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Symplectic spinors and Hodge theory

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Contents

1 Introduction

I have said the 21st century might be the era of quantum mathematics or, if you like, of infinite-dimensional mathematics. What could this mean? Quantum mathematics could mean, if we get that far, 'understanding properly the analysis, geometry, topology, algebra of various non-linear function spaces', and by 'understanding properly' I mean understanding it in such a way as to get quite rigorous proofs of all the beautiful things the physicists have been speculating about.

Sir Michael Atiyah

In the literature, topics contained in this thesis are treated rather separately. From a philosophical point of view, a common thread of themes that we consider is represented by the above quotation of M. Atiyah. We are inspired by mathematical and theoretical physics.

The content of the thesis concerns analysis, differential geometry and representation theory on infinite dimensional objects. A specific infinite dimensional object which we consider is the Segal–Shale–Weil representation of the metaplectic group. This representation originated in number theory and theoretical physics. We analyze its tensor products with finite dimensional representations, induce it to metaplectic structures defined over symplectic and contact projective manifolds obtaining differential operators whose properties we study. From other point of view, differential geometry uses the infinite dimensional algebraic objects to obtain vector bundles and differential operators, that we investigate by generalizing analytic methods known currently – namely by a Hodge theory for complexes in categories of specific modules over C^*-algebras.

Results described in the thesis arose from the year 2003 to the year 2016. At the beginning, we aimed to classify all **first order invariant differential operators** acting between bundles over contact projective manifolds that are induced by those irreducible representations of the metaplectic group which have bounded multiplicities. See Krýsl [41] for a result. Similar results were achieved by Fegan [14] in the case of irreducible finite rank bundles over Riemannian manifolds equipped with a conformal structure. In both cases, for any such two bundles, there is at most one first order invariant differential operator up to a scalar multiple.[1] The condition for the existence in the case of contact projective manifolds is expressed by the highest weight of the induced representation considered as a module over a suitable simple group, by a conformal weight, and by the action of $-1 \in \mathbb{R}^\times$. The result is based on a decomposition of the tensor product of an irreducible highest weight $\mathfrak{sp}(2n,\mathbb{C})$-module that has bounded multiplicities with the defining representation of $\mathfrak{sp}(2n,\mathbb{C})$ into irreducible submodules. See Krýsl [40]. For similar classification results in the case of more general parabolic geometries and bundles induced by finite dimensional modules, see Slovák, Souček [71].

Our next research interest, described in the thesis, was the **de Rham sequence tensored by the Segal–Shale–Weil representation**. From the algebraic point of view, the Segal–Shale–Weil representation (SSW representation) is an L^2-globalization of the direct sum of two specific infinite dimensional Harish-Chandra $(\mathfrak{g}, K)$-modules with bounded multiplicities over the metaplectic Lie algebra, which are called completely pointed. We decompose the de Rham sequence with values in the mentioned direct sum into sections of irreducible bundles, i.e., bundles induced by irreducible representations. See Krýsl [38]. For a $2n$ dimensional symplectic manifold

[1] and up to operators of the zeroth order. See section 4.3.

with a metaplectic structure, the bundle of exterior forms of degree i, $i \leq n$, with values in the Segal–Shale–Weil representation decomposes into $2(i+1)$ irreducible bundles. For $i \geq n$, the number of such bundles is $2(2n-i+1)$. It is known that the decomposition structure of completely reducible representations is connected to the so-called Schur–Weyl–Howe dualities. See Howe [29] and Goodman, Wallach [20]. We use the decomposition of the twisted de Rham sequence to obtain a duality for the metaplectic group which acts in this case, on the exterior forms with values in the SSW representation. See Krýsl [46]. The dual partner to the metaplectic group appears to be the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2)$.

Any Fedosov connection (i.e., a symplectic and torsion-free connection) on a symplectic manifold with a metaplectic structure defines a covariant derivative on the symplectic spinor bundle which is the bundle induced by the Segal–Shale–Weil representation. We prove that twisted de Rham derivatives map sections of an irreducible subbundle into sections of at most three irreducible subbundles. Next, we are interested in the quite fundamental question on the structure of the curvature tensor of the symplectic spinor covariant derivative similarly as one does in the study of the curvature of a Levi-Civita or a Riemannian connection. Inspired by results of Vaisman in [75] on the curvature tensors of Fedosov connections, we derive a decomposition of the curvature tensor on symplectic spinors. See Krýsl [42]. Generalizing this decomposition, we are able to find certain subcomplexes of the twisted de Rham sequence, that are called symplectic twistor complexes in a parallel to the spin geometry. These complexes exist under specific restrictions on the curvature of the Fedosov connection. Namely, the connection is demanded to be of Ricci-type. See Krýsl [43]. Further results based on the decomposition of the curvature concern a relation of the spectrum of the symplectic spinor Dirac operator to the spectrum of the symplectic spinor Rarita–Schwinger operator. See Krýsl [39]. The symplectic Dirac operator was introduced by K. Habermann. See [23]. The next result is on symplectic Killing spinors. We prove that if there exists a non-trivial (i.e., not covariantly constant) symplectic Killing spinor, the connection is not Ricci-flat. See [45].

Since the classical theories on analysis of elliptic operators on compact manifolds are not applicable in the case of the de Rham complex twisted by the Segal–Shale–Weil representation, we tried to develop a **Hodge theory for infinite rank bundles**. We use and elaborate ideas of Mishchenko and Fomenko ([58] and [59]) on generalizations of the Atiyah–Singer index theorem to investigate cohomology groups of infinite rank elliptic complexes concerning their topological and algebraic properties. We work in the categories PH^*_A and H^*_A whose objects are pre-Hilbert C^*-modules and Hilbert C^*-modules, respectively, and whose morphisms are adjointable maps between the objects. These notions go back to the works of Kaplansky [31], Paschke [62] and Rieffel [63]. Analyzing proofs of the classical Hodge theory, we are lead to the notion of a Hodge-type complex in an additive and dagger category. We introduce a class of self-adjoint parametrix possessing complexes, and prove that any self-adjoint parametrix possessing complex in PH^*_A is of Hodge-type. Further, we prove that in H^*_A the category of self-adjoint parametrix possessing complexes coincides with the category of the Hodge-type ones. Using these results, we show that an elliptic complex on sections of finitely generated projective Hilbert C^*-bundles over compact manifolds are of Hodge-type if the images of the Laplace operators of the complex are closed. The cohomology groups of such complexes are isomorphic to the kernels of the Laplacians and they are Banach spaces with respect to the quotient topology. Further, we prove that the cohomology groups are finitely generated projective Hilbert C^*-modules. See Krýsl [47]. Using the result of Bakić and Guljaš [2] for modules over a C^*-algebra of compact operators K, we are able to remove the condition on the closed image. We prove in [51] that elliptic complexes of differential operators on finitely generated projective K-Hilbert bundles are of Hodge-type and that their cohomology groups are finitely generated projective Hilbert K-modules. See Krýsl [48], [49] and [50] for a possible application connected to the quotation of Atiyah.

We find it more appropriate to mention author's results and their context in Introduction, and treat motivations, development of important notions, and most of the references in Chapters 2 and 3. In the 2nd Chapter, we recall a definition, realization and characterization of the Segal–Shale–Weil representation. In Chapter 3, symplectic manifolds, Fedosov connections, metaplectic structures, symplectic Dirac and certain related operators are introduced. Results of K. and L. Habermann on global analysis related to these operators are mentioned in this part as well. Chapter 4 of the thesis contains own results of the applicant. We start with the appropriate representation theory and Howe-type duality. Then we treat results on the twisted de Rham derivatives, curvature of the symplectic spinor derivative and twistor complexes. Symplectic Killing spinors are defined in this part as well. We give a classification of the invariant operators for contact projective geometries together with results on the decomposition of the appropriate tensor products in the third subsection. The fourth subsection is devoted to the Hodge theory. The last part of the thesis consists of selected articles published in the period 2003–2016.

2 Symplectic spinors

The discovery of symplectic spinors as a rigorous mathematical object is attributed to I. E. Segal, D. Shale and A. Weil. See Shale [66] and Weil [80]. Segal and Shale considered the real symplectic group as a group of canonical transformations and were interested in a certain quantization of Klein–Gordon fields. Weil was interested in number theory connected to theta functions, so that he considered more general symplectic groups than the ones over the real numbers. Let us notice, that this representation appeared in works of van Hove (see Folland [15], p. 170) at the Lie algebra level and can be found in certain formulas of Fresnel in wave optics already (see Guillemin, Sternberg [22], p. 71).

When doing quantization, one has to assign to "any" function defined on the phase space of a considered classical system an operator acting on a certain function space – a Hilbert space by a rule. Usually, smooth functions are considered to represent the right class for the set of quantized functions. The prescription shall assign to the Poisson bracket of two smooth functions a multiple of the commutator of the operators assigned to the individual functions. The multiple is determined by "laws of nature". It equals to $(\imath\hbar)^{-1}$, where $\imath$ is a fixed root of -1 and $\hbar$ is the Planck constant over 2π. Thus, in the first steps, the quantization map is demanded to be a Lie algebra homomorphism up to a multiple. In further considerations, there is a freedom allowed in the sense that the image of a Poisson bracket need not be the $(\imath\hbar)^{-1}$ multiple of the commutator precisely, but the so-called deformations are allowed. (See Waldmann [77] and also Markl, Stasheff [54] for a framework of quite general deformations.) This tolerance is mainly because of the Groenewold–van Hove no go theorem (see Waldmann [77]). Analytically, deformations are convergent series in the small variable $\hbar$. Their connection to the formal deformations is given by the so called Borel lemma [77].

The state space of a classical system with finite degrees of freedom is modeled by a symplectic manifold (M, ω). The state space of a point particle moving in an n-dimensional vector space or n point particles on a line is the space $\mathbb{R}^{2n}$ or the intersection of open half-spaces in it, respectively. Considering the coordinates $q^1, \ldots, q^n$, and $p_1, \ldots, p_n$ on $\mathbb{R}^{2n}$ where q^i projects onto the i-th coordinate and p_i onto the $(n+i)$-th one, ω equals to $\sum_{i=1}^n dq^i \wedge dp_i$, or to its restriction to the intersection, respectively.

The group of all linear maps Φ on $\mathbb{R}^{2n}$ which preserve the symplectic form, i.e., $\Phi^*\omega = \omega$, is called the **symplectic group**. Elements of this group do not change the form of dynamic equations governing non-quantum systems – the Hamilton's equations. In this way, they coincide with linear canonical transformations used in Physics.[2] See, e.g., Arnold [1] or Marsden, Ratiu [55].

The symplectic group $G = Sp(2n, \mathbb{R})$ is an $n(2n+1)$ dimensional Lie group. Its maximal compact subgroup is isomorphic to the unitary group $U(n)$ determined by choosing a compatible positive linear complex structure, i.e., an $\mathbb{R}$-linear map $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that 1) $J^2 = -1_{\mathbb{R}^{2n}}$ and 2) the bilinear map $g : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ given by $g(v, w) = \omega(v, Jw)$ is symmetric and positive definite, i.e., a scalar product. The unitary group can be proved diffeotopic to the circle S^1, and consequently, its first fundamental group is isomorphic to $\mathbb{Z}$. Thus, for each $m \in \mathbb{N}$, $Sp(2n, \mathbb{R})$ has a unique non-branched m-folded covering $\lambda(m) : {}^m\widetilde{Sp(2n, \mathbb{R})} \to Sp(2n, \mathbb{R})$ up to a covering isomorphism. We fix an element e in the preimage of the neutral element in $Sp(2n, \mathbb{R})$ on the two fold covering. The unique Lie group such that its neutral element is e and such that the

[2] The system is supposed to be non-dissipative, i.e., its Hamiltonian function does not depend on time.

covering map is a Lie group homomorphism is called the **metaplectic group**, or if we wish, the real metaplectic group. We set $\lambda = \lambda(2)$ and $\widetilde{G} = Mp(2n, \mathbb{R}) = {}^2\,\widetilde{Sp(2n}, \mathbb{R})$. We denote the λ-preimage of $U(n)$ by $\widetilde{K}$.

So far, the construction of the metaplectic group was rather abstract. One of the basic results of the structure theory of Lie groups is that this is unavoidable indeed. By this we mean that there is no faithful representation of $Mp(2n, \mathbb{R})$ by matrices on a **finite dimensional** vector space. Otherwise said, the metaplectic group cannot be realized as a Lie subgroup of any finite dimensional general linear group. Following Knapp [32], we prove this statement.

Theorem 1: The connected double cover $Mp(2n, \mathbb{R})$ does not have a realization as a Lie subgroup of $GL(W)$ for a finite dimensional vector space W.

