_{kj} \wedge \overline{x}_j$ for all $j \in J$, we obtain that $b_k(\tilde{x}) = \beta_k = a_k(\tilde{x})$, i.e. the equality with index $k \in I^{=}(\alpha)$, which holds at point $\overline{x}$, remains satisfied also at point $\tilde{x}$.

It remains to prove that $\tilde{x}$ is the maximum element satisfying system (5.5), (5.6). Let us assume for this purpose that x is any point such that $\tilde{x} \leq x \leq \overline{x}, x \neq \tilde{x}$ so that there exists an index $r \in J$ such that $\tilde{x}_r < x_r \leq \overline{x}_r$. Therefore it must be $r \in J(\alpha)$ and there exists an index $i \in I^{<}(\alpha)$ such that $a_(\tilde{x}) = \alpha < b_{ir} \wedge x_r \leq b_i(x)$ and according to the considerations above $a_{ir} = \alpha = a_{ir} \wedge \tilde{x}_r = a_{ir} \wedge x_r$. Since this equality holds for any index r with the property $\tilde{x}_r < x_r \leq \overline{x}_r$ and for the other indices $j \in J$ we have $x_j = \tilde{x}_j = \overline{x}_j$, we obtain that $a_i(x) = a_i(\tilde{x}) = \alpha < b_{ir} \wedge x_r \leq b_i(x)$, and therefore x does not satisfy system (5.5), (5.6), which completes the proof.

□

Summarizing the considerations above, we propose the following procedure to find the maximum element $x^{\max}$ of set $M(\overline{x})$. Using (5.4), we find $\tilde{x}$, which is according to Theorem 5.1.1 the maximum element of the set of all solutions of system (5.5), (5.6). Therefore, if $\tilde{x} \in M(\overline{x})$, then $\tilde{x} = x^{\max}$ and we stop. Otherwise, we use $\tilde{x}$ as the new upper bound and repeat the procedure.

Let us assume that we have changed the notation in such a way that $a_i(\tilde{x}) \leq b_i(\tilde{x})$ for all $i \in I$. and let $\tilde{x} \notin M(\overline{x})$. Let us return to the notation $\alpha(\overline{x})$ for any upper bound $\overline{x}$ and let $\alpha(\tilde{x}) \equiv \min_{i \in I^{<}(\tilde{x})} a_i(\tilde{x})$. Since after possibly changing the notation such that $a_i(\tilde{x}) \leq b_i(\tilde{x})$ for all $i \in I$, $\alpha(\tilde{x}) \equiv \min_{i \in I^{<}(\tilde{x}} a_(\tilde{x}) \geq \alpha(\overline{x})$, we will have $I^{=}(\alpha(\overline{x})) \subseteq I^{=}(\alpha(\tilde{x}))$. Since $b_{ij} \wedge \overline{x}_j \leq \alpha(\overline{x})$ for all $j \in J(\alpha(\overline{x}))$, we have $J(\alpha(\tilde{x}) \cap J(\alpha(\overline{x}) = \emptyset$. Therefore, if we use $\tilde{x}$ as a new upper bound on the next iteration, we will decrease in (5.4) at least one new variable.Therefore we will have at most n such iterations.

Besides, since $\alpha(\tilde{x}) \geq \alpha(\overline{x})$, all already satisfied equations with indices $i \in I^{<}(\alpha(\overline{x})) \cup I^{=}(\alpha(\overline{x}))$ will remain satisfied in accordance with Theorem 5.1.1. It follows that in the next iteration with the new upper bound $\tilde{x}$ after applying formula (5.4), the already satisfied equations remain satisfied and at least one new equation with index $i \in I^{<}(\alpha(\tilde{x})$ will be satisfied. Therefore the number of iterations does not exceed $\min(n, m)$. We will describe now the corresponding algorithm explicitly step by step.

Algorithm 5.1.1

$\boxed{0}$ *Input* $I,\ J,\ \underline{x}, \overline{x},\ a_{ij}$ *and* b_i *for all* $i \in I$ *and* $j \in J$;

$\boxed{1}$ *If* $\overline{x} \in M(\overline{x})$, *then* $x^{\max} := \overline{x}$, *STOP;*

$\boxed{2}$ *Change of notation such that* $a_i(\overline{x}) \leq b_i(\overline{x})$ *for all* $i \in I$;

$\boxed{3}$ *Find* $\alpha(\overline{x}),\ I^{<}(\alpha(\overline{x})),\ I^{=}(\alpha(\overline{x}))$;

$\boxed{4}$ *Set* $\tilde{x}_j := \alpha(\overline{x})$ *if* $j \in J(\alpha(\overline{x}))$, $\tilde{x}_j := \overline{x}_j$ *otherwise;*

$\boxed{5}$ *If* $\tilde{x} \in M(\overline{x})$, *then* $x^{\max} := \tilde{x}$, *STOP;*

$\boxed{6}$ *Set* $\overline{x} := \tilde{x}$, *go to* $\boxed{2}$;

5.2 Solving Optimization Problems under Two-Sided $(\max, \min)-$ Linear Equation Constraints - an Iteration Method

In this section we consider an optimization problem that is a combination of the problems solved in Section 4.2 and Section 5.1. In other words, we solve the following optimization problem

$$f(x) \equiv \max_{j \in J} f_j(x_j) \quad \longrightarrow \quad \min \tag{5.7}$$

subject to

$$\max_{j \in J}(a_{ij} \wedge x_j) \ = \max_{j \in J}(b_{ij} \wedge x_j), \quad \text{for all} \quad i \in I, \tag{5.8}$$

$$\underline{x} \le x \le \overline{x}, \tag{5.9}$$

where $\underline{x},\ \overline{x} \in R^n,\ I \equiv \{1,\ \dots,\ m\},\ J \equiv \{1,\ \dots,\ ...n\},\ a_{ij}, b_{ij} \in R$ for all $i \in I, j \in J$ are given and $f_j(x_j),\ j \in J$ are continuous and unimodal functions on $[\underline{x}_j\ ,\ \overline{x}_j]$.

We use the notation introduced in Section 5.1. In particular, M denotes the set of all solutions of (5.8), moreover, we use notation

$$M(\underline{x}, \overline{x}) \equiv \{x | x \in M \ \&\ \underline{x} \le x \le \overline{x}\}.$$

We assume that $f_j(x_j),\ j \in J$ are continuous and unimodal functions on $[\underline{x}_j\ ,\ \overline{x}_j]$ and denote $x_j^* \equiv \text{argmin}(f_j(x_j))\ ;\ x_j \in R$ so that f_j is strictly decreasing on $[-\infty, x_j^*]$ and strictly increasing on $[x_j^*, \infty]$. To simplify the explanation, we will assume that $x_j^* \in [\underline{x}_j, \overline{x}_j]$ for all $j \in J$ and set $\underline{f}_j \equiv f(x_j^*),\ \overline{f_j} \equiv \max\{f_j(\underline{x}_j)\ ,\ f_j(\overline{x}_j)\}$.

In what follows we will propose an iteration algorithm for solving minimization problem (5.7), (5.8), (5.9).

Lemma 5.2.1 Let $\underline{x} \leq \overline{x}$, let $x^{\max}$ be the maximum element of $M(\overline{x})$. Then $M(\underline{x}, \overline{x}) \neq \emptyset$ if and only if $\underline{x} \leq x^{\max}$.

Proof: If $\underline{x} \leq x^{\max}$, then $x^{\max} \in M(\underline{x}, \overline{x})$ and thus $M(\underline{x}, \overline{x}) \neq \emptyset$. If $\underline{x} \not\leq x^{\max}$, then $M(\underline{x}, \overline{x}) = \emptyset$, since any element of $M(\underline{x}, \overline{x})$ would have to satisfy the inequalities $\underline{x} \leq x \leq x^{\max}$, which is impossible under the assumption that $\underline{x} \not\leq x^{\max}$.

□

Definition 5.2.1 Let x^{opt} be the optimal solution of problem (5.7), (5.8), (5.9). An element $x(\varepsilon)$ is called an $\varepsilon-$approximation of x^{opt} if $f(x(\varepsilon)) - f(x^{\text{opt}}) < \varepsilon$ and $x(\varepsilon) \in M(\underline{x}, \overline{x})$.

Lemma 5.2.2 Let $j \in J$ and $\alpha \in R$. Let us set $V_j(\alpha) \equiv \{x_j \in [\underline{x}_j\ ,\ \overline{x}_j]\ ;\ f_j(x_j) \leq \alpha\}$. Then either $V_j(\alpha) = \emptyset$ or $V_j(\alpha) = [\underline{x}_j(\alpha)\ ,\ \overline{x}_j(\alpha)]$, where $\underline{x}_j(\alpha) \geq \underline{x}_j$, $\overline{x}_j(\alpha) \leq \overline{x}_j$.

Proof: Since we assumed that functions $f_j,\ j \in J$ are continuous and unimodal and the turning point x_j^* is contained in interval $[\underline{x}_j, \overline{x}_j]$, we have $\text{argmin}(f_j(x_j)) = x_j^*$. Therefore if $f_j(x_j^*) > \alpha$, then $V_j(\alpha) = \emptyset$ and otherwise $V_j(\alpha) = [\underline{x}_j(\alpha), \overline{x}_j(\alpha)]$ is a subinterval of interval $[\underline{x}_j, \overline{x}_j]$ with the property $f_j(\underline{x}_j(\alpha)) = f_j(\overline{x}_j(\alpha)) = \alpha$.

□

Algorithm 5.2.1

[1] *Input* $\underline{x},\ \overline{x},\ \varepsilon > 0,\ x_j^* := \text{argmin}\{f_j(x_j)\ ;\ \underline{x}_j \leq x_j \leq \overline{x}_j\}$ *for all* $j \in J$, $\underline{f} := f(x^*),\ \overline{f} := \max(f(\underline{x}), f(\overline{x}))$*;*

[2] *Find the maximum element* $\tilde{x}$ *of* $M(\overline{x})$ *using the method from [17];*

[3] *If* $\underline{x} \not\leq \tilde{x}$*, then* $M(\underline{x}, \overline{x}) = \emptyset$*, STOP.*

[4] $\overline{f} := \max(f(\underline{x}), f(\tilde{x}))$*, find* $\underline{x}(\alpha), \overline{x}(\alpha)$*;*

[5] *If* $(f(\tilde{x}) - \underline{f}) \leq \varepsilon$, *then* $\tilde{x}$ *is the* ε*-approximation of the optimal solution, STOP.*

[6] $\alpha := \underline{f} + (\overline{f} - \underline{f})/2$, *find* $\underline{x}(\alpha),\ \overline{x}(\alpha),\ \underline{f} := \min\{f(x);\ \underline{x}(\alpha) \leq x \leq \overline{x}(\alpha)\}$;

[7] *Find the maximum element* $\tilde{x}^{(1)}$ *of* $M(\overline{x}(\alpha))$ *using the method from [17];*

[8] *If* $\underline{x}(\alpha) \not\leq \tilde{x}^{(1)}$, *then set* $\underline{f} := f(\overline{x}(\alpha))$, *go to* [6];

[9] *Set* $\tilde{x} := \tilde{x}^{(1)}$, *go to* [4].

5.3 Applications and Numerical Examples

Example 5.3.1

Let us consider a situation, in which transportation means of different size provide transporting goods from places $i \in I$ to one terminal T. The goods are unloaded in T and the transportation means (possibly with other goods uploaded in T) have to return to i. We assume that the connection between i and T is only possible via one of the places (e.g. cities) $j \in J$ the roads between i and j are one-way roads, and the capacity of the road between $i \in I$ and $j \in J$ is equal to a_{ij}. We have to join each place j with T by a two-way road with a capacity x_j in both directions. The total capacity of the connection between i and T is therefore equal to $max_{j\in J}(a_{ij} \wedge x_j)$.

In the opposite direction, the transport from T to i is carried out via other one-way roads between places $j \in J$ and $i \in I$ with (in general, different) capacities between j and i equal to b_{ij}. Since the roads between T and j are two-way roads, the total capacity of the connection between T and i is equal to $max_{j\in J}(b_{ij} \wedge x_j)$, for all $i \in I$.

We assume that the transportation means can only pass through some roads with the capacity which is not smaller than the capacity of the transportation mean and our task is to choose appropriate capacities $x_j, j \in J$. In order that each of the transportation means may return to i, it is natural to require for each i that the maximal attainable capacity of connections between i and T via j is equal to maximal attainable capacity of connections between T and i on the way back. In other words, we have to choose $x_j, j \in J$ in such a way that in the next problems, (see [17]).

$$\max_{j\in J}(a_{ij} \wedge x_j) = \max_{j\in J}(b_{ij} \wedge x_j) \quad \text{for all } \; i \in I\,.$$

This system of two-sided (max, min)-linear systems of equations with the same variable on both sides.
If in a practical problem, we have to join places j with terminal T by a one-way road with a capacity x_j in direction from places j to terminal T and a capacity y_j in direction from terminal T to places j. The total capacity of the connection between i and T is therefore equal to $max_{j\in J}(a_{ij} \wedge x_j)$. The transport from T to i is carried out via other one-way roads between places $j \in J$ and $i \in I$ with (in general, different) capacities between j and i equal to b_{ij}. Since the roads between T and j are one-way roads, the total capacity of the connection between T and i is equal to $max_{j\in J}(b_{ij} \wedge y_j)$, for all $i \in I$. We assume that the transportation means can only pass through some roads with the capacity which is not smaller than the capacity of the transportation mean and our task is to choose appropriate capacities x_j & $y_j, j \in J$. In order that each of the transportation means may return to i, it is natural to require for each i that the maximal attainable capacity of connections between i and T via j is equal to maximal attainable capacity of connections between T and i on the way back. In other words, we have to choose x_j & $y_j, j \in J$ in such a way that in the next problems,

$$\max_{j\in J}(a_{ij} \wedge x_j) = \max_{j\in J}(b_{ij} \wedge y_j) \quad \forall \quad i \in I\,. \tag{5.10}$$

This system with different variables on every side. It is easy to change it to the system with the same variables on both sides by introducing new variables $z = (x, y)$ and appropriate sufficiently small coefficients a_{ij} for $j > n$ on the left side and sufficiently small coefficients b_{ik} for $k < n$ on the right side, we obtain the system

$$\max_{j\in K}(a_{ij} \wedge z_j) = \max_{j\in K}(b_{ij} \wedge z_j) \quad \forall \quad i \in I,$$

where $K = \{1,\ 2,\ \dots,\ 2n\}$. This system has the form as system of two-sided (max, min)-linear systems of equations with the same variable in both sides.
Let us assume further that the choice of $x_j,\ y_k$ is connected with penalties $f_j(x_j),\ g_k(y_k)$ respectively. The penalties may be connected e.g. with some eco-

nomic or ecologic requirements (costs, air pollution) so that it is quite natural to accept that $f_j,\ g_k$ with $j \in J, k \in K$ are continuous strictly increasing functions. The problem of minimizing the maximum penalty under the constraints given by (5.10) and by some lower and upper bounds on x, y, can be easily transformed to an optimization problem of the form (5.7), (5.8)), (5.9).

Example 5.3.2

Let us consider the following optimization problem:
minimize

$$f(x) \equiv \max\big(f_1(x_1), f_2(x_2), f_3(x_3)\big)$$

subject to

$$\max(4 \wedge x_1, 5 \wedge x_2, 0 \wedge x_3) = \max(0 \wedge x_1, 4 \wedge x_2, 5 \wedge x_3),$$

$$\underline{x} \le x \le \overline{x},$$

where $\underline{x} = (5,3,4),\ \overline{x} = (6,6,6)$ and functions $f_j(x_j), j \in J \equiv \{1,2,3\}$ are defined by formulas

$$\begin{aligned} f_1(x_1) &= \max(-x_1 + 21/4, x_1 - 21/4),\\ f_2(x_2) &= \max(-x_2 + 4, x_2 - 4),\\ f_3(x_3) &= (x_3 - 5)^2. \end{aligned}$$

Thus, $x^* = (21/4,\ 4,\ 5)$.

