Charles University in Prague

Faculty of Mathematics and Physics

MASTER THESIS

Bc. Lada Peksová

Algebras over operads and properads

Mathematical Institute of Charles University

Supervisor of the master thesis: Ing. Branislav Jurčo, CSc., DSc.

Study programme: Mathematics

Study branch: Mathematical Structures

Prague 2016

I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources.

I understand that my work relates to the rights and obligations under the Act No. 121/2000 Sb., the Copyright Act, as amended, in particular the fact that the Charles University in Prague has the right to conclude a license agreement on the use of this work as a school work pursuant to Section 60 subsection 1 of the Copyright Act.

In .... date .... signature of the author

Title: Algebras over operads and properads

Author: Bc. Lada Peksová

Department: Mathematical Institute of Charles University

Supervisor: Ing. Branislav Jurčo, CSc., DSc., Mathematical Institute of Charles University

Abstract: Operads are objects that model operations with several inputs and one output. We define such structures in the context of graphs, namely oriented trees. Then we generalize operads to properads and modular operads by taking general graphs with, or without, orientation.

Further we construct the cobar complex of operads and properads and illustrate the construction on the examples of the associative operad Ass and the Frobenius properad $Frob$. Algebras over the cobar complex of operads correspond to certain homotopy algebras, for our example of Ass it is A_∞. We find its Maurer-Cartan equation and convert it from coderivations to derivations. Similarly we find the Maurer-Cartan equation for $C(Frob)$.

Keywords: operads, properads, algebras over operads, Maurer-Cartan equation, cobar complex

I would like to express my gratitude to my supervisor Braňo Jurčo and advisor Martin Doubek for introducing me to the topic, for explaining me the abstract concepts and for many useful comments.

I would like to thank to my family, whose supported me throughout entire process, and I would also like to thank to Víťa Kala for encouraging me in difficult times

Contents

Introduction **3**

Conventions and notation **5**

1 Operads **7**
1.1 Axioms of operad structure . . . 7
1.2 Construction of operads via graphs . . . 9
1.3 Cyclic operads . . . 15
1.4 Modular operads . . . 16
1.5 Differential graded operads . . . 18

2 The cobar complex of operads **21**
2.1 General construction . . . 21
2.2 Cobar complex of operad Ass . . . 27

3 The duality of derivations and coderivations **31**
3.1 Algebras and coalgebras . . . 31
3.2 Description of algebras as dual of coalgebras . . . 32
3.3 A_∞-algebra . . . 39
3.4 Maurer-Cartan equation in derivations . . . 41

4 Properads **45**
4.1 PROPs and properads . . . 45
4.2 Cobar complex of properads . . . 51
4.3 Algebras over $C(Frob)$ . . . 56

Bibliography **61**

Introduction

Operads are objects that model operations with several inputs and one output. The operad captures the composition of operations and the permutation of variables. Such a structure can be considered in the context of graphs. The composition of m-ary operation and n-ary operation gives us $(m+n-1)$-ary operation. In the graph context it is the joining of two directed trees, which produces a new directed tree.

Classical algebras, e.g., associative algebras and commutative algebras are algebras over operads. Such algebras are coded by operads Ass and Com.

If we take instead of directed trees undirected trees, we get the notion of a cyclic operad. The distinction between inputs and the output disappears. And by replacing trees by even more general undirected graphs we get modular operads.

An essential example of modular operads are homeomorphy classes of two dimensional compact orientable surfaces with labeled boundary components – for short Riemann surfaces with punctures (holes). Two Riemann surfaces S_1 with n_1 holes and genus g_1 and S_2 with n_2 holes and genus g_2 can be 'glued' together by identifying a puncture from S_1 with a puncture from S_2. This gives us a new Riemann surface with $n_1 + n_2 - 1$ holes and genus $g_1 + g_2$. If we identify two punctures on one Riemann surface with n holes and genus g we get a surface with holes $n - 2$ holes and genus $g + 1$.

There is another way of generalizing this. If we take instead of directed trees directed graphs we get the notion of PROPs. PROPs admit operations with several inputs and several outputs. Well known examples of algebras over PROPs are bialgebras.

In contrast to operads, even a couple of generators with several identities can lead to free PROPs with infinite dimensional components. To handle this combinatorial explosion we introduce properads. A properad is the connected part of PROP. Connected graphs then allow us to split infinite dimensional components of properad into the union of infinitely many finite dimensional components. Our main example of a properad is the Frobenius properad, motivated by the Frobenius bialgebra.

It is obvious that the example of Riemann surfaces with holes can be adapted as an example of properad structure. If we specify which holes serve as inputs and which as outputs, we get the structure of a properad.

The construction of cobar complex is a useful general construction. For some operads, namely for Koszul operads, the cobar complex can be used to construct a minimal model of this operad. We will not use this aspect. Instead of this we will show that the cobar complex of a general operad P with trivial components $P(0)$ and $P(1)$ forms a new differential graded operad, $C(P)$. If we consider an algebra over this new differential graded operad we get a homotopy algebra, where the required identities from operad hold up to homotopy. For example, the cobar complex of the operad Ass gives us an A_∞ operad - strong homotopy associative algebra, where the associativity holds up to homotopy.

Homomorphisms of differential graded operads that respect the differentials satisfy certain condition. The condition can be interpreted as certain Maurer-Cartan equation. Hence algebras over the cobar complex of an operad P are in bijection with solutions of certain classical master equation within some (in general) non-commutative symplectic geometry. The homotopy algebra can be therefore eqivalently expressed as square zero coderivation on appropriate vector space.

For example, A_∞-algebra on a graded vector space V is equivalently expressed as coderivation C on the tensor coalgebra T^cV such that $C^2 = 0$.

The idea can be generalized to properads as well.

The thesis is organized as follows. In the first chapter we review definitions of operads, cyclic operads, modular operads and differential graded operads. We recall the notions of graphs, oriented graphs and trees. We describe free operads as algebras over some monad and show some basic examples as quotients of free operad.

In the second chapter we outline the idea of the construction of the cobar complex and show an example of such a construction for the operad Ass. In the following we explicitly show that an algebra over cobar construction of operad Ass corresponds to some Maurer-Cartan equation.

The third chapter is devoted to the study of the duality of algebras and coalgebras. We prove that by the dualization of any coalgebra V with coproduct Δ and coderivation C we obtain algebra structure on the dual space $V^\#$ with product μ and derivation D. We show that the Maurer-Cartan equation for A_∞-algebra is equivalent to a coderivation identity on tensor coalgebra.

Then we translate Maurer-Cartan equation for finite dimensional space to a derivation identity through the use of previous results of this chapter.

In the last chapter, we introduce the notions of PROPs and properads. First as an enriched category and then as decorated oriented graphs. We present an example of the Frobenius properad and show that we need to introduce the 'grading' by the genus of the graph.

We generalize the construction of cobar complex for properads and show it explicitly on the example of Frobenius properad. Then we describe the algebra over this cobar complex and show that it also corresponds to some Maurer-Cartan equation.

Conventions and notation

- $\mathbb{K}$ is a field of characteristic 0. Multiplication in $\mathbb{K}$ is denoted by the symbol $\cdot$ or omitted.

  $\mathbb{K}$ will be fixed throughout the thesis, and so we sometimes just say vector space instead of a $\mathbb{K}$-vector space, etc.

- For a $\mathbb{K}$-vector space V, $V^{\#}$ denotes its linear dual.

- $[n]$ is the set $\{1, 2, \ldots n\}$.

- Σ_n is the symmetric group of permutations of the set $[n]$.

  For a right Σ_n-module P, $f \in P$, $\sigma \in \Sigma_n$, we denote the right action of σ on the element f as $f\sigma$.

- δ_i^j is a Kronecker delta. Function is equal to 1 if the variables (indices) are the same, and 0 otherwise.

- $\otimes$ is the tensor product.

- $\odot$ is the unordered tensor product (defined in 1.2.10).

- $\wedge$ is the exterior product defined for an n-tuple $x_1, x_2, ..x_n$ as

$$x_{\sigma(1)} \wedge w_{\sigma(2)} \wedge \ldots \wedge x_{\sigma(n)} = \operatorname{sgn}(\sigma) x_1 \wedge x_2 \wedge \ldots \wedge x_n$$

  where $\operatorname{sgn}(\sigma)$ is the signature of the permutation $\sigma \in \Sigma_n$.

- $\circ$ is the composition of two maps.

  If the pairing of outputs and inputs is not unambiguous, we use the symbol ${}_{i_1,\ldots i_n}\circ_{j_1,\ldots j_n}$. In that case we pair the input labeled by index i_k with the output labeled by index j_k.

1. Operads

1.1 Axioms of operad structure

There are two different ways of approaching operads. They can be defined as collections of objects with morphisms satisfying axioms of associativity, equivariance and eventually existence of unit. This description is useful for computations but it is hard to generalize. For this reason it is convenient to think also about more abstract description.

Since every operad can be considered as the quotient of a free operad, we introduce the notion of free operad whose construction can be viewed as taking certain coproduct over the set of trees to form a semigroup structure. Then by replacing trees by another types of graph we can easily get more general concepts as cyclic operads, modular operads and properads.

For our next purposes is necessary to translate the definitions of operads and properads into language of differential graded vector spaces, which is the subject of the last chapter.

Let us first review some basic definitions and examples. The notions and definition are mainly taken from [Mar06], [MSS02] and [LV12]. For more abstract approach is recommended [Get09]. Some details can be also found in [YJ15] and [Val14].

Definition 1.1.1. A **non-unital operad** in category of $\mathbb{K}$-modules is a collection $P = \{P(n)\}_{n\geq 0}$ of right $\mathbb{K}[\Sigma_n]$-modules where Σ_n is symmetric group together with $\mathbb{K}$-linear maps called composition maps

$$\circ_i : P(m) \otimes P(n) \to P(m+n-1) \tag{1.1}$$

(where $1 \leq i \leq m$ and $0 \leq n$) such that the following two axioms are satisfied:

- **Associativity:** For each $1 \leq j \leq m$, $0 \leq n, 0 \leq k$ and $f \in P(m)$, $g \in P(n)$, $h \in P(k)$

$$(f \circ_i g) \circ_j h = \begin{cases} (f \circ_j h) \circ_{i+k-1} g & \text{if } 1 \leq j < i \\ f \circ_i (g \circ_{j-i+1} h) & \text{if } i \leq j < n+i \\ (f \circ_{j-n+1} h) \circ_i g & \text{if } i+n \leq j \leq m+n-1 \end{cases} \tag{1.2}$$

- **Equivariance:** For each $1 \leq i \leq m$, $0 \leq n$, $\tau \in \Sigma_m$ and $\sigma \in \Sigma_n$ let $\tau \circ_i \sigma$ be the permutation where pairs

$$(i, \tau \circ_i \sigma(i)), (i+1, \tau \circ_i \sigma(i+1)), \ldots (i+n, \tau \circ_i \sigma(i+n))$$

  corresponds to σ inserted on i-th place of τ[1]. Then for $f \in P(m), g \in P(n)$ we require

$$(f\tau) \circ_i (g\sigma) = \big(f \circ_{\tau(i)} g\big)(\tau \circ_i \sigma) \tag{1.3}$$

  where the action of $\tau \in \Sigma_m$ on an element $f \in P(m)$ is denoted as $f\tau$.

[1] For example if we take permutation $\tau = (4,1,3,2) \in \Sigma_4$ and $\sigma = (2,1,3) \in \Sigma_3$ and insert σ as second argument of τ we get $\tau \circ_2 \sigma = (2,5,4,6,3,1) \in \Sigma_6$.

For **unital operads** there is one more axiom

- **Unitality:** There exists $e \in P(1)$ such that

  (1.4) $$f \circ_i e = f$$

  for $f \in P(m)$ and $1 \leq i \leq m$

  (1.5) $$e \circ_1 g = g$$

  for $g \in P(n)$.

Remark 1.1.2. The constructions and definitions from this section can be in most cases done for a general commutative ring **k**. We suggestively denote this ring in the same way as it is usual for characteristic zero field $\mathbb{K}$. This is because in the following sections we will work with special $\mathbb{K}$-modules, vector spaces.

In the following we will be talking mainly about unital operads. The non-unital version can be easily thought of by omitting requirements involving unit.

Similarly one can define non-Σ operads. In this case the axiom of equivariance is omitted and each component is just a $\mathbb{K}$-module.

Definition 1.1.3. Let $P = \{P(n)\}_{n\geq 0}$ and $Q = \{Q(n)\}_{n\geq 0}$ be two operads. Then a **homomorphism of operads** $h : P \to Q$ is a collection of maps $h_n : P(n) \to Q(n)$ such that these maps are equivariant, commute (or intertwine) with operadic composition and preserve the unit.

In other words, if $f \in P(n), g \in P(m), \sigma \in \Sigma_n$ then

$$\begin{aligned} h_n(f\sigma) &= h_n(f)\sigma \\ h_n(f \circ_i g) &= h_n(f) \circ_i h_n(g) \\ h_1(e_P) &= e_Q \end{aligned}$$

where $e_P \in P(1), e_Q \in Q(1)$ are the units of P and Q, respectively.

Definition 1.1.4. An **ideal** I in an operad P is a collection

$$I = \{I(n) | I(n) \subset P(n)\}_{n\geq 0}$$

of Σ_n-invariant subspaces such that for all $f, g \in P$ $f \circ_i g$ is in I if $f \in I$ or $g \in I$.

There are some basic examples we will use in the following.

Example 1.1.5. A *commutative operad* is a collection $Com = \{Com(n)\}_{n\geq 1}$ such that $Com(n) = \mathbb{K}$ with trivial Σ_n-action for every n.

Example 1.1.6. An *associative operad* is $Ass = \{Ass(n)\}_{n\geq 1} = \{\mathbb{K}[\Sigma_n]\}_{n\geq 1}$.

Notice that any element $f \in Ass(n)$ such that $f \in \Sigma_n$ can be expressed as $f = e_n\sigma$ where e_n denotes the identity permutation in Σ_n as element of operad and $\sigma \in \Sigma_n$ denotes the right Σ_n-action. Since the axiom of equivariance requires

$$(e_m\tau \circ_i e_n\sigma) = \left(e_m \circ_{\tau(i)} e_n\right)(\tau \circ_i \sigma)$$

and this is for identity permutations $\tau \in \Sigma_m, \sigma \in \Sigma_n$ defined as

$$e_n \circ_i e_m = e_{m+n-1}$$

the operadic compositions $\circ_i$ are uniquely determined.

Example 1.1.7. An *endomorphism operad* is a collection $End_V = \{End_V(n)\}_{n\geq 0}$ for a $\mathbb{K}$-module V (vector space) such that $End_V(n) = Hom_{\mathbb{K}}(V^{\otimes n}, V)$. For elements $f \in End_V(m)$ and $g \in End_V(n)$ is the composition defined as

$$f \circ_i g = f\left(\underbrace{\mathbb{1}_V \otimes \ldots \otimes \mathbb{1}_V}_{i-1 \text{ times}} \otimes g \otimes \mathbb{1}_V \otimes \ldots \otimes \mathbb{1}_V\right)$$

where $\mathbb{1}_V$ denotes identity morphism on V. The symmetric group action is defined as

$$(f\sigma)(v_1, v_2, \ldots v_m) = f\left(v_{\sigma^{-1}(1)}, v_{\sigma^{-1}(2)}, \ldots v_{\sigma^{-1}(m)}\right)$$

where $v_1, v_2, \ldots v_m \in V$ and $\sigma \in \Sigma_m$.

Definition 1.1.8. An **algebra** A **over operad** P (or P-algebra) is a homomorphism of operads $h : P \to End_V$ for some $\mathbb{K}$-module V

Remark 1.1.9. From adjunction of functors *Hom* and $\otimes$ we can equivalently describe a P-algebra structure as

$$Hom_{\mathbb{K}}\left(P, Hom_{\mathbb{K}}\left(V^{\otimes n}, V\right)\right) \cong Hom_{\mathbb{K}}\left(P \otimes_{\mathbb{K}} V^{\otimes n}, V\right)$$

Remark 1.1.10. The concept of endomorphism operad can be defined even more generally for a symmetric[2] monoidal category with internal hom functor

$$[\cdot,\cdot] : C^{op} \times C \to C$$

For example one can take the category **Set** with cartesian product '$\times$' as the monoidal symmetric bifunctor. But for our purposes it mostly suffices to consider the case of vector spaces (or later of differential graded vector spaces).

1.2 Construction of operads via graphs

Let us now show the other other definition.

Definition 1.2.1. A **graph** G is a set of **vertices** $Vert(G)$ together with a set of **edges** $Edge(G)$ which are two-elements subsets of $Vert(G)$.

Let us denote the set of all edges adjacent to (containing) $v \in Vert(G)$ as $Edge(v)$. Furthermore, we define **external vertices** to be vertices with only one adjacent edge. The other vertices are called **internal** and their set is denoted as $IntVert(G)$. The edges containing some external vertex are called **legs**.

Definition 1.2.2. A graph G is **connected** when there exists for every two vertices $u, v \in Vert(G)$ a sequence of edges $\{u, w_1\}, \{w_1, w_2\}, \{w_2, w_3\}, \ldots \{w_n, v\}$ such that every two consecutive edges have one vertex in common and the vertices u, v are elements of some edges in this sequence.

If there exist for some vertices $u, v \in Vert(G)$ two such sequences of edges that every vertex contained in some edge of sequence is contained in exactly two edges of this sequence, we call these sequences together as a **cycle**. An **acyclic** graph is a graph with no cycles.

[2]For non-Σ operads the category does not have to be symmetric.

Remark 1.2.3. It is possible to define so-called **multiple edges** if we take $Edge(G)$ as a multiset (set with repetition). Two multiple edges are adjacent to the same two vertices and are also considered as cycle.

Definition 1.2.4. One can assign to edge formed by vertices u, v an orientation (u, v). The oriented edge (u, v) is **incoming** to vertex v and **outgoing** from vertex u. We denote the set of edges incoming to vertex v as $In(v)$. If all edges in graph G have assigned orientation, the graph is called **directed (or oriented)**.

A **directed cycle** in directed graph is a sequence of consecutive edges

$$(u_1, u_2), (u_2, u_3), \dots (u_{n-1}, u_n), (u_n, u_1)$$

(every two consecutive edges have in common vertex u_i and only one of these edges is incoming into this vertex).

Graphs without specified orientation are called **undirected**.

Definition 1.2.5. Let T be a finite non-empty connected acyclic graph. Then T is a **tree**. Trees where all internal vertices have exactly three adjacent edges are called *binary trees.*

Rooted tree is a directed tree where each vertex has exactly one outgoing edge and each inner vertex has at least one incoming edge. The whole tree has therefore exactly one outgoing leg, called the **root**. The incoming legs are called **leaves**.

Non-planar rooted tree means that an embedding of the tree into the plane is not given, and therefore we have not chosen certain ordering of incoming edges for each vertex[3]. Hence there is no ordering of leaves.

Remark 1.2.6. We will work both with directed and undirected graphs, and so we will always be careful to specify what kind of graph we are working with. For operads in this section it will be directed trees, for cyclic operads in Section 1.3 it will be undirected trees, for modular operads in Section 1.4 undirected graphs and for properads in Section 4.1 directed graphs.

Definition 1.2.7. Let $\mathbf{Tree}_n$ be the category of pairs (T, l) where T is a rooted (directed) non-planar tree and l is a bijection

$$l : \{\text{leaves of } T\} \to [n]$$

Morphisms in this category are graph isomorphisms preserving orientation of edges.

Remark 1.2.8. Notice that we can take the union of categories of trees

$$Tree = \bigcup_{n \geq 1} \mathbf{Tree}_n$$

with operation between trees $T_1, T_2 \in Tree$ defined by identifying the root of T_1 with one of the leafs of T_2, which gives us again a non-planar rooted directed tree. The operation is obviously associative (hence we get a structure of a semigroup).

[3]For example there exist two different planar binary trees with three legs but only one non-planar.

Our next goal is to define an action of Σ_n on $\mathbf{Tree}_n$. We will make this by extending Σ_n-module structure to a functor from the category $\mathbf{Tree}_n$ to the category of $\mathbb{K}$-modules $\mathbf{Mod}_{\mathbb{K}}$.

Definition 1.2.9. Let $E = \{E(n)\}_{n\geq 0}$ be a collection of right $\mathbb{K}[\Sigma_n]$-modules. Then E is a **Σ-module**.

From now on E will always denote some Σ-module.

Let us denote the set of bijections from finite set X to set Y as $Bij(Y, X)$. Then for finite set S such that $|S| = n$ define

$$E(S) = E(n) \otimes_{\Sigma_n} Bij([n], S) \tag{1.6}$$

where $[n] = \{1, 2, \dots n\}$. We can easily see that $Bij([n], S)$ is a left $Bij([n], [n])$-module and $Bij([n], [n])$ is just Σ_n.

If we want to use the finite set X as an index set we have to choose an special ordering of X. To avoid this choice let us introduce the following notion.

Definition 1.2.10. Let X be a finite set such that $|X| = k$ and let us denote by $Ord(X)$ the set of all orderings of this set, i.e., $Ord(X) = Bij([k], X)$. For every $x \in X$ let V_x be a vector space. The **unordered tensor product** of vector spaces V_x is

$$\bigodot_{x\in X} V_x := \operatorname*{colim}_{\tau \in \Sigma_k} \left(\bigoplus_{\sigma \in Ord(X)} V_{\sigma^{-1}(1)} \otimes \dots \otimes V_{\sigma^{-1}(k)} \to \bigoplus_{\sigma \in Ord(X)} V_{(\tau\sigma)^{-1}(1)} \otimes \dots \otimes V_{(\tau\sigma)^{-1}(k)} \right) =$$
$$= \left(\bigoplus_{\sigma \in Ord(X)} V_{\sigma^{-1}(1)} \otimes \dots \otimes V_{\sigma^{-1}(k)} \right)_{\Sigma_k}$$

where the right subscript Σ_n denotes a Σ_n-coinvariants[4] For short let us write

$$V_1 \odot \dots \odot V_n := \bigodot_{i=1}^{n} V_i$$

For $T \in \mathbf{Tree}_n$, E defined in 1.2.9 and (1.6) let us define the unordered tensor product over internal vertices of T as

$$\widetilde{E}(T) = \bigodot_{v \in IntVert(T)} E(In(v)) \tag{1.7}$$

It is easily seen that $\widetilde{E} : \mathbf{Tree}_n \to \mathbf{Mod}_{\mathbb{K}}$ is a functor and that two isomorphic trees have isomorphic $\mathbb{K}$-modules.

