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Positioning of Orlicz Space and Optimality

DEPARTMENT OF MATHEMATICAL ANALYSIS

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Název Práce: Positioning of Orlicz Space and Optimality

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Abstrakt: Řešíme problém, kdy k danému Banachovu prostoru funkcí s normou invariantní vůči nerostoucímu přerovnání $Y(\Omega)$ existuje optimální (největší) Orliczův prostor $L^A(\Omega)$ splňující Sobolevovo vnoření $W^m L^A(\Omega) \hookrightarrow Y(\Omega)$. V práci předkládáme kompletní charakterizaci tohoto problému pro třídu Marcinkiewiczových koncových prostorů a ukazujeme některé důležité příklady.

Klíčová slova: Optimální doména, Optimální cílový prostor, Orliczův prostor, prostor s normou invariantní vůči nerostoucímu přerovnání, Sobolevův prostor, oblast s lipschitzovskou hranicí, fundamentální funkce, Marcinkiewiczův koncový prostor, redukční věta.

Title: Positioning of Orlicz Space and Optimality

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Abstract: Given a rearrangement-invariant Banach function space $Y(\Omega)$, we consider the problem of the existence of an optimal (largest) domain Orlicz space $L^A(\Omega)$ satisfying the Sobolev embedding $W^m L^A(\Omega) \hookrightarrow Y(\Omega)$. We present a complete solution of this problem within the class of Marcinkiewicz endpoint spaces which covers several important examples.

Keywords: Optimal domain space, optimal range space, Orlicz space, rearrangement-invariant space, Sobolev space, Lipschitz domain, fundamental function, Marcinkiewicz endpoint space, reduction theorem.

Contents

1. Introduction

For a given rearrangement-invariant (r.i.) Banach function space $Y(\Omega)$, we ask whether there exists an optimal (i.e. largest) Orlicz space $L^A(\Omega)$ satisfying the embedding

$$W^m L^A(\Omega) \hookrightarrow Y(\Omega),$$

where Ω stands for a bounded Lipschitz domain in $\mathbb{R}^n$ and $W^m L^A(\Omega)$ is an Orlicz-Sobolev space. By optimality we mean that the space $L^A(\Omega)$ cannot be replaced by a strictly bigger Orlicz space, i.e., every embedding of an Orlicz-Sobolev space to $Y(\Omega)$ factorizes through the space $W^m L^A(\Omega)$.

R. Kerman and L. Pick [6] solved this problem in the general setting of r.i. spaces. They developed a tool to reduce the Sobolev embedding to the boundedness of a certain weighted Hardy operator and they used it to characterize the optimal source and target spaces in the class of r.i. spaces.

If we restrict ourselves only to Orlicz spaces, the situation becomes more complicated. Consider the well known classical Sobolev embedding $W^1 L^p(\Omega) \hookrightarrow L^{p^*}(\Omega)$, where $1 < p < n$ and $p^* = np/(n-p)$. The optimal r.i. range space is the Lorentz space $L^{p^*,p}(\Omega)$, and in the embedding

$$W^1 L^p(\Omega) \hookrightarrow L^{p*,p}(\Omega)$$

the domain $L^p(\Omega)$ is the optimal r.i. space and also the optimal Orlicz space.

On the other hand, if we start with the space $L^\infty(\Omega)$, then, as A. Cianchi and L. Pick showed in [4], an optimal Orlicz space does not exist at all. They presented a method that for a given Young function A such that the corresponding space $L^A(\Omega)$ satisfies the embedding $W^1 L^A(\Omega) \hookrightarrow L^\infty(\Omega)$ constructs a Young function B which grows essentially more slowly than A and the embedding $W^1 L^B(\Omega) \hookrightarrow L^\infty(\Omega)$ still holds.

In this work we present a generalization of this method to the class of Marcinkiewicz endpoint spaces. The main result (Theorem 5.1) gives a complete characterization when the optimal Orlicz domain exists and how to construct it. To put it simply, to a given Marcinkiewicz endpoint space M we construct an "optimal Orlicz candidate" $L^B(\Omega)$ in terms of the fundamental function. If the embedding $W^m L^B(\Omega) \hookrightarrow M(\Omega)$ holds, then $L^B(\Omega)$ is the optimal Orlicz domain, otherwise the optimal Orlicz domain does not exist at all.

Our approach is carried out in two steps. In chapter 3 we reduce the embedding

$$W^m L^A(\Omega) \hookrightarrow M(\Omega) \tag{1.1}$$

to the one-dimensional condition

$$\int_1^t \widetilde{A}(s)\, s^{\frac{n}{m-n}-1}\, \mathrm{d}s \lesssim t^{\frac{n}{m-n}}\, \widetilde{B}(Ct), \quad t \in (2, \infty), \tag{1.2}$$

where C is a constant and B is the Young function depending only on the dimension, the order of the derivative and the fundamental function of the space M. Here $\widetilde{A}$ and $\widetilde{B}$ denote the complementary Young functions to A and B, respectively.

In chapter 4 we characterize those Young functions B for which there exists another Young function $\widetilde{A}_1$ that grows essentially faster than $\widetilde{A}$ and still satisfies (1.2), with possibly different constants. It turns out that these functions are exactly those that satisfy the condition

$$\limsup_{t\to\infty} \frac{1}{G(Nt)} \int_1^t \frac{G(s)}{s}\,\mathrm{d}s = \infty \quad \text{for every } N \geq 1, \tag{1.3}$$

where $G(t) = t^{\frac{n}{m-n}}\,\widetilde{B}(t)$. The condition (1.3) exactly says that $W^m L^B(\Omega)$ does not embed into the space $M(\Omega)$. If the condition (1.3) is not satisfied, then $L^B(\Omega)$ is the optimal Orlicz domain.

Next we investigate when the Young function $\widetilde{B}$ satisfies the Δ_2 condition and in such cases we prove the equivalence between the condition (1.3) and the condition

$$\liminf_{t\to\infty} \frac{G(Nt)}{G(t)} = 1 \quad \text{for every } N \geq 1. \tag{1.4}$$

This requirement is similar to a characterization of slowly varying functions, but still quite weaker.

We also compute several important examples for various target spaces (Examples 3.8 and 5.2). For instance, if we start with $L^n(\Omega)$ in the place of a domain in (1.1) then, as J. A. Hempel, G. R. Morris and N. S. Trudinger in [5] (see also a more general result by A. Cianchi in [3]) showed, the optimal Orlicz range space is $\exp L^{n'}(\Omega)$, where $n' = n/(n-1)$. For this target, the right hand side of (1.2) is equivalent to $\log(t)$ which clearly satisfies conditions Δ_2 and (1.4). Therefore, in this case the optimal Orlicz domain does not exist.

At the end, we describe a simple generalization of the sufficiency for the existence of the optimal Orlicz domain for any target r.i. space. It can be stated as follows: for any r.i. space $Y(\Omega)$ compute optimal r.i. domain $X_Y(\Omega)$ and set $L^A(\Omega)$ in a way that its fundamental function coincides with that of $X_Y(\Omega)$. If $L^A(\Omega) \subseteq X_Y(\Omega)$ then $L^A(\Omega)$ is optimal.

The question if the converse is true in general still remains unanswered.

2. Preliminaries

Let us now recall and fix the notation which will be used in this work.

By $A \lesssim B$ and $A \gtrsim B$ we mean that $A \le C\,B$ and $A \ge C\,B$, respectively, where C is a positive constant independent of the appropriate quantities involved in A and B. We shall write $A \simeq B$ when both of the estimates $A \lesssim B$ and $A \gtrsim B$ are satisfied. We shall use the convention $0 \cdot \infty = 0$, $\frac{0}{0} = 0$ and $\frac{\infty}{\infty} = 0$.

When X and Y are Banach spaces, we say that X is embedded into Y, and write $X \hookrightarrow Y$, if $X \subseteq Y$ and there exists a positive constant C, such that $\|f\|_Y \le C\,\|f\|_X$ for every $f \in X$.

We say that a function $G\colon [0,\infty) \to (0,\infty)$ satisfies the Δ_2 condition at infinity if there exists $K > 0$ and $T \ge 0$ such that $G(2t) \le K\,G(t)$ for every $t \ge T$. We will use only Δ_2 condition at infinity, hence we shall shortly say Δ_2 condition and write $G \in \Delta_2$.

For a nonnegative function f we shall write $\int_0 f < \infty$ when there exists some $c > 0$ such that the integral $\int_0^c f$ converges. By integral we always mean the Lebesgue integral.

2.1 Rearrangement-invariant spaces

In this section we recall definitions and some basic facts concerning the rearrangement-invariant spaces, which we will need in the following text. We shall not prove well-known results; all of these can be found in the monograph by C. Bennett and R. Sharpley [1].

Suppose Ω is a domain in $\mathbb{R}^n$. Let $\mathcal{M}(\Omega)$ be a class of real-valued measurable functions on Ω and $\mathcal{M}^+(\Omega)$ the class of nonnegative functions in $\mathcal{M}(\Omega)$. Given $f \in \mathcal{M}$ we define its *nonincreasing rearrangement* on $(0,|\Omega|)$ as

$$f^*(t) := \inf\bigl\{\lambda > 0,\ \mu_f(\lambda) \le t\bigr\}, \quad 0 < t < |\Omega|,$$

where μ_f is the *distribution function* of f, i.e.,

$$\mu_f(\lambda) := \bigl|\bigl\{x \in \Omega,\ |f(x)| > \lambda\bigr\}\bigr|, \quad \lambda > 0,$$

where the $|\cdot|$ stands for the Lebesgue measure. The *Hardy average* f^{**} is defined on $(0,|\Omega|)$ as

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,\mathrm{d}s, \quad 0 < t < |\Omega|.$$

Let f, $g \in \mathcal{M}^+(\Omega)$. Then we have the *Hardy-Littlewood inequality*

$$\int_\Omega f(x)\,g(x)\,\mathrm{d}x \le \int_0^{|\Omega|} f^*(t)\,g^*(t)\,\mathrm{d}t.$$

When $E \subseteq \Omega$ is measurable, we denote by χ_E the *characteristic function* of E defined by

$$\chi_E(x) = \begin{cases} 1 & x \in E, \\ 0 & x \in \Omega \setminus E. \end{cases}$$

A *simple function* is a finite sum $\sum_j \lambda_j \chi_{E_j}$, where $\lambda_j \neq 0$ is a real number and $E_j \subseteq \Omega$ has finite measure for every index j.

Denote by I the interval $(0,1)$. A mapping $\varrho\colon \mathcal{M}^+(I) \to [0,\infty]$ is called a *rearrangement-invariant (r.i.) Banach function norm* on $\mathcal{M}^+(I)$, if for all f, g, f_n $(n \in \mathbb{N})$ in $\mathcal{M}^+(I)$, for all constants $a \geq 0$ and for every measurable subset E of I, the following properties hold:

(P1) $\varrho(f) = 0 \;\leftrightarrow\; f = 0$ a.e.; $\varrho(af) = a\varrho(f)$; $\varrho(f+g) \leq \varrho(f) + \varrho(g)$;

(P2) $0 \leq f \leq g$ a.e. implies $\varrho(f) \leq \varrho(g)$;

(P3) $0 \leq f_n \uparrow f$ a.e. implies $\varrho(f_n) \uparrow \varrho(f)$;

(P4) $\varrho(\chi_I) < \infty$;

(P5) $\int_0^1 f(x)\,\mathrm{d}x \lesssim \varrho(f)$;

(P6) $\varrho(f) = \varrho(f^*)$.