In accordance with algorithm 5.2.1 described in Section 5.2, we proceed as follows

1. $\underline{x} = (5,3,4),\ \overline{x} = (6,6,6),\ ,\ x^* = (21/4,4,5),\ \underline{f} := 0,\ \overline{f} := 2,\ \varepsilon = 1/2;$

2. $\tilde{x} = \overline{x} = (6,6,6);$

3. $\underline{x} \le \tilde{x};$

4. $\overline{f} := f(\tilde{x}) = 2,\ \alpha := 0 + (2\ -\ 0)/2 = 1,\ \underline{x} := \underline{x}(\alpha) = (17/4,\ 3,\ 4)),\ \overline{x}(\alpha) = (6,\ 5,\ 6);$

5. $\tilde{x}^{(1)}\ =\ (6,5,5);$

6 $\underline{x} \le \tilde{x}^{(1)}$, $\tilde{x} := \tilde{x}^{(1)}$;

8 $f(\tilde{x}) - \underline{f} = 1 > \varepsilon = 1/2$;

4 $\overline{f} := f(\tilde{x}) = 1,\ \alpha = 0 + (1-0)/2 = 1/2,\ \underline{x}(\alpha) = (19/4,\ ,\ 7/2,\ 9/2),\ \underline{x} := \overline{x}(\alpha) = (23/4,\ 9/2,\ 11/2)$;

5 $\tilde{x}^{(1)} = (23/4,\ 9/2,\ 9/2)$;

6 $\underline{x} \le \tilde{x}^{(1)}$, $\tilde{x} := \tilde{x}^{(1)}$;

8 $f(\tilde{x}) - \underline{f} = 1/2 - 0 = 1/2 = \varepsilon$;

9 $\tilde{x} = (23/4,\ 9/2,\ 9/2)$ *is the* ε*-optimal solution, STOP;*

Example 5.3.3

In this example we consider a modification of the above problem - now the penalty functions are increasing. The problem sounds as follows:
minimize

$$f(x) \equiv \max(2x_1, 3x_2, x_3)$$

subject to

$$\max(4 \wedge x_1, 5 \wedge x_2, 0 \wedge x_3) = \max(0 \wedge x_1, 4 \wedge x_2, 5 \wedge x_3),$$

$$\underline{x} \le x \le \overline{x},$$

where $\underline{x} = (5,3,5)$, $\overline{x} = (6,6,6)$. For the modified objective function f the computation gives a different result shown below.

1 $\underline{x} = (5,3,5),\ \overline{x} = (6,6,6),\ \underline{f} := 10,\ \overline{f} := 18,\ \varepsilon = 1$;

2 $\tilde{x} = \overline{x} = (6,6,6)$;

3 $\underline{x} \le \tilde{x}$;

4 $\overline{f} := f(\tilde{x}) = 18,\ f^{(1)} := 10 + (18 - 10)/2 = 14,\ \overline{x}^{(1)} := (6, 14/3, 6)$;

5 $\tilde{x} = (6, 14/3, 14/3)$;

6 $\underline{x} \not\le \tilde{x}$;

$\boxed{7}$ $\underline{f} := f(\overline{x}^{(1)}) = 14,\ \tilde{x} := \overline{y}$, go to $\boxed{4}$;

$\boxed{4}$ $\overline{f} := f(\tilde{x}) = 18,\ f^{(1)} := 14 + (18\ -\ 14)/2 = 16,\ \overline{x}^{(1)} := (6, 16/3, 6)$;

$\boxed{5}$ $\tilde{x} = (6, 16/3, 5)$;

$\boxed{6}$ $\underline{x}\ \leq\ \tilde{x}^{(1)}, \tilde{x}\ :=\ \tilde{x}^{(1)} = (6, 16/3, 5)$;

$\boxed{8}$ $f(\tilde{x})\ -\ \underline{f} = 16 - 14 = 2 > \varepsilon$;

$\boxed{4}$ $\overline{f} := f(\tilde{x}) = 16,\ f^{(1)} := 14 + (16 - 14)/2 = 15,\ \overline{x}^{(1)} := (6, 5, 5)$;

$\boxed{5}$ $\tilde{x} = (6, 5, 5)$;

$\boxed{6}$ $\underline{x}\ \leq\ \tilde{x}$;

$\boxed{8}$ $f(\tilde{x})\ -\ \underline{f} = 15 - 14 = 1 = \varepsilon$;

$\boxed{9}$ $\tilde{x}$ is the ε-optimal solution, STOP;

6

Optimization Problems under Two-Sided (max, min)-Linear Inequality Constraints - a Threshold Method

We consider optimization problems, the objective function of which is equal to the maximum of a finite number of continuous and unimodal functions of one variable. The set of feasible solutions is described by a system of $(max, min)-$linear (or using an alternative notation $(max, \wedge)-$linear) inequalities with variables on both sides. Let us note that any $(\max, \min)$-linear inequality can be transformed to equation by introducing a slack variable having a sufficiently high upper bound (e.g. greater than all coefficients of the system) on the appropriate side. Therefore the algorithm described in Chapter 3 can be used also for solving systems of $(\max, \min)-$linear inequalities. Let us consider the simple example to clarify this technique.

Example 6.0.4

Consider the system of inequalities

$$\max\{5 \wedge x_1, 7 \wedge x_2, 10 \wedge x_3\} \;\leq\; \max\{3 \wedge x_1, 5 \wedge x_2, 6 \wedge x_3\}$$

$$\max\{9 \wedge x_1, 8 \wedge x_2, 1 \wedge x_3\} \;\geq\; \max\{5 \wedge x_1, 1 \wedge x_2, 2 \wedge x_3\}$$

$$x_j \leq 15 \quad \forall j \in j = \{1, 2, 3\}$$

We can replace this system by the following system of equations by introducing slack variables $x_4,\ x_5$*:*

$$\max\{5 \wedge x_1, 7 \wedge x_2, 10 \wedge x_3\, 15 \wedge x_4\} = \; \max\{3 \wedge x_1, 5 \wedge x_2, 6 \wedge x_3\}$$

$$\max\{9 \wedge x_1, 8 \wedge x_2, 1 \wedge x_3\} \;=\; \max\{5 \wedge x_1, 1 \wedge x_2, 2 \wedge x_3\, 15 \wedge x_5\}$$

$$x_j \leq 15 \quad \forall j \in j = \{1, 2, 3, 4, 5\}$$

The maximum element of the first system is $x^{\max} = (15, 6, 6)$*. The second system has the maximum element* $x^{\max} = (15, 6, 6, 15, 9)$*. Let us note that if the upper bound of* x_5 *were not high enough (e.g. less than* 9*), then the second inequality could have not been transformed to equality by an appropriate choice of the value* x_5 *and we would have to choose* $x_1 < 15$ *and the two systems would not be equivalent.*

The transformation of systems of $(\max, \min)$-linear inequalities to equivalent systems of $(\max, \min)$-linear equations shows that the systems of inequalities have similar properties like systems of equations:

(1) If no lower bound and a finite upper bound on variables is given, the system is always solvable and the set of solutions of the inequality system has the unique finite maximum element $x^{\max}$;

(2) If $\underline{x}$ is a finite lower bound of the variables, then the system of inequalities is solvable if and only if $\underline{x} \leq x^{\max}$.

Let us note that if we have variables x on the left hand sides and different variables y on the right hand sides, the system can be processed like the one-sided system as in the first Chapter. Including lower and upper bounds on $x,\ y$ is only a technical problem.

The practical application of the problem that will be presented in this chapter, it will be like the problem that was presented such as Example 5.3.1 in the previous section with a simple change so that we will choose appropriate capacities $x_j, j \in J$. In order that each of the transportation means may return to i, we may e.g. require for each i that the maximal attainable capacity of connections between i and T via j is greater than or equal to maximal attainable capacity of connections between T and i on the way back. In other words, we have to choose $x_j, j \in J$, which satisfy relation (6.1) below.

In what follows, assume that we have the same variables on the left hand sides and right hand sides of the inequality system.

6.1 Systems of Two-Sided $(\max, \min)-$ Linear Inequality

Let us consider the following system of inequalities:

$$a_i(x) \geq b_i(x), i \in I, \tag{6.1}$$

where similarly as in the preceding section $a_i(x) = \max_{j \in J}(a_{ij} \wedge x_j),\ b_i(x) = \max_{j \in J}(b_{ij} \wedge x_j)$, and $a_{ij},\ b_{ij} \in R,\ i \in I,\ j \in J$ be given numbers. Let $M^{\geq}$ denote the set of all solutions of system (6.1). We will set for any $x, y \in R^n : x \leq y \Leftrightarrow x_j \leq y_j \quad \forall j \in J$. Let us set

$$M^{\geq}(\underline{x}, \overline{x}) = \{x \ ; \ x \in M^{\geq} \ \& \ \underline{x} \leq x \leq \overline{x}\}$$

for any finite $\underline{x} \leq \overline{x}$ and let $x^{\max}$ denote the maximum element of $M^{\geq}(\underline{x}, \overline{x})$. So that $M^{\geq}(\underline{x}, \overline{x}) \subset M^{\geq}$, and $M^{\geq}(\underline{x}, x^{\max}) \subset M^{\geq}$, also it is clear $M^{\geq}(\underline{x}, x^{\max}) \subseteq M^{\geq}(\underline{x}, \overline{x})$.
To prove $M^{\geq}(\underline{x}, \overline{x}) \subseteq M^{\geq}(\underline{x}, x^{\max})$ there are two cases: the first one, if $\overline{x} \notin M^{\geq}$, then $x^{\max} < \overline{x}$. Therefore $\forall\ x \in M^{\geq}(\underline{x}, \overline{x})$, the inequality $x \leq x^{\max}$ verified, i.e. $x_j \leq x_j^{\max}\ \forall\ j \in J$ and if $x^* \in (x^{\max}, \overline{x}]$, (i.e. $x^{\max} < x^* \leq \overline{x}$, i.e. $x_{j_0}^{\max} < x_{j_0}^* \leq \overline{x}_{j_0}$ for at least one $j_0 \in J$ and $x_j^{\max} \leq x_j^* \leq \overline{x}_j$ for $j \in J \ \& \ j \neq j_0$)

then $x^* \notin M^{\geq}$, otherwise x^* is the maximum element of $M^{\geq}(\underline{x}, \overline{x})$, but this contradicts the hypothesis $x^{\max}$ is the maximum element of $M^{\geq}(\underline{x}, \overline{x})$. So that for any $x \in M^{\geq}(\underline{x}, \overline{x})$, we have $x \leq x^{\max}$, and $x \in M^{\geq}(\underline{x}, x^{\max})$, then $M^{\geq}(\underline{x}, \overline{x}) \subseteq M^{\geq}(\underline{x}, x^{\max})$. The second case, if $\overline{x} \in M^{\geq}$, then $x^{\max} = \overline{x}$, $M^{\geq}(\underline{x}, \overline{x}) \subseteq M^{\geq}(\underline{x}, x^{\max})$. Then we have

$$M^{\geq}(\underline{x}, x^{\max}) = M^{\geq}(\underline{x}, \overline{x}) \subset M^{\geq}.$$

In this section we will propose an algorithm, which find the maximum element of the set $M^{\geq}(\underline{x}, \overline{x})$, and calculates the maximum solution of system (6.1), take in account $\underline{x} \leq x \leq \overline{x}$. Note that, since any equation can be replaced by two inequalities, therefor we can use the next algorithm to find the maximum element of the set $M^{=}(\underline{x}, \overline{x})$, which is the set of all solutions of a system of equations, $(a_i(x) = b_i(x), i \in I)$. So that we will adjust the algorithm for systems of equations described chapter 5, which find the maximum element of the set $M^{=}$. This can be done simply by leaving out step $\boxed{3}$ of the algorithm in the preceding chapter and make little change in that algorithm. We will provide this algorithm after adjusted as the following:

Algorithm 6.1.1 *We will provide algorithm, which find the maximum element of the set of all solutions of system (6.1) with the boundary conditions* $\underline{x} \leq x \leq \overline{x}$.

$\boxed{0}$ *Input* $I,\ J,\ \overline{x},\ a_{ij}$ *and* b_{ij} *for all* $i \in I$ *and* $j \in J$.

$\boxed{1}$ *Find* $I^{<}(\overline{x}) \equiv \{i \in I\ ;\ a_i(\overline{x}) < b_i(\overline{x})\}$.

$\boxed{2}$ *If* $I^{<}(\overline{x}) = \emptyset$, *then* $x^{\max} := \overline{x}$, *STOP.*

$\boxed{3}$ *Find* $\alpha(\overline{x}) \equiv \min_{i \in I^{<}(\overline{x})} a_i(\overline{x})$.

$\boxed{4}$ *Find* $I^{<}(\alpha(\overline{x})) \equiv \{i \in I^{<}(\overline{x})\ ;\ a_i(\overline{x}) = \alpha(\overline{x})\}$.

$\boxed{5}$ *Find* $H_i^{<}(\overline{x}) \equiv \{j \in J\ ; b_{ij} \wedge \overline{x}_j > \alpha(\overline{x})\}\ ,\ \forall\ i \in I^{<}(\alpha(\overline{x}))$.

$\boxed{6}$ *Set* $H^{<}(\overline{x}) := \bigcup_{i \in I^{<}(\alpha(\overline{x}))} H_i^{<}(\overline{x})$.

$\boxed{7}$ *Set* $\overline{x}_j := \alpha(\overline{x})$ *for all* $j \in H^{<}(\overline{x})$ *go to* $\boxed{1}$.

We will illustrate the performance of this algorithm by the following small numerical example.

Example 6.1.1
Let $J = \{1, 2, 3, 4\}$*,* $I = \{1, 2, 3\}$*,* $\overline{x} = (10, 10, 10, 10)$*, and consider system (6.1) of inequalities where* a_{ij} & b_{ij} $\forall$ $i \in I$ *and* $j \in J$ *are given by the matrices* A *and* B *as follows:*

$$A = \begin{pmatrix} 7 & 5 & 3 & 0 \\ 4 & 3 & 1 & 2 \\ 10 & 20 & 10 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 6 & 13 & 10 & -1 \\ 8 & 0 & 3 & 1 \\ 1 & 1 & 1 & -8 \end{pmatrix}$$

By substitution for these values in system (6.1) and using Algorithm (6.1.1):

Iteration 1:

$\boxed{1}$ $I^{<}(\overline{x}) = \{1, 2\}$.
$\boxed{2}$ $I^{<}(\overline{x}) \neq \emptyset$.
$\boxed{3}$ $\alpha(\overline{x}) = \min(7, 4) = 4$.
$\boxed{4}$ $I^{<}(\alpha(\overline{x})) = \{2\}$.
$\boxed{5}$ $H_2^{<}(\overline{x}) = \{1\}$.
$\boxed{6}$ $H^{<}(\overline{x}) = \{1\}$.
$\boxed{7}$ $\overline{x}_1 = 4,\ \overline{x} = (4, 10, 10, 10)$ *go to* $\boxed{1}$.

Iteration 2:

$\boxed{1}$ $I^{<}(\overline{x}) = \{1\}$.
$\boxed{2}$ $I^{<}(\overline{x}) \neq \emptyset$.
$\boxed{3}$ $\alpha(\overline{x}) = 5$.
$\boxed{4}$ $I^{<}(\alpha(\overline{x})) = \{1\}$.
$\boxed{5}$ $H_i^{<}(\overline{x}) = \{2, 3\}$.
$\boxed{6}$ $H^{<}(\overline{x}) = \{2, 3\}$.
$\boxed{7}$ $\overline{x}_2 = 5, \overline{x}_3 = 5,\ \overline{x} = (4, 5, 5, 10)$ *go to* $\boxed{1}$.

Iteration 3:

$\boxed{1}$ $I^{<}(\overline{x}) = \emptyset$*, then* $x^{\max} = (4, 5, 5, 10)$ *STOP.*

In the next part of this section we will introduce a method which finds the minimum upper bound $\tilde{x}$ for solution of system (6.1) such that $\tilde{x} \geq \underline{x}$. In other words $\tilde{x}$ has the following properties:

(1) $\tilde{x} \in M^{\geq}(\underline{x}, x^{\max})$

(2) If $\underline{x} \leq \tilde{x}, \underline{x} \neq \tilde{x}$, then there exists $x^* \in M^{\geq}(\underline{x}, x^{\max})$ such that $x^* \not\leq \tilde{x}$.
It will be clear that $\tilde{x} \in M^{\geq}(\underline{x}, x^{\max})$ and this element is suitable to find the optimal solution of the minimization problem as we will see in the next section.