Finally, we can define for $n \geq 0$

$$\Psi(E)(n) = \operatorname*{colim}_{T \in \mathbf{Tree}_n} \widetilde{E}(T) \tag{1.8}$$

$$\Psi(E) = \{\Psi(E)(n)\}_{n\geq 0} \tag{1.9}$$

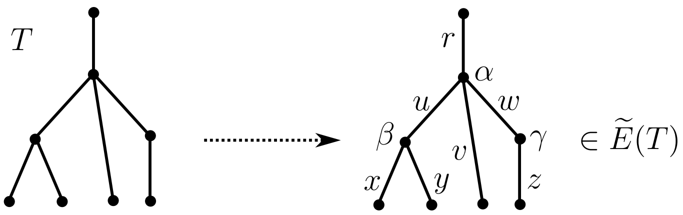


Figure 1.1: Element $\alpha \odot \beta \odot \gamma \in \widetilde{E}(T) = E(3) \odot E(2) \odot E(1)$

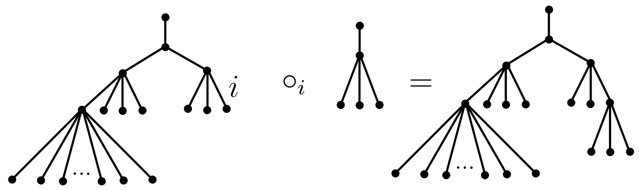


Figure 1.2: Composition $\circ_i$

Such a structure corresponds to the free non-unital operad. The composition $\circ_i$ is represented by joining two trees, as it is indicated on figure 1.2, using semigroup structure of *Tree* and action of symmetric group as relabeling of leaves. The associativity axiom, for example, can be visualized as two-step attaching of two trees independently on order of these steps.

Remark 1.2.11. If we enlarge category $\mathbf{Tree}_n$ by one special degenerate tree with no internal vertices, only one edge, and denote this category as $U\mathbf{Tree}_n$, then

$$\Gamma(E)(n) = \operatorname*{colim}_{T \in U\mathbf{Tree}_n} E(T), \qquad \Gamma(E) = \{\Gamma(E)(n)\}_{n \geq 0} \tag{1.10}$$

represents unital free operad.

We have described the operad structure (composition $\circ_i$ and Σ-actions) explicitly for directed rooted trees. This could be taken as a definition of operad but then we would have to repeat this also for cyclic operads over general trees and modular operads over general graphs. To make the definition more compact but still mathematically precise let us describe it more categorically.

Definition 1.2.12. A **monad** T over a category C is an endofunctor $T : C \to C$ with two natural transformations called multiplication $\mu : T \circ T \to T$ and unit morphism $\eta : 1_C \to T$ (where 1_C represents identity functor) such that the following axioms are satisfied:

1. $\mu \circ T\mu = \mu \circ \mu T$ as natural transformations $T \circ T \circ T \to T$
2. $\mu \circ T\eta = \mu \circ \eta T = 1_T$ as natural transformations $T \to T$, where 1_T denotes the identity transformation $T \to T$

Remark 1.2.13. The composition of two adjoint functors, for example, the composition of

$$\Psi : \Sigma\text{-}\mathbf{Mod}_C \to \{\text{Non-unital operads in } C\} =: \{\text{n.-u.op. in } C\}$$

[4]The element v is identified with element $v\sigma$ for $\sigma \in \Sigma_n$).

with the forgetful functor $U_\Psi : \{\text{n.-u.op. in } C\} \to \Sigma\text{-}\mathbf{Mod}_C$ gives us a monad. The endofunctor T is just $U_\Psi \circ \Psi$, the unit morphism η is a unit (a natural transformation given from adjunction) $u : 1_{\Sigma\text{-}\mathbf{Mod}_C} \to U_\Psi \circ \Psi$ and the multiplication μ is a counit $c : \Psi \circ U_\Psi \to 1_{\text{n.-u.op.}}$.

The same can be done for $\Gamma : \Sigma\text{-}\mathbf{Mod}_C \to \{\text{Operads in C}\}$ with forgetful functor $U_\Gamma : \{\text{Operads in C}\} \to \Sigma\text{-}\mathbf{Mod}_C$.

Definition 1.2.14. An **algebra over monad** T is an object $A \in C$ with structure morphism $\alpha : T(A) \to A$ satisfying

$$\alpha \circ T(\alpha) = \alpha \circ \mu_A$$

$$\alpha \circ \eta_A = 1_A$$

Then a Theorem 40 from [Mar06] shows

Theorem 1.2.15. A Σ-module P is a non-unital operad if and only if it is an algebra over the monad $U_\Psi \circ \Psi$, and it is an operad if and only if it is an algebra over the monad $U_\Gamma \circ \Gamma$.

Remark 1.2.16. We defined the free operad over the category of trees specified in Definition 1.2.5. But sometimes it is more convenient to change the properties of these graphs a little bit.

Since we are not using the external vertices at all we can omit them. Hence instead of edges with external vertices we can think about graphs with *half-edges*. The joining of two trees is then just putting together two half-edges into one edge as indicated on Figure 1.3.

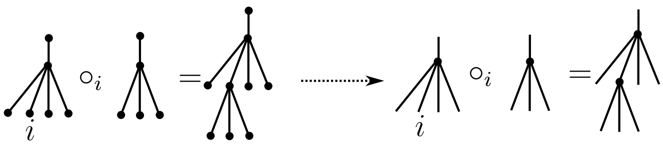


Figure 1.3: Formal omitting of external vertices

Example 1.2.17. The operad Com from Example 1.1.5 is in this language the free operad generated by a Σ-module E_{Com} factored by an ideal R_{Com}.

$$E_{Com} = \begin{cases} \mathbb{K} \cdot \mu & \text{if } n = 2 \\ 0 & \text{if } n \neq 2 \end{cases}$$

where μ is a trivial representation of Σ_2, i.e., for any $\tau \in \Sigma_2$ we have $\mu \circ \tau = \mu$. Elements from $\Gamma(E_{Com})(3)$ then must satisfy the relations displayed on Figure 1.4.

The component $\Gamma(E_{Com})(3)$ is therefore given by three elements as indicated on Figure 1.5 which can be shortly written as

$$\Gamma(E_{Com})(3) = \text{Span}\{(\mu \circ_1 \mu)(1, 2, 3), (\mu \circ_1 \mu)(1, 3, 2), (\mu \circ_1 \mu)(2, 3, 1)\}$$

The ideal $R_{Com} \subset \Gamma(E_{Com})(3)$ is generated by the relation

$$(\mu \circ_1 \mu)(i, j, k) = (\mu \circ_2 \mu)(i, j, k)$$

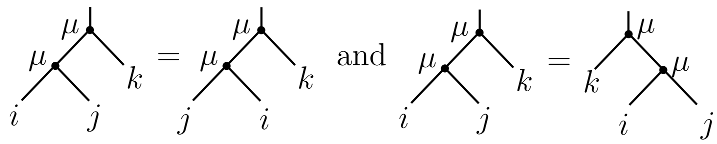


Figure 1.4: Relations betweeen elements of $\Gamma(E_{Com})(3)$

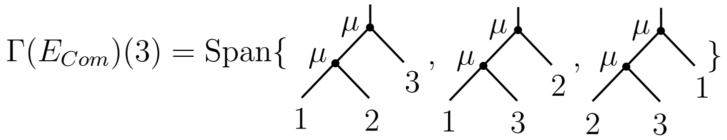


Figure 1.5: Component $\Gamma(E_{Com})(3)$

displayed on figure **??** But this relation identifies all three generators of $\Gamma(E_{Com})(3)$ into one element.

Similarly we get $dim((\Gamma(E_{Com})/R_{Com})(n) = 1$. Hence we can think about elements of operad Com as planar trees with only one internal vertex decorated by an element from $\Gamma(E_{Com})(n)$ with n legs, where the labeling of them play no role.

Example 1.2.18. In the same way, the operad Ass from Example 1.1.6 is a free operad generated by a Σ-module

$$E_{Ass} = \begin{cases} \mathbb{K}[\Sigma_2] & \text{if } n = 2 \\ 0 & \text{if } n \neq 2 \end{cases}$$

where α is the generator of the regular representation of $\mathbb{K}[\Sigma_2]$, factored by an ideal $R_{Ass} \subset \Gamma(E_{Ass})(3)$.

Elements from $\Gamma(E_{Ass})(3)$ must satisfy relations displayed on Figure 1.7, where $e = \alpha^2 \in \Sigma_2$ denotes the identity permutation.

Hence $\Gamma(E_{Ass})(3) = \text{Span}\{(\alpha \circ_1 \alpha)(i,j,k), (\alpha \circ_2 \alpha)(i,j,k) | i,j,k \in [3]\}$ has dimension 12. The ideal R_{Ass} is generated by the relation

$$(\alpha \circ_1 \alpha)(i,j,k) = (\alpha \circ_2 \alpha)(i,j,k)$$

Hence $dim((\Gamma(E_{Ass})/R_{Ass})(n) = n!$. Thus we can consider elements of operad $Ass = \Gamma(E_{Ass})/R_{Ass}$ as planar trees with only one internal vertex decorated by an element from $\Gamma(E_{Ass})(n)$ with n legs labeled by set $[n]$.

Remark 1.2.19. Unfortunately it is very hard to describe the operad End_V in this language for a general vector space V. For example, if one considers two vector spaces $V = \text{Span}\{v\}, W = \text{Span}\{v_1, v_2\}$ then $\Gamma(E_{End_V})/R_{End_V}(n)$ is one-dimensional and $\Gamma(E_{End_W})/R_{End_W}(n)$ is $(2 \cdot 2^n)$-dimensional for every $n \in \mathbb{N}$.

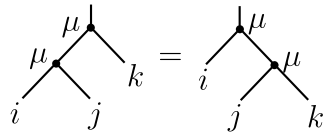


Figure 1.6: Ideal R_{Com}

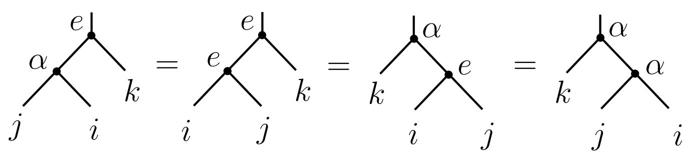


Figure 1.7: Relations between elements of $\Gamma(E_{Ass})(3)$

Despite these problems with specifying Σ-module for endomorphism operad End_V, if $E(0) = E(1) = 0$ and $E(n)$ is finite dimensional for every $n \geq 2$, then the components of free operads $\Psi(E)(n)$ and $\Gamma(E)(n)$ are finite dimensional.

Now we can use the second definition to easily introduce a generalization of operad.

1.3 Cyclic operads

Whereas for operads there is a distinction between 'input' leafs and 'output' root, for cyclic ones there is not such distinction. Hence for cyclic operads and modular operads the action of permutation group affects also the output leg.

Definition 1.3.1. Let us denote the set $\{0, 1, \dots n\}$ as $[n]^+$. Then $\mathbf{Tree}_n^+$ is the category of trees (without orientation) with legs labeled by the set $[n]^+$. Such trees are sometimes called **cyclic**.

Let Σ_n^+ be the permutation group of the set $[n]^+$ and let **cyclic Σ-module** (shortly written as Σ^+-module) be a collection $W = \{W(n)\}_{n\geq 0}$ of right $\mathbb{K}[\Sigma_n^+]$-modules.

The functor from $\mathbf{Tree}_n^+$ to $\mathbf{Mod}_{\mathbb{K}}$ is defined analogously as for $\mathbf{Tree}_n$. Let $W \in \Sigma^+$-mod and S^+ be a set such that $|S^+| = n + 1$, then

$$W((S^+)) = W(n) \otimes_{\Sigma_n^+} Bij\left([n]^+, S^+\right)$$

where double parentheses remind us that $\mathbb{K}[\Sigma_n^+]$-module $W(n)$ is associated with a set with $n + 1$ elements. Then

$$W((T)) = \bigotimes_{v \in IntVert(T)} W((Edge(v)))$$

for a cyclic tree T. Then for $n \geq 0$ let

$$\Psi_c : \Sigma^+\text{-}\mathbf{Mod}_{\mathbb{K}} \to \{\text{non-unital cyclic operads}\}$$

be a functor such that

$$\Psi_c(W)(n) = \operatorname*{colim}_{T \in \mathbf{Tree}_n^+} W((T))$$

Similarly by enlarging the category $\mathbf{Tree}_n^+$ by the trees with only one edge and no internal vertices we get the category $U\mathbf{Tree}_n^+$ for which we can define a functor

$$\Gamma_c : \Sigma^+\text{-}\mathbf{Mod}_{\mathbb{K}} \to \{\text{unital cyclic operads}\}$$

as

$$\Gamma_c(W)(n) = \operatorname*{colim}_{T \in U\mathbf{Tree}_n^+} W((T))$$

Definition 1.3.2. A **non-unital cyclic operad** (**cyclic operad** respectively) is an algebra over the monad $U_{\Psi_c} \circ \Psi_c$ (over monad $U_{\Gamma_c} \circ \Gamma_c$, respectively), where U_{Ψ_c} (U_{Γ_c}) is the appropriate forgetful functor.

1.4 Modular operads

We will use undirected finite connected graphs. We omit external vertices (as was already mentioned in 1.2.16) and consider edges adjacent to external vertices as only half-edges. Unlike for cyclic operads, cycles and multiple edges are allowed[5]. Hence there can be 'holes' in the graphs. Moreover the vertices of the graphs have assigned a non-negative integer, called *genus* of the vertex.

But not all such graphs are suitable for defining modular operads. To exclude the unsuitable ones we use the *stability condition*.

The notion is taken from [Mar06]. But the aspects of graphs are adapted for [DJM13] (and [GK98]).

Definition 1.4.1. The **genus of a graph** G is defined as

$$g(G) = b_1(G) + \sum_{v \in Vert(G)} g(v) \tag{1.11}$$

where $b_1(G)$ is a first Betti number and $g(v)$ is the genus of the vertex (a non-negative number assigned to every vertex).

Remark 1.4.2. The Betti number $b_1(G)$ is defined as the rank of the first homology $H_1(G)$ of the graph. But for graphs it can be equivalently defined as

$$b_1(G) = |Edge(G)| - |Vert(G)| + k \tag{1.12}$$

where k is the number of components of the graph G (hence for connected graphs $k = 1$) and the half-edges are not counted in the edges $Edge(G)$.

Definition 1.4.3. The graph is **stable** if it is a finite connected graph such that for all vertices $v \in Vert(G)$

$$0 < 2(g(v) - 1) + |HEdge(v)| \tag{1.13}$$

where $HEdge(v)$ is the set of edges and half-edges connected to a vertex v.

Let $M\mathbf{Gr}(n, g)$ be the category of pairs (G, l) where G is a stable graph of genus g and l is a bijection

$$l : \{\text{half-edges of } G\} \to [n]$$

Morphisms of $M\mathbf{Gr}(n, g)$ are isomorphisms of stable graph preserving labeling of half-edges.

Remark 1.4.4. As shown in [GK98] there are only finitely many isomorphism classes of stable graphs in $M\mathbf{Gr}(n, g)$.

[5]See Definition 1.2.2 and Remark 1.2.3.

Let us consider a stable graph $G(X,g) \in M\mathbf{Gr}(n,g)$, $|X| = n$. We have the inequality

$$0 < \sum_{v \in Vert(G)} (2(g(v)-1) + |HEdge(v)|) =$$

$$= \left(\sum_{v \in Vert(G)} 2g(v) \right) - 2|Vert(G)| + 2|Edge(G)| + n$$

We use here the fact that every edge is adjacent to two different vertices or twice to one vertex. Using (1.12) and then (1.11) we get

$$0 < \left(\sum_{v \in Vert(G)} 2g(v) \right) + 2b_1(G) - 2 + n$$

$$0 < 2g - 2 + n \tag{1.14}$$

The inequality (1.14) will be called the **stability condition**.

Definition 1.4.5. A **modular Σ-module** (shortly written as MΣ-$\mathbf{Mod}_{\mathbb{K}}$) is a collection $P = \{P(n,g)\}_{n,g \geq 0}$ of right $\mathbb{K}[\Sigma_m]$-modules satisfying the stability condition

$$0 < 2g - 2 + n$$

Hence, for example, we exclude the cases $(n,g) \in \{(0,0),(0,1),(1,0),(2,0)\}$.

If P is a modular Σ-module and X a set with n elements, define

$$P(X,g) = P(n,g) \otimes_{\Sigma_n} Bij([n],X)$$

for $g \geq 0$. Then similarly as in (1.7) we can define unordered tensor product over the vertices of G as

$$\widetilde{P}(G) = \bigodot_{v \in Vert(G)} P(HEdge(v), g)$$

where $\widetilde{P} : \mathbf{Gr}(X,g) \to \mathbf{Mod}_{\mathbb{K}}$ is a functor. Finally we get functor

$$\Gamma_m : \mathrm{M\Sigma}\text{-}\mathbf{Mod}_{\mathbb{K}} \to \{\text{modular operads}\}$$

$$\Gamma_m(P)(n,g) = \operatorname*{colim}_{G \in \mathbf{Gr}(n,g)} \widetilde{P}(G)$$

and

$$\Gamma_m(P) = \{\Gamma_m(P)(n,g)\}_{n,g \geq 0, s.c.}$$

where *s.c.* means that n,g satisfy stability condition.

Definition 1.4.6. A **modular operad** is an algebra over the monad $U_{\Gamma_m} \circ \Gamma_m$, where U_{Γ_m} is appropriate forgetful functor.

Remark 1.4.7. Note that modular operads do not have unit, because such unit ought to be an element from (2,0), but this space is excluded by stability condition.

Remark 1.4.8. Axiomatic definition of modular operads(for example in [MSS02]) similar to 1.1.1 introduces two composition. The first one is similar to the one from definition 1.1.1

$$
{}_i\odot_j : P(n_1, g_1) \otimes P(n_2, g_2) \to P(n_1 + n_2, g_1 + g_2) \tag{1.15}
$$

corresponding to joining two half-edges adjacent to two distinct vertices decorated by modular Σ-module P. The second one

$$
\xi_{i,j} : P(n, g) \to P(n - 2, g + 1) \tag{1.16}
$$

corresponds to joining two half-edges adjacent to one vertex decorated by P. These compositions must satisfy axioms listed in Definition 2 in [DJM13].

1.5 Differential graded operads

Let us first recall the basic notions of a differential graded vector space and operad.

Definition 1.5.1. Differential graded vector space (a dg vector space for short) $V = (V, d)$ is a collection of vector spaces $\{V^n\}$ such that $V = \bigoplus_{n\in\mathbb{Z}} V^n$ together with collection of homomorphisms $d^n : V^n \to V^{n+1}$ called differential d, such that $d^{n+1} \circ d^n = 0$, shortly written as $d^2 = 0$ (such structure is called cochain complex).

A **morphism** f **of degree** r between two differential graded vector spaces $f : (V, d_V) \to (W, d_W)$ is a collection of maps $f^n : V^n \to W^{n+r}$ such that

$$
d_W \circ f = (-1)^r f \circ d_V
$$

These cochain complexes of vector spaces together with morphisms of degree 0 form a category denoted as $dgVec$.

Remark 1.5.2. The tensor product of two dg vector spaces V, W is a new dg vector space

$$
(V \otimes W)^n = \bigoplus_{i+j=n} V^i \otimes W^j
$$

with differential

$$
d = d_V \otimes 1_W + 1_V \otimes d_W
$$

Note that the evaluation of the mapping $(1_V \otimes d_W)$ on elements from $V \otimes W$ is nontrivial because of the grading of the vector spaces V, W. The sign is defined by the *Koszul convention*

$$
(1_V \otimes d_W)(v \otimes w) = (-1)^{|v|\cdot|d_W|} v \otimes d_W(w) \tag{1.17}
$$

where $v \in V, w \in W$ and the vertical bars $|\cdot|$ denote the degree of elements (operation, respectively), in the case of differential defined as $|d_W| = 1$.

Remark 1.5.3. The symmetric structure is defined via the switching map

$$\tau : V \otimes W \to W \otimes V$$

for some elements $v \in V^i, w \in W^j$ as

$$\tau\left(v \otimes w\right) = (-1)^{|v|\cdot|w|}\, w \otimes v$$

The map is then linearly extended to other elements of $V \otimes W$. Using this map we can also define the sign of an arbitrary permutation of elements of $V^{\otimes n}$ as

$$\sigma\left(v_1 \otimes v_2 \otimes \ldots \otimes v_n\right) = (-1)^I \left(v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \ldots v_{\sigma^{-1}(n)}\right)$$

where $I = \sum_{\substack{i<j \\ \sigma(i)>\sigma(j)}} |v_i| \cdot |v_j|$ and $\sigma \in \Sigma_n$.

Definition 1.5.4. A **differential graded Σ-module** (a dg Σ-module for short) is a collection $A = \{A(n)\}_{n\geq 1}$ of right $\mathbb{K}\left[\Sigma_n\right]$-modules such that $A(n) \in dgVec$ and differentials $d(n)^i : A(n)^i \to A(n)^{i+1}$ which are Σ_n-equivariant, in other words

$$d(n)^i\left(v\sigma\right) = \left(d(n)^i\left(v\right)\right)\sigma$$

where $v \in A(n), \sigma \in \Sigma_n$. These Σ-modules with appropriate equivariant maps form a category denoted as *dg*-Σ-*Mod*.

Now we can adapt Definition 1.1.1 from Section 1.1.

Definition 1.5.5. A **differential graded operad** (a dg operad for short), is a differential graded Σ-module $A = \{A(n)\}_{n\geq 1}$ with composition maps

$$\circ_i : A(m)^k \otimes A(n)^l \to A(m+n)^{k+l}$$

satisfying the axioms of associativity, equivariance and optionally unitality.

2. The cobar complex of operads

2.1 General construction

The cobar complex is the result of the action of a functor $\mathcal{C} : (P, d_P) \to (\mathcal{C}(P), \delta)$ from an operad with differential d_P to an operad $C(P)$ with differential δ.

In a nutshell, the whole construction comes from dualization. More precisely, the new differential comes from the dualization of composition maps $\circ_i$ of the operad P and the new composition maps are defined by connection of two tree graphs. But to make this construction match to our present notion of differential it is necessary to introduce the suspension term.