The *associate norm* of an r.i. norm ϱ is another such norm ϱ' defined as

$$\varrho'(g) := \sup_{\varrho(f)\leq 1} \int_0^1 g(t)\, f(t)\,\mathrm{d}t, \quad f, g \in \mathcal{M}^+(I).$$

It obeys the *Principle of Duality*; that is,

$$\varrho'' := (\varrho')' = \varrho.$$

Furthermore, the *Hölder inequality*

$$\int_0^1 f(t)\, g(t)\,\mathrm{d}t \leq \varrho(f)\, \varrho'(g)$$

holds for every $f, g \in \mathcal{M}^+(I)$.

The corresponding *rearrangement-invariant Banach function space* or, for short, *r.i. space* is the collection

$$L_\varrho(I) := \big\{ f \in \mathcal{M}(I),\ \varrho(|f|) < \infty \big\}$$

endowed with r.i. norm

$$\|f\|_{L_\varrho(I)} := \varrho(|f|), \quad f \in L_\varrho(I).$$

Next, given a bounded domain Ω in $\mathbb{R}^n$, we define the r.i. space

$$L_\varrho(\Omega) := \big\{ f \in \mathcal{M}(\Omega),\ \varrho\big(f^*(t|\Omega|)\big) < \infty \big\}$$

with

$$\|f\|_{L_\varrho(\Omega)} := \varrho\big(f^*(t|\Omega|)\big), \quad f \in L_\varrho(\Omega).$$

If ϱ_1 and ϱ_2 are two r.i. norms, then $L_{\varrho_1}(\Omega) \subseteq L_{\varrho_2}(\Omega)$ implies $L_{\varrho_1}(\Omega) \hookrightarrow L_{\varrho_2}(\Omega)$.

Let φ be a nonnegative function defined on the interval $[0,\infty)$. If

(i) $\varphi(t)=0$ iff $t=0$,
(ii) $\varphi(t)$ is nondecreasing on $(0,\infty)$,
(iii) $\varphi(t)/t$ is nonincreasing on $(0,\infty)$,

then φ is said to be *quasiconcave*. We also say that a function φ defined on bounded interval $[0,R]$, for $R\in(0,\infty)$, is quasiconcave if the continuation by constant value $\varphi(R)$ is quasiconcave on $[0,\infty)$.

The *fundamental function* of an r.i. norm ϱ on $\mathcal{M}^+(I)$ is defined by

$$\varphi_\varrho(t) := \varrho\big(\chi_{(0,t)}\big), \quad t\in I, \quad \varphi_\varrho(0)=0.$$

The fundamental function is quasiconcave on $[0,1)$, continuous except perhaps at the origin and satisfies

$$\varphi_\varrho(t)\,\varphi_{\varrho'}(t)=t, \quad t\in I.$$

Quasiconcave functions need not be concave, however, every r.i. space can be equivalently renormed so that its fundamental function is concave.

Let φ be a concave function. We define the *Lorentz endpoint space* $\Lambda_\varphi(\Omega)$ by the function norm

$$\varrho_{\Lambda_\varphi}(f) := \int_0^1 f^*(t)\,\mathrm{d}\varphi(t), \quad f\in\mathcal{M}^+(I),$$

where $\mathrm{d}\varphi$ stands for the Lebesgue-Stieltjes measure associated with φ. We define the *Marcinkiewicz endpoint space* $M_\varphi(\Omega)$ by the function norm

$$\varrho_{M_\varphi}(f) := \sup_{0<t<1} f^{**}(t)\,\varphi(t), \quad f\in\mathcal{M}^+(I).$$

The endpoint spaces $\Lambda_\varphi(\Omega)$ and $M_\varphi(\Omega)$ are r.i. spaces with the fundamental function φ. If $X(\Omega)$ is an r.i. space with the fundamental function φ, then

$$\Lambda_\varphi(\Omega)\hookrightarrow X\hookrightarrow M_\varphi(\Omega).$$

In other words, $\Lambda_\varphi(\Omega)$ and $M_\varphi(\Omega)$ are respectively the smallest and the largest r.i. spaces having the fundamental function equivalent to φ.

The associate space of a Lorentz endpoint space Λ_φ is the Marcinkiewicz endpoint space M_ψ where both φ and ψ are concave and $\varphi(t)\,\psi(t)=t$ on I.

If $|\Omega|<\infty$, then for every r.i. space $X(\Omega)$

$$L^\infty(\Omega)\hookrightarrow X(\Omega)\hookrightarrow L^1(\Omega).$$

Assume either $1<p,q<\infty$ or $p=q=1$ or $p=q=\infty$. The *Lorentz space* $L^{p,q}(\Omega)$ is defined by the functional

$$\varrho_{p,q}(f)=\varrho_q\left(t^{\frac{1}{p}-\frac{1}{q}}f^*(t)\right), \quad f\in\mathcal{M}^+(I),$$

where

$$\varrho_q(f)=\begin{cases}\left(\int_0^1 f(t)^q\,\mathrm{d}t\right)^{\frac{1}{q}}, & 1\le q<\infty,\\ \operatorname*{ess\,sup}_{0<t<1} f(t), & q=\infty,\end{cases}$$

stands for the Banach function norm of the *Lebesgue space* $L^q(\Omega)$. The functional $\varrho_{p,q}$ is a Banach function norm if and only if $1 \le q \le p$. However, for $1 < p < \infty$, $\varrho_{p,q}$ can be equivalently replaced by Banach function norm

$$\varrho_{(p,q)}(f) = \varrho_q\left(t^{\frac{1}{p}-\frac{1}{q}} f^{**}(t)\right).$$

The fundamental function of the norm $\varrho_{(p,q)}$ satisfies

$$\varphi_{\varrho_{(p,q)}}(t) = t^{\frac{1}{p}}, \quad t \in [0,1).$$

The spaces $L^{p,1}(\Omega)$ and $L^{p,\infty}(\Omega)$ are equal to the Lorentz and Marcinkiewicz endpoint spaces $\Lambda_\varphi(\Omega)$ and $M_\varphi(\Omega)$, respectively, with $\varphi(t) = t^{1/p}$. If the first parameter is fixed then the Lorentz spaces are nested, i.e., we have $L^{p,q}(\Omega) \hookrightarrow L^{p,r}(\Omega)$ whenever $1 < p < \infty$ and $1 \le q \le r \le \infty$.

2.2 Orlicz Spaces

We also need to know definitions and all the basic facts about Young functions and Orlicz Spaces. All of these can be found for instance in the book by L. Pick, A. Kufner, O. John and S. Fučík [7].

We shall say that A is a *Young function* if there exists a function $a\colon [0,\infty) \to [0,\infty)$ such that

$$A(t) = \int_0^t a(s)\,\mathrm{d}s, \quad t \in [0,\infty),$$

and a has the following properties:

(i) $a(s) > 0$ for $s > 0$, $a(0) = 0$;
(ii) a is right-continuous;
(iii) a is nondecreasing;
(iv) $\lim_{s\to\infty} a(s) = \infty$.

Every Young function is continuous, nonnegative, strictly increasing, convex on $[0,\infty)$ and satisfies

$$\lim_{t\to 0^+} \frac{A(t)}{t} = \lim_{t\to\infty} \frac{t}{A(t)} = 0.$$

Furthermore, one has

$$A(\alpha t) \le \alpha\, A(t), \quad \alpha \in [0,1], \quad t \ge 0,$$

and

$$A(\beta t) \ge \beta\, A(t), \quad \beta \in (1,\infty), \quad t \ge 0.$$

Moreover $A(t)/t$ is increasing on $(0,\infty)$ and we have the estimates

$$A(t) \le a(t)\, t \le A(2t), \quad t \in (0,\infty).$$

A Young function satisfies the Δ_2 condition at infinity if and only if

$$\limsup_{t\to\infty} \frac{t\,a(t)}{A(t)} < \infty.$$

For a Young function A and a domain $\Omega \subseteq \mathbb{R}^n$, the *Orlicz space* $L^A = L^A(\Omega)$ is the collection of all functions $f \in \mathcal{M}(\Omega)$ for which there exists a $\lambda > 0$ such that

$$\int_\Omega A\left(\frac{|f(x)|}{\lambda}\right) \mathrm{d}x < \infty.$$

The Orlicz Space $L^A(\Omega)$ is endowed with the *Luxemburg norm*

$$\|f\|_{L^A} := \inf\left\{\lambda > 0,\ \int_\Omega A\left(\frac{|f(x)|}{\lambda}\right) \mathrm{d}x \leq 1\right\}.$$

The *complementary function* $\widetilde{A}$ of a Young function A is given by

$$\widetilde{A}(t) := \sup_{s>0}\big(st - A(s)\big), \quad t \in [0, \infty).$$

The complementary function $\widetilde{A}$ is a Young function as well and the complementary function of $\widetilde{A}$ is once more A. For any Young function A and its complementary function $\widetilde{A}$ there is the relation

$$t \leq A^{-1}(t)\, \widetilde{A}^{-1}(t) \leq 2t, \quad t \in [0, \infty).$$

With the help of the complementary function, we can define an alternative *Orlicz norm* on an Orlicz space by

$$\|f\|_{(L^A)} := \sup\left\{\int_\Omega \big|f(x)\, g(x)\big|\, \mathrm{d}x\right\},$$

where the supremum is taken over all functions $g \in \mathcal{M}(\Omega)$ such that

$$\int_\Omega \widetilde{A}\big(|g(x)|\big)\, \mathrm{d}x < \infty.$$

The Luxemburg and Orlicz norms are equivalent, namely,

$$\|f\|_{L^A} \leq \|f\|_{(L^A)} \leq 2\, \|f\|_{L^A}.$$

When $L^A(\Omega)$ is an Orlicz space endowed with the Luxemburg norm then the associate space is $L^{\widetilde{A}}(\Omega)$ with the Orlicz norm. In particular, the sharp Hölder inequality for Orlicz spaces has the form

$$\int_\Omega \big|f(x)\, g(x)\big|\, \mathrm{d}x \leq \|f\|_{L^A}\, \|f\|_{(L^{\widetilde{A}})}.$$

The Orlicz space $L^A(\Omega)$ is an r.i. space and

$$\|\chi_E\|_{L^A} = \frac{1}{A^{-1}\big(\frac{1}{|E|}\big)}, \quad E \subseteq \Omega,$$

thus, for a bounded domain Ω, the fundamental function for the Luxemburg norm is

$$\varphi_{L^A}(t) = \frac{1}{A^{-1}\big(\frac{1}{t|\Omega|}\big)}, \quad t \in I, \quad \varphi_{L^A}(0) = 0.$$

An Orlicz space $L^A(I)$ with fundamental function φ coincides with the Marcinkiewicz endpoint space $M_\varphi(I)$ if there exists $\delta \in (0, 1)$ such that

$$\int_0^1 A\big(\delta\, A^{-1}\big(\tfrac{1}{t}\big)\big)\, \mathrm{d}t < \infty.$$

For $|\Omega| < \infty$, the inclusion relation between Orlicz spaces is governed by inequalities involving the corresponding Young functions. If A and B are Young functions then $L^A(\Omega) \hookrightarrow L^B(\Omega)$ if and only if there exist $c > 0$ and $T \geq 0$ such that

$$B(t) \leq A(ct), \quad t \geq T,$$

which we denote by $B \prec A$ or $A \succ B$. If both $A \prec B$ and $A \succ B$ hold, we say that A and B are equivalent and write $A \approx B$. When $|\Omega| < \infty$, the inclusion $L^A(\Omega) \subseteq L^B(\Omega)$ is proper if and only if

$$\limsup_{t \to \infty} \frac{B(t)}{A(\lambda t)} = 0$$

for every $\lambda > 0$. We write $B \prec\prec A$ or $A \succ\succ B$.

