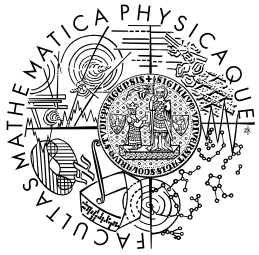

FACULTY
OF MATHEMATICS
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MASTER THESIS

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Microscopic nuclear models for open-shell nuclei

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Study programme:	Physics
Study branch:	Nuclear and Particle Physics

Prague 2017

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Abstract: Since the nucleus is a quantum many-body system consisting of constituents whose mutual interaction is not satisfactorily known, it is necessary to use approximate methods when describing the nucleus. Basic approximate approaches in the microscopic theory of the nucleus are the Hartree-Fock theory, Tamm-Dancoff approximation and random phase approximation. They are described in the first chapter of this thesis. The main aim was to develop microscopic models for open-shell nuclei with two valence particles or holes. They are described in the second chapter, which contains detailed derivations of the relevant formulae. These methods have been numerically implemented. The results of the calculations of the nuclear spectra and the electromagnetic transition probabilities are presented in the third chapter.

Keywords: Tamm-Dancoff approximation, random phase approximation, open-shell nuclei, nuclear spectra, electromagnetic transition probabilities

I would like to thank my supervisor František Knapp for his guiding, explanation of many physical problems ant time he dedicated to me.

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Introduction

Theoretical description of the atomic nucleus deals with two fundamental problems. The first one is that the nucleus is a quantum many-body system, for which it is impossible to solve the Schrödinger equation exactly. The second one is that the mutual interaction between the constituents (protons and neutrons) is still not satisfactorily known. The fundamental theory of strong interactions is the quantum chromodynamics, which exhibits a nonperturbative behavior in the energy region relevant for the description of the nuclei. Therefore, a derivation of the nucleon-nucleon interaction from the quantum chromodynamics is still not available. The consequence of these facts is that we have to treat the nuclei using various approximate approaches and introduce different seemingly inconsistent nuclear models for the description of diffrent nuclear phenomena.

One of the first nuclear models was the liquid drop model [1] describing the nucleus as a drop of a nucleon liquid, which may perform various collective motions. This model is suitable for a description of phenomena which can be understood without a detailed knowledge of the inner structure of the nucleus (nucleon degrees of freedom). A quite different approach is the microscopic theory describing the nucleus as a composite object consisting of nucleons interacting via strong interaction. A basis of the microscopic concept is the shell model or, in other words, the model of independent particles. Within this model, the individual nucleons occupy discrete energy levels similarly as the electrons in the atom and the nucleon states can be obtained from quantum-mechanical calculations by solving the Schrödinger equation.

An exact nucleon-nucleon potential is not known, but it is possible to gain an information about it from nucleon-nucleon scattering experiments. The models of the nucleon-nucleon interaction with parameters adjusted so that they reproduce accurately the scattering experimental data and properties of the deuteron are called the realistic potentials. In recent years a progress in the derivation of the nuclear forces from the chiral perturbation theory [2], which represents an effective theory of the strong interactions in the low energy region, was achieved. Properties of a chiral realistic potential, namely the $\mathrm{NNLO_{opt}}$ (Optimized Chiral Interaction at Next-to-Next-to-Leading Order) [3], in many-body calculations are object of research in this thesis.

One of our aims was to perform systematical calculations of spectra and electromagnetic transition probabilities of some doubly-magic nuclei in the framework of the Tamm-Dancoff approximation (TDA) and the random phase approximation (RPA). The main aim was to develop microscopical models for nuclei with two nucleons added to or removed from a doubly-magic core and carry out calculations for such open-shell nuclei within this models.

The first chapter describes basic microscopical models for closed-shell nuclei, namely the Hartree-Fock theory, TDA and RPA. The models treating open-shell nuclei with two valence particles or holes based on an analogy to the TDA and RPA are described in the second chapter, which contains detailed derivations of relevant formulae. Results of numerical calculations are presented in the third chapter.

1. Microscopic models for closed-shell nuclei

In this chapter we describe the ideas and techniques of the mean-field shell model, an approach considering non-interacting nucleons moving in an external field, with emphasis on the basic microscopic approach, namely the Hartree-Fock method. In the following sections two standard methods, which include part of the residual interaction neglected in the mean field approach and describe the collective excitations in nuclei, namely Tamm-Dancoff and random phase approximations, are discussed. The methods presented in this chapter are valid only for closed-shell nuclei, however, they can be extended to open-shell nuclei by introducing the formalism of quasi-particles (see e.g. [4]).

1.1 Nuclear mean field

Microscopic nuclear models consider the nucleus as a composite object consisting of A strongly interacting nucleons. Assuming that the nucleons interact via two-body force only, their mutual interaction is described by the potential $V(\vec{r}_i, \vec{r}_j)$, where $\vec{r}_i$ and $\vec{r}_j$ are the coordinates of the nucleons. This interaction can generally be very complicated and also depend on spins, isospins and momenta of the nucleons. The corresponding A-nucleon Schrödinger equation is

$$i\hbar\frac{\partial}{\partial t}\Psi(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_A) = H\Psi(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_A), \tag{1.1}$$

with Hamiltonian

$$H = \sum_{i=1}^{A} \frac{\vec{p}_i^{\,2}}{2m_i} + \sum_{\substack{i,j=1 \\ i<j}}^{A} V(\vec{r}_i, \vec{r}_j), \tag{1.2}$$

where m_i is the mass of the nucleon and $\vec{p}_i$ is the nucleon momentum operator. The first term in the Hamiltonian H represents the kinetic energy and the second one represents the potential energy. The many-body Schrödinger equation (1.1) cannot be solved exactly and, therefore, it is necessary to use approximate methods.

The mean field approximation converts the problem of mutually interacting nucleons into a problem of non-interacting nucleons occupying discrete energy levels (shells) in the mean field. In the nuclear many-body Hamiltonian (1.2) a summed single-particle potential energy in an external field $U(\vec{r}_i)$ can formally be added and subtracted:

$$H = \sum_{i=1}^{A} \frac{\vec{p}_i^{\,2}}{2m_i} + \sum_{i=1}^{A} U(\vec{r}_i) + \sum_{\substack{i,j=1 \\ i<j}}^{A} V(\vec{r}_i, \vec{r}_j) - \sum_{i=1}^{A} U(\vec{r}_i) = H_{\mathrm{mf}} + V_{\mathrm{res}}, \tag{1.3}$$

where

$$H_{\mathrm{mf}} = \sum_{i=1}^{A} \frac{\vec{p}_i^{\,2}}{2m_i} + \sum_{i=1}^{A} U(\vec{r}_i) = \sum_{i=1}^{A} h(\vec{r}_i) \tag{1.4}$$

is the nuclear mean-field Hamiltonian and

$$V_{\text{res}} = \sum_{\substack{i,j=1 \\ i<j}}^{A} V(\vec{r}_i, \vec{r}_j) - \sum_{i=1}^{A} U(\vec{r}_i) \tag{1.5}$$

is the residual interaction, which can be neglected in the first approximation. The problem remains how to determine the mean field, in particular, an optimal potential U that minimizes the residual interaction. Often one just selects a phenomenological mean-field potential (e. g. the three-dimensional harmonic oscillator, more realistic Woods-Saxon potential [5] or Nilsson potential for deformed nuclei [6])[1], which is a practical shortcut taken at the expense of theoretical preciseness. The other option is to attempt to calculate the mean field from the nucleon-nucleon interaction.

Within the mean-field shell model[2] we solve the problem of non-interacting nucleons in an external potential $U(\vec{r})$. For this potential we can obtain the single-particle stationary states ϕ_i by solving the Schrödinger equation

$$h(\vec{r})\phi_i(\vec{r}) = \varepsilon_i \phi_i(\vec{r}), \tag{1.6}$$

with single-particle Hamiltonian

$$h(\vec{r}) = \frac{\vec{p}^2}{2m_i} + U(\vec{r}). \tag{1.7}$$

Since nucleons are fermions, we should construct the A-nucleon wave functions in the form of antisymmetrized products of the single-particle wave functions

$$\Psi(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_A) = \mathcal{A}\left[\prod_{i=1}^{A} \phi_i(\vec{r}_i)\right] = \frac{1}{\sqrt{A!}} \begin{vmatrix} \phi_1(\vec{r}_1) & \phi_1(\vec{r}_2) & \ldots & \phi_1(\vec{r}_A) \\ \phi_2(\vec{r}_1) & \phi_2(\vec{r}_2) & & \\ \vdots & \vdots & \ddots & \\ \phi_A(\vec{r}_1) & \phi_A(\vec{r}_2) & & \phi_A(\vec{r}_A) \end{vmatrix} \tag{1.8}$$

which are called the Slater determinants[3].

1.2 Hartree-Fock method and particle-hole formalism

The Hartree-Fock (HF) method[4] allows us to obtain an optimal mean-field potential together with the corresponding single-particle states and their energies. The HF equations can be derived using the variational method in which we seek an optimal set of single-particle states $\{\phi_i(\vec{r})\}$ that minimize the ground-state energy of the nucleus

$$E_{\text{gs}} = \langle \Psi | H | \Psi \rangle, \tag{1.9}$$

[1] It turns out that for correct description of the energy levels which agrees with the observation of magic numbers, the potential should contain a spin-orbital term, as was shown by Mayer [7] and Haxel, Jensen and Suess [8]

[2] For more on the shell model see e. g. [9]

[3] For more on the quantum mechanics of many-body systems see e. g. [10]

[4] This method was originally developed for atomic physics (see [11] and [12]).

with

$$H = H_{\mathrm{mf}} + V_{\mathrm{res}}. \tag{1.10}$$

According to the variational method, the variation of the ground-state energy should vanish for small variations of the single-particle states $\phi_i(\vec{r}) \to \phi_i(\vec{r}) + \delta\phi_i(\vec{r})$. The energy (1.9) has to be varied under the constraint that the normalization of Ψ is preserved, i. e. $\langle\Psi|\Psi\rangle = 1$. This leads to the constrained variational problem

$$\delta\left(\frac{\langle\Psi|H|\Psi\rangle}{\langle\Psi|\Psi\rangle}\right) = 0, \tag{1.11}$$

which can be solved by using the method of Lagrange multipliers. It turns out that the Lagrange multipliers are the single-particle energies ε_i. The result is the HF equation

$$\frac{-\hbar^2}{2m_i}\Delta\phi_i(\vec{r}) + V_{\mathrm{HF}}(\{\phi_j(\vec{r})\})\phi_i(\vec{r}) = \varepsilon_i\phi_i(\vec{r}), \tag{1.12}$$

where

$$\begin{aligned} j &= 1, 2, \ldots, A, \\ i &= 1, 2, \ldots, \infty. \end{aligned} \tag{1.13}$$

This equation is a Schrödinger-like equation except that the potential is replaced with a functional of the unknown wave functions $V_{\mathrm{HF}}(\{\phi_j(\vec{r})\})$. The potential V_{HF} represents the HF mean field and acts on $\phi_i(\vec{r})$ in the following way

$$V_{\mathrm{HF}}(\{\phi_j(\vec{r})\})\phi_i(\vec{r}) = V_{\mathrm{H}}(\vec{r})\phi_i(\vec{r}) - \int \mathrm{d}^3\vec{r}\,' V_{\mathrm{F}}(\vec{r}\,', \vec{r})\phi_i(\vec{r}\,'). \tag{1.14}$$

The first term with the local potential

$$V_{\mathrm{H}}(\vec{r}) = \sum_{j=1}^{A} \int \mathrm{d}^3\vec{r}\,' \phi_j^*(\vec{r}\,')V(\vec{r}\,', \vec{r})\phi_j(\vec{r}\,'), \tag{1.15}$$

where $V(\vec{r}\,', \vec{r})$ is the potential of the nucleon-nucleon interaction, is called the Hartree term, and the second term with the non-local potential

$$V_{\mathrm{F}}(\vec{r}\,', \vec{r}) = \sum_{j=1}^{A} \phi_j^*(\vec{r}\,')V(\vec{r}\,', \vec{r})\phi_j(\vec{r}) \tag{1.16}$$

is called the Fock, or exchange term.

The equation system (1.12) for unknown functions $\phi_i(\vec{r})$ and corresponding energies ε_i is self-consistent because $V_{\mathrm{H}}(\vec{r})$ and $V_{\mathrm{F}}(\vec{r}\,', \vec{r})$ depend on the functions $\phi_j(\vec{r})$ which are the solutions of the equation system. Such equation system is usually solved by iterative way.[5] This means that we start with a set of guessed single-particle wave functions $\{\phi_j^0(\vec{r})\}_{j=1}^{A}$ (e. g. the harmonic oscillator basis) and use them to calculate the HF mean-field potential $V_{\mathrm{HF}}(\{\phi_j^0\})$. Then we substitute this potential into the equation (1.12). By solving this equation we obtain a new set of wave functions $\{\phi_i^1(\vec{r})\}_{i=1}^{\infty}$ with eigenenergies ε_i^1. With this new set of wave functions we generate a new potential $V_{\mathrm{HF}}(\{\phi_j^1\})$. Then we use this potential to solve the equation (1.12) and obtain another wave functions and eigenenergies.

[5] See also [13]

We repeat this procedure until we obtain wave functions (or eigenenergies) which do not differ from those of the previous iteration more than a preset limit. In such a way we obtain the self-consistent mean field $V_{\rm HF}(\vec{r})$ and the corresponding single-particle states ϕ_i (so called Hartree-Fock basis) with energies ε_i.

For description of many-body systems it is useful to introduce the formalism of creation and annihilation operators. So far, we have considered only one type of nucleon. Now we distinguish between protons and neutrons. Let $a^\dagger_\alpha(b^\dagger_\alpha)$ be a proton (neutron) creation operator creating proton (neutron) in the state $|\alpha\rangle$. The corresponding annihilation operators are a_α and b_α. These operators satisfy the following anticommutation and commutation relations

$$\begin{aligned}
\{a^\dagger_\alpha, a^\dagger_\beta\} = 0\,,\ \{a_\alpha, a_\beta\} = 0\,,\ \{a_\alpha, a^\dagger_\beta\} = \delta_{\alpha\beta} \quad & \forall\alpha,\beta, \\
\{b^\dagger_\alpha, b^\dagger_\beta\} = 0\,,\ \{b_\alpha, b_\beta\} = 0\,,\ \{b_\alpha, b^\dagger_\beta\} = \delta_{\alpha\beta} \quad & \forall\alpha,\beta, \\
[a^\dagger_\alpha, b^\dagger_\beta] = [a_\alpha, b_\beta] = [a^\dagger_\alpha, b_\beta] = [a_\alpha, b^\dagger_\beta] = 0 \quad & \forall\alpha,\beta.
\end{aligned} \tag{1.17}$$

The nuclear Hamiltonian, which has the form (1.2) in the coordinate representation, can be written in the formalism of creation and annihilation operators as

$$\begin{aligned}
H =& \sum_{\alpha\beta}\langle\alpha|t|\beta\rangle_\pi a^\dagger_\alpha a_\beta + \frac{1}{4}\sum_{\alpha\beta\gamma\delta}\langle\alpha\beta|V|\overline{\gamma\delta}\rangle_\pi a^\dagger_\alpha a^\dagger_\beta a_\delta a_\gamma + \sum_{\alpha\beta}\langle\alpha|t|\beta\rangle_\nu b^\dagger_\alpha b_\beta \\
&+ \frac{1}{4}\sum_{\alpha\beta\gamma\delta}\langle\alpha\beta|V|\overline{\gamma\delta}\rangle_\nu b^\dagger_\alpha b^\dagger_\beta b_\delta b_\gamma + \sum_{\alpha\beta\gamma\delta}\langle\alpha\beta|V|\gamma\delta\rangle_{\pi\nu} a^\dagger_\alpha b^\dagger_\beta a_\gamma b_\delta,
\end{aligned} \tag{1.18}$$

where

$$\langle\alpha|t|\beta\rangle_{\pi/\nu} = \int \mathrm{d}^3\vec{r}\,\phi^*_\alpha(\vec{r})\frac{\vec{p}^{\,2}}{2m}\phi_\beta(\vec{r}) \tag{1.19}$$

is the one-body matrix element of the kinetic energy operator (indices π and ν distinguish between protons and neutrons) and

$$\langle\alpha\beta|V|\gamma\delta\rangle_{\pi/\nu/\pi\nu} = \int \mathrm{d}^3\vec{r}\,'\mathrm{d}^3\vec{r}\,\phi^*_\alpha(\vec{r})\phi^*_\beta(\vec{r}\,')V(\vec{r},\vec{r}\,')\phi_\gamma(\vec{r})\phi_\delta(\vec{r}\,') \tag{1.20}$$

is the two-body interaction matrix element. We have introduced the antisymmetrized two-body matrix elements

$$\langle\alpha\beta|V|\overline{\gamma\delta}\rangle_{\pi/\nu} = \langle\alpha\beta|V|\gamma\delta\rangle_{\pi/\nu} - \langle\alpha\beta|V|\delta\gamma\rangle_{\pi/\nu} \tag{1.21}$$

with useful symmetry properties

$$\begin{aligned}
\langle\alpha\beta|V|\overline{\gamma\delta}\rangle_{\pi/\nu} &= -\langle\beta\alpha|V|\overline{\gamma\delta}\rangle_{\pi/\nu} = -\langle\alpha\beta|V|\overline{\delta\gamma}\rangle_{\pi/\nu} \\
&= \langle\beta\alpha|V|\overline{\delta\gamma}\rangle_{\pi/\nu} = \langle\gamma\delta|V|\overline{\alpha\beta}\rangle^*_{\pi/\nu}.
\end{aligned} \tag{1.22}$$

Proton-neutron two-body interaction matrix elements $\langle\alpha\beta|v|\gamma\delta\rangle_{\pi\nu}$ cannot be antisymmetrized because we consider protons and neutrons as distinguishable particles. Therefore, they have only symmetry property

$$\langle\alpha\beta|V|\gamma\delta\rangle_{\pi\nu} = \langle\gamma\delta|V|\alpha\beta\rangle^*_{\pi\nu}. \tag{1.23}$$

In the case of spherical nuclei we suppose the rotational symmetry of the mean field. Then the eigenstates $|\alpha\rangle = |n_a l_a j_a m_\alpha\rangle \equiv |a, m_\alpha\rangle$ of the mean field are characterized by four quantum numbers satisfying

$$\begin{aligned} \vec{l}^{\,2}|n_a l_a j_a m_\alpha\rangle &= \hbar^2 l_a(l_a+1)|n_a l_a j_a m_\alpha\rangle, \\ \vec{j}^{\,2}|n_a l_a j_a m_\alpha\rangle &= \hbar^2 j_a(j_a+1)|n_a l_a j_a m_\alpha\rangle, \\ j_z|n_a l_a j_a m_\alpha\rangle &= \hbar m_\alpha|n_a l_a j_a m_\alpha\rangle, \end{aligned} \tag{1.24}$$

where $\vec{l}$ is the orbital angular momentum operator, $\vec{j} = \vec{l} + \vec{s}$ is the nucleon total angular momentum operator and $\vec{s}$ is the spin 1/2 operator.

Now we introduce the particle-hole (ph) formalism and define the Hartree-Fock (particle-hole) vacuum $|\mathrm{HF}\rangle$. This vacuum is an approximation of the ground state of a closed-shell nucleus. In this state, the energy levels for protons and neutrons are fully occupied up to the Fermi level and the levels above the Fermi level are empty. If a nucleon is excited from a state below the Fermi level to a state above the Fermi level, a hole occurs in the previous state and a particle occurs in the new state. The single-particle states below the Fermi level are called hole states and those above the Fermi level are called particle states. The HF vacuum is annihilated by the operators a_α, where $|\alpha\rangle$ is a state whose energy ε_α is greater than the energy ε_F of the Fermi level, and by the operators $a_\beta^\dagger$, where $|\beta\rangle$ is a state with energy $\varepsilon_\beta \leq \varepsilon_\mathrm{F}$. Thus

$$\begin{aligned} a_\alpha|\mathrm{HF}\rangle = 0 \ , &\quad \varepsilon_\alpha > \varepsilon_\mathrm{F}, \\ a_\beta^\dagger|\mathrm{HF}\rangle = 0 \ , &\quad \varepsilon_\beta \leq \varepsilon_\mathrm{F} \end{aligned} \tag{1.25}$$

(the same holds for the neutron operators), where

$$|\mathrm{HF}\rangle = a_1^\dagger a_2^\dagger \ldots a_Z^\dagger b_1^\dagger b_2^\dagger \ldots b_N^\dagger|0\rangle, \tag{1.26}$$

which means that the HF vacuum consists of Z protons occupying the Z lowest proton states and N neutrons occupying the N lowest neutron states. The highest occupied levels define proton and neutron Fermi levels.

It is convenient to introduce the hole creation and annihilation operators $h_\beta^{\pi\dagger}, h_\beta^\pi, h_\beta^{\nu\dagger}, h_\beta^\nu$. The notation is

$$h_\beta^{\pi\dagger} = \tilde{a}_\beta \ , \quad h_\beta^\pi = \tilde{a}_\beta^\dagger \ , \quad h_\beta^{\nu\dagger} = \tilde{b}_\beta \ , \quad h_\beta^\nu = \tilde{b}_\beta^\dagger \, , \tag{1.27}$$

where

$$\tilde{a}_\beta = (-1)^{j_b+m_\beta} a_{-\beta} \ , \quad \tilde{b}_\beta = (-1)^{j_b+m_\beta} b_{-\beta} \, , \tag{1.28}$$

with

$$|-\beta\rangle = |n_b l_b j_b, -m_\beta\rangle \equiv |b, -m_\beta\rangle. \tag{1.29}$$

The operators $\tilde{a}_\beta, \tilde{b}_\beta, a_\beta^\dagger$ and $b_\beta^\dagger$ are spherical tensors of rank j_b[6], thus the hole operators $h_\beta^{\pi\dagger}, h_\beta^{\nu\dagger}, \tilde{h}_\beta^\pi = -a_\beta^\dagger$ and $\tilde{h}_\beta^\nu = -b_\beta^\dagger$ are also spherical tensors of rank j_b.

[6]They satisfy the commutations relations

$$[J_z, T_{\lambda\mu}] = \hbar\mu T_{\lambda\mu}, \tag{1.30}$$

$$[J_\pm, T_{\lambda\mu}] = \hbar\sqrt{\lambda(\lambda+1) - \mu(\mu\pm1)}\, T_{\lambda,\mu\pm1} \tag{1.31}$$

defining a spherical tensor $T_{\lambda\mu}$ of rank λ (see e.g. [14]). This can be verified using the expressions

The simplest excitation of the HF vacuum is a one-particle-one-hole (1-p-1-h) configuration

$$|\mathrm{p}_\alpha \mathrm{h}_\beta\rangle_\pi = a^\dagger_\alpha h^{\pi\dagger}_\beta |\mathrm{HF}\rangle \quad \text{or} \quad |\mathrm{p}_\alpha \mathrm{h}_\beta\rangle_\nu = b^\dagger_\alpha h^{\nu\dagger}_\beta |\mathrm{HF}\rangle, \tag{1.35}$$

where $|\alpha\rangle$ is a particle state and $|\beta\rangle$ is a hole state. Analogously, we can excite n nucleons and create n-p-n-h configuration. The description of the excited states of a closed-shell nucleus relies on the treatment of this type of excitations within different approximation schemes, including their mixing through the residual interaction. Furthermore, we will be interested in nuclei with two nucleons added to or removed from a closed-shell nucleus. States of such nuclei can be described by means of particle-particle (pp) or hole-hole (hh) configurations

$$a^\dagger_\alpha a^\dagger_{\alpha'} |\mathrm{HF}\rangle \ , \quad b^\dagger_\alpha b^\dagger_{\alpha'} |\mathrm{HF}\rangle \ , \quad h^{\pi\dagger}_\beta h^{\pi\,\dagger}_{\beta'} |\mathrm{HF}\rangle \ , \quad h^{\nu\dagger}_\beta h^{\nu\,\dagger}_{\beta'} |\mathrm{HF}\rangle. \tag{1.36}$$

This will be discussed in the next chapter.

In the formalism of creation and annihilation operators, there is an alternative way to derive the HF equations which uses Wick's theorem (see e. g. [4]) in the ph representation. This means that we use normal ordering and contractions with respect to the HF vacuum. Starting from the Hamiltonian (1.18) and using Wick's theorem we obtain the Hamiltonian in a form consisting of one-body operators (for protons and neutrons), two-body operators and a constant term. Then we change the single-particle basis so that the set of creation operators $\{a^\dagger_\alpha\}$ is transformed to a new set $\{a^{\dagger\prime}_\alpha\}$. This is accomplished by a unitary transformation

$$a^\dagger_\alpha = \sum_{\alpha'} U^*_{\alpha\alpha'} a^{\dagger\prime}_{\alpha'} \quad , \quad a_\alpha = \sum_{\alpha'} U_{\alpha\alpha'} a'_{\alpha'}. \tag{1.37}$$

We introduce an analogous transformation for the neutron operators. Moreover, we require the new basis to be such that one-body operators in the Hamiltonian are diagonal. Then we end up with the HF equations

$$\langle\alpha|t|\beta\rangle_\pi + \sum_h \langle h\alpha|V|\overline{h\beta}\rangle_\pi + \sum_h \langle \alpha h|V|\beta h\rangle_{\pi\nu} = \varepsilon^\pi_\alpha \delta_{\alpha\beta}, \tag{1.38}$$

$$\langle\alpha|t|\beta\rangle_\nu + \sum_h \langle h\alpha|V|\overline{h\beta}\rangle_\nu + \sum_h \langle h\alpha|V|h\beta\rangle_{\pi\nu} = \varepsilon^\nu_\alpha \delta_{\alpha\beta} \tag{1.39}$$

for the computation of the single-particle energies ε^π_α for protons and ε^ν_α for neutrons, where the index h denotes the hole states.

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of the nuclear angular momentum projection operator J_z and the ladder operators $J_\pm$ in the formalism of creation and annihilation operators

$$J_z = \hbar \sum_\alpha m_\alpha a^\dagger_\alpha a_\alpha + \hbar \sum_\alpha m_\alpha b^\dagger_\alpha b_\alpha, \tag{1.32}$$

$$J_\pm = \hbar \sum_\alpha m^\mp_\alpha a^\dagger_\alpha a_{\alpha\mp1} + \hbar \sum_\alpha m^\mp_\alpha b^\dagger_\alpha b_{\alpha\mp1}, \tag{1.33}$$

where

$$m^\pm_\alpha = \sqrt{j_a(j_a+1) - m_\alpha(m_\alpha \pm 1)} \ , \quad |\alpha \pm 1\rangle = |a, m_\alpha \pm 1\rangle. \tag{1.34}$$

If we choose set of single-particle states which satisfy the HF equations, the one-body parts of the Hamiltonian become diagonal and the Hamiltonian takes the form

$$H = H_{\rm mf} + V_{\rm res}, \tag{1.40}$$

where

$$\begin{aligned} H_{\rm mf} =& \sum_{\alpha} \varepsilon^{\pi}_{\alpha} a^{\dagger}_{\alpha} a_{\alpha} + \sum_{\alpha} \varepsilon^{\nu}_{\alpha} b^{\dagger}_{\alpha} b_{\alpha} - \frac{1}{2} \sum_{hh'} \langle hh' | V | \overline{hh'} \rangle_{\pi} \\ & - \frac{1}{2} \sum_{hh'} \langle hh' | V | \overline{hh'} \rangle_{\nu} - \sum_{hh'} \langle hh' | V | hh' \rangle_{\pi\nu} \end{aligned} \tag{1.41}$$

is the HF mean-field Hamiltonian and

$$\begin{aligned} V_{\rm res} =& \frac{1}{4} \sum_{\alpha\beta\gamma\delta} \langle \alpha\beta | V | \overline{\gamma\delta} \rangle_{\pi} : a^{\dagger}_{\alpha} a^{\dagger}_{\beta} a_{\delta} a_{\gamma} : + \frac{1}{4} \sum_{\alpha\beta\gamma\delta} \langle \alpha\beta | V | \overline{\gamma\delta} \rangle_{\nu} : b^{\dagger}_{\alpha} b^{\dagger}_{\beta} b_{\delta} b_{\gamma} : \\ & + \sum_{\alpha\beta\gamma\delta} \langle \alpha\beta | V | \gamma\delta \rangle_{\pi\nu} : a^{\dagger}_{\alpha} b^{\dagger}_{\beta} a_{\gamma} b_{\delta} : \end{aligned} \tag{1.42}$$

is the residual interaction, with normal ordering with respect to the HF vacuum.

Since the HF vacuum represents an approximation of the nuclear ground state, the ground-state energy calculated as the expectation value of the Hamiltonian is

$$\begin{aligned} E_{\rm HF} = \langle {\rm HF} | H | {\rm HF} \rangle =& \sum_{h} \varepsilon^{\pi}_{h} + \sum_{h} \varepsilon^{\nu}_{h} - \frac{1}{2} \sum_{hh'} \langle hh' | V | \overline{hh'} \rangle_{\pi} \\ & - \frac{1}{2} \sum_{hh'} \langle hh' | V | \overline{hh'} \rangle_{\nu} - \sum_{hh'} \langle hh' | V | hh' \rangle_{\pi\nu}. \end{aligned} \tag{1.43}$$

1.3 Tamm-Dancoff Approximation

In the previous sections the nucleus was described as a system of non-interacting nucleons in the mean-field potential. The wave function of a nuclear state was given by a Slater determinant corresponding to a particular occupation of the mean-field single-particle states by nucleons. The ground state of a doubly-magic nucleus was approximated by the HF vacuum and the excited states were described as n-p-n-h configurations. In this section we take into account a part of the residual interaction, neglected in the mean-field approximation, and introduce the concept of configuration mixing. This means that due to the residual interaction nucleon configurations are mixed and the nuclear wave function becomes a linear combination of many Slater determinants.

The simplest scheme of configuration mixing of ph excitations in doubly magic nuclei is the Tamm-Dancoff approximation (TDA), which takes into account only 1-p-1-h configurations. Within the TDA the ground state is the HF vacuum and the excited states are linear combinations of 1-p-1-h excitations of $|{\rm HF}\rangle$. The linear combinations are obtained by diagonalizing the nuclear Hamiltonian in a basis of these excitations.

In the angular-momentum-coupled representation, the ph basis states are

$$|ab^{-1}; JM\rangle = \sum_{m_{\alpha} m_{\beta}} (j_a m_{\alpha} j_b m_{\beta} | JM) c^{\dagger}_{\alpha} h^{\dagger}_{\beta} | {\rm HF} \rangle, \tag{1.44}$$

where $|\alpha\rangle$ is a particle state, $|\beta\rangle$ is a hole state, J and M are quantum numbers related to the square and projection of the total angular momentum of the ph state, $(j_a m_\alpha j_b m_\beta|JM)$ is the Clebsch-Gordan coefficient and $c^\dagger_\alpha, h^\dagger_\beta$ are $a^\dagger_\alpha, h^{\pi\dagger}_\beta$ in the case of a proton excitation or $b^\dagger_\alpha, h^{\nu\dagger}_\beta$ in the case of a neutron excitation (this notation is used throughout the thesis). Now we derive the matrix elements $\langle ab^{-1}; JM|H|cd^{-1}; JM\rangle$ of the Hamiltonian

$$H = H_{\rm mf} + V_{\rm res}, \tag{1.45}$$

which is given by (1.41) and (1.42).

If a, b, c, d denote proton states, the matrix element of $H_{\rm mf}$ is

$$\begin{aligned}
&\langle ab^{-1};JM|H_{\rm mf}|cd^{-1};JM\rangle \\
&\quad= \sum_{\alpha'} \varepsilon^\pi_{\alpha'} \sum_{\substack{m_\alpha m_\beta \\ m_\gamma m_\delta}} (j_a m_\alpha j_b m_\beta|JM)(j_c m_\gamma j_d m_\delta|JM)\langle {\rm HF}|h^\pi_\beta a_\alpha a^\dagger_{\alpha'} a_{\alpha'} a^\dagger_\gamma h^{\pi\dagger}_\delta|{\rm HF}\rangle \\
&\quad+ \sum_{\alpha'} \varepsilon^\nu_{\alpha'} \sum_{\substack{m_\alpha m_\beta \\ m_\gamma m_\delta}} (j_a m_\alpha j_b m_\beta|JM)(j_c m_\gamma j_d m_\delta|JM)\langle {\rm HF}|h^\pi_\beta a_\alpha b^\dagger_{\alpha'} b_{\alpha'} a^\dagger_\gamma h^{\pi\dagger}_\delta|{\rm HF}\rangle \\
&\quad+ K \sum_{\substack{m_\alpha m_\beta \\ m_\gamma m_\delta}} (j_a m_\alpha j_b m_\beta|JM)(j_c m_\gamma j_d m_\delta|JM)\langle {\rm HF}|h^\pi_\beta a_\alpha a^\dagger_\gamma h^{\pi\dagger}_\delta|{\rm HF}\rangle,
\end{aligned} \tag{1.46}$$

where

$$K \equiv -\frac{1}{2}\sum_{hh'}\langle hh'|V|\overline{hh'}\rangle_\pi - \frac{1}{2}\sum_{hh'}\langle hh'|V|\overline{hh'}\rangle_\nu - \sum_{hh'}\langle hh'|V|hh'\rangle_{\pi\nu}. \tag{1.47}$$

Using the anticommutation and commutation relations (1.17) one gets

$$\begin{aligned}
\langle {\rm HF}|h^\pi_\beta a_\alpha a^\dagger_{\alpha'} a_{\alpha'} a^\dagger_\gamma h^{\pi\dagger}_\delta|{\rm HF}\rangle &= \delta_{\beta\delta}\delta_{\alpha\alpha'}\delta_{\alpha'\gamma} - \delta_{-\beta\alpha'}\delta_{\alpha\gamma}\delta_{\alpha',-\delta} + \delta_{\beta\delta}\delta_{\alpha\gamma}\theta(\varepsilon_{\rm F} - \varepsilon_{\alpha'}), \\
\langle {\rm HF}|h^\pi_\beta a_\alpha b^\dagger_{\alpha'} b_{\alpha'} a^\dagger_\gamma h^{\pi\dagger}_\delta|{\rm HF}\rangle &= \delta_{\beta\delta}\delta_{\alpha\gamma}\theta(\varepsilon_{\rm F} - \varepsilon_{\alpha'}), \\
\langle {\rm HF}|h^\pi_\beta a_\alpha a^\dagger_\gamma h^{\pi\dagger}_\delta|{\rm HF}\rangle &= \delta_{\beta\delta}\delta_{\alpha\gamma},
\end{aligned}$$

where

$$\theta(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \tag{1.48}$$

is the Heaviside step function. Substituting into (1.46) and using the orthogonality relation for Clebsch-Gordan coefficients

$$\sum_{m_\alpha m_\beta} (j_a m_\alpha j_b m_\beta|JM)(j_a m_\alpha j_b m_\beta|J'M') = \delta_{JJ'}\delta_{MM'} \tag{1.49}$$

one gets

$$\begin{aligned}
\langle ab^{-1}; JM|H_{\rm mf}|cd^{-1}; JM\rangle &= \delta_{ac}\delta_{bd}(\varepsilon_a - \varepsilon_b + \sum_h \varepsilon^\pi_h + \sum_h \varepsilon^\nu_h + K) \\
&= \delta_{ac}\delta_{bd}(\varepsilon_a - \varepsilon_b + E_{\rm HF}).
\end{aligned} \tag{1.50}$$

Since this matrix element is diagonal and we are interested in the excitation energies, the ground-state energy $E_{\rm HF}$ can be omitted. If a, b, c, d are neutron

states, we analogously get the same result. If a, b are proton states and c, d are neutron states or vice versa, it can be easily shown that the matrix element of H_{mf} vanishes.