Usually (as for example in [LV12] or [Mil11]), the cobar complex is defined as a functor Ω from the category of coaugmented differential graded cooperads to the category of augmented differential graded operads. To avoid the new definitions of cooperads, cocomposition maps, cooperad morphism etc., we use the approach from [MSS02] or [GK94].

The dualization of operad actually gives us a cooperads with cocomposition maps $\circ^{\#}$. But in contrast with the second approach we are working with operads such that their components $P(n)$ are only finite dimensional.

Remark 2.1.1. As already mentioned in Remark 1.2.16, we will consider edges adjacent to external vertices as half-edges and we omit the external vertices. Hence the set $Vert(G)$ of vertices of the graph G contains only 'internal' vertices. Similarly the set $Edge(G)$ contains only inner edges (not half-edges).

Remark 2.1.2. Although we showed in 1.2.11 how to define the free unital operad, in the following we will consider only non-unital[1] operads with$P(0) = P(1) = 0$. Then $P^{\#}(0) = P^{\#}(1) = 0$.

The problem with unital operads arises from the need to identify decorated trees. The trees by themselves are not isomorphic in $U\mathbf{Tree}_n$, but we want to identify their decorations in the operad.

By this restriction we also omit trees containing a vertex with only one incoming edge and one outgoing edge. Trees without such vertices are called *reduced trees* and their category is denoted as $R\mathbf{Tree}_n$.

We also require $P(n)$ to be a finite dimensional dg vector space for all n. Then the spaces $(P(n))^{\#}$ are also finite dimensional and more importantly, there is an isomorphism of vector spaces

$$(P(n) \otimes P(m))^{\#} \cong (P(n))^{\#} \otimes (P(m))^{\#}$$

We abreviate the notation for the dual of operad $P = \{P(n)\}$ as $P^{\#} = \{(P(n))^{\#}\}$.

Remark 2.1.3. Since the spaces $P(n)$ and $(P(n))^{\#}$ are non-canonically isomorphic, the spaces $(P(n))^{\#}$ are also Σ_n-modules. Moreover the action on $(P(n))^{\#}$ corresponds canonically to action on $P(n)$ in the sense of the following observation.

[1] More precisely, it is enough to work with an operad in which $P(1)$ is a direct sum $\widetilde{P}(1) \bigoplus \mathbb{K}$ such that the part $\mathbb{K}$ can be omitted. This sort of operad is called *augmented*.

Let $\{e_i\}_{i\in I}$ be a basis of $P(n)$ and $\{e^i\}_{i\in I}$ a dual basis of $(P(n))^{\#}$ (where I is a finite index set). Then $f \in P(n)$ can be expressed as

$$f = \sum_{i\in I} f^i \cdot e_i = f^i \cdot e_i$$

where f^i are coefficients and we use the *Einstein summation convention* (when there is twice the same index variable in one term, we sum over this index over all possible variables). The same holds for $\lambda \in P^{\#}(n)$ with coefficients λ_i

$$\lambda = \sum_{i\in I} \lambda_i \cdot e^i = \lambda_i \cdot e^i$$

Then for $\sigma \in \Sigma_n$ we have

$$\lambda(f\sigma) = \lambda_j \cdot e^j(f^i \cdot e_i\sigma) = \lambda_j \cdot f^i \cdot e^j(e_i\sigma)$$

Note that $e_i\sigma = e_{\sigma^{-1}(i)}$, since we ahve $e_i(\sigma\tau) = (e_i\sigma)\tau$. Hence

$$\lambda_j \cdot f^i \cdot e^j(e_i\sigma) = \lambda_j \cdot f^i \cdot \delta^j_{\sigma^{-1}(i)}$$

where $\delta^j_{\sigma^{-1}(i)}$ stands for Kronecker delta. If $j = \sigma^{-1}(i)$, then $\sigma(j) = i$, and therefore

$$\lambda_j \cdot f^i \cdot \delta^j_{\sigma^{-1}(i)} = \lambda_j \cdot f^i \cdot e^{\sigma(j)}(e_i) = \lambda_j \cdot (e^j\sigma^{-1})(f^i \cdot e_i) = (\lambda\sigma^{-1})(f)$$

For example, if the action of Σ_n is trivial on $P(n)$, it must be trivial also on $P^{\#}(n)$.

Definition 2.1.4. The **dual complex** $V^{\#} = \left(V^{\#}, d^{\#}\right)$ of complex $V = (V, d)$ is defined as

$$\left(V^{\#}\right)^n = Hom\left(V^{-n}, \mathbb{K}\right)$$
$$d^{\#}(\alpha) = (-1)^{|\alpha|}\, \alpha \circ d$$

where $\alpha \in V^{\#}$.

Remark 2.1.5. Notice that $d^{\#}$ raises the degree of elements of $V^{\#}$ by 1, since for $\alpha \in (V^{\#})^n$, the evaluation of $d^{\#}(\alpha)$ is given as $d^{\#}(\alpha)(v) = (-1)^{|\alpha|}(\alpha \circ d)(v)$, where $d(v)$ must be in V^{-n}. Hence $v \in V^{-n-1}$, $d^{\#}(\alpha) \in \left(V^{\#}\right)^{n+1}$ and $\left(V^{\#}, d^{\#}\right)$ is a cochain complex.

Remark 2.1.6. To get an idea what dualization is in our case, let us for a moment denote the symbol for the map $P(m) \otimes P(n) \to P(m+n-1)$ as $\circ_i^{m,n}$ to distinguish it from the map $P(k) \otimes P(l) \to P(k+l-1) = P(m+n-1)$ when $k + l = m + n$. If we add this into the notation, then the dual map is denoted as

$$(\circ_i^{m,n})^{\#} : (P(m+n-1))^{\#} \to (P(m) \otimes P(n))^{\#}$$

which for $\lambda \in P^{\#}(m+n-1)$ and $\alpha \in P(m), \beta \in P(n)$ gives

$$(\circ_i^{m,n})^{\#}(\lambda)(\alpha \otimes \beta) = \lambda(\alpha \circ_i \beta)$$

Under the identification $(P(m) \otimes P(n))^{\#} \cong (P(m))^{\#} \otimes (P(n))^{\#}$ we can write

$$(\circ_i^{m,n})^{\#}(\lambda) = \sum_j \lambda_{1,j} \otimes \lambda_{2,j}$$

where $\lambda_{1,j} \in (P(m)^{\#})$, $\lambda_{2,j} \in (P(n))^{\#}$, and then

$$(\circ_i^{m,n})^{\#}(\lambda)(\alpha \circ_i \beta) = \sum_j (-1)^{|\alpha|\cdot|\lambda_{2,j}|} \lambda_{1,j}(\alpha)\lambda_{2,j}(\beta)$$

But in our definition of $\circ_i$ in 1.1.1 we did not specify m and n. Hence if we want in general define $(\circ_i)^{\#}$ acting on the space $P(N)$ for $N \geq 2$, we should sum over all possible splittings of the number N into pairs (m, n) such that $m+n = N+1$ and $m, n \geq 2$. Our next goal is to extend these 'splittings' by a graded Leibniz rule to an arbitrary element of the operad $\Psi(P^{\#})$.

Let us denote an arbitrary element of the free operad $\Psi(P^{\#})$ as $P^{\#}(T)$ for $T \in R\mathbf{Tree}_n$ as in (1.7). Let $T, T' \in R\mathbf{Tree}_n$ be trees such that T is equivalent to T' after the contraction of an edge e. Then the map

$$\Delta_e^{T,T'} : P^{\#}(T) \to P^{\#}(T')$$

acts by raising the number of inner edges by one, as it adds one new edge e into tree T as shown in Figure 2.1.

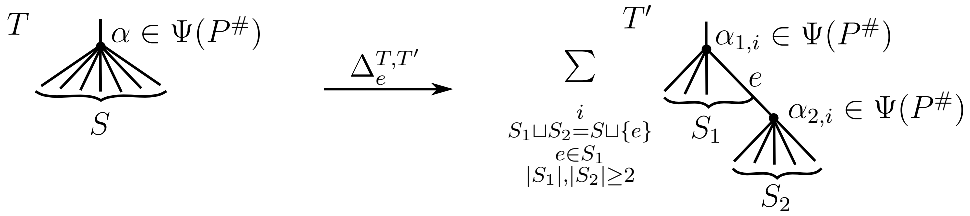


Figure 2.1: Action of $\Delta_e^{T,T'}$

Since we want to define $\Delta_e^{T,T'}$ satisfying graded Leibniz rule, we need to specify the signs generated by this map on all possible elements of $\Psi(P^{\#})$, as indicated in Figure 2.2.

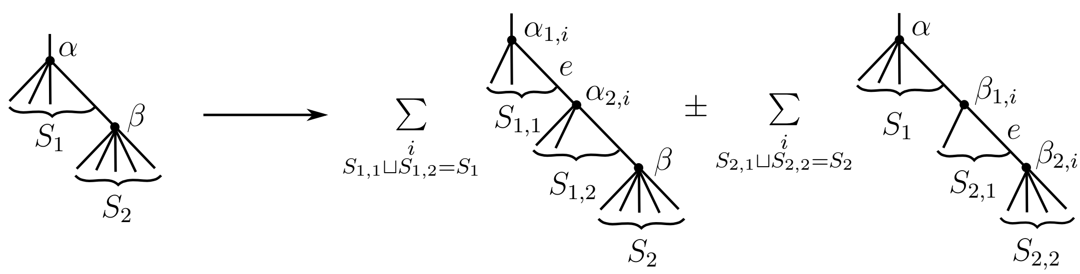


Figure 2.2: Action of Δ on arbitrary element of $\Psi(P^{\#})$

Definition 2.1.7. For $V \in dgVec$, the **suspension** $\uparrow$ is defined as $(\uparrow V)^i = V^{i-1}$. Similarly the **desuspension** is defined as $(\downarrow V)^i = V^{i+1}$.

Definition 2.1.8. Let S be a finite set, then $\mathbb{K}^S$ can be considered as a dg vector space in degree 0 and $\uparrow \mathbb{K}^S$ as a dg vector space in degree 1. The **determinant of the set** S is the one dimensional dg vector space in degree $|S|$

$$det(S) = \bigwedge^{|S|} \left(\uparrow \mathbb{K}^S\right)$$

where $\wedge$ stands for the exterior product.

For any tree T, let us denote the cardinality of the set of inner edges $Edge(T)$ as $|T|$. The **determinant of the tree** is

$$det(T) = \uparrow det\left(Edge(T)\right)$$

Remark 2.1.9. Notice that $det(\cdot)$ assigns to a tree T a one dimensional vector space of degree $|T| + 1$. The '+1' corresponds to the root edge.

Definition 2.1.10. The **cobar bicomplex** of an operad (P, d_P) in arity n is a Σ_n-module bigraded complex

$$\mathcal{C}(P)(n)^{*,*} = \bigoplus_{i\geq 1, j\in\mathbb{Z}} \mathcal{C}(P)(n)^{i,j}$$

such that $\mathcal{C}(P)(1) = 0$ and for $n \geq 2$

$$\mathcal{C}(P)(n)^{i,j} = \operatorname*{colim}_{\substack{T\in R\mathbf{Tree}_n \\ |T|+1=i}} P^{\#}(T)^j \otimes det(T)$$

The grading denoted by i represents the number of edges such that the differential $\delta : \mathcal{C}(P)(n)^{i,*} \to \mathcal{C}(P)(n)^{i+1,*}$ act as

$$\delta : \operatorname*{colim}_{|T|+1=i} P^{\#}(T) \otimes det(T) \to \operatorname*{colim}_{|T'|+1=i+1} P^{\#}(T') \otimes det(T') \tag{2.1}$$

which is the unique extension of the map[2] $\Delta_e^{T,T'}(\cdot) \otimes (e \wedge \cdot)$, where for short we denote $\uparrow \mathbb{K}^e \wedge det(T)$ as $e \wedge det(T)$. The map $\Delta_e^{T,T'}$ is compatible with colimits in the category of reduced trees.

The grading denoted by j comes from the dualization of P with grading such that

$$\delta_P = (d_P)^{\#} \otimes \mathbb{1} : P^{\#}(T)^j \otimes det(T) \to P^{\#}(T)^{j+1} \otimes det(T)$$

where $(d_P)^{\#}$ is the extension of $d_P^{\#} : P^{\#} \to P^{\#}$ to $d_P^{\#} : P^{\#}[T] \to P^{\#}[T]$. The total differential is then given as

$$d = \delta + (-1)^{|T|}\,\delta_P$$

Remark 2.1.11. Since in the following we will consider only cobar bicomplex of an operad with the trivial differential $d_P = 0$, we define the **cobar complex** (a dg Σ-module) $\mathcal{C}(P) = \{C(P)(n)\}_{n\geq 1}$ as

$$\mathcal{C}(P)(n)^* = \bigoplus_{i\geq 1} \mathcal{C}(P)(n)^i$$

such that $\mathcal{C}(P)(1) = 0$ with differential $d = \delta : \mathcal{C}(P)(n)^i \to \mathcal{C}(P)(n)^{i+1}$ defined in (2.1).

[2] Introduced in 2.1.6 and 2.1.8.

Lemma 2.1.12. The cobar complex $\mathcal{C}(P)$ is a cochain complex.

Proof. The condition $d^2 = 0$ follows from the dualization of coassociativity and the change of grading by determinant.

We will show the idea of the proof of $d^2(P^{\#}(T) \otimes det(T)) = 0$ only for $T \in R\mathbf{Tree}_4$, $|T| = 0$ with half-edges labeled by i, j, k, l. For arbitrary tree $T \in R\mathbf{Tree}_n$, $|T| \geq 0$, the idea is the same, we only have to work with more indices and more sums in the expansions.

$d(T)$ gives us four different sums[3] of decorated trees of type T_1 and six sums of decorated trees of type T_2 as indicated in Figure 2.3. The tree of type T_1 with specified labeling of half-edges and decoration of vertices will be denoted as $T_{1,\alpha}$, similarly the tree of type T_2 with specified labeling and decorations will be denoted as $T_{2,\alpha}$.

All elements of $d\left(P^{\#}(T) \otimes det(T)\right)$ have determinant $det(T_1) = det(T_2) = e$, where e denotes the new edge. Shortly we can write this relation as

$$d(P^{\#}(T)) = P^{\#}(T_1) \otimes e + P^{\#}(T_2) \otimes e \tag{2.2}$$

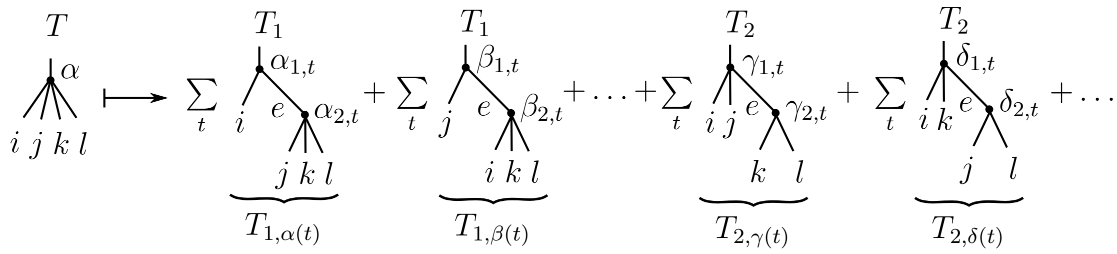


Figure 2.3: $d(T)$

Four sums of decorated trees of type T_1 will give us after the action of second d twelve sums of decorated trees of type T_3. Six sums of decorated trees of type T_2 will give us twelve sums of decorated trees of type T_3 and moreover six sums of decorated trees of type T_4 as indicated in Figures 2.4 and 2.5. All elements of these sums have determinant $f \wedge e$.

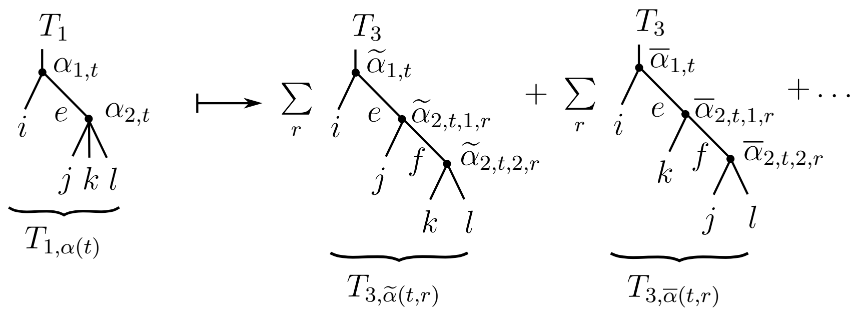


Figure 2.4: Action of d on decorated tree T_1

Decorated trees T_3 given by the action of d on trees T_1 are isomorphic with trees T_3 given by the action of d on trees T_2 but with swapped roles of edges e

[3]There are four different sums since there are four different labelings of half-edges of tree T_1.

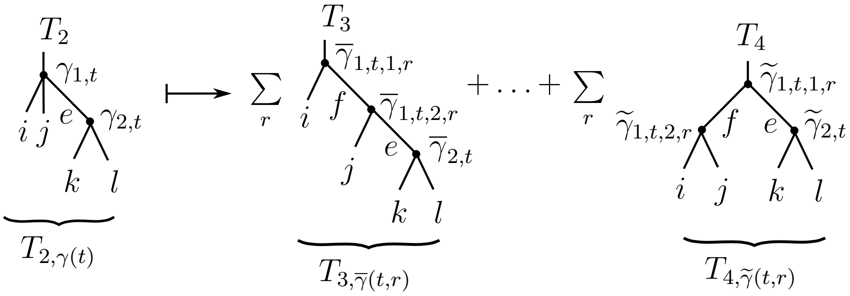


Figure 2.5: Action of d on decorated tree T_2

and f. For example,

$$d\left(\sum_t T_{2,\gamma(t)}\right) = \sum_{t,r} T_{3,\overline{\gamma}(t,r)} \otimes (f\wedge e) + \sum_{t,r} T_{4,\widetilde{\gamma}(t,r)} \otimes (f \wedge e) =$$
$$= -\sum_{t,r} T_{3,\widetilde{\alpha}(t,r)} \otimes (e \wedge f) + \sum_{t,r} T_{4,\widetilde{\gamma}(t,r)} \otimes (f\wedge e)$$

Hence $d^2(T)$ will contain no decorated trees of type T_3.

The six sums of decorated trees T_4 given by the action d on T_2 contain pairs of trees with same labels of half-edges but swapped roles of edges e and f. Hence these sums together are also zero and therefore

$$d^2(P^{\#}(T)) = 0$$

□

The Theorem 3.11 from [MSS02] shows that we can define the structure of an operad on $C(P)$.

Theorem 2.1.13. There is an isomorphism of vector spaces between $\mathcal{C}(P)(n)$ and the free non-unital operad $\Psi\left(\uparrow P^{\#}\right)(n)$ for $n \geq 2$.

Remark 2.1.14. The composition maps $\circ_i$ in the free operad $\psi\left(\uparrow P^{\#}\right)$ can be therefore defined also on $\mathcal{C}(P)(n)$. The differential d is then the derivation with respect to compositions $\circ_i$ (represented by joining two trees decorated by elements of $P^{\#}$).

Furthermore, the degree of determinant over edges of tree T with vertices decorated by elements of $P^{\#}$ can be equivalently introduced as the sum of degrees over tree vertices decorated by elements of $\uparrow P^{\#}$.

Remark 2.1.15. Cobar complex is sometimes, defined with operadic suspension s (each component of arity n is suspended $(n-1)$-times and tensored by signum representation of Σ_n), which requires to define determinant as one dimensional space of degree $|T|+2-n$ for $T \in R\mathbf{Tree}_n$. The cobar complex has therefore non-positive gradation.

The homology groups $H_*(\Psi(s\uparrow P^{\#}))$ for Koszul operad P are then exactly the *Koszul dual* of the operad P, i.e., the Koszul resolution of the Koszul operad P is the cobar complex over the Koszul dual of P. This fact is used in for example [DL14] and [DCV13].

Since we do not use this property in this text we will always work with the positively graded complex defined in 2.1.11.

2.2 Cobar complex of operad Ass

The dimension of the component $Ass(n)$ of arity n of the operad Ass, introduced in 1.1.6, is $n!$ and Σ_n acts transitively on $Ass(n)$. Hence we can consider $Ass^{\#}(n)$ as planar tree with only one vertex decorated by the dual of the identity permutation $e_n \in \Sigma_n$, α_n, and with n half-edges labeled by the set $[n]$ from left to right. The rest of elements of $Ass^{\#}(n)$ are generated by the action of the group Σ_n. As in the previous section we omit the component $Ass(1)$ and set it equal to 0.

Definition 2.2.1. The cobar complex over the operad Ass is called A_∞**-operad**

$$A_\infty = \mathcal{C}(Ass)$$

Remark 2.2.2. Let us consider the operad End_W over a dg vector space (W, d_W). The degree of a map $f \in End_W(n)$, i.e., $f : W^{\otimes n} \to W$, acting as

$$f\,(v_1 \otimes v_2 \otimes \ldots \otimes v_n) = v$$

is defined as $|f| = |v| - \sum_{i=1}^{n} |v_i|$. Then we can define the differential d_{End} of the operad End_W as

$$d_{Env}(f) = d_W \circ_1 f - (-1)^{|f|} \sum_{i=1}^{n} f \circ_i d_W \tag{2.3}$$

(the sign in front of the sum is determined by the requirement that $d_{End}(f) = 0$ for a chain map f).

It is easy to check that $d^2_{End} = 0$ (since $d^2_W = 0$). Hence End_W over a dg vector space W is a dg operad.

Definition 2.2.3. The degree 0 homomorphism $h : A_\infty \to End_W$ of dg operads (A_∞, d_{A_∞}) and (End_W, d_{End}), i.e.,

$$h \circ d_{A_\infty} = d_{End} \circ h \tag{2.4}$$

is called A_∞**-algebra**.

Theorem 2.2.4. The A_∞-algebra corresponds to Maurer-Cartan equation

$$\sum_{j=1}^{n} \sum_{i=1}^{n-j+1} m_{n-j+1} \circ_i m_j = 0$$

where $m_n : V^{\otimes n} \to V$ is a map of degree 1.