Proof. Let us suppose that there exists a faithful representation $\eta' : \widetilde{G} \to \mathrm{Aut}(W)$ of the metaplectic group on a finite dimensional space W. This representation gives rise to a faithful representation $\eta : \widetilde{G} \to \mathrm{Aut}(W^{\mathbb{C}})$ on the complexified vector space $W^{\mathbb{C}}$. This map is injective by its construction. The corresponding Lie algebra representation, i.e., the map $\eta_* : \widetilde{\mathfrak{g}} \to \mathrm{End}(W^{\mathbb{C}})$ is well defined because of the finite dimension of $W^{\mathbb{C}}$. Consequently, we have the commutative diagram

$$\begin{tikzcd} \widetilde{\mathfrak{g}} \arrow[r, "\eta_*"] \arrow[d, "\lambda_*"] & \mathrm{End}(W^{\mathbb{C}}) \\ \mathfrak{g} \arrow[r, "j'"] & \mathfrak{sp}(2n, \mathbb{C}) \arrow[u, "\phi'"'] \end{tikzcd}$$

where $\mathfrak{g}$ is the Lie algebra of the appropriate symplectic group, j' is the natural inclusion and ϕ' is defined by $\phi'(A + \imath B) = \eta_* \lambda_*^{-1}(A) + \imath \eta_* \lambda_*^{-1}(B)$, $A, B \in \mathfrak{g}$. Taking the exponential of the Lie algebra $\mathfrak{sp}(2n, \mathbb{C}) \subseteq \mathrm{End}(\mathbb{C}^{2n})$, we get the group $Sp(2n, \mathbb{C})$. Because this group is simply connected, we get a representation $\phi : Sp(2n, \mathbb{C}) \to \mathrm{Aut}(W^{\mathbb{C}})$ which integrates ϕ' in the sense that $\phi_* = \phi'$. Because λ_*, η_* and also ϕ_* are derivatives at 1 of the corresponding Lie groups representations, and j' is the derivative at $1 \in G$ of the canonical inclusion $j : G \to Sp(2n, \mathbb{C})$, we obtain a corresponding diagram at Lie groups level which is commutative as well. Especially, we have $\eta = \phi \circ j \circ \lambda$. However, the right hand side of this expression is not injective whereas the left hand side is. This is a contradiction. $\square$

Remark: The complex orthogonal group $SO(n, \mathbb{C})$ is not simply connected, so that the above proof does not apply for $G = SO(n, \mathbb{R})$ and its connected double cover $\widetilde{G} = Spin(n, \mathbb{R})$. If it applied, $Spin(n, \mathbb{R})$ would not have any faithful finite dimensional representation.

2.1 The Segal–Shale–Weil representation

For the canonical symplectic vector space $(\mathbb{R}^{2n}, \omega)$, a group $H(n) = \mathbb{R}^{2n} \times \mathbb{R}$ is assigned in which the group law is given by

$$(v, t) \cdot (w, s) = (v + w, s + t + \frac{1}{2}\omega(v, w))$$

where $(v, t), (w, s) \in H(n)$. This is the so called **Heisenberg group** $H(n)$ of dimension $2n + 1$. Let us set $L = \mathbb{R}^n \times \{0\} \times \{0\} \subseteq H(n)$ and $L' = \{0\} \times \mathbb{R}^n \times \{0\}$ for the vector space of initial space and for the vector space of initial impulse conditions, respectively. In particular, the symplectic vector space $\mathbb{R}^{2n}$ is isomorphic to the direct sum $L \oplus L'$.

For any $u' \in H(n)$, we have a unique $t \in \mathbb{R}$ and $q \in L$, $p \in L'$ such that $u' = (q, p, t)$. The **Schrödinger representation** Sch of the Heisenberg group $Sch : H(n) \to U(L^2(L))$ is given by

$$(Sch((q,p,t))f)(x) = e^{2\pi\imath t + \pi\imath\omega(q,p) + 2\pi\imath\omega(x,p)} f(x+q)$$

where $q, x \in L$, $p \in L'$, $t \in \mathbb{R}$, and $f \in L^2(L)$. It is an irreducible representation. See Folland [15]. (By $\omega(x,p)$, we mean $\omega((x,0),(0,p))$ and similarly for $\omega(q,p)$.)

We may "twist" this representation in the following way. For any $g \in G$, we set $l_g : H(n) \to H(n)$, $l_g(u,t) = (gu, t)$, where $u \in \mathbb{R}^{2n}$ and $t \in \mathbb{R}$. Consequently, we obtain a family of representations $Sch \circ l_g$ of the Heisenberg group $H(n)$ parametrized by elements g of the symplectic group G. The action of the center of $H(n)$ is the same for each member of the family $(Sch \circ l_g)_{g \in G}$. Namely, $(Sch \circ l_g)(0,0,t) = e^{2\pi\imath t}$, $t \in \mathbb{R}$. Let us recall the Stone–von Neumann theorem. For a proof, we refer to Folland [15] or Wallach [78].

Theorem 2 (Stone–von Neumann): Let T be an irreducible unitary representation of the Heisenberg group on a separable infinite dimensional Hilbert space H. Then T is unitarily equivalent to the Schrödinger representation.

By Stone–von Neumann theorem, we find a unitary operator $C_g : L^2(L) \to L^2(L)$ that intertwines $Sch \circ l_g$ and Sch for each $g \in G$.[3] By Schur lemma for irreducible unitary representations (see Knapp [33]), we see that there is a function $m : G \times G \to U(1)$ such that $C_g C_{g'} = m(g,g') C_{gg'}$, $g, g' \in G$. In particular, $g \mapsto C_g$ is a projective representation of $Sp(2n, \mathbb{R})$ on the Hilbert space $L^2(L)$. It was proved by Shale in [66] and Weil in [80] that it is possible to lift the cocycle m and the projective representation $g \mapsto C_g$ of G to the metaplectic group to obtain a true representation of the 2-fold cover. We denote this representation by σ and call it the **Segal–Shale–Weil representation**. Note that some authors call it the symplectic spinor, metaplectic or oscillator representation. The representation is unitary and faithful. See, e.g., Weil [80], Borel, Wallach [5], Folland [15], Moeglin et al. [60], Habermann, Habermann [26] or Howe [30].

The "essential" uniqueness of the Segal–Shale–Weil representation with respect to the choice of a representation of the Heisenberg group is expressed in the next theorem.

Theorem 3: Let $T : H(n) \to U(W)$ be an irreducible unitary representation of the Heisenberg group on a Hilbert space W and $\sigma' : \widetilde{G} \to U(W)$ be a non-trivial unitary representation of the metaplectic group such that for all $(v,t) \in H(n)$ and $g \in \widetilde{G}$

$$\sigma'(g) T(v,t) \sigma'(g)^{-1} = T(\lambda(g)v, t).$$

Then there exists a deck transformation γ of λ, such that σ' is equivalent either to $\sigma \circ \gamma$ or to $\sigma^* \circ \gamma$, where $\sigma^*(g) = \tau\sigma(g)\tau$ and $(\tau(f))(x) = \overline{f(x)}$, $x \in \mathbb{R}^n$, $g \in \widetilde{G}$ and $f \in L^2(\mathbb{R}^n)$.
Proof. See Wallach [78], p. 224. $\square$

Remark: Let us recall that a deck transformation γ is any continuous map which satisfies $\lambda \circ \gamma = \lambda$. Note that in the case of the symplectic group covered by the metaplectic group, a deck transformation is either the identity map or the map "interchanging" the leaves of the metaplectic group.

[3] We say that $C : W \to W$ intertwines a representation $T : H \to \mathrm{Aut}(W)$ of the group H if $C \circ T(h) = T(h) \circ C$ for each $h \in H$.

2.2 Realization of symplectic spinors

There are several different objects that one could call a symplectic basis. We choose the one which is convenient for considerations in projective contact geometry. (See Yamaguchi [82] for a similar choice.) If (V,ω) is a symplectic vector space of dimension $2n$ over a field $\mathbb{k}$ of characteristic zero, we call a basis $(e_i)_{i=1}^{2n}$ of V a **symplectic basis** if $\omega(e_i,e_j)=\delta_{i,2n+1-j}$ for $1\leq i\leq n$ and $1\leq j\leq 2n$, and $\omega(e_i,e_j)=-\delta_{i,2n+1-j}$ for $n+1\leq i\leq 2n$ and $1\leq j\leq 2n$. Thus, with respect to a symplectic basis, the matrix of the symplectic form is

$$(\omega_{ij})=\left(\begin{array}{c|c}0 & K\\ \hline -K & 0\end{array}\right)$$

where K is the following $n\times n$ matrix

$$K=\begin{pmatrix}0 & \dots & 0 & 1\\ 0 & \dots & 1 & 0\\ \vdots & ⋰ & \vdots & \vdots\\ 1 & \dots & 0 & 0\end{pmatrix}.$$

Further, we denote by ω^{ij}, $i,j=1,\dots,2n$, the coordinates which satisfy $\sum_{k=1}^{2n}\omega_{ik}\omega^{jk}=\delta_i^j$ for each $i,j=1,\dots,2n$. They define a bilinear form $\omega^*:V^*\times V^*\to\mathbb{k}$, e.g., by setting $\omega^*=\sum_{i,j=1}^{2n}\omega^{ij}e_i\wedge e_j$. We use ω_{ij} and ω^{ij} to **rise** and **lower** indices of tensors over V. For coordinates $K_{ab\dots c\dots d}{}^{rs\dots t\dots u}$ of a tensor K on V, we denote the expression $\sum_{c=1}^{2n}\omega^{ic}K_{ab\dots c\dots d}{}^{rs\dots t}$ by $K_{ab\dots}{}^{i}{}_{\dots d}{}^{rs\dots t}$ and $\sum_{t=1}^{2n}K_{ab\dots c}{}^{rs\dots t\dots u}\omega_{ti}$ by $K_{ab\dots c}{}^{rs\dots}{}_{i}{}^{\dots u}$ and similarly for other types of tensors and in the geometric setting when we consider tensor fields on symplectic manifolds.

Remark: Let $(\mathbb{R}^{2n},\omega)$ be the canonical symplectic vector space introduced at the beginning of this Chapter. Then the canonical arithmetic basis of $\mathbb{R}^{2n}$ is not a symplectic basis according to our definition unless $n=1$.

Let us denote the λ-preimage of $g\in Sp(2n,\mathbb{R})$ by $\widetilde{g}$. Suppose $A,B\in M_n(\mathbb{R})$, A is invertible and $B^t=B$. We define the following representation of $\widetilde{G}$ on $L^2(\mathbb{R}^n)$

$$\begin{aligned}(\sigma(h_1)f)(x) &= \pm e^{-\pi\imath g_0(Bx,x)/2}f(x)\text{ for any }h_1\in\widetilde{g_1},\ g_1=\left(\begin{array}{c|c}1 & 0\\ \hline B & 1\end{array}\right)\\ (\sigma(h_2)f)(x) &= \sqrt{\det A^{-1}}f(A^{-1}x)\text{ for any }h_2\in\widetilde{g_2},\ g_2=\left(\begin{array}{c|c}A & 0\\ \hline 0 & A^{-1t}\end{array}\right)\\ (\sigma(h_3)f)(x) &= \pm\imath^n e^{\pi\imath n/4}(\mathcal{F}f)(x)\text{ for any }h_3\in\widetilde{g_3},\ g_3=J_0=\left(\begin{array}{c|c}0 & -1\\ \hline 1 & 0\end{array}\right)\end{aligned}$$

where $f\in L^2(\mathbb{R}^n)$ and $x\in\mathbb{R}^n$. The $\pm$ signs and the square roots in the definition of $\sigma(h_i)$ depend on the specific element in the preimage of g_i. The coordinates of g_i, $i=1,2,3$, are considered with respect to the canonical basis of $\mathbb{R}^{2n}$. See Folland [15]. Notice that we use the Fourier transform defined by $(\mathcal{F}f)(y)=\int_{x\in\mathbb{R}^n}e^{-2\pi\imath g_0(x,y)}f(x)\mathrm{d}x$, $y\in\mathbb{R}^n$, with respect to the Lebesgue measure $\mathrm{d}x$ on $\mathbb{R}^n$ induced by the scalar product $g_0(x,y)=\omega(x,J_0y)$, $(x,0),(y,0)\in\mathbb{R}^n\times\{0\}\simeq L$. Elements of type g_1,g_2 and g_3 generate $Sp(V,\omega)$. See Folland [15]. Note that in Habermann, Habermann [26], a different convention for the Fourier transform is used. Note that $L^2(\mathbb{R}^n)$ decomposes into the direct sum $L^2(\mathbb{R}^n)_+\oplus L^2(\mathbb{R}^n)_-$ of irreducible $\widetilde{G}$-modules of the even and of the odd functions in $L^2(\mathbb{R}^n)$. For a proof that σ is a representation, see Folland [15] or Wallach [78] for instance. For a proof that σ intertwines the Schrödinger representation of the Heisenberg group, see Wallach [78], Habermann, Habermann [26] or Folland [15]. A proof that $L^2(L)_\pm$ are irreducible is contained in Folland [15].