If a, b, c, d are proton states, the matrix element of V_{res} is

$$\begin{aligned}
&\langle ab^{-1}; JM|V_{\mathrm{res}}|cd^{-1}; JM\rangle_\pi \\
&\quad = \frac{1}{4}\sum_{\alpha'\beta'\gamma'\delta'} \langle \alpha'\beta'|V|\overline{\gamma'\delta'}\rangle_\pi \sum_{\substack{m_\alpha m_\beta \\ m_\gamma m_\delta}} (j_a m_\alpha j_b m_\beta|JM)(j_c m_\gamma j_d m_\delta|JM) \\
&\quad \times \langle \mathrm{HF}|h^\pi_\beta a_\alpha : a^\dagger_{\alpha'} a^\dagger_{\beta'} a_{\delta'} a_{\gamma'} : a^\dagger_\gamma h^{\pi\dagger}_\delta|\mathrm{HF}\rangle \\
&\quad + \frac{1}{4}\sum_{\alpha'\beta'\gamma'\delta'} \langle \alpha'\beta'|V|\overline{\gamma'\delta'}\rangle_\nu \sum_{\substack{m_\alpha m_\beta \\ m_\gamma m_\delta}} (j_a m_\alpha j_b m_\beta|JM)(j_c m_\gamma j_d m_\delta|JM) \\
&\quad \times \langle \mathrm{HF}|h^\pi_\beta a_\alpha : b^\dagger_{\alpha'} b^\dagger_{\beta'} b_{\delta'} b_{\gamma'} : a^\dagger_\gamma h^{\pi\dagger}_\delta|\mathrm{HF}\rangle \\
&\quad + \sum_{\alpha'\beta'\gamma'\delta'} \langle \alpha'\beta'|V|\gamma'\delta'\rangle_{\pi\nu} \sum_{\substack{m_\alpha m_\beta \\ m_\gamma m_\delta}} (j_a m_\alpha j_b m_\beta|JM)(j_c m_\gamma j_d m_\delta|JM) \\
&\quad \times \langle \mathrm{HF}|h^\pi_\beta a_\alpha : a^\dagger_{\alpha'} b^\dagger_{\beta'} a_{\gamma'} b_{\delta'} : a^\dagger_\gamma h^{\pi\dagger}_\delta|\mathrm{HF}\rangle. \qquad (1.51)
\end{aligned}$$

Since

$$\langle \mathrm{HF}|h^\pi_\beta a_\alpha : b^\dagger_{\alpha'} b^\dagger_{\beta'} b_{\delta'} b_{\gamma'} : a^\dagger_\gamma h^{\pi\dagger}_\delta|\mathrm{HF}\rangle = 0, \qquad (1.52)$$

$$\langle \mathrm{HF}|h^\pi_\beta a_\alpha : a^\dagger_{\alpha'} b^\dagger_{\beta'} a_{\gamma'} b_{\delta'} : a^\dagger_\gamma h^{\pi\dagger}_\delta|\mathrm{HF}\rangle = 0, \qquad (1.53)$$

the last two terms in (1.51) vanish. After some tedious manipulations, which are described in [15], one ends up with

$$\begin{aligned}
&\langle ab^{-1}; JM|V_{\mathrm{res}}|cd^{-1}; JM\rangle_\pi \\
&\quad = \sum_{J'} (2J'+1)(-1)^{j_b+j_c+J'} \langle ad; J'|V|bc; J'\rangle_\pi \begin{Bmatrix} j_a & j_b & J \\ j_c & j_d & J' \end{Bmatrix}, \qquad (1.54)
\end{aligned}$$

where

$$\langle ad; J'|V|bc; J'\rangle_\pi = \sum_{\substack{m_\alpha m_\beta \\ m_\gamma m_\delta}} (j_a m_\alpha j_d m_\delta|J'M')(j_b m_\beta j_c m_\gamma|J'M')\langle \alpha\delta|V|\overline{\beta\gamma}\rangle_\pi \qquad (1.55)$$

is the angular-momentum-coupled two-body interaction matrix element and $\begin{Bmatrix} j_a & j_b & J \\ j_c & j_d & J' \end{Bmatrix}$ is the 6j symbol[7]. Using the symmetry property

$$\langle ad; J'|V|bc; J'\rangle_\pi = (-1)^{j_b+j_c+J'+1} \langle ad; J'|V|cb; J'\rangle_\pi, \qquad (1.56)$$

one gets

$$\langle ab^{-1}; JM|V_{\mathrm{res}}|cd^{-1}; JM\rangle_\pi = -\sum_{J'} (2J'+1) \begin{Bmatrix} j_a & j_b & J \\ j_c & j_d & J' \end{Bmatrix} \langle ad; J'|V|cb; J'\rangle_\pi. \qquad (1.57)$$

[7] See [16]

This result is called the Pandya transformation and it is valid also in the case, when a, b, c, d are neutron states, except that the matrix element $\langle ad; J'|V|cb; J'\rangle_\pi$ is replaced with the analogous matrix element $\langle ad; J'|V|cb; J'\rangle_\nu$.

If a, b are proton states and c, d are neutron states, it can be easily shown that only the third term in $V_{\rm res}$ given by (1.42) contributes and, therefore, the matrix element of $V_{\rm res}$ is

$$\langle ab^{-1}; JM|V_{\rm res}|cd^{-1}; JM\rangle_{\pi\nu} = \sum_{\alpha'\beta'\gamma'\delta'} \langle \alpha'\beta'|V|\gamma'\delta'\rangle_{\pi\nu} \sum_{\substack{m_\alpha m_\beta \\ m_\gamma m_\delta}} (j_a m_\alpha j_b m_\beta|JM) \times (j_c m_\gamma j_d m_\delta|JM) \langle {\rm HF}|h^\pi_\beta a_\alpha : a^\dagger_{\alpha'} b^\dagger_{\beta'} a_{\gamma'} b_{\delta'} : b^\dagger_\gamma h^{\nu\dagger}_\delta|{\rm HF}\rangle. \quad (1.58)$$

For a non-zero contribution the states α', δ' have to be particle states and the states β', γ' have to be hole states. With this observation one gets

$$\langle {\rm HF}|h^\pi_\beta a_\alpha : a^\dagger_{\alpha'} b^\dagger_{\beta'} a_{\gamma'} b_{\delta'} : b^\dagger_\gamma h^{\nu\dagger}_\delta|{\rm HF}\rangle = (-1)^{j_b+m_\beta+j_d+m_\delta} \delta_{-\delta\beta'} \delta_{\gamma\delta'} \delta_{-\beta\gamma'} \delta_{\alpha\alpha'}. \quad (1.59)$$

Substitution into (1.58) yields

$$\langle ab^{-1}; JM|V_{\rm res}|cd^{-1}; JM\rangle_{\pi\nu} = \sum_{\substack{m_\alpha m_\beta \\ m_\gamma m_\delta}} (-1)^{j_b+m_\beta+j_d+m_\delta} (j_a m_\alpha j_b m_\beta|JM)(j_c m_\gamma j_d m_\delta|JM) \langle \alpha\ -\delta|V|-\beta\,\gamma\rangle_{\pi\nu}.$$

Performing the same manipulations that led to the formula (1.54), we obtain analogous result

$$\langle ab^{-1}; JM|V_{\rm res}|cd^{-1}; JM\rangle_{\pi\nu} = \sum_{J'} (2J'+1)(-1)^{j_b+j_c+J'} \langle ad; J'|V|bc; J'\rangle_{\pi\nu} \begin{Bmatrix} j_a & j_b & J \\ j_c & j_d & J' \end{Bmatrix}, \quad (1.60)$$

where the matrix elements $\langle ad; J'|V|bc; J'\rangle_{\pi\nu}$ don't have any symmetry properties analogous to (1.56).

If a, b are neutron states and c, d are proton states, the matrix element of $V_{\rm res}$ is

$$\langle ab^{-1}; JM|V_{\rm res}|cd^{-1}; JM\rangle_{\nu\pi} = \sum_{J'} (2J'+1)(-1)^{j_a+j_d+J'} \langle cb; J'|V|da; J'\rangle_{\pi\nu} \begin{Bmatrix} j_c & j_d & J \\ j_a & j_b & J' \end{Bmatrix}, \quad (1.61)$$

which was obtained from (1.60) using the symmetry of the TDA Hamiltonian matrix (any Hamiltonian matrix is Hermitian and our particle-hole matrix elements are real).

To solve the TDA eigenvalue problem for a given angular momentum and parity J^π, we form all possible ph states $|ab^{-1}; J^\pi M\rangle$ with a common value of M (the matrix elements of the Hamiltonian are independent of M) in the given single-particle valence space, using the HF self-consistent basis[8]. Then we construct the Hamiltonian matrix in the basis of these states using the formulae above.

[8]Since HF calculations require computers, the first TDA calculations were performed with phenomenological shell model wave functions [17].

Diagonalization of this matrix yields the eigenenergies E_ν and the corresponding eigenstates

$$|\nu; J^\pi M\rangle = \sum_{ab} X^\nu_{ab} |ab^{-1}; J^\pi M\rangle, \tag{1.62}$$

which fulfill the orthonormality condition

$$\langle \nu; J^\pi M | \nu'; J^\pi M\rangle = \sum_{ab} X^{\nu *}_{ab} X^{\nu'}_{ab} = \delta_{\nu\nu'}. \tag{1.63}$$

This task can be formulated in the form of the TDA equations

$$\sum_{cd} \langle ab^{-1}; J^\pi M | H | cd^{-1}; J^\pi M\rangle X^\nu_{cd} = E_\nu X^\nu_{ab}. \tag{1.64}$$

The TDA represents the most simple microscopical description of collectivity in nuclei, which is nicely explained by a schematic model with a separable interaction (see [18] or [19]).

Let us consider the electromagnetic transitions from the ground state $|\mathrm{HF}\rangle$ to an excited state $|\nu\rangle$ with the angular momentum J, which is given by (1.62). Since the angular momentum of the ground states of even-even nuclei is zero, the reduced transition probability of the type X (electric or magnetic) and multipolarity λ is

$$B(X\lambda; 0^+_{\mathrm{gs}} \to \nu) = \delta_{\lambda J} |\langle \nu || M^{(X)}_J || \mathrm{HF}\rangle|^2, \tag{1.65}$$

where (for derivation see [15])

$$\langle \nu || M^{(X)}_J || \mathrm{HF}\rangle = \sum_{ab} X^\nu_{ab} \langle a || M^{(X)}_J || b\rangle \tag{1.66}$$

is the reduced matrix element of the multipole operator of the type X and multipolarity J.

1.4 Random Phase Approximation

In the TDA framework we admit configuration mixing for the excited states, while the ground state $|\mathrm{HF}\rangle$ remains unchanged. This complete omission of the residual interaction in the ground state is a severe drawback of the TDA method. The random phase approximation[9] (RPA) is a sophisticated *ph* theory which extends the TDA by including correlations in the nuclear ground state. This means that the ground state is no more given by the HF vacuum. In this theory the ground state is a correlated state containing the *ph* vacuum and a part of *n-p-n-h* configurations. These correlations are responsible for enhancement of some electromagnetic transition probabilities. Typically, the correlations in the RPA ground state lead to strong collectivity of the electric octupole excitation to the first 3^- state.

[9]This method was originally developed by Bohm and Pines [20] in the theory of the plasma oscillations of the electron gas. The first applications of the RPA to nuclear physics were made by Baranger [21] and Sawicki [22].

The RPA equations are usually derived by using the equation-of-motion method [23], which can also be used to derive the HF equations or the TDA. The aim is to find the eigenenergies and eigenvectors of the Hamiltonian H:

$$H|\nu\rangle = E_\nu|\nu\rangle. \tag{1.67}$$

The eigenvectors $|\nu\rangle$ can be expressed by means of the so called phonon creation operators $Q^\dagger_\nu$:

$$|\nu\rangle = Q^\dagger_\nu|0\rangle, \tag{1.68}$$

where $|0\rangle$ is the ground state, which is defined as the vacuum for phonons. This means that the annihilation operator Q_ν, which is the Hermitian conjugate of the creation operator $Q^\dagger_\nu$, annihilates the vacuum, i.e.

$$Q_\nu|0\rangle = 0 \quad \forall\nu. \tag{1.69}$$

Using the above relations we convert the Schrödinger equation (1.67) to the equation of motion

$$[H, Q^\dagger_\nu]|0\rangle = (E_\nu - E_0)Q^\dagger_\nu|0\rangle, \tag{1.70}$$

where E_0 is the ground state energy, i.e.

$$H|0\rangle = E_0|0\rangle. \tag{1.71}$$

Multiplying the equation of motion (1.70) from the left by the state $\langle 0|\delta Q$, where $\delta Q^\dagger$ is the variation of the phonon operator $Q^\dagger_\nu$, we get

$$\langle 0|\delta Q[H, Q^\dagger_\nu]|0\rangle = (E_\nu - E_0)\langle 0|\delta Q Q^\dagger_\nu|0\rangle. \tag{1.72}$$

Since $\langle 0|Q^\dagger_\nu = \langle 0|HQ^\dagger_\nu = 0$, we can write this equation in the commutator form

$$\langle 0|\big[\delta Q, [H, Q^\dagger_\nu]\big]|0\rangle = (E_\nu - E_0)\langle 0|[\delta Q, Q^\dagger_\nu]|0\rangle. \tag{1.73}$$

Until now the derivation was exact. To proceed we have to choose a concrete form of the phonon creation operator and, in the RPA case, replace the unknown vacuum $|0\rangle$ with some approximate vacuum state.

Since it is admitted, in the RPA, that the true ground state is not simply the HF vacuum, we can not only create a *ph* pair but also annihilate one. Therefore, in the angular-momentum coupled representation, the RPA phonon creation operator is (see [4] or [15])

$$Q^\dagger_\nu = \sum_{ab} \big[X^\nu_{ab} A^\dagger_{ab}(JM) - Y^\nu_{ab}\tilde{A}_{ab}(JM)\big], \tag{1.74}$$

where

$$A^\dagger_{ab}(JM) \equiv [c^\dagger_a h^\dagger_b]_{JM} = \sum_{m_\alpha m_\beta} (j_a m_\alpha j_b m_\beta|JM) c^\dagger_\alpha h^\dagger_\beta \tag{1.75}$$

is the *ph* creation operator ($|\alpha\rangle$ is a particle state and $|\beta\rangle$ is a hole state),

$$\tilde{A}_{ab}(JM) = (-1)^{J+M}\big(A^\dagger_{ab}(J\ -M)\big)^\dagger \tag{1.76}$$

is the adjoint tensor operator and X^ν_{ab}, Y^ν_{ab} are the amplitudes. The minus sign in (1.74) has been chosen for convenience. The corresonding phonon annihilation operator obtained by Hermitian conjugation is

$$Q_\nu = \sum_{ab} \left[X^{\nu *}_{ab} A_{ab}(JM) - Y^{\nu *}_{ab} \tilde{A}^\dagger_{ab}(JM) \right]. \tag{1.77}$$

We have two kinds of variations δQ, namely

$$\delta Q = A_{ab}(JM) \ , \quad \delta Q = \tilde{A}^\dagger_{ab}(JM). \tag{1.78}$$

Substituting these variations and (1.74) into the equation (1.73) and using the quasi-boson approximation, i. e. the replacement of the unknown vacuum state $|0\rangle$ with the HF vacuum in the vacuum expectation values of the commutators, we can obtain (for derivation see [15])

$$\sum_{cd} A_{ab,cd} X^\nu_{cd} + \sum_{cd} B_{ab,cd} Y^\nu_{cd} = \hbar\Omega_\nu X^\nu_{ab}, \tag{1.79}$$

$$-\sum_{cd} B^*_{ab,cd} X^\nu_{cd} - \sum_{cd} A^*_{ab,cd} Y^\nu_{cd} = \hbar\Omega_\nu Y^\nu_{ab}, \tag{1.80}$$

or in matrix form

$$AX^\nu + BY^\nu = \hbar\Omega_\nu X^\nu, \tag{1.81}$$

$$-B^* X^\nu - A^* Y^\nu = \hbar\Omega_\nu Y^\nu, \tag{1.82}$$

where

$$A_{ab,cd} = \langle \mathrm{HF} | A_{ab}(JM) H A^\dagger_{cd}(JM) | \mathrm{HF} \rangle = \langle ab^{-1}; JM | H | cd^{-1}; JM \rangle \tag{1.83}$$

is the TDA matrix,

$$B_{ab,cd} = \langle \mathrm{HF} | A_{ab}(JM) \tilde{A}_{cd}(JM) H | \mathrm{HF} \rangle \tag{1.84}$$

is the so called correlation matrix and $\hbar\Omega_\nu = E_\nu - E_0$ is the excitation energy of the state $|\nu\rangle$. The equations (1.81) and (1.82) can be combined into one matrix equation

$$\begin{pmatrix} A & B \\ -B^* & -A^* \end{pmatrix} \begin{pmatrix} X^\nu \\ Y^\nu \end{pmatrix} = \hbar\Omega_\nu \begin{pmatrix} X^\nu \\ Y^\nu \end{pmatrix} \tag{1.85}$$

whose elements themselves are matrices. The TDA matrix A is Hermitian and the correlation matrix B is symmetric, but the "supermatrix" in the RPA matrix equation (1.85) is non-Hermitian. Thus the corresponding eigenvalues are not necessarily real.

We know how to construct the TDA matrix A from the previous section. Now we derive the matrix elements of the correlation matrix B. The formula (1.84) gives

$$B_{ab,cd} = (-1)^{J+M} \sum_{\substack{m_\alpha m_\beta \\ m_\gamma m_\delta}} (j_a m_\alpha j_b m_\beta | JM)(j_c m_\gamma j_d m_\delta | J - M) \langle \mathrm{HF} | h_\beta c_\alpha h_\delta c_\gamma H | \mathrm{HF} \rangle, \tag{1.86}$$

where the Hamiltonian H is given by (1.40). It is easy to deduce that contribution of the mean-field Hamiltonian H_{mf} given by (1.41) vanishes. If a, b, c, d are

proton states, only the first term in the residual interaction $V_{\rm res}$ given by (1.42) contributes and, therefore, we have

$$\begin{aligned} B^{\pi}_{ab,cd} =&(-1)^{J+M} \sum_{\substack{m_\alpha m_\beta \\ m_\gamma m_\delta}} (j_a m_\alpha j_b m_\beta | JM)(j_c m_\gamma j_d m_\delta | J - M) \\ &\times \frac{1}{4} \sum_{\alpha'\beta'\gamma'\delta'} \langle \alpha'\beta' | V | \overline{\gamma'\delta'} \rangle_\pi \langle {\rm HF} | h^\pi_\beta a_\alpha h^\pi_\delta a_\gamma : a^\dagger_{\alpha'} a^\dagger_{\beta'} a_{\delta'} a_{\gamma'} : | {\rm HF} \rangle. \end{aligned} \tag{1.87}$$

After some tedious manipulations, which are described in [15], one ends up with

$$\begin{aligned} B^{\pi}_{ab,cd} =&(-1)^{j_b+j_c+J} \sqrt{(1+\delta_{ac})(1+\delta_{bd})} \\ &\times \sum_{J'} (-1)^{J'} (2J'+1) \begin{Bmatrix} j_a & j_b & J \\ j_d & j_c & J' \end{Bmatrix} \langle ac; J' | V | bd; J' \rangle_\pi. \end{aligned} \tag{1.88}$$

This result is valid also in the case, when a, b, c, d are neutron states, except that the matrix element $\langle ac; J' | V | bd; J' \rangle_\pi$ is replaced with the analogous matrix element $\langle ac; J' | V | bd; J' \rangle_\nu$.

If a, b are proton states and c, d are neutron states, only the third term in the residual interaction $V_{\rm res}$ given by (1.42) contributes to $B_{ab,cd}$ given by (1.86) and, therefore, we have

$$\begin{aligned} B^{\pi\nu}_{ab,cd} =&(-1)^{J+M} \sum_{\substack{m_\alpha m_\beta \\ m_\gamma m_\delta}} (j_a m_\alpha j_b m_\beta | JM)(j_c m_\gamma j_d m_\delta | J - M) \\ &\times \sum_{\alpha'\beta'\gamma'\delta'} \langle \alpha'\beta' | V | \gamma'\delta' \rangle_{\pi\nu} \langle {\rm HF} | h^\pi_\beta a_\alpha h^\nu_\delta b_\gamma : a^\dagger_{\alpha'} b^\dagger_{\beta'} a_{\gamma'} b_{\delta'} : | {\rm HF} \rangle. \end{aligned} \tag{1.89}$$

For a non-zero contribution the states α', β' have to be particle states and the states γ', δ' have to be hole states. With this observation one gets

$$\langle {\rm HF} | h^\pi_\beta a_\alpha h^\nu_\delta b_\gamma : a^\dagger_{\alpha'} b^\dagger_{\beta'} a_{\gamma'} b_{\delta'} : | {\rm HF} \rangle = (-1)^{j_b+m_\beta+j_d+m_\delta} \delta_{-\delta\delta'} \delta_{\gamma\beta'} \delta_{-\beta\gamma'} \delta_{\alpha\alpha'}. \tag{1.90}$$

Substitution into (1.89) yields

$$\begin{aligned} B^{\pi\nu}_{ab,cd} =&(-1)^{J+M} \sum_{\substack{m_\alpha m_\beta \\ m_\gamma m_\delta}} (-1)^{j_b+m_\beta+j_d+m_\delta} (j_a m_\alpha j_b m_\beta | JM) \\ &\times (j_c m_\gamma j_d m_\delta | J - M) \langle \alpha\gamma | V | - \beta \; - \delta \rangle_{\pi\nu}. \end{aligned} \tag{1.91}$$

Performing the same manipulations that led to the formula (1.88), we obtain analogous result

$$B^{\pi\nu}_{ab,cd} = (-1)^{j_b+j_c+J} \sum_{J'} (-1)^{J'} (2J'+1) \begin{Bmatrix} j_a & j_b & J \\ j_d & j_c & J' \end{Bmatrix} \langle ac; J' | V | bd; J' \rangle_{\pi\nu}. \tag{1.92}$$

If a, b are neutron states and c, d are proton states, the matrix element of the correlation matrix B is

$$B^{\nu\pi}_{ab,cd} = (-1)^{j_a+j_d+J} \sum_{J'} (-1)^{J'} (2J'+1) \begin{Bmatrix} j_c & j_d & J \\ j_b & j_a & J' \end{Bmatrix} \langle ca; J' | V | db; J' \rangle_{\pi\nu}, \tag{1.93}$$

which was obtained from (1.92) using the symmetry of the correlation matrix.

Let us now discuss some properties of the RPA solutions.The RPA ground state $|RPA\rangle$ is defined by analogy to (1.69) by

$$Q_\nu|RPA\rangle = 0 \quad \forall\nu. \tag{1.94}$$

The excited state is

$$|\nu; JM\rangle = Q^\dagger_\nu|RPA\rangle \tag{1.95}$$

and it is interpreted as a quantum of vibration of the nuclear surface. The RPA phonons $Q^\dagger_\nu$ contain amplitudes X^ν_{ab} analogous to the amplitudes of the TDA eigenstates (1.62), and amplitudes Y^ν_{ab} generated by the correlation matrix B. We obtain these amplitudes together with the excitation energies $\hbar\Omega_\nu$ by solving the RPA equations (1.85). The orthonormality relation derived using the quasi-boson approximation is (see [15] or [24])

$$\langle\nu; JM|\nu'; JM\rangle = \sum_{ab}\left(X^{\nu*}_{ab}X^{\nu'}_{ab} - Y^{\nu*}_{ab}Y^{\nu'}_{ab}\right) = \delta_{\nu\nu'} \tag{1.96}$$

and contains the normalization condition

$$\langle\nu; JM|\nu; JM\rangle = \sum_{ab}\left(|X^\nu_{ab}|^2 - |Y^\nu_{ab}|^2\right) = 1, \tag{1.97}$$

which concerns only the physical solutions of the RPA equations with positive energy (see below). The RPA equations can be reduced to the TDA equations by putting all amplitudes Y^ν_{ab} equal to zero. Thus these amplitudes are a measure of the ground-state correlations. The TDA results can be reproduced by putting the correlation matrix B equal to zero matrix.

The set of the solutions of the RPA equations is overcomplete. It turns out that for every solution $|\nu; JM\rangle$ with positive energy E_ν and amplitudes X^ν, Y^ν there exists another solution $|\nu_-; JM\rangle$ with negative energy $E_{\nu_-} = -E_\nu$ and amplitudes $X^{\nu_-} = Y^{\nu*}, Y^{\nu_-} = X^{\nu*}$. Furthermore, it can be shown that the solutions with negative energy have negative squared norm, i.e. $\langle\nu_-; JM|\nu_-; JM\rangle = -1$. Thus we consider these solutions as unphysical and accept only the physical solutions with positive energy, which constitute a complete set of eigenstates.

The RPA is relevant when the ground-state correlations quantified by the amplitudes Y^ν_{ab} are significant and lead to collective enhancement of the transition probabilities, which the TDA cannot describe. On the other hand, we expect the amplitudes Y^ν_{ab} to be small, otherwise the quasi-boson approximation, which is based on the assumption that the correlated ground state does not differ much from the HF vacuum, would not be justified. This condition is violated by the negative energy solutions mentioned above, which is another demonstration of their unphysical nature.

According to the Thouless theorem [25] the RPA ground state can be expressed as (see [15] or [4])

$$|RPA\rangle = Ne^S|\mathrm{HF}\rangle, \tag{1.98}$$

where N is a normalization factor and

$$S = \frac{1}{2}\sum_{JM}\sum_{abcd} C_{abcd}(J)A^\dagger_{ab}(JM)\tilde{A}^\dagger_{cd}(JM) \tag{1.99}$$

with amplitudes $C_{abcd}(J)$ fulfilling the set of linear equations[10]

$$\sum_{ab} X^{\nu *}_{ab} C_{abcd}(J) = Y^{\nu *}_{cd}. \tag{1.100}$$

Method of solving these equations can be found in [27]. It is obvious that the RPA ground state contains not only $|\mathrm{HF}\rangle$ but also n-p-n-h configurations with $n = 2, 4, 6, ...$ Thus the RPA excited states consist of 1-p-1-h, 3-p-3-h etc. configurations. Knowing the explicit form of the ground state, one can use it to derive the RPA equations and the matrices A, B from (1.73) again avoiding the quasi-boson approximation, then solve the equations and obtain another ground state using the Thouless theorem (1.98). This procedure can be repeated iteratively until self-consistency is achieved. A detailed description of such self-consistent method extending the RPA and avoiding the drawbacks of the quasi-boson approximation can be found in [28].

Another methods going beyond the RPA are so called higher RPA's, which don't limit the phonon operator to 1-p-1-h configurations and include also 2-p-2-h[11] and higher configurations (see [30] and [31]).

Now we briefly describe a method of numerical solution of the RPA equations which is derived in [24]. We assume that the matrices A and B are real, which is usually the case of practical computations. It is possible to derive

$$(A + B)(A - B)P^{\nu} = \hbar^2 \Omega^2_{\nu} P^{\nu}, \tag{1.101}$$

where P^{ν} is a vector whose explicit form is not important (and can be found in [24]). If the matrix $(A - B)$ is positive definite, we can decompose it like this

$$(A - B) = T^T T, \tag{1.102}$$

where T is a triangular matrix, i.e. $T_{ik} = 0$ for $i > k$. The non-Hermitian RPA eigenvalue problem ca be reduced to the symmetric eigenvalue problem of half the dimension

$$T(A + B)T^T R^{\nu} = \hbar^2 \Omega^2_{\nu} T^{\nu}, \tag{1.103}$$

which provides only the positive (physical) energies $\hbar\Omega_{\nu}$ and the normalized eigenvectors R^{ν}. Then we can obtain the correctly normalized phonon amplitudes from

$$\begin{pmatrix} X^{\nu} \\ Y^{\nu} \end{pmatrix} = \frac{1}{2}\big((\hbar\Omega_{\nu})^{-1/2} T^T R^{\nu} \pm (\hbar\Omega_{\nu})^{1/2} T^{-1} R^{\nu}\big). \tag{1.104}$$

If the matrix $(A - B)$ is not positive definite, it is impossible to perform the decomposition (1.102) and obtain the phonon amplitudes from (1.104). However, the eigenenergies can still be obtained from the eigenvalue problem (1.101), which may provide negative squared energy $\hbar^2\Omega^2_{\nu}$ and thus imaginary energy $\hbar\Omega_{\nu}$. This is consequence of the non-Hermiticity of the RPA "supermatrix". It was shown by Thouless [32] that the appearance of an imaginary energy implies the instability of the HF vacuum, which means that $|\mathrm{HF}\rangle$ doesn't minimize the energy expectation.

[10] For further details see [26]

[11] A detailed description of the second RPA including 2-p-2-h configurations can be found in [29].

Let us now consider the electromagnetic transitions from the ground state $|\mathrm{RPA}\rangle$ to an excited RPA state $|\nu\rangle$ with the angular momentum J. The reduced transition probability is

$$B(X\lambda; 0^+_{\mathrm{gs}} \to \nu) = \delta_{\lambda J} |\langle \nu || M^{(X)}_J || \mathrm{RPA} \rangle|^2, \tag{1.105}$$

where (for derivation see [15])

$$\langle \nu || M^{(X)}_J || \mathrm{RPA} \rangle = \sum_{ab} \langle a || M^{(X)}_J || b \rangle \left[(-1)^J X^\nu_{ab} + Y^\nu_{ab} \right]. \tag{1.106}$$

We see that if the amplitudes Y^ν_{ab} are zero, the TDA result (1.66) is reproduced. For a collective state $|\nu\rangle$ the RPA result (1.106) can lead to significant enhancement of the reduced transition probability. This occurs when the products of phonon amplitudes and reduced single-particle matrix elements of $M^{(X)}_J$ sum coherently.

2. Microscopic models for open-shell nuclei

One can move from doubly-magic nuclei by filling nucleons into the next open shell. Since a doubly-magic core is supposed to be quite stable, the correlations among the valence nucleons should be important in explanation of phenomena observed in experimental study of open-shell nuclei. Several calculations for such nuclei based on the shell-model have been performed (see [33], [34] and [35]). In this chapter we focus on methods treating spherical nuclei created from a doubly-magic core by adding or removing two nucleons of the same type. These methods are based on the analogy to the *ph* TDA and *ph* RPA.

2.1 Particle-particle and hole-hole TDA

Within the particle-particle or hole-hole TDA (*pp*TDA or *hh*TDA) the nuclear states are linear combinations of *pp* or *hh* configurations obtained by diagonalizing the nuclear Hamiltonian in such a basis.

Let us start with the *pp*TDA. In analogy to the TDA, we seek the expression for the matrix elements $\langle p_1p_2;JM|H|p_3p_4;JM\rangle$ of the Hamiltonian in the basis of angular-momentum-coupled *pp* configurations

$$|p_1p_2;JM\rangle = \mathcal{N}_{p_1p_2}(J)\sum_{m_1m_2}(j_1m_1j_2m_2|JM)c^\dagger_{\pi_1}c^\dagger_{\pi_3}|\mathrm{HF}\rangle, \tag{2.1}$$

where $|p_i\rangle = |n_il_ij_i\rangle$ and $|\pi_i\rangle = |n_il_ij_im_i\rangle$ are particle states[1] and

$$\mathcal{N}_{ab}(J) = \frac{\sqrt{1+\delta_{ab}(-1)^J}}{1+\delta_{ab}} \tag{2.2}$$

is the normalization factor. We have

$$\begin{aligned}\langle p_1p_2;JM|H|p_3p_4;JM\rangle &= \mathcal{N}_{p_1p_2}(J)\mathcal{N}_{p_3p_4}(J)\\ &\times\sum_{\substack{m_1m_2\\m_3m_4}}(j_1m_1j_2m_2|JM)(j_3m_3j_4m_4|JM)\langle\mathrm{HF}|c_{\pi_2}c_{\pi_1}Hc^\dagger_{\pi_3}c^\dagger_{\pi_4}|\mathrm{HF}\rangle,\end{aligned} \tag{2.3}$$

where the Hamiltonian H is given by (1.40).

Let us consider proton *pp* configurations (for neutrons the derivation is analogous). The contribution of the mean-field Hamiltonian H_{mf}, which is given

[1] This notation is used throughout this thesis.

by (1.41), is

$$\begin{aligned}\langle p_1p_2;JM|H_{\rm mf}|p_3p_4;JM\rangle &= \mathcal{N}_{p_1p_2}(J)\mathcal{N}_{p_3p_4}(J)\\ \times\Bigg[\sum_\alpha \varepsilon^\pi_\alpha \sum_{\substack{m_1m_2\\m_3m_4}} &(j_1m_1j_2m_2|JM)(j_3m_3j_4m_4|JM)\langle{\rm HF}|a_{\pi_2}a_{\pi_1}a^\dagger_\alpha a_\alpha a^\dagger_{\pi_3}a^\dagger_{\pi_4}|{\rm HF}\rangle\\ +\sum_\alpha \varepsilon^\nu_\alpha \sum_{\substack{m_1m_2\\m_3m_4}} &(j_1m_1j_2m_2|JM)(j_3m_3j_4m_4|JM)\langle{\rm HF}|a_{\pi_2}a_{\pi_1}b^\dagger_\alpha b_\alpha a^\dagger_{\pi_3}a^\dagger_{\pi_4}|{\rm HF}\rangle\\ +K\sum_{\substack{m_1m_2\\m_3m_4}} &(j_1m_1j_2m_2|JM)(j_3m_3j_4m_4|JM)\langle{\rm HF}|a_{\pi_2}a_{\pi_1}a^\dagger_{\pi_3}a^\dagger_{\pi_4}|{\rm HF}\rangle\Bigg],\end{aligned}\tag{2.4}$$

where K is given by (1.47). Using the anticommutation and commutation relations (1.17) one gets

$$\begin{aligned}\langle{\rm HF}|a_{\pi_2}a_{\pi_1}a^\dagger_\alpha a_\alpha a^\dagger_{\pi_3}a^\dagger_{\pi_4}|{\rm HF}\rangle &= \delta_{\alpha\pi_3}\delta_{\alpha\pi_1}\delta_{\pi_2\pi_4}-\delta_{\alpha\pi_3}\delta_{\alpha\pi_2}\delta_{\pi_1\pi_4}-\delta_{\alpha\pi_1}\delta_{\alpha\pi_4}\delta_{\pi_2\pi_3}\\ &\quad+\delta_{\alpha\pi_4}\delta_{\alpha\pi_2}\delta_{\pi_1\pi_3}+\delta_{\pi_1\pi_3}\delta_{\pi_2\pi_4}\theta(\varepsilon_F-\varepsilon_\alpha)-\delta_{\pi_1\pi_4}\delta_{\pi_2\pi_3}\theta(\varepsilon_F-\varepsilon_\alpha),\\ \langle{\rm HF}|a_{\pi_2}a_{\pi_1}b^\dagger_\alpha b_\alpha a^\dagger_{\pi_3}a^\dagger_{\pi_4}|{\rm HF}\rangle &= \delta_{\pi_1\pi_3}\delta_{\pi_2\pi_4}\theta(\varepsilon_F-\varepsilon_\alpha)-\delta_{\pi_1\pi_4}\delta_{\pi_2\pi_3}\theta(\varepsilon_F-\varepsilon_\alpha),\\ \langle{\rm HF}|a_{\pi_2}a_{\pi_1}a^\dagger_{\pi_3}a^\dagger_{\pi_4}|{\rm HF}\rangle &= \delta_{\pi_1\pi_3}\delta_{\pi_2\pi_4}-\delta_{\pi_1\pi_4}\delta_{\pi_2\pi_3}.\end{aligned}$$

Substitution into (2.4), the relation

$$(j_2m_2j_1m_1|JM)=(-1)^{j_1+j_2-J}(j_1m_1j_2m_2|JM)\tag{2.5}$$

and the orthogonality of Clebsch-Gordan coefficients (1.49) yield

$$\begin{aligned}&\langle p_1p_2;JM|H_{\rm mf}|p_3p_4;JM\rangle\\ &\quad=\mathcal{N}_{p_1p_2}(J)\mathcal{N}_{p_3p_4}(J)\Big[\delta_{p_1p_3}\delta_{p_2p_4}\big(\varepsilon^\pi_{p_1}+\varepsilon^\pi_{p_2}+\sum_h\varepsilon^\pi_h+\sum_h\varepsilon^\nu_h+K\big)\\ &\qquad-\delta_{p_2p_3}\delta_{p_1p_4}(-1)^{j_1+j_2-J}\big(\varepsilon^\pi_{p_1}+\varepsilon^\pi_{p_2}+\sum_h\varepsilon^\pi_h+\sum_h\varepsilon^\nu_h+K\big)\Big].\end{aligned}\tag{2.6}$$

Thus the result is

$$\begin{aligned}\langle p_1p_2;JM|H_{\rm mf}|p_3p_4;JM\rangle &= \mathcal{N}_{p_1p_2}(J)\mathcal{N}_{p_3p_4}(J)\\ \times\Big[\delta_{p_1p_3}\delta_{p_2p_4}(\varepsilon^\pi_{p_1}+\varepsilon^\pi_{p_2}&+E_{\rm HF})-\delta_{p_2p_3}\delta_{p_1p_4}(-1)^{j_1+j_2-J}(\varepsilon^\pi_{p_1}+\varepsilon^\pi_{p_2}+E_{\rm HF})\Big].\end{aligned}\tag{2.7}$$

Since this matrix element is diagonal and we are interested in the excitation energies, the constant term $E_{\rm HF}$, which affects the energy eigenvalues by a common energy shift, can be omitted. If p_1,p_2,p_3,p_4 are neutron states, the result is the same except that the single-particle energies correspond to neutron states.