Let us set $T_{ij},\ i \in I,\ j \in J$ be defined as follows:

$$T_{ij} = \{x_j\ ; x_j \leq x_j^{\max} \&\ a_{ij} \wedge x_j\ \geq\ b_i(\underline{x})\ \vee\ \underline{x}_j\},\ \forall\ i \in I,\ j \in J.$$

Note that if $i_1,\ i_2$ are two different indices of I, $j \in J$, and $b_{i_2}(\underline{x}) \vee \underline{x}_j \leq b_{i_1}(\underline{x}) \vee \underline{x}_j$, then evidently $T_{i_1 j} \subseteq T_{i_2 j}$. It follows that for any subset of r indices of I, there exists such permutation $i_1,\ ,\ldots,\ i_r$ of these indices that the inclusions $T_{i_1 j}\ \subseteq\ T_{i_2 j}\ \subseteq\ \ldots\ \subseteq T_{i_r j}$ hold so that $\bigcap_{h=1}^{r} T_{i_h j} = T_{i_1 j}$.
The sets T_{ij} have the following properties:

$$T_{ij} \neq \emptyset\ \Leftrightarrow\ a_{ij} \geq b_i(\underline{x}) \vee \underline{x}_j,$$

$$T_{ij} \neq \emptyset\ \Rightarrow\ T_{ij} = [b_i(\underline{x}) \vee \underline{x}_j\ ,\ x_j^{\max}].$$

Since we assumed that $\underline{x} \leq x^{\max}$, set $M^{\geq}(\underline{x}, x^{\max})$ is nonempty. Let us note that for any $x \in M^{\geq}(\underline{x}, x^{\max})$ and any $i \in I$, the inequalities

$$b_i(x) \geq b_i(\underline{x})\ \&\ x_j \geq \underline{x}_j\ \forall\ j \in J$$

hold and further there exists for each $i \in I$ an index $j(i) \in J$ such that $T_{ij(i)} \neq \emptyset$ (otherwise set $M^{\geq}(\underline{x}, x^{\max})$ would be empty, because we would have $a_{ij} < b_i(\underline{x}) \vee \underline{x}_j\ \forall j \in J$ and therefore $a_i(x) < b_i(x)$ for any $x \in R^n$ and we have $\underline{x} \leq x^{\max}$ so that $M^{\geq}(\underline{x}, x^{\max}) \neq \emptyset$). Let us note further, that if $a_{ij} \wedge x_j < b_i(\underline{x}) \vee \underline{x}_j\ \forall j \in J$, then we have $a_i(x) < b_i(\underline{x})$ and thus $x \notin M^{\geq}(\underline{x}, x^{\max})$. If for some fixed $j \in J$ the inequalities $a_{ij} < b_i(\underline{x}) \vee \underline{x}_j$ hold , then $a_{ij} \wedge x_j < b_i(\underline{x}) \vee \underline{x}_j\ \forall\ x_j \in R$ so that $T_{ij} = \emptyset$ and x_j will never be "active" in $a_i(x)$ or $b_i(x)$ if $x \in M^{\geq}$ (i.e. it will never determine the values of $a_i(x)$ or $b_i(x)$). Therefore we will exclude such variables from our considerations and assume that

$$\forall i \in I\ \ \exists\ \ j(i) \in J\ \ a_{ij(i)} \geq b_i(\underline{x}) \vee \underline{x}_j$$

In such a case we have either $a_{ij(i)} \geq b_i(\underline{x}) > \underline{x}_j$ or $a_{ij(i)} \geq \underline{x}_j \geq b_i(\underline{x})$. Assume that for each $j \in J$ there exists at least one "row" index $i \in I$ such that $a_{ij} \geq b_i(\underline{x}) \vee \underline{x}_j$. Let us choose for each $i \in I$ an index $j(i) \in J$ such that $T_{ij(i)} \neq \emptyset$. Let us set $V_j,\ j \in J$ be defined as follows:

$$V_j = \{i \in I; a_{ij} \geq b_i(\underline{x}) \vee \underline{x}_j\},$$

in other way $V_j = \{i \in I\ ;\ j(i) = j\}$. It means that V_j is the set of those row indices $i \in I$, for which nonempty set T_{ij} was chosen in column $j \in J$. Note that some of the sets V_j may be empty and further

$$\bigcup_{j \in J} V_j = I$$

and

$$\bigcap_{i \in V_j} T_{ij} = [\max_{k \in V_j}(b_k(\underline{x})) \vee \underline{x}_j, x_j^{\max}],$$

where we set the maximum equal to $-\infty$ if $V_j = \emptyset$. Let us denote $\max_{k \in V_j}(b_k(\underline{x})) = b_{k(j)}(\underline{x})$. Let a vector $\tilde{x}$ will be defined as follows:

$$\tilde{x}_j = \max_{k \in V_j}(b_k(\underline{x})) \vee \underline{x}_j = b_{k(j)}(\underline{x}) \vee \underline{x}_j \quad \forall j \in J, \tag{6.2}$$

so that if $V_j = \emptyset$, then $\tilde{x}_j = \underline{x}_j$.

The element $\tilde{x}$ defined by (6.2) has the following properties:

(1) $M^{\geq}(\underline{x}, \tilde{x}) \neq \emptyset,\ \&\ \tilde{x} \in M^{\geq}(\underline{x}, \tilde{x})$.

(2) $\xi \in M^{\geq}(\underline{x}, \tilde{x}) \Rightarrow \underline{x} \leq \xi \leq \tilde{x}$.

(3) There may exist elements $\eta \in M^{\geq}(\underline{x}, \tilde{x})$ such that $\eta \neq \tilde{x}$.

If $\tilde{x}$ is the minimum element of $M^{\geq}(\underline{x}, x^{\max})$, then it would be $\tilde{x} \in M^{\geq}(\underline{x}, x^{\max})$ and for any $x \in M^{\geq}(\underline{x}, x^{\max}) \Rightarrow x \geq \tilde{x}$. Therefore, because of the property (3) $\tilde{x}$ is not the minimum element of $M^{\geq}(\underline{x}, x^{\max})$, but we can say that $\tilde{x}$ is the minimum upper bound of $M^{\geq}(\underline{x}, x^{\max})$ such that $M^{\geq}(\underline{x}, \tilde{x}) \neq \emptyset$. Let us choose $\tau \leq x^{\max},\ \&\ \tau \neq x^{\max}$, and $\check{x} \in M^{\geq}(\underline{x}, \tau) \ \Rightarrow\ \check{x} \leq x^{\max}$ and $a_i(\check{x}) \geq b_i(\check{x})\ \forall\ i \in I$ and

$\underline{x} \leq \breve{x} \leq \tau$. Let $H = \{x^{\max}(\tau) \mid x^{\max}(\tau) \textit{ is the maximum element of } M^{\geq}(\underline{x}, \tau)\}$, then $\tilde{x}$ is the minimum element of H.

Theorem 6.1.1
Let $\tilde{x}$ be defined as in (6.2). Then $\tilde{x} \in M^{\geq}(\underline{x}, x^{\max})$.

Proof:

Since evidently $\tilde{x} \geq \underline{x}$, we have to prove that only $a_i(\tilde{x}) \geq b_i(\tilde{x}),\ \forall i \in I$. Let $i \in I$ be arbitrarily chosen. We have

$$b_i(\tilde{x}) = \max_{j \in J}(b_{ij} \wedge \tilde{x}_j) = \max_{j \in J}(b_{ij} \wedge (\max_{k \in V_j}(b_k(\underline{x}) \vee \underline{x}_j))) = \max_{j \in J}(b_{ij} \wedge (b_{k(j)}(\underline{x}) \vee \underline{x}_j))$$

Let us assume that

$$b_i(\tilde{x}) = \max_{j \in J}(b_{ij} \wedge \tilde{x}_j) = b_{ij(i)} \wedge \tilde{x}_{j(i)}.$$

so that

$$b_i(\tilde{x}) = b_{k(j(i))}(\tilde{x}) \vee \underline{x}_{j(i)}.$$

Since in this case $i \in V_{j(i)}$, we have $a_{ij(i)} \geq \tilde{x}_{j(i)}$ and we obtain

$$a_i(\tilde{x}) \geq a_{ij(i)} \wedge \tilde{x}_{j(i)} = \tilde{x}_{j(i)} \geq b_{ij(i)} \wedge \tilde{x}_{j(i)} = b_i(\tilde{x}).$$

Since $i \in I$ was arbitrarily chosen, the theorem is proved.

□

We will clarify the previous discussion about the properties of $\tilde{x}$ through the following simple example.

Example 6.1.2
Let $J = \{1, 2, \ldots, 6\}$, $I = \{1, 2\}$, $\underline{x} = (3, 5, 1, 2, 9, 9)$ and consider system (6.1) of inequalities where a_{ij} & b_{ij} $\forall$ $i \in I$ and $j \in J$ are given by the matrices A and B as follows:

$$A = \begin{pmatrix} -100 & -100 & 10 & 20 & -10 & -20 \\ -50 & -50 & 100 & 200 & -30 & -40 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 7 & 50 & -10 & -20 & -30 & -40 \\ 8 & 9 & -15 & -25 & 10 & 11 \end{pmatrix}$$

By substitution for these values in system (6.1) we have

$$a_1(\underline{x}) = \max_{1 \le j \le 6}(a_{1j} \wedge \underline{x}_j) = 2 \quad , \quad b_1(\underline{x}) = \max_{1 \le j \le 6}(b_{1j} \wedge \underline{x}_j) = 5$$

$$a_2(\underline{x}) = \max_{1 \le j \le 6}(a_{2j} \wedge \underline{x}_j) = 2 \quad , \quad b_2(\underline{x}) = \max_{1 \le j \le 6}(b_{2j} \wedge \underline{x}_j) = 9$$

therefore $\underline{x} \notin M^{\geq}(\underline{x}, x^{\max})$, *and*
$V_1 = V_2 = \emptyset, V_3 = \{1, 2\}, V_4 = \{1, 2\}, V_5 = V_6 = \emptyset.$
We will find $\tilde{x}$ *as follows:*
$\tilde{x}_1 = \underline{x}_1 = 3,\ \&\ \tilde{x}_2 = \underline{x}_2 = 5\ \&\ \tilde{x}_3 = \max_{1 \le i \le 2}(b_i(\underline{x})) \vee \underline{x}_3 = \max(5, 9) \vee 1 = 9$
and similarly $\tilde{x}_4 = 9$, *farther* $\tilde{x}_5 = \underline{x}_5 = 9,\ \&\ \tilde{x}_6 = \underline{x}_6 = 9,$
i.e $\tilde{x} = (3, 5, 9, 9, 9, 9)$ *and we have:*

$$a_1(\tilde{x}) = 9 \ \ \& \ \ b_1(\tilde{x}) = 5$$

$$a_2(\tilde{x}) = 9 \ \ \& \ \ b_2(\tilde{x}) = 9$$

so that $\tilde{x} \in M^{\geq}(\underline{x}, x^{\max})$. *let us choose* $\xi = (3, 5, 1, 9, 9, 9) \le \tilde{x}$ *and* $\xi \neq \tilde{x}$ *also we have*

$$a_1(\xi) = 9 \ \ \& \ \ b_1(\xi) = 5$$

$$a_2(\xi) = 9 \ \ \& \ \ b_2(\xi) = 9$$

then $\xi \in M^{\geq}(\underline{x}, x^{\max})$. *Therefore* $\tilde{x}$ *because of the property (3) is not the minimum element of* $M^{\geq}(\underline{x}, x^{\max})$, *but it is the minimum upper bound of* $M^{\geq}(\underline{x}, x^{\max})$.

Element $\tilde{x}$ defined by (6.2) shows that the given lower bound $\underline{x}$ might not be an element of $M^{\geq}(\underline{x}, x^{\max})$. Moreover we obtained an explicit dependence of $\tilde{x}$ on the given lower bound $\underline{x}$ (compare (6.2)), which can be used for sensitivity analysis of the set $M^{\geq}(\underline{x}, \overline{x})$ or for a post optimal analysis of optimization problems, the set of feasible solutions of which is equal to $M^{\geq}(\underline{x}, x^{\max})$. The properties of $\tilde{x}$ enable us to solve some of the optimization problems mentioned above explicitly.

6.2 Optimization Problems under Two-Sided (max, min)-Linear Inequality Constraints

In this section we consider an optimization problem that is a combination of the problems solved in the above chapters but with a different feasible set. In other words, let us consider for instance the optimization problem:

$$f(x) \equiv \max_{j \in J} f_j(x_j) \quad \longrightarrow \quad \min \tag{6.3}$$

subject to

$$x \in M^{\geq}(\underline{x}, x^{\max}),$$

where $f_j,\ j \in J$ are increasing functions. Let indices $j(i) \in J$ will be chosen for each $i \in I$ such that

$$\min_{j \in J} f_j(x_j^{(i)}) = f_{j(i)}(x_{j(i)}),$$

where

$$f_j(x_j^{(i)}) = \min_{x_j \in T_{ij}} f_j(x_j).$$

Let $\tilde{x}$ be defined as in (6.2) and then we have to proceed as follows:

$$\tilde{T}_{ij} = \begin{cases} \emptyset & \text{if} \quad a_{ij} < b_i(\underline{x}), \\ b_i(\underline{x}) & \text{if} \quad a_{ij} > b_i(\underline{x}), \\ [\underline{x}_j, \tilde{x}] & \text{if} \quad a_{ij} = b_{ij}. \end{cases}$$

Let us set $f_j(\tilde{x}_j^{(i)}) = \min_{x_j \in \tilde{T}_{ij}} f_j(x_j)$, (if $\tilde{T}_{ij} = \emptyset$, we set minimum equal to $+\infty$). Let us set

$$\min_{j \in J} f_j(\tilde{x}_j^{(i)}) = f_{j(i)}(\tilde{x}_{j(i)}^{(i)}).$$

And $\tilde{R}_j = \{i \in I \mid j(i) = j\},\ \forall\ j \in J$, (it may be $\tilde{R}_j = \emptyset$ for some j). Then we have

$$f_k(x_k^{opt}) = \max_{i \in \tilde{R}_k} f_k(\tilde{x}_k^{(i)}),$$

if $\tilde{R}_k \neq \emptyset$, but when $\tilde{R}_k = \emptyset$, we set

$$f_k(x_k^{opt}) = f_k(\underline{x}_k).$$

The proof can be carried out in the same way as in the one sided case in Chapter 3. Direct exact solving of such optimization problems as well as their sensitivity and parametric analysis are beyond the scope of this chapter and will be the subject of further research. It is not necessary to pressed this algorithm in the next section explicitly. It remains to say that we use the same method as for equation constraints in the above chapter, only $x^{\max}$ is the maximum element of $M^{\geq}(\underline{x}, \overline{x})$, (instead of the the maximum element of $M^{=}(\underline{x}, \overline{x})$).

We mentioned above that a system of inequalities can be transformed to a system of equations by making use of slack variables. Let us note that the other way round, systems of equations considered can be solved alternatively by the methods in this section, if we replace the equation system by the system of inequalities of the form

$$\begin{aligned} a_i(x) &\geq b_i(x), \ i \in I \\ b_i(x) &\geq a_i(x), \ i \in I \\ x_j &\geq \underline{x}_j, \ j \in J. \end{aligned}$$

In concrete terms, we will describe now the corresponding algorithm explicitly step by step.

Algorithm 6.2.1 *We will provide algorithm, which summarizes the above discussion and finds the optimal solution x^{opt} of problem (6.3).*

$\boxed{0}$ *Input* $m, n, \underline{x}, \overline{x}, A, B, f(x)$.

$\boxed{1}$ *Find* $x^{\max} \in M^{\geq}(\underline{x}, \overline{x})$.

$\boxed{2}$ *If* $\underline{x} \not\leq x^{\max}$, *then* $M^{\geq}(\underline{x}, \overline{x}) = \emptyset$, *STOP.*

$\boxed{3}$ $V_j := \{i \in I \ ; \ a_{ij} > b_i(\underline{x}) \vee \underline{x}_j\} \quad \forall j \in J.$

$\boxed{4}$ $x_j^{(i)} := (b_i(\underline{x}) \vee \underline{x}_j) \quad \forall i \in V_j$ *for all* $j \in J$ *such that* $V_j \neq \emptyset$.

$\boxed{5}$ *Set* $\tilde{x}_j := \max_{i \in V_j}(x_j^{(i)})$ *if* $V_j \neq \emptyset$, $\tilde{x}_j := \underline{x}_j$ *if* $V_j = \emptyset$.