Proof. From Theorem 2.1.13 we know that A_∞ can be viewed as the free operad over the Σ-module $\uparrow Ass^{\#}$, and therefore we can define the homomorphism just on elements of this Σ-module. Moreover, since h is Σ_n-equivariant on every n-ary component, it is enough to define the homomorphism on $\uparrow \alpha_n$ for $n \geq 2$. Let us denote it as $h(\uparrow \alpha_n) = m_n$. Then (2.5) give us

(2.5)

$$h \circ d_{A_\infty}(\uparrow \alpha_n) = d_{End} \circ h(\uparrow \alpha_n) = d_{End}(m_n) = d_W \circ_1 m_n - (-1)^{|m_n|} \sum_{i=1}^{n} m_n \circ_i d_W$$

The differential d_{A_∞} was originally defined in (2.1) for the cobar complex $\mathcal{C}(A)$. For elements from $\uparrow Ass^{\#}$ we can adapt it as the composition $(\uparrow \otimes \uparrow) \circ d_{A_\infty} \circ \downarrow$ without taking the determinant into account. Since all elements of $Ass^{\#}$ have degree zero, we have

$$(\uparrow \otimes \uparrow)(\alpha_1 \otimes \alpha_2) = (-1)^{|\uparrow| \cdot |\alpha_1|} (\uparrow \alpha_1 \otimes \uparrow \alpha_2) = (\uparrow \alpha_1 \otimes \uparrow \alpha_2)$$

Hence

$$d_{A_\infty}(\uparrow \alpha_n) = \sum_{j=2}^{n-1} \sum_{i=1}^{n-j+1} \uparrow \widetilde{\alpha}_{n-j+1} \odot \uparrow \widetilde{\alpha}_j \tag{2.6}$$

where $\widetilde{\alpha}_{n-j+1}$ denotes the tree with one vertex decorated by element α_{n-j+1} and with half-edges labeled by the set $S_1 = \{1, 2, \dots i, i+j, \dots n\}$, which is isomorphic with the set $[n-j+1]$ via the bijection $b_1 : S_1 \to [n]$

$$b_1(k) = \begin{cases} k & \text{if } k \leq i \\ k-j+1 & \text{if } k > i \end{cases}$$

and $\widetilde{\alpha}_j$ denotes the tree with one vertex decorated by element α_j and with half-edges labeled by the set $S_2 = \{i, i+1, \dots i+j-1\}$, which is isomorphic with the set $[j]$ via the bijection $b_2 : S_2 \to [j]$

$$b_2(k) = k - i + 1$$

Hence the right hand side of (2.6) corresponds to all possible joinings of two planar trees, both with half-edges labeled from left to right, such that the whole composition is the planar tree with half-edges labeled from left to right as indicated on Figure 2.6.

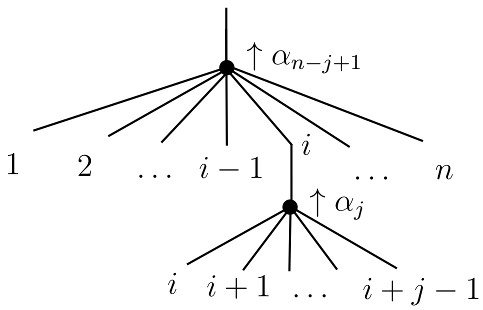


Figure 2.6: Composition of trees decorated by $\uparrow \widetilde{\alpha}_{n-j+1} \odot \uparrow \widetilde{\alpha}_j$

The left hand side of (2.5) is then

$$h \circ d_{A_\infty}(\alpha_n) = \sum_{j=2}^{n-1} \sum_{i=1}^{n-j+1} m_{n-j+1} \circ_i m_j$$

Since the elements $\uparrow \alpha_n$ have degree 1 and h is of degree 0, each m_n has also degree 1. If we denote $d_W = -m_1$, then (2.5) can be shortly written as

$$m_1 \circ_1 m_n + \sum_{i=1}^{n} m_n \circ_i m_1 + \sum_{j=2}^{n-1} \sum_{i=1}^{n-j} m_{n-j+1} \circ_i m_j = \sum_{j=1}^{n} \sum_{i=1}^{n-j+1} m_{n-j+1} \circ_i m_j = 0 \tag{2.7}$$

□

Remark 2.2.5. The result of 'cobar functor' C on dg operads is again an ordinary dg operad. But for modular operads is necessary to introduce analogous functor F, called *Feynman transform*. The result of Feynman transform is not the modular operad, but a certain type of *twisted* modular operad.

The conceptual difference comes from the dualization of composition $\xi_{i,j}$ defined in 1.16. In operads, roughly speaking, is the change of degree encoded in the increase of the number of vertices of the graph. But the cocomposition $\xi_{i,j}^{\#}$ does not allow such practice. The details of construction and its consequences are more explained in [GK98] and [MSS02]. We will just reproduce the results from Theorem 16 in [DJM13] in convention introduced in section 1.4.

The algebra over Feynman transform $F(P)$ of a modular operad P on a dg vector space V is equivalently determined by a collection

$$\{\alpha(X,g) : (P(X,g))^{\#} \to End_V(X,g) | 0 < 2g-2+|X|\}$$

of degree 0 linear maps (since we define the morphism $\alpha : F(P) \to End_V$ only on generators of this free operad $F(P)$, the $\alpha(X,g)$ do not have to be compatible with differential on $(P(G))^{\#}$), where $End_V(X,g) = Hom_{\mathbb{K}}(\bigodot_X V, \mathbb{K})$, such that

$$End_V(\rho) \circ \alpha(X,g) = \alpha(X',g) \circ (P(\rho^{-1}))^{\#} \tag{2.8}$$

for $\rho \in Bij(X',X)$, and

(2.9)

$$d_{End_V} \circ \alpha(X,g) = \alpha(X,g) \circ (d_P)^{\#} + (\xi_{i,j})_{End} \circ \alpha\left(X \sqcup \{i,j\}, g-1\right) \circ (\xi_{i,j})_P^{\#} +$$
$$+\frac{1}{2} \sum_{\substack{X_1 \sqcup X_2 = X \\ g_1+g_2=g}} ({}_i\odot_j)_{End} \circ (\alpha(X_1 \sqcup \{i\}, g_1) \otimes \alpha(X_2 \sqcup \{j\}, g_2)) \circ ({}^{X_1\sqcup\{i\},g_1}{}_i\odot_j^{X_2\sqcup\{i\},g_2})_P^{\#}$$

where the map

$$({}^{X_1\sqcup\{i\},g_1}{}_i\odot_j^{X_2\sqcup\{i\},g_2})_P^{\#} : (P(X,g))^{\#} \to (P(X_1 \sqcup \{i\}, g_1))^{\#} \otimes (P(X_2 \sqcup \{j\}, g_2))^{\#}$$

is analogical to the map $\Delta_e^{T,T'}$ from the beginning of this section. Note that (2.8) corresponds to our observation in Remark 2.1.3.

3. The duality of derivations and coderivations

3.1 Algebras and coalgebras

In this section we introduce the basic notions of coalgebras and algebras. The motivation and notion is mostly taken from articles of Kajiura [Kaj02], [Kaj07] and Kajiura with Stasheff [KS06b], [KS06a].

Definition 3.1.1. Let V be a graded vector space with a mapping $\Delta : V \to V \otimes V$ of degree 0 which is *coassociative*, i.e.,

$$(\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta, \tag{3.1}$$

where 1 denotes the identity mapping and $\circ$ denotes the composition of mappings.
Then V is **coalgebra** with **coproduct** Δ.

Definition 3.1.2. Let V be a coalgebra with coproduct Δ. A linear operator $C : V \to V$ of degree 1 is called **coderivation** if

$$\Delta \circ C = (C \otimes 1) \circ \Delta + (1 \otimes C) \circ \Delta \tag{3.2}$$

Definition 3.1.3. Let V, W be two coalgebras. We call $F : V \to W$ a **cohomomorphism** if

$$\Delta \circ F = (F \otimes F) \circ \Delta \tag{3.3}$$

Remark 3.1.4. An important point to note here is the equivalence of Definitions 3.1.1, 3.1.2 and 3.1.3 to the condition that following diagrams commute.

$$\begin{array}{ccc} V & \xrightarrow{\Delta} & V \otimes V \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle\Delta\otimes 1} \\ V \otimes V & \xrightarrow{1\otimes\Delta} & V \otimes V \otimes V \end{array} , \quad \begin{array}{ccc} V & \xrightarrow{C} & V \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle\Delta} \\ V \otimes V & \xrightarrow{1\otimes C + C\otimes 1} & V \otimes V \end{array} , \quad \begin{array}{ccc} V & \xrightarrow{F} & W \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle\Delta} \\ V \otimes V & \xrightarrow{F\otimes F} & W \otimes W \end{array}$$

The advantage of using commutative diagrams is that it is easier understand the notions of algebra, derivation and homomorphism as objects and morphisms in the opposite category.

In this way we can define **algebra** V with **product** μ as a graded vector space V with mapping $\mu : V \otimes V \to V$ which is *associative*, i.e.,

$$\mu \circ (\mu \otimes 1) = \mu \circ (1 \otimes \mu)$$

or equivalently requiring that the following diagram commutes

$$\begin{array}{ccc} V & \xleftarrow{\mu} & V \otimes V \\ {\scriptstyle\mu}\uparrow & & \uparrow{\scriptstyle\mu\otimes 1} \\ V \otimes V & \xleftarrow{1\otimes\mu} & V \otimes V \otimes V \end{array}$$

The definition of **derivation** $D : V \to V$ and **homomorphism** $G : V \to W$ are introduced similarly. In terms of commutative diagrams they mean

$$
\begin{array}{ccc}
V & \xleftarrow{\quad D \quad} & V \\
\mu \big\uparrow & & \big\uparrow \mu \\
V \otimes V & \xleftarrow{1 \otimes D + D \otimes 1} & V \otimes V
\end{array}
\quad , \quad
\begin{array}{ccc}
V & \xleftarrow{\quad G \quad} & W \\
\mu \big\uparrow & & \big\uparrow \mu \\
V \otimes V & \xleftarrow{G \otimes G} & W \otimes W
\end{array}
$$

For now we can define the degree of derivation as $|D| = 1$. In the next section we will see the connection between the degree of derivation and coderivation.

3.2 Description of algebras as dual of coalgebras

This section is devoted to the study of duals of vector spaces in order to introduce the notion of an algebra from the notion of a coalgebra.

Remark 3.2.1. Note that the other direction (from algebras to coalgebras) is for infinitely dimensional vector spaces much more complicated. The problems arise from the following observation.

Let us define mapping $\mu : V \otimes V \to V$ and denote by $V^{\#} = Hom\,(V, \mathbb{K})$ the dual space. Then the dual mapping acts as $\mu^{\#} : V^{\#} \to (V \otimes V)^{\#}$

There is also an injective mapping $i : V^{\#} \otimes V^{\#} \to (V \otimes V)^{\#}$ such that

$$i\,(\alpha \otimes \beta)\,(v \otimes w) = (\alpha, \beta)\,(v \otimes w) = (-1)^{|\beta| \cdot |v|} \alpha\,(v) \cdot \beta\,(w)$$

for $\alpha, \beta \in V^{\#}, v, w \in V$, where $\alpha\,(v)$, or equivalently $\langle \alpha | v \rangle$, is the evaluation of the form on the vector, i.e., the duality between V and $V^{\#}$. If we pick another two elements $\delta, \gamma \in V^{\#}$ such that

$$(\alpha, \beta)\,(v \otimes w) = (\delta, \gamma)\,(v \otimes w)\,,$$

then we can choose w such that $\beta\,(w) \neq 0$ and $\gamma\,(w) \neq 0$. From this we get

$$\alpha\,(v) = c \cdot \delta\,(v)\,,$$

where $c \in \mathbb{K}$ is some non-zero constant. Now we can conversely choose v such that $\alpha\,(v) \neq 0$ and $\delta\,(v) \neq 0$ but we already know that $\alpha\,(v) = c \cdot \delta\,(v)$ hence

$$c \cdot \delta\,(v)\,\beta\,(w) = \delta\,(v)\,\gamma\,(w)$$

and therefore $\beta\,(w) = c^{-1} \cdot \gamma\,(w)$.

This means that $(\alpha \otimes \beta) = (\delta \otimes \gamma)$ since there exists constant c such that $\alpha = c \cdot \delta$ and $\beta = c^{-1} \cdot \gamma$, so the mapping i is injective.

Hence if we want to define some mapping[1] $\phi : V^{\#} \to V^{\#} \otimes V^{\#}$ we cannot just restrict the mapping $\mu^{\#}$ to some subspace.

$$
\begin{array}{ccc}
(V \otimes V)^{\#} & \xleftarrow{\ \mu^{\#}\ } & V^{\#} \\
i \big\uparrow & \swarrow \phi & \\
V^{\#} \otimes V^{\#} & &
\end{array}
$$

[1]We want to define coalgebra structure from algebra structure.

This can be done only in the finitely-dimensional cases where the dual of a vector space is isomorphic to the space itself.[2] And thus an injective mapping between two finite dimensional spaces of the same dimension is an isomorphism. In the infinite dimensional case the vector space $\left(V^{\#}\right)^{\#}$ is never isomorphic to the space V, and thus the construction can be done only in one direction.

Remark 3.2.2. Let us make one more observation. For $\alpha \in V^{\#}, \phi^{\#} : V^{\#} \to V^{\#}$

$$(\phi^{\#}(\alpha))(v) = (-1)^{|\alpha|\cdot|\phi|}\alpha(\phi(v))$$

given by Koszul convention. Degrees of elements (or more accurately functions acting on them) on both sides must be the same and therefore

$$|\phi^{\#}(\alpha)| = |\alpha \circ \pi| \tag{3.4}$$

We can consider the left hand side of (3.4) as $\phi^{\#}$ acting on element $\alpha \in V^{\#}$, and hence

$$|\phi^{\#}(\alpha)| = |\phi^{\#}| + |\alpha|$$

and the right hand side as the composition of two functions acting on V, thus

$$|\alpha \circ \phi| = |\alpha| + |\phi|$$

Together we get $|\phi^{\#}| + |\alpha| = |\phi| + |\alpha|$. Hence $|\phi^{\#}| = |\phi|$.

Lemma 3.2.3. Let V be a coalgebra with a coproduct Δ. Then we can construct an algebra $V^{\#}$ with product μ such that

$$\langle \mu(\alpha \otimes \beta) | v \rangle = \langle i(\alpha \otimes \beta) | \Delta(v) \rangle, \tag{3.5}$$

where i is an injective mapping $i : V^{\#} \otimes V^{\#} \hookrightarrow (V \otimes V)^{\#}$ and $v \in V$, $\alpha, \beta \in V^{\#}$.

Proof. By the definition of a coalgebra with coassociative coproduct we know that the diagram

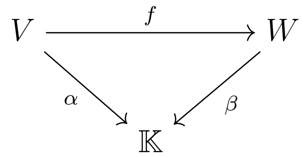
$$\begin{array}{ccc} V & \xrightarrow{\Delta} & V \otimes V \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle\Delta\otimes 1} \\ V \otimes V & \xrightarrow{1\otimes\Delta} & V \otimes V \otimes V \end{array}$$

commutes and by dualization we get the commutative diagram

$$\begin{array}{ccc} V^{\#} & \xleftarrow{\Delta^{\#}} & (V \otimes V)^{\#} \\ {\scriptstyle\Delta^{\#}}\uparrow & & \uparrow{\scriptstyle(\Delta\otimes 1)^{\#}} \\ (V \otimes V)^{\#} & \xleftarrow{(1\otimes\Delta)^{\#}} & (V \otimes V \otimes V)^{\#} \end{array}$$

Here for $f : V \to W$, the dual map $f^{\#} : W^{\#} \to V^{\#}$

[2]In this case $V \simeq V^{\#}$ and hence $(V \otimes V)^{\#} \simeq V \otimes V \simeq V^{\#} \otimes V^{\#}$.



is given by $f^{\#}(\beta) = \beta \circ f = \alpha$

Now by the injectivity of the map $i : V^{\#} \otimes V^{\#} \hookrightarrow (V \otimes V)^{\#}$ or by the injectivity of a similar map $i : V^{\#} \otimes V^{\#} \otimes V^{\#} \hookrightarrow (V \otimes V \otimes V)^{\#}$ (for short also denoted as i) we see

$$\Delta^{\#} \circ (1 \otimes \Delta)^{\#} \circ i = \Delta^{\#} \circ (\Delta \otimes 1)^{\#} \circ i. \tag{3.6}$$

Let us now compute the image of map $(1 \otimes \Delta)^{\#} \circ i$. Since for any $\alpha, \beta, \gamma \in V^{\#}$ we have

$$i(\alpha \otimes \beta \otimes \gamma) = (\alpha, \beta, \gamma)$$

so that

$$(\alpha, \beta, \gamma)(v \otimes w \otimes u) = (-1)^{|\beta| \cdot |v|}(-1)^{|\gamma| \cdot (|v| + |w|)} \alpha(v) \cdot \beta(w) \cdot \gamma(u)$$

and for any $v, w, u \in V$, the mapping $(1 \otimes \Delta)^{\#}(\alpha, \beta, \gamma)$ acts as

(3.7)
$$\Big((1 \otimes \Delta)^{\#}(\alpha, \beta, \gamma)\Big)(v \otimes w) = (-1)^{(|\alpha| + |\beta| + |\gamma|) \cdot |\Delta|}(\alpha, \beta, \gamma)((1 \otimes \Delta)(v \otimes w)) =$$
$$= (-1)^{(|\beta| + |\gamma|) \cdot |v|} \alpha(v) \cdot (\beta, \gamma)(\Delta w)$$

where from the definition of Δ and $\Delta^{\#}$ we have

$$(\beta, \gamma)(\Delta w) = (-1)^{(|\beta| + |\gamma|) \cdot |\Delta|} \Delta^{\#}(\beta, \gamma)(w) = \Delta^{\#}(\beta, \gamma)(w)$$

and $\Delta^{\#}(\beta, \gamma) \in V^{\#}$. Thus the image of $(1 \otimes \Delta)^{\#} \circ i : V^{\#} \otimes V^{\#} \otimes V^{\#} \to (V \otimes V)^{\#}$ is in the image of $i : V^{\#} \otimes V^{\#} \to (V \otimes V)^{\#}$.

Therefore we can consider a map $a : V^{\#} \otimes V^{\#} \otimes V^{\#} \to V^{\#} \otimes V^{\#}$ such that

$$(1 \otimes \Delta)^{\#} \circ i = i \circ a, \tag{3.8}$$

and similarly for the map $(\Delta \otimes 1)^{\#}$ let us consider the map

$$b : V^{\#} \otimes V^{\#} \otimes V^{\#} \to V^{\#} \otimes V^{\#}$$

such that

$$(\Delta \otimes 1)^{\#} \circ i = i \circ b \tag{3.9}$$

Together the equations (3.6), (3.8) and (3.9) give us

$$\Delta^{\#} \circ i \circ a = \Delta^{\#} \circ i \circ b \tag{3.10}$$

If we denote the mapping $\Delta^{\#} \circ i : V^{\#} \otimes V^{\#} \to V^{\#}$ as μ, we get a new commutative diagram

$$
\begin{array}{ccc}
V^{\#} & \xleftarrow{\ \mu\ } & V^{\#}\otimes V^{\#} \\
{\scriptstyle\mu}\uparrow & & \uparrow{\scriptstyle a} \\
V^{\#}\otimes V^{\#} & \xleftarrow{\ b\ } & V^{\#}\otimes V^{\#}\otimes V^{\#}
\end{array}
$$

Our next goal is to find out the properties of mappings a, b in relation to μ.

Now let us show that $a = 1 \otimes \mu$, or equivalently show that

$$(1 \otimes \Delta)^{\#} \circ i = i \circ \left(1 \otimes \Delta^{\#} \circ i\right) \tag{3.11}$$

We already showed in (3.7) the left hand side applied on an element

$$\alpha \otimes \beta \otimes \gamma \in V^{\#} \otimes V^{\#} \otimes V^{\#}$$

If we similarly apply the right hand side of (3.11) to the same element, we first get $(1 \otimes i)(\alpha, \beta, \gamma) = \alpha \otimes (\beta, \gamma)$, so that

$$(\alpha \otimes (\beta, \gamma))(v \otimes w \otimes u) = \alpha(v) \otimes (\beta, \gamma)(w \otimes u) = \alpha(v) \otimes (\beta(w) \cdot \gamma(u))$$

and then we get $\left(1 \otimes \Delta^{\#}\right)(\alpha \otimes (\beta, \gamma)) = \alpha \otimes \Delta^{\#}(\beta, \gamma)$. By the action of map i we finally get $\left(\alpha, \Delta^{\#}(\beta, \gamma)\right)$, so that

$$\begin{aligned}\left(\alpha, \Delta^{\#}(\beta, \gamma)\right)(v \otimes w) = (-1)^{|\Delta^{\#}(\beta,\gamma)|\cdot|v|}\alpha(v) \cdot \Delta^{\#}(\beta, \gamma)(w) = \\ = (-1)^{(|\Delta|+|\beta|+|\gamma|)\cdot|v|}\alpha(v) \cdot \Delta^{\#}(\beta, \gamma)(w) = (-1)^{(|\beta|+|\gamma|)\cdot|v|}\alpha(v) \cdot (\beta, \gamma)(\Delta w)\end{aligned}$$

where we use observation from Remark 3.2.2. Therefore the equality in (3.11) holds.

Analogously we get the relation $b = \mu \otimes 1$ and therefore we obtain four mappings such that the following diagram commutes.

$$
\begin{array}{ccc}
V^{\#} & \xleftarrow{\ \mu\ } & V^{\#}\otimes V^{\#} \\
{\scriptstyle\mu}\uparrow & & \uparrow{\scriptstyle \mu\otimes 1} \\
V^{\#}\otimes V^{\#} & \xleftarrow{\ 1\otimes\mu\ } & V^{\#}\otimes V^{\#}\otimes V^{\#}
\end{array}
$$

We can easily see, that the commutative diagram corresponds to the associativity of the map μ, and therefore μ is a product. From definition of μ we finally get

$$\langle \mu(\alpha \otimes \beta) | v\rangle = \langle \Delta^{\#} \circ i(\alpha \otimes \beta) | v\rangle = \langle i(\alpha \otimes \beta) | \Delta v\rangle$$

and therefore the equation (3.5) holds. □

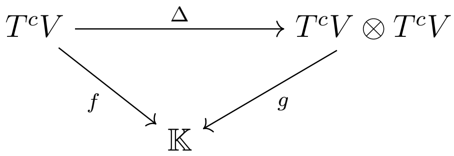
Example 3.2.4. Let us explain the relation between the coproduct Δ and the product μ, and how to get an expression for μ from an expression for Δ in one important example.