The contribution of the residual interaction $V_{\rm res}$ given by (1.42) is

$$\begin{aligned}\langle p_1p_2;JM|V_{\rm res}|p_3p_4;JM\rangle &= \mathcal{N}_{p_1p_2}(J)\mathcal{N}_{p_3p_4}(J)\\ &\times\Bigg[\frac{1}{4}\sum_{\alpha\beta\gamma\delta}\langle\alpha\beta|V|\overline{\gamma\delta}\rangle_\pi\sum_{\substack{m_1m_2\\m_3m_4}}(j_1m_1j_2m_2|JM)(j_3m_3j_4m_4|JM)\\ &\times\langle{\rm HF}|a_{\pi_2}a_{\pi_1}:a^\dagger_\alpha a^\dagger_\beta a_\delta a_\gamma:a^\dagger_{\pi_3}a^\dagger_{\pi_4}|{\rm HF}\rangle\\ &+\frac{1}{4}\sum_{\alpha\beta\gamma\delta}\langle\alpha\beta|V|\overline{\gamma\delta}\rangle_\nu\sum_{\substack{m_1m_2\\m_3m_4}}(j_1m_1j_2m_2|JM)(j_3m_3j_4m_4|JM)\\ &\times\langle{\rm HF}|a_{\pi_2}a_{\pi_1}:b^\dagger_\alpha b^\dagger_\beta b_\delta b_\gamma:a^\dagger_{\pi_3}a^\dagger_{\pi_4}|{\rm HF}\rangle\\ &+\sum_{\alpha\beta\gamma\delta}\langle\alpha\beta|V|\gamma\delta\rangle_{\pi\nu}\sum_{\substack{m_1m_2\\m_3m_4}}(j_1m_1j_2m_2|JM)(j_3m_3j_4m_4|JM)\\ &\times\langle{\rm HF}|a_{\pi_2}a_{\pi_1}:a^\dagger_\alpha b^\dagger_\beta a_\gamma b_\delta:a^\dagger_{\pi_3}a^\dagger_{\pi_4}|{\rm HF}\rangle\Bigg].\end{aligned}\tag{2.8}$$

Since

$$\langle{\rm HF}|a_{\pi_2}a_{\pi_1}:b^\dagger_\alpha b^\dagger_\beta b_\delta b_\gamma:a^\dagger_{\pi_3}a^\dagger_{\pi_4}|{\rm HF}\rangle=0,\tag{2.9}$$

$$\langle{\rm HF}|a_{\pi_2}a_{\pi_1}:a^\dagger_\alpha b^\dagger_\beta a_\gamma b_\delta:a^\dagger_{\pi_3}a^\dagger_{\pi_4}|{\rm HF}\rangle=0,\tag{2.10}$$

the last two terms in (2.8) vanish. In the first term the states $\alpha,\beta,\gamma,\delta$ have to be particle states to get a non-zero contribution. Thus we get

$$\begin{aligned}&\langle p_1p_2;JM|V_{\rm res}|p_3p_4;JM\rangle\\ &=\mathcal{N}_{p_1p_2}(J)\mathcal{N}_{p_3p_4}(J)\frac{1}{4}\sum_{\substack{\pi_5\pi_6\\\pi_7\pi_8}}\langle\pi_5\pi_6|V|\overline{\pi_7\pi_8}\rangle_\pi\sum_{\substack{m_1m_2\\m_3m_4}}(j_1m_1j_2m_2|JM)(j_3m_3j_4m_4|JM)\\ &\quad\times\langle{\rm HF}|a_{\pi_2}a_{\pi_1}a^\dagger_{\pi_5}a^\dagger_{\pi_6}a_{\pi_8}a_{\pi_7}a^\dagger_{\pi_3}a^\dagger_{\pi_4}|{\rm HF}\rangle.\end{aligned}\tag{2.11}$$

Using the anticommutation relations (1.17) one gets

$$\begin{aligned}\langle{\rm HF}|a_{\pi_2}a_{\pi_1}a^\dagger_{\pi_5}a^\dagger_{\pi_6}a_{\pi_8}a_{\pi_7}a^\dagger_{\pi_3}a^\dagger_{\pi_4}|{\rm HF}\rangle &= \delta_{\pi_1\pi_5}\delta_{\pi_2\pi_6}\delta_{\pi_3\pi_7}\delta_{\pi_4\pi_8}-\delta_{\pi_1\pi_5}\delta_{\pi_2\pi_6}\delta_{\pi_3\pi_8}\delta_{\pi_4\pi_7}\\ &\quad-\delta_{\pi_1\pi_6}\delta_{\pi_2\pi_5}\delta_{\pi_3\pi_7}\delta_{\pi_4\pi_8}+\delta_{\pi_1\pi_6}\delta_{\pi_2\pi_5}\delta_{\pi_3\pi_8}\delta_{\pi_4\pi_7}.\end{aligned}$$

Substitution into (2.11) yields

$$\begin{aligned}&\langle p_1p_2;JM|V_{\rm res}|p_3p_4;JM\rangle\\ &=\mathcal{N}_{p_1p_2}(J)\mathcal{N}_{p_3p_4}(J)\frac{1}{4}\sum_{\substack{m_1m_2\\m_3m_4}}(j_1m_1j_2m_2|JM)(j_3m_3j_4m_4|JM)\\ &\quad\times\big(\langle\pi_1\pi_2|V|\overline{\pi_3\pi_4}\rangle_\pi-\langle\pi_1\pi_2|V|\overline{\pi_4\pi_3}\rangle_\pi-\langle\pi_2\pi_1|V|\overline{\pi_3\pi_4}\rangle_\pi+\langle\pi_2\pi_1|V|\overline{\pi_4\pi_3}\rangle_\pi\big).\end{aligned}$$

Using the symmetry properties (1.22) one gets

$$\begin{aligned}&\langle p_1p_2;JM|V_{\rm res}|p_3p_4;JM\rangle\\ &\quad=\mathcal{N}_{p_1p_2}(J)\mathcal{N}_{p_3p_4}(J)\sum_{\substack{m_1m_2\\m_3m_4}}(j_1m_1j_2m_2|JM)(j_3m_3j_4m_4|JM)\langle\pi_1\pi_2|V|\overline{\pi_3\pi_4}\rangle_\pi,\end{aligned}$$

which is the angular-momentum-coupled two-body interaction matrix element. Thus the result is

$$\langle p_1 p_2; JM|V_{\rm res}|p_3 p_4; JM\rangle = \langle p_1 p_2, J|V|p_3 p_4, J\rangle_\pi. \tag{2.12}$$

If p_1, p_2, p_3, p_4 are neutron states, the result is the same except that the coupled two-body interaction matrix element is of the neutron type.

Now we proceed to an analogous derivation for the hhTDA. We seek the formula for the matrix elements $\langle h_1 h_2; JM|H|h_3 h_4; JM\rangle$ of the Hamiltonian in the basis of hh configurations

$$|h_1 h_2; JM\rangle = \mathcal{N}_{h_1 h_2}(J) \sum_{m_1 m_2} (j_1 m_1 j_2 m_2|JM) h^\dagger_{\eta_1} h^\dagger_{\eta_2}|{\rm HF}\rangle, \tag{2.13}$$

where $|h_i\rangle = |n_i l_i j_i\rangle$ and $|\eta_i\rangle = |n_i l_i j_i m_i\rangle$ are hole states (this notation is used throughout this thesis).

Let us consider proton hh configurations (for neutron states the derivation is analogous). The contribution of the mean-field Hamiltonian is

$$\begin{aligned}
\langle h_1 h_2; JM|H_{\rm mf}|h_3 h_4; JM\rangle &= \mathcal{N}_{h_1 h_2}(J)\mathcal{N}_{h_3 h_4}(J) \\
&\times \Bigg[\sum_\alpha \varepsilon^\pi_\alpha \sum_{\substack{m_1 m_2 \\ m_3 m_4}} (j_1 m_1 j_2 m_2|JM)(j_3 m_3 j_4 m_4|JM) \langle {\rm HF}| h^\pi_{\eta_2} h^\pi_{\eta_1} a^\dagger_\alpha a_\alpha h^{\pi\,\dagger}_{\eta_3} h^{\pi\,\dagger}_{\eta_4}|{\rm HF}\rangle \\
&+ \sum_\alpha \varepsilon^\nu_\alpha \sum_{\substack{m_1 m_2 \\ m_3 m_4}} (j_1 m_1 j_2 m_2|JM)(j_3 m_3 j_4 m_4|JM) \langle {\rm HF}| h^\pi_{\eta_2} h^\pi_{\eta_1} b^\dagger_\alpha b_\alpha h^{\pi\,\dagger}_{\eta_3} h^{\pi\,\dagger}_{\eta_4}|{\rm HF}\rangle \\
&+ K \sum_{\substack{m_1 m_2 \\ m_3 m_4}} (j_1 m_1 j_2 m_2|JM)(j_3 m_3 j_4 m_4|JM) \langle {\rm HF}| h^\pi_{\eta_2} h^\pi_{\eta_1} h^{\pi\,\dagger}_{\eta_3} h^{\pi\,\dagger}_{\eta_4}|{\rm HF}\rangle \Bigg].
\end{aligned} \tag{2.14}$$

Using the anticommutation and commutation relations (1.17) one gets

$$\begin{aligned}
&\langle {\rm HF}| h^\pi_{\eta_2} h^\pi_{\eta_1} a^\dagger_\alpha a_\alpha h^{\pi\,\dagger}_{\eta_3} h^{\pi\,\dagger}_{\eta_4}|{\rm HF}\rangle \\
&\qquad = (-1)^{j_1+m_1+j_2+m_2+j_3+m_3+j_4+m_4} \langle {\rm HF}| a^\dagger_{-\eta_2} a^\dagger_{-\eta_1} a^\dagger_\alpha a_\alpha a_{-\eta_3} a_{-\eta_4}|{\rm HF}\rangle \\
&\qquad = -\delta_{-\eta_1\alpha}\delta_{-\eta_3\alpha}\delta_{\eta_2\eta_4} + \delta_{-\eta_1\alpha}\delta_{-\eta_4\alpha}\delta_{\eta_2\eta_3} - \delta_{\eta_1\eta_3}\delta_{-\eta_2\alpha}\delta_{-\eta_4\alpha} \\
&\qquad\quad + \delta_{-\eta_2\alpha}\delta_{\eta_1\eta_4}\delta_{-\eta_3\alpha} + \delta_{\eta_1\eta_3}\delta_{\eta_2\eta_4}\theta(\varepsilon_F - \varepsilon_\alpha) - \delta_{\eta_1\eta_4}\delta_{\eta_2\eta_3}\theta(\varepsilon_F - \varepsilon_\alpha), \\
&\langle {\rm HF}| h^\pi_{\eta_2} h^\pi_{\eta_1} b^\dagger_\alpha b_\alpha h^{\pi\,\dagger}_{\eta_3} h^{\pi\,\dagger}_{\eta_4}|{\rm HF}\rangle \\
&\qquad = (-1)^{j_1+m_1+j_2+m_2+j_3+m_3+j_4+m_4} \langle {\rm HF}| a^\dagger_{-\eta_2} a^\dagger_{-\eta_1} b^\dagger_\alpha b_\alpha a_{-\eta_3} a_{-\eta_4}|{\rm HF}\rangle \\
&\qquad = \delta_{\eta_1\eta_3}\delta_{\eta_2\eta_4}\theta(\varepsilon_F - \varepsilon_\alpha) - \delta_{\eta_1\eta_4}\delta_{\eta_2\eta_3}\theta(\varepsilon_F - \varepsilon_\alpha), \\
&\langle {\rm HF}| h^\pi_{\eta_2} h^\pi_{\eta_1} h^{\pi\,\dagger}_{\eta_3} h^{\pi\,\dagger}_{\eta_4}|{\rm HF}\rangle \\
&\qquad = (-1)^{j_1+m_1+j_2+m_2+j_3+m_3+j_4+m_4} \langle {\rm HF}| a^\dagger_{-\eta_2} a^\dagger_{-\eta_1} a_{-\eta_3} a_{-\eta_4}|{\rm HF}\rangle \\
&\qquad = \delta_{\eta_1\eta_3}\delta_{\eta_2\eta_4} - \delta_{\eta_1\eta_4}\delta_{\eta_2\eta_3},
\end{aligned}$$

where it was easy to deduce that the phase factors could be omitted. Substitution into (2.14) and the relations (2.5) and (1.49) yield similarly as in the previous case the result

$$\begin{aligned}
\langle h_1 h_2; JM|H_{\rm mf}|h_3 h_4; JM\rangle &= \mathcal{N}_{h_1 h_2}(J)\mathcal{N}_{h_3 h_4}(J) \\
&\times \Big[\delta_{h_1 h_3}\delta_{h_2 h_4}(-\varepsilon^\pi_{h_1} - \varepsilon^\pi_{h_2} + E_{\rm HF}) - \delta_{h_2 h_3}\delta_{h_1 h_4}(-1)^{j_1+j_2-J}(-\varepsilon^\pi_{h_1} - \varepsilon^\pi_{h_2} + E_{\rm HF}) \Big],
\end{aligned} \tag{2.15}$$

where $E_{\rm HF}$ can be omitted as in the previous case. We see that the result is analogous to the ppTDA result except that the single-particle energies change sign. If h_1, h_2, h_3, h_4 are neutron states, the result is the same except that the single-particle energies correspond to neutron states.

The contribution of residual interaction is

$$\begin{aligned}\langle h_1h_2;JM|V_{\rm res}|h_3h_4;JM\rangle &= \mathcal{N}_{h_1h_2}(J)\mathcal{N}_{h_3h_4}(J)\\ &\times\Bigg[\frac{1}{4}\sum_{\alpha\beta\gamma\delta}\langle\alpha\beta|V|\overline{\gamma\delta}\rangle_\pi\sum_{\substack{m_1m_2\\m_3m_4}}(j_1m_1j_2m_2|JM)(j_3m_3j_4m_4|JM)\\ &\times\langle{\rm HF}|h^\pi_{\eta_2}h^\pi_{\eta_1}:a^\dagger_\alpha a^\dagger_\beta a_\delta a_\gamma:h^{\pi\,\dagger}_{\eta_3}h^{\pi\,\dagger}_{\eta_4}|{\rm HF}\rangle\\ &+\frac{1}{4}\sum_{\alpha\beta\gamma\delta}\langle\alpha\beta|V|\overline{\gamma\delta}\rangle_\nu\sum_{\substack{m_1m_2\\m_3m_4}}(j_1m_1j_2m_2|JM)(j_3m_3j_4m_4|JM)\\ &\times\langle{\rm HF}|h^\pi_{\eta_2}h^\pi_{\eta_1}:b^\dagger_\alpha b^\dagger_\beta b_\delta b_\gamma:h^{\pi\,\dagger}_{\eta_3}h^{\pi\,\dagger}_{\eta_4}|{\rm HF}\rangle\\ &+\sum_{\alpha\beta\gamma\delta}\langle\alpha\beta|V|\gamma\delta\rangle_{\pi\nu}\sum_{\substack{m_1m_2\\m_3m_4}}(j_1m_1j_2m_2|JM)(j_3m_3j_4m_4|JM)\\ &\times\langle{\rm HF}|h^\pi_{\eta_2}h^\pi_{\eta_1}:a^\dagger_\alpha b^\dagger_\beta a_\gamma b_\delta:h^{\pi\,\dagger}_{\eta_3}h^{\pi\,\dagger}_{\eta_4}|{\rm HF}\rangle\Bigg].\end{aligned}\tag{2.16}$$

Since

$$\langle{\rm HF}|h^\pi_{\eta_2}h^\pi_{\eta_1}:b^\dagger_\alpha b^\dagger_\beta b_\delta b_\gamma:h^{\pi\,\dagger}_{\eta_3}h^{\pi\,\dagger}_{\eta_4}|{\rm HF}\rangle=0,\tag{2.17}$$

$$\langle{\rm HF}|h^\pi_{\eta_2}h^\pi_{\eta_1}:a^\dagger_\alpha b^\dagger_\beta a_\gamma b_\delta:h^{\pi\,\dagger}_{\eta_3}h^{\pi\,\dagger}_{\eta_4}|{\rm HF}\rangle=0,\tag{2.18}$$

the last two terms in (2.16) vanish. In the first term the states $\alpha,\beta,\gamma,\delta$ have to be hole states for a non-zero contribution. Thus we get

$$\begin{aligned}\langle h_1h_2;JM|V_{\rm res}|h_3h_4;JM\rangle &= \mathcal{N}_{h_1h_2}(J)\mathcal{N}_{h_3h_4}(J)\\ &\times\frac{1}{4}\sum_{\substack{\eta_5\eta_6\\\eta_7\eta_8}}\langle\eta_5\eta_6|V|\overline{\eta_7\eta_8}\rangle_\pi\sum_{\substack{m_1m_2\\m_3m_4}}(j_1m_1j_2m_2|JM)(j_3m_3j_4m_4|JM)\\ &\times\langle{\rm HF}|a^\dagger_{-\eta_2}a^\dagger_{-\eta_1}:a^\dagger_{\eta_5}a^\dagger_{\eta_6}a_{\eta_8}a_{\eta_7}:a_{-\eta_3}a_{-\eta_4}|{\rm HF}\rangle(-1)^{j_1+m_1+j_2+m_2+j_3+m_3+j_4+m_4}.\end{aligned}\tag{2.19}$$

Since

$$\begin{aligned}&\langle{\rm HF}|a^\dagger_{-\eta_2}a^\dagger_{-\eta_1}:a^\dagger_{\eta_5}a^\dagger_{\eta_6}a_{\eta_8}a_{\eta_7}:a_{-\eta_3}a_{-\eta_4}|{\rm HF}\rangle\\ &=\langle{\rm HF}|a^\dagger_{-\eta_2}a^\dagger_{-\eta_1}a_{\eta_8}a_{\eta_7}a^\dagger_{\eta_5}a^\dagger_{\eta_6}a_{-\eta_3}a_{-\eta_4}|{\rm HF}\rangle=\delta_{-\eta_3\eta_6}\delta_{-\eta_1\eta_8}\delta_{-\eta_4\eta_5}\delta_{-\eta_2\eta_7}\\ &\quad-\delta_{-\eta_3\eta_6}\delta_{-\eta_4\eta_5}\delta_{-\eta_1\eta_7}\delta_{-\eta_2\eta_8}-\delta_{-\eta_1\eta_8}\delta_{-\eta_4\eta_6}\delta_{-\eta_3\eta_5}\delta_{-\eta_2\eta_7}+\delta_{-\eta_4\eta_6}\delta_{-\eta_3\eta_5}\delta_{-\eta_1\eta_7}\delta_{-\eta_2\eta_8},\end{aligned}$$

where we have used the anticommutation relations (1.17), we obtain

$$\begin{aligned}&\sum_{\substack{\eta_5\eta_6\\\eta_7\eta_8}}\langle\eta_5\eta_6|V|\overline{\eta_7\eta_8}\rangle_\pi\langle{\rm HF}|a^\dagger_{-\eta_2}a^\dagger_{-\eta_1}:a^\dagger_{\eta_5}a^\dagger_{\eta_6}a_{\eta_8}a_{\eta_7}:a_{-\eta_3}a_{-\eta_4}|{\rm HF}\rangle\\ &\qquad=\langle-\eta_4\ -\eta_3|V|\overline{-\eta_2\ -\eta_1}\rangle_\pi-\langle-\eta_4\ -\eta_3|V|\overline{-\eta_1\ -\eta_2}\rangle_\pi\\ &\qquad-\langle-\eta_3\ -\eta_4|V|\overline{-\eta_2\ -\eta_1}\rangle_\pi+\langle-\eta_3\ -\eta_4|V|\overline{-\eta_1\ -\eta_2}\rangle_\pi\\ &\qquad=4\langle-\eta_1\ -\eta_2|V|\overline{-\eta_3\ -\eta_4}\rangle_\pi,\end{aligned}\tag{2.20}$$

where we have used the symmetry properties (1.22). Substituting this into (2.19) we get

$$\langle h_1h_2;JM|V_{\rm res}|h_3h_4;JM\rangle = \mathcal{N}_{h_1h_2}(J)\mathcal{N}_{h_3h_4}(J)\sum_{\substack{m_1m_2\\m_3m_4}}(j_1m_1j_2m_2|JM)$$
$$\times(j_3m_3j_4m_4|JM)\langle -\eta_1\ -\eta_2|V|\overline{-\eta_3\ -\eta_4}\rangle_\pi(-1)^{j_1+m_1+j_2+m_2+j_3+m_3+j_4+m_4}.$$

Since the Clebsch-Gordan coefficient are non-zero only for $m_2 = M - m_1$ and $m_4 = M - m_3$, we effectively have

$$(-1)^{j_1+m_1+j_2+m_2+j_3+m_3+j_4+m_4} = (-1)^{j_1+m_1+j_2+M-m_1+j_3+m_3+j_4+M-m_3}$$
$$= (-1)^{j_1+j_2+j_3+j_4+2M} = (-1)^{j_1+j_2+j_3+j_4}.$$

Thus the result is

$$\langle h_1h_2;JM|V_{\rm res}|h_3h_4;JM\rangle = (-1)^{j_1+j_2+j_3+j_4}\langle h_1h_2,J|V|h_3h_4,J\rangle_\pi. \tag{2.21}$$

If h_1, h_2, h_3, h_4 are neutron states, the result is the same except that the coupled two-body interaction matrix element is of the neutron type.

The procedure of solving the ppTDA eigenvalue problem for a given angular momentum and parity J^π is following. We form the basis consisting of all possible pp configurations $|p_1p_2;J^\pi M\rangle$ with a common value of M in the given valence space. For these basis states we adopt the convention $p_1 \le p_2$ to avoid counting the same physical states twice. Then we construct the Hamiltonian matrix in this basis using the formulae derived above. Finally, we diagonalize this matrix and obtain the eigenenergies E_ν and the corresponding eigenstates

$$|\nu;J^\pi M\rangle = \sum_{p_1\le p_2} C^\nu_{p_1p_2}|p_1p_2;J^\pi M\rangle \tag{2.22}$$

obeying the orthonormality condition

$$\langle\nu;J^\pi M|\nu';J^\pi M\rangle = \sum_{p_1\le p_2} C^{\nu*}_{p_1p_2}C^{\nu'}_{p_1p_2} = \delta_{\nu\nu'}. \tag{2.23}$$

The problem can be formulated in the form of the ppTDA equations

$$\sum_{p_3\le p_4}\langle p_1p_2;J^\pi M|H|p_3p_4;J^\pi M\rangle C^\nu_{p_3p_4} = E_\nu C^\nu_{p_1p_2}. \tag{2.24}$$

In the case of the hhTDA the procedure is analogous.

2.2 Electromagnetic transitions within *pp*TDA and *hh*TDA

Let us now consider electric transitions from the ground state to some excited state within ppTDA and hhTDA. The general formula for the reduced probability of a transition of the electric type and multipolarity J from an initial state i to a final state f with angular momenta j_i and j_f is

$$B(elJ;i\to f) = \frac{|\langle\alpha_fj_f||M^{(el)}_J||\alpha_ij_i\rangle|^2}{2j_i+1}, \tag{2.25}$$

where α_i and α_f are additional quantum numbers. In our case the initial state is the lowest 0^+ ppTDA or hhTDA eigenstate[2]. Thus we seek $\langle \nu_f; j_f || M_J^{(el)} || \nu_{\rm gs}; 0 \rangle$, where ν_f and $\nu_{\rm gs}$ denote the final and the initial ppTDA or hhTDA eigenstate, respectively. Using the Wigner-Eckart theorem we get

$$\langle \nu_f; j_f m_f | M_{JM}^{(el)} | \nu_{\rm gs}; 00 \rangle = \frac{1}{\sqrt{2.0+1}} (00JM|j_f m_f) \langle \nu_f; j_f || M_J^{(el)} || \nu_{\rm gs}; 0 \rangle, \quad (2.26)$$

where $(00JM|j_f m_f) = \delta_{Jj_f} \delta_{Mm_f}$. Therefore, j_f must be equal to J and m_f must be equal to M. Thus we seek

$$\langle \nu_f; J || M_J^{(el)} || \nu_{\rm gs}; 0 \rangle = \langle \nu_f; JM | M_{JM}^{(el)} | \nu_{\rm gs}; 00 \rangle, \quad (2.27)$$

where the value of M is arbitrary (we can choose e.g. $M = 0$). Knowing the expression for (2.27) we can calculate the reduced transition probability

$$B(elJ; \nu_{\rm gs} 0 \to \nu_f J) = |\langle \nu_f; JM | M_{JM}^{(el)} | \nu_{\rm gs}; 00 \rangle|^2. \quad (2.28)$$

The electric multipole operator in the long-wave approximation is

$$M_{JM}^{(el)} = \sum_{i=1}^{A} e_i^{(\rm eff)} r_i^J Y_{JM}(\theta_i, \varphi_i), \quad (2.29)$$

where Y_{JM} is the spherical harmonic. In the formalism of the creation and annihilation operators we have

$$\begin{aligned} M_{JM}^{(el)} &= e_{\rm p}^{(\rm eff)} \sum_{\alpha\beta} \langle \alpha | r^J Y_{JM} | \beta \rangle a_\alpha^\dagger a_\beta + e_{\rm n}^{(\rm eff)} \sum_{\alpha\beta} \langle \alpha | r^J Y_{JM} | \beta \rangle b_\alpha^\dagger b_\beta \\ &= e_{\rm p}^{(\rm eff)} \sum_{\alpha\beta} \frac{1}{\sqrt{2j_b+1}} (j_b m_\beta JM | j_a m_\alpha) \langle a || r^J Y_J || b \rangle a_\alpha^\dagger a_\beta \\ &+ e_{\rm n}^{(\rm eff)} \sum_{\alpha\beta} \frac{1}{\sqrt{2j_b+1}} (j_b m_\beta JM | j_a m_\alpha) \langle a || r^J Y_J || b \rangle b_\alpha^\dagger b_\beta, \end{aligned} \quad (2.30)$$

where the Wigner-Eckart theorem was used. We have introduced the effective charge

$$e_i^{(\rm eff)} = \begin{cases} e_{\rm p}^{(\rm eff)} & \text{for protons,} \\ e_{\rm n}^{(\rm eff)} & \text{for neutrons.} \end{cases} \quad (2.31)$$

In the case of ppTDA, the ground state is the lowest 0^+ ppTDA eigenstate

$$|\nu_{\rm gs}; 00\rangle = \sum_{p_3 \le p_4} C_{p_3 p_4}^{\nu_{\rm gs}} |p_3 p_4; 00\rangle \quad (2.32)$$

and the final excited state is

$$|\nu_f; JM\rangle = \sum_{p_1 \le p_2} C_{p_1 p_2}^{\nu_f} |p_1 p_2; JM\rangle, \quad (2.33)$$

[2]The angular momentum and parity of the ground states of eve-even nuclei are 0^+.

with $|p_1p_2;JM\rangle$ given by (2.1). Substituting (2.32), (2.33) and (2.30) into (2.27) we get

$$\langle\nu_f;JM|M^{(el)}_{JM}|\nu_{\rm gs};00\rangle=\sum_{p_1\le p_2}\sum_{p_3\le p_4}C^{\nu_f*}_{p_1p_2}C^{\nu_{\rm gs}}_{p_3p_4}\langle p_1p_2;JM|M^{(el)}_{JM}|p_3p_4;00\rangle. \quad (2.34)$$

Let us consider proton pp configurations (for neutrons the derivation is analogous). Then

$$\begin{aligned}
&\langle p_1p_2;JM|M^{(el)}_{JM}|p_3p_4;00\rangle\\
&\quad=e^{(\rm eff)}_{\rm p}\mathcal{N}_{p_1p_2}(J)\mathcal{N}_{p_3p_4}(0)\sum_{\substack{m_1m_2\\m_3m_4}}(j_1m_1j_2m_2|JM)(j_3m_3j_4m_4|00)\\
&\qquad\times\sum_{\alpha\beta}\frac{1}{\sqrt{2j_b+1}}(j_bm_\beta JM|j_am_\alpha)\langle a||r^JY_J||b\rangle\langle{\rm HF}|a_{\pi_2}a_{\pi_1}a^\dagger_\alpha a_\beta a^\dagger_{\pi_3}a^\dagger_{\pi_4}|{\rm HF}\rangle\\
&\qquad+e^{(\rm eff)}_{\rm n}\mathcal{N}_{p_1p_2}(J)\mathcal{N}_{p_3p_4}(0)\sum_{\substack{m_1m_2\\m_3m_4}}(j_1m_1j_2m_2|JM)(j_3m_3j_4m_4|00)\\
&\qquad\times\sum_{\alpha\beta}\frac{1}{\sqrt{2j_b+1}}(j_bm_\beta JM|j_am_\alpha)\langle a||r^JY_J||b\rangle\langle{\rm HF}|a_{\pi_2}a_{\pi_1}b^\dagger_\alpha b_\beta a^\dagger_{\pi_3}a^\dagger_{\pi_4}|{\rm HF}\rangle.
\end{aligned} \quad (2.35)$$

Using the anticommutation and commutation relations (1.17) we obtain

$$\begin{aligned}
\langle{\rm HF}|a_{\pi_2}a_{\pi_1}a^\dagger_\alpha a_\beta a^\dagger_{\pi_3}a^\dagger_{\pi_4}|{\rm HF}\rangle&=\delta_{\alpha\pi_2}\delta_{\beta\pi_4}\delta_{\pi_1\pi_3}+\delta_{\alpha\pi_1}\delta_{\beta\pi_3}\delta_{\pi_2\pi_4}-\delta_{\alpha\pi_2}\delta_{\beta\pi_3}\delta_{\pi_1\pi_4}\\
&\quad-\delta_{\alpha\pi_1}\delta_{\beta\pi_4}\delta_{\pi_2\pi_3}+\delta_{\alpha\beta}\delta_{\pi_2\pi_4}\delta_{\pi_1\pi_3}\theta(\varepsilon_F-\varepsilon_\alpha)-\delta_{\alpha\beta}\delta_{\pi_2\pi_3}\delta_{\pi_1\pi_4}\theta(\varepsilon_F-\varepsilon_\alpha),\\
\langle{\rm HF}|a_{\pi_2}a_{\pi_1}b^\dagger_\alpha b_\beta a^\dagger_{\pi_3}a^\dagger_{\pi_4}|{\rm HF}\rangle&=\delta_{\alpha\beta}\delta_{\pi_2\pi_4}\delta_{\pi_1\pi_3}\theta(\varepsilon_{\rm F}-\varepsilon_\alpha)-\delta_{\alpha\beta}\delta_{\pi_2\pi_3}\delta_{\pi_1\pi_4}\theta(\varepsilon_{\rm F}-\varepsilon_\alpha).
\end{aligned}$$

Substitution into (2.35), the relation (2.5) and the orthogonality of Clebsch-Gordan coefficients (1.49) yield

$$\begin{aligned}
&\langle p_1p_2;JM|M^{(el)}_{JM}|p_3p_4;00\rangle=e^{(\rm eff)}_{\rm p}\mathcal{N}_{p_1p_2}(J)\mathcal{N}_{p_3p_4}(0)\times\\
&\Bigg[\delta_{p_1p_3}\frac{1}{\sqrt{2j_4+1}}\langle p_2||r^JY_J||p_4\rangle\sum_{\substack{m_1m_2\\m_4}}(j_1m_1j_2m_2|JM)(j_1m_1j_4m_4|00)(j_4m_4JM|j_2m_2)\\
&+\delta_{p_2p_4}\frac{1}{\sqrt{2j_3+1}}\langle p_1||r^JY_J||p_3\rangle\sum_{\substack{m_1m_2\\m_3}}(j_1m_1j_2m_2|JM)(j_3m_3j_2m_2|00)(j_3m_3JM|j_1m_1)\\
&-\delta_{p_1p_4}\frac{1}{\sqrt{2j_3+1}}\langle p_2||r^JY_J||p_3\rangle\sum_{\substack{m_1m_2\\m_3}}(j_1m_1j_2m_2|JM)(j_3m_3j_1m_1|00)(j_3m_3JM|j_2m_2)\\
&-\delta_{p_2p_3}\frac{1}{\sqrt{2j_4+1}}\langle p_1||r^JY_J||p_4\rangle\sum_{\substack{m_1m_2\\m_4}}(j_1m_1j_2m_2|JM)(j_2m_2j_4m_4|00)(j_4m_4JM|j_1m_1)\Bigg]
\end{aligned} \quad (2.36)$$

The second term in (2.35) has vanished because of the relation (1.49). Thus the contribution of the neutron part of $M^{(el)}_{JM}$ is zero. The sums of the products of three Clebsch-Gordan coefficients can be reduced to sums with only one summation index (angular momentum projection) using the fact that $(j_1m_1j_2m_2|JM)=0$,

if $m_1+m_2 \neq M$. In the case of neutron ppTDA eigenstates the result is the same except that the effective charge is replaced with $e_{\rm n}^{\rm (eff)}$.

In the case of hhTDA, the ground state is the lowest 0^+ hhTDA eigenstate

$$|\nu_{\rm gs};00\rangle = \sum_{h_3\leq h_4} C_{h_3h_4}^{\nu_{\rm gs}} |h_3h_4;00\rangle \tag{2.37}$$

and the final excited state is

$$|\nu_f;JM\rangle = \sum_{h_1\leq h_2} C_{h_1h_2}^{\nu_f} |h_1h_2;JM\rangle, \tag{2.38}$$

with $|h_1h_2;JM\rangle$ given by (2.13). Substituting (2.37), (2.38) and (2.30) into (2.27) we get

$$\langle\nu_f;JM|M_{JM}^{(el)}|\nu_{\rm gs};00\rangle = \sum_{h_1\leq h_2}\sum_{h_3\leq h_4} C_{h_1h_2}^{\nu_f *} C_{h_3h_4}^{\nu_{\rm gs}} \langle h_1h_2;JM|M_{JM}^{(el)}|h_3h_4;00\rangle. \tag{2.39}$$

Let us consider proton hh configurations (for neutron states the derivation is analogous). Then

$$\begin{aligned}
&\langle h_1h_2;JM|M_{JM}^{(el)}|h_3h_4;00\rangle \\
&= e_{\rm p}^{\rm (eff)} \mathcal{N}_{h_1h_2}(J)\mathcal{N}_{h_3h_4}(0) \sum_{\substack{m_1m_2\\ m_3m_4}} (j_1m_1j_2m_2|JM)(j_3m_3j_4m_4|00) \\
&\times \sum_{\alpha\beta} \frac{1}{\sqrt{2j_b+1}} (j_bm_\beta JM|j_am_\alpha)\langle a||r^JY_J||b\rangle \langle{\rm HF}|h_{\eta_2}^\pi h_{\eta_1}^\pi a_\alpha^\dagger a_\beta h_{\eta_3}^{\pi\,\dagger} h_{\eta_4}^{\pi\,\dagger}|{\rm HF}\rangle,
\end{aligned} \tag{2.40}$$

where we have omitted the neutron part of $M_{JM}^{(el)}$ because it gives zero contribution similarly as in the case of ppTDA. Using the anticommutation relations (1.17) we obtain

$$\begin{aligned}
&\langle{\rm HF}|h_{\eta_2}^\pi h_{\eta_1}^\pi a_\alpha^\dagger a_\beta h_{\eta_3}^{\pi\,\dagger} h_{\eta_4}^{\pi\,\dagger}|{\rm HF}\rangle \\
&=(-1)^{j_1+m_1+j_2+m_2+j_3+m_3+j_4+m_4}\langle{\rm HF}|a_{-\eta_2}^\dagger a_{-\eta_1}^\dagger a_\alpha^\dagger a_\beta a_{-\eta_3} a_{-\eta_4}|{\rm HF}\rangle \\
&=(-1)^{j_1+m_1+j_2+m_2+j_3+m_3+j_4+m_4}\big[\delta_{-\eta_4\alpha}\delta_{-\eta_1\beta}\delta_{\eta_2\eta_3} + \delta_{-\eta_3\alpha}\delta_{-\eta_2\beta}\delta_{\eta_1\eta_4} - \delta_{-\eta_3\alpha}\delta_{-\eta_1\beta}\delta_{\eta_2\eta_4} \\
&\quad - \delta_{-\eta_4\alpha}\delta_{-\eta_2\beta}\delta_{\eta_1\eta_3} + \delta_{\alpha\beta}\delta_{\eta_1\eta_3}\delta_{\eta_2\eta_4}\theta(\varepsilon_{\rm F}-\varepsilon_\alpha) - \delta_{\alpha\beta}\delta_{\eta_1\eta_4}\delta_{\eta_2\eta_3}\theta(\varepsilon_{\rm F}-\varepsilon_\alpha)\big].
\end{aligned}$$

Substitution into (2.40) and the relations (2.5) and (1.49) yield

$$
\begin{aligned}
&\langle h_1 h_2; JM|M^{(el)}_{JM}|h_3 h_4; 00\rangle = e_{\rm p}^{(\rm eff)}\mathcal{N}_{h_1h_2}(J)\mathcal{N}_{h_3h_4}(0)\\
&\times\Bigg[\delta_{h_2h_3}\frac{1}{\sqrt{2j_1+1}}\langle h_4||r^JY_J||h_1\rangle\\
&\times\sum_{\substack{m_1m_2\\m_4}}(j_1m_1j_2m_2|JM)(j_2m_2j_4m_4|00)(j_1-m_1JM|j_4-m_4)(-1)^{j_1+m_1+j_4+m_4}\\
&+\delta_{h_1h_4}\frac{1}{\sqrt{2j_2+1}}\langle h_3||r^JY_J||h_2\rangle\\
&\times\sum_{\substack{m_1m_2\\m_3}}(j_1m_1j_2m_2|JM)(j_3m_3j_1m_1|00)(j_2-m_2JM|j_3-m_3)(-1)^{j_2+m_2+j_3+m_3}\\
&-\delta_{h_2h_4}\frac{1}{\sqrt{2j_1+1}}\langle h_3||r^JY_J||h_1\rangle\\
&\times\sum_{\substack{m_1m_2\\m_3}}(j_1m_1j_2m_2|JM)(j_3m_3j_2m_2|00)(j_1-m_1JM|j_3-m_3)(-1)^{j_1+m_1+j_3+m_3}\\
&-\delta_{h_1h_3}\frac{1}{\sqrt{2j_2+1}}\langle h_4||r^JY_J||h_2\rangle\\
&\times\sum_{\substack{m_1m_2\\m_4}}(j_1m_1j_2m_2|JM)(j_1m_1j_4m_4|00)(j_2-m_2JM|j_4-m_4)(-1)^{j_2+m_2+j_4+m_4}\Bigg].
\end{aligned}
\tag{2.41}
$$

Similarly as in the previous case, the sums of the products of three Clebsch-Gordan coefficients can be again reduced to sums with only one summation index. In the case of neutron hhTDA eigenstates the result is the same except that the effective charge is replaced with $e_{\rm n}^{(\rm eff)}$.

Let us now discuss the effective charges. They represent an effective way of taking into account effects not explicitly included in the model, e.g. ph excitations of the core. The effective charges can be introduced in this way (see [36]):

$$
e_{\rm p}^{(\rm eff)} = (1+\chi)e\,, \quad e_{\rm n}^{(\rm eff)} = \chi e\,, \tag{2.42}
$$

where e is the "bare" charge of a single proton and χ is the electric polarization constant. More on the microscopic origin of the effective charges can be found in [37].

2.3 Particle-particle and hole-hole RPA

Similarly as in the case of ppTDA and hhTDA, we modify the RPA in order to describe open-shell nuclei with two valence particles or holes added to a doubly-magic core. We derive the corresponding ppRPA and hhRPA equations using the equation-of-motion method described at the beginning of Section 1.4.