$\boxed{6}$ $Q := \{k \in J \ ; \ f(\tilde{x}) = f_k(\tilde{x}_k)\}, \quad P := \{j \in J \ ; \ \tilde{x}_j = \underline{x}_j\}.$

$\boxed{7}$ *If* $Q \cap P \neq \emptyset$, *then set* $x^{opt} := \tilde{x}$, *STOP.*

$\boxed{8}$ $P_k := \{i \in I \ ; \ \tilde{x}_k = x_k^{(i)}\} \ \ \forall k \in Q.$

$\boxed{9}$ $V_k := V_k \setminus P_k \ \ \forall k \in Q.$

$\boxed{10}$ *If* $\bigcup_{j \in J} V_j = I$, *go to* $\boxed{4}$.

$\boxed{11}$ *Set* $x^{opt} := \tilde{x}$, *STOP.*

We will illustrate the performance of this algorithm by the following numerical examples.

Example 6.2.1

Let $J = \{1, 2, \ldots, 5\}$, $I = \{1, 2, 3\}$, $\overline{x} = (10, 10, 10, 10, 10)$, $\underline{x} = (0, 3, 0, 0, 1)$ *and consider system (6.1) of inequalities where* a_{ij} & b_{ij} $\forall$ $i \in I$ *and* $j \in J$ *are given by the matrices* A *and* B *as follows:*

$$A = \begin{pmatrix} -10 & 10 & 15 & -9 & -8 \\ 5 & -8 & 10 & 20 & 7 \\ 3 & 4 & -18 & 19 & 11 \end{pmatrix}, \qquad B = \begin{pmatrix} 7 & 2 & -10 & -20 & 6 \\ 8 & 9 & -15 & -25 & 5 \\ 13 & -17 & 12 & 10 & 9 \end{pmatrix}$$

and consider the objective function

$$f(x) = max(x_1, x_2 - 3, x_3, x_4, x_5).$$

By substitution for these values in system (6.1) and using algorithm (6.1.1) we have:

$$a_1(\overline{x}) = \max_{1 \le j \le 5}(a_{1j} \wedge \overline{x}_j) = 10 \ \ , \ \ b_1(\overline{x}) = \max_{1 \le j \le 5}(b_{1j} \wedge \overline{x}_j) = 7$$

$$a_2(\overline{x}) = \max_{1 \le j \le 5}(a_{2j} \wedge \overline{x}_j) = 10 \ \ , \ \ b_2(\overline{x}) = \max_{1 \le j \le 5}(b_{2j} \wedge \overline{x}_j) = 9$$

$$a_3(\overline{x}) = \max_{1 \le j \le 5}(a_{3j} \wedge \overline{x}_j) = 10 \ \ , \ \ b_3(\overline{x}) = \max_{1 \le j \le 5}(b_{3j} \wedge \overline{x}_j) = 10$$

Then $x^{\max} = \overline{x} = (10, 10, 10, 10, 10)$, *therefore* $\underline{x} \le x^{\max}$, *and by using the algorithm (6.2.1) we have*

$\boxed{1}$ $x^{\max} = \overline{x} = (10, 10, 10, 10, 10).$

$\boxed{2}$ $\underline{x} \le x^{\max}.$

$\boxed{3}$ $V_1 = \{2\},\ V_2 = \{1,3\}, V_3 = \{1,2\}, V_4 = \{2,3\}, V_5 = \{2,3\}.$

$\boxed{4}$ $x_1^{(1)} = 2,\quad x_2^{(1)} = 3,\quad x_3^{(1)} = 2,\quad x_4^{(1)} = 2,\quad x_5^{(1)} = 2,\ x_1^{(2)} = 3,\quad x_2^{(2)} = 3,\quad x_3^{(2)} = 3,\quad x_4^{(2)} = 3,\quad x_5^{(2)} = 3, x_1^{(3)} = 1,\quad x_2^{(3)} = 3,\quad x_3^{(3)} = 1,\quad x_4^{(3)} = 1,\quad x_5^{(3)} = 1.$

$\boxed{5}$ $\tilde{x} = (3,3,2,3,3).$

$\boxed{6}$ $Q = \{1,4,5\},, f(\tilde{x}) = 3,\ P = \{2\}$ *then* $Q \cap P = \emptyset.$

$\boxed{8}$ $P_1 = \{2\},\ P_2 = \{1,2,3\}, P_3 = \{1\}, P_4 = \{2\}, P_5 = \{2\}.$

$\boxed{9}$ $V_1 = \emptyset,\ V_2 = \emptyset, V_3 = \{2\}, V_4 = \{3\}, V_5 = \{3\}.$

$\boxed{10}$ $\bigcup_{j \in J} V_j = \{2,3\} \neq I.$

$\boxed{11}$ $x^{opt} = \tilde{x},\ STOP.$

Then $x^{opt} = (3,3,2,3,3)$ *is the optimal solution of the set* $M^{\geq}(\underline{x}, \overline{x})$ *and* $f\left(x^{opt}\right) = \max(3,0,2,3,3)$, *then the objective function is equal to* 3.

Example 6.2.2

Let $J = \{1,2,\ldots,5\},\ I = \{1,2,\ldots,6\},\ \overline{x} = (20,20,20,20,20),\ \underline{x} = (0,3,0,0,0)$ *and consider system (6.1) of inequalities where* $a_{ij}\ \ \&\ \ b_{ij}\ \ \forall\ \ i \in I$ *and* $j \in J$ *are given by the matrices* A *and* B *as follows:*

$$A = \begin{pmatrix} 2 & 2 & 6 & 0 & 13 \\ 8 & 11 & 10 & 7 & 7 \\ 4 & 3 & 0 & 13 & 8 \\ 14 & 3 & 3 & 13 & 2 \\ 1 & 3 & 13 & 4 & 2 \\ 12 & 15 & 7 & 3 & 14 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 10 & 9 & -1 & 5 \\ 3 & -3 & 1 & -6 & -7 \\ 4 & -8 & 2 & -14 & 11 \\ 14 & -7 & 7 & -3 & 4 \\ 6 & -8 & 12 & 2 & 0 \\ 0 & -11 & 2 & -3 & 5 \end{pmatrix}$$

and consider the objective function

$$f(x) = max_{j \in J}(f_j(x_j),$$

where $f_j(x_j) = c_j x_j + d_j$, $c = (6,3,7,3,7)$ *and* $d = (10,0,5,1,7)$. *By substitution for these values in system (6.1) and using we have:*

$$a_1(\overline{x}) = \max_{1\le j\le 5}(a_{1j} \wedge \overline{x}_j) = 13 \quad , \quad b_1(\overline{x}) = \max_{1\le j\le 5}(b_{1j} \wedge \overline{x}_j) = 10$$

$$a_2(\overline{x}) = \max_{1\le j\le 5}(a_{2j} \wedge \overline{x}_j) = 11 \quad , \quad b_2(\overline{x}) = \max_{1\le j\le 5}(b_{2j} \wedge \overline{x}_j) = 3$$

$$a_3(\overline{x}) = \max_{1\le j\le 5}(a_{3j} \wedge \overline{x}_j) = 13 \quad , \quad b_3(\overline{x}) = \max_{1\le j\le 5}(b_{3j} \wedge \overline{x}_j) = 11$$

$$a_4(\overline{x}) = \max_{1\le j\le 5}(a_{4j} \wedge \overline{x}_j) = 14 \quad , \quad b_4(\overline{x}) = \max_{1\le j\le 5}(b_{4j} \wedge \overline{x}_j) = 14$$

$$a_5(\overline{x}) = \max_{1\le j\le 5}(a_{5j} \wedge \overline{x}_j) = 13 \quad , \quad b_5(\overline{x}) = \max_{1\le j\le 5}(b_{5j} \wedge \overline{x}_j) = 12$$

$$a_6(\overline{x}) = \max_{1\le j\le 5}(a_{6j} \wedge \overline{x}_j) = 15 \quad , \quad b_6(\overline{x}) = \max_{1\le j\le 5}(b_{6j} \wedge \overline{x}_j) = 5$$

Then $x^{\max} = \overline{x} = (20, 20, 20, 20, 20)$, *therefore* $\underline{x} \le x^{\max}$, *and by using the algorithm (6.2.1) we have:*

$\boxed{1}$ $x^{\max} = \overline{x} = (20, 20, 20, 20, 20)$.

$\boxed{2}$ $\underline{x} \le x^{\max}$.

$\boxed{3}$ $V_1 = \{2, 3, 4, 5, 6\},\ V_2 = \{2, 6\}, V_3 = \{1, 2, 4, 5, 6\},$

$V_4 = \{2, 3, 4, 5, 6\}, V_5 = \{1, 2, 3, 4, 5, 6\}$.

$\boxed{4}$ *find* $x_j^{(i)}$.

$\boxed{5}$ $\tilde{x} = (0, 3, 3, 0, 3)$.

$\boxed{6}$ $Q = \{5\}, f(\tilde{x}) = 28,\ P = \{1, 2, 4\}$ *then* $Q \cap P = \emptyset$.

$\boxed{10}$ $\bigcup_{j\in J} V_j = \{1, 2, 3, 4, 5, 6\} = I$ *go to* $\boxed{4}$.

$\boxed{4}$ *find* $x_j^{(i)}$.

$\boxed{5}$ $\tilde{x} = (0, 3, 3, 0, 0)$.

$\boxed{6}$ $Q = \{3\}, f(\tilde{x}) = 26,\ P = \{1, 2, 4, 5\}$ *then* $Q \cap P = \emptyset$.

$\boxed{10}$ $\bigcup_{j\in J} V_j = \{1, 2, 4, 5, 6\} \ne I$.

$\boxed{11}$ $x^{opt} = \tilde{x}$, *STOP.*

Then $x^{opt} = (0, 3, 3, 0, 0)$ *is the optimal solution of the set* $M^{\ge}(\underline{x}, \overline{x})$ *and* $f\left(x^{opt}\right) = \max\left(10, 9, 26, 1, 7\right)$, *then the objective function is equal to* 26.

7

Optimization Problems on Attainable Sets of Systems of $(\max, \min)$-Linear Equations

In preceding sections we studied systems of $(\max, \min)-$ linear equations and inequalities and optimization problems under constraints described by such systems. The first question which has to be raised before solving such problems is the solvability of given system of equations and inequalities. If the system has a solution, we can continue and find a special solution of such system (e.g. maximum solution, optimal solution etc...). the question, which arises in connection with practical applications is what to do if the given system of equations and inequalities has no solution. The possible practical applications of such problems mentioned in the examples above in the preceding chapters show that in case that the system has no solution, we will have to modify the original system (i.e. to modify its input coefficients) in such a way that the new problem has a solution. In such a situation it is natural to try to modify the problems in such a way that the original aims of the given system (e.g. bounds on costs or arrival times) will be violated as little as possible. In this section we propose an approach to solving some of such problems in connection with one - sided $(\max, \min)-$ linear equation systems. Let us note that problems, the original formulation of which has no solution were called sometimes in the literature "incorrectly posed problems" (see e.g. I. I. Eremin et al. [13]). The results in the literature concentrate on mostly incorrectly posed linear and convex optimization problems. This chapter can therefore be understood as a contribution to this part of operations research applied to problems, in which $(\max, \min)-$ linear systems occur. Such problems are neither linear or convex in usual algebraic sense. One such problem for $(\max, +)-$ linear equation system was considered using a different approach in [20]. Unlike to the results of [20], our purpose in this section is to present an approach to incorrectly posed $(\max, \min)-$ linear one-sided equation systems.

7.1 Notations, Problem Formulation - Case $(\max, \min)$

Let us introduce the following notations:
$I = \{1, 2, \ldots, m\}$, $J = \{1, 2, \ldots, n\}$. Let A be a matrix with finite elements $a_{ij} \in R = (-\infty, +\infty)$, $\forall\ i \in I$, $j \in J$, let $\alpha \wedge \beta \equiv \min(\alpha, \beta)$ for any $\alpha, \beta \in R$. Vector $A \otimes x \in R^m$ for $x = (x_1, \ldots, x_n)^T \in R^n$ will be defined as follows:

$$(A \otimes x)_i \equiv \max_{j \in J}(a_{ij} \wedge x_j) \quad \forall i \in I.$$

The system of $(\max, \min)$-linear equations with right-hand side $b \in R^m$ is an equation system of the following form:

$$A \otimes x = b.$$

The set of all solutions of the system will be denoted $M(b)$, i.e.

$$M(b) = \{x \in R^n\ ;\ A \otimes x = b\}.$$

Definition 7.1.1
Set

$$R(A) \equiv \{b \in R^m\ ;\ \exists\ \ x \in R^n \text{ such that } A \otimes x = b\}$$

is called attainable set of matrix A.

In what follows we will solve the following optimization problem:

PROBLEM I.
Minimize

$$\left|\left|b-\hat{b}\right|\right| = \max_{i\in I}\left|b_i-\hat{b}_i\right|$$

subject to

$$b \in R(A)$$

The optimal solution of **PROBLEM I** will be denoted b^{opt}. Let us note that if $\hat{b} \in R(A)$, it is evidently $b^{opt} = \hat{b}$. Therefore we will assume in what follows that $\hat{b} \notin R(A)$.

7.2 Properties of Attainable Sets and Analysis of PROBLEM I

In this section we will study in more detail some properties of attainable sets and analysis of **PROBLEM I**.

Lemma 7.2.1
Set $R(A)$ has the maximum element, i.e. an element $b^{\max} \in R(A)$ such that $b \le b^{\max}\ \ \forall\ \ b \in R(A)$.

Proof:
Let $\alpha_i = \max_{j\in J} a_{ij}\ \ \forall i \in I$. Let $x \in R^n$ be arbitrarily chosen. Then $a_{ij} \wedge x_j \le a_{ij}$ for all $i \in J,\ j \in J$. Therefore for any $i \in I$ we obtain that

$$\max_{j\in J}(a_{ij} \wedge x_j) = \max_{j\in J} a_{ij} = \alpha_i.$$

Therefore if we set $b_i^{\max} = \alpha_i\ \ \forall\ \ i \in I$, then $b^{\max} \in R(A)$, since e.g. if $\hat{x} \in R^n$ and $\hat{x}_j \ge \max_{i\in I}\alpha_i$ we have $\max_{j\in J}(a_{ij} \wedge \hat{x}_j) = \alpha_i = b_i^{\max}$. For an arbitrary $b \in R(A)$ there exists $x \in R^n$ such that $b = A \otimes x \le A \otimes \hat{x} = b^{\max}$, so that $b^{\max}$ is the maximum element of $R(A)$, which completes the proof.

□

Lemma 7.2.2

Let $b \in R^m,\ I_j^> = \{i \in I\ ;\ a_{ij} > b_i\}\ \forall\ j \in J$*. Let* $M(b) = \{x \in R^n\ ;\ A \otimes x = b\}$ *be nonempty. Let vector* $x(b) \in R^n$ *be defined as follows:*

$$x_j(b) = \min_{i \in I_j^>} b_i \ \ \forall\ \ j \in J \ \textit{if}\ I_j^> \neq \emptyset.$$

We set the minimum equal to infinity if $I_j^> = \emptyset$*. Then* $x(b)$ *is the maximum element of set* $M(b)$*.*

Proof:

Let us note that if $x \in M(b)$, then it must be $a_{ij} \wedge x_j \leq b_i$ for all $i \in I,\ j \in J$. Therefore it must be $x \leq x(b)\ \ \forall x \in M(b)$ so that $x(b)$ is the upper bound for elements of $M(b)$. It remains to prove that if set $M(b)$ is nonempty it must be $x(b) \in M(b)$.