Let V be a graded vector space and $T^c V = \bigoplus_{n \geq 1} V^{\otimes n}$ its tensor coalgebra with coproduct defined as

$$\Delta(c_1 \otimes \ldots \otimes c_n) = \sum_{k=1}^{n-1} (c_1 \otimes \ldots \otimes c_k) \otimes (c_{k+1} \otimes \ldots \otimes c_n) \tag{3.12}$$

Let us choose $f \in (T^cV)^{\#}, g \in (T^cV \otimes T^cV)^{\#}$ such that $g \circ \Delta = f$, or equivalently from Koszul convention (and $|\Delta| = 0$)

$$\Delta^{\#}(g) = (-1)^{|\Delta|\cdot|g|} f = f$$



Let us consider the restriction of $\Delta^{\#}$ on the space $(T^cV)^{\#} \otimes (T^cV)^{\#}$, i.e., the product $\mu := \Delta^{\#} \circ i$, and denote elements from corresponding space with a tilde, i.e. for $\tilde{g} \in (T^cV)^{\#} \otimes (T^cV)^{\#}$ we have $\tilde{g} \circ \Delta = \tilde{f}$, or equivalently $\mu(g) = f$. Our task is now to define μ corresponding to Δ from (3.12).

Since $\tilde{g} \in (T^cV)^{\#} \otimes (T^cV)^{\#}$, we can think about it as an element $\tilde{g} = g_1 \otimes g_2$. Notice that a general element $h \in (T^cV)^{\#}$ can be expressed as

$$h = \left(h_{j_1 \dots j_m} e^{j_1} \otimes \dots \otimes e^{j_m}\right)_{m, j_1, \dots j_m} \in \prod_m \left(V^{\#}\right)^{\otimes m},$$

and that a general element of T^cV is a finite sum of elements $a_{i_1} \otimes \dots \otimes a_{i_n} \in V^{\otimes n}$ such that

$$a_1 \otimes \dots \otimes a_n = \sum_{i_1} a_{1,i_1} e_{i_1} \otimes \sum_{i_2} a_{2,i_2} e_{i_2} \otimes \cdots \sum_{i_n} a_{n,i_n} e_{i_n} = \sum_{i_1, i_2 \dots i_n} a_{i_1 \dots i_n} e_{i_1} \otimes \dots \otimes e_{i_n},$$

where $a_{1,i_1}, \dots a_{n,i_n}, a_{i_1 \dots i_n} \in \mathbb{K}$, $e_{i_1}, \dots e_{i_n}$ are base elements of V, $e^{j_1}, \dots e^{j_m}$ base elements of $V^{\#}$ such that $e^j(e_i) = \delta_i^j$, where δ stands for the Kronecker delta. Since $e^j(e_i)$ has to be of degree 0, $|e^j| = -|e_i|$. Hence we can compute just the evaluation of generators $g_1 = e^{j_1} \otimes \dots \otimes e^{j_m}$, $g_2 = e^{j_{m+1}} \otimes \dots \otimes e^{j_l}$ of $(T^cV)^{\#}$ on the coproduct $\Delta(e_{i_1} \otimes \dots e_{i_n})$ of generators of (T^cV) as defined in (3.12).

Then we get

$$\langle \left(e^{j_1} \otimes \dots \otimes e^{j_m}\right) \otimes \left(e^{j_{m+1}} \otimes \dots \otimes e^{j_l}\right) | \Delta(e_{i_1} \otimes \dots \otimes e_{i_n}) \rangle = \tag{3.13}$$
$$= \sum_{k=1}^{n-1} (-1)^{\left(|e_{i_1}| + \dots |e_{i_k}|\right)\left(|e^{j_{m+1}}| + \dots |e^{j_l}|\right)} \langle e^{j_1} \otimes \dots e^{j_m} | e_{i_1} \otimes \dots e_{i_k} \rangle \langle e^{j_{m+1}} \otimes \dots e^{j_l} | e_{i_{k+1}} \otimes \dots e_{i_n} \rangle$$

For better readability let us introduce the following notation for basis elements with indices increasing by one

$$(-1)^{\left(|e_{i_1}| + \dots + |e_{i_k}|\right) \cdot \left(|e^{j_{m+1}}| + \dots + |e^{j_l}|\right)} = (-1)^{[i_1 \dots i_k | j_{m+1} \dots j_l]} \tag{3.14}$$

Now from the evaluation of $\langle e^{j_1} \otimes \dots \otimes e^{j_m} | e_{i_1} \otimes \dots \otimes e_{i_k} \rangle$ we get $m = k$ and

$$\langle e^{j_1} \otimes \dots \otimes e^{j_m} | e_{i_1} \otimes \dots \otimes e_{i_k} \rangle = (-1)^{[i_1 | j_2 \dots j_m]} \cdot \delta_{i_1}^{j_1} \dots (-1)^{[i_{k-1} | j_m]} \cdot \delta_{i_{k-1}}^{j_{k-1}} \cdot \delta_{i_k}^{j_k} \tag{3.15}$$

and similarly the evaluation of

$$\langle e^{j_1} \otimes \dots \otimes e^{j_m} | e_{i_1} \otimes \dots \otimes e_{i_k} \rangle \langle e^{j_{m+1}} \otimes \dots \otimes e^{j_l} | e_{i_{k+1}} \otimes \dots \otimes e_{i_n} \rangle$$

will give us $l = n$ and

$$\langle e^{j_1} \otimes \ldots \otimes e^{j_m} | e_{i_1} \otimes \ldots \otimes e_{i_k} \rangle \langle e^{j_{m+1}} \otimes \ldots \otimes e^{j_l} | e_{i_{k+1}} \otimes \ldots \otimes e_{i_n} \rangle = \\ = (-1)^{[i_{k+1}|j_{m+2}\ldots j_l]} \cdot \delta^{j_{k+1}}_{i_{k+1}} \ldots (-1)^{[i_{n-1}|j_n]} \cdot \delta^{j_{n-1}}_{i_{n-1}} \cdot \delta^{j_n}_{i_n} \tag{3.16}$$

Hence there is only one non-zero element of the sum on the right hand side of (3.13) corresponding to conditions $m = k$ and $l = n$ such that $i_1 = j_1, \ldots i_n = j_n$. If we apply the results from (3.15) and (3.16) we get the following

$$\begin{aligned} &\left\langle \left(e^{i_1} \otimes \ldots \otimes e^{i_m}\right) \otimes \left(e^{i_{m+1}} \otimes \ldots \otimes e^{i_n}\right) | \Delta \left(e_{i_1} \otimes \ldots \otimes e_{i_n}\right) \right\rangle = \\ &= (-1)^{[i_1\ldots i_m|i_{m+1}\ldots i_n]} (-1)^{[i_1|i_2\ldots i_m]} \ldots (-1)^{[i_{m-1}|i_m]} (-1)^{[i_{m+1}|i_{m+2}\ldots i_n]} \ldots (-1)^{[i_{n-1}|i_n]} = \\ &= (-1)^{[i_1|i_2\ldots i_n]} \ldots (-1)^{[i_{m-1}|i_m\ldots i_n]} (-1)^{[i_m|i_{m+1}\ldots i_n]} (-1)^{[i_{m+1}|i_{m+2}\ldots i_n]} \ldots (-1)^{[i_{n-1}|i_n]} = \\ &= (-1)^{[i_1|i_2\ldots i_n]} \ldots (-1)^{[i_{n-1}|i_n]} \end{aligned} \tag{3.17}$$

We can easily see that this is the same as evaluating $f = e^{i_1} \otimes \ldots \otimes e^{i_n}$ on $e_{i_1} \otimes \ldots \otimes e_{i_n}$

$$\langle e^{i_1} \otimes \ldots \otimes e^{i_n} | e_{i_1} \otimes \ldots \otimes e_{i_n} \rangle = (-1)^{[i_1|i_2\ldots i_n]} \ldots (-1)^{[i_{n-1}|i_n]}$$

Hence finally

$$\mu \left(\left(e^{i_1} \otimes \ldots \otimes e^{i_m}\right) \otimes \left(e^{i_{m+1}} \otimes \ldots \otimes e^{i_n}\right) \right) = e^{i_1} \otimes \ldots \otimes e^{i_n} \tag{3.18}$$

For coproduct Δ defined in (3.12) on tensor coalgebra we computed the corresponding product μ on $(T^c V)^{\#}$ defined in (3.18).

In very a similar fashion as in Lemma 3.2.3 we can construct a derivation from a coderivation.

Lemma 3.2.5. Let V be a coalgebra with coproduct Δ and coderivation C. Then we can construct an algebra $V^{\#}$ with product μ and derivation D such that

$$\langle D(\alpha) | v \rangle = (-1)^{|\alpha| \cdot |C|} \langle \alpha | \, C(v) \rangle \tag{3.19}$$

for $v \in V$ and $\alpha \in V^{\#}$.

Proof. By the definition of coalgebra with coderivation we have the commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\quad C \quad} & V \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle\Delta} \\ V \otimes V & \xrightarrow{1 \otimes C + C \otimes 1} & V \otimes V \end{array}$$

and by dualization we get the commutative diagram

$$\begin{array}{ccc} V^{\#} & \xleftarrow{\quad C^{\#} \quad} & V^{\#} \\ {\scriptstyle\Delta^{\#}}\uparrow & & \uparrow{\scriptstyle\Delta^{\#}} \\ (V \otimes V)^{\#} & \xleftarrow{(1 \otimes C + C \otimes 1)^{\#}} & (V \otimes V)^{\#} \end{array}$$

As $i : V^{\#} \otimes V^{\#} \to (V \otimes V)^{\#}$ is injective composition with i will not violate the identity, hence

$$\Delta^{\#} \circ (C \otimes 1 + 1 \otimes C)^{\#} \circ i = C^{\#} \circ \Delta^{\#} \circ i \tag{3.20}$$

As in lemma 3.2.3 let us compute the image of map $(C \otimes 1 + 1 \otimes C)^{\#} \circ i$. Since for any $\alpha, \beta \in V^{\#}$ and $v, w \in V$ we have

$$(i (\alpha \otimes \beta)) (v \otimes w) = (\alpha, \beta) (v \otimes w) = (-1)^{|\beta| \cdot |v|} \alpha (v) \cdot \beta (w)$$

the mapping $(C \otimes 1 + 1 \otimes C)^{\#} (\alpha, \beta)$ acts as

$$\begin{aligned} &\Big((C \otimes 1 + 1 \otimes C)^{\#} (\alpha, \beta) \Big) (v \otimes w) = \\ &= (-1)^{|C| \cdot (|\alpha| + |\beta|)} (\alpha, \beta) \Big(C (v) \otimes w + (-1)^{|v| \cdot |C|} v \otimes C (w) \Big) = \\ &= (-1)^{|\alpha| + |\beta|} \Big((-1)^{|\beta| \cdot (|C| + |v|)} \alpha (C (v)) \cdot \beta (w) + (-1)^{|v| \cdot |C|} (-1)^{|\beta| \cdot |v|} \alpha (v) \cdot \beta (C (w)) \Big) \end{aligned} \tag{3.21}$$

Thus the image of $(C \otimes 1 + 1 \otimes C)^{\#} \circ i : V^{\#} \otimes V^{\#} \to (V \otimes V)^{\#}$ is in the image of $i : V^{\#} \otimes V^{\#} \to (V \otimes V)^{\#}$. Therefore we can consider a map

$$e : V^{\#} \otimes V^{\#} \to V^{\#} \otimes V^{\#}$$

such that

$$(C \otimes 1 + 1 \otimes C)^{\#} \circ i = i \circ e \tag{3.22}$$

If we plug in (3.22) into (3.20) we get

$$\Delta^{\#} \circ i \circ e = C^{\#} \circ \mu$$

which is the same as

$$\mu \circ e = C^{\#} \circ \mu$$

Let us denote the mapping $C^{\#}$ as D. Together we have commutative diagram

$$\begin{array}{ccc} V^{\#} & \xleftarrow{\quad D \quad} & V^{\#} \\ \mu \big\uparrow & & \big\uparrow \mu \\ V^{\#} \otimes V^{\#} & \xleftarrow{\quad e \quad} & V^{\#} \otimes V^{\#} \end{array}$$

Now let us show that $e = D \otimes 1 + 1 \otimes D$, or equivalently,

$$(C \otimes 1 + 1 \otimes C)^{\#} \circ i = i \circ \left(C^{\#} \otimes 1 + 1 \otimes C^{\#} \right) \tag{3.23}$$

In (3.21) we computed the left hand side applied to an element $\alpha \otimes \beta \in V^{\#} \otimes V^{\#}$, acting on elements $v, w \in V$. If we apply the right hand side of (3.23) on the same element we at first get

$$\left(C^{\#} \otimes 1 + 1 \otimes C^{\#} \right) (\alpha \otimes \beta) = C^{\#} (\alpha) \otimes \beta + (-1)^{|\alpha| \cdot |C^{\#}|} \alpha \otimes C^{\#} (\beta)$$

and after acting with i we get a mapping whose evaluation is

$$\left(\left(C^{\#}\left(\alpha\right),\beta\right)+(-1)^{|\alpha|\cdot|C^{\#}|}\left(\alpha,C^{\#}\left(\beta\right)\right)\right)\left(v\otimes w\right)= \tag{3.24}$$
$$=(-1)^{|\beta|\cdot|v|}C^{\#}(\alpha)\left(v\right)\cdot\beta\left(w\right)+(-1)^{|\alpha|\cdot|C^{\#}|}\left(-1\right)^{(|C^{\#}|+|\beta|)\cdot|v|}\alpha\left(v\right)\cdot C^{\#}(\beta)\left(w\right)$$
$$=(-1)^{|\beta|\cdot|v|}(-1)^{|\alpha|\cdot|C|}\alpha\left(C\left(v\right)\right)\cdot\beta\left(w\right)+(-1)^{|\alpha|}(-1)^{|\beta|\cdot|v|}(-1)^{|\beta|\cdot|C|}\alpha\left(v\right)\cdot\beta\left(C\left(w\right)\right)$$

From definition of coderivation $|C|=1$ and observation in Remark 3.2.2 we get $|C^{\#}|=1$, and hence the terms in (3.21) and (3.24) are the same, which shows the equality of terms in (3.23).

The diagram showing this property

$$\begin{array}{ccc} V^{\#} & \xleftarrow{\quad D \quad} & V^{\#} \\ {\scriptstyle\mu}\uparrow & & {\scriptstyle\mu}\uparrow \\ (V\otimes V)^{\#} & \xleftarrow{(1\otimes D+D\otimes 1)} & (V\otimes V)^{\#} \end{array}$$

also corresponds to the condition[3] that D is derivation.

Finally we also see that

$$\langle D\left(\alpha\right)|v\rangle=\langle C^{\#}\left(\alpha\right)|v\rangle=(-1)^{|\alpha|\cdot|C|}\langle\alpha|C\left(v\right)\rangle$$

□

3.3 A_∞-algebra

Definition 3.3.1. Let V be a (generally infinite dimensional) graded vector space and $T^cV=\bigoplus_{n=1}^{\infty}V^{\otimes n}$ its tensor coalgebra with coderivation C of degree 1. T^cV is an A_∞**-algebra** if

$$C^2=0 \tag{3.25}$$

Remark 3.3.2. We already defined A_∞ in Definition 2.2.3 and showed its equivalent expression via maps $m_n : V^{\otimes n}\to V$ of degree 1 in (3.26). Let us now show that this defines the same structure as 3.3.1.

Let us denote the projection $proj_n : T^cV\to V^{\otimes n}$ and $c_n=C\circ proj_n$. It is obvious that

$$C=\sum_{i=1}^{\infty}c_i$$

Note that although is the map defined as infinite sum of non-trivial maps, when we enumerate it on element $v\in\bigoplus_{i=1}^{m}V^{\otimes i}\subset T^cV$, only finite many members of the sum will be used.

Proposition 1.2.9 from [LV12] says that coderivation C is completely determined by

$$proj_1\circ C=\sum_{i=1}^{\infty}proj_1\circ c_i$$

[3] This property is usually called the Leibniz rule.

Definition 3.3.1 is therefore equivalently defined as a sequence of linear maps $\{proj_1 \circ c_n : V^{\otimes n} \to V\}_{i\geq 1}$. Let us denote these maps as $m_n = proj_1 \circ C \circ proj_n$. Lemma 9.2.2 ibid shows that this is equivalent[4] to

$$\sum_{\substack{k,l \\ k+l=n+1}} \sum_{i=1}^{k} m_k \circ_i m_l = 0 \tag{3.26}$$

where $m_i \circ m_j = \sum_{k=1}^{i} m_i \circ (1^{\otimes(k-1)} \otimes m_j \otimes 1^{\otimes(i-k)})$.

Hence the definitions are equivalent. This can be compared with Section 2.1 in [KS06b].

Remark 3.3.3. If we split the sequence of maps $\{m_n\}_{n\in\mathbb{N}}$ into a differential map $\overline{m} = m_1 : V \to V$ and the rest denoted as $\theta = \{m_n : V^{\otimes n} \to V\}_{n\geq 2}$ then equation (3.26) can be expressed as

$$\overline{m}^2 + \overline{m} \circ \theta + \theta \circ \overline{m} + \theta^2 = 0 \tag{3.27}$$

The term $\overline{m} \circ \theta + \theta \circ \overline{m}$ can be understood similarly as in (2.3) as the natural differential d on $Hom(T^cV, V)$, i.e.,

$$d(\theta) = \overline{m} \circ \theta + (-1)^{|\overline{m}|\cdot|\theta|} \theta \circ \overline{m} = [\overline{m}, \theta]$$

where $[\cdot,\cdot]$ denotes the Lie bracket on the vector space $Hom(T^cV, V)$. Then, since $\overline{m}^2 = 0$, we have equivalently

$$d(\theta) + \frac{1}{2} [\theta, \theta] = 0 \tag{3.28}$$

such equation corresponds to the *Maurer-Cartan equation*[5] in dg Lie algebra

$$(Hom(T^cV, V), d, [\cdot,\cdot])$$

Remark 3.3.4. The confusing designation 'algebra' for tensor coalgebra T^cV comes from their original motivation as generalization of dg algebra with differential m_1, with m_2 an associative multiplication up to homotopy given by the ternary operation m_3. But the composition of these two maps leads to introducing another operation m_4 and in general n-ary operations m_n.

It is usual to define the multiplication map m_2 as degree 0 operation. In our case it is of degree 1. To adapt our maps to usual setting it is necessary to introduce operad suspension, already mentioned in 2.1.15. Since we do not need the map m_2 to have degree 0, we skip this process and just work with maps of degree 1.

[4]The equivalence is actually proved up to sign. But this sign is given by different degree of m_n in the proof. In our case is the map $proj_1$ of degree 0 and the coderivation C is of degree 1, hence the map $proj_1 \circ C$ is of degree 1. The maps $m_n : V^{\otimes n} \to V$ used in Lemma 9.2.2 are of degree $n-2$.

[5]The solutions of (3.28) can be therefore seen as deformations of V.

3.4 Maurer-Cartan equation in derivations

We have seen in remark 3.3.2 that the coderivation C on A_∞-algebra can be equivalently expressed as a collection of degree one maps $\{m_n : V^{\otimes n} \to V\}_{n\in\mathbb{N}}$ such that

$$\sum_{\substack{r,s\\ r+s=n+1}} \sum_{t=1}^{r} m_r \circ_t m_s = 0 \tag{3.29}$$

And we proved in the Lemmas 3.2.3 and 3.2.5 that we can construct from coalgebra (V, Δ, C) an algebra $(V^\#, \mu, D)$. Let us now show that equation similar to (3.29) holds also for algebra.

Theorem 3.4.1. Let V be a finite dimensional vector space. Then equation (3.29) equivalent to $C^2 = 0$ for coderivation C on T^cV can be translated to equation

$$0 = \sum_{\substack{r,s\\ r+s=n+1}} \sum_{t=1}^{r} \widetilde{m}(s)\, {}_1{\circ}_t \widetilde{m}(r)$$

equivalent to $D^2 = 0$, where $\widetilde{m}(n) : V^\# \to (V^\#)^{\otimes n}$ and the symbol ${}_1{\circ}_t$ denotes the pairing of t-th output of $\widetilde{m}(r)$ with 1-st input of $\widetilde{m}(s)$.