Let us start with the ppRPA. In analogy to RPA phonon (1.74), in the ppRPA case the eigenstate ν of a system with $A+2$ nucleons in the angular-momentum-

coupled representation is

$$|A+2,\nu;JM\rangle = Q^\dagger_\nu|A,0\rangle \\ = \left(\sum_{p_1\le p_2} X^\nu_{p_1p_2}[c^\dagger_{p_1}c^\dagger_{p_2}]_{JM} - \sum_{h_3\le h_4} Y^\nu_{h_3h_4}[c^\dagger_{h_4}c^\dagger_{h_3}]_{JM}\right)|A,0\rangle, \quad (2.43)$$

where p_1 and p_1 denote single-particle states above the Fermi level, h_3 and h_4 are states below the Fermi level and

$$[c^\dagger_{p_1}c^\dagger_{p_2}]_{JM} = \mathcal{N}_{p_1p_2}(J)\sum_{m_1m_2}(j_1m_1j_2m_2|JM)c^\dagger_{\pi_1}c^\dagger_{\pi_2} \quad (2.44)$$

is the angular momentum coupling. The ground state (representing the system with A nucleons) is defined so that

$$Q_\nu|A,0\rangle = 0. \quad (2.45)$$

The variation of the phonon operator δQ, appearing in equation (1.73), is

$$\delta Q = \sum_{p_1\le p_2}\delta X_{p_1p_2}\big([c^\dagger_{p_1}c^\dagger_{p_2}]_{JM}\big)^\dagger - \sum_{h_3\le h_4}\delta Y_{h_3h_4}\big([c^\dagger_{h_4}c^\dagger_{h_3}]_{JM}\big)^\dagger, \quad (2.46)$$

with

$$\big([c^\dagger_{p_1}c^\dagger_{p_2}]_{JM}\big)^\dagger = \mathcal{N}_{p_1p_2}(J)\sum_{m_1m_2}(j_1m_1j_2m_2|JM)c_{\pi_2}c_{\pi_1} = (-1)^{j_1+j_2-J}[c_{p_2}c_{p_1}]_{JM}, \quad (2.47)$$

where the relation (2.5) was used. Substitution of (2.46) into equation (1.73) and comparison of the terms with $\delta X_{p_1p_2}$ and $\delta Y_{h_3h_4}$ on both sides of the equation yield two sets of equations:

$$(-1)^{j_1+j_2-J}\langle A,0|\big[[c_{p_2}c_{p_1}]_{JM},[H,Q^\dagger_\nu]\big]|A,0\rangle \\ = \hbar\Omega_\nu(-1)^{j_1+j_2-J}\langle A,0|\big[[c_{p_2}c_{p_1}]_{JM},Q^\dagger_\nu\big]|A,0\rangle, \\ (-1)^{j_3+j_4-J}\langle A,0|\big[[c_{h_3}c_{h_4}]_{JM},[H,Q^\dagger_\nu]\big]|A,0\rangle \\ = \hbar\Omega_\nu(-1)^{j_3+j_4-J}\langle A,0|\big[[c_{h_3}c_{h_4}]_{JM},Q^\dagger_\nu\big]|A,0\rangle, \quad (2.48)$$

where $\hbar\Omega_\nu$ is the excitation energy of the $A+2$ nucleus related to the ground state of the nucleus with A nucleons.

Since $|X^\nu_{p_1p_2}|^2$ is the probability of finding the state $[c^\dagger_{p_1}c^\dagger_{p_2}]_{JM}|A,0\rangle$ in the state $|A+2,\nu;JM\rangle$ and $|Y^\nu_{h_3h_4}|^2$ is the probability of finding the state $[c^\dagger_{h_4}c^\dagger_{h_3}]_{JM}|A,0\rangle$ in the state $|A+2,\nu;JM\rangle$, we have

$$X^\nu_{p_1p_2} = (-1)^{j_1+j_2-J}\langle A,0|[c_{p_2}c_{p_1}]_{JM}|A+2,\nu;JM\rangle \\ = (-1)^{j_1+j_2-J}\langle A,0|[c_{p_2}c_{p_1}]_{JM}Q^\dagger_\nu|A,0\rangle \\ = (-1)^{j_1+j_2-J}\langle A,0|\big[[c_{p_2}c_{p_1}]_{JM},Q^\dagger_\nu\big]|A,0\rangle \\ \approx (-1)^{j_1+j_2-J}\langle \mathrm{HF}|\big[[c_{p_2}c_{p_1}]_{JM},Q^\dagger_\nu\big]|\mathrm{HF}\rangle, \quad (2.49)$$

where we used the commutator, because $\langle A,0|Q^\dagger_\nu = 0$, and the quasi-boson approximation (in the last step), and similarly

$$Y^\nu_{h_3h_4} = (-1)^{j_3+j_4-J}\langle A,0|[c_{h_3}c_{h_4}]_{JM}|A+2,\nu;JM\rangle \\ \approx (-1)^{j_3+j_4-J}\langle \mathrm{HF}|\big[[c_{h_3}c_{h_4}]_{JM},Q^\dagger_\nu\big]|\mathrm{HF}\rangle. \quad (2.50)$$

Thus within the quasi-boson approximation the equations (2.48) can be written as

$$\sum_{p_1'\le p_2'} X^{\nu}_{p_1'p_2'}(-1)^{j_1+j_2-J}\langle \mathrm{HF}|\Big[[c_{p_2}c_{p_1}]_{JM},\big[H,[c^{\dagger}_{p_1'}c^{\dagger}_{p_2'}]_{JM}\big]\Big]|\mathrm{HF}\rangle$$
$$-\sum_{h_3'\le h_4'} Y^{\nu}_{h_3'h_4'}(-1)^{j_1+j_2-J}\langle \mathrm{HF}|\Big[[c_{p_2}c_{p_1}]_{JM},\big[H,[c^{\dagger}_{h_4'}c^{\dagger}_{h_3'}]_{JM}\big]\Big]|\mathrm{HF}\rangle=\hbar\Omega_{\nu}X^{\nu}_{p_1p_2}$$

and

$$\sum_{p_1'\le p_2'} X^{\nu}_{p_1'p_2'}(-1)^{j_3+j_4-J}\langle \mathrm{HF}|\Big[[c_{h_3}c_{h_4}]_{JM},\big[H,[c^{\dagger}_{p_1'}c^{\dagger}_{p_2'}]_{JM}\big]\Big]|\mathrm{HF}\rangle$$
$$-\sum_{h_3'\le h_4'} Y^{\nu}_{h_3'h_4'}(-1)^{j_3+j_4-J}\langle \mathrm{HF}|\Big[[c_{h_3}c_{h_4}]_{JM},\big[H,[c^{\dagger}_{h_4'}c^{\dagger}_{h_3'}]_{JM}\big]\Big]|\mathrm{HF}\rangle=\hbar\Omega_{\nu}Y^{\nu}_{h_3h_4}.$$

These equations can be written in the matrix form

$$AX^{\nu}+BY^{\nu}=\hbar\Omega_{\nu}X^{\nu}, \tag{2.51}$$
$$-B^{\dagger}X^{\nu}-CY^{\nu}=\hbar\Omega_{\nu}Y^{\nu} \tag{2.52}$$

or

$$\begin{pmatrix} A & B\\ -B^{\dagger} & -C\end{pmatrix}\begin{pmatrix}X^{\nu}\\ Y^{\nu}\end{pmatrix}=\hbar\Omega_{\nu}\begin{pmatrix}X^{\nu}\\ Y^{\nu}\end{pmatrix}, \tag{2.53}$$

with

$$A_{p_1p_2,p_1'p_2'}=(-1)^{j_1+j_2-J}\langle \mathrm{HF}|\Big[[c_{p_2}c_{p_1}]_{JM},\big[H,[c^{\dagger}_{p_1'}c^{\dagger}_{p_2'}]_{JM}\big]\Big]|\mathrm{HF}\rangle, \tag{2.54}$$

$$B_{p_1p_2,h_3'h_4'}=-(-1)^{j_1+j_2-J}\langle \mathrm{HF}|\Big[[c_{p_2}c_{p_1}]_{JM},\big[H,[c^{\dagger}_{h_4'}c^{\dagger}_{h_3'}]_{JM}\big]\Big]|\mathrm{HF}\rangle, \tag{2.55}$$

$$C_{h_3h_4,h_3'h_4'}=(-1)^{j_3+j_4-J}\langle \mathrm{HF}|\Big[[c_{h_3}c_{h_4}]_{JM},\big[H,[c^{\dagger}_{h_4'}c^{\dagger}_{h_3'}]_{JM}\big]\Big]|\mathrm{HF}\rangle. \tag{2.56}$$

In general the matrices A and B have different dimensions and B is a rectangular matrix.

Now we derive the formulae for the matrices A, B and C. We start with the matrix A. Since c_{π_1} annihilates the HF vacuum $|\mathrm{HF}\rangle$, we can write

$$\begin{aligned}A_{p_1p_2,p_1'p_2'}=&(-1)^{j_1+j_2-J}\langle \mathrm{HF}|[c_{p_2}c_{p_1}]_{JM}\big[H,[c^{\dagger}_{p_1'}c^{\dagger}_{p_2'}]_{JM}\big]|\mathrm{HF}\rangle\\ =&(-1)^{j_1+j_2-J}\langle \mathrm{HF}|[c_{p_2}c_{p_1}]_{JM}H[c^{\dagger}_{p_1'}c^{\dagger}_{p_2'}]_{JM}|\mathrm{HF}\rangle\\ &-(-1)^{j_1+j_2-J}\langle \mathrm{HF}|[c_{p_2}c_{p_1}]_{JM}[c^{\dagger}_{p_1'}c^{\dagger}_{p_2'}]_{JM}H|\mathrm{HF}\rangle\\ =&A^{(1)}_{p_1p_2,p_1'p_2'}-A^{(2)}_{p_1p_2,p_1'p_2'},\end{aligned} \tag{2.57}$$

where

$$A^{(1)}_{p_1p_2,p_1'p_2'}=(-1)^{j_1+j_2-J}\langle \mathrm{HF}|[c_{p_2}c_{p_1}]_{JM}H[c^{\dagger}_{p_1'}c^{\dagger}_{p_2'}]_{JM}|\mathrm{HF}\rangle=(-1)^{j_1+j_2-J}$$
$$\times\mathcal{N}_{p_1p_2}(J)\mathcal{N}_{p_1'p_2'}(J)\sum_{\substack{m_1m_2\\ m_1'm_2'}}(j_2m_2j_1m_1|JM)(j_1'm_1'j_2'm_2'|JM)\langle \mathrm{HF}|c_{\pi_2}c_{\pi_1}Hc^{\dagger}_{\pi_1'}c^{\dagger}_{\pi_2'}|\mathrm{HF}\rangle$$
$$=\mathcal{N}_{p_1p_2}(J)\mathcal{N}_{p_1'p_2'}(J)\sum_{\substack{m_1m_2\\ m_1'm_2'}}(j_1m_2j_2m_2|JM)(j_1'm_1'j_2'm_2'|JM)\langle \mathrm{HF}|c_{\pi_2}c_{\pi_1}Hc^{\dagger}_{\pi_1'}c^{\dagger}_{\pi_2'}|\mathrm{HF}\rangle,$$

where the relation (2.5) was used. Comparing this with (2.3) we find out that $A^{(1)}_{p_1p_2,p_1'p_2'}$ is the ppTDA matrix. Similarly, the second part of the matrix $A_{p_1p_2,p_1'p_2'}$ is

$$\begin{aligned}A^{(2)}_{p_1p_2,p_1'p_2'} &= (-1)^{j_1+j_2-J}\langle \mathrm{HF}|[a_{p_2}a_{p_1}]_{JM}[a^\dagger_{p_1'}a^\dagger_{p_2'}]_{JM}H|\mathrm{HF}\rangle \\ &= \mathcal{N}_{p_1p_2}(J)\mathcal{N}_{p_1'p_2'}(J)\sum_{\substack{m_1m_2\\m_1'm_2'}}(j_1m_1j_2m_2|JM)(j_1'm_1'j_2'm_2'|JM) \\ &\quad\times\langle \mathrm{HF}|a_{\pi_2}a_{\pi_1}a^\dagger_{\pi_1'}a^\dagger_{\pi_2'}H|\mathrm{HF}\rangle, \end{aligned} \tag{2.58}$$

where we consider proton pp configurations (for neutrons the derivation is analogous with the same result). The Hamiltonian H is given by (1.40), (1.41) and (1.42). It is obvious that the residual interaction V_{res} doesn't contribute to $A^{(2)}_{p_1p_2,p_1'p_2'}$, thus we substitute only the mean-field Hamiltonian H_{mf} into (2.58) and obtain

$$\begin{aligned}A^{(2)}_{p_1p_2,p_1'p_2'} &= \mathcal{N}_{p_1p_2}(J)\mathcal{N}_{p_1'p_2'}(J) \\ &\times\Big[\sum_\alpha \varepsilon^\pi_\alpha \sum_{\substack{m_1m_2\\m_1'm_2'}}(j_1m_1j_2m_2|JM)(j_1'm_1'j_2'm_2'|JM)\langle \mathrm{HF}|a_{\pi_2}a_{\pi_1}a^\dagger_{\pi_1'}a^\dagger_{\pi_2'}a^\dagger_\alpha a_\alpha|\mathrm{HF}\rangle \\ &+\sum_\alpha \varepsilon^\nu_\alpha \sum_{\substack{m_1m_2\\m_1'm_2'}}(j_1m_1j_2m_2|JM)(j_1'm_1'j_2'm_2'|JM)\langle \mathrm{HF}|a_{\pi_2}a_{\pi_1}a^\dagger_{\pi_1'}a^\dagger_{\pi_2'}b^\dagger_\alpha b_\alpha|\mathrm{HF}\rangle \\ &+K\sum_{\substack{m_1m_2\\m_1'm_2'}}(j_1m_1j_2m_2|JM)(j_1'm_1'j_2'm_2'|JM)\langle \mathrm{HF}|a_{\pi_2}a_{\pi_1}a^\dagger_{\pi_1'}a^\dagger_{\pi_2'}|\mathrm{HF}\rangle\Big],\end{aligned}$$

where K is given by (1.47). Since

$$\begin{aligned}\langle \mathrm{HF}|a_{\pi_2}a_{\pi_1}a^\dagger_{\pi_1'}a^\dagger_{\pi_2'}a^\dagger_\alpha a_\alpha|\mathrm{HF}\rangle &= \langle \mathrm{HF}|a_{\pi_2}a_{\pi_1}a^\dagger_{\pi_1'}a^\dagger_{\pi_2'}b^\dagger_\alpha b_\alpha|\mathrm{HF}\rangle \\ &= \theta(\varepsilon_{\mathrm{F}}-\varepsilon_\alpha)\langle \mathrm{HF}|a_{\pi_2}a_{\pi_1}a^\dagger_{\pi_1'}a^\dagger_{\pi_2'}|\mathrm{HF}\rangle \end{aligned} \tag{2.59}$$

and

$$\langle \mathrm{HF}|a_{\pi_2}a_{\pi_1}a^\dagger_{\pi_1'}a^\dagger_{\pi_2'}|\mathrm{HF}\rangle = \delta_{\pi_1\pi_1'}\delta_{\pi_2\pi_2'}-\delta_{\pi_1\pi_2'}\delta_{\pi_2\pi_1'}, \tag{2.60}$$

using the relations (2.5), (1.49) and (1.43) we obtain after some manipulations

$$A^{(2)}_{p_1p_2,p_1'p_2'} = \mathcal{N}_{p_1p_2}(J)\mathcal{N}_{p_1'p_2'}(J)[\delta_{p_1p_1'}\delta_{p_2p_2'}-\delta_{p_1p_2'}\delta_{p_2p_1'}(-1)^{j_1+j_2-J}]E_{\mathrm{HF}}. \tag{2.61}$$

We see that $A^{(2)}_{p_1p_2,p_1'p_2'}$ is a diagonal term giving the constant contribution E_{HF}. We omit this term because we are interested in the excitation energies. The conclusion is that the matrix A is nothing but the ppTDA matrix from Section 2.1.

Let us have a look at the matrix C. Since $\langle \mathrm{HF}|c_{\eta_3}=0$, from (2.56) we get

$$\begin{aligned}C_{h_3h_4,h_3'h_4'} &= -(-1)^{j_3+j_4-J}\langle \mathrm{HF}|\big[H,[c^\dagger_{h_4'}c^\dagger_{h_3'}]_{JM}\big][c_{h_3}c_{h_4}]_{JM}|\mathrm{HF}\rangle \\ &= -(-1)^{j_3+j_4-J}\langle \mathrm{HF}|H[c^\dagger_{h_4'}c^\dagger_{h_3'}]_{JM}[c_{h_3}c_{h_4}]_{JM}|\mathrm{HF}\rangle \\ &\quad+(-1)^{j_3+j_4-J}\langle \mathrm{HF}|[c^\dagger_{h_4'}c^\dagger_{h_3'}]_{JM}H[c_{h_3}c_{h_4}]_{JM}|\mathrm{HF}\rangle. \end{aligned} \tag{2.62}$$

Similarly as in the case of $A^{(2)}_{p_1p_2,p_1'p_2'}$, it can be shown that the firs term in (2.62) is diagonal and gives the contribution $-E_{\rm HF}$. Thus we omit this term because we are interested in the excitation energies. Thus

$$C_{h_3h_4,h_3'h_4'} = (-1)^{j_3+j_4-J}\langle \mathrm{HF}|[c^\dagger_{h_4'}c^\dagger_{h_3'}]_{JM}H[c_{h_3}c_{h_4}]_{JM}|\mathrm{HF}\rangle$$
$$= \mathcal{N}_{h_3'h_4'}(J)\mathcal{N}_{h_3h_4}(J)\sum_{\substack{m_4'm_3'\\ m_3m_4}}(j_4'm_4'j_3'm_3'|JM)(j_4m_4j_3m_3|JM)\langle \mathrm{HF}|c^\dagger_{\eta_4'}c^\dagger_{\eta_3'}Hc_{\eta_3}c_{\eta_4}|\mathrm{HF}\rangle,$$

where the relation (2.5) was used. It can be shown that this is exactly the hhTDA matrix from Section 2.1.

Next we derive the formula for the matrix B. Since c_{π_1} annihilates the HF vacuum $|\mathrm{HF}\rangle$, from (2.55) we get

$$B_{p_1p_2,h_3'h_4'} = -(-1)^{j_1+j_2-J}\langle \mathrm{HF}|[c_{p_2}c_{p_1}]_{JM}\big[H,[c^\dagger_{h_4'}c^\dagger_{h_3'}]_{JM}\big]|\mathrm{HF}\rangle. \tag{2.63}$$

Since $c^\dagger_{\eta_3'}|\mathrm{HF}\rangle = 0$, we get

$$B_{p_1p_2,h_3'h_4'} = (-1)^{j_1+j_2-J}\langle \mathrm{HF}|[c_{p_2}c_{p_1}]_{JM}[c^\dagger_{h_4'}c^\dagger_{h_3'}]_{JM}H|\mathrm{HF}\rangle$$
$$= \mathcal{N}_{p_1p_2}(J)\mathcal{N}_{h_3'h_4'}(J)\sum_{\substack{m_1m_2\\ m_3'm_4'}}(j_1m_1j_2m_2|JM)(j_4'm_4'j_3'm_3'|JM)\langle \mathrm{HF}|c_{\pi_2}c_{\pi_1}c^\dagger_{\eta_4'}c^\dagger_{\eta_3'}H|\mathrm{HF}\rangle,$$

where the relation (2.5) was used. The Hamiltonian H is given by (1.40) and it is obvious that the contribution of the mean-field Hamiltonian (1.41) is zero and only the residual interaction $V_{\rm res}$ contributes. Let us suppose that p_1, p_2, h_3', h_4' are proton states (for neutron states the derivation is analogous). Substitution from (1.42) yields

$$\begin{aligned}
B_{p_1p_2,h_3'h_4'} = {}& \mathcal{N}_{p_1p_2}(J)\mathcal{N}_{h_3'h_4'}(J)\\
&\times\Big[\frac{1}{4}\sum_{\alpha\beta\gamma\delta}\langle\alpha\beta|V|\overline{\gamma\delta}\rangle_\pi\sum_{\substack{m_1m_2\\ m_3'm_4'}}(j_1m_1j_2m_2|JM)(j_4'm_4'j_3'm_3'|JM)\\
&\times\langle \mathrm{HF}|a_{\pi_2}a_{\pi_1}a^\dagger_{\eta_4'}a^\dagger_{\eta_3'}:a^\dagger_\alpha a^\dagger_\beta a_\delta a_\gamma:|\mathrm{HF}\rangle\\
&+\frac{1}{4}\sum_{\alpha\beta\gamma\delta}\langle\alpha\beta|V|\overline{\gamma\delta}\rangle_\nu\sum_{\substack{m_1m_2\\ m_3'm_4'}}(j_1m_1j_2m_2|JM)(j_4'm_4'j_3'm_3'|JM)\\
&\times\langle \mathrm{HF}|a_{\pi_2}a_{\pi_1}a^\dagger_{\eta_4'}a^\dagger_{\eta_3'}:b^\dagger_\alpha b^\dagger_\beta b_\delta b_\gamma:|\mathrm{HF}\rangle\\
&+\sum_{\alpha\beta\gamma\delta}\langle\alpha\beta|V|\gamma\delta\rangle_{\pi\nu}\sum_{\substack{m_1m_2\\ m_3'm_4'}}(j_1m_1j_2m_2|JM)(j_4'm_4'j_3'm_3'|JM)\\
&\times\langle \mathrm{HF}|a_{\pi_2}a_{\pi_1}a^\dagger_{\eta_4'}a^\dagger_{\eta_3'}:a^\dagger_\alpha b^\dagger_\beta a_\gamma b_\delta:|\mathrm{HF}\rangle\Big].
\end{aligned}\tag{2.64}$$

It holds

$$\langle \mathrm{HF}|a_{\pi_2}a_{\pi_1}a^\dagger_{\eta_4'}a^\dagger_{\eta_3'}:b^\dagger_\alpha b^\dagger_\beta b_\delta b_\gamma:|\mathrm{HF}\rangle = 0, \tag{2.65}$$

$$\langle \mathrm{HF}|a_{\pi_2}a_{\pi_1}a^\dagger_{\eta_4'}a^\dagger_{\eta_3'}:a^\dagger_\alpha b^\dagger_\beta a_\gamma b_\delta:|\mathrm{HF}\rangle = 0 \tag{2.66}$$

and $\langle \mathrm{HF}|a_{\pi_2}a_{\pi_1}a^\dagger_{\eta_4'}a^\dagger_{\eta_3'} : a^\dagger_\alpha a^\dagger_\beta a_\delta a_\gamma : |\mathrm{HF}\rangle$ doesn't vanish only if α, β are particle states and δ, γ are hole states. Therefore, we obtain

$$B_{p_1p_2,h_3'h_4'} = \mathcal{N}_{p_1p_2}(J)\mathcal{N}_{h_3'h_4'}(J)\frac{1}{4}\sum_{\substack{\pi_3\pi_4\\ \eta_1\eta_2}}\langle\pi_3\pi_4|V|\overline{\eta_1\eta_2}\rangle_\pi \sum_{\substack{m_1m_2\\ m_3'm_4'}}(j_1m_1j_2m_2|JM) \times(j_4'm_4'j_3'm_3'|JM)\langle \mathrm{HF}|a_{\pi_2}a_{\pi_1}a^\dagger_{\eta_4'}a^\dagger_{\eta_3'}a^\dagger_{\pi_3}a^\dagger_{\pi_4}a_{\eta_2}a_{\eta_1}|\mathrm{HF}\rangle. \quad (2.67)$$

Using the anticommutation relations (1.17) one gets

$$\langle \mathrm{HF}|a_{\pi_2}a_{\pi_1}a^\dagger_{\eta_4'}a^\dagger_{\eta_3'}a^\dagger_{\pi_3}a^\dagger_{\pi_4}a_{\eta_2}a_{\eta_1}|\mathrm{HF}\rangle = \delta_{\eta_3'\eta_2}\delta_{\eta_4'\eta_1}\delta_{\pi_1\pi_3}\delta_{\pi_2\pi_4} - \delta_{\eta_3'\eta_2}\delta_{\eta_4'\eta_1}\delta_{\pi_1\pi_4}\delta_{\pi_2\pi_3} - \delta_{\eta_3'\eta_1}\delta_{\eta_4'\eta_2}\delta_{\pi_1\pi_3}\delta_{\pi_2\pi_4} + \delta_{\eta_3'\eta_1}\delta_{\eta_4'\eta_2}\delta_{\pi_1\pi_4}\delta_{\pi_2\pi_3}.$$

Substitution into (2.67) yields

$$B_{p_1p_2,h_3'h_4'} = \mathcal{N}_{p_1p_2}(J)\mathcal{N}_{h_3'h_4'}(J)\frac{1}{4}\sum_{\substack{m_1m_2\\ m_3'm_4'}}(j_1m_1j_2m_2|JM)(j_4'm_4'j_3'm_3'|JM) \times\Big[\langle\pi_1\pi_2|V|\overline{\eta_4'\eta_3'}\rangle_\pi - \langle\pi_2\pi_1|V|\overline{\eta_4'\eta_3'}\rangle_\pi - \langle\pi_1\pi_2|V|\overline{\eta_3'\eta_4'}\rangle_\pi + \langle\pi_2\pi_1|V|\overline{\eta_3'\eta_4'}\rangle_\pi\Big].$$

Using the symmetry properties (1.22) we obtain

$$B_{p_1p_2,h_3'h_4'} = -\mathcal{N}_{p_1p_2}(J)\mathcal{N}_{h_3'h_4'}(J)\sum_{\substack{m_1m_2\\ m_3'm_4'}}(j_1m_1j_2m_2|JM)(j_4'm_4'j_3'm_3'|JM)\langle\pi_1\pi_2|V|\overline{\eta_3'\eta_4'}\rangle_\pi.$$

Since $(j_4'm_4'j_3'm_3'|JM) = (-1)^{j_3'+j_4'-J}(j_3'm_3'j_4'm_4'|JM)$, the final result is

$$B_{p_1p_2,h_3'h_4'} = -(-1)^{j_3'+j_4'-J}\langle p_1p_2, J|V|h_3'h_4', J\rangle_\pi. \quad (2.68)$$

If p_1, p_2, h_3', h_4' are neutron states, the result is the same except that the index π is replaced with ν.

Now we proceed to analogous derivation of the hhRPA equations. Within the hhRPA the eigenstate ν of a system with $A-2$ nucleons in the angular-momentum-coupled representation is

$$|A-2,\nu;JM\rangle = Q^\dagger_\nu|A,0\rangle = \left(\sum_{h_1\le h_2}X^\nu_{h_1h_2}[h^\dagger_{h_1}h^\dagger_{h_2}]_{JM} - \sum_{p_3\le p_4}Y^\nu_{p_3p_4}[h^\dagger_{p_4}h^\dagger_{p_3}]_{JM}\right)|A,0\rangle \quad (2.69)$$

where h_1 and h_2 are hole states and p_3 and p_4 are particle states. The variation δQ of the phonon operator is

$$\delta Q = \left(\sum_{h_1\le h_2}\delta X_{h_1h_2}(-1)^{j_1+j_2-J}[h_{h_2}h_{h_1}]_{JM} - \sum_{p_3\le p_4}\delta Y_{p_3p_4}(-1)^{j_3+j_4-J}[h_{p_3}h_{p_4}]_{JM}\right)|A,0\rangle, \quad (2.70)$$

where the relation (2.47) was used. Substitution of (2.70) into (1.73) and comparison of the terms with $\delta X_{h_1h_2}$ and $\delta Y_{p_3p_4}$ on both sides of the equation yield two sets of equations:

$$
\begin{aligned}
(-1)^{j_1+j_2-J}\langle A,0|\big[[h_{h_2}h_{h_1}]_{JM},[H,Q^\dagger_\nu]\big]|A,0\rangle \\
&= \hbar\Omega_\nu(-1)^{j_1+j_2-J}\langle A,0|\big[[h_{h_2}h_{h_1}]_{JM},Q^\dagger_\nu\big]|A,0\rangle, \\
(-1)^{j_3+j_4-J}\langle A,0|\big[[h_{p_3}h_{p_4}]_{JM},[H,Q^\dagger_\nu]\big]|A,0\rangle \\
&= \hbar\Omega_\nu(-1)^{j_3+j_4-J}\langle A,0|\big[[h_{p_3}h_{p_4}]_{JM},Q^\dagger_\nu\big]|A,0\rangle, \quad (2.71)
\end{aligned}
$$

where $\hbar\Omega_\nu$ is the excitation energy of the the state $|A-2,\nu;JM\rangle$ related to the state $|A,0\rangle$. Since $|X^\nu_{h_1h_2}|^2$ is the probability of finding the state $[h^\dagger_{h_1}h^\dagger_{h_2}]_{JM}|A,0\rangle$ in the state $|A-2,\nu;JM\rangle$, we have

$$
\begin{aligned}
X^\nu_{h_1h_2} &= (-1)^{j_1+j_2-J}\langle A,0|[h_{h_2}h_{h_1}]_{JM}|A-2,\nu,JM\rangle \\
&= (-1)^{j_1+j_2-J}\langle A,0|[h_{h_2}h_{h_1}]_{JM}Q^\dagger_\nu|A,0\rangle \\
&= (-1)^{j_1+j_2-J}\langle A,0|\big[[h_{h_2}h_{h_1}]_{JM},Q^\dagger_\nu\big]|A,0\rangle \\
&\approx (-1)^{j_1+j_2-J}\langle \mathrm{HF}|\big[[h_{h_2}h_{h_1}]_{JM},Q^\dagger_\nu\big]|\mathrm{HF}\rangle, \quad (2.72)
\end{aligned}
$$

where we used the commutator, because $\langle A,0|Q^\dagger_\nu=0$, and the quasi-boson approximation. Since $|Y^\nu_{p_3p_4}|^2$ is the probability of finding the state $[h^\dagger_{p_4}h^\dagger_{p_3}]_{JM}|A,0\rangle$ in the state $|A-2,\nu;JM\rangle$, we similarly get

$$
\begin{aligned}
Y^\nu_{p_3p_4} &= (-1)^{j_3+j_4-J}\langle A,0|[h_{p_3}h_{p_4}]_{JM}|A-2,\nu,JM\rangle \\
&\approx (-1)^{j_3+j_4-J}\langle \mathrm{HF}|\big[[h_{p_3}h_{p_4}]_{JM},Q^\dagger_\nu\big]|\mathrm{HF}\rangle. \quad (2.73)
\end{aligned}
$$

Thus within the quasi-boson approximation the equations (2.71) can be written as

$$
\begin{aligned}
&\sum_{h'_1\le h'_2} X^\nu_{h'_1h'_2}(-1)^{j_1+j_2-J}\langle \mathrm{HF}|\Big[[h_{h_2}h_{h_1}]_{JM},\big[H,[h^\dagger_{h'_1}h^\dagger_{h'_2}]_{JM}\big]\Big]|\mathrm{HF}\rangle \\
&\quad -\sum_{p'_3\le p'_4} Y^\nu_{p'_3p'_4}(-1)^{j_1+j_2-J}\langle \mathrm{HF}|\Big[[h_{h_2}h_{h_1}]_{JM},\big[H,[h^\dagger_{p'_4}h^\dagger_{p'_3}]_{JM}\big]\Big]|\mathrm{HF}\rangle = \hbar\Omega_\nu X^\nu_{h_1h_2}, \\
&\sum_{h'_1\le h'_2} X^\nu_{h'_1h'_2}(-1)^{j_3+j_4-J}\langle \mathrm{HF}|\Big[[h_{p_3}h_{p_4}]_{JM},\big[H,[h^\dagger_{h'_1}h^\dagger_{h'_2}]_{JM}\big]\Big]|\mathrm{HF}\rangle \\
&\quad -\sum_{p'_3\le p'_4} Y^\nu_{p'_3p'_4}(-1)^{j_3+j_4-J}\langle \mathrm{HF}|\Big[[h_{p_3}h_{p_4}]_{JM},\big[H,[h^\dagger_{p'_4}h^\dagger_{p'_3}]_{JM}\big]\Big]|\mathrm{HF}\rangle = \hbar\Omega_\nu Y^\nu_{p_3p_4},
\end{aligned}
$$

which can be written in the matrix form

$$
\begin{aligned}
AY^\nu + BX^\nu &= -\hbar\Omega_\nu Y^\nu, \quad (2.74)\\
-B^\dagger Y^\nu - CX^\nu &= -\hbar\Omega_\nu X^\nu \quad (2.75)
\end{aligned}
$$

or

$$
\begin{pmatrix} A & B \\ -B^\dagger & -C \end{pmatrix}\begin{pmatrix} Y^\nu \\ X^\nu \end{pmatrix} = -\hbar\Omega_\nu \begin{pmatrix} Y^\nu \\ X^\nu \end{pmatrix}, \quad (2.76)
$$

with

$$A_{p_3p_4,p_3'p_4'} = (-1)^{j_3+j_4-J}\langle \mathrm{HF}|\Big[[h_{p_3}h_{p_4}]_{JM},\big[H,[h^\dagger_{p_4'}h^\dagger_{p_3'}]_{JM}\big]\Big]|\mathrm{HF}\rangle, \tag{2.77}$$

$$C_{h_1h_2,h_1'h_2'} = (-1)^{j_1+j_2-J}\langle \mathrm{HF}|\Big[[h_{h_2}h_{h_1}]_{JM},\big[H,[h^\dagger_{h_1'}h^\dagger_{h_2'}]_{JM}\big]\Big]|\mathrm{HF}\rangle, \tag{2.78}$$

$$B_{p_3p_4,h_1'h_2'} = -(-1)^{j_3+j_4-J}\langle \mathrm{HF}|\Big[[h_{p_3}h_{p_4}]_{JM},\big[H,[h^\dagger_{h_1'}h^\dagger_{h_2'}]_{JM}\big]\Big]|\mathrm{HF}\rangle. \tag{2.79}$$

Let us now have a look at these matrices. Since $\langle \mathrm{HF}|h_{\pi_3} = 0$, the matrix A is

$$\begin{aligned} A_{p_3p_4,p_3'p_4'} =& -(-1)^{j_3+j_4-J}\langle \mathrm{HF}|\big[H,[h^\dagger_{p_4'}h^\dagger_{p_3'}]_{JM}\big][h_{p_3}h_{p_4}]_{JM}|\mathrm{HF}\rangle \\ =& -(-1)^{j_3+j_4-J}\langle \mathrm{HF}|H[h^\dagger_{p_4'}h^\dagger_{p_3'}]_{JM}[h_{p_3}h_{p_4}]_{JM}|\mathrm{HF}\rangle \\ &+(-1)^{j_3+j_4-J}\langle \mathrm{HF}|[h^\dagger_{p_4'}h^\dagger_{p_3'}]_{JM}H[h_{p_3}h_{p_4}]_{JM}|\mathrm{HF}\rangle, \end{aligned} \tag{2.80}$$

where the first term is diagonal and gives the constant contribution E_{HF}. We omit this term because we are interested in the excitation energies. It can be shown that the second term is the ppTDA matrix. Thus the matrix A is exactly the same matrix A which is in the "supermatrix" in the ppRPA equations (2.53). Since $h_{\eta_1}|\mathrm{HF}\rangle = 0$, the matrix C is

$$\begin{aligned} C_{h_1h_2,h_1'h_2'} =&(-1)^{j_1+j_2-J}\langle \mathrm{HF}|[h_{h_2}h_{h_1}]_{JM}\big[H,[h^\dagger_{h_1'}h^\dagger_{h_2'}]_{JM}\big]|\mathrm{HF}\rangle \\ =&(-1)^{j_1+j_2-J}\langle \mathrm{HF}|[h_{h_2}h_{h_1}]_{JM}H[h^\dagger_{h_1'}h^\dagger_{h_2'}]_{JM}|\mathrm{HF}\rangle \\ &-(-1)^{j_1+j_2-J}\langle \mathrm{HF}|[h_{h_2}h_{h_1}]_{JM}[h^\dagger_{h_1'}h^\dagger_{h_2'}]_{JM}H|\mathrm{HF}\rangle, \end{aligned} \tag{2.81}$$

where the second term is diagonal and gives the contribution $-E_{\mathrm{HF}}$, thus we drop it. It is obvious that the first term is the hhTDA matrix. Thus the matrix C is the same matrix C as in the ppRPA case.