Let us set

$$S_j(x_j) \equiv \{k \in I\ ;\ a_{kj} \wedge x_j = b_k\}\ \ \forall j \in J.$$

If $I_j^> \neq \emptyset$, then

$$S_j(x_j(b)) = \{k \in I\ ;\ x_j(b) = b_k = \min_{i \in I_j^>}(b_i)\}.$$

If $I_j^> = \emptyset$, then $x_j(b) = \infty$ and

$$S_j(x_j(b)) = \{k \in I\ ;\ a_{kj} = b_k\}.$$

We will show further that

$$x(b) \in M(b) \iff \bigcup_{j \in J} S_j(x_j(b)) = I.$$

Really if $\bigcup_{j \in J} S_j(x_j(b)) = I$ and $p \in I$ is arbitrary, then there exists index $j(p) \in J$ such that $p \in S_{j(p)}(x_{j(p)}(b))$ and therefore $a_{pj} \wedge x_j(b) \leq b_p$ for all $j \in J$ and $a_{pj(p)} \wedge x_{j(p)}(b) = b_p$ so that $\max_{j \in J}(a_{pj} \wedge x_j(b)) = b_p$. Since p was arbitrary, we obtain that $x(b) \in M(b)$. To prove the opposite implication let us assume that $\bigcup_{j \in J} S_j(x_j(b)) \neq I$ so that there exists index $i_0 \in I$ such that $i_0 \notin \bigcup_{j \in J} S_j(x_j(b))$

and therefore $a_{i_0 j} \wedge x_j(b) \neq b_{i_0} \quad \forall j \in J$ and therefore $\max_{j \in J}(a_{i_0 j} \wedge x_j(b)) \neq b_{i_0}$ and thus $x(b) \notin M(b)$.
Let us note that if $x_j \leq x_j(b)$ for any $j \in J$, then $S_j(x_j) \subseteq S_j(x_j(b))$. Therefore if $\bigcup_{j \in J} S_j(x_j(b)) \subset I$, then for any $x \leq x(b)$ we have

$$\bigcup_{j \in J} S_j(x_j) \subseteq \bigcup_{j \in J} S_j(x_j(b)) \subset I$$

and thus $M(b) = \emptyset$, since all elements of $M(b)$ must satisfy the inequality $x \leq x(b)$ if $x \overset{>}{\nless} x(b)$, $x \in M(b)$. It follows that

$$M(b) \neq \emptyset \iff x(b) \in M(b).$$

In other words if $M(b) \neq \emptyset$, then $x(b) \in M(b)$ and $x \leq x(b)$ for all $x \in M(b)$, so that $x(b)$ in the maximum element of $M(b)$, what was to be proved.

□

Lemma 7.2.3
Let $b^{\max}$ be the maximum element of $R(A)$, $\hat{b} \in R^m$ such that $\hat{b}_p \geq b_p^{\max}$ for some $p \in I$, b an arbitrary element of $R(A)$. Then

$$\left| b_p - \hat{b}_p \right| \geq \left| b_p^{\max} - \hat{b}_p \right|.$$

Proof:
Since $b^{\max}$ is the maximum element of $R(A)$, and $\hat{b}_p \geq b_p^{\max}$, the following inequalities hold for any $b \in R(A)$:

$$b_p \leq b_p^{\max} \leq \hat{b}_p.$$

It follows that $\quad b_p - \hat{b}_p \leq b_p^{\max} - \hat{b}_p \leq 0 \quad$ so that we obtain

$$\left| b_p - \hat{b}_p \right| \geq \left| b_p^{\max} - \hat{b}_p \right|.$$

□

As a consequence of Lemma 7.2.3 we obtain that if $\hat{b} \geq b^{\max}$, (i.e. $\hat{b}_s \geq b_s^{\max}$, $\forall s \in I$) then $b^{opt} = b^{\max}$.

Example 7.2.1
Let $m = n = 3,\ \hat{b} = (8, 8, 8)^T$,

$$A = \begin{pmatrix} 3 & 1 & 5 \\ 4 & 4 & 6 \\ 7 & 7 & 3 \end{pmatrix}$$

In this case $b^{\max} = (5, 6, 7)^T \leq \hat{b} = (8, 8, 8)^T$. *Taking into account Lemma 7.2.3, we obtain that the optimal solution of* ***PROBLEM I*** *is equal to* b^{max}. *The optimal value of the objective function of* ***PROBLEM I*** *is therefore* $\left\|b^{\max} - \hat{b}\right\| = \max(3, 2, 1) = 3$.

7.3 Algorithm - A Parametric Version

In what follows we will replace **PROBLEM I** with the following parametric optimization problem:

PROBLEM II
Minimize t
subject to $\left\|b - \hat{b}\right\| \leq t,\ b \in R(A)$
where $\left\|b - \hat{b}\right\| = \max_{i \in I} \left|b_i - \hat{b}_i\right|$
this problem is equivalent to the next problem:

PROBLEM III
Minimize t
subject to $\hat{b}_i - t \leq \max_{j \in J}(a_{ij} \wedge x_j) \leq \hat{b}_i + t, \quad \forall \ i \in I.$

Let $M(t)$ denote the set of feasible solutions of PROBLEM II. We have then

$$M(t) = \{x \ ; \ \hat{b}_i - t \leq x_j \leq \hat{b}_i + t,\ \forall\ i \in I\}.$$

And let us set for all $i \in I,\ j \in J$.

$$T_{ij}(t) \equiv \{x_j \mid \hat{b}_i - t \leq a_{ij} \wedge x_j \leq \hat{b}_i + t\}. \tag{7.1}$$

Note that $\max_{j\in J}(a_{ij}\wedge x_j) \le \hat{b}_i + t, \ \forall \ i \in I,$ implies that for each fixed $j \in J$ it is $a_{ij} \wedge x_j \le \hat{b}_i + t, \ \forall \ i \in I,$ so that for each fixed $j \in J$ and t it must be

$$x_j \le x_j(\hat{b} + t) \equiv min_{i \in I_j(t)}(\hat{b}_i + t) \tag{7.2}$$

where $I_j(t) \equiv \{i \in I \mid a_{ij} > \hat{b}_i + t\}$, and we set the minimum equal to infinity if $I_j^>(t) = \emptyset$. Let us note that if $a_{ij} > \hat{b}_i + t$ i.e. $t < a_{ij} - \hat{b}_i$, then $\hat{b}_i + t$ is the upper bound for $x_j \in T_{ij}(t)$ and if $t \ge a_{ij} - \hat{b}_i$, then $a_{ij} \le \hat{b}_i + t$ so that also $a_{ij} \wedge x_j \le \hat{b}_i + t$ and $\hat{b}_i + t$ is no more an upper bound for x_j, i.e. the upper bound for x_j is higher. Let us note further that if $T_{ij}(t) \ne \emptyset$, then it must be fulfilled two inequalities

$$a_{ij} \ge \hat{b}_i - t \ \text{ and } \ \hat{b}_i - t \le x_j(\hat{b}_i + t) \tag{7.3}$$

If either $a_{ij} < \hat{b}_i - t$ or $\hat{b}_i - t > x_j(\hat{b}_i + t)$, then $T_{ij}(t)$ is empty.
We will find minimum value of t, for which the inequalities of (7.3) hold. The minimum value of t, for which $a_{ij} \ge \hat{b}_i - t$ holds is evidently

$$\tau_{ij}^{(1)} \equiv \hat{b}_i - a_{ij}.$$

To define the minimal value of t, for which $\hat{b}_i - t \le x_j(\hat{b}_i + t)$ holds, we will investigate $x_j(\hat{b}_i + t)$ as a function of t. We have for any fixed $j \in J$ and $t \ge 0$:

$$x_j(\hat{b} + t) = \min_{i \in I_j(t)} (\hat{b}_i + t) = \hat{b}_{k(j,t)} + t, \tag{7.4}$$

where we set $x_j(\hat{b} + t) = \infty$, if $I_j(t) = \emptyset$. Note that $x_j(\hat{b} + t) = \infty$, for all $t \ge \max_{i\in I}(a_{ij} - \hat{b}_i)$. We will consider therefore only values $t \le \max_{i\in I}(a_{ij} - \hat{b}_i)$.
Let us set

$$I_j^{(1)} \equiv \left\{ k \mid \max_{k \in I}(a_{ij} - \hat{b}_i) = a_{kj} - \hat{b}_k = \alpha_j^{(1)} \right\},$$

$$I_j^{(2)} \equiv \left\{ k \mid \max_{k \in I \setminus I_j^{(1)}}(a_{ij} - \hat{b}_i) = a_{kj} - \hat{b}_k = \alpha_j^{(2)} \right\},$$

$$I_j^{(3)} \equiv \left\{ k \mid \max_{k \in I \setminus (I_j^{(1)} \cup I_j^{(1)})}(a_{ij} - \hat{b}_i) = a_{kj} - \hat{b}_k = \alpha_j^{(3)} \right\},$$

$$\vdots \qquad \vdots$$

$$I_j^{(p)} \equiv \left\{ k \mid \max_{k \in I \setminus \bigcup_{h=1}^{p-1} I_j^{(h)}} (a_{ij} - \hat{b}_i) = a_{kj} - \hat{b}_k = \alpha_j^{(p)} \right\},$$

where $I \setminus \bigcup_{h=1}^{p-1} I_j^{(h)} \neq \emptyset$, and $\bigcup_{h=1}^{p} I_j^{(h)} = I$. Values $\alpha_j^{(p)}$, $h = 1, \cdots, p$ are therefore different values, which occur in the set $a_{ij} - \hat{b}_i$, $i \in I$ and holds $\alpha_j^{(1)} > \alpha_j^{(2)} > \cdots > \alpha_j^{(p)}$, $1 \leq p \leq m$. The following numerical example enlightens the definition of $I_j^{(k)}$, $k = 1, ..., p$.

Example 7.3.1

Let $m = 5$, $(a_{1j}, a_{2j}, a_{3j}, a_{4j}, a_{5j})^T = (5, 8, 8, 16, 20)^T$, $\hat{b} = (3, 5, 5, 12, 14)$ *so that* $(a_{1j} - \hat{b}_1, a_{2j} - \hat{b}_2, a_{3j} - \hat{b}_3, a_{4j} - \hat{b}_4, a_{5j} - \hat{b}_5)^T = (2, 3, 3, 4, 6)^T$, *and we obtain* $p = 4$ *and* $I_j^{(1)} = \{5\}$, *with* $\alpha_j^{(1)} = 6$, $I_j^{(2)} = \{4\}$, *with* $\alpha_j^{(2)} = 4$ $I_j^{(3)} = \{2, 3\}$, *with* $\alpha_j^{(3)} = 3$ $I_j^{(4)} = \{1\}$, *with* $\alpha_j^{(4)} = 2$.

Having determined values $\alpha_j^{(1)}, \alpha_j^{(2)}, \cdots, \alpha_j^{(p)}$, we can find explicitly $I_j(t)$ in dependence of t :

$$I_j(t) = \emptyset \quad \text{if} \quad t \geq \alpha_j^{(1)},$$

$$I_j(t) = I_j^{(1)} \quad \text{if} \quad \alpha_j^{(2)} \leq t < \alpha_j^{(1)},$$

$$I_j(t) = I_j^{(1)} \cup I_j^{(2)} \quad \text{if} \quad \alpha_j^{(3)} \leq t < \alpha_j^{(2)},$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$I_j(t) = \bigcup_{h=1}^{p-1} I_j^{(h)} \quad \text{if} \quad \alpha_j^{(p)} \leq t < \alpha_j^{(p-1)}$$

$$I_j(t) = \bigcup_{h=1}^{p} I_j^{(h)} = I \quad \text{if} \quad t < \alpha_j^{(p)}.$$

Now we can find the explicit form of $x_j(\hat{b} + t)$ as a function of t :

$$x_j(\hat{b} + t) = \infty \quad \text{where} \quad t \geq \alpha_j^{(1)},$$

$$x_j(\hat{b} + t) = min_{i \in I_j(t)} \hat{b}_i + t = \hat{b}_{k(j,t)} + t \quad \text{where} \quad \alpha_j^{(2)} \leq t < \alpha_j^{(1)},$$

$$\text{and} \quad k(j, t) \in I_j^{(1)}$$

$$x_j(\hat{b}+t) = min_{i \in I_j(t)} \hat{b}_i + t = \hat{b}_{k(j,t)} + t \quad \text{where} \quad \alpha_j^{(3)} \le t < \alpha_j^{(2)},$$
$$\text{and} \quad k(j,t) \in I_j^{(1)} \cup I_j^{(2)}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$x_j(\hat{b}+t) = min_{i \in I_j(t)} \hat{b}_i + t = \hat{b}_{k(j,t)} + t \quad \text{where} \quad \alpha_j^{(p)} \le t < \alpha_j^{(p-1)},$$
$$\text{and} \quad k(j,t) \in \bigcup_{h=1}^{p-1} I_j^{(h)}$$

$$x_j(\hat{b}+t) = min_{i \in I_j(t)} \hat{b}_i + t = \hat{b}_{k(j,t)} + t \quad \text{where} \quad 0 \le t < \alpha_j^{(p)},$$
$$\text{and} \quad k(j,t) \in \bigcup_{h=1}^{p} I_j^{(h)} = I$$

Example 7.3.1 (continued)

Let us find $I_j(t)$ and $x_j(\hat{b}+t)$ for the numerical data of this example, we obtain

$$I_j(t) = \emptyset, \quad x_j(\hat{b}+t) = \infty \quad \text{if} \quad t \ge 6,$$

$$I_j(t) = \{5\}, \quad x_j(\hat{b}+t) = 14+t \quad \text{if} \quad 4 \le t < 6,$$
$$I_j(t) = \{4,5\}, \quad x_j(\hat{b}+t) = 12+t \quad \text{if} \quad 3 \le t < 4,$$
$$I_j(t) = \{2,3,4,5\}, \quad x_j(\hat{b}+t) = 5+t \quad \text{if} \quad 2 \le t < 3,$$
$$I_j(t) = \{1,2,3,4,5\} = I, \quad x_j(\hat{b}+t) = 3+t \quad \text{if} \quad 0 \le t < 2.$$

It follows that $x_j(\hat{b}+t)$ is for each $j \in J$ a strictly increasing, partially continuous function of t with at most m discontinuity points, in which it is continuous from above). Graph of $x_j(\hat{b}+t)$ is as Figure 6.1.

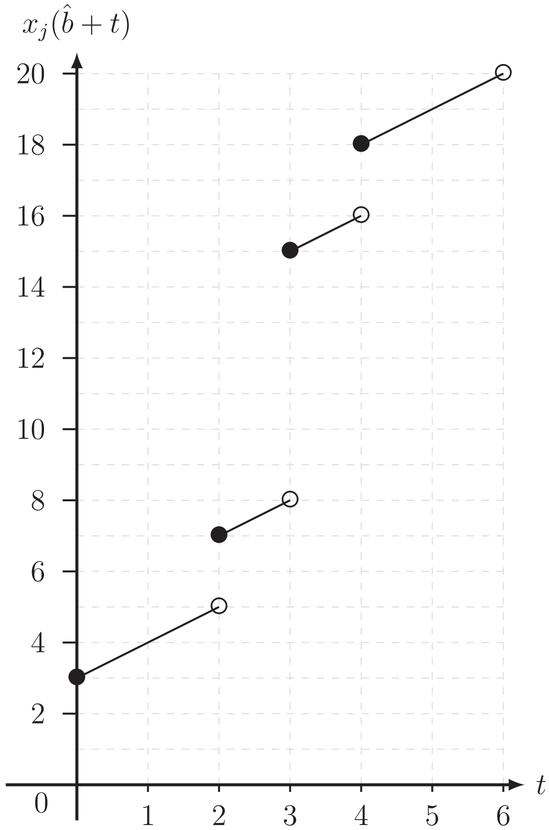

Figure 6.1: Graph of $x_j(\hat{b}+t)$

The explicit expression of $x_j(\hat{b}+t)$ makes possible to find $\tau_{ij}^{(2)}$ such that $\hat{b}_i - \tau_{ij}^{(2)} \leq x_j(\hat{b}+\tau_{ij}^{(2)})$ and $\hat{b}_i - t > x_j(\hat{b}+t)$ if $t < \tau_{ij}^{(2)}$. In detail the following ideas as Figure 6.2 a , b:
Possibility (1) as in Figure 6.2 a: in this case $\hat{b}_i - \tau_{ij}^{(2)} = x_j(\hat{b}+\tau_{ij}^{(2)})$.
Possibility (2) as in Figure 6.2 b: in this case $\hat{b}_i - \tau_{ij}^{(2)} < x_j(\hat{b}+\tau_{ij}^{(2)})$ and $\hat{b}_i - t > x_j(\hat{b}+t)$ if $t < \tau_{ij}^{(2)}$.