Proof. Since V is finite dimensional, we have $(V^{\otimes n})^\# \cong \left(V^\#\right)^{\otimes n}$ and

$$(T^cV)^\# = \left(\bigoplus_{n\in\mathbb{N}} V^{\otimes n}\right)^\# \cong \prod_n \left(V^\#\right)^{\otimes n}$$

Also we can think of maps $m_n : V^{\otimes n} \to V$ as 1-contravariant and of n-covariant tensor $m(n) \in V \otimes \underbrace{V^\# \otimes \ldots \otimes V^\#}_{n}$ given in coordinates as

$$\begin{aligned} m(n)(e_{k_1} \otimes \ldots \otimes e_{k_n}) &= [m(n)(\,ht]^i_{j_1,j_2\cdots j_n} e_i \otimes e^{j_1} \otimes \ldots \otimes e^{j_n}(e_{k_1} \otimes \ldots \otimes e_{k_n}) = \\ &= [m(n)]^i_{j_1,j_2\cdots j_n}\, e_i \cdot e^{j_1}(e_{k_1}) \cdot (-1)^{[k_1|j_2,\ldots j_n]} \cdot e^{j_2}(e_{k_2}) \cdot (-1)^{[k_2|j_3,\ldots j_n]} \ldots \\ &\qquad \ldots e^{j_{n-1}}(e_{k_{n-1}}) \cdot (-1)^{[k_{n-1}|j_n]} \cdot e^{j_n}(e_{k_n}) \end{aligned} \tag{3.30}$$

where we use Einstein summation convention and the notation introduced in (3.14), e_i, e^j denote the base vectors of V ($V^\#$, respectively) and we use the observations from Example 3.2.4. Therefore

$$m(n)(e_{k_1} \otimes \ldots \otimes e_{k_n}) = [m(n)]^i_{k_1,k_2\cdots k_n}\, e_i (-1)^{[k_1|k_2\ldots k_n]} \cdot (-1)^{[k_2|k_3\ldots k_n]} \ldots (-1)^{[k_{n-1}|k_n]}$$

Note that the evaluation of $\phi \in W^\#$ on $f(v)$ such that $f : V \to W$ (and $f^\# : W^\# \to V^\#$), $v \in V$ is given by

$$\langle \phi | f(v) \rangle = \phi(f(v)) = \phi \circ f(v) = \langle \phi \circ f | v \rangle = (-1)^{|f|\cdot|\phi|} \langle f^\#(\phi) | v \rangle \tag{3.31}$$

Therefore the evaluation of e^r on an element from (3.30) is

$$\langle e^r | m(n)(e_{k_1} \otimes \ldots \otimes e_{k_n}) \rangle = (-1)^{|m(n)|\cdot|e^r|} \langle m^\#(n)(e^r) | e_{k_1} \otimes \ldots \otimes e_{k_n} \rangle$$

and since for all n the degree is $|m(n)| = 1$, we can omit it in the exponent. Therefore the map $m^{\#}(n) : V^{\#} \to (V^{\#})^{\otimes n}$ is given in coordinates as

(3.32)

$$m^{\#}(n)(e^r) = [m(n)]^i_{j_1,j_2\dots j_n}\, e_i \otimes e^{j_1} \otimes \ldots \otimes e^{j_n}(e^r) = [m(n)]^r_{j_1,j_2\dots j_n}\, e^{j_1} \otimes \ldots \otimes e^{j_n}$$

From (2.7) we get the left hand side of the next equation for arbitrary $e^l \in V^{\#}$ and $e_{k_1}, \ldots e_{k_n} \in V$

(3.33)

$$0 = \langle e^l| \sum_{\substack{r,s \\ r+s=n+1}} \sum_{t=1}^{r} m(r) \left(1^{\otimes t-1} \otimes m(s) \otimes 1^{n-s-t+1}\right) (e_{k_1} \otimes e_{k_2} \otimes \ldots \otimes e_{k_n})\rangle =$$

$$= \sum_{\substack{r,s \\ r+s=n+1}} \sum_{i=1}^{r} (-1)^{|e^m|} \langle m^{\#}(r)(e^m)| \left(1^{\otimes t-1} \otimes m(s) \otimes 1^{n-s-t+1}\right) (e_{k_1} \otimes e_{k_2} \otimes \ldots \otimes e_{k_n})\rangle$$

We can use (3.32) but we have to determine what exactly is

$$\left(1^{\otimes t-1} \otimes m(s) \otimes 1^{n-s-t+1}\right)^{\#}$$

Obviously this map is of degree $|m(s)| = 1$ and we know that

(3.34)

$$\langle e^{j_1} \otimes \ldots e^{j_r} | e_{k_1} \otimes \ldots e_{k_{t-1}} \otimes m(s) \left(e_{k_t} \otimes \ldots \otimes e_{k_{t+s-1}}\right) \otimes e_{k_{t+s}} \otimes \ldots e_{k_n}\rangle =$$
$$= (-1)^{a(t)} \langle e^{j_1} \otimes \ldots \otimes e^{j_r}| \left(1^{\otimes t-1} \otimes m(s) \otimes 1^{n-s-t+1}\right) (e_{k_1} \otimes \ldots \otimes e_{k_n})\rangle =$$
$$= (-1)^{a(t)} (-1)^{|e^{j_1}|+\ldots|e^{j_r}|} \langle \left(1^{\otimes t-1} \otimes m(s) \otimes 1^{n-s-t+1}\right)^{\#} \left(e^{j_1} \otimes \ldots \otimes e^{j_r}\right) | e_{k_1} \otimes \ldots \otimes e_{k_n}\rangle$$

where $(-1)^{a(t)} = (-1)^{|e_{k_1}|+\ldots+|e_{k_{t-1}}|}$ is given by the Koszul convention. Now we step by step evaluate the left hand side of (3.34) (multiplied by $(-1)^{a(t)}$)

$$\langle e^{j_1} \otimes \ldots e^{j_r} | e_{k_1} \otimes \ldots e_{k_{t-1}} \otimes m(s) \left(e_{k_t} \otimes \ldots \otimes e_{k_{t+s-1}}\right) \otimes e_{k_{t+s}} \otimes \ldots e_{k_n}\rangle =$$
$$= (-1)^{[k_1|j_2,\ldots j_r]} \langle e^{j_1}|e_{k_1}\rangle \ldots (-1)^{[k_{t-1}|j_t,\ldots j_r]} \langle e^{j_{t-1}}|e_{k_{t-1}}\rangle (-1)^{|m(s)|+[k_t,\ldots k_{t+s-1}|j_{t+1},\ldots j_r]}$$
$$\cdot \langle e^{j_t}|m(s) \left(e_{k_t} \otimes \ldots e_{k_{t+s-1}}\right)\rangle \cdot (-1)^{[k_{t+1}|j_{t+2},\ldots j_r]} \langle e^{j_{t+1}}|e_{k_{t+s}}\rangle \cdot \ldots \cdot \langle e^{j_r}|e_{k_n}\rangle$$

From the observation in (3.31) we see that

$$\langle e^{j_t}|m(s) \left(e_{k_t} \otimes \ldots e_{k_{t+s-1}}\right)\rangle = (-1)^{|e^{j_t}|} \langle m^{\#}(s) e^{j_t} | e_{k_t} \otimes \ldots e_{k_{t+s-1}}\rangle$$

and if we now rewrite the evaluations back, we get

$$\langle e^{j_1} \otimes \ldots e^{j_r} | e_{k_1} \otimes \ldots e_{k_{t-1}} \otimes m(s) \left(e_{k_t} \otimes \ldots \otimes e_{k_{t+s-1}}\right) \otimes e_{k_{t+s}} \otimes \ldots e_{k_n}\rangle =$$
$$= (-1)^{a(t)} (-1)^{|e^{j_t}|+\ldots+|e^{j_r}|} \langle e^{j_1} \otimes \ldots \otimes m^{\#}(s) \left(e^{j_t}\right) \otimes e^{j_{t+1}} \otimes \ldots \otimes e^{j_r} | e_{k_1} \otimes \ldots \otimes e_{k_n}\rangle$$

Hence the right hand side of (3.34) is given as

$$(-1)^{a(t)} (-1)^{|e^{j_1}|+\ldots|e^{j_r}|} \langle \left(1^{\otimes t-1} \otimes m(s) \otimes 1^{n-s-t+1}\right)^{\#} \left(e^{j_1} \otimes \ldots \otimes e^{j_r}\right) | e_{k_1} \otimes \ldots \otimes e_{k_n}\rangle =$$
$$= (-1)^{a(t)} (-1)^{|e^{j_t}|+\ldots+|e^{j_r}|} \langle e^{j_1} \otimes \ldots \otimes m^{\#}(s) \left(e^{j_t}\right) \otimes e^{j_{t+1}} \otimes \ldots \otimes e^{j_r} | e_{k_1} \otimes \ldots \otimes e_{k_n}\rangle$$

Finally we get

$$\langle e^{j_1} \otimes \ldots \otimes e^{j_r} | \left(1^{\otimes t-1} \otimes m(s) \otimes 1^{n-s-t+1}\right) (e_{k_1} \otimes \ldots \otimes e_{k_n})\rangle = \tag{3.35}$$
$$= (-1)^{|e^{j_t}|+\ldots+|e^{j_r}|} \cdot \langle e^{j_1} \otimes \ldots \otimes m^{\#}(s) \left(e^{j_t}\right) \otimes e^{j_{t+1}} \otimes \ldots \otimes e^{j_r} | e_{k_1} \otimes \ldots \otimes e_{k_n}\rangle$$

Now we can use the results from (3.35) and the coordinate expression of $m^{\#}(r)(e^m)$ and rewrite Equation (3.33) as

$$\sum_{\substack{r,s \\ r+s=n+1}} \sum_{t=1}^{r} (-1)^{|e^m|} \langle m^{\#}(r)(e^m) | \left(1^{\otimes t-1} \otimes m(s) \otimes 1^{n-s-t+1}\right) (e_{k_1} \otimes e_{k_2} \otimes \ldots \otimes e_{k_n})\rangle =$$
$$= \sum_{\substack{r,s \\ r+s=n+1}} \sum_{t=1}^{r} (-1)^{|e^m|} (-1)^{|e^{j_1}|+\ldots+|e^{j_r}|} \left[m(r)\right]^m_{j_1,\ldots j_r}$$
$$\cdot \langle \left(1^{\otimes t-1} \otimes m(s) \otimes 1^{n-s-t+1}\right)^{\#} (e^{j_1} \otimes \ldots \otimes e^{j_r}) | e_{k_1} \otimes e_{k_2} \otimes \ldots \otimes e_{k_n}\rangle =$$
$$= \sum_{\substack{r,s \\ r+s=n+1}} \sum_{t=1}^{r} (-1)^{|e^m|} (-1)^{|e^{j_1}|+\ldots+|e^{j_r}|} (-1)^{|e_{j_t}|+\ldots+|e_{j_r}|} \left[m(r)\right]^m_{j_1,\ldots j_r} \tag{3.36}$$
$$\cdot \langle e^{j_1} \otimes \ldots \otimes e^{j_{t-1}} \otimes m^{\#}(s)(e^{j_t}) \otimes e^{j_{t+1}} \otimes \ldots \otimes e^{j_r} | e_{k_1} \otimes e_{k_2} \otimes \ldots \otimes e_{k_n}\rangle =$$
$$= \sum_{\substack{r,s \\ r+s=n+1}} \sum_{t=1}^{r} (-1)^{|e^m|} (-1)^{|e^{j_1}|+\ldots+|e^{j_{t-1}}|} \left[m(r)\right]^m_{j_1,\ldots j_r} \left[m(s)\right]^p_{q_1,\ldots q_s}$$
$$\cdot \langle e^{j_1} \otimes \ldots \otimes e^{j_{t-1}} \otimes \left(e_p \otimes e^{q_1} \otimes \ldots e^{q_s}\right) \left(e^{j_t}\right) \otimes e^{j_{t+1}} \otimes \ldots \otimes e^{j_r} | e_{k_1} \otimes e_{k_2} \otimes \ldots \otimes e_{k_n}\rangle =$$
$$= \sum_{\substack{r,s \\ r+s=n+1}} \sum_{t=1}^{r} (-1)^{|e^m|} (-1)^{|e^{k_1}|+\ldots+|e^{k_{t-1}}|} \left[m(r)\right]^m_{k_1,\ldots k_{t-1}, j_t, k_{t+s},\ldots k_r} \left[m(s)\right]^{j_t}_{k_t,\ldots k_{t+s-1}}$$
$$\cdot (-1)^{[k_1|k_2\ldots k_n]} \ldots (-1)^{[k_{n-1}|k_n]}$$

The equation (3.36) holds for arbitrary $e^m \in V^{\#}, e_{k_1}, \ldots e_{k_n} \in V$ hence

$$0 = \sum_{\substack{r,s \\ r+s=n+1}} \sum_{t=1}^{r} (-1)^{|e^{k_1}|+\ldots+|e^{k_{t-1}}|} \left[m(r)\right]^m_{k_1,\ldots k_{t-1}, j_t, k_{t+s},\ldots k_r} \left[m(s)\right]^{j_t}_{k_t,\ldots k_{t+s-1}} \tag{3.37}$$

Equation (3.37) clearly corresponds to the composition of maps

$$\widetilde{m}(n) : V^{\#} \to (V^{\#})^{\otimes n}$$

satisfying a relation similar to (2.7)

$$0 = \sum_{\substack{r,s \\ r+s=n+1}} \sum_{t=1}^{r} \widetilde{m}(s) \, {}_1\circ_t \widetilde{m}(r)$$

where the symbol ${}_1\circ_t$ denotes pairing of t-th output of $\widetilde{m}(r)$ with 1-st input of $\widetilde{m}(s)$. □

4. Properads

4.1 PROPs and properads

In this section we want to generalize the concept of operads. The structure of operads was motivated by functions with n inputs and 1 output. The idea of the generalization is to take ‘functions’ with n inputs and m outputs.

We can again define composition maps[1] connecting outputs of one element to inputs of another one. But this time it is possible to define new composition maps, called *horizontal*, taking inputs of two different elements as the input of one new element and their outputs as the outputs of the new one. Similarly as operads this structure can be visualized. The elements correspond to decorated directed non-planar (not necessarily connected) graphs with actions of symmetric groups on inputs and outputs. Such structure is usually called PROPs.

We do not need such a general structure – for us is enough to use only connected parts without horizontal compositions as was introduced in [Val07]. Nevertheless, it turns out that it is easier to define PROPs and then just to restrict the definition to define properads as in [Mar06].

The definitions related to categories are mainly taken from nLab websites [Sch16b], [Sch16a] and [Dun15].

Remark 4.1.1. Similarly as in 1.1.1 we can define *PROP* in category $\mathbf{Mod}_{\mathbb{K}}$ as a collection $P = \{P(m,n)\}_{m,n\geq 0}$ of (Σ_m, Σ_n)-bimodules such that the left action of Σ_m commutes with the right action of Σ_n together with two types of composition maps called *vertical*

$$\circ : P(m,n) \otimes P(n,k) \to P(m,k)$$

and *horizontal*

$$\boxplus : P(m_1,n_1) \otimes \ldots \otimes P(m_s,n_s) \to P(m_1 + \ldots + m_s, n_1 + \ldots + n_s)$$

and a *unital element* $e \in P(1,1)$. This all together is required to satisfy axioms similar to the axioms from the definition of operad. It is easy to see what should be the meaning of these axioms. But to write them down requires work with many indices which makes them hard to read. Hence let us show them in a more compact form. As first let us recall some notions from category theory.

Remark 4.1.2. Let us denote objects of a category C as $\mathrm{Ob}(C)$ and its morphisms as $C(\cdot,\cdot)$.

A category C is a **monoidal category** if it is equipped with a bifunctor $\otimes : C \times C \to C$ called **tensor product**, special object called the **identity object** $I \in C$ and three natural isomorphisms

$$\alpha_{a,b,c} : (a \otimes b) \otimes c \cong a \otimes (b \otimes c)$$

$$\lambda_a : I \otimes a \cong a$$

$$\rho_a : a \otimes I \cong a$$

[1] Let us call them a moment as *vertical*.

for any $a, b, c \in \mathrm{Ob}(C)$ such that some additional identities[2] hold. For more see [Sch16b].

In a **strict** monoidal category, the natural isomorphisms α, λ, ρ are identites. A **symmetric** monoidal category has moreover one natural isomorphism

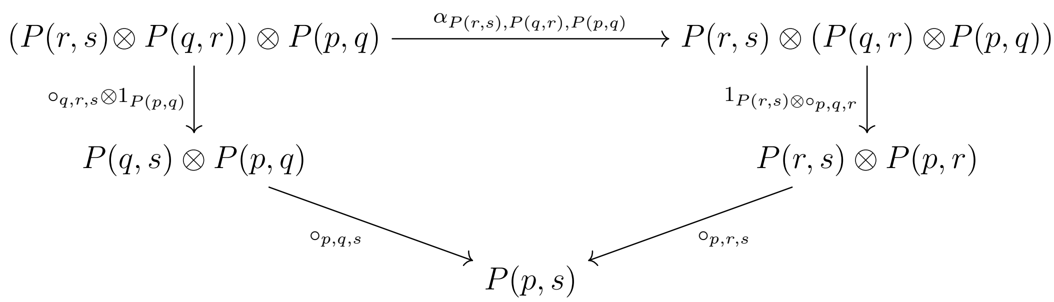
$$S_{a,b} : a \otimes b \to b \otimes a$$

again satisfying some additional identites.

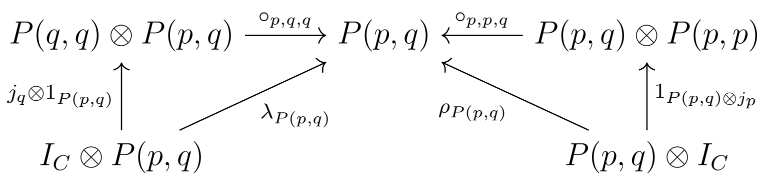
Definition 4.1.3. A category P is **enriched in a monoidal category** C if for all pairs of objects $(p, q) \in \mathrm{Ob}(P) \times \mathrm{Ob}(P)$ the hom-set $P(p, q)$ is an object of category C, $P(p, q) \in \mathrm{Ob}(C)$, called **hom-object** such that for all triples $(p, q, r) \in \mathrm{Ob}(P) \times \mathrm{Ob}(P) \times \mathrm{Ob}(P)$ there exists a morphism

$$\circ_{p,q,r} : P(q, r) \otimes P(p, q) \to P(p, r)$$

called the **composition morphism**, and the following diagram expressing associativity of this composition commutes



and further for any object $p \in \mathrm{Ob}(P)$ there exists a morphism $j_p : I_C \to P(p, p)$ called the **identity morphism** such that the following diagram expressing properties of the unit also commutes



Definition 4.1.4. A **PROP** is a symmetric monoidal category $P = (P, \boxplus, S_P, I_P)$ (with tensor product denoted as $\boxplus$) enriched over symmetric strict monoidal category $\mathbf{Mod}_{\mathbb{K}} = (\mathbf{Mod}_{\mathbb{K}}, \otimes_{\mathbb{K}}, S_{Mod}, \mathbb{K})$ such that the objects of P can be identified with the set $\mathbb{N}_0$ and the tensor product of P satisfies

$$m \boxplus n = m + n$$

Let us now make an observation that this definition already contains everything we need.

[2]These identities can be expressed via commutative diagrams sometimes called *pentagonal* and *triangular*.

Remark 4.1.5. First notice that in the identification we have $I_P = 0$ and since any object of P can be identified as $m = 1^{\boxplus m}$, we have a structure of (Σ_m, Σ_n)-bimodule on any hom-object $P(m,n)$ induced by natural isomorphism S_P.

Vertical compositions of hom-objects are given from the definition of enrichment over category $\mathbf{Mod}_{\mathbb{K}}$ and the horizontal composition is given by the bifunctor $\boxplus$. Associativity of both of them is obvious.

The unit elements $P(n,n)$ for vertical composition are defined as horizontal composition of identity morphism $e \in P(1,1)$.

Definition 4.1.6. Let $P = \{P(m,n)\}_{m,n\geq 0}, Q = \{Q(m,n)\}_{m,n\geq 0}$ be two PROPs. A **homomorphism of PROPs** $h : P \to Q$ is a collection of bi-equivariant[3] maps $h = \{h_{m,n} : P(m,n) \to Q(m,n)\}_{m,n\geq 0}$ commuting with vertical and horizontal compositions.

Definition 4.1.7. An **ideal** in PROP is a collection $I = \{I(m,n)\}_{m,n\geq 0}$ of Σ_m-left invariant and Σ_n-right invariant subspaces closed under vertical and horizontal compositions with, i.e., for $f \in P(m,n), g \in P(n,k)$ is $f \circ g \in I(m,k)$ if $f \in I(m,n)$ or $g \in I(n,k)$.

Example 4.1.8. An *endomorphism* PROP over a $\mathbb{K}$-module V is a collection $End_V = \{End_V(m,n)\}_{m,n\geq 0}$ of multilinear maps $V^{\otimes n} \to V^{\otimes m}$. Vertical composition corresponds to composition of linear maps and horizontal composition to tensor product of linear maps. The unit element is the identity map $1_V \in End_V(1,1)$.

Definition 4.1.9. An **algebra over PROP** P (also denoted as a P-algebra) is a homomorphism of PROPs $h : P \to End_V$ for some $\mathbb{K}$-module V.

Similarly as we already discussed in the beginning of section 1.1, PROPs can be defined in two ways. We showed above the definition using categorical approach to an axiomatic definition. Let us now outline the second one using oriented graphs.

Definition 4.1.10. Let $E = \{E(m,n)\}_{m,n\geq 0}$ be a system of (Σ_m, Σ_n)-bimodules. Then E is a **Σ-bimodule**.

For finite sets X, Y such that $|Y| = m, |X| = n$ define

$$E(Y,X) = Bij(Y,[m]) \otimes_{\Sigma_m} E(m,n) \otimes_{\Sigma_n} Bij([n],X)$$

Definition 4.1.11. A **directed** (m,n)**-graph**, $m,n \geq 1$, is a directed graph where can be multiple edges but no directed cycles. Obviously the legs (half-edges) can be divided into two disjoint sets – the incoming legs, called **inputs**, and the outgoing legs, called **outputs**. An example of such (m,n)-graph is given on figure 4.1[4].

Let $\mathbf{Gr}(m,n)$ be the category of triplets (G, l_1, l_2) where G is an (m,n)-graph, l_1 is a bijection

$$\{\text{incoming half-edges of } G\} \to [n]$$

[3] $h(m,n)$ is 'left' equivariant for left action of Σ_m and 'right' equivariant for right action of Σ_n, hence for $p \in P(m,n), \sigma \in \Sigma_m, \tau \in \Sigma_n$ we have $h(m,n)(\sigma p \tau) = \sigma h(m,n)(p)\tau$.

[4] As was already observed in remark 1.2.16 and indicated in Figure 1.3 we can omit the external vertices. And again, we depict the direction of each edge as bottom-up orientation of its graphical representation.

and l_2 is a bijection

$$\{\text{outgoing half-edges of } G\} \to [m]$$

Morphisms of $\mathbf{Gr}(m, n)$ are isomorphism of graphs preserving labeling of half-edges.

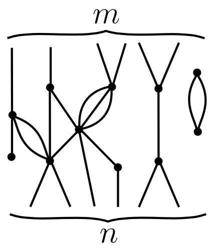


Figure 4.1: Example of (m, n)-graph

Let us consider an extension of the category $\mathbf{Gr}(m, m)$ by a special graph e_m with m-components having no internal vertices, visualized in Figure 4.2. The categories enlarged by such graphs are denoted as $U\mathbf{Gr}(m, n)$ (when $m \neq n$, then $U\mathbf{Gr}(m, n) = \mathbf{Gr}(m, n)$).