Taking the derivative σ_* at the unit element of $\widetilde{G}$ of the representation σ restricted to smooth vectors in $L^2(L)$, we get the representation $\sigma_* : \widetilde{\mathfrak{g}} \to \mathrm{End}\,(S)$ of the Lie algebra of the metaplectic group on the vector space $S = S(L)$ of Schwartz functions on L. See Borel, Wallach [5] and Folland [15] where the smooth vectors are determined. Note that, we have $S \simeq S_+ \oplus S_-$ similarly as in the previous decomposition. For $n \times n$ real matrices $B = B^t$, $C = C^t$ and A, we have (see Folland [15])

$$\sigma_*(X) = \frac{1}{4\pi\imath}\sum_{i,j=1}^{n} B_{ij}\frac{\partial^2}{\partial x^i \partial x^j} \text{ for } X = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$$

$$\sigma_*(Y) = -\pi\imath\sum_{i,j=1}^{n} C_{ij}x^i x^j \text{ for } Y = \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}$$

$$\sigma_*(Z) = -\sum_{i,j=1}^{n} A_{ij}x^j\frac{\partial}{\partial x^i} - \frac{1}{2}\sum_{i=1}^{n} A_{ii} \text{ for } Z = \begin{pmatrix} A & 0 \\ 0 & -A^t \end{pmatrix}.$$

It follows that

$$\sigma_*(J_0) = \imath\sum_{i=1}^{n}\left(\frac{1}{4\pi}\frac{\partial^2}{\partial (x^i)^2} - \pi (x^i)^2\right).$$

Definition 1: For any $m \in \mathbb{N}_0$, we set $h_m(x) = \frac{2^{1/4}}{\sqrt{m!}}(\frac{-1}{2\sqrt{\pi}})^m e^{\pi x^2}\frac{d^m}{dx^m}(e^{-2\pi x^2})$. For $n \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^n$, we define the Hermite function h_α with index $\alpha = (\alpha_1, \ldots, \alpha_n)$ by $h_\alpha(x^1, \ldots, x^n) = h_{\alpha_1}(x^1)\ldots h_{\alpha_n}(x^n)$, $(x^1, \ldots, x^n) \in \mathbb{R}^n$.

Remark: For Hermite functions, see Whittaker, Watson [81] and Folland [15]. We use the convention of Folland [15]. Especially, $h_0(x) = 2^{1/4}e^{-\pi x^2}$.

Well known properties of Hermite functions make us able to derive that for any $\alpha = (\alpha_1, \ldots$ $\ldots, \alpha_n) \in \mathbb{N}_0^n$

$$\sigma_*(J_0)h_\alpha = -\imath(|\alpha| + \frac{n}{2})h_\alpha$$

where $|\alpha| = \sum_{i=1}^n \alpha_i$. Thus, the Hermite functions are eigenfunctions of $\sigma_*(J_0)$.

2.3 Weyl algebra and Symplectic spinor multiplication

Let $\Bbbk$ be a field of characteristic zero. For any $n \in \mathbb{N}$, the **Weyl algebra** W_n over $\Bbbk$ is the associative algebra generated by elements $1 \in \Bbbk$, $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ satisfying to the relations $1a_i = a_i 1$, $1b_i = b_i 1$, $a_i b_j - b_j a_i = -\delta_{ij}1$, $a_i a_j = a_j a_i$, $b_i b_j = b_j b_i$, $1 \leq i, j \leq n$. It is known that W_n has a faithful representation on the space of polynomials $\Bbbk[q^1, \ldots, q^n]$ given by $1 \mapsto 1$ (multiplication by 1), $a_i \mapsto q^i$ and $b_i \mapsto \frac{\partial}{\partial q^i}$, where q^i denotes the multiplication of a polynomial by q^i and $\frac{\partial}{\partial q^i}$ is the partial derivative in the i-th variable. See, e.g., Björk [3].

Any associative algebra A over field $\Bbbk$ can be equipped with the commutator

$$[\,,\,] : A \times A \to A$$

defined by $[x, y] = xy - yx,\ x, y \in A$, making it a Lie algebra. The **Heisenberg Lie algebra** H_n is the real vector space $\mathbb{R}^{2n+1}[q^1, \dots, q^n, p_1, \dots, p_n, t]$ with the Lie bracket

$$[\,,\,] : H_n \times H_n \to \{0\} \times \{0\} \times \mathbb{R} \subseteq H_n$$

prescribed on basis by $[\partial_t, \partial_{q^i}] = [\partial_t, \partial_{p_i}] = [\partial_{q^i}, \partial_{q^j}] = [\partial_{p_i}, \partial_{p_j}] = 0$ and $[\partial_{q^i}, \partial_{p_j}] = -\delta_{ij}\partial_t$, $1 \leq i, j \leq n$. Note that $[\,,\,]$ is not the Lie bracket of vector fields in this case. It is the Lie algebra of the Heisenberg group $H(n)$ and it is isomorphic (as a Lie algebra) to

$$W_n(1) = \{\tau 1 + \sum_{i=1}^{n} (\alpha_i a_i + \beta_i b_i) |\, \tau, \alpha_i, \beta_i \in \mathbb{R},\, i = 1, \dots, n\} \subseteq W_n$$

equipped with the commutator as the Lie algebra bracket. An isomorphism can be given on a basis by $\partial_t \mapsto 1$, $\partial_{q^i} \mapsto a_i$, $\partial_{p_i} \mapsto b_i$, $i = 1, \dots, n$.

For a symplectic vector space (V, ω) of dimension $2n$ over $\mathbb{R}$, let us choose a symplectic basis $(e_i)_{i=1}^{2n}$ and consider the tensor algebra

$$A = T(V^{\mathbb{C}}) = \mathbb{C} \oplus V^{\mathbb{C}} \oplus (V^{\mathbb{C}} \otimes V^{\mathbb{C}}) \oplus \cdots .$$

Let us set $sCliff(V, \omega) = A/I$, where I is the two sided ideal generated over A by elements $v \otimes w - w \otimes v + \imath\omega(v, w)$, $v, w \in V^{\mathbb{C}}$. The complex associative algebra $sCliff(V, \omega)$ is called the **symplectic Clifford algebra**. Let us consider the map $1 \mapsto 1$, $e_{n+i} \mapsto -a_i$ and $e_{n+1-i} \mapsto \imath b_i$, $i = 1, \dots, n$, which extends to a homomorphism of associative algebras $sCliff(V, \omega)$ and W_n for $\Bbbk = \mathbb{C}$. It is not difficult to see that this map is an isomorphism onto W_n. Summing up, W_n and $sCliff(V, \omega)$ are isomorphic as associative algebras. The Heisenberg Lie algebra H_n embeds homomorphically into $sCliff(V, \omega)$ (considered as a Lie algebra with respect to the commutator) via $\partial_t \mapsto 1$, $\partial_{q^i} \mapsto -e_{n+i}$ and $\partial_{p_i} \mapsto \imath e_{n+1-i}$, $i = 1, \dots, n$.

Remark: Note that there is an isomorphism of the Heisenberg Lie algebra H_n with $\Bbbk_1[q^1, \dots,$ $\dots q^n, p_1, \dots, p_n]$, the space of degree one polynomials in q^i, p_i $(i = 1, \dots, n)$, equipped with the Poisson bracket

$$\{f, g\}_P = \sum_{i,j=1}^{n} \left(\frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} \right)$$

where $f, g \in \Bbbk_1[q^1, \dots, q^n, p_1, \dots, p_n]$.

We come to the following important definition.

Definition 2: Let $(e_i)_{i=1}^{2n}$ be a symplectic basis of (V, ω). For $i = 1, \dots, n$ and $f \in S$, we set

$$e_i \cdot f = \imath x^i f \quad \text{and} \quad e_{i+n} \cdot f = \frac{\partial f}{\partial x^{n-i+1}}$$

and extend it linearly to V. The map $\cdot : V \times S \to S$ is called the **symplectic spinor multiplication**.

Remark: In the preceding definition, $f \in S(\mathbb{R}^n)$ and x^i denotes the projection onto the i-th coordinate in $\mathbb{R}^n$. Note that the symplectic spinor multiplication depends on the choice of a symplectic basis. Because of its equivariant properties (see Habermann [26], p. 13), one can use the multiplication on the level of bundles. In this case, we denote it by the dot as well. Note that the equivariance of the symplectic Clifford multiplication with respect to the Segal–Shale–Weil representation makes the definitions of the symplectic spinor Dirac, the second symplectic spinor Dirac and the associated operator correct.

3 Symplectic spinors in differential geometry

Let us recall that a **symplectic manifold** is a manifold equipped with a closed non-degenerate exterior differential 2-form ω.

One of the big achievements of Bernhard Riemann in geometry is a definition of the curvature (Krümmungsmaß) in an arbitrary dimension. After publishing of his Habilitationsschrift, Levi-Civita and Riemannian connections became fundamental objects for metric geometries. Intrinsic notions and properties (such as straight lines, angle deficits, parallelism etc.) of many geometries known in that time can be defined and investigated by means of them. Using these connections, one can find out quite easily, whether the given manifold is locally isometric to the Euclidean space.

Definition 3: Let (M, ω) be a symplectic manifold. An affine connection ∇ on M is called symplectic if $\nabla\omega = 0$. Such a connection is called a **Fedosov connection** if it is torsion-free.

For symplectic connections, see, e.g., Libermann [52], Tondeur [74], Vaisman [75] and Gelfand, Retakh, Shubin [19]. In contrast to Riemannian geometry, we have the following theorem which goes back to Tondeur [74]. See Vaisman [75] for a proof.

Theorem 4: The space of Fedosov connections on a symplectic manifold (M, ω) is isomorphic to an affine space modeled on the infinite dimensional vector space $\Gamma(S^3TM)$, where S^3TM denotes the third symmetric product of the tangent bundle of M.

Remark: Note that due to a theorem of Darboux (see McDuff, Salamon [56]), all symplectic manifolds of equal dimension are locally equivalent. In particular, symplectic connections cannot serve for distinguishing of symplectic manifolds in the local sense. From the eighties of the last century, symplectic connections gained an important role in mathematical physics. They became crucial for quantization procedures. See Fedosov [13] and Waldmann [77].

Let (M^{2n}, ω) be a symplectic manifold and ∇ be a Fedosov connection. The **curvature tensor field** of ∇ is defined by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

where $X, Y, Z \in \mathfrak{X}(M)$. A **local symplectic frame** $(U, (e_i)_{i=1}^{2n})$ of (M, ω) is an open subset U in M and a sequence of vector fields e_i on U such that $((e_i)_m)_{i=1}^{2n}$ is a symplectic basis of (T_mM, ω_m) for each $m \in U$.

Let $(U, (e_i)_{i=1}^{2n})$ be a local symplectic frame. For $X = \sum_{i=1}^{2n} X^i e_i, Y = \sum_{i=1}^{2n} Y^i e_i, Z =$

$\sum_{i=1}^{2n} Z^i e_i, V = \sum_{i=1}^{2n} V^i e_i \in \mathfrak{X}(M)$, $X^i, Y^i, Z^i, V^i \in \mathcal{C}^\infty(U)$, and $i, j, k, l = 1, \ldots, 2n$, we set

$$\begin{aligned}
R_{ijkl} &= \omega(R(e_k, e_l)e_j, e_i) \\
\sigma(X, Y) &= \mathrm{Tr}(V \mapsto R(V, X)Y),\ V \in \mathfrak{X}(M) \\
\sigma_{ij} &= \sigma(e_i, e_j) \\
\sigma_{ijkl} &= \frac{1}{2(n+1)}(\omega_{il}\sigma_{jk} - \omega_{ij}\sigma_{lk} + \omega_{jl}\sigma_{ik} - \omega_{jl}\sigma_{ik} + 2\sigma_{ij}\omega_{kl}) \\
\widetilde{\sigma}(X, Y, Z, V) &= \sum_{i,j,k,l=1}^{2n} \sigma_{ijkl} X^i Y^j Z^k V^l \\
W &= R - \widetilde{\sigma}
\end{aligned}$$

where at the last row, R represents the $(4,0)$-type tensor field $\sum_{i,j,k,l=1}^{2n} R_{ijkl}\epsilon^i \otimes \epsilon^j \otimes \epsilon^k \otimes \epsilon^l$ and $(\epsilon^i)_{i=1}^{2n}$ is the frame dual to $(e_i)_{i=1}^{2n}$.

Definition 4: We call W the **symplectic Weyl curvature**. The $(2,0)$-type tensor field σ is called the **symplectic Ricci curvature**. A symplectic manifold with a Fedosov connection is called of Ricci-type if $W = 0$ and it is called Ricci-flat if $\sigma = 0$.