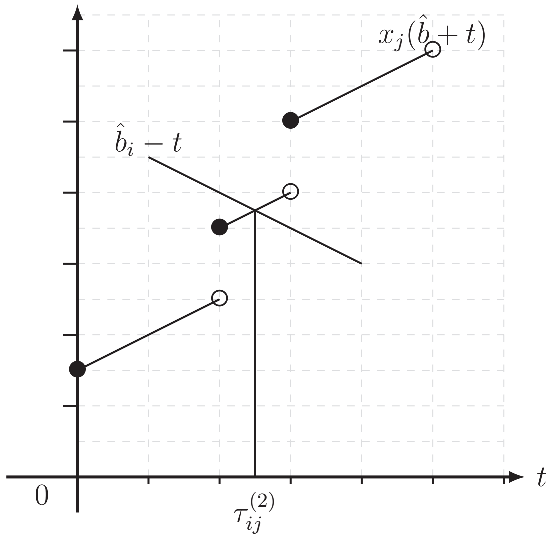

Figure 6.2 b: Graph of $x_j(\hat{b}+t)$ and $\hat{b}_i - t$

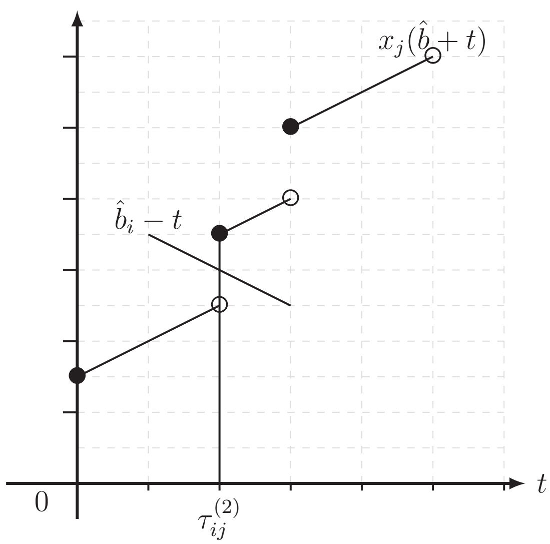

Figure 6.2 b: Graph of $x_j(\hat{b}+t)$ and $\hat{b}_i - t$

Let us set

$$\tau_{ij} \equiv \max(\tau_{ij}^{(1)}, \tau_{ij}^{(2)}),$$

Since we obtained that

$$T_{ij}(t) \neq \emptyset \text{ if and only if } t \geq \tau_{ij}.$$

In other words τ_{ij} is the optimal solution of the minimization problem

$$\text{Minimize} \quad t \quad \text{subject to} \quad T_{ij}(t) \neq \emptyset.$$

Note that it follows from Lemma 3.1.4 of chapter 2 that for any fixed t ,

$$M(t) \neq \emptyset \text{ if and only if } \forall\, i \in I \;\exists\; j(i) \in J \text{ such that } T_{ij(i)}(t) \neq \emptyset \tag{7.5}$$

Which leads us to provide the next lemma.

Lemma 7.3.1 *Let us set* $I_j^{>} = \{i \in I \;;\; a_{ij} > \hat{b}_i + t\}$, $T_{ij}(t) = \{x_j \;;\; \hat{b}_i - t \leq a_{ij} \wedge x_j \leq \min_{k \in I_j^{>}} \hat{b}_k + t = \hat{b}_{k(j)} + t\}$ *for any* $i \in I, j \in J$. *Then*

$$M(t) \neq \emptyset \iff \forall i \in I \;\exists\; j(i) \in J,$$

such that

$$T_{ij(i)}(t) \neq \emptyset \;\;\&\;\; x \leq x(\hat{b} + t),$$

where we set $\hat{b} + t = (\hat{b}_1 + t, \cdots, \hat{b}_m + t)$.

Proof:

Let t be arbitrary and fixed. Since according to Lemma 7.2.2 $x(\hat{b} + t)$ is the maximum element of set M , then $M(t) \neq \emptyset$ if and only if $x_j \leq x_j(\hat{b}+t) \;\forall\, x_j \in T_{ij}(t)$ or in other words the upper bound of $T_{ij}(t)$ must not be violated if x is in $M(t)$. Let us assume know that $x \in M(t)$ and at the same time there exists index $k \in I$ such that $T_{kj}(t) = \emptyset \;\; \forall\; j \in J$. Since $x \in M(t)$, it must be $x_j \leq x_j(\hat{b} + t)$ for all $j \in J$ and therefore if $T_{kj}(t) \;\forall\; j \in J$ is empty, we have $a_{kj} \wedge x_j < \hat{b}_k - t \;\; \forall\; j \in J$ and therefore $\max_{j \in J}(a_{kj} \wedge x_j) < \hat{b}_k - t$ and $x \notin M(t)$, which is a cotradiction. To prove the oppsite assertion, we assume that for each $i \in I$, there exists at least one index $j(i) \in J$ such that $T_{ij(i)}(t) \neq \emptyset$ and $x \leq x(\hat{b}+t)$. We will prove that $M(t) \neq \emptyset$. In this case it is e.g. $\max_{j \in J}(a_{ij} \wedge x_j(\hat{b}+t)) \geq \hat{b}_i - t$. Since $x(\hat{b}+t)$ evidently satisfies the upper bound condition $x_j \leq x(\hat{b}+t)$, we obtain that $x(\hat{b}+t) \in M(t)$ and thus $M(t) \neq \emptyset$, which completes the proof.

□

As a consequeance of (7.5) and lemma 7.3.1 we obtain:
$M(t) \neq \emptyset$ if and only if $t \geq \tau \equiv \max_{i\in I} \min_{j\in J} \tau_{ij}$.
Therefore the necessary and sufficient condition of Lemma 7.3.1 will be satisfied for $t \geq \max_{i\in I} \min_{j\in J} \tau_{ij}$. Therefore the optimal solution t^{opt} of PROBLEM II is

$$t^{opt} = \max_{i\in I} \min_{j\in J} \tau_{ij}.$$

We will illustrate the theoretical result by a small numerical example.

Example 7.3.2
Let $m = n = 3$, $\hat{b} = (0, 1, 1)^T$,

$$A = \begin{pmatrix} 3 & 1 & 5 \\ 4 & 4 & 6 \\ 7 & 7 & 3 \end{pmatrix}$$

In this case $b^{\max} = (5, 6, 7)^T$ *and*
$\min_{j\in J} \tau_{1j} = \min(0, 0, 0) = 0$,
$\min_{j\in J} \tau_{2j} = \min(1/2, 1/2, 1/2) = 1/2$,
$\min_{j\in J} \tau_{3j} = \min(1/2, 1/2, 1/2) = 1/2$,
so that $t^{opt} = \max_{i\in I} \min_{j\in J} \tau_{ij} = \max(0, 1/2, 1/2) = 1/2$ *and the optimal solution of PROBLEM I is:* $b^{opt} = A \otimes (0, 1/2, 1/2)^T = (1/2, 1/2, 1/2)^T$. *Note that since for* $\tilde{x} = (1/2, 1/2, 1/2)^T$ *we have* $A \otimes \tilde{x} = b^{opt}$, *we obtain that* $b^{opt} \in R(A)$. *The optimal value of the objective function of* ***PROBLEM I*** *is* $\left\| b^{opt} - \hat{b} \right\| = \max(1/2, 1/2, 1/2) = 1/2$
In the next section, we will propose an algorithm for solving ***PROBLEM I***.

7.4 The Algorithm - Case $(\max, \min)$ - Threshold Version.

Let us introduce the following notations (we assume that $b \in R(A)$, $\hat{b} \notin R(A)$, $i \in I$*):*

$$H^+(b) = \{i \in I \ ; \ b_i > \hat{b}_i\}, \ H^-(b) = I \setminus H^+(b),$$

Let us define $\hat{x}(b) \in R^n$ *as follows:*

$$\hat{x}_j(b) = \begin{cases} \min_{i \in I_j^>} b_i, & \text{if } I_j^> \neq \emptyset, \\ \max_{i \in I_j^=} b_i, & \text{if } I_j^= \neq \emptyset \text{ and } I_j^> = \emptyset, \\ \infty & \text{otherwise,} \end{cases}$$

we set $I_j^> = \{i \in I \ ; \ a_{ij} > b_i\} \ \ \forall j \in J$, $I_j^= = \{i \in I \ ; \ a_{ij} = b_i\} \ \ \forall j \in J$ *such that* $I_j^> = \emptyset$.
Let us set further

$$G_i(b) = \{h \in J \ ; \ a_{ih} \wedge \hat{x}_h(b) = \max_{j \in J}(a_{ij} \wedge \hat{x}_j(b))\},$$

$$P(b) = \{i \in I \ ; \ a_{ih} \wedge \hat{x}_h(b) = \hat{x}_h(b) \ \forall h \in G_i(b).\}$$

We will call terms $a_{ih} \wedge \hat{x}_h(b)$ *in the definition of* $G_i(b)$ *"active terms of* $G_i(b)$*". If* $i_0 \in P(b)$*, then all active terms of* $G_{i_0}(b)$ *are equal to* $\hat{x}_h(b)$*, or in other words* $a_{i_0 h} \wedge \hat{x}_h(b) = \hat{x}_h(b)$*, which means that* $a_{i_0 h} \geq \hat{x}_h(b) \ \forall h \in G_{i_0}(b)$*.*

$$F^+(b) = \{k \in H^+(b) \cap P(b) \ ; \ \left|b_k - \hat{b}_k\right| = \max_{i \in I} \left|b_i - \hat{b}_i\right|\},$$

$$F^-(b) = \{k \in H^-(b) \cap P(b) \ ; \ \left|b_k - \hat{b}_k\right| = \max_{i \in I} \left|b_i - \hat{b}_i\right|\},$$

The main idea of the proposed algorithm is that we will begin the calculations with the maximum element $b^{\max}$ *and will try to decrease the value of the objective function of PROBLEM I. by decreasing components of* b *in such a way that we stay within attainable set* $R(A)$*. We will assume* $I_j^> \neq \emptyset$ *for all* $j \in J$ *to avoid infinite components of* $\hat{x}(b)$*. Let*

$$\overline{G} \equiv \bigcup_{k \in F^+(\bar{b})} G_k(\bar{b})$$

Algorithm 7.4.1

$\boxed{0}$ *Input* $m,\ n\ ,\ I,\ J,\ A,\ \hat{b},\ b^{\max},\ \bar{b} := b^{\max}$

$\boxed{1}$ *Determine* $H^+(\bar{b}),\ H^-(\bar{b}),\ \hat{x}(\bar{b}),\ G_i(\bar{b})\ \forall i \in I,\ P(\bar{b}),\ F^+(\bar{b}),\ F^-(\bar{b})$.

$\boxed{2}$ *If* $F^-(\bar{b}) \neq \emptyset$, *go to* $\boxed{8}$.

$\boxed{3}$ $\tilde{F}(\bar{b}) := \{i \in (I \setminus F^+(\bar{b})) \cap P(\bar{b})\ ;\ G_i(\bar{b}) \subseteq \bigcup_{k\in(F^+(\bar{b})\cap P(\bar{b}))} G_k(\bar{b})\}$.

$\boxed{4}$ $T(\bar{b}) := F^+(\bar{b}) \cup \tilde{F}(\bar{b})$; *if* $T(\bar{b}) = \emptyset$, *go to* $\boxed{8}$.

$\boxed{5}$ *set for* $t \geq 0$: $b_i(t) := \bar{b}_i - t \quad \forall i \in T(\bar{b})$, $b_i(t) := \bar{b}_i$ *otherwise.*

$\boxed{6}$ *Increase t until a value* $\tau > 0$, *for which for the first time one of the following events will occur:*

(a) $b_i(\tau) = \hat{b}_i$ *for some* $i \in T(\bar{b})$;

(b) $b_i(\tau) = \bar{b}_p$ *for some* $i \in T(\bar{b}),\ p \in I \setminus T(\bar{b})$;

(c) $\left|b_i(\tau) - \hat{b}_i\right| = \max_{k\in(I\setminus T(\bar{b}))} \left|\bar{b}_k - \hat{b}_k\right|$ *for some* $i \in T(\bar{b})$. *It may happen that* $G_p(\bar{b}) \nsubseteq \overline{G}$ *so that* $p \notin \tilde{F}(\bar{b})$, *but* $G_p(\bar{b} - t) \subseteq \overline{G}$ *so that* $p \in \tilde{F}(\bar{b} - t) \cap P(\bar{b} - t)$;

(d) $\left|b_i(\tau) - \hat{b}_i\right| = \left|b_k(\tau) - \hat{b}_k\right|$, *where* $i \in \hat{F}^+(\bar{b})$ *and* $k \in T(\bar{b}) \cap H^-(\bar{b})$.

(e) $P(\bar{b})$ *may change, i.e. until for some* $t = \tau$ *may be* $P(b(\tau)) \neq P(\bar{b})$, $\bar{b}_i - t = \max_{j\in J\setminus G_i(\bar{b})} a_{ij}$ *for some* $i \in T(\bar{b})$.

– *Find* τ *by making use of algorithm (7.4.2).*

$\boxed{7}$ *Set* $\bar{b} := b(\tau)$, *go to* $\boxed{1}$.

$\boxed{8}$ *Set* $b^{opt} := \bar{b}$, *STOP.*

In what follows we will bring an algorithm (7.4.2) for determining τ *in step* $\boxed{7}$ *of algorithm (7.4.1). For this purpose we will introduce the following simplifying notations:*

$$\alpha(\bar{b}) \equiv \left\|\bar{b} - \hat{b}\right\|,$$

$$\beta(\bar{b}) \equiv \max_{i\in I\setminus T(\bar{b})} \left|\bar{b}_i - \hat{b}_i\right|.$$

Let us recall that $b_i(t) = \bar{b}_i - t \ \forall i \in T(\bar{b}),\ b_i(t) = \bar{b}_i \ \forall i \in I \setminus T(\bar{b})$. *We have then:*
$\left| b_i(t) - \hat{b}_i \right| = b_i(t) - \hat{b}_i = \bar{b}_i - t - \hat{b}_i \quad \forall i \in F^+(\bar{b})$
$\left| b_i(t) - \hat{b}_i \right| = \hat{b}_i - b_i(t) = \hat{b}_i - \bar{b}_i + t \ \forall i \in H^-(\bar{b}) \cap T(\bar{b})$.
We will analyze in detail cases (a) - (d) from step $\boxed{6}$.
Case (a)
We have for $i \in T(\bar{b})$
$b_i(t) = \hat{b}_i$ *if* $t = \tau_i^{(1)} \equiv (\bar{b}_i - \hat{b}_i)$. *Case (a) takes place for the first time if*

$$t = \tau^{(1)} \equiv \min_{i \in T(\bar{b})} \tau_i^{(1)}.$$

Case (b)
We have $b_i(t) = \bar{b}_p$ *for some* $i \in T(\bar{b}),\ p \in I \setminus T(\bar{b})$ *if* $t = \tau_{ip}^{(2)} \equiv \bar{b}_i - \bar{b}_p$. *Case (b) takes place for the first time if*

$$t = \tau^{(2)} = \min_{i \in T(\bar{b}),\ p \notin T(\bar{b})} \tau_{ip}^{(2)}.$$

Case (c)
We have $\left\| b(t) - \hat{b} \right\| = \alpha(\bar{b}) - t$, *so that* $\left\| b(t) - \hat{b} \right\| = \beta(\bar{b})$ *if*

$$t = \tau^{(3)} \equiv (\alpha(\bar{b}) - \beta(\bar{b})).$$

Case (d)
We have for $k \in T(\bar{b}) \cap H^-(\bar{b})$ *the equality* $\left| b_k(t) - \hat{b}_k \right| = \hat{b}_k - \bar{b}_k + t$ *so that* $\left\| b(t) - \hat{b} \right\| = \left| b_k(t) - \hat{b}_k \right|$ *if* $\alpha(\bar{b}) - t = (\hat{b}_k - \bar{b}_k + t)$, *i.e. if* $t = \tau_k^{(4)} \equiv (\alpha(\bar{b}) - \hat{b}_k + \bar{b}_k)/2$. *Case (d) takes place for the first time if*

$$t = \tau^{(4)} \equiv \min_{k \in (T(\bar{b}) \cap H^-(\bar{b}))} \tau_k^{(4)}.$$

Case (e)

$$\bar{b}_i - t = \max_{j \in J \setminus G_i(\bar{b})} (a_{ij}), \textit{for some } i \in T(\bar{b})$$

i.e.

$$t = \tau_i^5 = \bar{b}_i - \max_{j \in J \setminus G_i(\bar{b})} (a_{ij})$$

for some $i \in T(\bar{b})$.
We set then $\tau^{(5)} = \min_{i \in T(\bar{b})} \tau_i^{(5)}$.
One of the Cases (a) - (e) takes place for the first time if

$$t = \tau \equiv \min_{1 \le k \le 5} \tau^{(k)}$$

Value $\tau > 0$ *will be inserted in step* $\boxed{6}$ *of algorithm (7.4.1). We will summarize these considerations in the following*

Algorithm 7.4.2

$\boxed{1}$ *Input* $\hat{b},\ \bar{b},\ \alpha(\bar{b}),\ \beta(\bar{b})$.