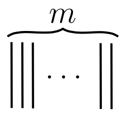


Figure 4.2: Graph $e_m \in U\mathbf{Gr}(m, m)$

For every vertex v we have a disjoint decomposition of adjacent edges into incoming edges $In(v)$ and outgoing edges $Out(v)$. Let us define for a graph $G \in U\mathbf{Gr}(m, n)$ and a Σ-bimodule E

(4.1) $$E(G) = \bigodot_{v \in vert(G)} E(Out(v), In(v))$$

$$\Gamma_p(E)(m, n) = \operatorname*{colim}_{G \in U\mathbf{Gr}(m,n)} E(G)$$

and functor $\Gamma_p : \Sigma\text{-}\mathbf{BiMod}_{\mathbb{K}} \to \{\text{PROPs}\}$

(4.2) $$\Gamma_p(E) = \{\Gamma_p(E)(m, n)\}_{m,n \geq 0}$$

Then $\Gamma_p(E)$ gives us a structure of PROP. Horizontal composition corresponds to disjoint union of graphs and vertical composition to joining graphs by identifying some incoming leaves with some outgoing leaves of two different graphs.

Remark 4.1.12. Note that graphs $G_1 \in U\mathbf{Gr}(m, n_1), G_2 \in U\mathbf{Gr}(n_2, k)$ can be composed, since we can first use the 'horizontal composition' with graphs e_p, e_q (of appropriate numbers of components $p, q \geq 0$) and then horizontal composition.

Definition 4.1.13. PROPs are algebras over the monad $U_{\Gamma_p} \circ \Gamma_p$, where U_{Γ_p} denotes the appropriate forgetful functor.

Now let us consider only *connected* graphs in $U\mathbf{Gr}(m,n)$ such that every vertex has at least one incoming edge and at least one outgoing edge. Let us denote the category of such graphs as $U\mathbf{Gr}^C(m,n)$. Then

$$\Gamma_p^C(E)(m,n) = \underset{G \in U\mathbf{Gr}^C(m,n)}{\text{colim}} E(G)$$

$$\Gamma_p^C(E) = \{\Gamma_p^C(E)(m,n)\}_{m,n\geq 0} \tag{4.3}$$

and we finally get

Definition 4.1.14. Properads are algebras over the monad $U_{\Gamma_p^C} \circ \Gamma_p^C$, where $U_{\Gamma_p^C}$ denotes the forgetful functor.

Similarly as in operads we can define a non-unital properad. Let us consider connected graphs in $\mathbf{Gr}(m,n)$ (such that every vertex has at least one incominng edge and at least one outgoing edge) denoted as $\mathbf{Gr}^C(m,n)$. Then

$$\Psi_p^C(E)(m,n) = \underset{G \in \mathbf{Gr}^C(m,n)}{\text{colim}} E(G)$$

and functor $\Gamma_p^C : \Sigma\text{-}\mathbf{BiMod}_{\mathbb{K}} \to \{\text{properads}\}$

$$\Psi_p^C(E) = \{\Psi_p^C(E)(m,n)\}_{m,n\geq 0} \tag{4.4}$$

Definition 4.1.15. Non-unital properads are algebras over the monad $U_{\Psi_p^C} \circ \Psi_p^C$, where $U_{\Psi_p^C}$ denotes the appropriate forgetful functor.

Remark 4.1.16. Since properads are over connected graphs, horizonal composition makes no sense for them. Vertical composition will be denoted as usualy by the symbol '$\circ$' and

$$\circ : P(m_1,n_1) \otimes P(m_2,n_2) \to P(k,l)$$

where $m_1 \leq k \leq m_1 + m_2 - 1, n_2 \leq l \leq n_1 + n_2 - 1$.

Similarly as for PROP, we can define the homorphism of properads, ideal of properad, endomorphism properad.

Remark 4.1.17. It is obvious that operads are just special cases of properads for $m = 1$.

Example 4.1.18. A *Frobenius bialgebra* can be defined (as in [Dun15]) as a vector space V with associative product $\widetilde{\alpha} : V \otimes V \to V$ and coassociative coproduct $\widetilde{\beta} : V \to V \otimes V$ satisfying Frobenius law

$$(1_A \otimes \widetilde{\alpha}) \circ (\widetilde{\beta} \otimes 1_A) = \widetilde{\beta} \circ \widetilde{\alpha} = (\widetilde{\alpha} \otimes 1_A) \circ (1_A \otimes \widetilde{\beta})$$

If we have moreover mapping $S : V \otimes V \to V \otimes V$ such that $S(x \otimes y) = y \otimes x$ for every $x, y \in V$ and $\widetilde{\alpha} \circ S = \widetilde{\alpha}$, $S \circ \widetilde{\beta} = \widetilde{\beta}$, then we have *symmetric* Frobenius bialgebra.

Symmetric Frobenius algebra can be considered as an algebra over Frobenius properad (defined in [CMW14]). Let us denote this properad as $Frob$. Then $Frob = \Gamma_p^C(E_{Frob})/R_{Frob}$ where E_{Frob} is a (Σ_m, Σ_n)-bimodule

$$E_{Frob}(m,n) = \begin{cases} \mathbb{K}\cdot\alpha & \text{if } m=1, n=2 \\ \mathbb{K}\cdot\beta & \text{if } m=2, n=1 \\ 0 & \text{otherwise} \end{cases}$$

where α, β are trivial representations of Σ_2 and the ideal R_{Frob} is displayed on Figure 4.3. The ideal R_{Frob} is shortly written as

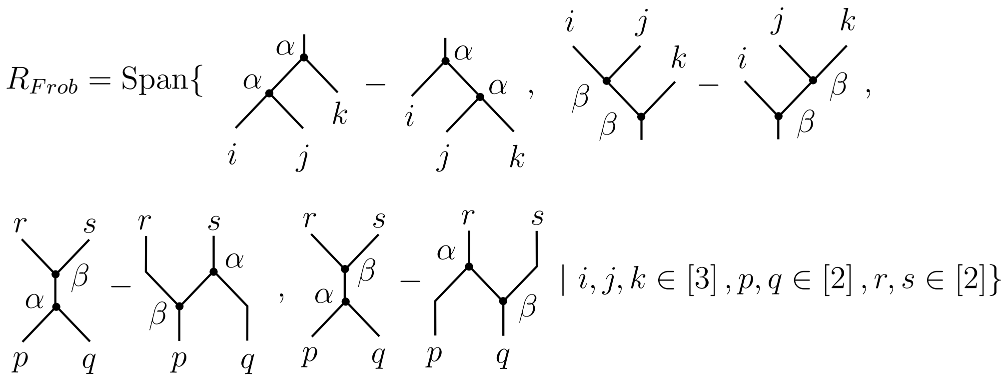


Figure 4.3: Ideal R_{Frob}

$$R_{Frob} = \text{Span}\{(\alpha\,{}_1\!\circ\alpha)-(\alpha\,{}_2\!\circ\alpha), (\beta\circ_1\beta)-(\beta\circ_2\beta), (\beta\circ\alpha)-(\alpha\,{}_1\!\circ_2\beta), (\beta\circ\alpha)-(\alpha\,{}_2\!\circ_1\beta)\}$$

where we use indices if we need to distinguish into which input we paste the result from the previous operation or which output of the previous operation is used.

Remark 4.1.19. Note that $Frob(m,n)$ is infinite dimensional for every $m, n \geq 1$. For example for $m = 1 = n$ we can construct elements as composition

$$(\alpha\,{}_1\!\circ\alpha\,{}_1\!\circ\ldots{}_1\!\circ\alpha)\,{}_{1,2\ldots n}\!\circ_{1,2,\ldots}(\beta\circ_1\beta\circ_1\ldots\circ_1\beta)$$

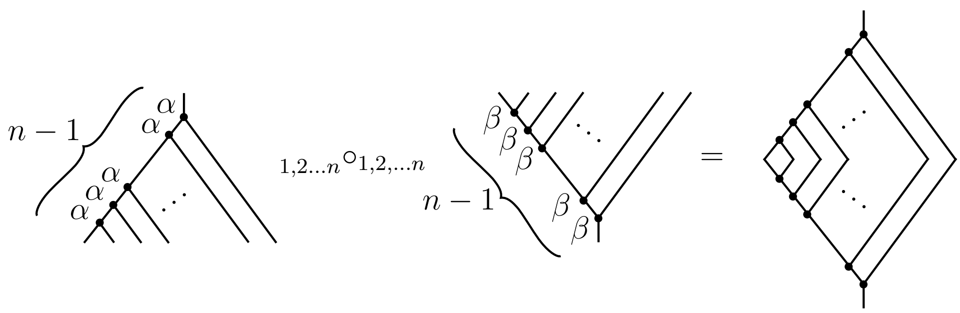


Figure 4.4: Element of $Frob(1,1)$

The replacement, for example, $\alpha\,{}_1\!\circ\alpha$ by element $\alpha\,{}_2\!\circ\alpha$ is only ‘deformation’ of the graph and it will not change the genus. Then by composition with an appropriate number of graphs decorated by $\mathbb{K}\cdot\alpha$ and $\mathbb{K}\cdot\beta$ we can get an element of arbitrary component $Frob(m, n)$. Hence every component $Frob(m, n)$ is infinite dimensional.

But we can define the genus of these graphs in the same way as for modular operads in 1.11. Then every component $Frob(m, n)$ can be considered as the disjoint union of components $Frob(m, n, g)$, where g denotes the genus of the underlying graph. As was mentioned in [AMT14] every component $Frob(m, n, g)$ is one dimensional for every $m, n \geq 1, g \geq 0$. The idea of identification of such elements from the same component $Frob(m, n, g)$ for different graphs is shown in Figure 4.5.

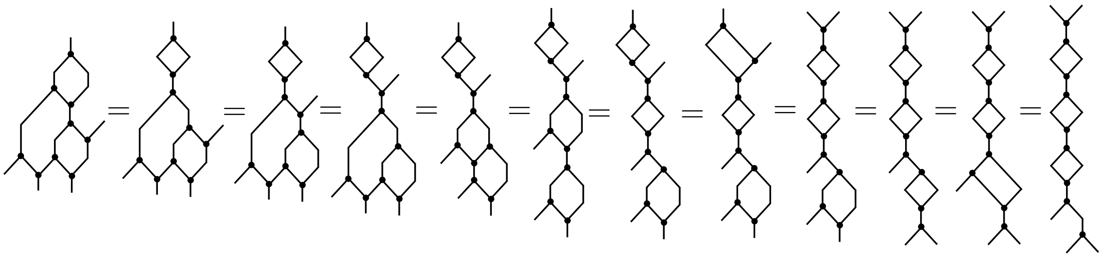


Figure 4.5: Equivalence of some elements of $Frob(2, 3, 3)$

Therefore we take as an definition of Frobenius properad the following theorem.

Theorem 4.1.20. Frobenius properad is a bimodule

$$Frob = \{Frob(m, n, g)\}_{m,n \geq 1, g \geq 0}$$

such that $dim(Frob(m, n, g)) = 1$ such that every component $Frob(m, n, g)$ has trivial left action of Σ_m and trivial right action of Σ_n

4.2 Cobar complex of properads

The idea of cobar complex construction is the same as for operads in section 2.1. The construction comes from dualization and we want to omit the properad unit in the construction. However to make clear how the construction works, we show it explicitly on Frobenius properad.

We will again use the definition of cobar complex as a result of the functor C. We are following mainly [DCTT10] and [Val07]. Some details can be found also in [MV09].

Remark 4.2.1. In the construction of the operad cobar complex we required components $P(n)$ to be finite dimensional and we omitted the components $P(0)$ and $P(1)$. Then we dualized the rest of components separately and the 'dual' was defined as $P^{\#} = \{(P(n))^{\#}\}$ for $n \geq 2$.

We showed in 4.1.19 that every component $Frob(m, n)$ is infinite dimensional but we can split it into one-dimensional components $Frob(m, n, g)$.

We define the dual of properad $Frob$ as

$$Frob^{\#} = \begin{cases} Frob(m, n, g)^{\#} & \text{if } (m, n, g) \neq (1, 1, 0) \\ 0 & \text{if } (m, n, g) = (1, 1, 0) \end{cases}$$

In the next we will use only $\mathbf{Gr}^C(m,n)$, which is the category of connected directed graphs without directed cycles such that every vertex has at least one incoming edge and at least one outgoing edge (without the special graphs e_m).

Remark 4.2.2. Left Σ_m and right Σ_n actions on the bimodule $Frob(m,n,g)$ correspond to left Σ_m and right Σ_n actions on $Frob^{\#}(m,n,g)$ in the same way as we have shown in 2.1.3. Since the actions are trivial on $Frob(m,n,g)$, they are trivial also on $(Frob(m,n,g))^{\#}$.

Similarly as for operads we want to transform the map $\circ$ from the properad $Frob$ into a differential of degree one on the cobar complex $C(Frob)$ of the properad $Frob$. Hence we want to implement grading on elements $Frob^{\#}(G)$ (defined for Σ-bimodule $Frob^{\#}$ as in (4.1)).

For operads the grading was introduced via the set of internal edges. For operads, the map $\circ^{\#}$ gave us always only one new edge. Therefore the raising of degree was in correspondence with raising of number of edges.

In properads, the map $\circ$ can join two vertices by more than one edge simultaneously, and therefore the map $\circ^{\#}$ can split one vertex into two new vertices connected with more than one edge as indicated in Figure 4.6[5]. But there are some restrictive conditions which the map $\circ^{\#}$ must satisfy.

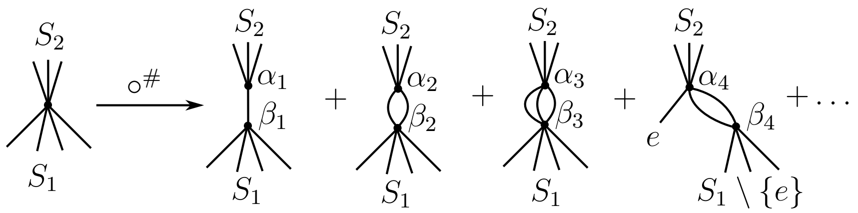


Figure 4.6: Map $\circ^{\#}$

Similarly as in Section 2.1, let us denote by the symbol $\circ^{\#}_{G,G'}$ a map given by the composition of the map $\circ^{\#}$ acting on $Frob^{\#}(G)$ with projection $proj_{G'}$ to one component $Frob^{\#}(G')$ of $\circ^{\#}(Frob^{\#}(G))$.

Remark 4.2.3. First, after the action of the map $\circ^{\#}_{G,G'}$ 'on vertex' decorated by an element from $Frob^{\#}(m,n,g)$ we will get graph G' with two new vertices instead of the old one, as indicated in Figure 4.7. These vertices will be connected by k edges and decorated by elements from $Frob^{\#}(m_1,n_1,g_1)$ and $Frob^{\#}(m_2,n_2,g_2)$.

Obviously $m_1+m_2=m+k$ and $n_1+n_2=n+k$. But it is not generally true that $g_1+g_2=g$. If $k>1$, then the new k edges create $k-1$ holes in the graph. But we want the genus of the graph G' to be the same as genus of the graph G. Hence we should have $g_1+g_2+k-1=g$. For this reason we introduce the following definition.

Definition 4.2.4. The **character** $\chi(v)$ **of vertex** v is defined as

$$\chi(v)=2g(v)+m+n-2$$

where $g(v)$ is the genus of the vertex v.

[5]Note that since $Frob^{\#}(m,n,g)$ are one dimensional with trivial (Σ_m,Σ_n) actions, we can omit sums over the basis. In the general case we have to consider sums over combinations of basis elements as in the Figure 2.1.

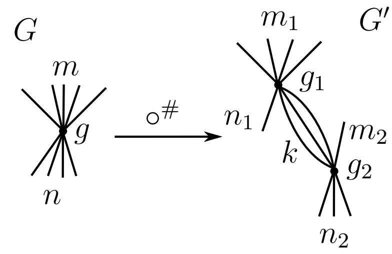


Figure 4.7: New vertices decorated by elements from $Frob^{\#}(m_1, n_1, g_1)$, $Frob^{\#}(m_2, n_2, g_2)$ created by the map $\circ^{\#}_{G,G'}$ from the vertex decorated by an element from $Frob^{\#}(m, n, g)$

If we add characters of vertices, we get

$$\chi_1 + \chi_2 = 2g(v_1) + m_1 + n_1 - 2 + 2g(v_2) + m_2 + n_2 - 2 =$$
$$= 2(g(v_1) + g(v_2) + k - 1) + m + n - 2 = 2g(v) + m + n - 2 = \chi$$

From this reason we will work in the following with characters (instead of genus). Obviously we can change the notation from $Frob^{\#}(m, n, g)$ to

$$Frob^{\#}(m, n, \chi) = Frob^{\#}(m, n, 2g + m + n - 2)$$

such that $\chi \geq m + n - 2$ and χ is of the same parity as $m + n$. The components $Frob^{\#}(m, n, \chi)$ are then defined unambiguously and without loss of uniqueness.

Remark 4.2.5. Note that we have already worked with this property of characters in Section 1.4. The characterization of stable graph in (1.13) is equivalent to the condition $0 < \chi(v)$ for all vertices v.

Remark 4.2.6. Secondly, the two new vertices v_1, v_2 must be connected by at least one edge. Hence we cannot get the situation from Figure 4.9.

Since the two new vertices are connected with at least one directed edge and the directed cycles are not allowed, we can determine one of these two new vertices as the 'incoming' one (in our figures it will be the lower one). We denote it as v_i. The 'outgoing' vertex is denoted as v_o.

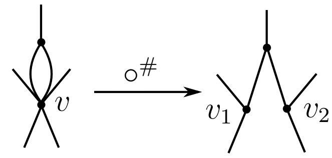


Figure 4.8: Impossible situation

Remark 4.2.7. The addition of k edges causes that we cannot use the determinant of graph in the same way as it was defined for trees in the case of operads

$$det(T) = \uparrow det(edge(T)) = \bigwedge^{|S_e|}(\uparrow \mathbb{K}^{S_e})$$

where S_e denotes a set of inner edges of tree T. But we can define the determinant of a graph $G \in \mathbf{Gr}^C(m, n, g)$ as

$$\widetilde{det}(G) = det(Vert(G)) = \uparrow \bigwedge^{|Vert(G)|} (\uparrow \mathbb{K}^{Vert(G)})$$

Notice that a graph with only one vertex has degree 1. If we take operads as a special case of properads, then a tree with p internal edges (and therefore $p+1$ vertices) has degree $p+1$ defined via determinant det (the additional 1 is for the root edge, which is not counted as internal edge) and $p+1$ via determinant $\widetilde{det}$.

Remark 4.2.8. The final observation is about the partition of half-edges and edges adjacent to a vertex v among vertices v_1, v_2.

(Σ_m, Σ_n) actions are trivial on $Frob^{\#}(m,n)$. Therefore if we have a graph with only one vertex, we can take it as planar graph where the labeling of half-edges plays no role. Hence we can take n incoming half-edges as labeled by the set $[n]$ from left to right. And similarly for outgoing half-edges.

The map $\circ^{\#}_{G,G'}$ divides the set S_1 of incoming half-edges and the set S_2 of outgoing half-edges between the vertices v_1, v_2. But these vertices are again decorated by elements of $Frob^{\#}$. Hence the labeling should not play role. Hence the set of incoming half-edges $\{e_{i_1}, e_{i_2}, \ldots e_{i_l}\}$ adjacent to v_2 can be arbitrarily permuted and we can take it as ordered from left to right. And the same holds for the set $\{e_{l+1}, \ldots e_k\}$ of incoming half-edges adjacent to v_1. Such an idea is recorded in concept of *unshuffles*.

Definition 4.2.9. An **unshuffle** τ of type (k,l) for $l \leq k$, denoted as $\tau \in UnSh(k,l)$, is an element of Σ_k such that for $i_1 < i_2 < \ldots < i_l$, $i_{l+1} < \ldots < i_k$ we have $\tau(i_j) = j$.

The same idea should hold also for outgoing half-edges but we need an inverse process of labeling, hence we use *shuffles*.

Definition 4.2.10. A **shuffle** σ of type (k,l) for $l \leq k$, denoted as $\sigma \in Sh(k,l)$, is an element of Σ_k such that $\sigma(1) < \sigma(2) < \ldots < \sigma(l)$ and $\sigma(l+1) < \ldots \sigma(k)$.

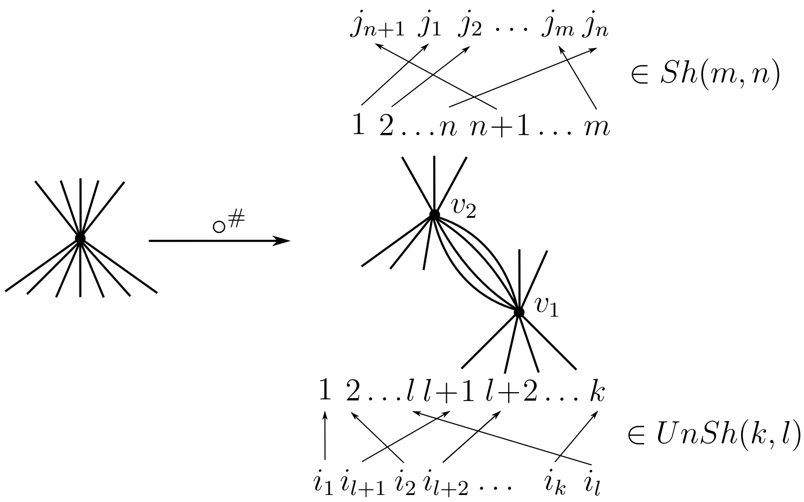


Figure 4.9: Partitions of half-edges

Definition 4.2.11. The **cobar complex** of the properad $Frob$ is for a triple (m,n,χ) defined as a graded (Σ_m, Σ_n)-bimodule

$$C(Frob)(m,n,\chi) = \bigoplus_{i=1}^{\infty} C(Frob)(m,n,\chi)^i$$

such that $C(Frob)(1,1,0) = 0$ and for the rest of combinations of $m, n \geq 1, \chi \geq 0$

$$C(Frob)(m,n,\chi) = \operatorname*{colim}_{\substack{G \in \mathbf{Gr}^C(m,n) \\ Vert(G)=i}} Frob^{\#}(G) \otimes \widetilde{det}(G)$$

otherwise[6] $C(Frob)(m,n,\chi) = 0$, such that the differential

$$d : \operatorname*{colim}_{\substack{G \in \mathbf{Gr}^C(m,n) \\ Vert(G)=i}} Frob^{\#}(G) \otimes \widetilde{det}(G) \to \operatorname*{colim}_{\substack{G' \in \mathbf{Gr}^C(m,n) \\ Vert(G')=i+1}} Frob^{\#}(G') \otimes \widetilde{det}(G')$$

is the unique extension of $\circ^{\#}(\cdot) \otimes (v_i \wedge \cdot)$ compatible with colimits over category $\mathbf{Gr}^C(m,n)$ where v_i is the 'incoming' vertex from the pair of new vertices and the 'outgoing' vertex v_o is used instead of the original vertex v in the set $Vert(G)$ in $\widetilde{det}(G)$.