Let (M, ω) be a symplectic manifold. We set

$$Q = \{f \text{ is a symplectic basis of } (T_m M, \omega_m) |\, m \in M\}$$

and call it the **symplectic repère bundle**. For any $f = (e_1, \ldots, e_{2n}) \in Q$, we denote by $\pi_Q(f)$ the unique point $m \in M$ such that each vector in f belongs to $T_m M$. The topology on Q is the coarsest one for which π_Q is continuous. It can be seen that $\pi_Q : Q \to M$ is a principal $Sp(2n, \mathbb{R})$-bundle.

Definition 5: A pair (P, Λ) is called a **metaplectic structure** if $\pi_P : P \to M$ is a principal $Mp(2n, \mathbb{R})$-bundle over M and $\Lambda : P \to Q$ is a principal bundle homomorphism such that the following diagram commutes. The horizontal arrows denote the actions of $\widetilde{G}$ and G, respectively.

$$\begin{tikzcd}
P \times \widetilde{G} \arrow[r] \arrow[dd, "\Lambda \times \lambda"] & P \arrow[dd, "\Lambda"] \arrow[dr, "\pi_P"] & \\
& & M \\
Q \times G \arrow[r] & Q \arrow[ur, "\pi_Q"'] &
\end{tikzcd}$$

A **compatible positive almost complex structure** J on a symplectic manifold (M, ω) is any endomorphism $J : TM \to TM$ such that $J^2 = -1_{TM}$ and such that $g(X, Y) = \omega(X, JY)$, $X, Y \in \mathfrak{X}(M)$, is a Riemannian metric. In particular, g is a symmetric tensor field. Note that J is an isometry and a symplectomorphism as well. A compatible positive almost complex structure always exists on a symplectic manifold (M, ω). See, e.g., McDuff, Salamon [56], pp. 63 and 70, for a proof.

Remark: Note that a Kähler manifold can be defined as a symplectic manifold equipped with a Fedosov connection ∇ and a compatible positive almost complex structure J such that $\nabla J = 0$, i.e., J is ∇-flat. Especially, any Kähler manifold is symplectic. The first example of a compact symplectic manifold which does not admit any Kähler structure was given by Thurston [72]. He was inspired by a review note of Libermann [53] who comments a mistake in an article of Guggenheimer [21]. See also the review [28] of the Guggenheimer's article by Hodge.

In the following theorem, a condition for the existence of a metaplectic structure is given.

Theorem 5: Let (M, ω) be a symplectic manifold and J be a compatible positive almost complex structure. Then (M, ω) possesses a metaplectic structure if and only if the second Stiefel-Whitney class $w_2(TM)$ of TM vanishes if and only if the first Chern class $c_1(TM) \in H^2(M, \mathbb{Z})$ of (TM, J) is even.
Proof. See Kostant [36] and Forger, Hess [16], p. 270. □

Remark: An element $a \in H^2(M, \mathbb{Z})$ is called even if there is an element $b \in H^2(M, \mathbb{Z})$ such that $a = 2b$. By a Chern class of (TM, J), we mean the Chern class of the complexification $TM^{\mathbb{C}}$ defined with the help of the compatible positive almost complex structure J. See Milnor, Stasheff [57].

3.1 Habermann's symplectic Dirac and associated second order operator

We introduce the symplectic Dirac operators and the associated second order operator of K. Habermann. Note that there exists a complex version of the metaplectic structure (so-called Mp^c-structure), and also of the mentioned operators. See Robinson, Rawnsley [64] and Cahen, Gutt, La Fuente Gravy and Rawnsley [10]. Let us notice that Mp^c structures exist globally on any symplectic manifold (see [64]). Generalizations of many results of Habermann, Habermann in [26] to the Mp^c-case are straightforward (see [10]).

Definition 6: Let (M^{2n}, ω) be a symplectic manifold admitting a metaplectic structure (P, Λ). The associated bundle $\mathcal{S} = P \times_\sigma S$ is called the **symplectic spinor** or the **Kostant's bundle**. Its smooth sections are called **symplectic spinor fields**.

After introducing the Kostant's bundle, we can set up definitions of the differential operators.

Definition 7: Let ∇ be a symplectic connection on a symplectic manifold (M, ω) admitting a metaplectic structure (P, Λ). Consider the principal connection $TQ \to \mathfrak{sp}(2n, \mathbb{R})$ induced by ∇ and its lift $Z : TP \to \widetilde{\mathfrak{g}}$ to the metaplectic structure. The associated covariant derivative $\nabla^S : \Gamma(\mathcal{S}) \to \Gamma(\mathcal{S} \otimes T^*M)$ on symplectic spinor fields is called the **symplectic spinor covariant derivative**. Let $(U, (e_i)_{i=1}^{2n})$ be a local symplectic frame. The operator $D : \Gamma(\mathcal{S}) \to \Gamma(\mathcal{S})$ defined for any $\phi \in \Gamma(\mathcal{S})$ by

$$D\phi = \sum_{i,j=1}^{2n} \omega^{ij} e_i \cdot \nabla^S_{e_j} \phi$$

is called the **(Habermann's) symplectic spinor Dirac operator**.

Let J be a compatible positive almost complex structure on a symplectic manifold (M, ω). A **local unitary frame** is a local symplectic frame which is orthogonal with respect to the associated Riemann tensor $g(X, Y) = \omega(X, JY)$, $X, Y \in \mathfrak{X}(M)$.

Definition 8: Let J be a compatible positive almost complex structure on a symplectic manifold which admits a metaplectic structure and $(U, (e_i)_{i=1}^{2n})$ be a local unitary frame. The operator $\widetilde{D} : \Gamma(\mathcal{S}) \to \Gamma(\mathcal{S})$ defined for any $\phi \in \Gamma(\mathcal{S})$ by

$$\widetilde{D}\phi = \sum_{i=1}^{2n} (Je_i) \cdot \nabla^S_{e_i} \phi$$

is called the **second symplectic spinor Dirac operator**. The operator $\mathfrak{P} = \imath[\widetilde{D}, D]$ is called the **associated second order operator**.

Remark: The associated second order operator $\mathfrak{P}$ is elliptic in the sense that its principal symbol $\sigma(\mathfrak{P}, \xi) : \mathcal{S} \to \mathcal{S}$ is a bundle isomorphism for any non-zero cotangent vector $\xi \in T^*M$. See Habermann, Habermann [26], p. 68.

For symplectic spinor covariant derivative ∇^S and a chosen compatible positive almost complex structure, one defines the formal adjoint $(\nabla^S)^* : \Gamma(\mathcal{S} \otimes T^*M) \to \Gamma(\mathcal{S})$ of ∇^S. See Habermann, Habermann [26].

Definition 9: The **Bochner-Laplace operator** on symplectic spinors $\Delta^S : \Gamma(\mathcal{S}) \to \Gamma(\mathcal{S})$ is the composition $\Delta^S = (\nabla^S)^* \circ \nabla^S$.

Definition 10: The curvature tensor field R^S on symplectic spinors induced by a Fedosov connection ∇ is defined by

$$R^S(X,Y)\phi = \nabla^S_X \nabla^S_Y \phi - \nabla^S_Y \nabla^S_X \phi - \nabla^S_{[X,Y]}\phi$$

where $X, Y \in \mathfrak{X}(M)$, $\phi \in \Gamma(\mathcal{S})$ and ∇^S is the symplectic spinor derivative.

In the next theorem, a relation of the associated second order operator $\mathfrak{P}$ to the Bochner-Laplace operator Δ^S on symplectic spinors is described. It is derived by K. Habermann, and it is a parallel to the well known Weitzenböck's and Lichnerowicz's formulas for the Laplace operator of the de Rham differentials (Hodge-Laplace) and the Laplace operator of a Levi-Civita connection (Bochner-Laplace); and for the square of the Dirac operator and the Laplace operator of a Lichnerowicz connection on spinors (Lichnerowicz-Laplace), respectively. See, e.g., Friedrich [18] for the latter formula. We present a version of the Habermann's theorem for Kähler manifold. See Habermann, Habermann [26] for more general versions.

Theorem 6: Let (M, ω, J) be a Kähler manifold and $(U, (e_i)_{i=1}^{2n})$ be a local unitary frame. Then for any $\phi \in \Gamma(\mathcal{S})$

$$\mathfrak{P}\phi = \Delta^S \phi + \imath \sum_{i,j=1}^{2n} (Je_i) \cdot e_j \cdot R^S(e_i, e_j)\phi.$$

Proof. See Habermann, Habermann [26]. □

For complex manifolds of complex dimension one[4], Habermann obtains the following consequence of the formula in Theorem 6.

[4]i.e., Riemann surfaces

Theorem 7: If M is a Riemann surface of genus $g \geq 2$, ω is a volume form on M, and (P, Λ) is a metaplectic structure, then the kernel of the associated second order operator is trivial.
Proof. See Habermann [25]. $\square$

Remark: In [24] and [25], Habermann proves that for T^2 ($g = 1$) and the trivial metaplectic structure, the null space for $\mathfrak{P}$ is isomorphic to the Schwartz space $S = S(\mathbb{R})$. In the case of the (trivial) metaplectic structure on the sphere, the kernel of the associated second order operator is rather complicated. See Habermann [25] or Habermann, Habermann [26]. In the case of genus $g = 1$ and non-trivial metaplectic structures, the kernel of $\mathfrak{P}$ is trivial as well. For it, see Habermann [25].

For further results on spectra and null-spaces of the introduced operators, see Brasch, Habermann, Habermann [6], Cahen, La Fuente Gravy, Gutt, Rawnsley [10] and Korman [35]. The key features used are the Weitzenböck-type formula (Theorem 6) and an orthogonal decomposition of the Kostant's bundle. To our knowledge, this decomposition was used firstly by Habermann in this context. It is derived from a $\widetilde{K}$-isomorphism between $L^2(\mathbb{R}^n)$ and the Hilbert orthogonal sum $\bigoplus_{m=0}^{\infty} \mathfrak{H}_m$ of the spaces

$$\mathfrak{H}_m = \bigoplus_{\alpha, |\alpha| \leq m} \mathbb{C}h_\alpha, \ m \in \mathbb{N}_0.$$

Recall that $\widetilde{K}$ denotes the preimage in the metaplectic group of the unitary group $U(n)$ by the covering λ. (See Habermann, Habermann [26], p. 18 for a description of the isomorphism.)

3.2 Quantization by symplectic spinors

For a symplectic manifold (M, ω) and a smooth function f on M, we denote by X_f the vector field ω-dual to df, i.e., such a vector field for which

$$\omega(X_f, Y) = (df)Y$$

for any $Y \in \mathfrak{X}(M)$. It is called the Hamiltonian vector field of f. A vector field is called symplectic if its flow preserves the symplectic form. Any Hamiltonian vector field is symplectic but not vice versa. For a study of these notions, we refer to the monograph McDuff, Salamon [56]. Note that in this formalism, a Poisson bracket of two smooth functions f, g on M is defined by

$$\{f, g\}_P = \omega(X_f, X_g).$$

Let (M, ω) be a symplectic manifold admitting a metaplectic structure. For a symplectic vector field Y, let L_Y denote the Lie derivative on the sections of the Kostant's bundle in direction Y. See Habermann, Klein [27] and Kolář, Michor, Slovák [34].

Definition 11: Let (M, ω) be a symplectic manifold admitting a metaplectic structure. For a smooth function f on M, we define a map $\mathfrak{q}(f) : \Gamma(\mathcal{S}) \to \Gamma(\mathcal{S})$ by

$$\mathfrak{q}(f)\phi = -\imath\hbar L_{X_f}\phi$$

for any $\phi \in \Gamma(\mathcal{S})$. We call $\mathfrak{q} : f \mapsto \mathfrak{q}(f)$ the **Habermann's map**.

Due to the properties of L_X, it is clear that $\mathfrak{q}$ maps into the vector space endomorphisms of $\Gamma(\mathcal{S})$

$$\mathfrak{q} : \mathcal{C}^\infty(M) \to \mathrm{End}(\Gamma(\mathcal{S})).$$

Moreover, Habermann derives the following theorem.

Theorem 8: Let (M,ω) be a symplectic manifold admitting a metaplectic struture. Then for any $f,g \in \mathcal{C}^\infty(M)$, the Habermann's map satisfies

$$[\mathfrak{q}(f),\mathfrak{q}(g)] = \imath\hbar\, \mathfrak{q}(\{f,g\}_P).$$

Proof. See Habermann, Habermann [26]. □

Remark: The Habermann's map $\mathfrak{q}$ satisfies the quantization condition (see Introduction) and thus, it gives an example of a non-deformed quantization. By this we mean that $\mathfrak{q}$ is a morphism of Poisson algebras $(\mathcal{C}^\infty(M), \{,\})$ and $(\mathrm{End}\,(\Gamma(\mathcal{S})), [,])$ up to a multiple. However notice that usually, a quantization is demanded to be a map on smooth functions $\mathcal{C}^\infty(M)$ defined on the phase space M into the space of operators on the vector space $L^2(N)$ of L^2-functions or L^2-sections of a line bundle over N where N denotes the Riemannian manifold of the configuration space. See Souriau [70] and Blau [4] for conditions on quantization maps, their constructions and examples.