Finally, we derive the formula for the matrix B. Since $\langle \mathrm{HF}|h_{\pi_3} = 0$, we get

$$B_{p_3p_4,h_1'h_2'} = (-1)^{j_3+j_4-J}\langle \mathrm{HF}|\big[H,[h^\dagger_{h_1'}h^\dagger_{h_2'}]_{JM}\big][h_{p_3}h_{p_4}]_{JM}|\mathrm{HF}\rangle. \tag{2.82}$$

Since $\langle \mathrm{HF}|h^\dagger_{\eta_1'} = 0$, we get

$$B_{p_3p_4,h_1'h_2'} = (-1)^{j_3+j_4-J}\langle \mathrm{HF}|H[h^\dagger_{h_1'}h^\dagger_{h_2'}]_{JM}[h_{p_3}h_{p_4}]_{JM}|\mathrm{HF}\rangle, \tag{2.83}$$

where the Hamiltonian H is given by (1.40). It is obvious that the mean-field Hamiltonian H_{mf} given by (1.41) doesn't contribute. Let us suppose that the states p_3, p_4, h_1', h_2' are proton states (for neutron states the derivation is analogous). Then it easy to deduce that the contribution of the last two terms in the residual interaction V_{res} given by (1.42) vanishes. Thus we obtain

$$\begin{aligned} B_{p_3p_4,h_1'h_2'} =&(-1)^{j_3+j_4-J}\mathcal{N}_{h_1'h_2'}(J)\mathcal{N}_{p_3p_4}(J)\sum_{\substack{m_1'm_2'\\ m_3m_4}}(j_1'm_1'j_2'm_2'|JM)(j_3m_3j_4m_4|JM) \\ &\times\frac{1}{4}\sum_{\alpha\beta\gamma\delta}\langle\alpha\beta|V|\overline{\gamma\delta}\rangle_\pi(-1)^{j_1'+m_1'+j_2'+m_2'+j_3+m_3+j_4+m_4} \\ &\times\langle \mathrm{HF}|:a^\dagger_\alpha a^\dagger_\beta a_\delta a_\gamma: a_{-\eta_1'}a_{-\eta_2'}a^\dagger_{-\pi_3}a^\dagger_{-\pi_4}|\mathrm{HF}\rangle, \end{aligned} \tag{2.84}$$

where α, β have to be hole states and γ, δ have to be particle states for a non-zero contribution. Since the Clebsch-Gordan coefficients are non-zero only if $m'_1 + m'_2 = M$ and $m_3 + m_4 = M$, we effectively have

$$(-1)^{j_3+j_4-J}(-1)^{j'_1+m'_1+j'_2+m'_2+j_3+m_3+j_4+m_4} = (-1)^{j'_1+j'_2-J}.$$

Thus

$$B_{p_3p_4,h'_1h'_2} = (-1)^{j'_1+j'_2-J}\mathcal{N}_{h'_1h'_2}(J)\mathcal{N}_{p_3p_4}(J)\sum_{\substack{m'_1m'_2\\m_3m_4}}(j'_1m'_1j'_2m'_2|JM)(j_3m_3j_4m_4|JM)$$
$$\times\frac{1}{4}\sum_{\substack{\eta_5\eta_6\\\pi_7\pi_8}}\langle\eta_5\eta_6|V|\overline{\pi_7\pi_8}\rangle_\pi\langle\mathrm{HF}|a^\dagger_{\eta_5}a^\dagger_{\eta_6}a_{\pi_8}a_{\pi_7}a_{-\eta'_1}a_{-\eta'_2}a^\dagger_{-\pi_3}a^\dagger_{-\pi_4}|\mathrm{HF}\rangle, \quad (2.85)$$

where $\langle\eta_5\eta_6|V|\overline{\pi_7\pi_8}\rangle_\pi = \langle\pi_7\pi_8|V|\overline{\eta_5\eta_6}\rangle_\pi$ according to the symmetry properties (1.22) (our two-body interaction matrix elements are real). Using the anticommutation relations (1.17) one gets

$$\langle\mathrm{HF}|a^\dagger_{\eta_5}a^\dagger_{\eta_6}a_{\pi_8}a_{\pi_7}a_{-\eta'_1}a_{-\eta'_2}a^\dagger_{-\pi_3}a^\dagger_{-\pi_4}|\mathrm{HF}\rangle = \delta_{-\pi_3\pi_7}\delta_{-\eta'_1\eta_6}\delta_{-\pi_4\pi_8}\delta_{-\eta'_2\eta_5}$$
$$-\delta_{-\pi_3\pi_7}\delta_{-\pi_4\pi_8}\delta_{-\eta'_2\eta_6}\delta_{-\eta'_1\eta_5} - \delta_{-\eta'_1\eta_6}\delta_{-\pi_4\pi_7}\delta_{-\pi_3\pi_8}\delta_{-\eta'_2\eta_5} + \delta_{-\pi_4\pi_7}\delta_{-\pi_3\pi_8}\delta_{-\eta'_2\eta_6}\delta_{-\eta'_1\eta_5}.$$

Substitution into (2.85) yields

$$B_{p_3p_4,h'_1h'_2} =(-1)^{j'_1+j'_2-J}\mathcal{N}_{h'_1h'_2}(J)\mathcal{N}_{p_3p_4}(J)\sum_{\substack{m'_1m'_2\\m_3m_4}}(j'_1m'_1j'_2m'_2|JM)(j_3m_3j_4m_4|JM)$$
$$\times\frac{1}{4}\Big(\langle-\pi_3-\pi_4|V|\overline{-\eta'_2-\eta'_1}\rangle_\pi - \langle-\pi_3-\pi_4|V|\overline{-\eta'_1-\eta'_2}\rangle_\pi$$
$$-\langle-\pi_4-\pi_3|V|\overline{-\eta'_2-\eta'_1}\rangle_\pi + \langle-\pi_4-\pi_3|V|\overline{-\eta'_1-\eta'_2}\rangle_\pi\Big). \quad (2.86)$$

Using the symmetry properties (1.22) we obtain

$$B_{p_3p_4,h'_1h'_2} = -(-1)^{j'_1+j'_2-J}\mathcal{N}_{h'_1h'_2}(J)\mathcal{N}_{p_3p_4}(J)\sum_{\substack{m'_1m'_2\\m_3m_4}}(j'_1m'_1j'_2m'_2|JM)(j_3m_3j_4m_4|JM)$$
$$\times\langle-\pi_3-\pi_4|V|\overline{-\eta'_1-\eta'_2}\rangle_\pi$$
$$= -(-1)^{j'_1+j'_2-J}\langle p_3p_4,J|V|h'_1h'_2,J\rangle_\pi. \quad (2.87)$$

If p_3, p_4, h'_1, h'_2 are neutron states, the result is the same except that the index π is replaced with ν. Comparing the result (2.87) with (2.68) we find out that the matrix B is the same as in the ppRPA case.

Finally, we can conclude that the "supermatrices" in the ppRPA equations (2.53) and hhRPA equations (2.76) are the same. Thus the diagonalization of this "supermatrix" simultaneously yields both ppRPA and hhRPA phonons, but in the case of hhRPA phonons excitation energies have opposite sign and the amplitudes X^ν and Y^ν have opposite meaning. In practice, we distinguish the ppRPA and hhRPA phonons in the following way. We solve the equation (2.53) and obtain eigenenergies $\hbar\Omega_\nu$ and amplitudes X^ν and Y^ν. If the amplitudes X^ν are large and the amplitudes Y^ν are small, the corresponding phonon is of the ppRPA type. If the amplitudes X^ν are small and the amplitudes Y^ν are large,

the corresponding phonon is of the hhRPA type and the amplitudes X^ν and Y^ν have opposite meaning.

Next we derive the orthonormality relation for the ppRPA eigenstates (2.43). We require

$$\begin{aligned}\delta_{\nu\nu'} &= \langle A+2,\nu;JM|A+2,\nu';J'M'\rangle = \langle A,0|Q_\nu Q^\dagger_{\nu'}|A,0\rangle \\ &= \langle A,0|[Q_\nu,Q^\dagger_{\nu'}]|A,0\rangle \approx \langle \mathrm{HF}|[Q_\nu,Q^\dagger_{\nu'}]|\mathrm{HF}\rangle \\ &= \sum_{p_1\le p_2}\sum_{p_1'\le p_2'} X^{\nu *}_{p_1p_2}X^{\nu'}_{p_1'p_2'}(-1)^{j_1+j_2-J}\langle \mathrm{HF}|\left[[c_{p_2}c_{p_1}]_{JM},[c^\dagger_{p_1'}c^\dagger_{p_2'}]_{J'M'}\right]|\mathrm{HF}\rangle \\ &+ \sum_{h_3\le h_4}\sum_{h_3'\le h_4'} Y^{\nu *}_{h_3h_4}Y^{\nu'}_{h_3'h_4'}(-1)^{j_3+j_4-J}\langle \mathrm{HF}|\left[[c_{h_3}c_{h_4}]_{JM},[c^\dagger_{h_4'}c^\dagger_{h_3'}]_{J'M'}\right]|\mathrm{HF}\rangle, \quad (2.88)\end{aligned}$$

where the quasi-boson approximation and the relation (2.47) were used. In the first term we have

$$\begin{aligned}&\langle \mathrm{HF}|\left[[c_{p_2}c_{p_1}]_{JM},[c^\dagger_{p_1'}c^\dagger_{p_2'}]_{J'M'}\right]|\mathrm{HF}\rangle = \langle \mathrm{HF}|[c_{p_2}c_{p_1}]_{JM}[c^\dagger_{p_1'}c^\dagger_{p_2'}]_{J'M'}|\mathrm{HF}\rangle \\ &= \mathcal{N}_{p_1p_2}(J)\mathcal{N}_{p_1'p_2'}(J')\sum_{\substack{m_1m_2\\ m_1'm_2'}}(j_2m_2j_1m_1|JM)(j_1'm_1'j_2'm_2'|J'M')\langle \mathrm{HF}|c_{\pi_2}c_{\pi_1}c^\dagger_{\pi_1'}c^\dagger_{\pi_2'}|\mathrm{HF}\rangle.\end{aligned}$$

Substitution of

$$\langle \mathrm{HF}|c_{\pi_2}c_{\pi_1}c^\dagger_{\pi_1'}c^\dagger_{\pi_2'}|\mathrm{HF}\rangle = \delta_{\pi_1\pi_1'}\delta_{\pi_2\pi_2'} - \delta_{\pi_1\pi_2'}\delta_{\pi_2\pi_1'} \quad (2.89)$$

and the relations (2.5) and (1.49) yield

$$\begin{aligned}&\langle \mathrm{HF}|\left[[c_{p_2}c_{p_1}]_{JM},[c^\dagger_{p_1'}c^\dagger_{p_2'}]_{JM}\right]|\mathrm{HF}\rangle \\ &\qquad = \delta_{JJ'}\delta_{MM'}\left[\mathcal{N}_{p_1p_2}(J)\right]^2\left[\delta_{p_1p_1'}\delta_{p_2p_2'}(-1)^{j_1+j_2-J} - \delta_{p_1p_2'}\delta_{p_2p_1'}\right]. \quad (2.90)\end{aligned}$$

Analogously we obtain

$$\begin{aligned}&\langle \mathrm{HF}|\left[[c_{h_3}c_{h_4}]_{JM},[c^\dagger_{h_4'}c^\dagger_{h_3'}]_{JM}\right]|\mathrm{HF}\rangle \\ &\qquad = -\delta_{JJ'}\delta_{MM'}\left[\mathcal{N}_{h_3h_4}(J)\right]^2\left[\delta_{h_3h_3'}\delta_{h_4h_4'}(-1)^{j_3+j_4-J} - \delta_{h_3h_4'}\delta_{h_4h_3'}\right]. \quad (2.91)\end{aligned}$$

Now we substitute (2.90) and (2.91) into (2.88). It is easy to deduce that the second term in (2.90) contributes only if $p_1 = p_2 = p_1' = p_2'$ and if this is the case, it gives the same contribution as the first term and this is compensated by the normalization factor $[\mathcal{N}_{p_1p_2}(J)]^2$, which is equal to $1/2$ in this case and 1 in other cases. The situation in (2.91) is analogous. Thus the substitution yields the ppRPA orthonormality relation

$$\langle A+2,\nu;JM|A+2,\nu';JM\rangle = \sum_{p_1\le p_2} X^{\nu *}_{p_1p_2}X^{\nu'}_{p_1p_2} - \sum_{h_3\le h_4} Y^{\nu *}_{h_3h_4}Y^{\nu'}_{h_3h_4} = \delta_{\nu\nu'}, \quad (2.92)$$

which contains the ppRPA normalization condition

$$\langle A+2,\nu;JM|A+2,\nu;JM\rangle = \sum_{p_1\le p_2}|X^\nu_{p_1p_2}|^2 - \sum_{h_3\le h_4}|Y^\nu_{h_3h_4}|^2 = 1. \quad (2.93)$$

In the case of the hhRPA, the derivation of the orthonormality and normalization relations is analogous and the results are

$$\langle A-2,\nu;JM|A-2,\nu';JM\rangle = \sum_{h_1\le h_2} X^{\nu *}_{h_1h_2}X^{\nu'}_{h_1h_2} - \sum_{p_3\le p_4} Y^{\nu *}_{p_3p_4}Y^{\nu'}_{p_3p_4} = \delta_{\nu\nu'}, \quad (2.94)$$

$$\langle A-2,\nu;JM|A-2,\nu;JM\rangle = \sum_{h_1\le h_2} |X^{\nu}_{h_1h_2}|^2 - \sum_{p_3\le p_4} |Y^{\nu}_{p_3p_4}|^2 = 1. \quad (2.95)$$

2.4 Electromagnetic transitions within *pp*RPA and *hh*RPA

In this section we derive the formula for the reduced probability of an electric transition of multipolarity J from the ground state with the angular momentum and parity 0^+ to some excited state with angular momentum J (in Section 2.2, it is shown that the angular momentum of the final state has to be equal to the multipolarity of the transition) within the ppRPA and hhRPA. The relation (2.28), in which the angular momentum projection M is arbitrary, is still valid, but the final state $|\nu_f;JM\rangle$ is a ppRPA or hhRPA phonon as well as the ground state $|\nu_{\rm gs};00\rangle$, which is the lowest 0^+ phonon.

Let us start with the ppRPA case. We have

$$|\nu_f;JM\rangle = Q^\dagger_{\nu_f}|A,0\rangle = \left(\sum_{p_1\le p_2} X^{\nu_f}_{p_1p_2}[c^\dagger_{p_1}c^\dagger_{p_2}]_{JM} - \sum_{h_3\le h_4} Y^{\nu_f}_{h_3h_4}[c^\dagger_{h_4}c^\dagger_{h_3}]_{JM}\right)|A,0\rangle$$

and

$$|\nu_{\rm gs};00\rangle = Q^\dagger_{\nu_{\rm gs}}|A,0\rangle = \left(\sum_{p'_1\le p'_2} X^{\nu_{\rm gs}}_{p'_1p'_2}[c^\dagger_{p'_1}c^\dagger_{p'_2}]_{00} - \sum_{h'_3\le h'_4} Y^{\nu_{\rm gs}}_{h'_3h'_4}[c^\dagger_{h'_4}c^\dagger_{h'_3}]_{00}\right)|A,0\rangle.$$

At first, we calculate the commutator $[Q_\nu, Q^\dagger_{\nu'}]$, where (see the relation (2.47))

$$Q_\nu = \sum_{p_1\le p_2} X^{\nu *}_{p_1p_2}(-1)^{j_1+j_2-J}[c_{p_2}c_{p_1}]_{JM} - \sum_{h_3\le h_4} Y^{\nu *}_{h_3h_4}(-1)^{j_3+j_4-J}[c_{h_3}c_{h_4}]_{JM}. \quad (2.96)$$

Using the relation

$$[AB,CD] = A\{B,C\}D - AC\{B,D\} + \{A,C\}DB - C\{A,D\}B \quad (2.97)$$

we obtain

$$\begin{aligned}
[Q_\nu, Q^\dagger_{\nu'}] &= \sum_{p_1\le p_2}\sum_{p'_1\le p'_2} X^{\nu *}_{p_1p_2}X^{\nu'}_{p'_1p'_2}(-1)^{j_1+j_2-J}\left[[c_{p_2}c_{p_1}]_{JM},[c^\dagger_{p'_1}c^\dagger_{p'_2}]_{J'M'}\right] \\
&+ \sum_{h_3\le h_4}\sum_{h'_3\le h'_4} Y^{\nu *}_{h_3h_4}Y^{\nu'}_{h'_3h'_4}(-1)^{j_3+j_4-J}\left[[c_{h_3}c_{h_4}]_{JM},[c^\dagger_{h'_4}c^\dagger_{h'_3}]_{J'M'}\right] \\
&\approx \sum_{p_1\le p_2}\sum_{p'_1\le p'_2} X^{\nu *}_{p_1p_2}X^{\nu'}_{p'_1p'_2}(-1)^{j_1+j_2-J}\langle{\rm HF}|\left[[c_{p_2}c_{p_1}]_{JM},[c^\dagger_{p'_1}c^\dagger_{p'_2}]_{J'M'}\right]|{\rm HF}\rangle \\
&+ \sum_{h_3\le h_4}\sum_{h'_3\le h'_4} Y^{\nu *}_{h_3h_4}Y^{\nu'}_{h'_3h'_4}(-1)^{j_3+j_4-J}\langle{\rm HF}|\left[[c_{h_3}c_{h_4}]_{JM},[c^\dagger_{h'_4}c^\dagger_{h'_3}]_{J'M'}\right]|{\rm HF}\rangle,
\end{aligned}$$

where another aspect of the quasi-boson approximation, namely replacing the commutator by its HF expectation value, was applied (see [15]). Comparing this with the relation (2.88) we find out that

$$[Q_\nu, Q^\dagger_{\nu'}] \approx \delta_{\nu\nu'}. \tag{2.98}$$

The matrix element in the relation (2.28) is

$$\begin{aligned}\langle \nu_f; JM|M^{(el)}_{JM}|\nu_{\rm gs};00\rangle &= \langle A,0|Q_{\nu_f}M^{(el)}_{JM}Q^\dagger_{\nu_{\rm gs}}|A,0\rangle \\ &= \langle A,0|Q_{\nu_f}[M^{(el)}_{JM},Q^\dagger_{\nu_{\rm gs}}]|A,0\rangle + \langle A,0|Q_{\nu_f}Q^\dagger_{\nu_{\rm gs}}M^{(el)}_{JM}|A,0\rangle,\end{aligned}$$

where the second term is

$$\langle A,0|Q^\dagger_{\nu_{\rm gs}}Q_{\nu_f}M^{(el)}_{JM}|A,0\rangle = 0 \tag{2.99}$$

because of the commutation relation (2.98). Since $Q_{\nu_f}|A,0\rangle = 0$, we can write

$$\begin{aligned}\langle \nu_f; JM|M^{(el)}_{JM}|\nu_{\rm gs};00\rangle =& \langle A,0|\big[Q_{\nu_f},[M^{(el)}_{JM},Q^\dagger_{\nu_{\rm gs}}]\big]|A,0\rangle \\ \approx& \langle {\rm HF}|\big[Q_{\nu_f},[M^{(el)}_{JM},Q^\dagger_{\nu_{\rm gs}}]\big]|{\rm HF}\rangle \\ =& \langle {\rm HF}|Q_{\nu_f}[M^{(el)}_{JM},Q^\dagger_{\nu_{\rm gs}}]|{\rm HF}\rangle - \langle {\rm HF}|[M^{(el)}_{JM},Q^\dagger_{\nu_{\rm gs}}]Q_{\nu_f}|{\rm HF}\rangle \\ =& \langle {\rm HF}|Q_{\nu_f}M^{(el)}_{JM}Q^\dagger_{\nu_{\rm gs}}|{\rm HF}\rangle - \langle {\rm HF}|Q_{\nu_f}Q^\dagger_{\nu_{\rm gs}}M^{(el)}_{JM}|{\rm HF}\rangle \\ & - \langle {\rm HF}|M^{(el)}_{JM}Q^\dagger_{\nu_{\rm gs}}Q_{\nu_f}|{\rm HF}\rangle + \langle {\rm HF}|Q^\dagger_{\nu_{\rm gs}}M^{(el)}_{JM}Q_{\nu_f}|{\rm HF}\rangle,\end{aligned} \tag{2.100}$$

where the quasi-boson approximation was used. Now we substitute the phonon creation and annihilation operators into (2.100). Several terms of the phonon operators don't contribute (because $[c^\dagger_{h'_4}c^\dagger_{h'_3}]_{00}|{\rm HF}\rangle = 0$, for instance). It can be deduced that

$$\begin{aligned}&\langle \nu_f;JM|M^{(el)}_{JM}|\nu_{\rm gs};00\rangle \\ &\quad= \sum_{p_1\le p_2}\sum_{p'_1\le p'_2} X^{\nu_f *}_{p_1p_2}X^{\nu_{\rm gs}}_{p'_1p'_2}(-1)^{j_1+j_2-J}\langle {\rm HF}|[c_{p_2}c_{p_1}]_{JM}M^{(el)}_{JM}[c^\dagger_{p'_1}c^\dagger_{p'_2}]_{00}|{\rm HF}\rangle \\ &\quad- \sum_{p_1\le p_2}\sum_{p'_1\le p'_2} X^{\nu_f *}_{p_1p_2}X^{\nu_{\rm gs}}_{p'_1p'_2}(-1)^{j_1+j_2-J}\langle {\rm HF}|[c_{p_2}c_{p_1}]_{JM}[c^\dagger_{p'_1}c^\dagger_{p'_2}]_{00}M^{(el)}_{JM}|{\rm HF}\rangle \\ &\quad- \sum_{h_3\le h_4}\sum_{h'_3\le h'_4} Y^{\nu_f *}_{h_3h_4}Y^{\nu_{\rm gs}}_{h'_3h'_4}(-1)^{j_3+j_4-J}\langle {\rm HF}|M^{(el)}_{JM}[c^\dagger_{h'_4}c^\dagger_{h'_3}]_{00}[c_{h_3}c_{h_4}]_{JM}|{\rm HF}\rangle \\ &\quad+ \sum_{h_3\le h_4}\sum_{h'_3\le h'_4} Y^{\nu_f *}_{h_3h_4}Y^{\nu_{\rm gs}}_{h'_3h'_4}(-1)^{j_3+j_4-J}\langle {\rm HF}|[c^\dagger_{h'_4}c^\dagger_{h'_3}]_{00}M^{(el)}_{JM}[c_{h_3}c_{h_4}]_{JM}|{\rm HF}\rangle.\end{aligned} \tag{2.101}$$

Let us suppose that the valence particles are protons (for neutrons the derivation is analogous). Then the substitution of $M^{(el)}_{JM}$ given by (2.30) into the matrix

element in the first term in (2.101) yields

$$\begin{aligned}
&\langle \mathrm{HF}|[a_{p_2}a_{p_1}]_{JM}M^{(el)}_{JM}[a^\dagger_{p'_1}a^\dagger_{p'_2}]_{00}|\mathrm{HF}\rangle \\
&\quad = e^{(\mathrm{eff})}_{\mathrm{p}}\mathcal{N}_{p_1p_2}(J)\mathcal{N}_{p'_1p'_2}(0)\sum_{\substack{m_1m_2\\ m'_1m'_2}}(j_2m_2j_1m_1|JM)(j'_1m'_1j'_2m'_2|00) \\
&\qquad \times\sum_{\alpha\beta}\frac{1}{\sqrt{2j_b+1}}(j_bm_\beta JM|j_am_\alpha)\langle a||r^JY_J||b\rangle\langle \mathrm{HF}|a_{\pi_2}a_{\pi_1}a^\dagger_\alpha a_\beta a^\dagger_{\pi'_1}a^\dagger_{\pi'_2}|\mathrm{HF}\rangle \\
&\qquad + e^{(\mathrm{eff})}_{\mathrm{n}}\mathcal{N}_{p_1p_2}(J)\mathcal{N}_{p'_1p'_2}(0)\sum_{\substack{m_1m_2\\ m'_1m'_2}}(j_2m_2j_1m_1|JM)(j'_1m'_1j'_2m'_2|00)\times \\
&\qquad \times\sum_{\alpha\beta}\frac{1}{\sqrt{2j_b+1}}(j_bm_\beta JM|j_am_\alpha)\langle a||r^JY_J||b\rangle\langle \mathrm{HF}|a_{\pi_2}a_{\pi_1}b^\dagger_\alpha b_\beta a^\dagger_{\pi'_1}a^\dagger_{\pi'_2}|\mathrm{HF}\rangle.
\end{aligned}$$

If we compare this with (2.35) and use the relation (2.5), we find out that this is equal to the result (2.36), valid for electromagnetic transitions within the ppTDA, multiplied by $(-1)^{j_1+j_2-J}$, except that the indices p_3 and p_3 are replaced with p'_1 and p'_2.

The matrix element in the last term in (2.101) is

$$\begin{aligned}
&\langle \mathrm{HF}|[a^\dagger_{h'_4}a^\dagger_{h'_3}]_{00}M^{(el)}_{JM}[a_{h_3}a_{h_4}]_{JM}|\mathrm{HF}\rangle \\
&\quad = e^{(\mathrm{eff})}_{\mathrm{p}}\mathcal{N}_{h_3h_4}(J)\mathcal{N}_{h'_3h'_4}(0)\sum_{\substack{m_3m_4\\ m'_3m'_4}}(j_3m_3j_4m_4|JM)(j'_4m'_4j'_3m'_3|00) \\
&\qquad \times\sum_{\alpha\beta}\frac{1}{\sqrt{2j_b+1}}(j_bm_\beta JM|j_am_\alpha)\langle a||r^JY_J||b\rangle\langle \mathrm{HF}|a^\dagger_{\eta'_4}a^\dagger_{\eta'_3}a^\dagger_\alpha a_\beta a_{\eta_3}a_{\eta_4}|\mathrm{HF}\rangle,
\end{aligned} \tag{2.102}$$

where we skipped the neutron part of the electric multipole operator $M^{(el)}_{JM}$ because, similarly as in the case of electromagnetic transitions within the ppTDA, it gives zero contribution due to the orthogonality of Clebsch-Gordan coefficients (1.49). Using the anticommutation relations (1.17) one gets

$$\begin{aligned}
\langle \mathrm{HF}|a^\dagger_{\eta'_4}a^\dagger_{\eta'_3}a^\dagger_\alpha a_\beta a_{\eta_3}a_{\eta_4}|\mathrm{HF}\rangle &= \delta_{\eta_4\alpha}\delta_{\eta'_3\beta}\delta_{\eta'_4\eta_3}+\delta_{\eta_3\alpha}\delta_{\eta'_4\beta}\delta_{\eta'_3\eta_4}-\delta_{\eta_3\alpha}\delta_{\eta'_3\beta}\delta_{\eta'_4\eta_4} \\
&\quad -\delta_{\eta_4\alpha}\delta_{\eta'_4\beta}\delta_{\eta'_3\eta_3}+\delta_{\alpha\beta}\delta_{\eta'_3\eta_3}\delta_{\eta'_4\eta_4}\theta(\varepsilon_{\mathrm{F}}-\varepsilon_\alpha)-\delta_{\alpha\beta}\delta_{\eta'_3\eta_4}\delta_{\eta'_4\eta_3}\theta(\varepsilon_{\mathrm{F}}-\varepsilon_\alpha),
\end{aligned} \tag{2.103}$$

where the last two terms don't contribute because of the relation (1.49). Substitution of (2.103) and $(j_3m_3j_4m_4|JM)=(-1)^{j_3+j_4-J}(j_4m_4j_3m_3|JM)$ into (2.102)

yield

$$\langle \mathrm{HF}|[a^\dagger_{h'_4}a^\dagger_{h'_3}]_{00}M^{(el)}_{JM}[a_{h_3}a_{h_4}]_{JM}|\mathrm{HF}\rangle = (-1)^{j_3+j_4-J}e^{(\mathrm{eff})}_{\mathrm{p}}\mathcal{N}_{h_3h_4}(J)\mathcal{N}_{h'_3h'_4}(0)\times$$

$$\Big[\delta_{h'_4h_3}\frac{1}{\sqrt{2j'_3+1}}\langle h_4||r^JY_J||h'_3\rangle\sum_{\substack{m_3m_4\\m'_3}}(j_4m_4j_3m_3|JM)(j_3m_3j'_3m'_3|00)(j'_3m'_3JM|j_4m_4)$$

$$+\delta_{h'_3h_4}\frac{1}{\sqrt{2j'_4+1}}\langle h_3||r^JY_J||h'_4\rangle\sum_{\substack{m_3m_4\\m'_4}}(j_4m_4j_3m_3|JM)(j'_4m'_4j_4m_4|00)(j'_4m'_4JM|j_3m_3)$$

$$-\delta_{h'_4h_4}\frac{1}{\sqrt{2j'_3+1}}\langle h_3||r^JY_J||h'_3\rangle\sum_{\substack{m_3m_4\\m'_3}}(j_4m_4j_3m_3|JM)(j_4m_4j'_3m'_3|00)(j'_3m'_3JM|j_3m_3)$$

$$-\delta_{h'_3h_3}\frac{1}{\sqrt{2j'_4+1}}\langle h_4||r^JY_J||h'_4\rangle\sum_{\substack{m_3m_4\\m'_4}}(j_4m_4j_3m_3|JM)(j'_4m'_4j_3m_3|00)(j'_4m'_4JM|j_4m_4)\Big]$$

The matrix element in the second term in (2.101) is (as in the previous case, the neutron part of $M^{(el)}_{JM}$ is omitted)

$$\begin{aligned}\langle \mathrm{HF}|[a_{p_2}a_{p_1}]_{JM}[a^\dagger_{p'_1}a^\dagger_{p'_2}]_{00}M^{(el)}_{JM}|\mathrm{HF}\rangle&\\ =\mathcal{N}_{p_1p_2}(J)\mathcal{N}_{p'_1p'_2}(0)\sum_{\substack{m_1m_2\\m'_1m'_2}}(j_2m_2j_1m_1|JM)(j'_1m'_1j'_2m'_2|00)&\\ \times\sum_{\alpha\beta}\frac{1}{\sqrt{2j_b+1}}(j_bm_\beta JM|j_am_\alpha)\langle a||r^JY_J||b\rangle\langle \mathrm{HF}|a_{\pi_2}a_{\pi_1}a^\dagger_{\pi'_1}a^\dagger_{\pi'_2}a^\dagger_\alpha a_\beta|\mathrm{HF}\rangle,&\end{aligned}$$

with

$$\langle \mathrm{HF}|a_{\pi_2}a_{\pi_1}a^\dagger_{\pi'_1}a^\dagger_{\pi'_2}a^\dagger_\alpha a_\beta|\mathrm{HF}\rangle = \delta_{\alpha\beta}\delta_{\pi_1\pi'_1}\delta_{\pi_2\pi'_2}\theta(\varepsilon_{\mathrm{F}}-\varepsilon_\alpha)-\delta_{\alpha\beta}\delta_{\pi_1\pi'_2}\delta_{\pi_2\pi'_1}\theta(\varepsilon_{\mathrm{F}}-\varepsilon_\alpha),$$

which gives zero contribution due to the relation (1.49). Thus

$$\langle \mathrm{HF}|[a_{p_2}a_{p_1}]_{JM}[a^\dagger_{p'_1}a^\dagger_{p'_2}]_{00}M^{(el)}_{JM}|\mathrm{HF}\rangle = 0. \tag{2.104}$$

Similarly, we find out that the matrix element in the third term in (2.101) is

$$\langle \mathrm{HF}|M^{(el)}_{JM}[a^\dagger_{h'_4}a^\dagger_{h'_3}]_{00}[a_{h_3}a_{h_4}]_{JM}|\mathrm{HF}\rangle = 0. \tag{2.105}$$

Substitution of all the matrix elements into (2.101) yields the final result

$$\langle\nu_f;JM|M^{(el)}_{JM}|\nu_{\rm gs};00\rangle = e^{(\rm eff)}_{\rm p}\sum_{p_1\le p_2}\sum_{p'_1\le p'_2} X^{\nu_f *}_{p_1p_2}X^{\nu_{\rm gs}}_{p'_1p'_2}\mathcal{N}_{p_1p_2}(J)\mathcal{N}_{p'_1p'_2}(0)\times$$

$$\Bigg[\delta_{p_1p'_1}\frac{1}{\sqrt{2j'_2+1}}\langle p_2||r^JY_J||p'_2\rangle\sum_{\substack{m_1m_2\\m'_2}}(j_1m_1j_2m_2|JM)(j_1m_1j'_2m'_2|00)(j'_2m'_2JM|j_2m_2)$$

$$+\delta_{p_2p'_2}\frac{1}{\sqrt{2j'_1+1}}\langle p_1||r^JY_J||p'_1\rangle\sum_{\substack{m_1m_2\\m'_1}}(j_1m_1j_2m_2|JM)(j'_1m'_1j_2m_2|00)(j'_1m'_1JM|j_1m_1)$$

$$-\delta_{p_1p'_2}\frac{1}{\sqrt{2j'_1+1}}\langle p_2||r^JY_J||p'_1\rangle\sum_{\substack{m_1m_2\\m'_1}}(j_1m_1j_2m_2|JM)(j'_1m'_1j_1m_1|00)(j'_1m'_1JM|j_2m_2)$$

$$-\delta_{p_2p'_1}\frac{1}{\sqrt{2j'_2+1}}\langle p_1||r^JY_J||p'_2\rangle\sum_{\substack{m_1m_2\\m'_2}}(j_1m_1j_2m_2|JM)(j_2m_2j'_2m'_2|00)(j'_2m'_2JM|j_1m_1)\Bigg]$$

$$+e^{(\rm eff)}_{\rm p}\sum_{h_3\le h_4}\sum_{h'_3\le h'_4} Y^{\nu_f *}_{h_3h_4}Y^{\nu_{\rm gs}}_{h'_3h'_4}\mathcal{N}_{h_3h_4}(J)\mathcal{N}_{h'_3h'_4}(0)\times$$

$$\Bigg[\delta_{h'_4h_3}\frac{1}{\sqrt{2j'_3+1}}\langle h_4||r^JY_J||h'_3\rangle\sum_{\substack{m_3m_4\\m'_3}}(j_4m_4j_3m_3|JM)(j_3m_3j'_3m'_3|00)(j'_3m'_3JM|j_4m_4)$$

$$+\delta_{h'_3h_4}\frac{1}{\sqrt{2j'_4+1}}\langle h_3||r^JY_J||h'_4\rangle\sum_{\substack{m_3m_4\\m'_4}}(j_4m_4j_3m_3|JM)(j'_4m'_4j_4m_4|00)(j'_4m'_4JM|j_3m_3)$$

$$-\delta_{h'_4h_4}\frac{1}{\sqrt{2j'_3+1}}\langle h_3||r^JY_J||h'_3\rangle\sum_{\substack{m_3m_4\\m'_3}}(j_4m_4j_3m_3|JM)(j_4m_4j'_3m'_3|00)(j'_3m'_3JM|j_3m_3)$$

$$-\delta_{h'_3h_3}\frac{1}{\sqrt{2j'_4+1}}\langle h_4||r^JY_J||h'_4\rangle\sum_{\substack{m_3m_4\\m'_4}}(j_4m_4j_3m_3|JM)(j'_4m'_4j_3m_3|00)(j'_4m'_4JM|j_4m_4)\Bigg] \tag{2.106}$$

where, similarly as in (2.36), the sums of the products of the Clebsch-Gordan coefficients can be reduced to sums with only one summation index. This result is valid also for valence neutrons except that the effective charge $e^{(\rm eff)}_{\rm p}$ is replaced with $e^{(\rm eff)}_{\rm n}$. We see that it reduces to the result (2.36) valid for electromagnetic transitions within the ppTDA, if the amplitudes $Y^{\nu_f}, Y^{\nu_{\rm gs}}$ are zero.

Now we proceed to analogous derivation of the matrix element $\langle\nu_f;JM|M^{(el)}_{JM}|\nu_{\rm gs};00\rangle$, where the final state and the ground state are hhRPA phonons

$$|\nu_f;JM\rangle = Q^\dagger_{\nu_f}|A,0\rangle = \left(\sum_{h_1\le h_2}X^{\nu_f}_{h_1h_2}[h^\dagger_{h_1}h^\dagger_{h_2}]_{JM} - \sum_{p_3\le p_4}Y^{\nu_f}_{p_3p_4}[h^\dagger_{p_4}h^\dagger_{p_3}]_{JM}\right)|A,0\rangle,$$

$$|\nu_{\rm gs};00\rangle = Q^\dagger_{\nu_{\rm gs}}|A,0\rangle = \left(\sum_{h'_1\le h'_2}X^{\nu_{\rm gs}}_{h'_1h'_2}[h^\dagger_{h'_1}h^\dagger_{h'_2}]_{00} - \sum_{p'_3\le p'_4}Y^{\nu_{\rm gs}}_{p'_3p'_4}[h^\dagger_{p'_4}h^\dagger_{p'_3}]_{00}\right)|A,0\rangle.$$

As in the ppRPA case, it holds:

$$[Q_\nu, Q^\dagger_{\nu'}] \approx \delta_{\nu\nu'}. \tag{2.107}$$

Therefore, as in the ppRPA case, we have

$$\begin{aligned}\langle \nu_f; JM|M^{(el)}_{JM}|\nu_{\rm gs};00\rangle \approx &\langle {\rm HF}|Q_{\nu_f} M^{(el)}_{JM} Q^\dagger_{\nu_{\rm gs}}|{\rm HF}\rangle - \langle {\rm HF}|Q_{\nu_f} Q^\dagger_{\nu_{\rm gs}} M^{(el)}_{JM}|{\rm HF}\rangle \\ &- \langle {\rm HF}|M^{(el)}_{JM} Q^\dagger_{\nu_{\rm gs}} Q_{\nu_f}|{\rm HF}\rangle + \langle {\rm HF}|Q^\dagger_{\nu_{\rm gs}} M^{(el)}_{JM} Q_{\nu_f}|{\rm HF}\rangle,\end{aligned} \tag{2.108}$$

where (see the relation (2.47))

$$Q_{\nu_f} = \sum_{h_1\le h_2} X^{\nu_f *}_{h_1h_2}(-1)^{j_1+j_2-J}[h_{h_2}h_{h_1}]_{JM} - \sum_{p_3\le p_4} Y^{\nu_f *}_{p_3p_4}(-1)^{j_3+j_4-J}[h_{p_3}h_{p_4}]_{JM}.$$

Now we substitute the phonon creation and annihilation operators into (2.108). Several terms of the phonon operators don't contribute (because $[h^\dagger_{p'_4}h^\dagger_{p'_3}]_{00}|{\rm HF}\rangle = 0$, for instance). It can be deduced that

$$\begin{aligned}&\langle \nu_f;JM|M^{(el)}_{JM}|\nu_{\rm gs};00\rangle \\ &= \sum_{h_1\le h_2}\sum_{h'_1\le h'_2} X^{\nu_f *}_{h_1h_2}X^{\nu_{\rm gs}}_{h'_1h'_2}(-1)^{j_1+j_2-J}\langle {\rm HF}|[h_{h_2}h_{h_1}]_{JM}M^{(el)}_{JM}[h^\dagger_{h'_1}h^\dagger_{h'_2}]_{00}|{\rm HF}\rangle \\ &- \sum_{h_1\le h_2}\sum_{h'_1\le h'_2} X^{\nu_f *}_{h_1h_2}X^{\nu_{\rm gs}}_{h'_1h'_2}(-1)^{j_1+j_2-J}\langle {\rm HF}|[h_{h_2}h_{h_1}]_{JM}[h^\dagger_{h'_1}h^\dagger_{h'_2}]_{00}M^{(el)}_{JM}|{\rm HF}\rangle \\ &- \sum_{p_3\le p_4}\sum_{p'_3\le p'_4} Y^{\nu_f *}_{p_3p_4}Y^{\nu_{\rm gs}}_{p'_3p'_4}(-1)^{j_3+j_4-J}\langle {\rm HF}|M^{(el)}_{JM}[h^\dagger_{p'_4}h^\dagger_{p'_3}]_{00}[h_{p_3}h_{p_4}]_{JM}|{\rm HF}\rangle \\ &+ \sum_{p_3\le p_4}\sum_{p'_3\le p'_4} Y^{\nu_f *}_{p_3p_4}Y^{\nu_{\rm gs}}_{p'_3p'_4}(-1)^{j_3+j_4-J}\langle {\rm HF}|[h^\dagger_{p'_4}h^\dagger_{p'_3}]_{00}M^{(el)}_{JM}[h_{p_3}h_{p_4}]_{JM}|{\rm HF}\rangle\end{aligned} \tag{2.109}$$

Similarly as in the ppRPA case,

$$\langle {\rm HF}|[h_{h_2}h_{h_1}]_{JM}[h^\dagger_{h'_1}h^\dagger_{h'_2}]_{00}M^{(el)}_{JM}|{\rm HF}\rangle \propto \delta_{J0}\delta_{M0} = 0, \tag{2.110}$$

$$\langle {\rm HF}|M^{(el)}_{JM}[h^\dagger_{p'_4}h^\dagger_{p'_3}]_{00}[h_{p_3}h_{p_4}]_{JM}|{\rm HF}\rangle \propto \delta_{J0}\delta_{M0} = 0 \tag{2.111}$$

and $\langle {\rm HF}|[h_{h_2}h_{h_1}]_{JM}M^{(el)}_{JM}[h^\dagger_{h'_1}h^\dagger_{h'_2}]_{00}|{\rm HF}\rangle$ is equal to the result (2.41), valid for the electromagnetic transitions within hhTDA, multiplied by $(-1)^{j_1+j_2-J}$ except that the indices h_3 and h_4 are replaced with h'_1 and h'_2.