$\boxed{2}$ $\tau_i^{(1)} \equiv (\bar{b}_i - \hat{b}_i) \ \ \forall i \in T(\bar{b})$,

$$\tau^{(1)} := \min_{i \in T(\bar{b})} \tau_i^{(1)}.$$

$\boxed{3}$ $\tau_{ip}^{(2)} \equiv \bar{b}_i - \bar{b}_p,\ \forall i \in T(\bar{b}),\ p \notin T(\bar{b})$,

$$t = \tau^{(2)} = \min_{i \in T(\bar{b}),\ p \notin T(\bar{b})} \tau_{ip}^{(2)}.$$

$\boxed{4}$ $\tau^{(3)} \equiv (\alpha(\bar{b}) - \beta(\bar{b}))$.

$\boxed{5}$ $\tau_k^{(4)} \equiv (\alpha(\bar{b}) - \hat{b}_k + \bar{b}_k)/2 \ \ \forall k \in T(\bar{b}) \cap H^-(\bar{b})$,

$$\tau^{(4)} \equiv \min_{k \in (T(\bar{b}) \cap H^-(\bar{b}))} \tau_k^{(4)}.$$

$\boxed{6}$

$$\tau^{(5)} \equiv \min_{i \in T(\bar{b})} (\bar{b}_i - \max_{j \in J \setminus G_i(\bar{b})} (a_{ij})).$$

$\boxed{7}$

$$\tau \equiv \min_{1 \le k \le 5} \tau^{(k)}.$$

Let us solve the same problem using the threshold algorithm from the preceding section. The iterations of this algorithm will be the following:

$\boxed{1}$ $m = n = 3, I, J, A, \hat{b}, \bar{b}; = b^{\max} = (5, 6, 7)^T$;

Iteration 1

$\boxed{2}$ $\bar{b} = (5, 6, 7),\ H^+(\bar{b}) = I, H^-(\bar{b}) = \emptyset,\ \hat{x}(\bar{b}) = (7, 7, 6)^T,\ G_1(\bar{b}) = \{3\},\ G_2(\bar{b}) = \{3\},\ G_3(\bar{b}) = \{1, 2\},\ P(\bar{b}) = \{2, 3\},\ F^+(\bar{b}) = \{3\}$;

$\boxed{3}$ $F^-(\bar{b}) = \emptyset$;

$\boxed{4}$ $\tilde{F}(\bar{b}) = \emptyset$;

$\boxed{5}$ $T(\bar{b}) = \{3\}$;

$\boxed{6}$ $b(t) = (5, 6, 7 - t)^T$

$\boxed{7}$ $\tau = \min(7, 1, 1, +\infty) = 1$;

$\boxed{8}$ $\bar{b} := b(\tau) = (5, 6, 6)^T$

Iteration 2

$\boxed{2}$ $\bar{b} = (5, 6, 6),\ H^+(\bar{b}) = I,\ H^-(\bar{b}) = \emptyset,\ \hat{x}(\bar{b}) = (6, 6, 6),\ \max_{j \in J}(a_{1j} \wedge x_j(\bar{b})) = a_{13} = 5$, *so that* $G_1(\bar{b}) = \{3\},\ \max_{j \in J}(a_{2j} \wedge x_j(\bar{b})) = x_3(\bar{b}) = 6,\ \ G_2(\bar{b}) = \{3\},\ \max_{j \in J}(a_{3j} \wedge x_j(\bar{b})) = x_1(\bar{b}) = x_2(\bar{b}) = 7$ *so that* $G_3(\bar{b}) = \{1, 2\}$, *futher we have* $P(\bar{b}) = \{2, 3\}$ *so that* $F^+(\bar{b}) = \{2, 3\},\ F^-(\bar{b}) = \emptyset,$; $\boxed{3}$ $F^-(\bar{b}) = \emptyset$;

$\boxed{4}$ $\tilde{F}(\bar{b}) = \emptyset$;

$\boxed{5}$ $T(\bar{b}) = \{2, 3\}$;

$\boxed{6}$ $b(t) = (5, 6 - t, 6 - t)$;

$\boxed{7}$ $\tau = 1$;

$\boxed{8}$ $\bar{b} = b(\tau) = (5, 5, 5)^T$;

Iteration 3

$\boxed{2}$ $\bar{b} = (5, 5, 5),\ H^+(\bar{b}) = I,\ H^-(\bar{b}),\ \hat{x}(\bar{b}) = (5, 5, 5),\ F^+(\bar{b}) = \{1\}$;

$\boxed{3}$ $F^-(\bar{b}) = \emptyset$;

$\boxed{4}$ $\tilde{F}(\bar{b}) = \{2\}$;

$\boxed{5}$ $T(\bar{b}) = \{1, 2\}$;

$\boxed{6}$ $b(t) = (5 - t, 5 - t, 5)$;

$\boxed{7}$ $\tau = 1$;

$\boxed{8}$ $\bar{b} = b(\tau) = (4, 4, 5)^T$;

Iteration 4

$\boxed{2}$ $\bar{b} = (4,4,5)^T$ $H^+(\bar{b}) = I, H^+(\bar{b}) = \emptyset$, $\hat{x}(\bar{b}) = (5,5,4)$, $G_1(\bar{b}) = \{3\}$, $G_2(\bar{b}) = \{1,2,3\}$, $G_3(\bar{b}) = \{1,2\}$, $P(\bar{b}) = \{1,3\}$, $F^+(\bar{b}) = \{1,3\}$;
$\boxed{3}$ $F^-(\bar{b}) = \emptyset$;
$\boxed{4}$ $\tilde{F}(\bar{b}) = \emptyset$;
$\boxed{5}$ $T(\bar{b}) = \{1,3\}$;
$\boxed{6}$ $b(t) = (4-t, 4, 5-t)$;
$\boxed{7}$ $\tau = 1$;
$\boxed{8}$ $\bar{b} = b(\tau) = (3,4,4)^T$;

Iteration 5

$\boxed{2}$ $\bar{b} = (3,4,4)$, $H^+(\bar{b}) = I$, $H^-(\bar{b}) = \emptyset$, $\hat{x}(\bar{b}) = (4,4,3)$, $P(\bar{b}) = \{2,3\}$, $F^+(\bar{b}) = \{2,3\}$;
$\boxed{3}$ $F^-(\bar{b}) = \emptyset$;
$\boxed{4}$ $\tilde{F}(\bar{b}) = \emptyset$
$\boxed{5}$ $T(\bar{b}) = \{2,3\}$;
$\boxed{6}$ $b(t) = (3, 4-t, 4-t)$;
$\boxed{7}$ $\tau = 1$;
$\boxed{8}$ $\bar{b} := b(\tau) = (3,3,3)$;

Iteration 6

$\boxed{2}$ $\bar{b} = (3,3,3)$, $H^+(\bar{b}) = I$, $H^-(\bar{b}) = \emptyset$, $\hat{x}(\bar{b}) = (3,3,3)$, $P(\bar{b}) = (1,2,3)$, $F^+(\bar{b}) = \{1\}$;
$\boxed{3}$ $F^-(\bar{b}) = \emptyset$;
$\boxed{4}$ $\tilde{F}(\bar{b}) = \emptyset$;
$\boxed{5}$ $T(\bar{b}) = \{1\}$;
$\boxed{6}$ $b(t) = (3-t, 3, 3)$;
$\boxed{7}$ $\tau = 1$;
$\boxed{8}$ $\bar{b}; = b(\tau) = (2,3,3)$;

Iteration 7.

$\boxed{2}$ $\bar{b} = (2,3,3)$, $H^+(\bar{b}) = I$, $H^-(\bar{b}) = \emptyset$, $\hat{x}(\bar{b}) = (2,3,2)$, $P(\bar{b}) = \{1,2,3\}$, $F^+(\bar{b}) = \{1,2,3\}$;
$\boxed{3}$ $F^-\bar{b} = \emptyset$;
$\boxed{4}$ $\tilde{F}(\bar{b}) = \emptyset$;
$\boxed{5}$ $T(\bar{b}) = \{1\}$;
$\boxed{6}$ $b(t) = (2-t, 3-t, 3-t)$;
$\boxed{7}$ $\tau = 1$;

$\boxed{8}$ $\bar{b} := b(\tau) = (1, 2, 2)$;

Iteration 8.

$\boxed{2}$ $\bar{b} = (1, 2, 2)$, $H^+(\bar{b}) = I$, $H^-(\bar{b}) = \emptyset$, $\hat{x}(\bar{b}) = (1, 2, 1)$, $G_1(\bar{b}) = \{1, 2, 3\}$, $G_2(\bar{b}) = \{2\}$, $G_3(\bar{b}) = \{2\}$, $P(\bar{b}) = \{1, 2, 3\}$, $F^+(\bar{b}) = \{2, 3\}$;

$\boxed{3}$ $F^-(\bar{b}) = \emptyset$;

$\boxed{4}$ $\tilde{F}(\bar{b}) = \emptyset$;

$\boxed{5}$ $T(\bar{b}) = \{1, 2, 3\}$;

$\boxed{6}$ $b(t) = (1, 2 - t, 2 - t)$;

$\boxed{7}$ $\tau = \tau^{(5)} = 1$;

$\boxed{8}$ $\bar{b} := b(\tau) = (1, 1, 1)$;

Iteration 9

$\boxed{2}$ $\bar{b} := (1, 1, 1)$, $H^+(\bar{b}) = \{1\}$, $H^-(\bar{b}) = \{2, 3\}$, $\hat{x}(\bar{b}) = (1, 1, 1)$, $G_i(\bar{b}) = I \ \forall \ i \in I$, $P(\bar{b}) = I$, $F^+(\bar{b}) = \{1\}$; $\boxed{3}$ $F^-(\bar{b}) = \emptyset$;

$\boxed{4}$ $\tilde{F}(\bar{b}) = \{2, 3\}$;

$\boxed{5}$ $T(\bar{b}) = \{1, 2, 3\}$;

$\boxed{6}$ $b(t) = (1 - t, 1 - t, 1 - t)$;

$\boxed{7}$ $\tau = \tau^{(4)} 1$;

$\boxed{8}$ $\bar{b} := b(\tau) = (1/2, 1/2, 1/2)$;

Iteration 10

$\boxed{2}$ $\bar{b} = (1/2, 1/2, 1/2)$, $H^+(\bar{b}) = \{1\}$, $H^-(\bar{b})\{2, 3\}$, $\hat{x}(\bar{b}) = (1/2, 1/2, 1/2)$, $F^+(\bar{b}) = \{1\}$;

$\boxed{3}$ $F^-(\bar{b}) = \{2, 3\} \neq \emptyset$;

$\boxed{9}$ $b^{opt} := \bar{b} = (1/2, 1/2, 1/2)$, *STOP.*

8

Generalization Optimization Problems under One-Sided $\max -$ Separable Equation and Inequality Systems

In this chapter, we will introduce a generalization of one-sided (max, min)-linear systems of inequalities where the unknowns appear only in the left side of inequalities and on the right side of these systems of inequalities we have constants.

Results obtained for one-sided $(\max, \min)-$ or $(\max, +)-$linear systems can be generalized for the system of the form:

$$\max_{j\in J}(a_{ij} \wedge r_{ij}(x_j)) \geq b_i,\ i \in I, \tag{8.1}$$

$$\max_{j\in J}(a_{ij} \wedge r_{ij}(x_j)) \leq b_i,\ i \in I_1, \tag{8.2}$$

where $J = \{1, ..., n\},\ I = \{1, ..., m\}$ and $I_1 = \{m + 1, ..., m + m_1\}$ are finite index sets, $b_i, a_{ij} \in R \forall i \in I \cup I_1, j \in J$, and $r_{ij} : R \to R$ are strictly increasing continuous functions with the range equal to R (i.e. $\{r_{ij}(x)|x_j \in R\} = R$). Systems (8.1) and (8.2) encompass also equalities. Note that the inequalities (8.2) can under our assumptions be replaced by upper bounds on variables x_j, i.e. we have

$$x_j \leq \overline{x}_j \equiv \min_{k \in I_{ij}} r_{ij}^{-1}(b_k) \ \ \forall \ \ j \in J, \tag{8.3}$$

where $I_{1j} = \{k \in I_1 \mid a_{kj} > b_k\} \ \ \forall \ \ j \in J$. Besides, an appropriate choice of $a_{ij}, r_{ij}(x_j)$ for $i \in I, j \in J$ makes possible to include also lower bounds on x_j. It remains to choose for some $i_0 \in I, j_0 \in J$ values $a_{i_0 j}$ sufficiently small for $j \neq j_0$ and $a_{i_0 j_0}$ sufficiently large and $r_{i_0 j_0}(x_{j_0}) = x_{j_0}$. Then we have

$$max_{j \in J}(a_{i_0 j} \wedge r_{i_0 j}(x_j)) = a_{i_0 j_0} \wedge r_{i_0 j_0}(x_{j_0}) = x_{j_0},$$

and the i_0-th inequality of (8.2) is equivalent with $x_{j_0} \geq b_{i_0}$. Therefore in what follows we will simplify system (8.1) and (8.2) and will consider only the system

$$\max_{j \in J}(a_{ij} \wedge r_{ij}(x_j)) \geq b_i, \ i \in I, \tag{8.4}$$

$$\underline{x}_j \leq x_j \leq \overline{x}_j, \ \ j \in J, \tag{8.5}$$

where $J = \{1, ..., n\}, \ \ I = \{1, ..., m\}$ and $\underline{x}, \ \overline{x}$ are finite elements of R^n and $b \in R^m$. The set of all solutions of system (8.4) and (8.5) will be denoted $M(b)$. In what follows, we will study properties of the set $M(b)$. For this purpose, we will define sets

$$T_{ij}(b_i) = \{x_j | \underline{x}_j \leq x_j \leq \overline{x}_j \ \ \& a_{ij} \wedge r_{ij}(x_j) \geq b_i\} \ \ \forall \ \ j \in J. \tag{8.6}$$

If $x_j \in T_{ij}(b_i)$, then it must be fulfilled the following conditions:

$$\overline{x}_j \geq x_j \geq \max(\underline{x}_j, r_{ij}^{-1}(b_i)), \ \ \& \ \ a_{ij} \geq b_i. \tag{8.7}$$

In other words inequalities (8.7) are necessary and sufficient for $T_{ij}(b_i) \neq \emptyset$ and if $T_{ij}(b_i) = \left[\max(\underline{x}_j, r_{ij}^{-1}(b_i)), \overline{x}_j\right]$ (i. e. $T_{ij}(b_i)$ is closed interval).

Theorem 8.0.1
$M(b) \neq \emptyset$ if and only if [$\forall \ i \in I \ \ \exists \ \ j(i) \in J$ such that $T_{ij(i)}(b_i) \neq \emptyset$]

Proof:
Let $M(b) \neq \emptyset$ and let the condition[$\forall \ i \in I \ \ \exists \ \ j(i) \in J$ such that $T_{ij(i)}(b_i) \neq \emptyset$

] is not fulfilled. Then there exists $i_0 \in I$ such that $T_{i_0j}(b_{i_0}) = \emptyset \quad \forall \quad j \in J$. In this case we have for any $x \in R^n$ and either there exists $j_0 \in J$ such that $x_{j_0} \notin \left[\underline{x}_{j_0}, \overline{x}_{j_0}\right]$ so that $x \notin M(b)$, or $a_{ij} \wedge r_{ij}(x_j) < b_{i_0} \quad \forall \quad j \in J$ so that $\max_{j\in J}(a_{i_0j} \wedge r_{i_0j}(x_j)) < b_{i_0}$ and thus $x \notin M(b)$ again. It follows that if the condition [$\forall\ i \in I \ \exists \ j(i) \in J$ such that $T_{ij(i)}(b_i) \neq \emptyset$] is not fulfilled it must be $M(b) = \emptyset$.