If $A \prec B$ or $A \prec\prec B$ then $\widetilde{A} \succ \widetilde{B}$ or $\widetilde{A} \succ\succ \widetilde{B}$ respectively.

2.3 Sobolev Spaces

Let Ω be a bounded open subset in $\mathbb{R}^n$, $n \geq 2$ and $1 \leq m \leq n-1$. Let $N = N(n,m)$ be the number of multiindices $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ satisfying $0 \leq |\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_k \leq m$.

Given a locally integrable function f on Ω having weak derivatives of all orders $|\alpha|$ for all $0 \leq |\alpha| \leq m$, denote the N-vector of all such derivatives by

$$D^m f := \left(\frac{\partial^\alpha f}{\partial x^\alpha} \right)_{0 \leq |\alpha| \leq m}$$

and by $|D^m f|$ the Euclidean length of this vector.

Let ϱ be an r.i. norm on $\mathcal{M}^+(I)$. The *Sobolev space* $W^m L_\varrho = W^m L_\varrho(\Omega)$ is the set

$$\left\{ f \colon \Omega \to \mathbb{R};\ D^m f \text{ is defined and } \varrho\big(|D^m f|^*(t|\Omega|)\big) < \infty \right\}$$

endowed with the norm

$$\|f\|_{W^m L_\varrho} := \varrho\big(|D^m f|^*(t|\Omega|)\big).$$

Let us recall the reduction theorem, the result of the work of R. Kerman and L. Pick [6].

Theorem 2.1. *Let ϱ_D and ϱ_R be r.i. norms on $\mathcal{M}^+(I)$. Let n, m be positive integers, $n \geq 2$ and $1 \leq m \leq n-1$. Then to each bounded domain $\Omega \subseteq \mathbb{R}^n$ with the Lipschitz boundary corresponds a constant $C > 0$, depending only on Ω, m and n, such that*

$$\|u\|_{L_{\varrho_R}} \leq C \, \|u\|_{W^m L_{\varrho_D}}, \quad u \in W^m L_{\varrho_D}(\Omega), \tag{2.1}$$

if and only if

$$\varrho_R \left(\int_t^1 f(s) s^{\frac{m}{n}-1} \, \mathrm{d}s \right) \lesssim \varrho_D(f), \quad f \in \mathcal{M}^+(I). \tag{2.2}$$

In the sequel the domain Ω will always stand for a Lipschitz domain such that $|\Omega| = 1$. This constitutes no loss of generality because if one deals with Ω having finite measure, one can simply write $t\,|\Omega|$ instead of t in the definition of an r.i. norm.

3. Reduction

The aim of this chapter is to prove the reduction of Sobolev embedding (1.1) to the one-dimensional condition (1.2). This will be done in several steps. Most of them are quite technical computations with Orlicz norms and Young functions and are stated as separated lemmas; the final step is reached in Theorem 3.5.

Lemma 3.1. *Let A be a Young function and α be nonzero real number. Assuming*

$$\int_0 A(s)\, s^{\frac{1}{\alpha}-1}\,\mathrm{d}s < \infty, \tag{3.1}$$

we define

$$E_\alpha(t) = |\alpha|^{-1} t^{-\frac{1}{\alpha}} \int_0^t A(s)\, s^{\frac{1}{\alpha}-1}\,\mathrm{d}s, \quad t \in (0,\infty).$$

Such E_α is an increasing mapping of $(0,\infty)$ onto itself. Moreover, if $R \in (0,\infty]$ then the following relations hold.

$$\|t^\alpha \chi_{(0,a)}(t)\|_{L^A(0,R)} = \frac{a^\alpha}{E_\alpha^{-1}\left(\frac{1}{a}\right)}, \quad a \in (0,R),\ \alpha > 0, \tag{3.2}$$

$$\|t^\alpha \chi_{(a,\infty)}(t)\|_{L^A(0,\infty)} = \frac{a^\alpha}{E_\alpha^{-1}\left(\frac{1}{a}\right)}, \quad a \in (0,\infty),\ \alpha < 0. \tag{3.3}$$

In addition, if $\varepsilon \in (0,R)$ and if $\alpha < 0$ then

$$\|t^\alpha \chi_{(a,R)}(t)\|_{L^A(0,R)} \simeq \|t^\alpha \chi_{(a,\infty)}(t)\|_{L^A(0,\infty)}, \quad a \in (0, R-\varepsilon). \tag{3.4}$$

Proof. Assume (3.1). By change of variables $s \mapsto ts$ we have

$$E_\alpha(t) = |\alpha|^{-1} \int_0^1 A(ts)\, s^{\frac{1}{\alpha}-1}\,\mathrm{d}s, \quad t \in (0,\infty),$$

hence E_α is increasing.

By definition of the Luxemburg norm, we have

$$\|t^\alpha \chi_{(0,a)}(t)\|_{L^A(0,R)} = \inf\left\{\lambda > 0,\ \int_0^a A\left(\frac{t^\alpha}{\lambda}\right)\,\mathrm{d}t \le 1\right\}.$$

Next, by change of variables we get for $\alpha > 0$

$$\begin{aligned}
\|t^\alpha \chi_{(0,a)}(t)\|_{L^A(0,R)} &= \inf\left\{\lambda > 0,\ \frac{\lambda^{\frac{1}{\alpha}}}{\alpha} \int_0^{\frac{a^\alpha}{\lambda}} A(s)\, s^{\frac{1}{\alpha}-1}\,\mathrm{d}s \le 1\right\} \\
&= \inf\left\{\lambda > 0,\ a\, E_\alpha\left(\tfrac{a^\alpha}{\lambda}\right) \le 1\right\} \\
&= \frac{a^\alpha}{E_\alpha^{-1}\left(\frac{1}{a}\right)}.
\end{aligned}$$

This proves the part (3.2). The proof of the relation (3.3) can be done in an analogous way and we omit it.

In order to prove the relation (3.4), we have to show two inequalities. Clearly

$$\|t^\alpha \chi_{(a,\infty)}(t)\|_{L^A(0,\infty)} \geq \|t^\alpha \chi_{(a,R)}(t)\|_{L^A(0,\infty)} = \|t^\alpha \chi_{(a,R)}(t)\|_{L^A(0,R)}$$

by the monotonicity of the norm. On the other hand, we have by the triangle inequality

$$\|t^\alpha \chi_{(a,\infty)}(t)\|_{L^A(0,\infty)} \leq \|t^\alpha \chi_{(a,R)}(t)\|_{L^A(0,R)} + \|t^\alpha \chi_{(R,\infty)}(t)\|_{L^A(0,\infty)}.$$

Using (3.3), the term $\|t^\alpha \chi_{(R,\infty)}(t)\|_{L^A(0,\infty)}$ equals $R^\alpha / E_\alpha^{-1}(\frac{1}{R})$ since $\alpha < 0$. Thanks to the assumptions, this quantity is finite, say K. The term $\|t^\alpha \chi_{(a,R)}(t)\|_{L^A(0,R)}$ is a decreasing function of the variable a, positive on $(0, R)$ and vanishing at R. Hence for every $\varepsilon \in (0, R)$ there exists a constant C such that

$$K \leq C\, \|t^\alpha \chi_{(a,R)}(t)\|_{L^A(0,R)}, \quad a \in (0, R - \varepsilon).$$

For those a we conclude that

$$\|t^\alpha \chi_{(a,\infty)}(t)\|_{L^A(0,\infty)} \leq (C + 1)\, \|t^\alpha \chi_{(a,R)}(t)\|_{L^A(0,R)}.$$

□

Lemma 3.2. *Let φ be a quasiconcave function on $(0, \infty)$ and $\alpha \in (0, 1)$. We define*

$$\overline{\varphi}(t) = t^\alpha \sup_{s \in (t,\infty)} \varphi(s)\, s^{-\alpha}, \quad t \in (0, \infty), \quad \overline{\varphi}(0) = 0.$$

Then $\overline{\varphi}(t)$ and $\overline{\varphi}(t)\, t^{1-\alpha}$ are quasiconcave.

Proof. Since φ is nondecreasing, we have for every $t \in (0, \infty)$

$$\begin{aligned} \overline{\varphi}(t) &= t^\alpha \sup_{s \in (t,\infty)} s^{-\alpha} \sup_{r \in (0,s)} \varphi(r) \\ &= t^\alpha \sup_{r \in (0,\infty)} \varphi(r) \sup_{s \in (\max\{r,t\},\infty)} s^{-\alpha} \\ &= t^\alpha \sup_{r \in (0,\infty)} \varphi(r) \min\left\{t^{-\alpha}, r^{-\alpha}\right\} \\ &= \sup_{r \in (0,\infty)} \varphi(r) \min\left\{1, (\tfrac{t}{r})^\alpha\right\}, \end{aligned}$$

hence $\overline{\varphi}$ is nondecreasing. Next, by definition we have

$$\frac{\overline{\varphi}(t)}{t} = t^{\alpha-1} \sup_{s \in (t,\infty)} \varphi(s)\, s^{-\alpha}, \quad t \in (0, \infty),$$

which is decreasing as a product of a decreasing function and a nonincreasing function. Surely $\overline{\varphi}(0) = 0$ and $\overline{\varphi}(t) > 0$ for positive t, therefore $\overline{\varphi}$ is quasiconcave.

The function $\overline{\varphi}(t)\, t^{1-\alpha}$ is quasiconcave because it is increasing as a product of a increasing function and nondecreasing function and because

$$\frac{\overline{\varphi}(t)\, t^{1-\alpha}}{t} = \sup_{s \in (t,\infty)} \varphi(s)\, s^{-\alpha}, \quad t \in (0, \infty),$$

is nonincreasing. The rest is trivial. □

Lemma 3.3. *Let $\alpha \in (0,1)$ and φ be a quasiconcave function on $(0,\infty)$ such that*

$$\sup_{t\in(0,\infty)} \varphi(t)\, t^{-\alpha} = \infty.$$

Let us define

$$F(t) = \overline{\varphi}\left(\tfrac{1}{t}\right) t^{\alpha}, \quad t \in (0,\infty),$$

where

$$\overline{\varphi}(t) = t^{\alpha} \sup_{s\in(t,\infty)} \varphi(s)\, s^{-\alpha}, \quad t \in (0,\infty), \quad \overline{\varphi}(0) = 0.$$

Then there exists a Young function B such that the fundamental function of the Orlicz space $L^B(I)$ is equivalent to $\overline{\varphi}(t)\, t^{1-\alpha}$ on $[0,1)$ and moreover

$$\widetilde{B}^{-1}(t) \simeq F(t), \quad t \in (0,\infty),$$

where $\widetilde{B}$ is the complementary function to B.