Now we calculate the matrix element in the last term in (2.109). We substitute the electric multipole operator $M^{(el)}_{JM}$ given by (2.30) into this matrix element and suppose that the states p_3, p_4, p'_3 and p'_4 are proton states (for neutron states the derivation is analogous). As in the previous derivations, we omit the neutron part of $M^{(el)}_{JM}$, because it gives zero contribution due to the relation (1.49), and obtain

$$\begin{aligned}&\langle {\rm HF}|[h^{\pi}_{p'_4}{}^\dagger h^{\pi}_{p'_3}{}^\dagger]_{00}M^{(el)}_{JM}[h^\pi_{p_3}h^\pi_{p_4}]_{JM}|{\rm HF}\rangle \\ &= e^{({\rm eff})}_{\rm p}\mathcal{N}_{p_3p_4}(J)\mathcal{N}_{p'_3p'_4}(0)\sum_{\substack{m_3m_4\\ m'_3m'_4}}(j_3m_3j_4m_4|JM)(j'_4m'_4j'_3m'_3|00) \\ &\quad\times\sum_{\alpha\beta}\frac{1}{\sqrt{2j_b+1}}(j_bm_\beta JM|j_am_\alpha)\langle a||r^JY_J||b\rangle\langle {\rm HF}|h^{\pi}_{\pi'_4}{}^\dagger h^{\pi}_{\pi'_3}{}^\dagger a^\dagger_\alpha a_\beta h^\pi_{\pi_3}h^\pi_{\pi_4}|{\rm HF}\rangle,\end{aligned} \tag{2.112}$$

with

$$\begin{aligned}
&\langle \mathrm{HF}|{h^{\pi}_{\pi'_4}}^{\dagger}{h^{\pi}_{\pi'_3}}^{\dagger}a^{\dagger}_{\alpha}a_{\beta}h^{\pi}_{\pi_3}h^{\pi}_{\pi_4}|\mathrm{HF}\rangle \\
&\qquad =(-1)^{j_3+m_3+j_4+m_4+j'_3+m'_3+j'_4+m'_4}\langle \mathrm{HF}|a_{-\pi'_4}a_{-\pi'_3}a^{\dagger}_{\alpha}a_{\beta}a^{\dagger}_{-\pi_3}a^{\dagger}_{-\pi_4}|\mathrm{HF}\rangle \\
&\qquad =(-1)^{j_3+m_3+j_4+m_4+j'_3+m'_3+j'_4+m'_4}\Big(\delta_{-\pi'_4\alpha}\delta_{-\pi_4\beta}\delta_{\pi'_3\pi_3}+\delta_{-\pi'_3\alpha}\delta_{-\pi_3\beta}\delta_{\pi'_4\pi_4} \\
&\qquad\quad -\delta_{-\pi'_4\alpha}\delta_{-\pi_3\beta}\delta_{\pi'_3\pi_4}-\delta_{-\pi'_3\alpha}\delta_{-\pi_4\beta}\delta_{\pi'_4\pi_3}+\delta_{\alpha\beta}\delta_{\pi'_3\pi_3}\delta_{\pi'_4\pi_4}\theta(\varepsilon_{\mathrm{F}}-\varepsilon_{\alpha}) \\
&\qquad\quad -\delta_{\alpha\beta}\delta_{\pi'_3\pi_4}\delta_{\pi'_4\pi_3}\theta(\varepsilon_{\mathrm{F}}-\varepsilon_{\alpha})\Big), \qquad (2.113)
\end{aligned}$$

where, as in the previous derivations, the last two terms give zero contribution due to the relation (1.49). Thus substitution of (2.113) into (2.112) yields

$$\begin{aligned}
&\langle \mathrm{HF}|[{h^{\pi}_{p'_4}}^{\dagger}{h^{\pi}_{p'_3}}^{\dagger}]_{00}M^{(el)}_{JM}[h^{\pi}_{p_3}h^{\pi}_{p_4}]_{JM}|\mathrm{HF}\rangle=(-1)^{j_3+j_4-J}e^{(\mathrm{eff})}_{\mathrm{p}}\mathcal{N}_{p_3p_4}(J)\mathcal{N}_{p'_3p'_4}(0) \\
&\times\Bigg[\delta_{p'_3p_3}\frac{1}{\sqrt{2j_4+1}}\langle p'_4||r^JY_J||p_4\rangle \\
&\times\sum_{\substack{m_3m_4\\ m'_4}}(j_4m_4j_3m_3|JM)(j'_4m'_4j_3m_3|00)(j_4-m_4JM|j'_4-m'_4)(-1)^{j_4+m_4+j'_4+m'_4} \\
&+\delta_{p'_4p_4}\frac{1}{\sqrt{2j_3+1}}\langle p'_3||r^JY_J||p_3\rangle \\
&\times\sum_{\substack{m_3m_4\\ m'_3}}(j_4m_4j_3m_3|JM)(j_4m_4j'_3m'_3|00)(j_3-m_3JM|j'_3-m'_3)(-1)^{j_3+m_3+j'_3+m'_3} \\
&-\delta_{p'_3p_4}\frac{1}{\sqrt{2j_3+1}}\langle p'_4||r^JY_J||p_3\rangle \\
&\times\sum_{\substack{m_3m_4\\ m'_4}}(j_4m_4j_3m_3|JM)(j'_4m'_4j_4m_4|00)(j_3-m_3JM|j'_4-m'_4)(-1)^{j_3+m_3+j'_4+m'_4} \\
&-\delta_{p'_4p_3}\frac{1}{\sqrt{2j_4+1}}\langle p'_3||r^JY_J||p_4\rangle \\
&\times\sum_{\substack{m_3m_4\\ m'_3}}(j_4m_4j_3m_3|JM)(j_3m_3j'_3m'_3|00)(j_4-m_4JM|j'_3-m'_3)(-1)^{j_4+m_4+j'_3+m'_3}\Bigg].
\end{aligned}$$

Substitution of all the matrix elements into (2.109) yields the final result

$$
\begin{aligned}
\langle \nu_f; JM|M^{(el)}_{JM}|\nu_{\rm gs};00\rangle &= e^{(\rm eff)}_{\rm p} \sum_{h_1\le h_2}\sum_{h'_1\le h'_2} X^{\nu_f *}_{h_1h_2} X^{\nu_{\rm gs}}_{h'_1h'_2} \mathcal{N}_{h_1h_2}(J)\mathcal{N}_{h'_1h'_2}(0) \\
&\times \Bigg[\delta_{h_2h'_1} \frac{1}{\sqrt{2j_1+1}} \langle h'_2||r^J Y_J||h_1\rangle \\
&\times \sum_{\substack{m_1m_2\\ m'_2}} (j_1m_1j_2m_2|JM)(j_2m_2j'_2m'_2|00)(j_1-m_1JM|j'_2-m'_2)(-1)^{j_1+m_1+j'_2+m'_2} \\
&+ \delta_{h_1h'_2} \frac{1}{\sqrt{2j_2+1}} \langle h'_1||r^J Y_J||h_2\rangle \\
&\times \sum_{\substack{m_1m_2\\ m'_1}} (j_1m_1j_2m_2|JM)(j'_1m'_1j_1m_1|00)(j_2-m_2JM|j'_1-m'_1)(-1)^{j_2+m_2+j'_1+m'_1} \\
&- \delta_{h_2h'_2} \frac{1}{\sqrt{2j_1+1}} \langle h'_1||r^J Y_J||h_1\rangle \\
&\times \sum_{\substack{m_1m_2\\ m'_1}} (j_1m_1j_2m_2|JM)(j'_1m'_1j_2m_2|00)(j_1-m_1JM|j'_1-m'_1)(-1)^{j_1+m_1+j'_1+m'_1} \\
&- \delta_{h_1h'_1} \frac{1}{\sqrt{2j_2+1}} \langle h'_2||r^J Y_J||h_2\rangle \\
&\times \sum_{\substack{m_1m_2\\ m'_2}} (j_1m_1j_2m_2|JM)(j_1m_1j'_2m'_2|00)(j_2-m_2JM|j'_2-m'_2)(-1)^{j_2+m_2+j'_2+m'_2} \Bigg] \\
&+ e^{(\rm eff)}_{\rm p} \sum_{p_3\le p_4}\sum_{p'_3\le p'_4} Y^{\nu_f *}_{p_3p_4} Y^{\nu_{\rm gs}}_{p'_3p'_4} \mathcal{N}_{p_3p_4}(J)\mathcal{N}_{p'_3p'_4}(0) \\
&\times \Bigg[\delta_{p'_3p_3} \frac{1}{\sqrt{2j_4+1}} \langle p'_4||r^J Y_J||p_4\rangle \\
&\times \sum_{\substack{m_3m_4\\ m'_4}} (j_4m_4j_3m_3|JM)(j'_4m'_4j_3m_3|00)(j_4-m_4JM|j'_4-m'_4)(-1)^{j_4+m_4+j'_4+m'_4} \\
&+ \delta_{p'_4p_4} \frac{1}{\sqrt{2j_3+1}} \langle p'_3||r^J Y_J||p_3\rangle \\
&\times \sum_{\substack{m_3m_4\\ m'_3}} (j_4m_4j_3m_3|JM)(j_4m_4j'_3m'_3|00)(j_3-m_3JM|j'_3-m'_3)(-1)^{j_3+m_3+j'_3+m'_3} \\
&- \delta_{p'_3p_4} \frac{1}{\sqrt{2j_3+1}} \langle p'_4||r^J Y_J||p_3\rangle \\
&\times \sum_{\substack{m_3m_4\\ m'_4}} (j_4m_4j_3m_3|JM)(j'_4m'_4j_4m_4|00)(j_3-m_3JM|j'_4-m'_4)(-1)^{j_3+m_3+j'_4+m'_4} \\
&- \delta_{p'_4p_3} \frac{1}{\sqrt{2j_4+1}} \langle p'_3||r^J Y_J||p_4\rangle \\
&\times \sum_{\substack{m_3m_4\\ m'_3}} (j_4m_4j_3m_3|JM)(j_3m_3j'_3m'_3|00)(j_4-m_4JM|j'_3-m'_3)(-1)^{j_4+m_4+j'_3+m'_3} \Bigg],
\end{aligned}
\tag{2.114}
$$

where, as in the previous cases, the sums of the products of Clebsch-Gordan coefficients can be reduced to sums with only one summation index. This result is valid also for transitions between neutron hhRPA eigenstates except that $e_{\mathrm{p}}^{(\mathrm{eff})}$ is replaced with $e_{\mathrm{n}}^{(\mathrm{eff})}$. We see that it reduces to the result (2.41) valid for electromagnetic transitions within the hhTDA, if the amplitudes $Y^{\nu_f}, Y^{\nu_{\mathrm{gs}}}$ are zero.

3. Numerical calculations

In this chapter we present the results of numerical calculations of energy spectra and reduced transition probabilities of electric type performed for selected nuclei within the TDA, RPA, *pp*TDA, *hh*TDA, *pp*RPA and *hh*RPA.

The harmonic oscillator wave functions (see Appendix A) have been used as the initial basis for solving the HF equations. We have used the program *HFB_DD* (Hartree-Fock-Bogoliubov Code with Density Dependent Interaction) [38], which represents a numerical implementation of the HF method and requires the nucleon-nucleon interaction matrix elements in the oscillator basis as the input.

We have used the realistic nucleon-nucleon potential $\mathrm{NNLO_{opt}}$ [3] with parameters optimized to minimize the effect of three-body interactions. The interaction matrix elements were generated by the computer program *vrenorm.exe*, which belongs to the package CENS (a Computational Environment for Nuclear Structure) [39]. The input parameters of this program are $\hbar\omega$, which is the parameter of the oscillator basis, and l_{max}, n_{max}, which determine the size of the basis and represent the maximal values of the quantum numbers l and n in the relation (A.4), which implies that $N_{max} = l_{max} = 2(n_{max} - 1)$. The parameter N_{max} is the maximal value of the quantum number N and thus determines the number of the used single-particle states.

The program *HFB_DD* provide all data needed for further calculations, namely the information about the HF single-particle basis, including the single-particle energies, and the transformed angular-momentum-coupled two-body interaction matrix elements in the HF basis.

For the description of excitation probability of the nucleus as a function of the excitation energy E it is convenient to introduce so called strength function. The strength function for given type of the excitation is defined as the energy distribution of the excitation probability

$$S_0(X\lambda) \equiv \sum_\nu B(X\lambda; 0^+_{\mathrm{gs}} \to \nu)\delta(E - E_\nu), \tag{3.1}$$

where the summation goes through all excited states $|\nu\rangle$ with excitation energy E_ν calculated in the framework of the used model (e. g. TDA and RPA) and $|0^+_{\mathrm{gs}}\rangle$ is the model ground state of a given eve-even nucleus. In order to simulate a finite width and effects of omitted configurations the delta-function in the definition (3.1) is usually substituted by the Lorentzian (see [40])

$$\xi_\Delta(E - E_\nu) = \frac{1}{2\pi}\frac{\Delta}{(E - E_\nu)^2 + \frac{\Delta^2}{4}}, \tag{3.2}$$

which satisfies

$$\lim_{\Delta\to 0} \xi_\Delta(E - E_\nu) = \delta(E - E_\nu). \tag{3.3}$$

The arbitrary parameter Δ characterizes the width of the Lorentzian.

3.1 Calculations within TDA and RPA

In this section we compare results of calculations performed for ^{16}O, ^{40}Ca and ^{208}Pb within the TDA and RPA. The TDA calculations were performed by the

program *HFB_DD*. In order to perform the RPA calculations, we have developed a new computer program. This program forms a basis of all possible *ph* states for a given angular momentum and parity from the HF single-particle states provided by the program *HFB_DD* in the given valence space. Then it constructs the TDA matrix using the formulae (1.50), (1.57), (1.60) and (1.61) and the correlation matrix using the formulae (1.88), (1.92) and (1.93) in this basis. Then it solves the RPA equations using the method described in Section 1.4, which is based on the equations (1.102), (1.103) and (1.104). If the decomposition (1.102) fails, the program doesn't compute the phonon amplitudes, but it still computes the eigenenergies solving the eigenvalue problem (1.101). The factorization (1.102), the diagonalization and the matrix inverse are performed by subroutines from the package LAPACK (Linear Algebra Package) [41]. This program reproduces the same results as the TDA calculations performed by the program *HFB_DD*, concerning the eigenenergies and the structure of the phonons, for vanishing correlation matrix. Thus we assume that the program works correctly. All TDA and RPA calculations have been performed in complete valence space for the given N_{max}.

The program computes also the reduced probabilities of electric transitions from the ground state using the formulae (1.105) and (1.106). The reduced single-particle matrix elements $\langle a||M_J^{(X)}||b\rangle$ of the transition operator are provided by the program *HFB_DD*. The electric multipole operator in the long-wave approximation is

$$M_{JM}^{(el)} = \sum_{i=1}^{A} e_i r_i^J Y_{JM}(\theta_i, \varphi_i). \tag{3.4}$$

Since the nucleon charge

$$e_i = e\left(\frac{1}{2} - (t_z)_i\right), \tag{3.5}$$

where $(t_z)_i$ is the isospin projection, the electric multipole operator can be decomposed to

$$M_{JM}^{(el)} = \frac{e}{2}\sum_{i=1}^{A} r_i^J Y_{JM}(\theta_i, \varphi_i) - e\sum_{i=1}^{A}(t_z)_i r_i^J Y_{JM}(\theta_i, \varphi_i). \tag{3.6}$$

The first term is the isoscalar operator and the second one is the isovector operator. Our RPA program computes the reduced matrix element $\langle\nu||M_J^{(X)}||\mathrm{RPA}\rangle_\pi$ given by (1.106), where a and b are only proton states, and $\langle\nu||M_J^{(X)}||\mathrm{RPA}\rangle_\nu$ given by (1.106), where a and b are only neutron states. Then it computes the isoscalar reduced transition probability of multipolarity J as the square of the absolute value of

$$\langle\nu||M_J^{(X)}||\mathrm{RPA}\rangle_{\mathrm{is}} = \frac{1}{2}\left(\langle\nu||M_J^{(X)}||\mathrm{RPA}\rangle_\pi + \langle\nu||M_J^{(X)}||\mathrm{RPA}\rangle_\nu\right), \tag{3.7}$$

the isovector reduced transition probability as the square of the absolute value of

$$\langle\nu||M_J^{(X)}||\mathrm{RPA}\rangle_{\mathrm{iv}} = \frac{1}{2}\left(\langle\nu||M_J^{(X)}||\mathrm{RPA}\rangle_\pi - \langle\nu||M_J^{(X)}||\mathrm{RPA}\rangle_\nu\right) \tag{3.8}$$

and the physical reduced transition probability as the square of the absolute value of $\langle\nu||M_J^{(X)}||\mathrm{RPA}\rangle_\pi$.

The spectra of the states of ^{16}O, ^{40}Ca and ^{208}Pb calculated within the TDA and RPA for $\hbar\omega = 16.3$ MeV in the case of ^{16}O and ^{40}Ca and $\hbar\omega = 11$ MeV in the case of ^{208}Pb[1] and for different values of N_{max} are shown in Fig. 3.1, 3.2 and 3.3 together with the experimental values taken from [43]. The figures contain five lowest states with positive and negative parity and the angular momenta from 0 to 4. We can see that the results of the TDA and RPA calculations are very similar except that the excitation energies of the lowest 3^- states calculated within the RPA are lower in comparison to the TDA values, which agrees with the results of Gillet et al. [44] (see Fig. 7 therein), who performed calculations for ^{208}Pb. This is a consequence of a strong enhancement of the collectivity of the lowest 3^- state in the RPA. Another aspect of this enhancement is a significant increase of the transition probability from the ground state to the lowest 3^- state (see below). Our results give also a large energy gap between the ground and the first excited state characteristic for doubly magic nuclei, if we don't consider the lowest 1^- state, which is a spurious state corresponding to the nucleus center-of-mass motion. We can also see that the configuration space corresponding to $N_{max} = 14$ is not sufficient to obtain converged results. However, a trend of a convergence can be observed. One can notice a significant disagreement with the experimental data.

In some cases, the RPA gives imaginary energy of the lowest 1^- state. The energy of this spurious state should be zero and in Fig. 3.4, 3.5 and 3.6 it is shown that the absolute value of the energy of the lowest 1^- state calculated within the RPA approaches zero with increasing N_{max}. A similar behavior can be noticed in the results of RPA calculations performed by Paar et al. [45] using the realistic interaction Argonne V18 [46] renormalized by the unitary correlation operator method (UCOM) (see references therein). The calculated excitation energies depend also on $\hbar\omega$, but this dependence becomes less significant with increasing N_{max}.

The excitation energies of the lowest 3^- states in ^{16}O, ^{40}Ca and ^{208}Pb calculated within the TDA and RPA are shown in Table 3.1 together with the experimental values taken from [43]. We can see that the values calculated within the RPA are closer to the experimental values than the TDA results. Thus the RPA leads to better agreement with the experiment concerning the collectivity of the lowest 3^- states, which can be seen also from the values of the reduced transition probabilities (see below).

Now we proceed to present the results of calculations of the reduced probabilities of the electric excitations of the ground state. The physical, isoscalar and isovector E0, E1, E2 and E3 strength functions calculated within the TDA and RPA for ^{16}O, ^{40}Ca and ^{208}Pb with $N_{max} = 14$ are shown in Fig. 3.7. 3.8, 3.9 and 3.10. We can see that the TDA and RPA results are very similar except that the probability of the excitation to the lowest 3^- state calculated within the RPA is much larger than the TDA result and the corresponding peak is shifted due to the energy shift discussed earlier. Thus the RPA leads to strong enhancement of the collectivity of the lowest 3^- state, which was already mentioned before. In the case of ^{208}Pb we observe an enhancement of the excitation probability calculated within the RPA also in the case of the lowest 0^+ and 2^+ state. We can also see

[1] We have chosen the values of $\hbar\omega$, for which we have observed fast convergence of the binding energies, calculated within the HF method, with increasing N_{max} (see [42]).

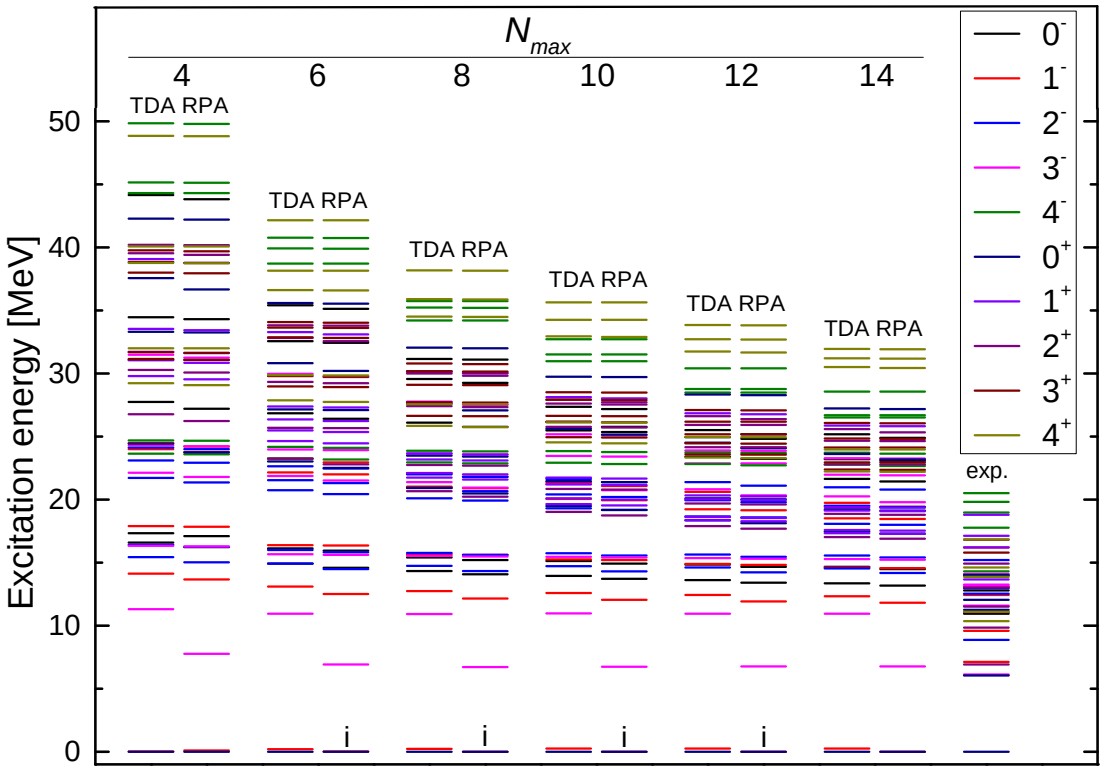

Figure 3.1: The spectra of five lowest states of ^{16}O with positive and negative parity and the angular momenta from 0 to 4 calculated within the TDA and RPA for $\hbar\omega = 16.3$ MeV and different values of N_{max} together with the experimental values taken from [43]. The label i above the lowest 1^- level means that the corresponding energy is imaginary. This level corresponds to the spurious state connected with the center-of-mass motion.

that the RPA leads to an enhancement of the E2 strength function for ^{208}Pb at the excitation energy close to 20 MeV.

Strong collectivity of the lowest 3^- RPA phonon can be seen also from the values of the corresponding phonon amplitudes. These amplitudes, calculated for ^{16}O and $N_{max} = 14$ are shown in Table 3.2. We can see a considerable size of some Y amplitudes and a fragmentation into several evenly sized amplitudes. This fragmentation leads to collectivity, which means that several *ph* components act in a coherent way to increase the transition probability.

All strength functions for ^{16}O were calculated for $\hbar\omega = 16.3$ MeV except the E1 strength function. This one was calculated for $\hbar\omega = 26$ MeV because for $\hbar\omega = 16.3$ MeV and $N_{max} = 14$ the factorization (1.102) in our RPA program fails in the case of the angular momentum and parity 1^-, which means that we were unable to calculate the amplitudes of the 1^- RPA phonons, which makes the calculation of $B(el\,1; 0^+_{\rm gs} \to 1^-)$ impossible. The E1 strength function for ^{16}O calculated within the RPA doesn't contain the peak corresponding to the excitation to the lowest 1^- state because the energy of the lowest 1^- RPA phonon calculated for $\hbar\omega = 26$ MeV and $N_{max} = 14$ is imaginary, which makes the calculations of the corresponding phonon amplitudes using the equation (1.104) impossible, which means that we were not able to calculate the corresponding

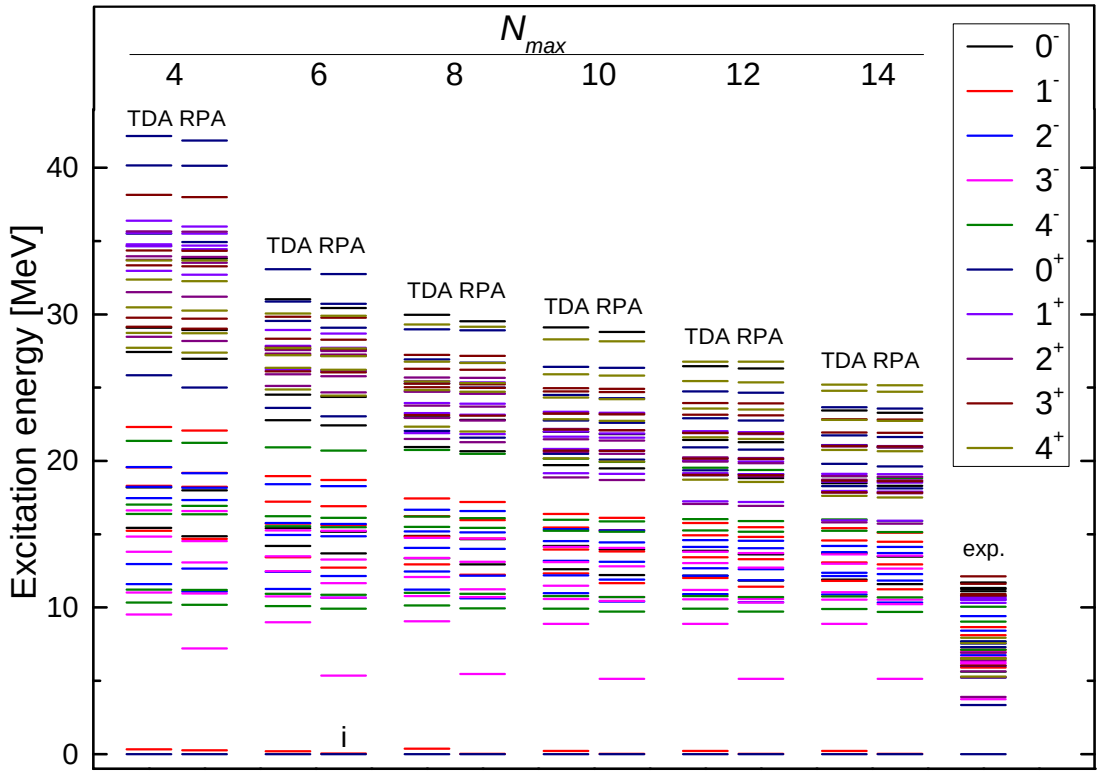

Figure 3.2: The same s in Fig. 3.1, but for ^{40}Ca. The oscillator parameter $\hbar\omega = 16.3$ MeV.

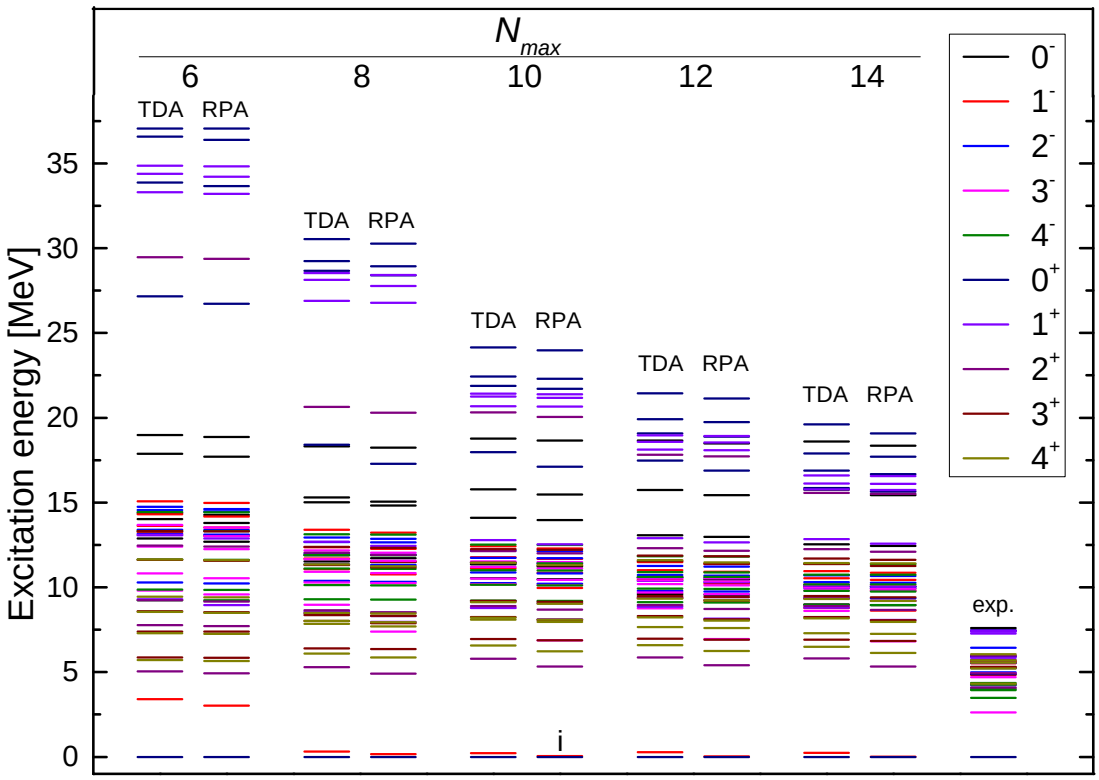

Figure 3.3: The same s in Fig. 3.1, but for ^{208}Pb. The oscillator parameter $\hbar\omega = 11$ MeV.

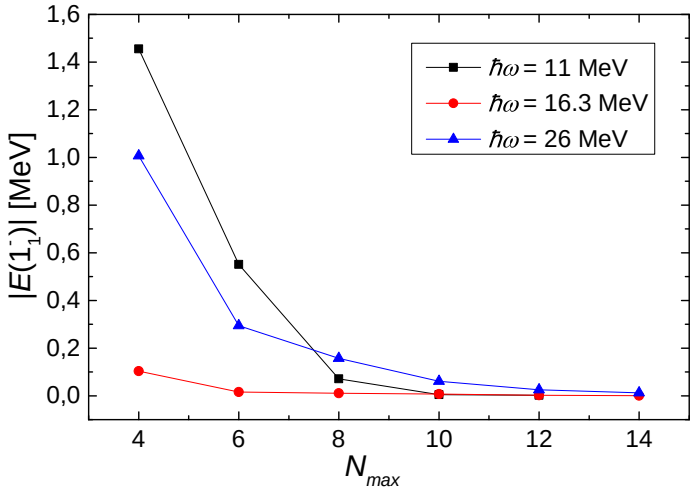

Figure 3.4: The dependence of the absolute value of the excitation energy of the lowest 1^- state in ^{16}O calculated within the RPA on N_{max} for different $\hbar\omega$.

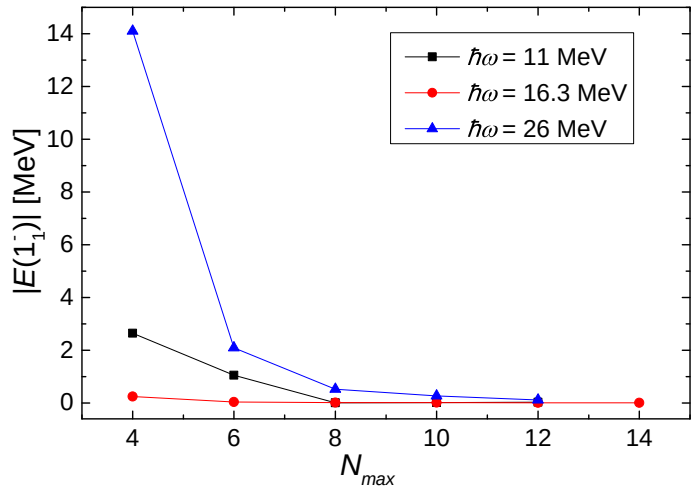

Figure 3.5: The same s in Fig. 3.4, but for ^{40}Ca.

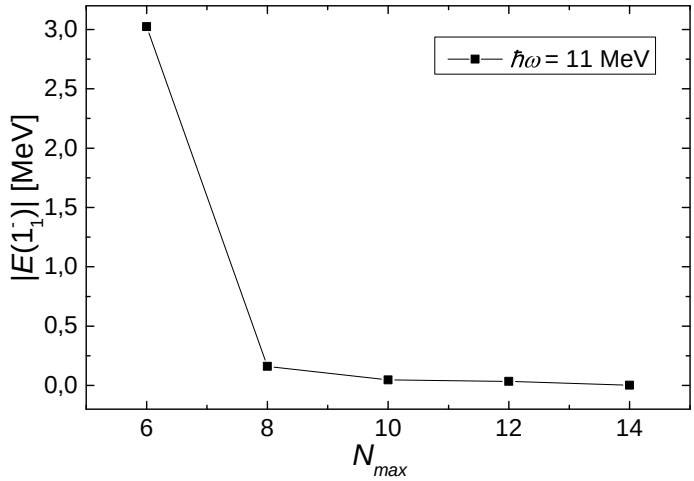

Figure 3.6: The same s in Fig. 3.4, but for ^{208}Pb.

Table 3.1: The excitation energies of the lowest 3^- states in ^{16}O, ^{40}Ca and ^{208}Pb calculated within the TDA and RPA for $N_{max} = 14$ together with the experimental values taken from [43]. The used values of $\hbar\omega$ are also shown.

nucleus	$\hbar\omega$ [MeV]	TDA value [keV]	RPA value [keV]	experiment [keV]
^{16}O	16.3	10951.74	6797.57	6129.89 ± 0.04
^{40}Ca	16.3	8878.86	5122.47	3736.69 ± 0.05
^{208}Pb	11	8603.695	6820.018	2614.522 ± 0.010

Table 3.2: The amplitudes of the lowest 3^- RPA phonon calculated for ^{16}O, $\hbar\omega = 16.3$ MeV and $N_{max} = 14$. Only amplitudes with the absolute value greater than 0.1 are shown. The first three columns contain the information about the *ph* configurations (we use the spectroscopic notation - see Appendix A).

excited nucleon	hole state	particle state	X amplitude	Y amplitude
proton	$1p_{3/2}$	$1d_{5/2}$	-0.230	0.122
proton	$1p_{3/2}$	$1d_{3/2}$	0.164	-0.0874
proton	$1p_{3/2}$	$2d_{3/2}$	-0.207	0.123
proton	$1p_{3/2}$	$3d_{3/2}$	0.126	-0.0846
proton	$1p_{1/2}$	$1d_{5/2}$	0.668	-0.258
proton	$1p_{1/2}$	$2d_{5/2}$	-0.178	0.104
neutron	$1p_{3/2}$	$1d_{5/2}$	0.125	0.125
neutron	$1p_{3/2}$	$2d_{3/2}$	0.118	0.118
neutron	$1p_{1/2}$	$1d_{5/2}$	-0.266	-0.266

$B(el\,1; 0^+_{\rm gs} \to 1^-_1)$. However, this peak corresponds to the spurious isoscalar E1 excitation connected with the nucleus center-of-mass motion, which is the only isoscalar E1 excitation (within the long-wave approximation) and can be removed (in the case of nuclei with $Z = N$) from the physical E1 strength function by adopting the effective charges (see [47])

$$e_{\rm p}^{\rm (eff)} = \frac{N}{A} e \ , \quad e_{\rm n}^{\rm (eff)} = -\frac{Z}{A} e \tag{3.9}$$

for the transition operator (3.4). In order to investigate the convergence with increasing size of the configuration space the E3 strength functions for ^{16}O, shown in Fig. 3.8, were calculated for several values of N_{max}. We can observe that with increasing N_{max} the probability of the excitation to the lowest 3^- state calculated within the TDA and RPA increases and then the convergence is achieved.

In order to reduce the spurious peak in the physical E1 strength function for ^{208}Pb, we have calculated this strength function also with the effective charges (3.9) adopted for the transition operator (3.4). The result is shown in Fig. 3.11., where the strength of the spurious state has been effectively subtracted from the strength function.

