Report on the doctoral thesis
"Computational Complexity in Graph Theory"
by
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The dissertation investigates two areas of algorithm theory: the computational complexity of constraint satisfaction problems and algorithmic questions related to geometric objects in the plane. Constraint satisfaction is a powerful framework for the investigation of many algorithmic problems and it has been studied extensively both in the theoretical and applied literature. An instance of the constraint satisfaction problem (CSP) consists of a set of variables that has to be assigned values subject to a set of constraints. Constraint satisfaction problems are hard in general, thus most of the research focuses on identifying tractable special cases. These special cases usually involve restrictions on the type of constraints that are allowed and/or on the structure induced by the constraints on the variables. In the past 10 years, a sophisticated algebraic approach was developed for the study of constraint satisfaction problems. This approach allows us to distinguish the tractable and hard problems based on the algebraic properties of the constraints.

Planar graphs and graphs related to planar objects have been investigated for a long time, motivated both by applications and pure mathematical interest. On one hand, the many different ways of representing and drawing graphs in the plane have been studied by the combinatorics and graph drawing communities. On the other hand, many applied problems can be formulated as a graph-theoretical question on planar objects such as segments and disks, thus the algorithmic questions related to planar geometric objects received considerable interest.

The main new scientific claims of the dissertation can be summarized in five points:

1. Chapter 3 gives a complete complexity characterization of constraint satisfaction with equality constraints over an infinite domain.

2. Chapter 4 gives a complete complexity characterization of CSP with temporal constraints.

3. Chapter 6 proves that every series-parallel graph can be represented as the contact graph of line segments with at most 3 directions.

4. Chapter 7 shows that the independent set problem for line segments in the plane is fixed-parameter tractable when parameterized by the size k of the independent set and the number d of different directions.

5. Chapter 8 proves that the PERFECT ORDERING problem (a special case of balanced ordering) is NP-hard for planar graphs with maximum degree 4.

In more detail:

1. Equality constraint languages contain constraints that are preserved under every permutation of the domain, that is, they can be described by a boolean combination of equalities. Chapter 3 proves a dichotomy theorem for equality constraint languages: for every such language, it is shown that CSP is either polynomial-time solvable or NP-complete. The literature on CSP contains several dichotomy theorems of this form and the result presented in this chapter is a nice addition to the previous results. The proof uses the algebraic theory of constraints which allows a clean treatment of the different types of constraints. The positive part of the dichotomy theorem (i.e., the algorithm for the polynomial-time solvable cases) is not very complicated, although the implementation details are explained at length. The interesting part of the proof is the different characterizations of the tractable cases (Lemma 3.1) and the way these characterizations are used to prove the hardness result (Lemma 3.4).

2. Chapter 4 proves a similar dichotomy theorem for temporal constraint languages, that is, for constraints that are the boolean combinations of comparison relations on the set of rational numbers. Without doubt, this result is the most significant part of the dissertation. The algebraic framework is used heavily to obtain the dichotomy. Several different types of polymorphisms are defined, and the result is obtained by investigating which combinations of these polymorphisms result in a tractable problem. This requires a lengthy and highly non-trivial case analysis.

3. Chapter 6 gives a short proof that every series-parallel graph can be represented as the intersection graph of segments in at most 3 directions. Furthermore, there is a representation where the segments only touch each other but they do not properly intersect. The proof is simple and the problem is not very well motivated, thus this is only a minor result compared to the previous chapters.

4. Chapter 7 investigates the maximum independent set problem of line segments from the fixed-parameter tractability point of view: it gives a $f(d,k)nm$ time algorithm for finding a independent set of size k in the intersection graph of segments in at most d directions. The algorithm does not require an explicit representation of the segments, only the intersection graph itself. The algorithm is based on reduction rules and bounded-depth search trees, both are standard techniques of parameterized complexity.

5. Chapter 8 investigates the balanced ordering problem, where the vertices

of the graph has to be ordered in such a way that the total inbalance of the vertices is minimized. In particular, in the PERFECT ORDERING problem we have to find an ordering where the total inbalance equals a natural lower bound. Tightening a previous result, the candidate shows that PERFECT ORDERING is NP-hard for graphs with maximum degree 4, and this holds also for planar graphs. As a positive result, it is shown that for every fixed k, it can be determined in polynomial time whether there is an ordering with at most k inbalanced vertices. The proofs use standard, but nontrivial techniques.

I would like to pose the following questions to the candidate:

- What is known about finite equality constraint languages? Can the results of Chapter 3 be applied for such problems?
- Is it meaningful to study temporal constraints on finite sets?
- Does the algorithm of Chapter 7 work for the intersection graphs of segments in more than 2 dimensions?

In summary, the work presented in the dissertation is interesting, well-motivated, technically difficult, and proves the ability of the candidate to do high-quality research. The research was done with various coauthors; in the evaluation of the dissertation, I assumed that the candidate had substantial contribution in the joint work. Based on the quality of the results, I recommend granting the PhD degree to the candidate.

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