4 Author's results in Symplectic spinor geometry

We present results achieved by the author in differential geometry concerning symplectic spinors that we consider important and relevant. We start with a chapter on representational theoretical, or if we wish equivariant, properties of exterior differential forms with values in symplectic spinors.

4.1 Decomposition of tensor products and a Howe-type duality

Let $\mathfrak{g}$ be the Lie algebra of symplectic group $Sp(2n,\mathbb{R})$, $\mathfrak{g}^{\mathbb{C}}$ the complexification of $\mathfrak{g}$, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$, Δ^+ a choice of positive roots, and $\{\varpi_i\}_{i=1}^n$ the set of fundamental weights with respect to these choices. Let us denote the irreducible complex highest weight module with highest weight $\lambda \in \mathfrak{h}^*$ by $L(\lambda)$. For any $\lambda = \sum_{i=1}^n \lambda_i \varpi_i$, we set $L(\lambda_1,\dots,\lambda_n) = L(\lambda)$. For $i = 0,\dots,2n$, we denote by σ^i the tensor product representation of the complexified symplectic Lie algebra $\mathfrak{g}^{\mathbb{C}}$ on $E^i = \bigwedge^i V^* \otimes S$, i.e., $\sigma^i : \mathfrak{g}^{\mathbb{C}} \to \mathrm{End}\,(E^i)$ and $\sigma^i(X)(\alpha \otimes s) = \lambda_*^{\wedge i}(X)\alpha \otimes s + \alpha \otimes \sigma_*(X)s$ for any $X \in \mathfrak{g}^{\mathbb{C}}, \alpha \in \bigwedge^i V^*$ and $s \in S$, where $\lambda_*^{\wedge i}$ denotes the action of $\mathfrak{g}^{\mathbb{C}}$ on $\bigwedge^i V^*$. We consider $E = \bigoplus_{i=0}^{2n} E^i$ equipped with the direct sum representation $\sigma^\bullet(X) = (\sigma^0(X),\dots,\sigma^{2n}(X))$, $X \in \mathfrak{g}^{\mathbb{C}}$. Let us notice that here, σ_* denotes the complex linear extension of the representation $\sigma_* : \mathfrak{g} \to \mathrm{End}(S)$ considered above.

Remark: Note that there is a misprint in Krýsl [46]. Namely, the "action" of $\mathfrak{g}$ on E (denoted by $\mathbb{W}$ there) is prescribed by $X(\alpha \otimes s) = \lambda_*^{\wedge i}(X)\alpha \otimes \sigma_*(X)s$ for $X \in \mathfrak{g}$, $\alpha \in \bigwedge^i V$, $s \in S$, and $i = 0,\dots,2n$. Actually, we meant the standard tensor product representation as given above, i.e., $X(\alpha \otimes s) = \lambda_*^{\wedge i}(X)\alpha \otimes s + \alpha \otimes \sigma_*(X)s$. However, the results in [46] are derived for the correct action $\sigma^\bullet$ defined above.

Definition 12: Let us set $\Xi = \{(i,j_i)|\, i = 0,\dots,n, j_i = 0,\dots,i\} \cup \{(i,j_i)|\, i = n+1,\dots,2n, j_i = 0,\dots,2n-i\}$, $\mathrm{sgn}(+) = 0$, $\mathrm{sgn}(-) = 1$, and

$$E_{\pm}^{ij} = L(\underbrace{\frac{1}{2},\cdots,\frac{1}{2}}_{j},\underbrace{-\frac{1}{2},\cdots,-\frac{1}{2}}_{n-j-1},-1+\frac{1}{2}(-1)^{i+j+\mathrm{sgn}(\pm)})$$

for $i = 0, \dots, n-1$, $j = 0, \dots, i$ and $i = n$, $j = 0, \dots, n-1$. For $i = j = n$, we set $E_+^{nn} = L(\frac{1}{2}, \cdots, \frac{1}{2})$ and $E_-^{nn} = L(\frac{1}{2}, \cdots, \frac{1}{2}, -\frac{5}{2})$. For $i = n+1, \dots, 2n$ and $j = 0, \dots, 2n-i$, we set $E_\pm^{ij} = E_\pm^{(2n-i)j}$. For any $(i,j) \in \mathbb{Z} \times \mathbb{Z} \setminus \Xi$, we define $E_\pm^{ij} = 0$. Finally for any $(i,j) \in \mathbb{Z} \times \mathbb{Z}$, we set $E^{ij} = E_+^{ij} \oplus E_-^{ij}$. For $(i,j) \in \Xi$, the $\mathfrak{g}^{\mathbb{C}}$-modules $E_\pm^{ij}$ are called **higher symplectic spinor modules** and their elements **higher symplectic spinors**.

Theorem 9: The following decomposition into irreducible $\mathfrak{g}^{\mathbb{C}}$-modules holds

$$\bigwedge^{\bullet} V^* \otimes S_\pm = \bigoplus_{(i,j) \in \Xi} E_\pm^{ij}.$$

Proof. Krýsl [46]. □

Remark: The decomposition holds also on the level of minimal and hyperfunction globalizations since the corresponding globalization functors are adjoint functors to the Harish-Chandra forgetful functor. See Vogan [76] and Casselmann [12]. It holds also for smooth Fréchet globalization $\widetilde{G} \to \mathrm{Aut}(S)$. By abuse of notation, we shall denote the tensor product representation of $\widetilde{G}$ on E by $\sigma^\bullet$ as well. The above decomposition holds also when V^* is replaced by V since the symplectic form gives an isomorphism of the appropriate representations of $\mathfrak{g}^{\mathbb{C}}$.

Definition 13: For $i = 0, \dots, 2n$, we denote the uniquely determined equivariant projections of $\bigwedge^i V \otimes S^\pm \to E_\pm^{ij} \subseteq \bigwedge^i V \otimes S_\pm$ by $p_\pm^{ij}$ and the projections $p_+^{ij} + p_-^{ij}$ onto E^{ij} by p^{ij}, $(i,j) \in \mathbb{Z} \times \mathbb{Z}$.

Let us recall a definition of the simple Lie superalgebra $\mathfrak{osp}(1|2)$. It is generated by elements e^+, e^-, h, f^+, f^- satisfying the following relations

$$\begin{aligned} [h, e^\pm] &= \pm e^\pm & [e^+, e^-] &= 2h \\ [h, f^\pm] &= \pm \frac{1}{2} f^\pm & \{f^+, f^-\} &= \frac{1}{2} h \\ [e^\pm, f^\mp] &= -f^\pm & \{f^\pm, f^\pm\} &= \pm \frac{1}{2} e^\pm \end{aligned}$$

where $\{,\}$ denotes the anticommutator, i.e., $\{a,b\} = ab + ba$, $a, b \in \mathfrak{osp}(1|2)$.

We give a $\mathbb{Z}_2$-grading to the vector space $E = \bigwedge^\bullet V \otimes S$ by setting $E_0 = \bigoplus_{i=0}^n \bigwedge^{2i} V \otimes S$, $E_1 = \bigoplus_{i=1}^n \bigwedge^{2i-1} V \otimes S$ and $E = E_0 \oplus E_1$. Further, we choose a symplectic basis $(e_i)_{i=1}^{2n}$ of (V, ω) and denote its dual basis by $(\epsilon^i)_{i=1}^{2n} \subseteq V^*$. The Lie superalgebra $\mathfrak{osp}(1|2)$ has a representation $\rho : \mathfrak{osp}(1|2) \to \mathrm{End}(E)$ on the superspace E given by

$$\rho(f^+)(\alpha \otimes s) = \frac{\imath}{2} \sum_{i=1}^{2n} \epsilon^i \wedge \alpha \otimes e_i \cdot s \quad \text{and} \quad \rho(f^-)(\alpha \otimes s) = \frac{1}{2} \sum_{i=1}^{2n} \omega^{ij} \iota_{e_i} \alpha \otimes e_j \cdot s$$

where $\alpha \in \bigwedge^\bullet V^*$, $s \in S$, and ι_v denotes the contraction by the vector v. Consequently, elements e^+, e^- and h act by

$$\rho(e^\pm) = \pm 2 \{\rho(f^\pm), \rho(f^\pm)\} \quad \text{and} \quad \rho(h) = \frac{1}{2} [\rho(e^+), \rho(e^-)]$$

where $\{\ ,\ \}$ and $[\ ,\ ]$ denote the anticommutator and the commutator on the associative algebra $\mathrm{End}(E)$, respectively.

The following theorem is parallel to the Schur and Weyl dualities for tensor representations of $GL(n,\mathbb{C})$ and $SO(n,\mathbb{C})$, respectively. See Howe [29] where they are treated.

Theorem 10: The following $\mathfrak{g}^{\mathbb{C}} \times \mathfrak{osp}(1|2)$-module isomorphism holds

$$\bigwedge^{\bullet} V^* \otimes S \simeq \bigoplus_{i=0}^{n} (E_+^{ii} \otimes F_i) \oplus \bigoplus_{i=0}^{n} (E_-^{ii} \otimes F_i)$$

where $F_i = \mathbb{C}^{2n-2i+1}$ and $\rho_i : \mathfrak{osp}(1|2) \to \mathrm{End}(F_i)$ is given on a basis $(b_j)_{j=i}^{2n-i}$ of F_i by prescriptions

$$\begin{aligned} &\rho_i(f^+)(b_j) = A(n, i+1, j) b_{j+1} && \rho_i(f^-)(b_j) = b_{j-1} \\ &\rho_i(h) = 2\{\rho_i(f^+), \rho_i(f^-)\} \quad \text{and} && \rho_i(e^\pm) = \pm 2\{\rho_i(f^\pm), \rho_i(f^\pm)\} \end{aligned}$$

where $i = 0, \ldots, n$ and $A(n,i,j) = \frac{(-1)^{i-j}+1}{16}(j-i) + \frac{(-1)^{i-j+1}+1}{16}(i+j-2n-1)$.
Proof. See Krýsl [46]. $\square$

Remark: In the preceding definition, if an index exceeds its allowed range, the object is considered to be zero. Thus, e.g., b_{2n-i+1} or b_{i-2} are zero vectors.

Theorem 11: For $i = 0, \ldots, n$, representations F_i are irreducible.
Proof. See Krýsl [46]. $\square$

Remark: Representations ρ_i in Theorem 10 depend on the choice of a basis, but not their equivalence class. As follows from Theorem 11, the multiplicity of $E_\pm^{ii}$ in the $\mathfrak{g}^{\mathbb{C}}$-module E is $2n - 2i + 1$ for $i = 0, \ldots, n$.

4.2 Differential geometry of higher symplectic spinors

For any symplectic manifold (M, ω) admitting a metaplectic structure (P, Λ), the decomposition from Theorem 9 can be lifted to the associated bundle $\mathcal{E} = P \times_{\sigma^\bullet} E$.

Remark: Since S is a smooth globalization, we may consider E as a representation of the metaplectic group as well.

Definition 14: Let (M, ω) be a symplectic manifold admitting a metaplectic structure (P, Λ). For any $(i,j) \in \mathbb{Z} \times \mathbb{Z}$, we set $\mathcal{E}^{ij} = P \times_{\widetilde{G}} E^{ij}$ and call it the **higher symplectic spinor bundle** and elements of its section spaces the **higher symplectic spinor fields** if $(i,j) \in \Xi$.

We keep denoting the lifts of the projections $\bigwedge^i V^* \otimes S \to E^{ij}$ to $\Gamma(\mathcal{E}^i) \to \Gamma(\mathcal{E}^{ij})$ by p^{ij}, where $\mathcal{E}^i = P \times_{\sigma^i} E^i$.

4.2.1 Curvature, higher curvature and symplectic twistor complexes

For a Fedosov connection ∇ on a symplectic manifold (M,ω) admitting a metaplectic structure, we consider the exterior covariant derivative d^{∇^S} for the induced symplectic spinor derivative ∇^S. See, e.g., Kolář, Michor, Slovák [34] for a general construction of such derivatives.