Our isoscalar electric octupole strength functions for ^{40}Ca and ^{208}Pb calculated within the RPA are in a qualitative agreement with those calculated by Liu and

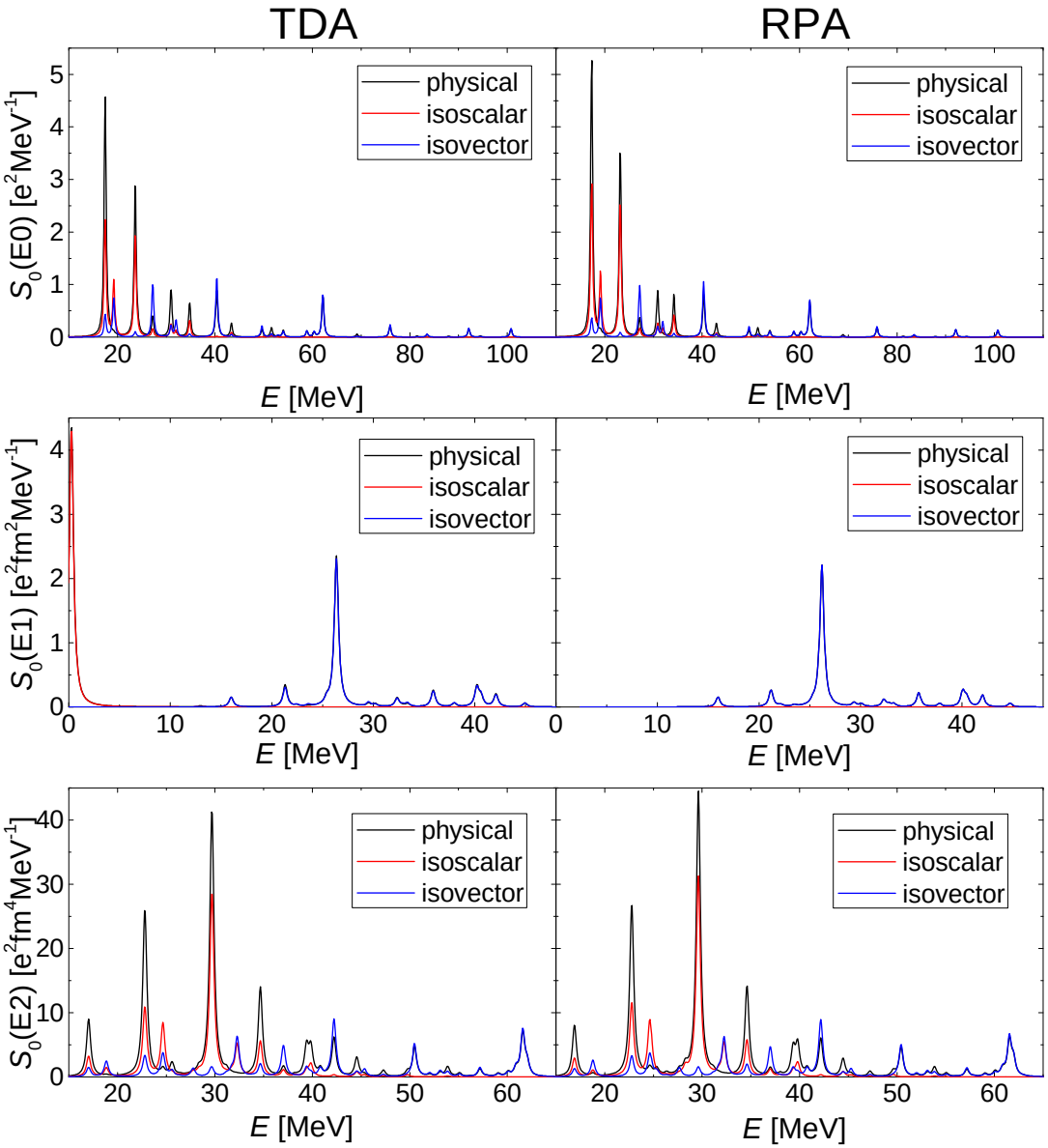

Figure 3.7: The physical, isoscalar and isovector E0, E1 and E2 strength functions for ^{16}O calculated within the TDA (left panel) and RPA (right panel) for $N_{max} = 14$ and $\hbar\omega = 16.3$ MeV in the case of E0 and E2 strength functions and $\hbar\omega = 26$ MeV in the case of E1 strength function. Width of the Lorentzian $\Delta = 0.5$ MeV.

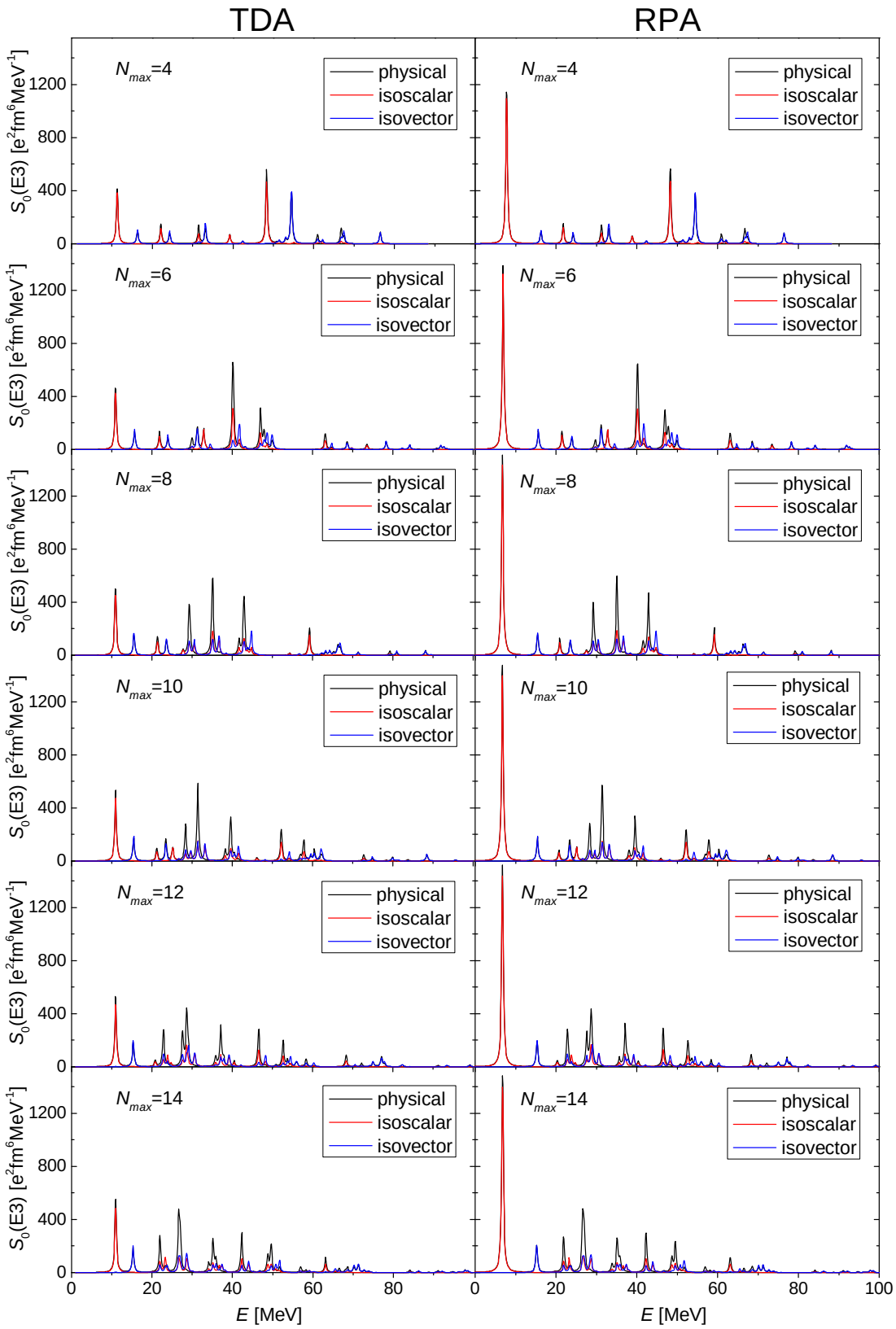

Figure 3.8: The physical, isoscalar and isovector E3 strength functions for ^{16}O calculated within the TDA (left panel) and RPA (right panel) for $\hbar\omega = 16.3$ MeV and different values of N_{max}. Width of the Lorentzian $\Delta = 0.5$ MeV.

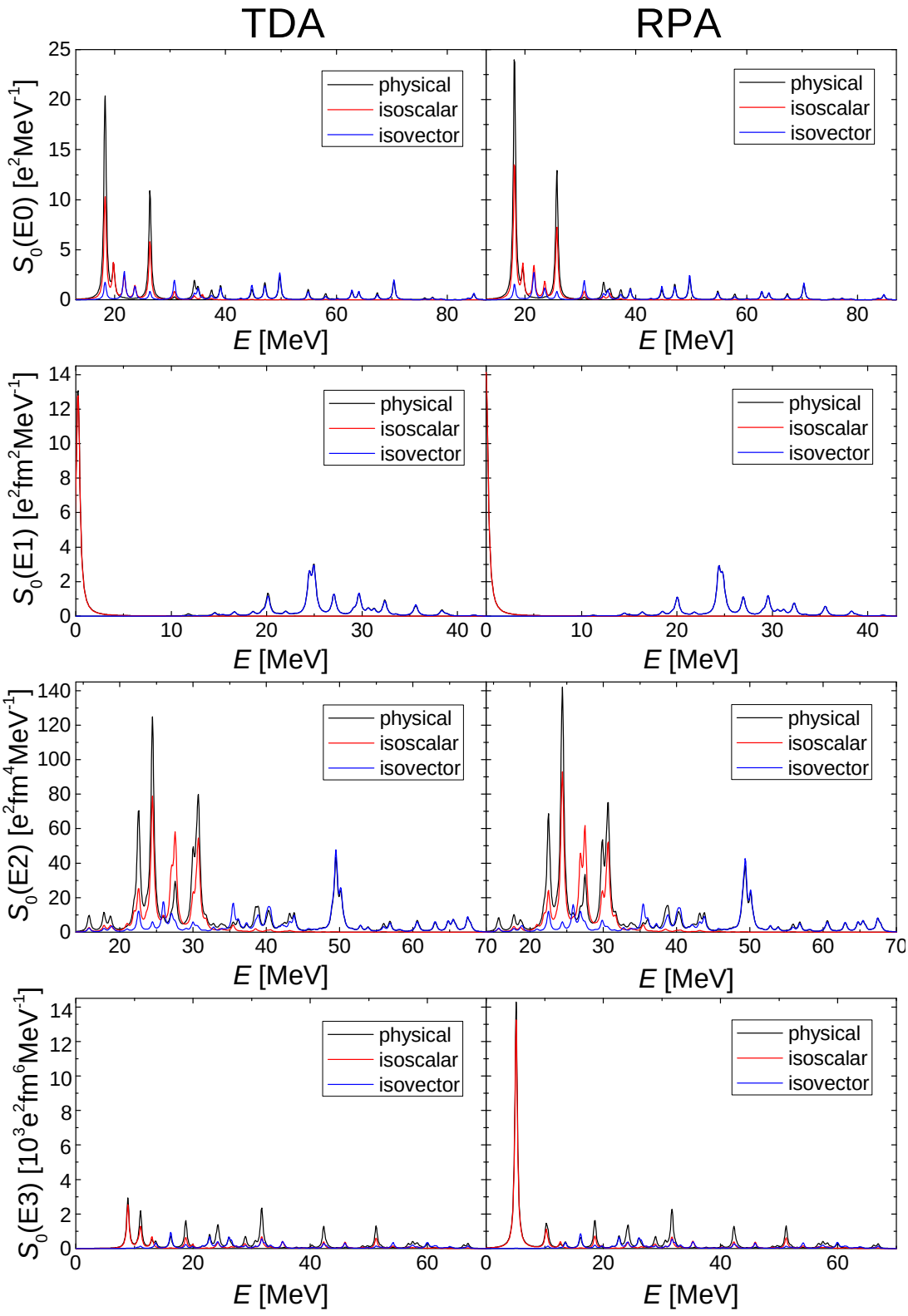

Figure 3.9: The physical, isoscalar and isovector E0, E1, E2 and E3 strength functions for ^{40}Ca calculated within the TDA (left panel) and RPA (right panel) for $\hbar\omega = 16.3$ MeV and $N_{max} = 14$. Width of the Lorentzian $\Delta = 0.5$ MeV.

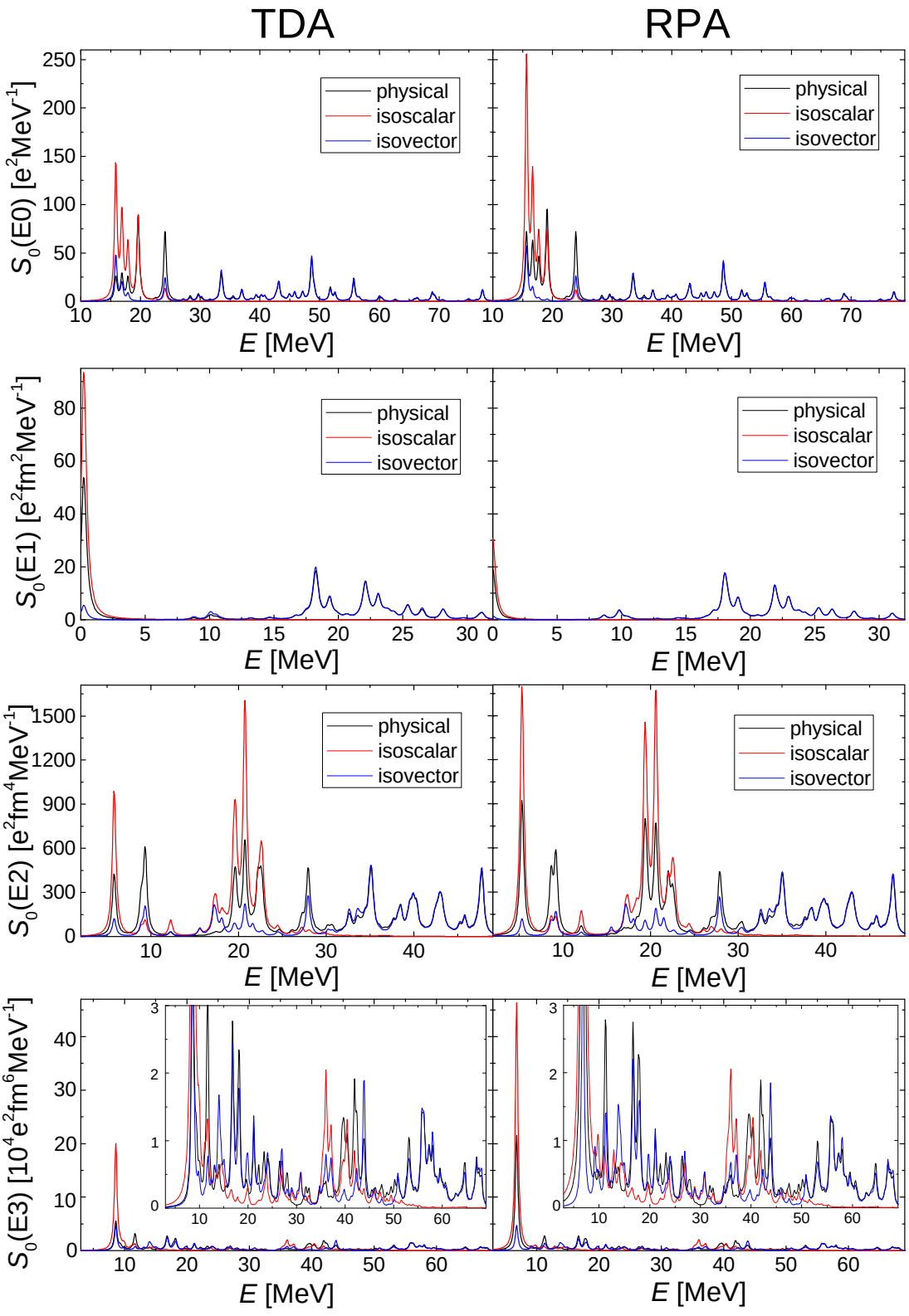

Figure 3.10: The physical, isoscalar and isovector E0, E1, E2 and E3 strength functions for ^{208}Pb calculated within the TDA (left panel) and RPA (right panel) for $\hbar\omega = 11$ MeV and $N_{max} = 14$. Width of the Lorentzian $\Delta = 0.5$ MeV. The bottom graphs contain inset graphs with smaller scale for S_0(E3) to show a detailed structure of the strength function.

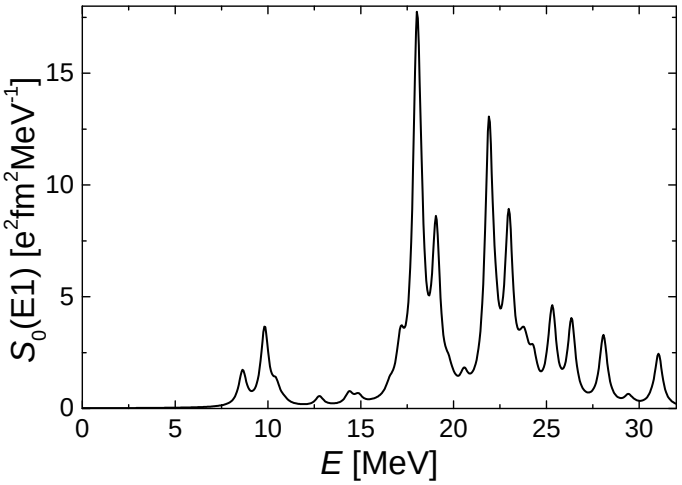

Figure 3.11: The physical E1 strength function for ^{208}Pb calculated within the RPA for $\hbar\omega = 11$ MeV and $N_{max} = 14$ with the effective charges (3.9) adopted for the transition operator (3.4). Width of the Lorentzian $\Delta = 0.5$ MeV.

Brown [48] (see Fig. 6 therein) within the RPA using the HF basis and the Skyrme interaction [49]. Their results show also a very strong excitation probability to the lowest 3^- state. Isoscalar monopole strength functions for ^{16}O, ^{40}Ca and ^{208}Pb calculated by Blaizot, Gogny and Grammaticos [50] (see Fig. 3 therein) are similar to our results.

The energy of centroid of a giant resonance $\bar{E}$ can be calculated as

$$\bar{E} = \sqrt{\frac{m_1(X\lambda)}{m_{-1}(X\lambda)}}, \tag{3.10}$$

where

$$m_k(X\lambda) \equiv \sum_\nu E_\nu^k B(X\lambda; 0^+_{\rm gs} \to \nu) \tag{3.11}$$

is the moment of the k-th order. According to Ref. [51], experimental energies of centroids of the observed electric isoscalar giant monopole resonances (ISGMR) are in the energy region 18-20 MeV for light nuclei ($A < 50$) and around 14 MeV for heavy nuclei ($A > 200$). This is in reasonable agreement with our calculated values shown in Table 3.3. The experimental energies of the centroids of the electric isoscalar (ISGQR) and isovector (IVGQR) giant quadrupole resonances can be approximated by expressions [47]

$$\bar{E}^{(ISGQR)}_{\rm exp}(A) \approx (61.0 \pm 1.7) A^{-1/3}\,{\rm MeV}, \tag{3.12}$$

$$\bar{E}^{(IVGQR)}_{\rm exp}(A) \approx (59.2 \pm 2.6) A^{-1/6}\,{\rm MeV} \tag{3.13}$$

obtained by the fitting to the experimental values of many spherical nuclei. Values for ^{16}O, ^{40}Ca and ^{208}Pb given by this formulae together with our results are shown in Table 3.3. We can notice a reasonable agreement between the theoretical and experimental values for ^{16}O, but for heavier systems there is a difference of several MeV.

Table 3.4 contains the reduced probabilities of electric excitations of multipolarity 3 from the ground state $0^+_{\rm gs}$ to the lowest 3^- state calculated for ^{16}O,

Table 3.3: The energies of centroids of some giant resonances calculated in the framework of the TDA ($\bar{E}_{\mathrm{TDA}}$) and RPA ($\bar{E}_{\mathrm{RPA}}$) for $N_{max} = 14$ together with the experimental values ($\bar{E}_{\mathrm{exp}}$). The used values of $\hbar\omega$ are also shown.

nucleus	$\hbar\omega$ [MeV]	type of giant resonance	$\bar{E}_{\mathrm{TDA}}$ [MeV]	$\bar{E}_{\mathrm{RPA}}$ [MeV]	$\bar{E}_{\mathrm{exp}}$ [MeV]
^{16}O	16.3	ISGMR	21.91	21.41	18-20
^{16}O	16.3	ISGQR	28.30	28.37	24.21 ± 0.67
^{16}O	16.3	IVGQR	41.74	41.41	37.3 ± 1.6
^{40}Ca	16.3	ISGMR	21.99	21.45	18-20
^{40}Ca	16.3	ISGQR	26.72	26.71	17.84 ± 0.50
^{40}Ca	16.3	IVGQR	42.54	42.17	32.0 ± 1.4
^{208}Pb	11	ISGMR	17.61	16.83	14
^{208}Pb	11	ISGQR	15.55	13.73	10.30 ± 0.29
^{208}Pb	11	IVGQR	31.13	30.47	24.3 ± 1.1

Table 3.4: The physical reduced probabilities $B(el\,3; 0^+_{\mathrm{gs}} \to 3^-_1)$ of electric excitations of multipolarity 3 from the ground state to the lowest 3^- state calculated for ^{16}O, ^{40}Ca and ^{208}Pb within the TDA and RPA with $N_{max} = 14$ together with the experimental values taken from [52]. The values of $\hbar\omega$ used in the calculations are also shown.

nucleus	$\hbar\omega$ [MeV]	TDA value [$e^2\mathrm{fm}^6$]	RPA value [$e^2\mathrm{fm}^6$]	experiment [$e^2\mathrm{fm}^6$]
^{16}O	16.3	437	1199	1550 ± 12
^{40}Ca	16.3	2288	11320	$(204 \pm 17) \cdot 10^2$
^{208}Pb	11	42654	169145	$(31 \pm 10) \cdot 10^4$

^{40}Ca and ^{208}Pb within the TDA and RPA with $N_{max} = 14$ together with the experimental values taken from [52] determined from lifetime measurements. We can see that the values of $B(el\,3; 0^+_{\mathrm{gs}} \to 3^-_1)$ calculated within the RPA are much closer to the experimental data than the TDA results. Thus the RPA gives better agreement with the experiment concerning the strong collectivity of the lowest 3^- states than the TDA.

Our values of $B(el\,3; 0^+_{\mathrm{gs}} \to 3^-_1)$ for ^{16}O and ^{208}Pb calculated within the RPA are similar to the results of RPA calculations performed by Krewald et al. [53] (see Tables 3 and 5 therein) using self-consistent basis and Skyrme interaction. Large values of $B(el\,3; 0^+_{\mathrm{gs}} \to 3^-_1)$ for ^{208}Pb have been calculated also by Ring and Speth [54], [55], who have performed RPA calculations using the Migdal interaction (see [24] and references therein) and phenomenological Woods-Saxon basis. Their values are even $55 \cdot 10^4\, e^2\mathrm{fm}^6$ in [54] and $546 \cdot 10^3\, e^2\mathrm{fm}^6$ in [55]. Their results overestimate the experimental value, whereas our result underestimates it significantly (by factor 1.8). The claculations of the transition probabilities between the ground state and the collective state 3^-_1 in ^{16}O, ^{40}Ca and ^{208}Pb performed by Blaizot and Gogny [56] show also that the values obtained in the RPA are much greater than those obtained in the TDA (see Table 7 therein).

The results of our TDA and RPA calculations for ^{16}O and ^{40}Ca, especially the behavior of the lowest 3^- states concerning the enhancement of the collectivity, are similar to results of TDA and RPA calculations performed by Suhonen [15] using the so called surface delta interaction (SDI) [57]. Calculations of Gillet and Sanderson [58], who have used a central effective interaction with a Gaussian radial dependence and phenomenological harmonic oscillator basis, have also shown that the energy of the lowest 3^- state in ^{40}Ca calculated within the RPA is lower and closer to the experimental value than the result obtained in the TDA (see Table 2a therein). Similar result has been obtained by Blomqvist and Kuo [59], who have used the Hamada-Johnston potential [60] and obtained the energy of the state 3^-_1 in ^{40}Ca equal to 5.57 MeV in the TDA and 3.37 MeV in the RPA. Improvement of the description of the collective state 3^-_1 in ^{16}O in the RPA was also shown by Mavromatis et al. [61], who have used the Kuo-Brown effective interaction [62] derived from the Hamada-Johnston potential and the oscillator basis and calculated that the energy of the state 3^-_1 obtained in the RPA is lower and closer to the experimental value than the TDA result (see Table 7 therein).

Giant dipole resonance is observed in photo-absorption experiments. The dominant contribution to the photo-absorption cross section comes from the E1 transitions and all other contributions are much smaller and usually neglected. Then the photo-absorption cross section σ is given by [63]

$$\sigma = \frac{16\pi^3\alpha}{9} \sum_{\nu} E_\nu B(el\,1; 0^+_{\rm gs} \to \nu)\delta(E - E_\nu), \tag{3.14}$$

where α is the fine structure constant and, similarly as in the case of the strength function (3.1), the delta-function is usually replaced with the Lorentzian (3.2). The photo-absorption cross sections for ^{16}O, ^{40}Ca and ^{208}Pb calculated within the TDA and RPA are shown in Fig. 3.12, 3.13 and 3.14 together with experimental data. Since in the isoscalar electric dipole strength function there are no excitations except spurious zero energy E1 excitation, the photo-absorption cross sections are calculated from the isovector E1 excitations. We can see that the cross-sections calculated within the TDA and RPA are similar (the cross-sections calculated within the RPA are reduced, but by a negligible amount) and they are in a reasonable agreement with the experimental cross sections.

3.2 Calculations in the framework of *pp*TDA, *hh*TDA, *pp*RPA and *hh*RPA

In this section we present results of calculations performed for ^{16}O, ^{40}Ca and ^{208}Pb within the *pp*TDA, *hh*TDA, *pp*RPA and *hh*RPA. These calculations provide data describing nuclei with two protons or neutrons added to or removed from the closed-shell nucleus. In order to perform these calculations, we have developed two computer programs. These programs form bases of all possible *pp* and *hh* states for a given angular momentum and parity from the HF single-particle states provided by the program *HFB_DD* in the given valence space. The first program constructs the *pp*TDA and *hh*TDA matrices using the formulae (2.7), (2.12), (2.15) and (2.21) in the corresponding bases, diagonalizes them

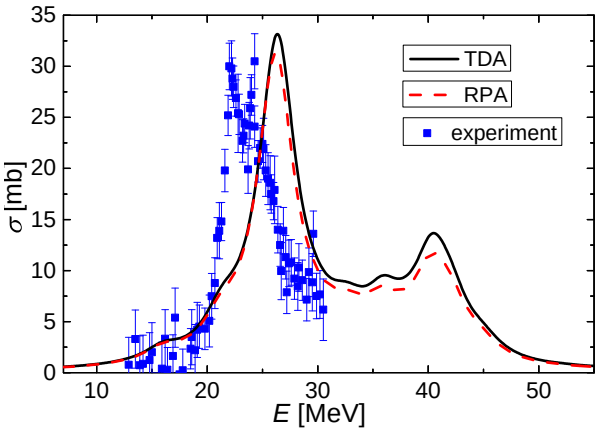

Figure 3.12: The photo-absorption cross section for ^{16}O calculated within the TDA and RPA for $\hbar\omega = 26$ MeV and $N_{max} = 14$ together with the experimental data taken from [64]. Width of the Lorentzian $\Delta = 4$ MeV.

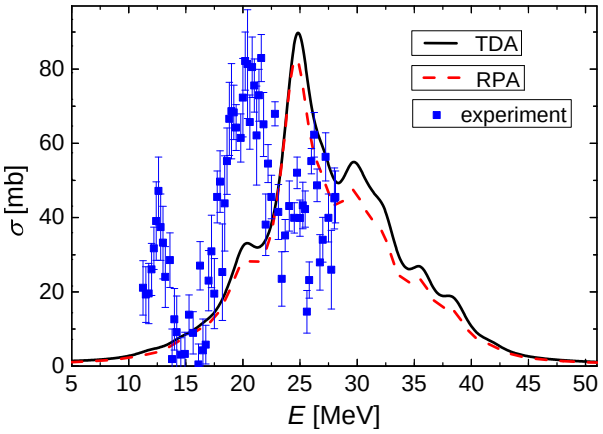

Figure 3.13: The same as in Fig. 3.12, but for ^{40}Ca, $\hbar\omega = 16.3$ MeV and $N_{max} = 14$. Experimental data taken from [65]. Width of the Lorentzian $\Delta = 3$ MeV.

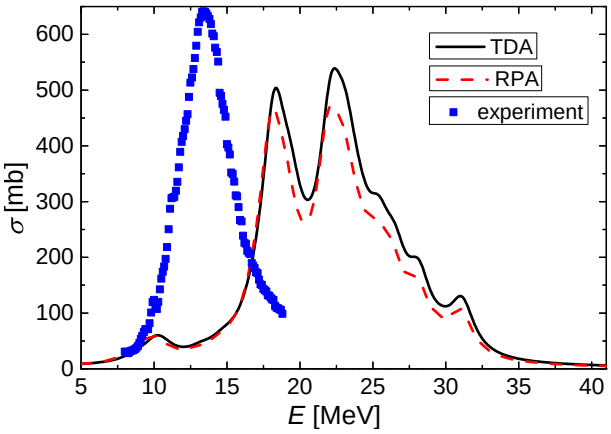

Figure 3.14: The same as in Fig. 3.12, but for ^{208}Pb, $\hbar\omega = 11$ MeV and $N_{max} = 14$. Experimental data taken from [66]. Width of the Lorentzian $\Delta = 2$ MeV.

using a subroutine from LAPACK and provides the eigenenergies and eigenvectors satisfying the orthonormality relation (2.23). The second program constructs the *pp*RPA "supermatrix" appearing in the *pp*RPA equations (2.53), where A is the *pp*TDA matrix, C is the *hh*TDA matrix and the matrix B is given by (2.68), and diagonalizes it using a subroutine from LAPACK for a general eigenvalue problem, which yields excitation energies of *pp*RPA and *hh*RPA phonons (with opposite sign in the case of *hh*RPA phonons). The program provides also the correponding phonon amplitudes satisfying the orthonormality relations (2.92) and (2.94). All calculations have been performed with the maximal possible valence space for the given N_{max}.

The programs compute also the reduced probabilities of electric transitions from the ground state using the formulae (2.28) and (2.34), (2.36) in the case of *pp*TDA, (2.39), (2.41) in the case of *hh*TDA, (2.106) in the case of *pp*RPA and (2.114) in the case of *hh*RPA. The reduced single-particle matrix elements $\langle a||r^J Y_J||b\rangle$ are provided by the program *HFB_DD*.

The program performing the *pp*TDA and *hh*TDA calculations gives the same results, concerning the excitation energies and the amplitudes of the eigenvectors, as the quasiparticle TDA calculations performed by the program *HFB_DD*, which reduce effectively to the TDA, *pp*TDA and *hh*TDA calculations in the case of closed-shell nuclei (see Appendix B). Thus we assume that the program works correctly.

The spectra of the energy levels of the open-shell nuclei ^{18}O, ^{18}Ne, ^{14}C, ^{14}O, ^{42}Ca, ^{42}Ti, ^{38}Ar, ^{38}Ca, ^{210}Pb, ^{210}Po, ^{206}Hg and ^{206}Pb calculated in the framework of the *pp*TDA, *hh*TDA, *pp*RPA and *hh*RPA applied to ^{16}O, ^{40}Ca and ^{208}Pb for $N_{max} = 14$ are shown in Fig. 3.15, 3.16 and 3.17 together with the experimental values taken from [43]. The energies of the lowest 0^+ states have been set to zero and the other energies are the excitation energies related to the these ground states. The figures contain several low-lying states with positive and negative parity and the angular momenta from 0 to 4.

We see that the results of the *pp*TDA and *hh*TDA calculations are very similar to those obtained within the *pp*RPA and *hh*RPA. In the case of ^{18}O, ^{18}Ne and ^{210}Pb the spin and parity of the first and the second excited state is 2^+ and 4^+, respectively, which agrees with the experiment, and the corresponding excitation energies calculated within the *pp*RPA are slightly closer to the experimental values than the results obtained within the *pp*TDA as shown in Table 3.5. In the case of ^{38}Ar and ^{38}Ca the calculations give the right spin and parity only for the first excited state 2^+_1 and, concerning the corresponding excitation energy, the *hh*RPA gives again a slightly better agreement with the experiment than the *hh*TDA as shown in Table 3.5. In the case of ^{42}Ca and ^{42}Ti only the *pp*RPA gives the right spin and parity of the first excited state 2^+_1, while the energy of the state 2^+_1 calculated within the *pp*TDA is below the energy of the ground state 0^+_1 (it is negative in Fig. 3.16 and Table 3.5). In the case of ^{14}C and ^{14}O the calculations don't give the right spins and parities of the lowest excited states. In the case of ^{210}Po, ^{206}Hg and ^{206}Pb the calculated energies of the states 2^+_1 and 4^+_1 are below the the energy of the ground state 0^+_1 (they are negative and the *pp*RPA and *hh*RPA values are less negative than the *pp*TDA and *hh*TDA values as shown in Table 3.5) except the energy of the state 4^+_1 in ^{206}Hg calculated within the *hh*RPA. The agreement with the experiment in the region of higher energies is

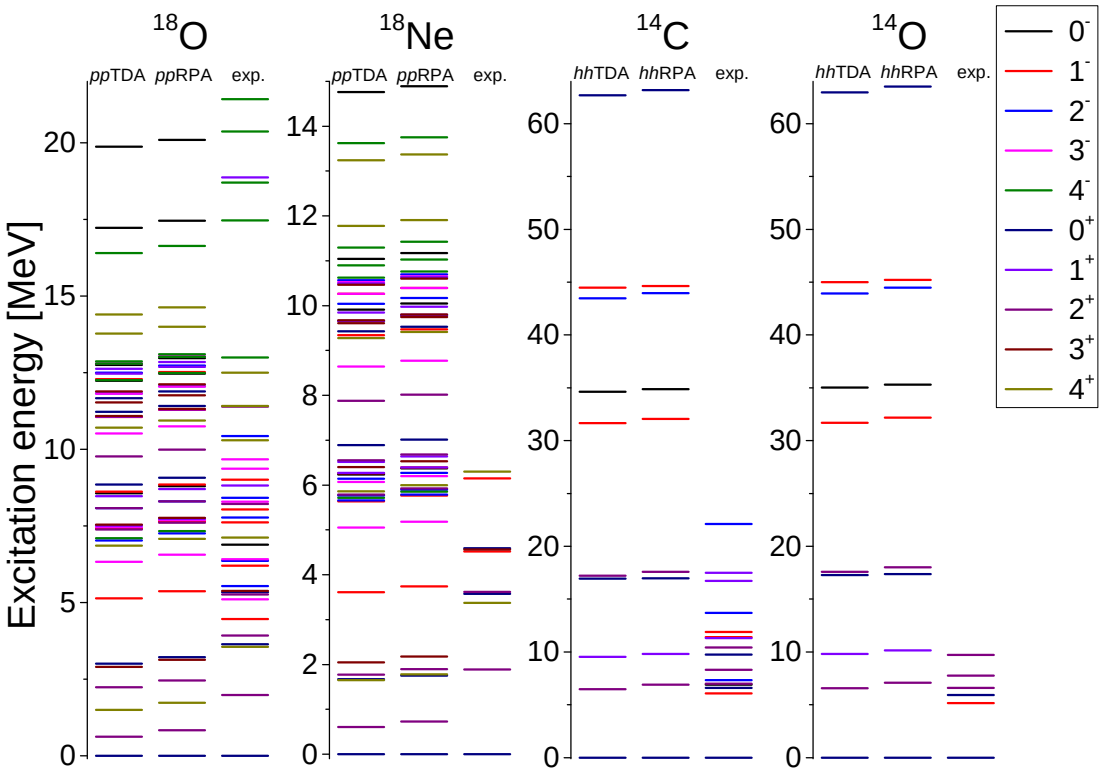

Figure 3.15: The spectra of states of ^{18}O, ^{18}Ne, ^{14}C and ^{14}O calculated within the ppTDA or hhTDA (the first column) and ppRPA or hhRPA (the second column) for $\hbar\omega = 16.3$ MeV and $N_{max} = 14$ together with the experimental values taken from [43] (the third column).

even worse.

In order to investigate the convergence with increasing N_{max}, in the case of ^{18}O we calculated the spectra for various values of N_{max}. The result is shown in Fig. 3.18. We can observe a trend of a convergence. The calculated excitation energies depend also on $\hbar\omega$, but this dependence becomes less significant with increasing N_{max} as shown in Fig. 3.19 which contains the excitation energies of the states 0_2^+ and 2_2^+ in ^{18}O calculated within the ppTDA for various $\hbar\omega$ and N_{max} as an example.

The low-lying part of our spectrum of ^{210}Pb calculated within the ppTDA is in a reasonable agreement with the results of similar calculations performed by Ma and True [67] (see Fig. 3 therein), Freed and Rhodes [68] (see Fig. 3 therein), Klemt and Speth [69] (see Fig. 3 therein) and Herling and Kuo [70], who performed ppTDA calculations using the Hamada-Johnston interaction and obtained the excitation energy of the first excited state 2_1^+ equal to 0.4 MeV which is similar to our result. Concerning the low-lying part of the spectrum of ^{210}Pb, our ppRPA calculations give results similar to results of ppRPA calculations performed by Vary and Ginocchio [71], who obtained the excitation energies of two lowest excited states 2_1^+ and 4_1^+ equal to 0.648 MeV and 0.742 MeV, respectively. The ppRPA calculations performed by Bouyssy and Vinh Mau [72] with the Tabakin interaction give the low-lying part of the spectrum of ^{18}O similar to our result (see Fig. 4 therein). Their excitation energies of the states 2_1^+

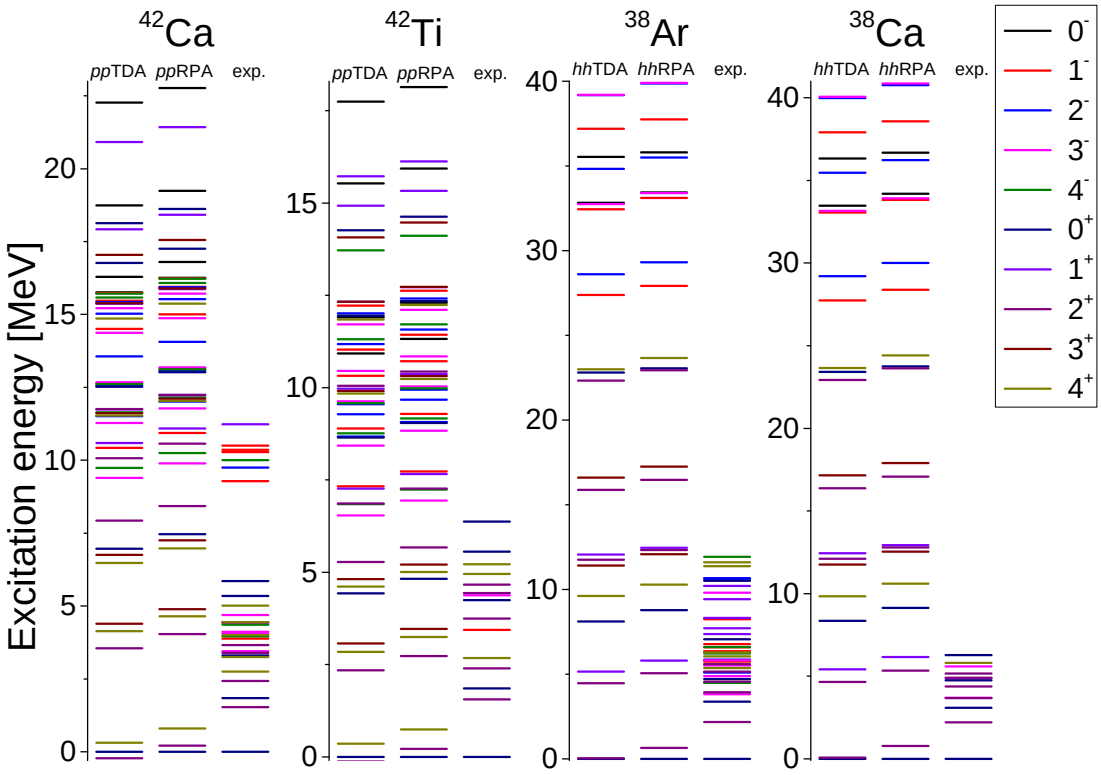

Figure 3.16: The same s in Fig. 3.15, but for ^{42}Ca, ^{42}Ti, ^{38}Ar and ^{38}Ca. The oscillator parameter $\hbar\omega = 16.3$ MeV and $N_{max} = 14$.