Proof. Define

$$u(t) = \overline{\varphi}(t)\, t^{1-\alpha}, \quad t \in [0,\infty).$$

Thanks to Lemma 3.2, u is quasiconcave and strictly increasing. Furthermore, define

$$b(s) = \frac{1}{s\, u^{-1}\left(\frac{1}{s}\right)}, \quad s \in (0,\infty),$$

and set $b(0) = 0$. Then define

$$B(t) = \int_0^t b(s)\, \mathrm{d}s, \quad t \in [0,\infty).$$

We claim that B is a Young function. The properties (i) and (ii) from the definition of Young function are clear. Let us prove that b is nondecreasing. The function $u(t)/t$ is nonincreasing and u itself is increasing, hence $s/u^{-1}(s)$ is nonincreasing and therefore $b(s) = \frac{1}{s\, u^{-1}(1/s)}$ is nondecreasing. It remains to show that $\lim_{s\to\infty} b(s) = \infty$. Indeed, suppose that there is $K > 0$ such that $b(s) < K$ for every nonnegative s. Then

$$\frac{1}{s\, u^{-1}\left(\frac{1}{s}\right)} \leq K, \quad s \in (0,\infty).$$

Since u maps $(0,\infty)$ onto the whole $(0,\infty)$, we can follow by

$$u(t) \leq K\, t, \quad t \in (0,\infty),$$

hence

$$\overline{\varphi}(t) \leq K\, t^{\alpha}$$

for all $t \in (0,\infty)$. We can rewrite this as

$$\sup_{t\in(0,\infty)} \overline{\varphi}(t)\, t^{-\alpha} \leq K$$

and by definition of $\overline{\varphi}$

$$\sup_{t\in(0,\infty)}\ \sup_{s\in(t,\infty)} \varphi(s)\, s^{-\alpha} \leq K,$$

that is,

$$\sup_{s\in(0,\infty)} \varphi(s)\, s^{-\alpha} \leq K,$$

which contradicts the assumption.

Now, since B is a Young function, we have that

$$B(t) \leq b(t)\, t \leq B(2t), \quad t \in [0,\infty).$$

It follows by definition of b that

$$B(t) \leq \frac{1}{u^{-1}\left(\frac{1}{t}\right)} \leq B(2t), \quad t \in (0,\infty).$$

Applying the increasing function B^{-1}, we get

$$t \leq B^{-1}\left(\frac{1}{u^{-1}\left(\frac{1}{t}\right)}\right) \leq 2t, \quad t \in (0,\infty),$$

that is, taking reciprocal values and $t \mapsto 1/s$,

$$\frac{s}{2} \leq \frac{1}{B^{-1}\left(\frac{1}{u^{-1}(s)}\right)} \leq s, \quad s \in (0,\infty).$$

Finally, since u is increasing on $(0,\infty)$ and $u(0,\infty) = (0,\infty)$, this implies

$$\frac{u(y)}{2} \leq \frac{1}{B^{-1}\left(\frac{1}{y}\right)} \leq u(y), \quad y \in (0,\infty).$$

Hence by the definition of the fundamental function for the Luxemburg norm we conclude that

$$\varphi_{L^B}(t) \simeq u(t), \quad t \in (0,1).$$

Finally define $\widetilde{B}$ as the associate function to B. Then

$$\widetilde{B}^{-1}(t) \simeq \frac{t}{B^{-1}(t)} \simeq t\, u\left(\tfrac{1}{t}\right) = \overline{\varphi}\left(\tfrac{1}{t}\right) t^{\alpha} = F(t), \quad t \in (0,\infty).$$

$\square$

The following theorem enables us to reduce an embedding to a Lorentz endpoint spaces only to testing on characteristic functions. The idea of the proof is based on [2, Theorem 8], where the Lorentz space $L^{p,1}(\Omega)$ occurs as a target space.

Theorem 3.4. *Let $Y(I)$ be a Banach function space and $\Lambda(I)$ be a Lorentz endpoint space over I. Suppose that T is a sublinear operator mapping $\Lambda(I)$ to $Y(I)$ and satisfying*

$$\|T\chi_E\|_Y \lesssim \|\chi_E\|_\Lambda \tag{3.5}$$

for every measurable set $E \subseteq I$. Then

$$\|Tf\|_Y \lesssim \|f\|_\Lambda$$

for every $f \in \Lambda(I)$.

Proof. Let f be a simple nonnegative function on I. Thus f can be written as a finite sum $f = \sum_j \lambda_j \chi_{E_j}$, where all lambdas are positive real numbers and the sets E_j are measurable subsets of I satisfying $E_1 \subseteq E_2 \subseteq \cdots$. Then, as readily seen, we have $f^* = \sum_j \lambda_j \chi^*_{E_j}$. Let φ be a fundamental function of $\Lambda(I)$. By the definition of the Lorentz norm we have

$$\|f\|_\Lambda = \int_0^1 f^* \,\mathrm{d}\varphi = \int_0^1 \sum_j \lambda_j \chi^*_{E_j} \,\mathrm{d}\varphi = \sum_j \lambda_j \int_0^1 \chi^*_{E_j} \,\mathrm{d}\varphi = \sum_j \lambda_j \|\chi_{E_j}\|_\Lambda.$$

On account of the sublinearity of T we have $|Tf| \leq \sum_j \lambda_j |T\chi_{E_j}|$, and consequently by (3.5) and by axioms (P1) and (P2) we obtain

$$\|Tf\|_Y \leq \sum_j \lambda_j \|T\chi_{E_j}\|_Y \lesssim \sum_j \lambda_j \|\chi_{E_j}\|_\Lambda = \|f\|_\Lambda.$$

Now if f is simple but no longer nonnegative, we use the same for the positive part of f and for the negative part of f.

Suppose that f is an arbitrary function in $\Lambda(I)$ and let f_n be a sequence of simple integrable functions converging to f in $\Lambda(I)$. Then

$$\|T(f_n) - T(f_m)\|_Y \leq \|T(f_n - f_m)\|_Y \lesssim \|f_n - f_m\|_\Lambda,$$

and Tf_n is Cauchy, hence convergent in $Y(I)$. Since limits are unique in $Y(I)$, it follows that $\lim Tf_n = Tf$ and

$$\|Tf\|_Y = \lim \|Tf_n\|_Y \lesssim \lim \|f_n\|_\Lambda = \|f\|_\Lambda$$

as we wished to show. □

Theorem 3.5. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, and $|\Omega| = 1$. Let m be an integer such that $1 \leq m \leq n-1$. Let $L^A(\Omega)$ be an Orlicz space with a Young function A and $M(\Omega)$ be a Marcinkiewicz endpoint space with a fundamental function φ satisfying*

$$\sup_{t\in(0,1)} \varphi(t)\, t^{\frac{m}{n}-1} = \infty. \tag{3.6}$$

Then the embedding

$$W^m L^A(\Omega) \hookrightarrow M(\Omega) \tag{3.7}$$

holds if and only if there exists $C > 0$ such that

$$\int_1^t \widetilde{A}(s)\, s^{\frac{n}{m-n}-1}\, \mathrm{d}s \lesssim t^{\frac{n}{m-n}}\, \widetilde{B}(Ct), \quad t \in (2, \infty), \tag{3.8}$$

where B is a Young function such that $\varphi_{L^B}(t) \simeq \overline{\varphi}(t)\, t^{\frac{m}{n}}$ and

$$\overline{\varphi}(t) = t^{1-\frac{m}{n}} \sup_{s \in (t,1)} \varphi(s)\, s^{\frac{m}{n}-1}.$$

Remark 3.6. Before proving this theorem, we show that the condition (3.6) does not cause any loss of generality. Indeed, suppose that (3.6) is not satisfied. Thus $\varphi(t) \lesssim t^{1-\frac{m}{n}}$, $t \in (0,1)$, and there is an inclusion between corresponding endpoint spaces

$$L^{\frac{n}{n-m},\infty}(\Omega) \hookrightarrow M_\varphi(\Omega).$$

Now recall the Sobolev embedding under the same assumptions on Ω. Consider the endpoint optimal r.i. embeddings

$$W^m L^1(\Omega) \hookrightarrow L^{\frac{n}{n-m},1}(\Omega)$$

and

$$W^m L^{\frac{n}{m},1}(\Omega) \hookrightarrow L^\infty(\Omega).$$

Therefore we can conclude that

$$W^m L^1(\Omega) \hookrightarrow L^{\frac{n}{n-m},1}(\Omega) \hookrightarrow L^{\frac{n}{n-m},\infty}(\Omega) \hookrightarrow M_\varphi(\Omega)$$

hence $W^m L^1(\Omega) \hookrightarrow M_\varphi(\Omega)$ and since $L^1(\Omega)$ is the largest r.i. space, every Orlicz space $L^A(\Omega)$ satisfies $W^m L^A(\Omega) \hookrightarrow M_\varphi(\Omega)$. Moreover, since $L^1(\Omega)$ is not an Orlicz space by definition, there is no optimal one.

Proof of Theorem 3.5. Using Theorem 2.1, the embedding (3.7) is equivalent to the inequality

$$\left\| \int_t^1 g(s)\, s^{\frac{m}{n}-1}\, \mathrm{d}s \right\|_M \lesssim \|g\|_{L^A}, \quad g \in L^A(I).$$

By the L^1 duality, this is the same as

$$\left\| t^{\frac{m}{n}-1} \int_0^t f(s)\, \mathrm{d}s \right\|_{L^{\widetilde{A}}} \lesssim \|f\|_{M'}, \quad f \in M'(I),$$

where $\widetilde{A}$ is the complementary function to A. This is equivalent to

$$\left\| t^{\frac{m}{n}-1} \int_0^t f^*(s)\, \mathrm{d}s \right\|_{L^{\widetilde{A}}} \lesssim \|f\|_{M'}, \quad f \in M'(I).$$

Indeed, one implication is just passing to only nonincreasing functions with the fact that $\|f\|_{M'} = \|f^*\|_{M'}$, and the other is thanks to Hardy-Littlewood inequality applied to functions f and $\chi_{(0,t)}$.

Using the fact that M' is a Lorentz endpoint space and passing to the characteristic functions while keeping Theorem 3.4 in mind, this is equivalent to

$$\left\| t^{\frac{m}{n}-1} \int_0^t \chi_{(0,a)}(s)\,\mathrm{d}s \right\|_{L^{\widetilde{A}}} \lesssim \varphi_{M'}(a), \quad a \in (0,1). \tag{3.9}$$

Let us compute the left hand side. Clearly

$$\begin{aligned}\left\| t^{\frac{m}{n}-1} \int_0^t \chi_{(0,a)}(s)\,\mathrm{d}s \right\|_{L^{\widetilde{A}}} &= \|t^{\frac{m}{n}-1}\chi_{(0,a)}(t)\cdot t + t^{\frac{m}{n}-1}\chi_{(a,1)}(t)\cdot a\|_{L^{\widetilde{A}}} \\ &\le \|t^{\frac{m}{n}}\chi_{(0,a)}(t)\|_{L^{\widetilde{A}}} + a\,\|t^{\frac{m}{n}-1}\chi_{(a,1)}(t)\|_{L^{\widetilde{A}}}.\end{aligned}$$

We suppose that $a \in (0, 1/2)$, since we are interested only in values of a near zero. We show that the second summand dominates the first one. Indeed,

$$\begin{aligned} a\|t^{\frac{m}{n}-1}\chi_{(a,1)}(t)\|_{L^{\widetilde{A}}} &\ge a\|t^{\frac{m}{n}-1}\chi_{(a,2a)}(t)\|_{L^{\widetilde{A}}} \ge a(2a)^{\frac{m}{n}-1}\|\chi_{(a,2a)}(t)\|_{L^{\widetilde{A}}} \\ &\simeq a^{\frac{m}{n}}\|\chi_{(0,a)}(t)\|_{L^{\widetilde{A}}} = \|a^{\frac{m}{n}}\chi_{(0,a)}(t)\|_{L^{\widetilde{A}}} \ge \|t^{\frac{m}{n}}\chi_{(0,a)}(t)\|_{L^{\widetilde{A}}}.\end{aligned}$$