Theorem 12: Let (M,ω) be a symplectic manifold admitting a metaplectic structure and ∇ be a Fedosov connection. Then for any $(i,j) \in \mathbb{Z}\times\mathbb{Z}$, the restriction of the exterior symplectic spinor derivative satisfies

$$d^{\nabla^S} : \Gamma(\mathcal{E}^{ij}) \to \Gamma(\mathcal{E}^{i+1,j-1}) \oplus \Gamma(\mathcal{E}^{i+1,j}) \oplus \Gamma(\mathcal{E}^{i+1,j+1}).$$

Proof. See Krýsl [38]. □

Remark: In particular, sections of each higher symplectic spinor bundle are mapped into sections of at most three higher symplectic spinor bundles. Note that in the case of orthogonal spinors in pseudo-Riemannian geometry, the target space structure of the exterior covariant derivative is similar. See Slupinski [68].

Let $(e_i)_{i=1}^{2n}$ be a local symplectic frame on (M,ω) and $(\epsilon^i)_{i=1}^{2n}$ be its dual symplectic coframe. Recall that above, we defined the symplectic Ricci and symplectic Weyl curvature tensor fields. Let us denote by σ^S the endomorphism of the symplectic spinor bundle defined for any $\phi \in \mathcal{S}$ by

$$\sigma^S\phi = \frac{\imath}{2}\sum_{i,j,k,l=1}^{2n} \sigma^{ij}{}_{kl}\epsilon^k \wedge \epsilon^l \otimes e_i \cdot e_j \cdot \phi.$$

Similarly we set

$$W^S\phi = \frac{\imath}{2}\sum_{i,j,k,l=1}^{2n} W^{ij}{}_{kl}\epsilon^k \wedge \epsilon^l \otimes e_i \cdot e_j \cdot \phi.$$

Recall that

$$\bigwedge^2 T^*M \otimes \mathcal{S} = \mathcal{E}^{20} \oplus \mathcal{E}^{21} \oplus \mathcal{E}^{22}$$

according to Theorem 9.

In the next theorem, components of R^S in $\mathcal{E}^{20}$, $\mathcal{E}^{21}$ and $\mathcal{E}^{22}$ are found. We notice that

1) we use the summation convention, i.e., if two indices occur which are labeled by the same letter, we sum over it without denoting the sum explicitly and
2) instead of $e_i \cdot e_j \cdot$, we write $e_{ij}\cdot$ and similarly for a higher number of indices.

Theorem 13: Let $n > 1$, (M^{2n},ω) be a symplectic manifold admitting a metaplectic structure and ∇ be a Fedosov connection. Then for any $\phi \in \Gamma(\mathcal{S})$, $\sigma^S\phi \in \Gamma(\mathcal{E}^{20}\oplus\mathcal{E}^{21})$ and $W^S\phi \in \Gamma(\mathcal{E}^{21}\oplus\mathcal{E}^{22})$. Moreover, we have the following projection formulas

$$\begin{aligned}
p^{20}R^S\phi &= \frac{\imath}{2n}\sigma^{ij}\omega_{kl}\epsilon^k \wedge \epsilon^l \otimes e_{ij}\cdot\phi \\
p^{21}R^S\phi &= \frac{\imath}{n+1}\sigma^{ij}\epsilon^k \wedge \epsilon^l \otimes (\omega_{il}e_{kj}\cdot - \frac{1}{2n}\omega_{kl}e_{ij}\cdot)\phi - \frac{\imath}{1-n}W^{ijk}{}_l\epsilon^m \wedge \epsilon^l \otimes e_{mkij}\cdot\phi \\
p^{22}R^S\phi &= \frac{\imath}{2}W^{ij}{}_{kl}\epsilon^k \wedge \epsilon^l \otimes e_{ij}\cdot\phi + \frac{\imath}{1-n}W^{ijk}{}_l\epsilon^m \wedge \epsilon^l \otimes e_{mkij}\cdot\phi.
\end{aligned}$$

Proof. See Krýsl [42]. □

Remark: Note that for $n = 1$, $E^{21} = E^{22} = 0$, so that there is no Weyl component of the curvature tensor of a Fedosov connection in this dimension. The formula for p^{20} holds also for $n = 1$.

Definition 15: For $(i, j), (i + 1, k) \in \Xi$, $a = 0, \dots, n - 1$ and $b = n, \dots, 2n - 1$, let us set

$$D^{ij}_{i+1,k} = p^{i+1,k} d^{\nabla^S}_{|\Gamma(\mathcal{E}^{ij})} : \Gamma(\mathcal{E}^{ij}) \to \Gamma(\mathcal{E}^{i+1,k}), \quad T_a = D^{aa}_{a+1,a+1} \quad \text{and} \quad T_b = D^{b,2n-b}_{b+1,2n-b-1}.$$

The operators T_i, $i = 0, \dots, 2n - 1$, are called the **symplectic twistor operators**.

Let (V, ω) be a symplectic vector space, $(e_i)_{i=1}^{2n}$ be a symplectic basis, $(\epsilon^i)_{i=1}^{2n}$ be a basis of V^* dual to $(e_i)_{i=1}^{2n}$, and $\sigma \in S^2V^*$ be a bilinear form. For $\alpha \in \bigwedge^\bullet V^*$ and $s \in S$, we set

$$\Sigma^\sigma(\alpha \otimes s) = \sum_{i,j=1}^{2n} \sigma^i{}_j \epsilon^j \wedge \alpha \otimes e_i \cdot s$$

and

$$\Theta^\sigma(\alpha \otimes s) = \sum_{i,j=1}^{2n} \alpha \otimes \sigma^{ij} e_i \cdot e_j \cdot s.$$

We keep denoting the corresponding tensors on symplectic spinor bundles by the same symbols. In this case, the the symplectic Ricci curvature tensor field plays the role of the tensor σ.

We use abbreviations

$$E^\pm = \rho(e^\pm) : E \to E \text{ and } F^\pm = \rho(f^\pm) : E \to E.$$

Let (M, ω) be a symplectic manifold which admits a metaplectic structure and ∇ be a Fedosov connection of Ricci-type. For a higher symplectic spinor field $\phi \in \Gamma(\mathcal{E})$, we have (see Krýsl [43]) the following formula

$$R^E \phi = \frac{1}{n+1}(E^+\Theta^\sigma + 2F^+\Sigma^\sigma)\phi.$$

Remark: By the higher curvature, we understand the curvature of ∇^S on higher symplectic spinors, i.e., $R^E = d^{\nabla^S} \circ d^{\nabla^S}$.

The above formula is used for proving the next theorem.

Theorem 14: Let $n > 1$, (M^{2n}, ω) be a symplectic manifold admitting a metaplectic structure and ∇ be a Fedosov connection of Ricci-type. Then

$$0 \longrightarrow \Gamma(\mathcal{E}^{00}) \xrightarrow{T_0} \Gamma(\mathcal{E}^{11}) \xrightarrow{T_1} \cdots \xrightarrow{T_{l-1}} \Gamma(\mathcal{E}^{nn}) \longrightarrow 0 \text{ and}$$

$$0 \longrightarrow \Gamma(\mathcal{E}^{nn}) \xrightarrow{T_n} \Gamma(\mathcal{E}^{n+1,n+1}) \xrightarrow{T_{n+1}} \cdots \xrightarrow{T_{2n-1}} \Gamma(\mathcal{E}^{2n,2n}) \longrightarrow 0$$

are complexes.
Proof. See Krýsl [43]. □

We call the complexes from Theorem 14 the **symplectic twistor complexes**.

Theorem 15: Let $n > 1$, (M^{2n}, ω) be a symplectic manifold admitting a metaplectic structure and ∇ be a Fedosov connection of Ricci-type. Then

$$0 \longrightarrow \Gamma(\mathcal{E}^{00}) \xrightarrow{T_0} \cdots \xrightarrow{T_{n-2}} \Gamma(\mathcal{E}^{n-1,n-1}) \xrightarrow{T_n T_{n-1}} \Gamma(\mathcal{E}^{n+1,n+1}) \xrightarrow{T_{n+1}} \cdots \xrightarrow{T_{2n-1}} \Gamma(\mathcal{E}^{2n,2n}) \longrightarrow 0$$

is a complex.
Proof. See Krýsl [43]. □

Definition 16: Let $(\mathcal{F}^i \to M)_{i \in \mathbb{Z}}$ be a sequence of vector bundles over a smooth manifold M, $D^\bullet = (\Gamma(\mathcal{F}^i), D_i : \Gamma(\mathcal{F}^i) \to \Gamma(\mathcal{F}^{i+1}))_{i \in \mathbb{Z}}$ be a complex of pseudodifferential operators and for each $\xi \in T^*M$, let $\sigma(D)(\xi)^\bullet = (\mathcal{F}_i, \sigma(D_i, \xi) : \mathcal{F}^i \to \mathcal{F}^{i+1})_{i \in \mathbb{Z}}$ be the complex of symbols evaluated in ξ which is associated to the complex $D^\bullet$. We call $D^\bullet$ **elliptic** if $\sigma(D)(\xi)^\bullet$ is an exact sequence in the category of vector bundles for any $\xi \in T^*M \setminus \{0\}$.

Remark: Note that in homological algebra, the above complexes are usually called cochain complexes.

Theorem 16: Let $n > 1$, (M^{2n}, ω) be a symplectic manifold admitting a metaplectic structure and ∇ be a Fedosov connection of Ricci-type. Then the complexes

$$0 \longrightarrow \Gamma(\mathcal{E}^0) \xrightarrow{T_0} \Gamma(\mathcal{E}^1) \xrightarrow{T_1} \cdots \xrightarrow{T_{n-2}} \Gamma(\mathcal{E}^{n-1}) \text{ and}$$

$$\Gamma(\mathcal{E}^n) \xrightarrow{T_n} \Gamma(\mathcal{E}^{n+1}) \xrightarrow{T_{n+1}} \cdots \xrightarrow{T_{2n-1}} \Gamma(\mathcal{E}^{2n}) \longrightarrow 0$$

are elliptic.
Proof. See Krýsl [44]. □

4.2.2 Symplectic spinor Dirac, twistor and Rarita–Schwinger operators

Definition 17: Let (M, ω) be a symplectic manifold admitting a metaplectic structure and ∇ be a Fedosov connection. The operators

$$\mathfrak{D} = F^- \circ D^{00}_{10} : \Gamma(\mathcal{S}) \to \Gamma(\mathcal{S}) \qquad \text{and } \mathfrak{R} = F^- \circ D^{11}_{21} : \Gamma(\mathcal{E}^{11}) \to \Gamma(\mathcal{E}^{11})$$

are called the **symplectic spinor Dirac** and the **symplectic spinor Rarita–Schwinger operator**, respectively.

Remark: $\mathfrak{D}$ is the 1/2 multiple of the Habermann's symplectic spinor Dirac operator.

Let us denote the set of eigenvectors of a vector space endomorphism $G : W \to W$ by eigen(G) and the set of its eigenvalues by *spec*(G). Recall that by an eigenvalue, we mean simply a complex number μ, for which there is a nonzero $w \in W$, such that $Gw = \mu w$. (We do not investigate spectra from the functional analysis point of view.)

Definition 18: A **symplectic Killing spinor field** is any not everywhere zero section $\phi \in \Gamma(\mathcal{S})$ for which there exists $\mu \in \mathbb{C}$ such that

$$\nabla^S_X \phi = \mu X \cdot \phi$$

for each $X \in \mathfrak{X}(M)$. (The dot denotes the symplectic Clifford multiplication.) The set of symplectic Killing spinor fields is denoted by kill. Number μ from the above equation is called the **symplectic Killing spinor number** and its set is denoted by *kill*.

Remark: The equation for a symplectic Killing spinor field can be written also as

$$\nabla^S \phi = -2\mu\imath F^+ \phi.$$

Remark: Note that there is a misprint in the abstract in Krýsl [39]. Namely, we write there that $-\imath l\lambda$ is not a symplectic Killing number instead of $\frac{\imath\lambda}{l}$ is not a symplectic Killing spinor number. In that paper, l denotes the half of the dimension of the corresponding symplectic manifold.

Theorem 17: If (M, ω) is a symplectic manifold admitting a metaplectic structure and ∇ is a Fedosov connection, then

$$\mathrm{kill} = \operatorname{Ker} T_0 \cap \operatorname{Ker} \mathfrak{D}.$$

Proof. See Krýsl [45]. $\square$

Theorem 18: Let (M^{2n}, ω) be a symplectic manifold admitting a metaplectic structure and ∇ be a Fedosov connection with Ricci tensor σ. Let ϕ be a symplectic Killing spinor field to the symplectic Killing spinor number μ. Then in a local symplectic frame $(U, (e_i)_{i=1}^{2n})$, we have

$$\Theta^\sigma \phi = 2\mu^2 n\phi.$$

Proof. See Krýsl [45]. $\square$

As a consequence of this theorem, we have

Theorem 19: Let (M, ω) be a symplectic manifold admitting a metaplectic structure and ∇ be a Ricci-flat Fedosov connection. Then $kill = \{0\}$ and any symplectic Killing spinor field on M is locally covariantly constant.
Proof. See Krýsl [45]. $\square$

Remark: By a locally covariantly constant field ϕ, we mean $\nabla^S \phi = 0$ which implies that ϕ is locally constant if the Kostant's bundle is trivial.