Let us show the idea, what is d^2 on $G \in \mathbf{Gr}^C(1,2)$ with only one vertex decorated by the element $Frob^{\#}(1,2,3)$ and incoming half-edges labeled by e_1, e_2 (there is only one outgoing half-edge, hence we do not need a labeling for it).

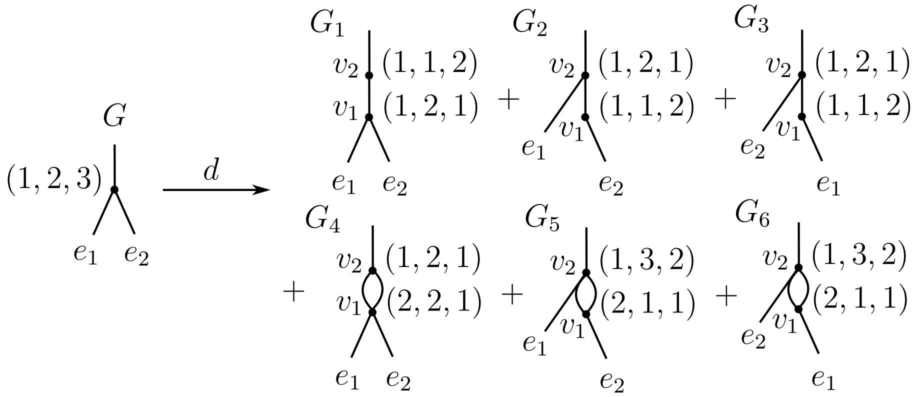


Figure 4.10: Action of d on decorated graph G

Acting by d gives us six different graphs as indicated on Figure 4.10[7]. All these graphs have determinant $\widetilde{det}(G_i) = v_1 \wedge v_2$.

Acting by d on these graphs gives us three different graphs shown on Figure 4.11. The graph G_1 gives us decorated graph H_1 with determinant $v_3 \wedge v_1 \wedge v_2$ and the same graph G_4 with determinant $v_1 \wedge v_2 \wedge v_3$. Hence they are together zero. Similarly G_2 and G_5 give us H_2 first with determinant $v_1 \wedge v_2 \wedge v_3$ and secondly with determinant $v_2 \wedge v_1 \wedge v_3$. Together zero. And finally G_3 and G_6 give us H_3, first with determinant $v_1 \wedge v_2 \wedge v_3$ and the second one with determinant $v_2 \wedge v_1 \wedge v_3$. Also together zero.

Lemma 4.2.12. The cobar complex $C(Frob)$ is a cochain complex.

Proof. The condition $d^2 = 0$ comes again from coassociativity of the map $\circ^{\#}$ on $Frob^{\#}$ and from the grading given by suspension.

□

[DCTT10] shows that a similar theorem as 2.1.13 holds.

[6] For cases when $m < 1, n < 1, \chi < 0, \chi < m + n - 2$ or χ is not of the same parity as $m + n$.

[7] For short we write only (m, n, χ) instead of $Frob^{\#}(m, n, \chi)$.

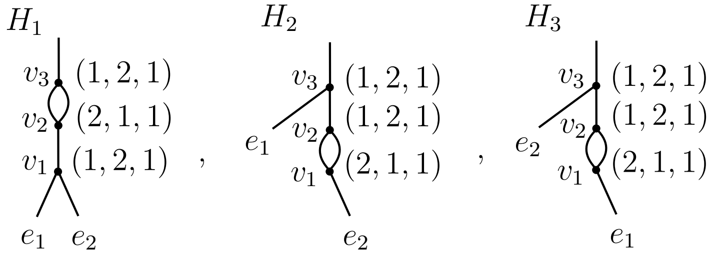


Figure 4.11: Action of d on decorated graphs $G_1, G_2, \dots G_6$

Theorem 4.2.13. There is an isomorphism of vector spaces between the free non-unital properad $\Psi_p^C(\uparrow Frob^{\#})(m,n,\chi)$ and $C(Frob)(m,n,\chi)$ for $m,n \geq 1$, $\chi \geq m+n-2$, χ of the same parity as $m+n$.

4.3 Algebras over $C(Frob)$

Similarly as in section 1.5 we can define differential graded properads.

Definition 4.3.1. A **differential graded Σ-bimodule** is a collection

$$A = \{A(m,n)\}_{m,n\geq 0}$$

of $(\mathbb{K}[\Sigma_m], \mathbb{K}[\Sigma_n])$-bimodules such that $A(m,n) \in dgVec$ and differentials

$$d(n)^i : A(m,n)^i \to A(m,n)^{i+1}$$

are (Σ_m, Σ_n)-biequivariant, i.e.

$$d(n)^i(\alpha v \beta) = \alpha \left(d(n)^i(v)\right) \beta$$

for $v \in A(m,n), \alpha \in \Sigma_m, \beta \in \Sigma_n$.

Definition 4.3.2. A **differential graded properad**[8] is a differential graded Σ-bimodule $A = \{A(m,n)\}_{m,n\geq 0}$ with composition maps defined in Section 4.1

$$\circ : A(m,n)^k \otimes A(n,p)^l \to A(m,p)^{k+l}$$

Remark 4.3.3. Similarly as in Remark 2.2.2 let us consider the about endomorphism properad $End_W^p = \{End_W^p(m,n)\}_{m,n\geq 1}$ over a dg vector space (W, d_W).

The degree of a map $f \in End_W^p(m,n)$, i.e. $f : W^{\otimes n} \to W^{\otimes m}$, is defined as $f : (W^{\otimes n})^k \to (W^{\otimes m})^{k+deg(f)}$.

Now we can define the differential d_{End^p} of the properad End_W^p as

$$d_{End^p}(f) = \sum_{j=1}^{m} d_W \,{}_1\circ_j f - (-1)^{|f|\cdot|d_W|} \sum_{i=1}^{n} f \,{}_i\circ_1 d_W \tag{4.5}$$

where

$$d_W \,{}_1\circ_j f = \left(1^{\otimes j-1} \otimes d_W \otimes 1^{\otimes m-j}\right) \circ f$$
$$f \,{}_i\circ_1 d_W = f \circ \left(1^{\otimes i-1} \otimes d_W \otimes 1^{\otimes n-i}\right)$$

It is easy to check that $d_{End^p}^2 = 0$. Hence End_W^p is a dg properad.

[8]Shortly dg properad

Definition 4.3.4. The degree 0 homomorphism $h : C(Frob) \to End^p_W$ of dg properads $(C(Frob), d)$ and (End^p_W, d_{End^p}), i.e.,

$$h \circ d = d_{End^p} \circ h \tag{4.6}$$

is called $C(Frob)$**-algebra**.

Remark 4.3.5. It is maybe not easy to see how many different graphs we can actually get by acting by d.

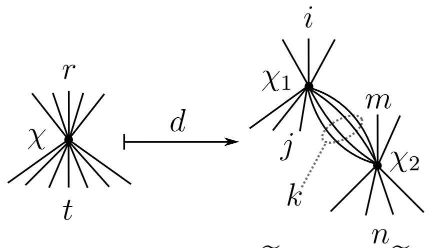
From Theorem 4.2.13 we know that $C(Frob)$ can be seen as the free properad over the Σ-bimodule $\uparrow Frob^{\#}$. The differential can be again considered as the composition of maps $(\uparrow \otimes \uparrow) \circ d \circ \downarrow$ without taking determinant into account. By similar argument as in Theorem 3.26, since all elements of $Frob^{\#}$ have degree zero, we have

$$(\uparrow \otimes \uparrow)(\alpha_1 \otimes \alpha_2) = (-1)^{|\uparrow|\cdot|\alpha_1|}(\uparrow \alpha_1 \otimes \uparrow \alpha_2) = (\uparrow \alpha_1 \otimes \uparrow \alpha_2)$$

for $\alpha_1, \alpha_2 \in Frob^{\#}$.

Since components of $Frob^{\#}$ are one dimensional, we can take $(Frob(r,t,\chi))^{\#}$ as a graph with only one vertex with incoming half-edges labeled by the set $[r]$ from left to right, outgoing half-edges labeled by the set $[t]$ from left to right and of characteristic χ. Such element will be denoted as $1_{r,t,\chi}$.

Our next goal is to determine the $d(\uparrow 1_{r,t,\chi})$. Acting by d on this element we get graphs with two vertices connected by k edges with corresponding decorations and labeling. Let us denote this for a moment as the unordered tensor product of two elements $\uparrow \widetilde{1}_{i,j,\chi_2} \odot \uparrow \widetilde{1}_{m,n,\chi_1}$ such that $\widetilde{1}_{i,j,\chi_2}$ is the decoration of outgoing vertex and $\widetilde{1}_{m,n,\chi_1}$ that of incoming vertex, but the order of labelings of half-edges is not specified. Let us determine now the restrictions on the numbers of half-edges and possible characteristics.



$\uparrow 1_{r,t,\chi}$ $\qquad \sum_{i,k,j,\chi_1} \uparrow \widetilde{1}_{i,j,\chi_2} \odot \uparrow \widetilde{1}_{m,n,\chi_1}$

Figure 4.12: Conditions on decorations of $d(1_{r,t,\chi})$

Obviously $i+m-k=r, j+n-k=t, \chi_1+\chi_2=\chi$. The number i of outgoing half-edges from an outgoing vertex must be greater than 1 but smaller than the number r of all outgoing half-edges. Hence the number of outgoing half-edges from outgoing vertex must be

$$1 \le i \le r$$

The new vertices must be connected, hence $k \ge 1$. k is maximal when the characteristics χ_1, χ_2 are minimal, this occurs when $\chi_1 = i+j-2$, $\chi_2 = m+n-2$. But we know $\chi = \chi_1 + \chi_2 = r+t+2k-4$. Hence the number k of connecting edges must satisfy

$$1 \le k \le \frac{1}{2}(\chi - r - t) + 2$$

Since the number of edges connecting the vertices is included in the number j of all incoming half-edges into an outgoing vertex, we get $k \leq j$. But there must be at least one incoming half-edge into an incoming vertex. Together this gives us the condition on j

$$k \leq j \leq t + k - 1$$

The characteristic χ_1 must be at least $i + j - 2$ by definition. In a graph with k connecting edges is χ_1 maximal for χ_2 minimal, i.e. for $\chi_2 = m + n - 2$. Hence the maximal χ_1 is given as $\chi_1 = \chi - \chi_2 \leq \chi - m - n + 2$. Similarly as in the previous we know that $m \geq k$ and $n \geq 1$. Therefore

$$i + j - 2 \leq \chi_1 \leq \chi - k + 1$$

for $k > 1$. If $k = 1$ then the incoming vertex would have decoration from $Frob^{\#}(1, 1, 0)$, which is not allowed. Hence we have the condition

$$i + j - 2 \leq \chi_1 \leq \min\{\chi - 1, \chi - k + 1\}$$

These all conditions together with the observation in 4.2.8 give us

$$d(\uparrow 1_{r,t,\chi}) = \tag{4.7}$$

$$= \sum_{i=1}^{r} \sum_{k=1}^{\left(\frac{1}{2}(\chi-r-t)+2\right)} \sum_{j=k}^{(t+k-1)} \sum_{\chi_1=i+j-2}^{\min\{\chi-1,\chi-k+1\}} \sum_{\substack{\tau\in Sh(r,r-i)\\ \sigma\in UnSh(t,t-j)}} \tau \left(\uparrow \widetilde{1}_{i,j,\chi_1} \odot \uparrow \widetilde{1}_{r+k-i,t+k-j,\chi-\chi_1}\right) \sigma$$

The element $\uparrow \widetilde{1}_{i,j,\chi_1}$ corresponds to a vertex decorated by the element $\uparrow 1_{i,j,\chi_1}$ with outgoing half-edges labeled by the set $[i]$ from left to right and incoming half-edges labeled from left to right by the set

$$S_1 = \{1, 2, \dots j - k, a_1, a_2 \dots a_k\}$$

which is isomorphic to the set $[j]$ via bijection $\beta_1 : S_1 \to [j]$

$$\beta_1(p) = \begin{cases} p & \text{if } p \in \{1, 2, \dots j - k\} \\ j - k + w & \text{if } p = a_w \end{cases}$$

The element $\uparrow \widetilde{1}_{r+k-i,t+k-j,\chi-\chi_1}$ corresponds to a vertex decorated by the element $\uparrow 1_{r+k-i,t+k-j,\chi-\chi_1}$ with outgoing half-edges labeled from left to right by set

$$S_2 = \{b_1, b_2, \dots b_k, i + 1, i + 2, \dots r\}$$

which is isomorphic to the set $[r + k - i]$ via bijection $\beta_2 : S_2 \to [r + k - i]$

$$\beta_2(p) = \begin{cases} w & \text{if } p = b_w \\ p - i + k & \text{if } p \in \{i + 1, i + 2, \dots r\} \end{cases}$$

and incoming edges labeled from left to right by the set $S_3 = \{j - k + 1, j - k + 2, \dots t\}$ which is isomorphic to the set $[t + k - j]$ via bijection $\beta_3 : S_3 \to [t + k - j]$

$$\beta_3(p) = p + k - j$$

We need not consider all possible connections of half-edges labeled by $a_1, \dots a_k$ and $b_1, \dots b_k$.

Theorem 4.3.6. The structure of $C(Frob)$-algebra corresponds to the Maurer-Cartan equation of form

$$0=\sum_{i,k,j}\sum_{\chi_1=i+j-2}^{\chi-k+1}\sum_{\substack{\tau\in Sh(r,r-i)\\ \sigma\in UnSh(t,t-j)}}\tau\left(\left(m_{i,j,\chi_1}\right)_{j-k+1,\dots j}\odot_{1,\dots k}\left(m_{r+k-i,t+k-j,\chi-\chi_1}\right)\right)\sigma$$

for $1\le i\le r, 1\le k\le\left(\frac{1}{2}(\chi-r-t)+2\right), k\le j\le(t+k-1)$ where the symbol ${}_{i_1,\dots i_n}\circ_{j_1,\dots j_n}$ denotes the pairing of output j_α of $m_{r+k-i,t+k-j,\chi-\chi_1}$ with input i_α of m_{i,j,χ_1}.

Proof. Since $C(Frob)$ can be considered as the free properad over the Σ-bimodule $\uparrow Frob^{\#}$ (see Theorem 4.2.13), it is enough to define the homomorphism only on generators of the free properad. Hence let us define

$$h(1_{r,t,\chi})=m_{r,t,\chi}:W^{\otimes t}\to W^{\otimes r}$$

The right hand side of (4.6) is

$$d_{End^p}\circ h\left(\uparrow 1_{r,t,\chi}\right)=d_{End^p}\left(m_{r,t,\chi}\right)= \tag{4.8}$$

$$=\sum_{j=1}^{r}d_{W\ 1}\circ_j m_{r,t,\chi}-(-1)^{|m_{r,t,\chi}|\cdot|d_W|}\sum_{i=1}^{t}m_{r,t,\chi\ i}\circ_1 d_W$$

Since $|\uparrow 1_{r,t,\chi}|=1$ and $|h|=0$, we get $|m_{r,t,\chi}|=1$.

In Remark 4.3.5 we showed what are the elements $d(\uparrow 1_{r,t,\chi})$. The action of the homomorphism h gives us the left hand side

$$h\circ d\left(1_{r,t,\chi}\right)=h\left(\sum_{i,k,j,\chi}\sum_{\substack{\tau\in Sh(r,r-i)\\ \sigma\in UnSh(t,t-j)}}\tau\left(\uparrow\widetilde{1}_{i,j,\chi_1}\odot\uparrow\widetilde{1}_{r+k-i,t+k-j,\chi-\chi_1}\right)\sigma\right)= \tag{4.9}$$

$$=\sum_{i,k,j,\chi}\sum_{\substack{\tau\in Sh(r,r-i)\\ \sigma\in UnSh(t,t-j)}}\tau\left(\left(m_{i,j,\chi_1}\right)_{j-k+1,\dots j-1,j}\odot_{1,2,\dots k}\left(m_{r+k-i,t+k-j,\chi-\chi_1}\right)\right)\sigma$$

If we denote $d_W=-m_{1,1,0}$, then (4.8) and (4.9) give us

$$0=\sum_{j=1}^{r}m_{1,1,0\ 1}\circ_j m_{r,t,\chi}+\sum_{i=1}^{t}m_{r,t,\chi\ i}\circ_1 m_{1,1,0}+ \tag{4.10}$$

$$+\sum_{i,k,j,\chi}\sum_{\substack{\tau\in Sh(r,r-i)\\ \sigma\in UnSh(t,t-j)}}\tau\left(\left(m_{i,j,\chi_1}\right)_{j-k+1,\dots j-1,j}\odot_{1,2,\dots k}\left(m_{r+k-i,t+k-j,\chi-\chi_1}\right)\right)\sigma=$$

$$=\sum_{i,k,j}\sum_{\chi_1=i+j-2}^{\chi-k+1}\sum_{\substack{\tau\in Sh(r,r-i)\\ \sigma\in UnSh(t,t-j)}}\tau\left(\left(m_{i,j,\chi_1}\right)_{j-k+1,\dots j}\odot_{1,\dots k}\left(m_{r+k-i,t+k-j,\chi-\chi_1}\right)\right)\sigma$$

for $1\le i\le r, 1\le k\le\left(\frac{1}{2}(\chi-r-t)+2\right), k\le j\le(t+k-1)$. □

Remark 4.3.7. The modular operad QC (quantum closed operad) defined in [DJM13] consists of homeomorphic classes of connected two dimensional compact orientable surfaces with labeled boundary components (for short Riemann surfaces).

The Riemann surface with the set X of the boundary components is determined by the genus of the surface, similarly as components $Frob(m, n, g)$ of Frobenius properad. The components of $QC(X, g)$ are one-dimensional spaces generated by X^g and the graph decorated by QC, $QC(X, g)$, corresponds to graph decorated by $Frob(m, n, g)$ (without component $Frob(1, 1, 0)$) for $|X| = m + n$ without taking an orientation into account. Hence we can compare the algebra over $F(QC)$ with algebra over $C(Frob)$.

Bibliography

[AMT14] H. Abbaspour, M. Marcolli, and T. Tradler. *Deformation Spaces: Perspectives on algebro-geometric moduli*. Aspects of Mathematics. Vieweg+Teubner Verlag, 2014.

[CMW14] R. Campos, S. Merkulov, and T. Willwacher. The frobenius properad is koszul. `http://arxiv.org/abs/1402.4048`, 2014.

[DCTT10] Gabriel C. Drummond-Cole, John Terilla, and Thomas Tradler. Algebras over $\Omega(\mathrm{co}\mathbf{Frob})$. *J. Homotopy Relat. Struct.*, 5(1):15–36, 2010.

[DCV13] Gabriel C. Drummond-Cole and Bruno Vallette. The minimal model for the Batalin-Vilkovisky operad. *Selecta Math. (N.S.)*, 19(1):1–47, 2013.

[DJM13] M. Doubek, B. Jurco, and K. Muenster. Modular operads and the quantum open-closed homotopy algebra. `http://arxiv.org/abs/1308.3223`, 2013.

[DL14] M. Doubek and T. Lada. Homotopy derivations. `http://arxiv.org/abs/1409.1691`, 2014.

[Dun15] Ross Duncan. Frobenius algebra. `https://ncatlab.org/nlab/show/Frobenius+algebra`, 2015. [Online; accessed 12-May-2016].

[Get09] Ezra Getzler. Operads revisited. In *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. I*, volume 269 of *Progr. Math.*, pages 675–698. Birkhäuser Boston, Inc., Boston, MA, 2009.

[GK94] Victor Ginzburg and Mikhail Kapranov. Koszul duality for operads. *Duke Math. J.*, 76(1):203–272, 1994.

[GK98] E. Getzler and M. M. Kapranov. Modular operads. *Compositio Math.*, 110(1):65–126, 1998.

[Kaj02] Hiroshige Kajiura. Homotopy algebra morphism and geometry of classical string field theories. *Nuclear Phys. B*, 630(3):361–432, 2002.

[Kaj07] Hiroshige Kajiura. Noncommutative homotopy algebras associated with open strings. *Rev. Math. Phys.*, 19(1):1–99, 2007.

[KS06a] Hiroshige Kajiura and Jim Stasheff. Homotopy algebras inspired by classical open-closed string field theory. *Comm. Math. Phys.*, 263(3):553–581, 2006.

[KS06b] Hiroshige Kajiura and Jim Stasheff. Open-closed homotopy algebra in mathematical physics. *J. Math. Phys.*, 47(2):023506, 28, 2006.

[LV12] Jean-Louis Loday and Bruno Vallette. *Algebraic operads*, volume 346 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2012.

[Mar06] M. Markl. Operads and props. http://arxiv.org/abs/math/0601129, 2006.

[Mil11] Joan Millès. André-Quillen cohomology of algebras over an operad. *Adv. Math.*, 226(6):5120–5164, 2011.

[MSS02] Martin Markl, Steve Shnider, and Jim Stasheff. *Operads in algebra, topology and physics*, volume 96 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002.

[MV09] Sergei Merkulov and Bruno Vallette. Deformation theory of representations of prop(erad)s. II. *J. Reine Angew. Math.*, 636:123–174, 2009.

[Sch16a] Urs Schreiber. Enriched category. https://ncatlab.org/nlab/show/enriched+category, 2016. [Online; accessed 12-May-2016].

[Sch16b] Urs Schreiber. Monoidal category. https://ncatlab.org/nlab/show/monoidal+category, 2016. [Online; accessed 12-May-2016].

[Val07] Bruno Vallette. A Koszul duality for PROPs. *Trans. Amer. Math. Soc.*, 359(10):4865–4943, 2007.

[Val14] Bruno Vallette. Algebra + homotopy = operad. In *Symplectic, Poisson, and noncommutative geometry*, volume 62 of *Math. Sci. Res. Inst. Publ.*, pages 229–290. Cambridge Univ. Press, New York, 2014.

[YJ15] Donald Yau and Mark W. Johnson. *A foundation for PROPs, algebras, and modules*, volume 203 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2015.