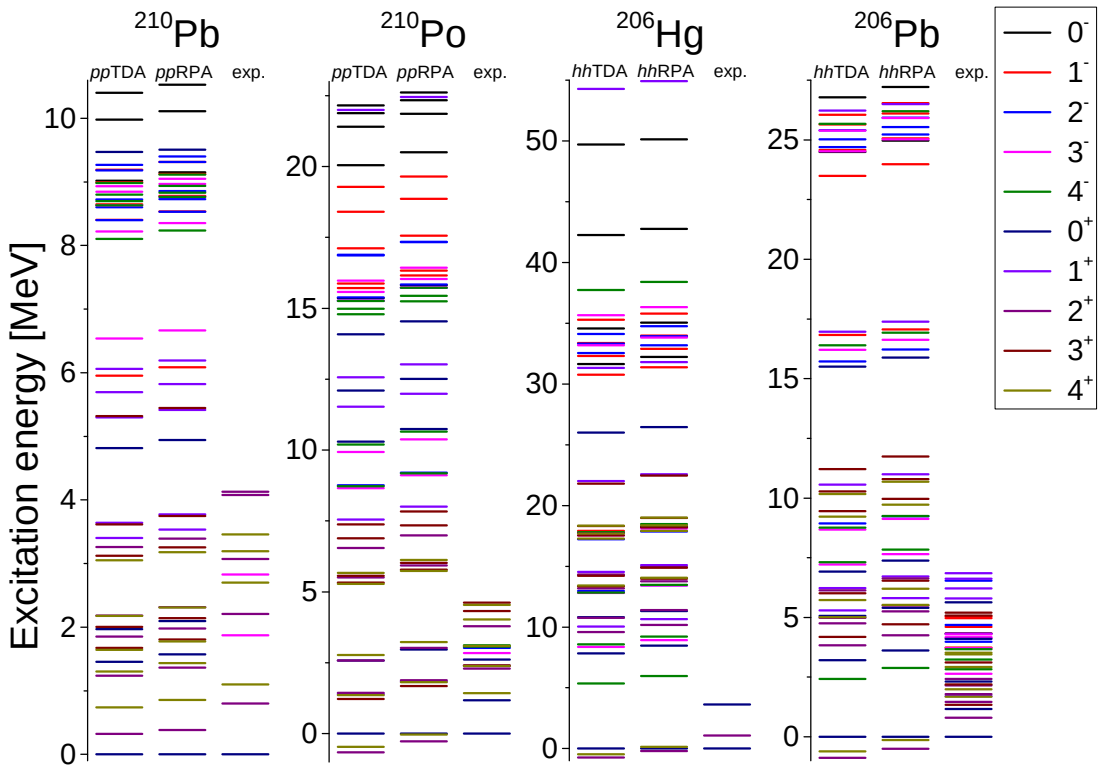

Figure 3.17: The same as in Fig. 3.15, but for ^{210}Pb, ^{210}Po, ^{206}Hg and ^{206}Pb. The oscillator parameter $\hbar\omega = 11$ MeV and $N_{max} = 14$.

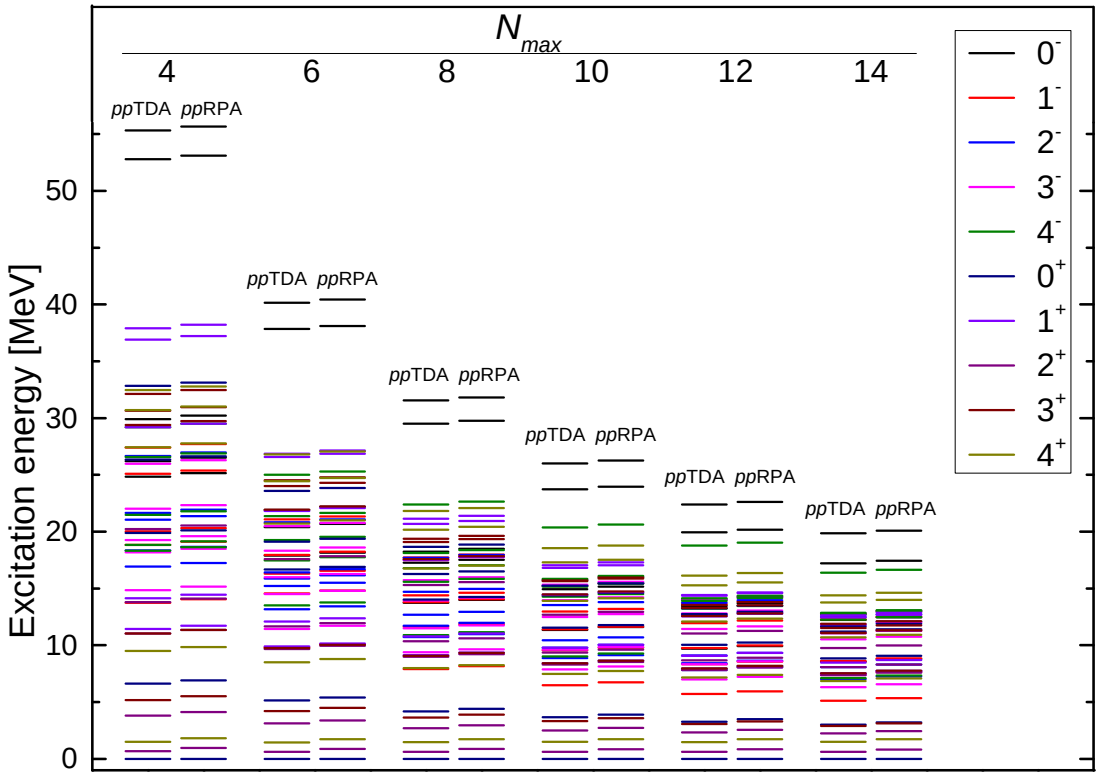

Figure 3.18: The spectra of five lowest states of ^{18}O with positive and negative parity and the angular momenta from 0 to 4 calculated within the ppTDA and ppRPA for $\hbar\omega = 16.3$ MeV and different values of N_{max}.

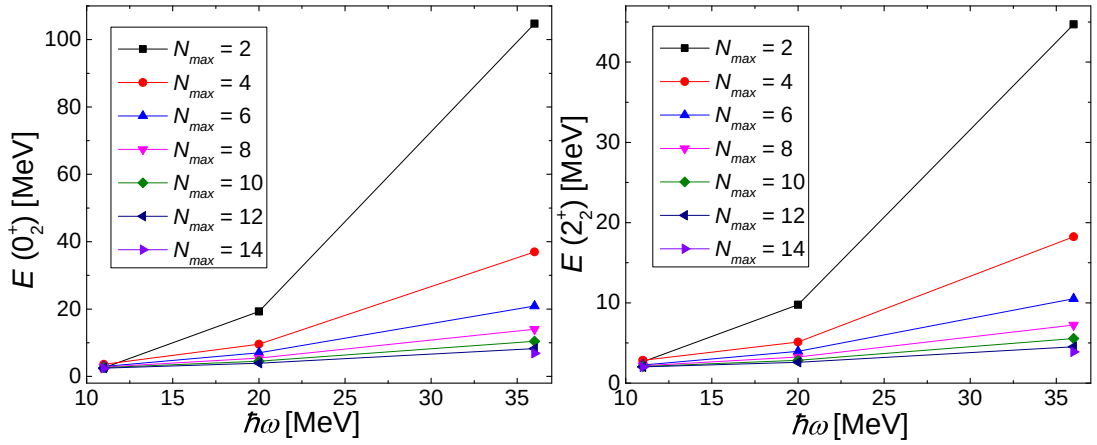

Figure 3.19: The excitation energies of the state 0_1^+ (left) and 2_1^+ (right) in ^{18}O calculated within the ppTDA for various $\hbar\omega$ and N_{max}.

Table 3.5: The excitation energies of selected states in a sample of open-shell nuclei calculated within the *pp*TDA, *hh*TDA, *pp*RPA and *hh*RPA for $N_{max} = 14$ and corresponding $\hbar\omega$ together with the experimental values taken from [43].

nucleus	$\hbar\omega$ [MeV]	state	$pp(hh)$TDA value [keV]	$pp(hh)$RPA value [keV]	experiment [keV]
^{18}O	16.3	2_1^+	613.50	830.78	1982.07 ± 0.09
^{18}O	16.3	4_1^+	1492.25	1722.35	3554.84 ± 0.40
^{18}Ne	16.3	2_1^+	603.0	730.4	1887.3 ± 0.2
^{18}Ne	16.3	4_1^+	1650.3	1781.8	3376.2 ± 0.4
^{42}Ca	16.3	2_1^+	-220.57	207.91	1524.71 ± 0.03
^{42}Ti	16.3	2_1^+	-123.7	213.2	1554.6 ± 0.3
^{38}Ar	16.3	2_1^+	26.27	640.63	2167.64 ± 0.05
^{38}Ca	16.3	2_1^+	74.54	780.43	2213.13 ± 0.10
^{210}Pb	11	2_1^+	320.2	384.8	799.7 ± 0.1
^{210}Pb	11	4_1^+	740.5	853.4	1097.7 ± 1.0
^{210}Po	11	2_1^+	-658.551	-269.427	1181.398 ± 0.010
^{210}Po	11	4_1^+	-462.870	-32.906	1426.701 ± 0.014
^{206}Hg	11	2_1^+	-725.92	-205.75	1068.20 ± 0.20
^{206}Pb	11	2_1^+	-880.903	-496.945	803.054 ± 0.025

and 4_1^+ are almost the same as ours. Many other *pp* and *hh* calculations were performed in the past (see e. g. [73], [74], [75] for calculations in the lead region and [76], [77]).

Now we proceed to present the results of our calculations of electric quadrupole and octupole strength functions for some open-shell nuclei performed within the *pp*TDA, *hh*TDA, *pp*RPA and *hh*RPA. The corresponding effective charges $e_{\mathrm{p/n}}^{(\mathrm{eff})}$ have been determined by fitting so that the reduced transition probabilities from the ground to the lowest 2^+ or 3^- state calculated within the *pp*RPA or *hh*RPA agree with the experimental values. They are shown in Table 3.6 together with the corresponding polarization constants χ (see the relation (2.42)) and the calculated and experimental values of $B(E2; 0_1^+ \to 2_1^+)$ and $B(E3; 0_1^+ \to 3_1^-)$ which were compared.

The E2 and E3 strength functions calculated using the effective charges in Table 3.6 and $N_{max} = 14$ are shown in Fig. 3.20 and 3.21. We can see that the results calculated within the *pp*TDA or *hh*TDA are very similar to those calculated within the *pp*RPA or *hh*RPA. In the case of light nuclei with two valence holes the strength functions contain only few transitions because of a limited space of the *hh* configurations with the given angular momentum and parity.

In order to investigate the convergence with increasing dimension of the configuration space, we have calculated the electric quadrupole strength function for ^{18}O using various values of N_{max}. The result is shown in Fig. 3.22. One can observe a trend of a convergence.

Table 3.6: The effective charges and polarization constatnts χ for E2 and E3 strength functions determined from the comparison of the values of $B(E2;0_1^+ \to 2_1^+)$ and $B(E3;0_1^+ \to 3_1^-)$ calculated within the ppRPA or hhRPA for $N_{max} = 14$ with the experimental values (denoted by exp) taken from [78] and [52]. The values of $\hbar\omega$ used in the calculations are also shown.

Effective charges and polarization constants for E2 strength functions					
nucleus	$\hbar\omega$ [MeV]	$B(E2;0_1^+ \to 2_1^+)$ $[e_{\mathrm{p/n}}^{(\mathrm{eff})^2}\mathrm{fm}^4]$	$B(E2;0_1^+ \to 2_1^+)_{\mathrm{exp}}$ $[e^2\mathrm{fm}^4]$	$e_{\mathrm{p/n}}^{(\mathrm{eff})}$ [e]	χ
^{18}O	16.3	22.7	45.1 ± 2.0	1.4	1.4
^{18}Ne	16.3	54	269 ± 26	2.2	1.2
^{14}C	16.3	1.9	18.7 ± 2.5	3.1	2.1
^{42}Ca	16.3	17	420 ± 30	5.0	5.0
^{42}Ti	16.3	21	870 ± 250	6.4	5.4
^{38}Ar	16.3	6	130 ± 10	4.7	3.7
^{38}Ca	16.3	5	96 ± 21	4.4	4.4
^{210}Pb	11	479	510 ± 15	1.03	1.03
^{210}Po	11	44	200 ± 40	2.1	1.1
^{206}Pb	11	61	1000 ± 20	4.0	4.0
Effective charges and polarization constants for E3 strength functions					
nucleus	$\hbar\omega$ [MeV]	$B(E3;0_1^+ \to 3_1^-)$ $[e_{\mathrm{p/n}}^{(\mathrm{eff})^2}\mathrm{fm}^6]$	$B(E3;0_1^+ \to 3_1^-)_{\mathrm{exp}}$ $[e^2\mathrm{fm}^6]$	$e_{\mathrm{p/n}}^{(\mathrm{eff})}$ [e]	χ
^{18}O	16.3	379	1120 ± 110	1.7	1.7
^{42}Ca	16.3	69	9100 ± 910	11.5	11.5
^{38}Ar	16.3	23	$(95 \pm 26) \cdot 10^2$	20.3	19.3
^{210}Pb	11	25103	$(40 \pm 10) \cdot 10^4$	4.0	4.0
^{210}Po	11	97	$(53 \pm 8) \cdot 10^4$	73.9	72.9
^{206}Pb	11	1230	$(65 \pm 4) \cdot 10^4$	23.0	23.0

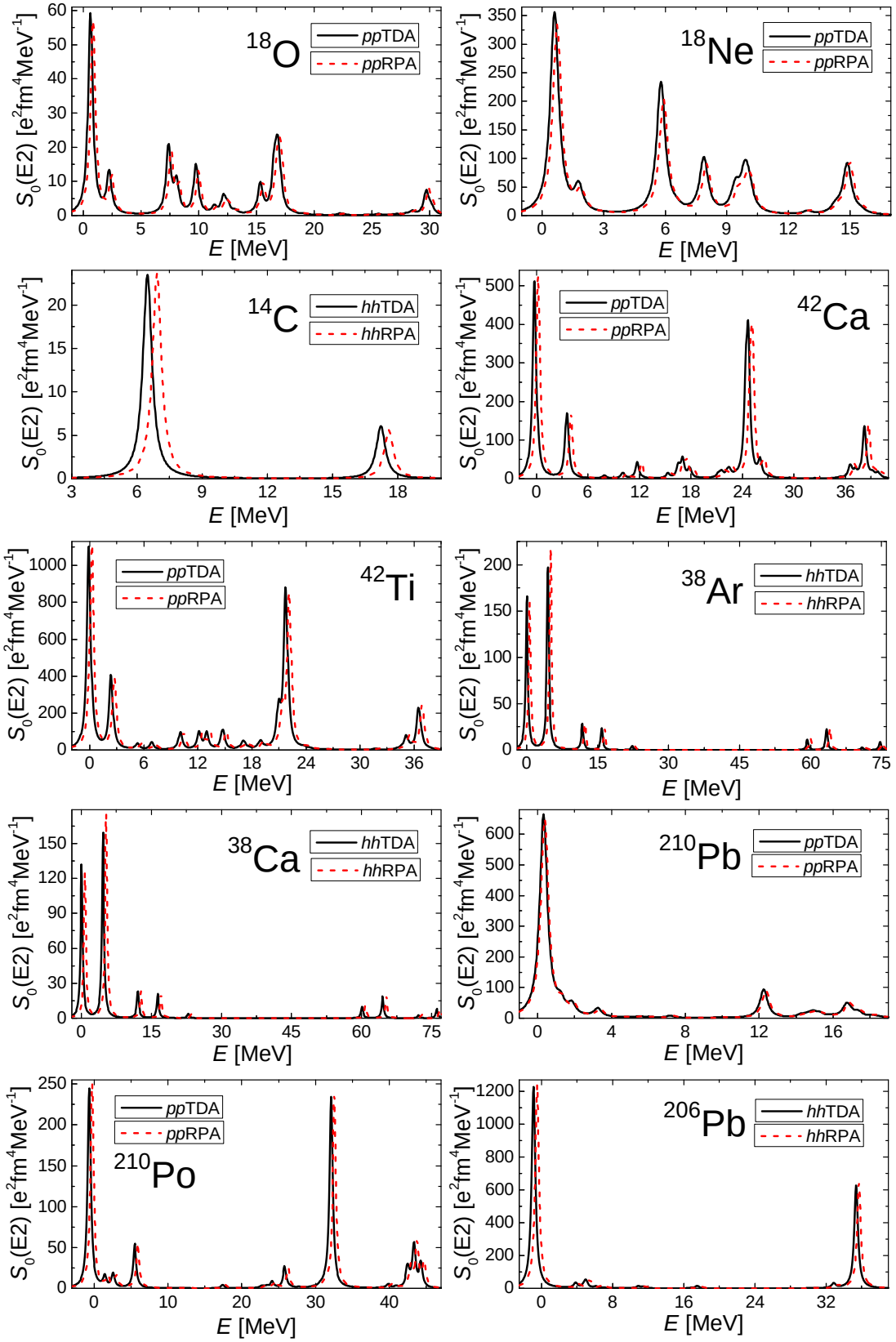

Figure 3.20: The electric quadrupole strength functions calculated within the ppTDA, hhTDA, ppRPA and hhRPA for various open-shell nuclei with $N_{max} = 14$ using the effective charges in Table 3.6. We have used $\hbar\omega = 11$ MeV for the nuclei in the lead region (^{210}Pb, ^{210}Po, ^{206}Pb) and $\hbar\omega = 16.3$ MeV for the remaining nuclei. Width of the Lorentzian $\Delta = 0.5$ MeV.

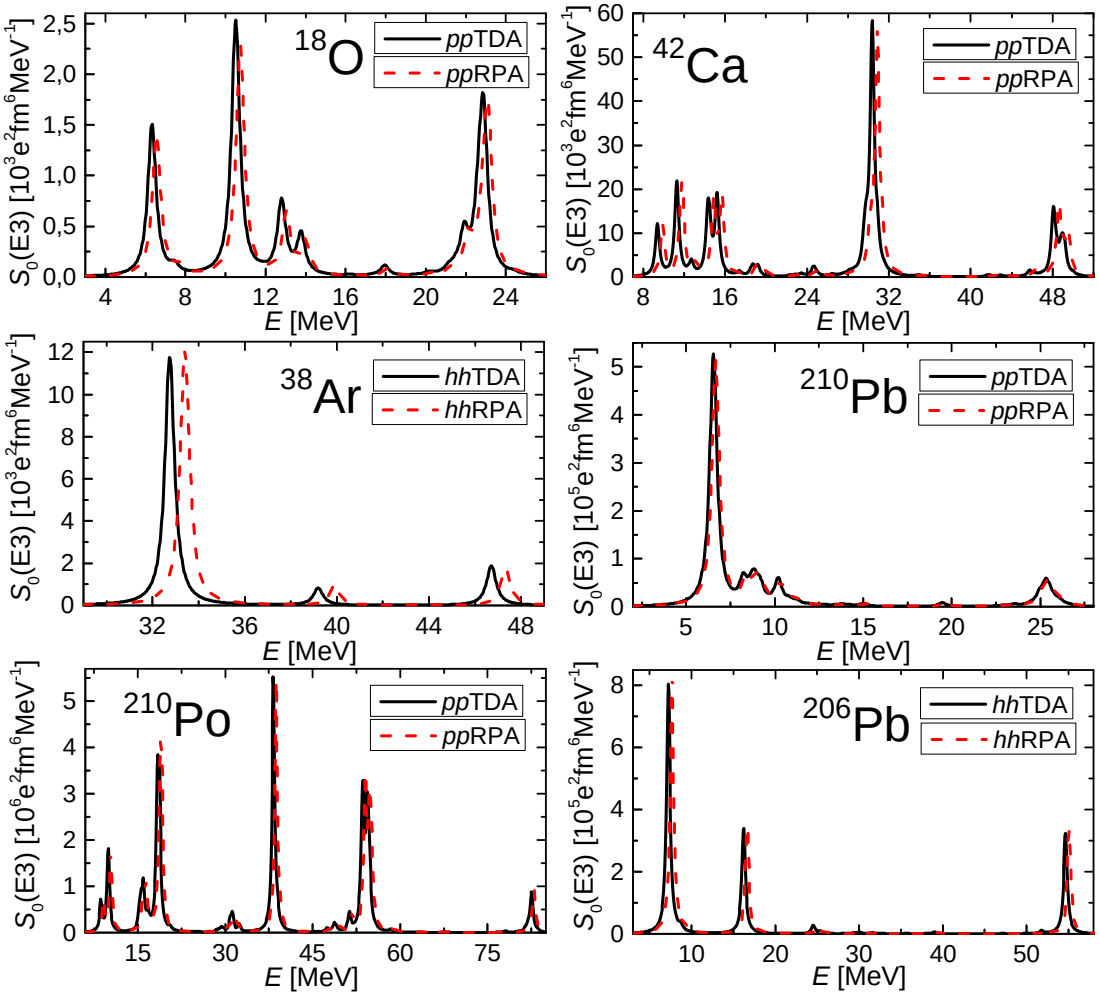

Figure 3.21: The electric octupole strength functions calculated in the framework of the ppTDA, hhTDA, ppRPA and hhRPA for various open-shell nuclei with $N_{max} = 14$ using the effective charges in Table 3.6. We have used $\hbar\omega = 11$ MeV for the nuclei in the lead region (^{210}Pb, ^{210}Po, ^{206}Pb) and $\hbar\omega = 16.3$ MeV for the remaining nuclei. Width of the Lorentzian $\Delta = 0.5$ MeV.

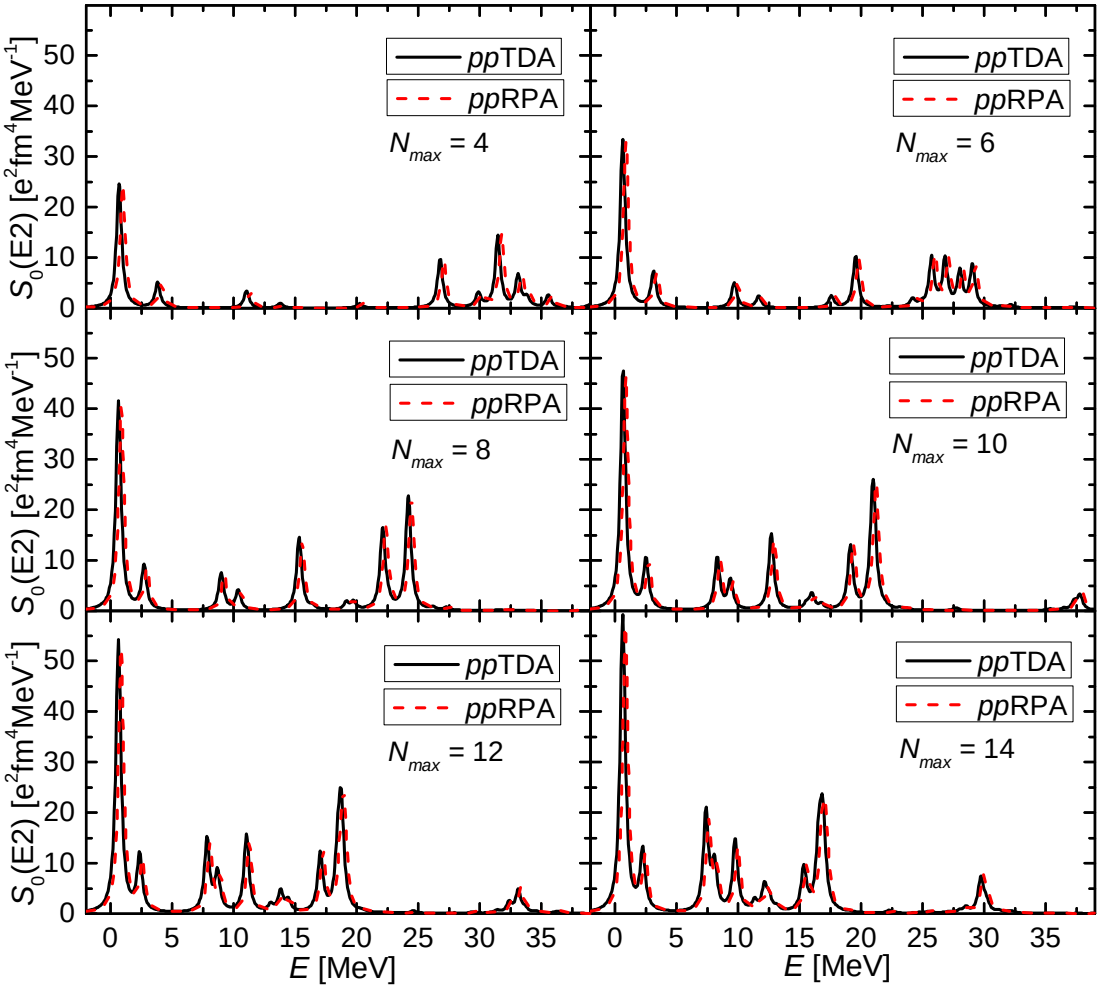

Figure 3.22: The electric quadrupole strength functions for ^{18}O calculated in the framework of the ppTDA and ppRPA for $\hbar\omega = 16.3$ MeV and various N_{max} using the effective charge $e_{\rm n}^{\rm (eff)} = 1.4e$. Width of the Lorentzian $\Delta = 0.5$ MeV.

Conclusion

We have carried out systematical calculations for ^{16}O, ^{40}Ca and ^{208}Pb within the TDA and RPA. The spectra, electric monopole, dipole, quadrupole and octupole strength functions and photo-absorption cross sections calculated in the framework of both approximations have been compared. The spectra, energies of centroids of some giant resonances, electric octupole transition probabilities and cross sections have been compared also with the experimental data. However, we were unable to reproduce them accurately. Disagreement of our results with the experiment was caused also by the fact that a bare realistic chiral nucleon-nucleon potential $\mathrm{NNLO_{opt}}$ has been used in all calculations. For better description of real nuclei an effective interaction relevant for nucleons in the nuclear matter should be used.

The results obtained in the TDA and RPA are very similar. The most significant difference between these methods concerns the lowest 3^- state. According to our calculations, the RPA gives lower excitation energy and much greater excitation probability of this state than the TDA and the results obtained in the RPA are closer to the experimental data. Thus the RPA leads to an improvement of the description of a strong collectivity of this state.

Moreover, the absolute value of the energy of the lowest 1^- state, which is a spurious state connected with the center-of-mass motion, calculated within the RPA is closer to zero and approaches zero with increasing dimension of the configurations space. We have found out that the configuration space corresponding to $N_{max} = 14$ is not sufficiently large to obtained fully converged results. However, a trend of a convergence has been observed.

Another aim of this thesis was to investigate open-shell nuclei. We have developed microscopical models for nuclei with two valence particles or holes added to a doubly-magic core based on an analogy to the TDA and RPA. We have derived the needed formulae and carried out their numerical implementation. In the framework of these models, namely the *pp*TDA, *hh*TDA, *pp*RPA and *hh*RPA, we have calculated the spectra and electric quadrupole and octupole strength functions of the nuclei formed from doubly-magic nuclei ^{16}O, ^{40}Ca and ^{208}Pb by adding or removing two nucleons of the same type.

The results of the *pp*TDA and *hh*TDA calculations are very similar to those obtained within the *pp*RPA and *hh*RPA. We have noticed a significant disagreement of the calculated spectra with the experiment, but we didn't expect that such simple models would reproduce the experimental data. In order to reproduce low-lying spectra of open-shell nuclei, realistic description of single-particle states is desirable. It is well known that $\mathrm{NNLO_{opt}}$ (or in general, chiral potentials) does not give proper description of single-particle states at the mean-field level. One has to supplement the interaction with corresponding three-body part. This can be done effectively by adding a phenomenological density-dependent potential [79].

We have investigated also the convergence of the results with increasing size of the configuration space and a trend of a convergence has been observed. The calculated excitation energies depend also on the parameter of the oscillator basis $\hbar\omega$, but this dependence becomes less significant with increasing N_{max}.

The effective charges determined for the strength functions of open-shell nuclei from comparison of the calculated and experimental transition probabilities between the ground and the first excited state have reasonable values in the case of the electric quadrupole excitations. On the other hand, the effective charges corresponding to the electric octupole excitation probabilities are very large. The explanation is in the following paragraph.

In the *ph* theories the lowest excitations are *ph* excitations from the highest closed major shell to the lowest open major shell. Since the parities of the major shells alternate, the *ph* theories yield a good description of the low-lying states with negative parity. On the other hand, in the *pp* theories the lowest excitations correspond to configurations with valence particles in the lowest major shell of the valence space. Thus the *pp* theories yield a good description of the low-lying states with positive parity (analogously for the *hh* case). For better description of the low-lying states with negative parity it is necessary to take into account *ph* excitations of the core. In our models effects of these omitted configurations are simulated by the effective charges.

The next step in improvement of the description of nuclei with two particles above a doubly-magic core could be taking into account the *ph* excitations of the core by coupling of *ph* and *pp* phonons in the framework of the multiphonon model, but this is beyond the scope of this work.

A. Isotropic harmonic oscillator

The states $|nlm_l\rangle$ of a particle in the isotropic harmonic oscillator potential

$$V(r) = \frac{1}{2}M\omega^2 r^2 \tag{A.1}$$

are characterized by three quantum numbers: n, l and m_l. The possible values of these quantum numbers are:

$$\begin{aligned} n &= 1, 2, 3, \ldots, \\ l &= 0, 1, 2, 3, \ldots, \\ m_l &= -l, \ldots, l. \end{aligned} \tag{A.2}$$

The energy of the state is

$$E = \hbar\omega \left(N + \frac{3}{2} \right), \tag{A.3}$$

with

$$N = 2(n-1) + l. \tag{A.4}$$

In the case of a nucleon, which has the spin 1/2, it is convenient to introduce the total angular momentum

$$\vec{j} = \vec{l} + \vec{s}. \tag{A.5}$$

Then the eigenstates of the isotropic harmonic oscillator are defined by the basis $|nljm\rangle$. The possible values of the quantum numbers are:

$$j = l + \frac{1}{2}, l + \frac{1}{2} - 1, \ldots, |l - \frac{1}{2}|, \tag{A.6}$$

$$m = -j, \ldots, j. \tag{A.7}$$

In the spectroscopic notation the $(2j+1)$-times degenerated state $|nlj\rangle$ is denoted by nX_j, where $X =$ s,p,d,f,... for $l = 0, 1, 2, 3, ...$

B. Quasiparticle TDA

The residual interaction is usually divided to two parts:

$$V_{\text{res}} = V_{\text{res}}^{(\text{short})} + V_{\text{res}}^{(\text{long})}, \tag{B.1}$$

where $V_{\text{res}}^{(\text{short})}$ is the short-range residual interaction, which is responsible for pairing of nucleons[1], and $V_{\text{res}}^{(\text{long})}$ is the long-range residual interaction, which is responsible for collective vibrations of the whole nucleus (correlated motion of many nucleons) and generates collective vibrational degrees of freedom of the nuclear motion. The pairing interaction modifies the mean field in open-shell nuclei and the result is the quasiparticle mean field instead of the single-particle one.

An effective method which takes into account the pairing short-range residual interaction is the Bardeen-Cooper-Schrieffer (BCS) theory[2]. Let us have a basis of the single-particle eigenstates of the mean field $|\alpha\rangle$. The starting point of the BCS theory is the assumption that the ground state of a system with an even number of fermions is

$$|\text{BCS}\rangle = \prod_{\alpha>0}(\mathcal{U}_a - \mathcal{V}_a c_\alpha^\dagger \tilde{c}_\alpha^\dagger), \tag{B.2}$$

where the summation goes through the states with a positive angular momentum projection and $\mathcal{U}_a$ and $\mathcal{V}_a$ are real parameters with the following meaning: $\mathcal{V}_a^2$ is the probability that a given pair of single-particle states $|\alpha\rangle$ and $|-\alpha\rangle$ is occupied by two particles and $\mathcal{U}_a^2$ is the probability that this pair is empty. As denoted, we have made the natural choice for spherical nuclei that the $\mathcal{U}$ and $\mathcal{V}$ parameters are independent of the projection quantum number m_α. They obey the normalization condition

$$\mathcal{U}_a^2 + \mathcal{V}_a^2 = 1 \ \ \forall \ \ a. \tag{B.3}$$

The BCS ground state $|\text{BCS}\rangle$ is the vacuum for new creation and annihilation operators, namely the quasiparticle operators $\beta_\alpha^\dagger$ and β_α, whose companion with good tensorial properties is $\tilde{\beta}_\alpha = (-1)^{j_a+m_\alpha}\beta_{-\alpha}$ according to (1.28). The quasiparticle operators are related to the particle operators via the Bogoliubov-Valatin transformation

$$\beta_\alpha^\dagger = \mathcal{U}_a c_\alpha^\dagger + \mathcal{V}_a \tilde{c}_\alpha, \tag{B.4}$$

$$\tilde{\beta}_\alpha = \mathcal{U}_a \tilde{c}_\alpha - \mathcal{V}_a c_\alpha^\dagger, \tag{B.5}$$

introduced by Bogoliubov(see [86] and [87]) and Valatin(see [88] and [89]). The quasiparticle operators also obey the fermion anticommutation relations. Each operator $\beta_\alpha^\dagger$ creates a quasiparticle that is a particle with probability amplitude

[1]The concept of nuclear pairing emerges from experimental observations (see [80] and references therein). It has been shown by Mayer [81] that the pairing is caused by a short-range attractive two-nucleon forces.

[2]The BCS theory was originally developed to explain the superconductivity of metals at low temperature [82], where the electrons form pairs behaving like bosons, which was shown by Cooper [83]. The application of this theory to the nuclear physics has been performed by Bohr, Mottelson and Pines [84] and Belyaev [85].

$\mathcal{U}_a$ and a hole with probability amplitude $\mathcal{V}_a$. In other words, the single-particle state α is empty with a probability $\mathcal{U}_a^2$ and occupied with a probability $\mathcal{V}_a^2$. The parameters $\mathcal{U}_a$ and $\mathcal{V}_a$ are obtained by solving the BCS equations, which are not presented here. For deeper study of the BCS theory see e.g. [90].

In analogy to the TDA we can introduce the quasiparticle TDA (QTDA), where we consider two-quasiparticle configurations. The QTDA equations can be derived using the equation-of-motion method described at the beginning of Section 1.4 with the phonon creation operator

$$Q_\nu^\dagger = \sum_{a \leq b} X_{ab}^\nu [\beta_a^\dagger \beta_b^\dagger]_{JM}, \tag{B.6}$$

where a and b both are either proton or neutron states and X_{ab}^ν are amplitudes, and $|\mathrm{BCS}\rangle$ as the corresponding vacuum, i.e. the ground state. For the derivation and the result see [15].

For closed-shell nuclei we obtain

$$\begin{array}{l} \mathcal{V}_a = 1, \ \mathcal{U}_a = 0 \quad \text{for} \quad \varepsilon_a < \varepsilon_\mathrm{F}, \\ \mathcal{V}_a = 0, \ \mathcal{U}_a = 1 \quad \text{for} \quad \varepsilon_a > \varepsilon_\mathrm{F}, \end{array} \tag{B.7}$$

thus the case of particles and holes is recovered and the vacuum $|\mathrm{BCS}\rangle$ is the HF vacuum $|\mathrm{HF}\rangle$. This means that the QTDA reduces effectively to the TDA, ppTDA and hhTDA.

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List of Abbreviations

abbreviation	meaning
NNLO_{opt}	Optimized chiral interaction at next-to-next-to-leading order
TDA	Tamm-Dancoff approximation
RPA	random phase approximation
HF	Hartree-Fock
ph	particle-hole
n-*p*-*n*-*h*	*n*-particle-*n*-hole
pp	particle-particle
hh	hole-hole
*pp*TDA	particle-particle Tamm-Dancoff approximation
*hh*TDA	hole-hole Tamm-Dancoff approximation
*pp*RPA	particle-particle random phase approximation
*hh*RPA	hole-hole random phase approximation
HFB_DD	Hartree-Fock-Bogoliubov code with density dependent interaction
CENS	Computational Environment for Nuclear Structure
LAPACK	Linear Algebra Package
UCOM	unitary correlation operator method
ISGMR	isoscalar giant monopole resonance
ISGQR	isoscalar giant quadrupole resonance
IVGQR	isovector giant quadrupole resonance
SDI	surface delta interaction
BCS	Bardee-Cooper-Schrieffer
QTDA	quasiparticle Tamm-Dancoff approximation