Theorem 20: Let $n > 1$, (M^{2n}, ω) be a symplectic manifold admitting a metaplectic structure and ∇ be a flat Fedosov connection. Then

(1) If $\mu \in spec(\mathfrak{D}) \setminus \frac{\imath n}{2} kill$, then $\frac{n-1}{n}\mu \in spec(\mathfrak{R})$.

(2) If $\phi \in \mathrm{eigen}(\mathfrak{D}) \setminus \mathrm{kill}$, then $T_0\phi \in \mathrm{eigen}(\mathfrak{R})$.

Proof. See Krýsl [39]. $\square$

Remark: For any $\lambda \in \mathbb{C}$, $\lambda kill$ denotes the number set $\{\lambda\alpha,\, \alpha \in kill\}$.

4.3 First order invariant operators in projective contact geometry

Some of the results described above can be modified to get information for contact projective manifolds which are more complicated objects to handle than the symplectic ones. Contact manifolds are models for time-dependent Hamiltonian mechanics. The adjective 'projective' is related to the fact that we want to deal with unparametrized geodesics rather than with the ones with a fixed parametrization. Connections that we consider are partial in the sense that they act on sections of the contact bundle only.

Definition 19: A **contact manifold** is a manifold M together with a corank one subbundle HM (contact bundle) of the tangent bundle TM which is not integrable in the Frobenius sense in any point of the manifold, i.e., for each $m \in M$, there are $\eta_m, \zeta_m \in H_m M$ such that $[\eta_m, \zeta_m] \notin HM$.

Equivalently, HM is a contact bundle if and only if the Levi bracket

$$L(X,Y) = q([X,Y])$$

is non-degenerate. Here $X, Y \in \Gamma(HM)$ and $q : TM \to QM = TM/HM$ denotes the quotient projection onto QM. The Levi bracket induces a tensor field which we denote by the same letter $L : \bigwedge^2 HM \to QM$.

Definition 20: For a contact manifold (M, HM), a partial connection $\nabla : \Gamma(HM) \times \Gamma(HM) \to \Gamma(HM)$ is called a **contact connection** if the associated exterior covariant derivative d^{∇} on $\Gamma(\bigwedge^2 HM)$ preserves the kernel of the Levi form, i.e., $d^{\nabla}_{\zeta}(\operatorname{Ker} L) \subseteq \operatorname{Ker} L$ for any $\zeta \in HM$. The set of contact connections is denoted by $\mathcal{C}_M$. A **contact projective manifold** is a contact manifold (M, HM) together with a set S_M of contact connections for which the following holds. If $\nabla^1, \nabla^2 \in S_M$, there exists a differential one-form $\Upsilon \in \Gamma(HM^*)$ such that for any $X, Y \in \Gamma(HM)$

$$\nabla^1_X Y - \nabla^2_X Y = \Upsilon(X)Y + \Upsilon(Y)X + \Upsilon^{\sharp}(L(X,Y))$$

where $\Upsilon^{\sharp} : QM \to HM$ is a bundle morphism defined by $L(\Upsilon^{\sharp}(\eta), \zeta) = \Upsilon(\zeta)\eta$, $\zeta \in QM$ and $\eta \in HM$. Morphisms between contact projective manifolds (M, HM, S_M) and (N, HN, S_N) are local diffeomorphisms $f : M \to N$ such that $f_*(HM) = HN$, and for any $\nabla \in S_N$, the pull-back connection $f^*\nabla \in S_M$.

Remark: For a contact projective manifold (M, HM, S_M), it is easy to see that the relation $R = S_M \times S_M \subseteq \mathcal{C}_M \times \mathcal{C}_M$ on the set of contact connections $\mathcal{C}_M$ is an equivalence.

Let (V, ω) be a real symplectic vector space of dimension $2n+2$ and $(e_i)_{i=1}^{2n+2}$ be a symplectic basis. The action of the symplectic group G' of (V, ω) on the projectivization of V is transitive and its stabilizer P' is a parabolic subgroup of G'. We denote the preimages of G' and P' by the covering $\lambda' : Mp(2n+2, \mathbb{R}) \to Sp(2n+2, \mathbb{R})$ by $\widetilde{G}'$ and $\widetilde{P}'$, respectively.

Definition 24: A **projective contact Cartan geometry** is a Cartan geometry $(\mathcal{G}', \vartheta)$ whose model is the Klein geometry $G' \to G'/P'$ with G' and P' as introduced above. We say that a Cartan geometry is a **metaplectic projective contact Cartan geometry** if it is modeled on the Klein geometry $\widetilde{G}'/\widetilde{P}'$.

Remark: For Cartan geometries, see Sharpe [67] and Čap, Slovák [11]. In Čap, Slovák [11], a theorem is proved on an equivalence of the category of the so-called regular normal projective contact Cartan geometries and the category of regular normal projective contact manifolds. See Čap, Slovák [11], pp. 277 and 410. See also Fox [17].

The Levi part $\widetilde{G_0}$ of $\widetilde{P}'$ is isomorphic $Mp(2n,\mathbb{R}) \times \mathbb{R}^\times$ with the semisimple part $\widetilde{G_0^{ss}} \simeq \widetilde{G} = Mp(2n,\mathbb{R})$ and the center isomorphic to the multiplicative group $\mathbb{R}^\times$. The Lie algebra $\mathfrak{p}'$ of $\widetilde{P}'$ is graded, $\mathfrak{p}' = (\mathfrak{sp}(2n,\mathbb{R}) \oplus \mathbb{R}) \oplus \mathbb{R}^{2n} \oplus \mathbb{R}$ with $\mathfrak{g}_0 \simeq \mathfrak{sp}(2n,\mathbb{R}) \oplus \mathbb{R}$, $\mathfrak{g}_1 \simeq \mathbb{R}^{2n}$ and $\mathfrak{g}_2 \simeq \mathbb{R}$. We denote the Lie algebra of $\widetilde{G}'$ by $\mathfrak{g}'$ and identify it with the Lie algebra $\mathfrak{sp}(2n+2,\mathbb{R})$. The semi-simple part $\mathfrak{g}_0^{ss}$ of $\mathfrak{g}_0$ is isomorphic $\mathfrak{sp}(2n,\mathbb{R})$. We denote it by $\mathfrak{g}$ in order to be consistent with the preceding sections. The grading of $\mathfrak{g}' = \bigoplus_{i=-2}^{2} \mathfrak{g}_i$, $\mathfrak{g}_{-2} \simeq \mathfrak{g}_2$ and $\mathfrak{g}_{-1} \simeq \mathfrak{g}_1$, can be visualized with respect to the basis $(e_i)_{i=1}^{2n+2}$ by the following block diagonal matrix of type $(1,n,1) \times (1,n,1)$

$$\mathfrak{g} = \left(\begin{array}{c|c|c} \mathfrak{g}_0 & \mathfrak{g}_1 & \mathfrak{g}_2 \\ \hline \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_1 \\ \hline \mathfrak{g}_{-2} & \mathfrak{g}_{-1} & \mathfrak{g}_0 \end{array}\right).$$

The center of the Lie algebra $\mathfrak{g}_0$ is generated by

$$Gr = \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & -1 \end{array}\right)$$

which is usually called the **grading element** because of the property $[Gr, X] = jX$ for each $X \in \mathfrak{g}_j$ and $j = -2, \ldots, 2$.

Let $\kappa : (\mathfrak{g}^{\mathbb{C}})^* \times (\mathfrak{g}^{\mathbb{C}})^* \to \mathbb{C}$ be the dual form to the Killing form of $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sp}(2n,\mathbb{C})$. We choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}^{\mathbb{C}}$ and a set of positive roots obtaining the set of fundamental weights $\{\varpi_i\}_{i=1}^n$ for $\mathfrak{g}^{\mathbb{C}}$. Further, we set $\langle X, Y\rangle = (4n+4)\kappa(X,Y)$, $X, Y \in (\mathfrak{g}^{\mathbb{C}})^*$, and define

$$c^{\mu}_{\lambda\nu} = \frac{1}{2}[\langle\lambda, \lambda+2\delta\rangle + \langle\nu, \nu+2\delta\rangle - \langle\mu, \mu+2\delta\rangle]$$

for any $\lambda, \mu, \nu \in \mathfrak{h}^*$, where δ is the sum of fundamental weights, or equivalently, the half-sum of positive roots. For any $\mu \in \mathfrak{h}^*$, we set

$$A = \{\sum_{i=1}^{n} \lambda_i\varpi_i |\, \lambda_i \in \mathbb{N}_0, i = 1, \ldots, n-1, \lambda_n + 2\lambda_{n-1} + 3 > 0, \lambda_n \in \mathbb{Z} + \frac{1}{2}\} \subseteq \mathfrak{h}^* \text{ and}$$

$$A_\mu = A \cap \{\mu + \nu |\, \nu = \pm\epsilon_i, i = 1, \ldots, n\}$$

where $\epsilon_1 = \varpi_1$, $\epsilon_i = \varpi_i - \varpi_{i-1}, i = 2, \ldots, n$.

Considering $\mathbb{C}^{2n}$ with the defining representation of $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sp}(2n,\mathbb{C})$, i.e., $\mathbb{C}^{2n} = L(\varpi_1)$, we have the following decomposition.

Theorem 21: For any $\mu \in A$, the following decomposition into irreducible $\mathfrak{g}^{\mathbb{C}}$-modules

$$L(\mu) \otimes \mathbb{C}^{2n} = \bigoplus_{\lambda \in A_\mu} L(\lambda)$$

holds.

Proof. See Krýsl [40]. □

Remark: The above decomposition has the same form when we consider the algebra $\mathfrak{sp}(2n,\mathbb{R})$ instead of $\mathfrak{sp}(2n,\mathbb{C})$.

The set $\{L(\lambda) | \lambda \in A\}$ coincides with the set of all infinite dimensional irreducible $\mathfrak{g}^{\mathbb{C}}$-modules with bounded multiplicities, i.e., those irreducible $\mathfrak{sp}(2n,\mathbb{C})$-modules W for which there exists a bound $l \in \mathbb{N}$ such that for any weight ν, $\dim W_\nu \leq l$.[5] See Britten, Hooper, Lemire [8] and Britten, Lemire [9].

In the next four steps, we define $\widetilde{P}$-modules $\mathbf{L}(\lambda, c, \gamma)$ for any $\lambda \in A$, $c \in \mathbb{C}$ and $\gamma \in \mathbb{Z}_2$.

1) Let $\mathbb{S}$ and $\mathbb{S}_+$ be the $\mathfrak{g}^{\mathbb{C}}$-modules of smooth $\widetilde{K}$-finite vectors of the $Mp(2n,\mathbb{R})$-modules $L^2(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)_+$, respectively. Recall that $L^2(\mathbb{R}^n)$ denotes the Segal–Shale–Weil module and $L^2(\mathbb{R}^n)_+$ is the submodule of even functions in $L^2(\mathbb{R}^n)$. For any $\lambda \in A$, there is an irreducible finite dimensional $\mathfrak{g}^{\mathbb{C}}$-module $F(\nu)$ with highest weight $\nu \in \mathfrak{h}^*$ such that $L(\lambda)$ is an irreducible summand in $\mathbb{S}_+ \otimes F(\nu) = \bigoplus_{i=1}^k \mathbb{S}_i$. For it, see Britten, Lemire [9]. Otherwise said, there exists a $j \in \{1,\dots,k\}$ such that $L(\lambda) \simeq \mathbb{S}_j$. The tensor product of the smooth globalization $S = S(\mathbb{R}^n)$ of $\mathbb{S}$ with $F(\nu)$ decomposes into a finite number of irreducible $\widetilde{G}$-submodules in the corresponding way
$$S_+ \otimes F(\nu) = \bigoplus_{i=1}^k S_i$$
i.e., $\mathbb{S}_i$ is the $\mathfrak{g}^{\mathbb{C}}$-module of smooth $\widetilde{K}$-finite vectors in S_i. We set $\mathbf{L}(\lambda) = S_j$, obtaining a $\widetilde{G}$-module.

2) We let the element $\exp(Gr) \in \widetilde{G_0}$ act by the scalar $\exp(c)$ (the conformal weight) on $\mathbf{L}(\lambda)$ and denote the resulting structure by $\mathbf{L}(\lambda, c)$.

3) Let us consider the element $(1,-1) \in Sp(2n,\mathbb{R}) \times \mathbb{R}^\times \subseteq \lambda'(\widetilde{G_0}) \subseteq P$ and the preimage $\Gamma = \lambda'^{-1}((1,-1)) \subseteq \widetilde{G_0} \simeq Mp(2n,\mathbb{R}) \times \mathbb{R}^\times$. Let us suppose that the element in Γ the first component of which is the neutral element $e \in Mp(2n,\mathbb{R})$ acts by $\gamma \in \mathbb{Z}_2$ on $\mathbf{L}(\lambda, c)$.