Therefore we can state that

$$\left\| t^{\frac{m}{n}-1} \int_0^t \chi_{(0,a)}(s)\,\mathrm{d}s \right\|_{L^{\widetilde{A}}} \simeq a\,\|t^{\frac{m}{n}-1}\chi_{(a,1)}(t)\|_{L^{\widetilde{A}}}.$$

At this moment, it is the time for using Lemma 3.1. We need the part (3.4) with (3.3) for $\alpha = m/n - 1 < 0$, $R = 1$ and $\varepsilon = 1/2$. The assumption (3.1) can be rendered as satisfied without any loss of generality since the domain Ω is of finite measure, hence the appropriate Young function can be redefined on $(0, 1)$ without any effect to the corresponding Orlicz space. Note also that we are using the associate function $\widetilde{A}$ instead of A. Hence we conclude that (3.9) is equivalent to

$$\frac{a^{\frac{m}{n}}}{E^{-1}_{\frac{m}{n}-1}\left(\frac{1}{a}\right)} \lesssim \varphi_{M'}(a), \quad a \in (0, 1/2).$$

Now we substitute $t = 1/a$ and use the fact that $\varphi_{M'}(a) = a/\varphi(a)$. We get

$$\varphi\left(\tfrac{1}{t}\right) t^{1-\frac{m}{n}} \lesssim E^{-1}_{\frac{m}{n}-1}(t), \quad t \in (2,\infty). \tag{3.10}$$

Let us define

$$F(t) = \overline{\varphi}\left(\tfrac{1}{t}\right) t^{1-\frac{m}{n}}, \quad t \in (0,\infty),$$

where the function $\overline{\varphi}(t)$ is taken from Lemma 3.2 for $\alpha = 1 - m/n$. Technically, φ is defined on $[0, 1)$ but we work with φ as with quasiconcave function on $[0,\infty)$, obtained as its continuation by the value $\varphi(1)$ on $[1,\infty)$.

We claim that $F(t)$ is the least nondecreasing majorant of $\varphi(1/t)\, t^{1-\frac{m}{n}}$. Indeed,

$$\overline{\varphi}(t) = t^{1-\frac{m}{n}} \sup_{s\in(t,\infty)} \varphi(s)\, s^{\frac{m}{n}-1}, \quad t \in (0,\infty),$$

hence

$$\overline{\varphi}\left(\tfrac{1}{t}\right) t^{1-\frac{m}{n}} = \sup_{s\in(0,t)} \varphi\left(\tfrac{1}{s}\right) s^{1-\frac{m}{n}}, \quad t \in (0,\infty),$$

and the claim follows.

Since the function $E_{\frac{m}{n}-1}$ is strictly increasing as well as its inverse, we can enlarge the left hand side of the inequality (3.10) by $F(t)$. Hence we can equivalently continue by

$$F(t) \lesssim E^{-1}_{\frac{n}{m}-1}(t), \quad t \in (2,\infty). \tag{3.11}$$

Now Lemma 3.3 applied to φ and $\alpha = 1 - m/n$ comes to play. We obtain that there exists a Young function B such that $\widetilde{B}^{-1}(t) \simeq F(t)$. Using this and passing to inverse functions, (3.11) is equivalent to the existence of some constant $C > 0$ such that

$$E_{\frac{m}{m}-1}(t) \le \widetilde{B}(Ct), \quad t \in (c,\infty),$$

where $c = E^{-1}_{\frac{m}{n}-1}(2) > 0$. This is however equivalent to

$$E_{\frac{m}{m}-1}(t) \lesssim \widetilde{B}(Ct), \quad t \in (2,\infty),$$

which is nothing but

$$\int_0^t \widetilde{A}(s)\, s^{\frac{n}{m-n}-1}\, \mathrm{d}s \lesssim t^{\frac{n}{m-n}}\, \widetilde{B}(Ct), \quad t \in (2,\infty).$$

Finally observe that the quantities $\int_0^t \widetilde{A}(s)\, s^{\frac{n}{m-n}-1}\, \mathrm{d}s$ and $\int_1^t \widetilde{A}(s)\, s^{\frac{n}{m-n}-1}\, \mathrm{d}s$ are comparable since $t \in (2,\infty)$. One can now immediately observe that the resulting inequality does not depend on the behavior of the Young function $\widetilde{A}$ on the interval $(0,1)$. $\square$

Remark 3.7. Note that Theorem 3.5 can be stated in a much simpler way in the case when the function

$$F(t) = \varphi\left(\tfrac{1}{t}\right) t^{1-\frac{m}{n}}, \quad t \in (1,\infty),$$

is strictly increasing. In such case, we do not have to define the envelope $\overline{\varphi}$, and not even the Young function B. Instead of that we just pass to the inverse functions straightaway and get thereby that (3.7) is equivalent to the inequality

$$\int_1^t \widetilde{A}(s)\, s^{\frac{n}{m-n}-1}\, \mathrm{d}s \lesssim t^{\frac{n}{m-n}}\, F^{-1}(Ct), \quad t \in (2,\infty).$$

We will see that this simplification is useful for computing the left hand side of the resulting inequality in many natural examples.

Examples 3.8. Let n, m be integers such that $n \geq 2$ and $1 \leq m \leq n-1$.

(i) Let $\Phi(t) = \exp(t^{\frac{n}{n-m}}) - 1$, $t \in [0,\infty)$. Then Φ is a Young function such that the space $L^\Phi(\Omega)$, denoted by $\exp L^{\frac{n}{n-m}}(\Omega)$, coincides with the Marcinkiewicz endpoint space $M_\varphi(\Omega)$, where

$$\varphi(t) = \log^{\frac{m}{n}-1}(\tfrac{2}{t}), \quad t \in (0,1).$$

Then by Remark 3.7 we have $F(t) = t^{1-\frac{m}{n}} \log^{\frac{m}{n}-1}(2t)$, $t \in (1,\infty)$, and $F^{-1}(t) \simeq t^{\frac{n}{n-m}} \log(t)$, $t \in (2,\infty)$, thus the embedding

$$W^m L^A(\Omega) \hookrightarrow \exp L^{\frac{n}{n-m}}(\Omega)$$

is equivalent to

$$\int_1^t \widetilde{A}(s)\, s^{\frac{n}{m-n}-1}\, \mathrm{d}s \lesssim \log(t), \quad t \in (2,\infty).$$

(ii) Let $\Phi(t) = \exp(t^{\frac{n}{n-m-q}}) - 1$, $t \in [0,\infty)$, $q < n-m$. Then Φ is a Young function such that the space $L^\Phi(\Omega)$, denoted by $\exp L^{\frac{n}{n-m-q}}(\Omega)$, coincides with the Marcinkiewicz endpoint space $M_\varphi(\Omega)$, where

$$\varphi(t) \simeq \log^{\frac{m+q}{n}-1}(\tfrac{2}{t}), \quad t \in (0,1).$$

Then $F(t) = t^{1-\frac{m}{n}} \log^{\frac{m+q}{n}-1}(2t)$, $t \in (1,\infty)$, $F^{-1}(t) \simeq t^{\frac{n}{n-m}} \log^{1-\frac{q}{n-m}}(t)$, $t \in (2,\infty)$, and the embedding

$$W^m L^A(\Omega) \hookrightarrow \exp L^{\frac{n}{n-m-q}}(\Omega)$$

is equivalent to

$$\int_1^t \widetilde{A}(s)\, s^{\frac{n}{m-n}-1}\, \mathrm{d}s \lesssim \log^{1-\frac{q}{n-m}}(t), \quad t \in (2,\infty).$$

(iii) Let $\Phi(t) = \exp\exp(t^{\frac{n}{n-m}}) - e$, $t \in [0,\infty)$. Then Φ is a Young function such that the space $L^\Phi(\Omega)$, denoted by $\exp\exp L^{\frac{n}{n-m}}(\Omega)$, coincides with the Marcinkiewicz endpoint space $M_\varphi(\Omega)$, where

$$\varphi(t) = \log^{\frac{m}{n}-1}\log(\tfrac{2}{t}), \quad t \in (0,1).$$

Then $F(t) = t^{1-\frac{m}{n}} \log^{\frac{m}{n}-1}\log(2t)$, $t \in (1,\infty)$, $F^{-1}(t) \simeq t^{\frac{n}{n-m}} \log\log(t)$, $t \in (2,\infty)$, and the embedding

$$W^m L^A(\Omega) \hookrightarrow \exp\exp L^{\frac{n}{n-m}}(\Omega)$$

is equivalent to

$$\int_1^t \widetilde{A}(s)\, s^{\frac{n}{m-n}-1}\, \mathrm{d}s \lesssim \log\log(t), \quad t \in (2,\infty).$$

(iv) If $M(\Omega) = L^\infty(\Omega)$, then $\varphi(t) = \chi_{(0,1]}(t)$. Hence $F(t) = t^{1-\frac{m}{n}}$, $t \in (1,\infty)$, and the embedding

$$W^m L^A(\Omega) \hookrightarrow L^\infty(\Omega)$$

is equivalent to

$$\int_1^t \widetilde{A}(s)\, s^{\frac{n}{m-n}-1}\, \mathrm{d}s \lesssim 1, \quad t \in (2,\infty).$$

4. Construction

In this section we study the reduced one-dimensional inequality (3.8). Note that all results in this section are independent of the Sobolev embeddings.

In Theorem 4.1 we find the sufficient condition (1.3) for $G(t) = t^{\frac{n}{m-n}}\,\widetilde{B}(t)$ so that A can be replaced by another essentially more slowly growing Young function still satisfying (3.8). The proof is partially constructive and is based on the idea of [4, Theorem 6.4], where $G(t) = 1$, which corresponds with L^∞ as the target space in (3.7).

We show in Theorem 4.2 that this condition is also necessary; we prove that if (1.3) is not satisfied, then there exists up to equivalence the $\succ\!\succ$-maximal Young function satisfying (3.8).

Finally we establish the equivalence between (1.3) and the simpler condition (1.4) and we compute several examples for right hand sides obtained in Examples 3.8.

Theorem 4.1. *Let Young functions A and B satisfy for integers m, n, $2 \le n$, $1 \le m \le n-1$ and some $C > 0$ the inequality*

$$\int_1^t \widetilde{A}(s)\, s^{\frac{n}{m-n}-1}\,\mathrm{d}s \lesssim t^{\frac{n}{m-n}}\,\widetilde{B}(Ct), \quad t \in (2,\infty). \tag{4.1}$$

Denote $G(t) = t^{\frac{n}{m-n}}\,\widetilde{B}(Ct)$. If

$$\limsup_{t\to\infty} G(t) = \infty \tag{4.2}$$

and

$$\limsup_{t\to\infty} \frac{1}{G(Mt)} \int_1^t \frac{G(s)}{s}\,\mathrm{d}s = \infty \tag{4.3}$$

for every $M \ge 1$, then there exists Young function A_1 satisfying $A_1 \prec\!\prec A$ and also

$$\int_1^t \widetilde{A}_1(s)\, s^{\frac{n}{m-n}-1}\,\mathrm{d}s \lesssim t^{\frac{n}{m-n}}\,\widetilde{B}(C_1 t), \quad t \in (2,\infty).$$

Proof. Let A and G be the functions from the assumptions. First, we can assume that G is nondecreasing, since otherwise we can pass to the greatest nondecreasing minorant which still majorizes the increasing left hand side of (4.1).