Let now the condition [$\forall\ i \in I \ \exists \ j(i) \in J$ such that $T_{ij(i)}(b_i) \neq \emptyset$] be fulfilled, let $i_0 \in I$ be arbitrary. We have $T_{i_0j(i_0)}(b_{i_0}) \neq \emptyset$ for some $j(i_0) \in J$. Let us set $p = j(i_0)$ and let $V_p \equiv \{i \in I \mid j(i) = p\}$, where $j(i)$ is defined by condition [$\forall\ i \in I \ \exists \ j(i) \in J$ such that $T_{ij(i)}(b_i) \neq \emptyset$]. Since $i_0 \in V_p$, we have $V_p \neq \emptyset$ and $T_p(b) \equiv \bigcap_{i\in V_p} T_{ip}(b_i) = \left[max_{i\in V_p}(\underline{x}_p, r_{ip}^{-1}(b_i)), \overline{x}_p\right] \neq \emptyset$. If we choose $x_p \in T_p(b)$ arbitrary, we have $x_j \in T_{i_0p}(b_{i_0}) \subseteq T_p(b_{i_0})$ so that a conditions (8.6) are fulfilled for $i = i_0, j = p$ and we obtain $\max_{j\in J}(a_{i_0j} \wedge r_{i_0j}(x_j)) \geq a_{i_0p} \wedge r_{i_0p}(x_p) \geq b_{i_0}$. Since $i_0 \in I$ was arbitrary chosen, it follows $M(b) \neq \emptyset$, which completes the proof.

□

Remark 8.0.1

Lut us note that if $M(b) \neq \emptyset$, then $\overline{x} \in M(b)$ and therefore $\overline{x}$ defined by (8.3) is the maximum element of $M(b)$ in the sense that $x \leq \overline{x}$ for all $x \in M(b)$. If relations (8.1) and (8.2) represent a system of equations (i.e. $I = I_1$), then it holds also that the system of equations is solvable if and only if $\overline{x} \in M(b)$.

We will use Theorem 8.0.1 to solve the following optimization problem:

Problem P:

$$f(x) \equiv \max_{j\in J} f_j(x_j) \quad \longrightarrow \quad \min$$

subject to

$$x \in M(b),$$

where $f_j : R \to R$ are for all $j \in J$ continuous functions.

In the sequel we will derive an explicit formula for the optimal solution of **Problem P:**. We will assume further that $\overline{x}$ is defined by (8.3), $T_{ij}(b_i)$ are defined by (8.5) and $M(b) \neq \emptyset$. The non-emptiness of $M(b)$ can be easily verified by Remark 8.0.1.

Theorem 8.0.2
Let $M(b) \neq \emptyset$, $x_j^{(i)} = \operatorname{argmin}\{f_j(x_j) \mid x_j \in T_{ij}(b_i)\}$ for all nonempty sets $T_{ij}(b_i)$, $i \in I$, $j \in J$. Let us set

$$J_i = \{j \mid T_{ij}(b_i) \neq \emptyset\} \quad \forall \quad i \in I \tag{8.8}$$

$$\min_{j \in J_i} f_j(x_j^{(i)}) = f_{j(i)}(x_{j(i)}^{(i)}) \quad \forall \quad i \in I \tag{8.9}$$

$$V_j = \{i \in I \mid j(i) = j\} \quad \forall \quad j \in J \tag{8.10}$$

$$T_j(b) = \begin{cases} \bigcap_{i \in V_j} T_{ij}(b_i) \quad \forall \quad j \in J, & \textit{such that } V_j \neq \emptyset, \\ \left[\underline{x}_j, \overline{x}_j\right] & \textit{otherwise,} \end{cases} \tag{8.11}$$

$$\hat{x}_j = argmin\{f_j(x_j) \mid x_j \in T_j(b)\} \quad \forall \quad j \in J \quad \textit{such that } V_j \neq \emptyset \tag{8.12}$$

$$\hat{x}_j \in T_j(b) \qquad \textit{arbitrary} \qquad \textit{if} \quad V_j = \emptyset. \tag{8.13}$$

Then $\hat{x}$ is the optimal solution of ***Problem P:***

Proof: It follows from the monotonicity of r_{ij} $'s$ that for any $j \in J$, for which $V_j \neq \emptyset$ there exists an index $k(j) \in I$, for which

$$T_j(b) = T_{k(j)j}(b_{k(j)}) \;\; \texttt{and} \;\; T_{ij}(b_i) \subseteq T_{k(j)j}(b_{k(j)}) = T_j(b).$$

Let $i_0 \in I$ be arbitrarily chosen, let $j(i_0)$ be defined as in (8.9). Let us set $p = j(i_0)$ to simplify the notation. Then we have $\hat{x}_{j(i_0)} = \hat{x}_p \in T_p = T_{k(p)p}(b_k(p)) \subseteq T_{i_0p}(b_{i_0})$ so that $\hat{x}_p \in T_{i_0p}(b_{i_0})$ and therefore $\hat{x}_p \in \left[\underline{x}_p, \overline{x}_p\right]$ and $a_{i_0p} \wedge \hat{x}_p \geq b_{i_0}$. It follows that

$$\max_{j \in J}(a_{i_0j} \wedge r_{i_0j}(\hat{x}_j)) \geq a_{i_0p} \wedge r_{i_0p}(\hat{x}_p) \geq b_{i_0}. \tag{8.14}$$

Since $i_0 \in I$ was chosen arbitrarily and $\hat{x}_j \in \left[\underline{x}_j, \overline{x}_j\right] \;\; \forall \;\; j$ it follows that $\hat{x} \in M(b)$. It remains to prove that $\hat{x}$ is the optimal solution of **Problem P:** i.e. $f(\hat{x}) \leq$

$f(x) \ \ \forall \ \ x \in M(b)$. Let us assume that $f(\hat{x}) = f_s(\hat{x}_s)$ and let us assume that there exists $\tilde{x} \in M(b)$ such that $f(\tilde{x}) < f(\hat{x})$. It is then $f_s(\tilde{x}_s) \leq f(\tilde{x}) < f_s(\hat{x}_s)$ so that $\tilde{x}_s \notin T_s = T_{k(s)s}(b_{k(s)})$, where $k(s) \in I$. Let us set $k(s) = h$ to simplify the notation. Then we have $\hat{x}_s = x^{(h)}_{j(h)} = argmin\{f_j(x^{(h)}_j) \mid j \in J_h\}$ according to (8.9) for $i = h = k(s)$ or in other words $s = j(h)$, where $j(h)$ is defined as in (8.9). Since $\tilde{x}_s \notin T_{hs}(b_h)$ and $\tilde{x} \in M(b)$, it must exist an index $v \in J, v \neq s$ such that $\tilde{x}_v \in T_{hv}(b_h)$ with $V_v \neq \emptyset$ (otherwise it would be $\max_{j\in J}(a_{hj} \wedge r_{hj}(\tilde{x}_j)) < b_h$). Then we obtain

$$\begin{aligned} f(\tilde{x}) &\geq f_v(\tilde{x}_v) \geq min_{x_v \in T_{hv}(b_h)} f_v(x_v) = f_v(x^{(h)}_v) \\ &\geq min_{j\in J_h} f_j(x^{(h)}_j) = f_s(x^{(h)}_s) = f_s(\hat{x}_s). \end{aligned} \tag{8.15}$$

It follows that $f(\tilde{x}) \geq f_s(\hat{x}_s) = f(\hat{x})$, which is contradiction with the assumed inequality $f(\tilde{x}) < f(\hat{x})$. This contradiction completes the proof.

□

Remark 8.0.2

Relations (8.12) and (8.13) give an explicit formula for $\hat{x}$ and $f(\hat{x})$ because we have for each $j \in J$ with $V_j \neq \emptyset$

$$T_j = \left[\max_{i\in V_j}(\underline{x}_j, r^{-1}_{ij}(b_i)), \overline{x}_j\right]$$

and if $V_j = \emptyset$

$$T_j = \left[\underline{x}_j, \overline{x}_j\right]$$

so that $T_j, \ \ j \in J$ are closed nonempty intervals in R^1 and $\hat{x}_j$ are points of minimum of continuous functions $f_j(x_j)$ on such intervals, which always exist. The concrete algorithms for finding $\hat{x}_j$ and their complexity depend on concrete form of functions f_j. For example:

- *If f_j is an increasing function, we have*

$$\hat{x}_j = \begin{cases} \max_{i\in V_j}(\underline{x}_j, r^{-1}_{ij}(b_i)), & \textit{if} \quad V_j \neq \emptyset, \\ \underline{x}_j & \textit{otherwise.} \end{cases}$$

- *If* f_j *is a decreasing function, then* $\hat{x}_j = \overline{x}_j$.
- *If* f_j *is a concave function,*

$$f_j(\hat{x}_j) = \begin{cases} \min\left(f_j(max_{i\in V_j}(\underline{x}_j, r_{ij}^{-1}(b_i)), f_j(\overline{x}_j)\right), & if \quad V_j \neq \emptyset, \\ \min\left(f_j(\underline{x}_j), f_j(\overline{x}_j)\right) & otherwise. \end{cases}$$

- *If* f_j *is a convex function, then* $\hat{x}_j$ *can be obtained by one of the known convex function minimization techniques (e.g. binary search on closed intervals for minimum of unimodal functions).*

Remark 8.0.3

The results of this section make possible to obtain some results of the previous sections as a special case. If we set a_{ij} *sufficiently large and* $r_{ij}(x_j) = c_{ij} + x_j$ *for* $c_{ij} \in R$, *we obtain* $(\max, +)-$ *linear systems. If we set* $r_{ij} = x_j$, *we obtain* $(\max, \min)-$ *linear systems.*

9

Conclusions

In this chapter we will introduce a summary of what has been accomplished in the previous chapters of the thesis.

In the first chapter, we have introduced the historical background since the beginning of this idea in the sixties of the last century and a brief summary of what has been done in previous studies on these topics about max-min algebra and optimization problems, where the set of feasible solutions of it is described by a system of $(\max, +)$ or (max, min) equations or inequalities with variables on one sided or both sides, also introduce some practical applications to these topics.

In the second chapter, we introduced for example a finite algorithm for finding the optimal solution of optimization problem in which the set of feasible solutions is described by a system of $(\max, +)-$ linear equations with variables on both sides. The main idea of the proposed algorithm consists in successive decreasing active variables in thresholds of the objective function f without leaving the feasible set until we cannot decrease the objective function without violating the constraint $\underline{x} \leq x$.

The actual beginning of the study of the topic of this thesis was in the third chapter, in which we study optimization problems under one-sided $(\max, \min)$-linear inequality constraints. We begin our studies by investigating properties of one-sided $(\max, \min)$-linear systems of inequalities where the unknowns appear in the left side only of inequalities and on the right side of these systems of inequalities we have constant variables only. We introduce Algorithm 3.1.1, which determines whether the set of all solutions of system (3.1) and (3.2), $M(\overline{x}) = \emptyset$ or finds the maximum element of set $M(\overline{x})$. As well as in the case if there is boundary conditions we introduce Algorithm 3.1.2, which performs the same process as in Algorithm 3.1.1. After that we solve optimization problems under one-sided $(\max, \min)$-linear inequality constraints and we introduce Algorithm 3.2.1, which finds an optimal solution of these optimization problems under the assumption that the set of all feasible solutions of one-sided $(\max, \min)$-linear systems of inequalities is not empty. We give Example 3.1.1, which shows one possible application of the system of one-sided $(\max, \min)$-linear systems of inequalities.

Since each equality can be replaced by two inequalities $\leq$, $\geq$, so that the system of equations in formulation (4.1), can be solved by the algorithm 3.1.2 given in chapter 3, but the disadvantage of the this technique is coming from the fact that we need to solve double the number of inequalities, which need more time and memory. So we introduce chapter four, in which we consider the optimization problems under one-sided $(\max, \min)$-linear equality constraints. First we study the structure of the set of all solutions of the given system of equations, which describe by (4.1) and (4.2) with finite entries a_{ij} & b_i, for all $i \in I$ &$j \in J$ and we determine the maximum element of this set. We propose Algorithm 4.2.1, which finds an optimal solution x^{opt} of problem (4.3) and (4.4), where $f_j(x_j)$ are continuous and monotone functions. The idea in Algorithm 4.2.1, was modified as in Algorithm 4.2.2 to be suitable to find the optimal solution for any general continuous functions $f_j(x_j)$.

The study has been expanded to include systems of two-sided $(\max, \min)-$ linear equations and inequalities, where the methods have been introduced in the previous chapters can not solve these systems since the $\max -$ and $\min -$ operations are only a semigroup operations, so that the variables can not be simply transferred from one-sided of the equations or inequalities to the other. Therefore two-sided $(\max, \min)-$linear equations and inequalities with variables on both sides of the relations have been studied and investigated in chapters fifth and sixth. In the fifth chapter of this thesis, we introduce Algorithm 5.2.1, which is depend on an iteration method to find the optimal solution of problem of optimization problems under two-sided $(\max, \min)-$linear equation constraints. Example 5.3.1 is an important practical application for the systems of two-sided (max, min)− linear equations and inequalities. In the sixth chapter, we introduce Algorithm 6.2.1, which is depend on a threshold method for find the optimal solution of optimization problems under two-sided (max, min)−linear inequality constraints. We can summarize the properties of the systems of (max, min)-linear inequalities studied in chapter 6 as follows:

(1) Any system of two-sided (max, min)-linear inequalities is solvable and has a unique maximum element $x^{\max}(A, B)$ depending on the matrices $A,\ B$ with finite elements $a_{ij},\ b_{ij}$ (note that including infinite elements can cause nonsolvability of the system).

(2) If we include an additional requirement $x \le \overline{x}$, then the system is also solvable and has the maximum element $x^{\max}(A,\ B,\ \overline{x}) \le x^{\max}(A, B)$ if no lower bound are given.

(3) The system with a finite lower bound on variables (i.e. with an additional constraint $x \ge \underline{x}$) is solvable if and only if $\underline{x} \le x^{\max}(A, B)$, or in case of the additional upper bound $\overline{x}$ if and only if $\underline{x} \le x^{\max}(A,\ B,\ \overline{x})$.

In the seventh chapter of the thesis, the concept of incorrectly posed is introduced for one-sided (max, min)−linear equations systems, where there is no solutions for the problem for given coefficients and we look for a close set (with respect to a given distance function) of coefficients generating a solvable problem. We introduce the concept the attainable set. Various approaches to solving such incorrectly posed problems are proposed. In section 7.3, we use a parametric version method to solve the incorrectly posed problem for one-sided (max, min)−linear equations systems. Also in section 7.4, we use a threshold version method to solve the same problem.

In the eighth chapter of the thesis, generalization optimization problems under one-sided max − separable equation and inequality systems have been studied.

Future studies

We will focus our studies in the future on applied problems and how can use the methods introduced in this study to solve practical problems.

We will try to make generalization for optimization problems under two-sided (max, min)-linear inequality and equalty constraints.

We will try to study:

- Parametric optimization problems under (max, min)-linear inequality and equality constraints.

- The sensitivity study of the methods that were introduced to solve optimization problems under (max, min)-linear inequality and equalty constraints.

- Inverse problems of systems of $max-$ Separable equation and inequality systems.

- Duality for optimization problems under (max, min)-linear inequality and equalty constraints.

List of papers constituting this thesis:

1. Zimmermann, K. and **Gad, M.**: Optimization Problems under One-Sided (max, +)-Linear Constraints,*International Conference Presentation of Mathematics '11 "ICPM'11"*, Liberec, on October 20 - 21, 2011.

2. Gavalec, M. , **Gad, M.** and Zimmermann, K.: Optimization Problems under (max, min)-Linear Equations and / or Inequality Constraints,*Journal of Mathematical Sciences* **193 (5)** (2013), 645–658. Translated from Russian Journal *Fundamentalnaya i Prikladnaya Matematika(Fundamental and Applied Mathematics)* **17 (6)**(2012), 3–21.

3. **Gad, M.**: Optimization Problems under One-Sided (max, min)-Linear Equality Constraints, *WDS'12 Proceedings of Contributed Papers Part I*, 1319, 2012. *21st Annual Student Conference, Week of Doctoral Students Charles University*, Prague, May 29 - June 1, 2012.

4. **Gad, M.**: Optimization Problems under Two-Sided (max, min)-Linear Inequalities Constraints, *Academic Coordination Centre "ACC" JOURNAL* **18 (4)**, 84–92, 2012. *International Conference Presentation of Mathematics '12 "Conference ICPM'12"*, Liberec, on June 21-22, 2012,

5. Zimmermann, K. and **Gad, M.**: Incorrectly Posed Optimization Problems under Extremally Linear Equation Constraints, *the scientific Journal of the Olomouc University* (to appear)

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