4) Finally, the preimage $\lambda'^{-1}(G_+) \subseteq \widetilde{P}$ of the unipotent part G_+ of P is supposed to act by the identity on $\mathbf{L}(\lambda, c)$. We denote the resulting admissible $\widetilde{P}$-module by $\mathbf{L}(\lambda, c, \gamma)$. (See Vogan [76] for the admissibility condition.)

For details on notions in the next definition, see Slovák, Souček [71].

Definition 25: Let $\mathfrak{G} = (\mathcal{G} \to M, \vartheta)$ be a Cartan geometry of type (G,H) and $\mathcal{E}, \mathcal{F} \to M$ be vector bundles associated to the principal H-bundle $\mathcal{G} \to M$. We call a vector space homomorphism $D : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{F})$ a **first order invariant differential operator** if there is a bundle homomorphism $\Phi : J^1\mathcal{E} \to \mathcal{F}$ such that $Ds = \Phi(s, \nabla^\vartheta s)$ for any section $s \in \Gamma(\mathcal{E})$, where $J^1\mathcal{E}$ denotes the first jet prolongation of $\mathcal{E} \to M$ and ∇^ϑ is the invariant derivative for $\mathfrak{G}$.

[5] By W_ν we mean the wight space $W_\nu = \{w \in W | H \cdot w = \nu(H)w \text{ for any } H \in \mathfrak{h}\}$.

It is convenient to divide the vector space of first order invariant differential operators by those bundle homomorphisms between $J^1\mathcal{E}$ and $\mathcal{F}$ which act trivially on the tangent space part of $J^1\mathcal{E}$. We call the resulting vector space the space of **first order invariant operators up to the zeroth order** and denote it by $\mathrm{Diff}^1_{\mathfrak{G}}(\mathcal{E}, \mathcal{F})$.

Remark: Between any bundles induced by irreducible bounded multiplicities representations introduced above, there is at most one such an invariant operator up to a multiple and up to the operators of zeroth order. An equivalent condition for its existence is given in the next theorem. The author obtained it at the infinitesimal level when writing his dissertation thesis already. See [37].

Theorem 22: Let $(\mathcal{G} \to M^{2n+1}, \vartheta)$ be a metaplectic contact projective Cartan geometry, (λ, c, γ), $(\mu, d, \gamma') \in A \times \mathbb{C} \times \mathbb{Z}_2$, and $\mathcal{E} = \mathcal{G} \times_{\widetilde{P}} \mathbf{L}(\lambda, c, \gamma)$ and $\mathcal{F} = \mathcal{G} \times_{\widetilde{P}} \mathbf{L}(\mu, d, \gamma')$ be the corresponding vector bundles over M. Then the space

$$\mathrm{Diff}^1_{(\mathcal{G}\to M,\vartheta)}(\mathcal{E}, \mathcal{F}) \simeq \begin{cases} \mathbb{C} & \text{if } \mu \in A_\lambda, c = d - 1 = c^{\mu}_{\lambda\varpi_1} \text{ and } \gamma = \gamma' \\ 0 & \text{in other cases.} \end{cases}$$

Proof. See Krýsl [41]. □

4.4 Hodge theory over C^*-algebras

An additive category is called **dagger** if it is equipped with a contravariant functor $*$ which is the identity on the objects, it is involutive on morphisms, $**F = F$, and it preserves the identity morphisms, i.e., $*\mathrm{Id}_C = \mathrm{Id}_C$ for any object C. No compatibility with the additive structure is demanded. See Brinkmann, Puppe [7]. For a morphism F, we denote $*F$ by F^*. For any additive category $\mathcal{C}$, we denote the category of its complexes by $\mathfrak{K}(\mathcal{C})$. If $\mathcal{C}$ is an additive and dagger category and $d^\bullet = (U^i, d_i)_{i\in\mathbb{Z}} \in \mathfrak{K}(\mathcal{C})$, we set $\Delta_i = d_i^* d_i + d_{i-1} d_{i-1}^*$, $i \in \mathbb{Z}$, and call it the i-th **Laplace operator**.

Definition 26: Let $\mathcal{C}$ be an additive and dagger category. We call a complex $d^\bullet = (U^i, d_i)_{i\in\mathbb{Z}} \in \mathfrak{K}(\mathcal{C})$ of **Hodge-type** if for each $i \in \mathbb{Z}$

$$U^i = \mathrm{Ker}\,\Delta_i \oplus \mathrm{Im}\,d_{i-1} \oplus \mathrm{Im}\,d_i^*.$$

We call $d^\bullet$ **self-adjoint parametrix possessing** if for each i, there exist morphisms $G_i : U^i \to U^i$ and $P_i : U^i \to U^i$ such that $\mathrm{Id}_{U^i} = G_i\Delta_i + P_i$, $\mathrm{Id}_{U^i} = \Delta_i G_i + P_i$, $\Delta_i P_i = 0$ and $P_i = P_i^*$.

Remark: In the preceding definition, we suppose that the images of the chain maps, the images of their adjoints, and the kernels of the Laplacians exist as objects in the additive and dagger category $\mathcal{C}$. The sign $\oplus$ denotes the biproduct in $\mathcal{C}$. See Weibel [79], p. 425.

The first two equations from the definition of a self-adjoint parametrix possessing complex are called the parametrix equations. Morphisms P_i from the above definition are idempotent as can be seen by composing the first equation with P_i from the right and using the equation $\Delta_i P_i = 0$. In particular, they are projections. The operators G_i are called the Green operators.

Definition 27: Let $(A, *_A, |\,|_A)$ be a C^*-algebra and A^+ be the positive cone of A, i.e., the set of all hermitian elements ($*_A a = a$) in A whose spectrum is contained in the non-negative real numbers. A tuple $(U, (,))$ is called a **pre-Hilbert A-module** if U is a right module over the complex associative algebra A, and $(,) : U \times U \to A$ is an A-sesquilinear map such that for all $u, v \in U$, $(u,v) = *_A(v,u)$, $(u,u) \in A^+$, and $(u,u) = 0$ implies $u = 0$. A pre-Hilbert module is called a **Hilbert A-module** if it is complete with respect to the norm $|u| = \sqrt{|(u,u)|_A}$, $u \in U$. A pre-Hilbert A-module morphism between $(U, (,)_U)$ and $(V, (,)_V)$ is any continuous A-linear map $F : U \to V$.

Remark: We consider that $(,)$ is antilinear in the left variable and linear in the right one as it is usual in physics.

An adjoint of a morphism $F : U \to V$ acting between pre-Hilbert modules $(U, (,)_U)$ and $(V, (,)_V)$ is a morphism $F^* : V \to U$ that satisfies the condition $(Fu, v)_V = (u, F^*v)_U$ for any $u \in U$ and $v \in V$. The category of pre-Hilbert and Hilbert C^*-modules and adjointable morphisms is an additive and dagger category. The dagger functor is the adjoint on morphisms. For any C^*-algebra A, we denote the categories of pre-Hilbert A-modules and Hilbert A-modules and adjointable morphisms by PH^*_A and H^*_A, respectively. In both of these cases, the dagger structure is compatible with the additive structure.

To any complex $d^\bullet = (U^i, d_i)_{i \in \mathbb{Z}} \in \mathfrak{K}(PH^*_A)$, the cohomology groups $H^i(d^\bullet) = \mathrm{Ker}\, d_i / \mathrm{Im}\, d_{i-1}$ are assigned which are A-modules and which we consider to be equipped with the canonical quotient topology. They are pre-Hilbert A-modules with respect to the restriction of $(,)_{U_i}$ to $\mathrm{Ker}\, d_i$ if and only if $\mathrm{Im}\, d_{i-1}$ has an A-orthogonal complement in $\mathrm{Ker}\, d_i$.

We have the following

Theorem 23: Let $d^\bullet = (U^i, d_i)_{i \in \mathbb{Z}}$ be a self-adjoint parametrix possessing complex in PH^*_A. Then for any $i \in \mathbb{Z}$

1) $d^\bullet$ is of Hodge-type
2) $H^i(d^\bullet)$ is isomorphic to $\mathrm{Ker}\, \Delta_i$ as a pre-Hilbert A-module
3) $\mathrm{Ker}\, d_i = \mathrm{Ker}\, \Delta_i \oplus \mathrm{Im}\, d_{i-1}$
4) $\mathrm{Ker}\, d^*_i = \mathrm{Ker}\, \Delta_{i+1} \oplus \mathrm{Im}\, d^*_{i+1}$
5) $\mathrm{Im}\, \Delta_i = \mathrm{Im}\, d_{i-1} \oplus \mathrm{Im}\, d^*_i$.

Proof. See Krýsl [50]. □

Remark: If the image of d_{i-1} is not closed, the quotient topology on the cohomology group $H^i(d^\bullet)$ is non-Hausdorff and in particular, it is not in PH^*_A. See, e.g., von Neumann [61] on the relevance of topology for state spaces. See also Krýsl [51] for further references and for a relevance of our topological observation (Theorem 23 item 2) to the basic principles of the so-called Becchi–Rouet–Stora–Tyutin (BRST) quantization.

Theorem 24: Let $d^\bullet = (U^i, d_i)_{i \in \mathbb{Z}}$ be a complex of Hodge-type in H^*_A, then $d^\bullet$ is self-adjoint parametrix possessing.
Proof. See Krýsl [51]. □

Definition 28: Let M be a smooth manifold, A be a C^*-algebra and $\mathcal{F} \to M$ be a Banach bundle with a smooth atlas such that each of its maps targets onto a fixed Hilbert A-module (the typical fiber). If the transition functions of the atlas are Hilbert A-module automorphisms, we call $\mathcal{F} \to M$ an A-Hilbert bundle. We call an A-Hilbert bundle $\mathcal{F} \to M$ **finitely generated projective** if the typical fiber is a finitely generated projective Hilbert A-module.

For further information on analysis on C^*-Hilbert bundles, we refer to Solovyov, Troitsky [69], Troitsky [73] and Schick [65]. In the paper of Troitsky, complexes are treated with an allowance of the so-called 'compact' perturbations.

Theorem 25: Let M be a compact manifold, A be a C^*-algebra and $D^\bullet = (\Gamma(\mathcal{F}^i), D_i)_{i\in\mathbb{Z}}$ be an elliptic complex on finitely generated projective A-Hilbert bundles over M. Let for each $i \in \mathbb{Z}$, the image of Δ_i be closed in $\Gamma(\mathcal{F}^i)$. Then for any $i \in \mathbb{Z}$

1) $D^\bullet$ is of Hodge-type
2) $H^i(D^\bullet)$ is a finitely generated projective Hilbert A-module isomorphic to $\operatorname{Ker}\Delta_i$ as a Hilbert A-module
3) $\operatorname{Ker} D_i = \operatorname{Ker}\triangle_i \oplus \operatorname{Im} D_{i-1}$
4) $\operatorname{Ker} D_i^* = \operatorname{Ker}\triangle_{i+1} \oplus \operatorname{Im} D_{i+1}^*$
5) $\operatorname{Im}\Delta_i = \operatorname{Im} D_{i-1} \oplus \operatorname{Im} D_i^*$.

Proof. See Krýsl [50]. □

Let H be a Hilbert space. Any C^*-subalgebra of the C^*-algebra of compact operators on H is called a **C^*-algebra of compact operators**.

For C^*-algebras of compact operators, we have the following analogue of the Hodge theory for elliptic complexes of operators on finite rank vector bundles over compact manifolds.

Theorem 26: Let M be a compact manifold, K be a C^*-algebra of compact operators and $D^\bullet = (\Gamma(\mathcal{F}^i), D_i)_{i\in\mathbb{Z}}$ be an elliptic complex on finitely generated projective K-Hilbert bundles over M. If $D^\bullet$ is elliptic, then for each $i \in \mathbb{Z}$

1) $D^\bullet$ is of Hodge-type
2) The cohomology group $H^i(D^\bullet)$ is a finitely generated projective Hilbert K-module isomorphic to the Hilbert K-module $\operatorname{Ker}\Delta_i$.
3) $\operatorname{Ker} D_i = \operatorname{Ker}\Delta_i \oplus \operatorname{Im} D_{i-1}$
4) $\operatorname{Ker} D_i^* = \operatorname{Ker}\Delta_{i+1} \oplus \operatorname{Im} D_{i+1}^*$
5) $\operatorname{Im}\Delta_i = \operatorname{Im} D_{i-1} \oplus \operatorname{Im} D_i^*$

Proof. See [51]. □

Remark: In particular, we see that the cohomology groups share properties of the fibers.

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5 Selected author's articles

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