Next we establish an upper bound for $\widetilde{A}$. Namely, for $t \in (1,\infty)$

$$\begin{aligned} G(2t) &\gtrsim \int_1^{2t} \widetilde{A}(s)\, s^{\frac{n}{m-n}-1}\,\mathrm{d}s \ge \int_t^{2t} \widetilde{A}(s)\, s^{\frac{n}{m-n}-1}\,\mathrm{d}s \\ &\ge \widetilde{A}(t) \int_t^{2t} s^{\frac{n}{m-n}-1}\,\mathrm{d}s \simeq \widetilde{A}(t)\, t^{\frac{n}{m-n}}. \end{aligned} \tag{4.4}$$

Using this, we obtain the existence of $\beta > 0$ such that

$$\beta\, G(2t) > \widetilde{A}(t)\, t^{\frac{n}{m-n}}, \quad t \in (1,\infty). \tag{4.5}$$

Now we fix this β and for every $t \in (1,\infty)$, we define the set

$$G_t = \left\{ s \in (1,\infty);\ \tfrac{\widetilde{A}(s)}{s} \geq \beta\, t^{\frac{n}{n-m}-1}\, G(2t) \right\}.$$

Since $\widetilde{A}(s)/s$ is a nondecreasing mapping from $(0,\infty)$ onto itself, the sets G_t are upper segments. In particular, G_t is nonempty for every $t \in (1,\infty)$. Let us define $\tau = \tau_t = \inf G_t$. Observe that for $t \in (1,\infty)$ and $s \in (1,t)$

$$\beta\, s^{\frac{n}{n-m}-1}\, G(2s) = \beta\, 2^{\frac{n}{n-m}}\, \frac{\widetilde{B}(2Cs)}{s} \leq \beta\, 2^{\frac{n}{n-m}}\, \frac{\widetilde{B}(2Ct)}{t} = \beta\, t^{\frac{n}{n-m}-1}\, G(2t)$$

and together with the estimate (4.5), we conclude that

$$\frac{\widetilde{A}(s)}{s} < \beta\, s^{\frac{n}{n-m}-1}\, G(2s) \leq \beta\, t^{\frac{n}{n-m}-1}\, G(2t)$$

for $s \in (1,t)$. Hence $\tau_t > t$ for every t. Moreover, since $\widetilde{A}(t)/t$ is continuous, we have the equality

$$\frac{\widetilde{A}(\tau)}{\tau} = \beta\, t^{\frac{n}{n-m}-1}\, G(2t), \quad t \in (1,\infty). \tag{4.6}$$

Let M be a real number such that $M \geq 1$. Then

$$\limsup_{t\to\infty} \frac{\widetilde{A}(\tau)}{\tau} \frac{t}{\widetilde{A}(2Mt)} = \infty. \tag{4.7}$$

Indeed, suppose that there exists $M \geq 1$ and some $K > 0$ such that there is for all $t \in (1,\infty)$ the estimate

$$\frac{\widetilde{A}(\tau)}{\tau} \frac{t}{\widetilde{A}(2Mt)} < K,$$

or equivalently

$$\frac{\widetilde{A}(2Mt)}{t} > K^{-1} \frac{\widetilde{A}(\tau)}{\tau}. \tag{4.8}$$

Now for $t > 2$ the following holds:

$$\begin{aligned}
G(Mt) &\gtrsim \int_1^{Mt} \widetilde{A}(s)\, s^{\frac{n}{m-n}-1}\, \mathrm{d}s \geq \int_M^{Mt} \widetilde{A}(s)\, s^{\frac{n}{m-n}-1}\, \mathrm{d}s \\
&\simeq \int_{1/2}^{t/2} \widetilde{A}(2Ms)\, s^{\frac{n}{m-n}-1}\, \mathrm{d}s && \text{(by change of variables)} \\
&\gtrsim \int_{1/2}^{t/2} \frac{\widetilde{A}(\tau_s)}{\tau_s}\, s^{\frac{n}{m-n}}\, \mathrm{d}s && \text{(by (4.8))} \\
&\simeq \int_{1/2}^{t/2} \frac{G(2s)}{s}\, \mathrm{d}s && \text{(by (4.6))} \\
&\simeq \int_1^t \frac{G(s)}{s}\, \mathrm{d}s.
\end{aligned}$$

This contradicts (4.3) for this M.

From estimate (4.7), we can take an increasing sequence $t_j \in (2, \infty)$, $j \geq 2$, such that

$$\lim_{j\to\infty} \frac{\widetilde{A}(\tau_j)}{\tau_j} \frac{t_j}{\widetilde{A}(jt_j)} = \infty, \tag{4.9}$$

where we define $\tau_j = \tau_{t_j}$. We claim that without loss of generality we can assume that $2t_j < \tau_j$ for every index $j \geq 2$. Indeed, suppose that there exists a subsequence j_k in $\mathbb{N}$ such that $\tau_{j_k} \leq 2t_{j_k}$. Then $\widetilde{A}(\tau_{j_k}) \leq \widetilde{A}(2t_{j_k})$ and

$$\frac{\widetilde{A}(\tau_{j_k})}{\tau_{j_k}} \frac{t_{j_k}}{\widetilde{A}(j_k t_{j_k})} \leq \frac{\widetilde{A}(2t_{j_k})}{t_{j_k}} \frac{t_{j_k}}{\widetilde{A}(\frac{j_k}{2} 2t_{j_k})} \leq \frac{\widetilde{A}(2t_{j_k})}{\widetilde{A}(2t_{j_k})} \frac{2}{j_k} = \frac{2}{j_k} \to 0 \text{ as } k \to \infty,$$

which is impossible due to (4.9).

At this moment, we can define a function $\widetilde{A}_1$ by the formula

$$\widetilde{A}_1(t) = \begin{cases} \widetilde{A}(t_j) + \frac{\widetilde{A}(\tau_j) - \widetilde{A}(t_j)}{\tau_j - t_j}(t - t_j), & t \in (t_j, \tau_j),\ j \in \mathbb{N}, \\ \widetilde{A}(t), & \text{otherwise.} \end{cases}$$

Obviously, $\widetilde{A}_1 \geq \widetilde{A}$ and $\widetilde{A}_1$ is a Young function. Moreover, for $j \in \mathbb{N}$, $j \geq 2$,

$$\begin{aligned} \frac{\widetilde{A}_1(2t_j)}{\widetilde{A}(jt_j)} &= \frac{\widetilde{A}(t_j) + \frac{\widetilde{A}(\tau_j) - \widetilde{A}(t_j)}{\tau_j - t_j} t_j}{\widetilde{A}(jt_j)} \\ &\geq \frac{\widetilde{A}(\tau_j) - \widetilde{A}(t_j)}{\widetilde{A}(jt_j)} \frac{t_j}{\tau_j} \\ &\geq \frac{\widetilde{A}(\tau_j) - \widetilde{A}(\frac{\tau_j}{2})}{\widetilde{A}(jt_j)} \frac{t_j}{\tau_j} && \text{(since } 2t_j < \tau_j) \\ &\geq \frac{1}{2} \frac{\widetilde{A}(\tau_j)}{\tau_j} \frac{t_j}{\widetilde{A}(jt_j)} && \text{(by } \widetilde{A}(\tau_j/2) \leq \widetilde{A}(\tau_j)/2), \end{aligned}$$

and the latter tends to infinity as $j \to \infty$ by (4.9). Therefore

$$\limsup_{t\to\infty} \frac{\widetilde{A}_1(t)}{\widetilde{A}(\lambda t)} = \infty$$

for every $\lambda > 2$, which is precisely $\widetilde{A}_1 \succ\succ \widetilde{A}$, hence $A_1 \prec\prec A$.

It remains to show that $\widetilde{A}_1$ satisfies the condition (4.1) with A replaced by A_1. Let $t \in (2, \infty)$ be fixed. We find $j \in \mathbb{N}$ such shat $t \in [t_j, t_{j+1})$. Then we have

$$\begin{aligned} \int_1^t \frac{\widetilde{A}_1(s)}{s^{\frac{n}{n-m}+1}} \, \mathrm{d}s &\leq \int_1^t \frac{\widetilde{A}(s)}{s^{\frac{n}{n-m}+1}} \, \mathrm{d}s \\ &\quad + \sum_{k=1}^{j} \int_{t_k}^{\tau_k} \left(\widetilde{A}(t_k) + \frac{\widetilde{A}(\tau_k) - \widetilde{A}(t_k)}{\tau_k - t_k} (s - t_k) \right) s^{\frac{n}{m-n}-1} \, \mathrm{d}s \\ &\leq 2 \int_1^t \frac{\widetilde{A}(s)}{s^{\frac{n}{n-m}+1}} \, \mathrm{d}s + \sum_{k=1}^{j} \frac{\widetilde{A}(\tau_k) - \widetilde{A}(t_k)}{\tau_k - t_k} \int_{t_k}^{\tau_k} (s - t_k) \, s^{\frac{n}{m-n}-1} \, \mathrm{d}s. \end{aligned}$$

We can follow with estimates of the latter integral. Since $\frac{n}{m-n} < -1$, we have for $k \in \mathbb{N}$ such that $1 \le k \le j$,

$$\int_{t_k}^{\tau_k} (s - t_k)\, s^{\frac{n}{m-n}-1}\, \mathrm{d}s \le \int_{t_k}^{\tau_k} s^{\frac{n}{m-n}}\, \mathrm{d}s \le \int_{t_k}^{\infty} s^{\frac{n}{m-n}}\, \mathrm{d}s \simeq t_k^{\frac{n}{m-n}+1}.$$

This together with the fact that $2t_k < \tau_k$ gives

$$\int_1^t \widetilde{A}_1(s)\, s^{\frac{n}{m-n}-1}\, \mathrm{d}s \lesssim 2 \int_1^t \frac{\widetilde{A}(s)}{s^{\frac{n}{n-m}+1}}\, \mathrm{d}s + 2 \sum_{k=1}^{j} \frac{\widetilde{A}(\tau_k)}{\tau_k}\, t_k^{\frac{n}{m-n}+1}.$$

Since (4.6) implies

$$\frac{\widetilde{A}(\tau_k)}{\tau_k}\, t_k^{\frac{n}{m-n}+1} = \beta\, G(2t_k),$$

we have

$$\int_1^t \widetilde{A}_1(s)\, s^{\frac{n}{m-n}-1}\, \mathrm{d}s \lesssim G(t) + \sum_{k=1}^{j} G(2t_k).$$

Because the sequence t_j could be taken arbitrarily fast growing, we can assume without loss of generality that $G(2t_i) \ge \sum_{k=1}^{i-1} G(2t_k)$ thanks to fact that G is increasing and unbounded by (4.2). Adding all the estimates together, we finally obtain that

$$\int_1^t \widetilde{A}_1(s) s^{\frac{n}{m-n}-1}\, \mathrm{d}s \lesssim G(t) + G(2t_j) \lesssim G(2t), \quad t \in (2, \infty),$$

which proves the theorem. □

Theorem 4.2. *Let m, n be integers such that $2 \le n$, $1 \le m \le n-1$. Let B be a Young function such that $G(t) = t^{\frac{n}{m-n}}\, \widetilde{B}(t)$ satisfies*

$$\limsup_{t \to \infty} \frac{1}{G(Mt)} \int_1^t \frac{G(s)}{s}\, \mathrm{d}s < \infty \tag{4.10}$$

for some $M \ge 1$. Then B is up to equivalence the $\succ\!\succ$-maximal element in the class of Young functions A satisfying

$$\int_1^t \widetilde{A}(s)\, s^{\frac{n}{m-n}+1}\, \mathrm{d}s \lesssim t^{\frac{n}{m-n}}\, \widetilde{B}(Ct), \quad t \in (2, \infty) \tag{4.11}$$

for some constant $C > 0$.

PROOF. Let us first show that the condition (4.11) is true for $A = B$. By (4.10), there is some $M \geq 1$ such that

$$\int_1^t \frac{G(s)}{s}\,\mathrm{d}s \lesssim G(Mt), \quad t \in (1,\infty). \tag{4.12}$$

Let us fix this M. By the definition of G, we have

$$\widetilde{B}(s)\, s^{\frac{n}{m-n}-1} = \frac{G(s)}{s}, \quad s \in (1,\infty).$$

Integrating over the interval $(1,t)$ and thanks to (4.12), we get

$$\int_1^t \widetilde{B}(s)\, s^{\frac{n}{m-n}-1}\,\mathrm{d}s = \int_1^t \frac{G(s)}{s}\,\mathrm{d}s \lesssim G(Mt) \simeq t^{\frac{n}{n-m}}\, \widetilde{B}(Mt), \quad t \in (1,\infty),$$

which implies the condition (4.11) holds with $A = B$.

Now suppose that (4.11) holds with $A = B_1$ for some Young function B_1. Then by the same calculation as in (4.4) we obtain that

$$\widetilde{B}_1(t) \lesssim G(2Mt)\, t^{\frac{n}{n-m}} \simeq \widetilde{B}(2Mt), \quad t \in (1,\infty).$$

This implies the relation $\widetilde{B}_1 \prec \widetilde{B}$ therefore, it cannot be true that $\widetilde{B}_1 \succ\!\succ \widetilde{B}$, and therefore B is $\succ\!\succ$-maximal. $\square$

The rest of this section is devoted to the condition (4.3). We show that under the assumption $G \in \Delta_2$, the integral criterion (4.3) is equivalent to a much simpler condition. Clearly $G \in \Delta_2$ is satisfied if and only if $\widetilde{B} \in \Delta_2$ and since B is Young function, we have some criterion to characterize it in the words of its fundamental function.

Let us start with an auxiliary lemma which will be needed in Theorem 4.4. The idea is based on the L'Hopital rule.

Lemma 4.3. *Let $c \in (0,\infty)$. Suppose that f and g are real functions having finite derivatives on (c,∞). If $g(x) \to \infty$ as $x \to \infty$, then*

$$\liminf_{x\to\infty} \frac{f'(x)}{g'(x)} \leq \liminf_{x\to\infty} \frac{f(x)}{g(x)}.$$

PROOF. Suppose that the left hand side of the inequality is finite and choose some constants L and r such that $-\infty < L < r < \liminf f'(x)/g'(x)$. We will show that $L \leq \liminf f(x)/g(x)$.

First, choose $c_1 > c$ such that $f'(x)/g'(x) > r$ for every $x > c_1$. For arbitrary $c_1 < y < x < \infty$ there is $\xi \in (y,x)$ satisfying

$$\frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f'(\xi)}{g'(\xi)} > r.$$

Now let y be fixed. Since $g(x) \to \infty$, there is $c_2 > c_1$ such that $g(x) > 0$ and $g(x) > g(y)$ for all $x > c_2$. For those x multiplying by $g(x) - g(y)$ and dividing by $g(x)$ we have the inequality

$$\frac{f(x) - f(y)}{g(x)} > r\frac{g(x) - g(y)}{g(x)},$$

which can be rewritten as

$$\frac{f(x)}{g(x)} > r + \frac{f(y) - rg(y)}{g(x)}.$$

Finally, we can find $c_3 > c_2$ such that $g(x) > \big(rg(y) - f(y)\big)/(r - L)$ for $x > c_3$. Then we obtain that

$$\frac{f(x)}{g(x)} > r - (r - L) = L$$

for $x > c_3$ hence $\liminf f(x)/g(x) \geq L$. □

Theorem 4.4. *Let $G\colon (0,\infty) \to (0,\infty)$ be a continuous nondecreasing function satisfying Δ_2 condition. Then the following are equivalent.*

(i)

$$\limsup_{t\to\infty} \frac{1}{G(Mt)} \int_1^t \frac{G(s)}{s}\,\mathrm{d}s = \infty \quad \textit{for every } M \geq 1;$$

(ii)

$$\limsup_{t\to\infty} \frac{1}{G(t)} \int_1^t \frac{G(s)}{s}\,\mathrm{d}s = \infty;$$

(iii)

$$\liminf_{t\to\infty} \frac{G(Mt)}{G(t)} = 1 \quad \textit{for every } M \geq 1.$$

Proof. The equivalence (ii)↔(i) is trivial, since the quantities $G(t)$ and $G(Mt)$ are comparable for every fixed $M \geq 1$ thanks to the fact that $G \in \Delta_2$.

Let us focus on the implication (iii)→(ii). Let $M \geq 1$ be fixed and suppose $t > 1$. Then

$$\int_1^{Mt} G(s)\frac{\mathrm{d}s}{s} \geq \int_t^{Mt} G(s)\frac{\mathrm{d}s}{s} \geq G(t)\int_t^{Mt} \frac{\mathrm{d}s}{s} = G(t)\log M.$$

Dividing both sides by $G(Mt)$ we obtain

$$\log M \frac{G(t)}{G(Mt)} \leq \frac{1}{G(Mt)} \int_1^{Mt} \frac{G(s)}{s}\,\mathrm{d}s.$$

Taking the limes superior as $t \to \infty$ on both sides of the inequality, we get

$$\log M = \log M\, \limsup_{t\to\infty} \frac{G(t)}{G(Mt)} \leq \limsup_{t\to\infty} \frac{1}{G(Mt)} \int_1^{Mt} \frac{G(s)}{s}\,\mathrm{d}s =: L,$$

where L is independent of M. Since $\log M \le L$ for arbitrary M, L has no other option but to equal infinity.

To prove (ii)→(iii), let $M \ge 1$ be fixed and let us define $f(t) = \int_1^t G(Ms)\frac{\mathrm{d}s}{s}$ and $g(t) = \int_1^t G(s)\frac{\mathrm{d}s}{s}$. Then both f and g are continuous and have derivatives, namely $f'(t) = G(Mt)/t$, $g'(t) = G(t)/t$. Since (ii) holds, it has to be $g(t) \to \infty$ as $t \to \infty$. Using Lemma 4.3, we get

$$\begin{aligned}
0 &\le \liminf_{t\to\infty} \frac{G(Mt)}{G(t)} - 1 \\
&\le \liminf_{t\to\infty} \frac{\int_1^t G(Ms)\frac{\mathrm{d}s}{s}}{\int_1^t G(s)\frac{\mathrm{d}s}{s}} - 1 \\
&\le \liminf_{t\to\infty} \frac{\int_M^{Mt} G(s)\frac{\mathrm{d}s}{s} - \int_1^t G(s)\frac{\mathrm{d}s}{s}}{\int_1^t G(s)\frac{\mathrm{d}s}{s}} \\
&\le \liminf_{t\to\infty} \frac{G(t)}{\int_1^t G(s)\frac{\mathrm{d}s}{s}} \frac{\int_t^{Mt} G(s)\frac{\mathrm{d}s}{s}}{G(t)}.
\end{aligned}$$

Since $\liminf_{t\to\infty} G(t)/\int_1^t G(s)\frac{\mathrm{d}s}{s} = 0$, it suffices to show that $\frac{1}{G(t)}\int_t^{Mt} G(s)\frac{\mathrm{d}s}{s}$ is bounded. To end this we use the fact that G is nondecreasing and, due to $G \in \Delta_2$, there is some $c > 0$ such that $G(Mt) \le c\,G(t)$ for big t. For such a t we have

$$\frac{1}{G(t)} \int_t^{Mt} G(s)\frac{\mathrm{d}s}{s} \le \frac{G(Mt)}{G(t)} \int_t^{Mt} \frac{\mathrm{d}s}{s} \le c\log M.$$

□

Remark 4.5. Let us mention that the situation when the Young function $\widetilde{B}$ and hence G satisfy the Δ_2 is quite common. Recall that a Young function A satisfies the Δ_2 condition if and only if

$$\limsup_{t\to\infty} \frac{t\,A'(t)}{A(t)} < \infty.$$

We can simply reformulate this condition in terms of its fundamental function φ as

$$\limsup_{t\to 0^+} \frac{\varphi(t)}{t\,\varphi'(t)} < \infty. \tag{4.13}$$

Now let φ be a quasiconcave function on $(0,\infty)$ such that

$$\sup_{t\in(0,\infty)} \varphi(t)\,t^{\frac{m}{n}-1} = \infty$$

and let B be the Young function from Lemma 3.3 corresponding to $\alpha = 1 - m/n$. We have that $\varphi_{L^B}(t) \simeq \overline{\varphi}(t)\,t^{\frac{m}{n}}$. Suppose for a time being that

$$\varphi_{L^B}(t) = \overline{\varphi}(t)\,t^{\frac{m}{n}}.$$

Then the fundamental function corresponding to the associate space $L^{\widetilde{B}}$ is

$$\varphi_{L^{\tilde{B}}}(t) = \frac{t^{1-\frac{m}{n}}}{\overline{\varphi}(t)}.$$

Since $\widetilde{B}$ is a Young function, $\widetilde{B}$ has a first order derivative everywhere in $(0, \infty)$ except perhaps at countably many points. Since $\widetilde{B}(s) = 1/\varphi^{-1}_{L^{\tilde{B}}}(1/s)$, the same is true for $\varphi^{-1}_{L^{\tilde{B}}}$ and also $\overline{\varphi}$.

Next, the derivative is

$$\varphi'_{L^{\tilde{B}}}(t) = \frac{1}{\big(\overline{\varphi}(t)\big)^2} \Big((1 - \tfrac{m}{n})\, t^{-\frac{m}{n}}\, \overline{\varphi}(t) - t^{1-\frac{m}{n}}\, \overline{\varphi}'(t) \Big).$$

Then using the criterion (4.13), we get that $\widetilde{B} \in \Delta_2$ is equivalent to

$$\limsup_{t \to 0^+} \frac{\varphi_{L^{\tilde{B}}}(t)}{t\, \varphi'_{L^{\tilde{B}}}(t)} < \infty,$$

that is, to

$$\limsup_{t \to 0^+} \frac{1}{(1 - \frac{m}{n}) - \frac{t\,\overline{\varphi}'(t)}{\overline{\varphi}(t)}} < \infty.$$

Therefore the expression $t\,\overline{\varphi}'(t)/\overline{\varphi}(t)$ always takes values in the interval $[0, 1 - m/n)$ and the fraction above can blow up only when

$$\limsup_{t \to 0^+} \frac{t\,\overline{\varphi}'(t)}{\overline{\varphi}(t)} = 1 - \frac{m}{n}.$$

Therefore, $\widetilde{B} \in \Delta_2$ if and only if

$$\limsup_{t \to 0^+} \frac{t\,\overline{\varphi}'(t)}{\overline{\varphi}(t)} < 1 - \frac{m}{n}.$$

This computation shows that the Marcinkiewicz endpoint spaces, which are far from the fundamental line $\frac{n}{n-m}$ in the sense described above, ensure the Δ_2 condition for $\widetilde{B}$. This shows that the characterization in Theorem 4.4 is useful for the target spaces near $L^\infty(\Omega)$.

5. Optimality

Let us return to the embedding $W^1X(\Omega) \hookrightarrow L^\infty(\Omega)$. We already know that the optimal Orlicz domain in this embedding does not exist, but we can compute the optimal r.i. domain. It is the Lorentz space $L^{n,1}(\Omega)$, that is, the optimal Sobolev embedding reads as

$$W^1L^{n,1}(\Omega) \hookrightarrow L^\infty(\Omega).$$

The fundamental function of the space $L^{n,1}(\Omega)$ is $\varphi_{L^{n,1}}(t) = t^{\frac{1}{n}}$, and, via the unique correspondence of the Young and fundamental functions in the class of Orlicz spaces, there is exactly one Orlicz space having this fundamental function: $L^n(\Omega)$. This is the only natural candidate for the optimal Orlicz domain but, as it is well known, it does not render the corresponding Sobolev embedding true.

Let us put this fact into the context of Theorem 3.5. If $M(\Omega) = L^\infty(\Omega)$ then the corresponding Young function B satisfies $B(t) = t^n$, $t \in (0, \infty)$.

The following theorem connects Theorems 3.5, 4.1 and 4.2.

Theorem 5.1. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$, and $|\Omega| = 1$. Let m be an integer such that $1 \leq m \leq n-1$. Let $M(\Omega)$ be a Marcinkiewicz endpoint space with a fundamental function φ satisfying*

$$\sup_{t\in(0,1)} \varphi(t)\, t^{\frac{m}{n}-1} = \infty.$$

Let B be a Young function such that

$$\varphi_{L^B}(t) \simeq t^{\frac{m}{n}}\, \overline{\varphi}(t), \quad t \in [0,1),$$

where

$$\overline{\varphi}(t) = t^{1-\frac{m}{n}} \sup_{s\in(t,\infty)} \varphi(s)\, s^{\frac{m}{n}-1}.$$

Define $G(t) = t^{\frac{n}{m-n}}\, \widetilde{B}(t)$, $t \in (0, \infty)$. Then the following are equivalent.

(i) *There exists an optimal Orlicz space $L^A(\Omega)$ satisfying the embedding*

$$W^mL^A(\Omega) \hookrightarrow M(\Omega);$$

(ii)

$$W^mL^B(\Omega) \hookrightarrow M(\Omega);$$

(iii) *$L^B(\Omega) \subseteq X(\Omega)$, where $X(\Omega)$ is the optimal r.i. domain in the embedding $W^mX(\Omega) \hookrightarrow M(\Omega)$;*

(iv) *there exists some* $C \geq 1$ *such that*

$$\limsup_{t\to\infty} \frac{1}{G(Ct)} \int_1^t \frac{G(s)}{s}\, \mathrm{d}s < \infty.$$

Moreover, if G *satisfies the* Δ_2 *condition, then each of the conditions* (i)–(iv) *is equivalent to following statement:*

(v) *there exists some* $C \geq 1$ *such that*

$$\liminf_{t\to\infty} \frac{G(Ct)}{G(t)} > 1.$$

Proof. Let us show (ii)$\leftrightarrow$(iv). Let B be the Young function from the assumptions of the theorem. Observe that, thanks to Theorem 3.5, the embedding $W^m L^B(\Omega) \hookrightarrow M(\Omega)$ holds if and only if

$$\int_1^t \widetilde{B}(s)\, s^{\frac{n}{m-n}-1}\, \mathrm{d}s \lesssim t^{\frac{n}{m-n}}\, \widetilde{B}(Ct), \quad t \in (2,\infty),$$

for some $C \geq 1$. Rewriting this, we obtain

$$\limsup_{t\to\infty} \frac{1}{G(Ct)} \int_1^t \frac{G(s)}{s}\, \mathrm{d}s < \infty$$

for $G(t) = t^{\frac{n}{m-n}}\, \widetilde{B}(t)$.

In order to show (ii)$\rightarrow$(i) we just use Theorem 4.2 which tells us that B is the $\succ\!\succ$-maximal Young function in class of all Young functions A satisfying

$$\int_1^t \widetilde{A}(s)\, s^{\frac{n}{m-n}-1}\, \mathrm{d}s \lesssim t^{\frac{n}{m-n}}\, \widetilde{B}(Ct), \quad t \in (2,\infty),$$

for some $C \geq 1$. This means that the space $L^B(\Omega)$ is the largest Orlicz space satisfying $W^m L^B(\Omega) \hookrightarrow M(\Omega)$.

To prove (i)$\rightarrow$(ii), we show that if (ii) is not satisfied then (i) is not either. Hence if $W^m L^B(\Omega) \hookrightarrow M(\Omega)$ does not hold then

$$\limsup_{t\to\infty} \frac{1}{G(Nt)} \int_1^t \frac{G(s)}{s}\, \mathrm{d}s = \infty$$

for every $N \geq 1$. If, in addition, $\limsup_{t\to\infty} G(t) = \infty$, then by Theorem 4.1 to a given Young function A satisfying

$$\int_1^t \widetilde{A}(s)\, s^{\frac{n}{m-n}-1}\, \mathrm{d}s \lesssim G(Ct), \quad t \in (2,\infty),$$

for some $C \geq 1$, there is another Young function A_1 satisfying $A_1 \prec\!\prec A$ and also

$$\int_1^t \widetilde{A}_1(s)\, s^{\frac{n}{m-n}-1}\, \mathrm{d}s \lesssim G(C_1 t), \quad t \in (2,\infty),$$

for some $C_1 \geq 1$. This, thanks to Theorem 3.5, says that to a given Orlicz space $L^A(\Omega)$ satisfying the Sobolev embedding $W^m L^A(\Omega) \hookrightarrow M(\Omega)$, there is another, strictly larger Orlicz space $L^{A_1}(\Omega)$ also satisfying $W^m L^{A_1}(\Omega) \hookrightarrow M(\Omega)$.

If G is bounded, then it is equivalent to a constant function on $(1,\infty)$, which means that $M(\Omega) = L^\infty(\Omega)$. (cf. Example 3.8, (iv)). This situation has already been described in [4, Theorem 6.4] and no optimal Orlicz domain exists. This is in accord with the fact that the expression

$$\limsup_{t\to\infty} \frac{1}{G(Ct)} \int_1^t \frac{G(s)}{s}\,\mathrm{d}s$$

is infinite for a constant function G and every $C \geq 1$.

The equivalence of (ii) and (iii) follows directly from the definition of the optimal r.i. space, and the equivalence of (iv) and (v) has been already stated in Theorem 4.4. $\square$

We recall that the growth assumption for the fundamental function φ has already been discussed in Remark 3.6. We have seen that we exclude only those spaces M for which the embedding $W^m L^1(\Omega) \hookrightarrow M(\Omega)$ is satisfied. It follows that Theorem 5.1 covers all reasonable situations.

Examples 5.2. Let n and m be integers such that $n \geq 2$ and $1 \leq m \leq n-1$. To a given Marcinkiewicz endpoint space $M(\Omega)$, let $G(t)$ be as in Theorem 5.1.

(i) If $M(\Omega) = \exp L^{\frac{n}{n-m}}(\Omega)$, then by Example 3.8 (i), $G(t) = \log(t)$, $t \in (1,\infty)$. Since

$$\liminf_{t\to\infty} \frac{\log(Ct)}{\log(t)} = 1 \quad \text{for every } C \geq 1,$$

and since G satisfies the Δ_2 condition, then, by Theorem 5.1, there is no largest Orlicz space $L^A(\Omega)$ in the embedding $W^m L^A(\Omega) \hookrightarrow \exp L^{\frac{n}{n-m}}(\Omega)$.

(ii) If $M(\Omega) = \exp L^{\frac{n}{n-m-q}}(\Omega)$, $q < n-1$, then by Example 3.8 (ii), $G(t) = \log^{1-\frac{q}{n-m}}(t)$, $t \in (1,\infty)$. Since

$$\liminf_{t\to\infty} \frac{\log^{1-\frac{q}{n-1}}(Ct)}{\log^{1-\frac{q}{n-1}}(t)} = 1 \quad \text{for every } C > 1,$$

and since G satisfies the Δ_2 condition, then, by Theorem 5.1, there is no largest Orlicz space $L^A(\Omega)$ in the embedding $W^m L^A(\Omega) \hookrightarrow \exp L^{\frac{n}{n-m-q}}(\Omega)$.

(iii) If $M(\Omega) = \exp\exp L^{\frac{n}{n-m}}(\Omega)$, then by Example 3.8 (iii), $G(t) = \log\log(t)$, $t \in (1,\infty)$. Since

$$\liminf_{t\to\infty} \frac{\log\log(Ct)}{\log\log(t)} = 1 \quad \text{for every } C \geq 1,$$

and since G satisfies the Δ_2 condition, then, by Theorem 5.1, there is no largest Orlicz space $L^A(\Omega)$ satisfying $W^m L^A(\Omega) \hookrightarrow \exp\exp L^{\frac{n}{n-m}}(\Omega)$.

At the end, we state a result which is in some sense a generalization of the implication (iii)→(i) in Theorem 5.1, no matter what the target space is like. In particular, it is not restricted to Marcinkiewicz endpoint spaces.

Theorem 5.3. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, and $|\Omega| = 1$. Let ϱ_R be an r.i. norm on $\mathcal{M}^+(I)$ and ϱ_D be the optimal r.i. domain norm in the Sobolev embedding $W^m L_{\varrho_D}(\Omega) \hookrightarrow L_{\varrho_R}(\Omega)$. Denote φ_D the fundamental function of the space $L_{\varrho_D}(\Omega)$ and $L^A(\Omega)$ the Orlicz space such that $\varphi_{L^A}(t) \simeq \varphi_D(t)$, $t \in [0,1)$. Suppose that*

$$L^A(\Omega) \subseteq L_{\varrho_D}(\Omega).$$

Then $W^m L^A(\Omega) \hookrightarrow L_{\varrho_R}(\Omega)$ and $L^A(\Omega)$ is the optimal Orlicz domain in this embedding.

Proof. Suppose that $W^m L^B(\Omega) \hookrightarrow L_{\varrho_R}(\Omega)$ where $L^B(\Omega)$ is an Orlicz space. Since $L_{\varrho_D}(\Omega)$ is the optimal domain, we have that $L^B(\Omega) \subseteq L_{\varrho_D}(\Omega)$ thus $L^B(\Omega) \hookrightarrow L_{\varrho_D}(\Omega)$. This means that $\varrho_D(f) \leq c\,\|f\|_{L^B}$ for every $f \in L^B(\Omega)$ and some c independent of f. Then

$$\varphi_D(t) = \varrho_D(\chi_{(0,t)}) \leq c\,\|\chi_{(0,t)}\|_{L^B} = c\,\varphi_{L^B}(t)$$

for all $t \in I$. By the definition of $L^A(\Omega)$ we have that $\varphi_{L^A} \simeq \varphi_D$, therefore

$$\varphi_{L^A}(t) \leq \tilde{c}\,\varphi_{L^B}(t), \quad t \in (0,1),$$

for some constant $\tilde{c} > 0$. Passing to inverse functions, we get that

$$\varphi_{L^B}^{-1}(s) \leq \varphi_{L^A}^{-1}(\tilde{c}s), \quad s \in \big(0, \varphi_{L^B}(1)\big),$$

that is, taking reciprocal values and $s \mapsto 1/(\tilde{c}t)$

$$\frac{1}{\varphi_{L^A}^{-1}\left(\frac{1}{t}\right)} \leq \frac{1}{\varphi_{L^B}^{-1}\left(\frac{1}{\tilde{c}t}\right)},$$

for every $t \in (T, \infty)$ where T is some positive constant. By the definition of fundamental function for Luxemburg norm we have

$$A(t) \leq B(\tilde{c}t),$$

for every $t \in (T, \infty)$, thus $A \prec B$. This implies that $L^B(\Omega) \hookrightarrow L^A(\Omega)$ hence $L^A(\Omega)$ is optimal. □

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