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To Whom It May Concern

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Dr. Jan Zemlicka

Habilitation thesis

Report by Alberto Facchini

The topics of this habilitation thesis are self-small modules, small modules, their relations with steady rings and strongly steady rings, the defect functor of homomorphisms, and the relations between categorical properties and the structure of a ring. The core of the habilitation thesis consists of nine articles. Five of these are joint papers (with Simion Breaz, Tomas Penk and Tamer Kosan). Most of the nine paper are published in journals with a particularly good reputation (Journal of Algebra, Algebra and Representation Theory, Journal of Pure and Applied Algebra,...).

The thesis is interesting and written well. One of the topics of the thesis is that there are some properties of the category of all modules over a ring that can be recognized from the structure of rings, for instance, the property of being a left perfect rings, but other classes of rings defined by natural conditions on their category of modules cannot be described by properties of the ring. For example, recall that an object c of an abelian category with products and coproducts is *compact* if the covariant functor Hom$(c, -)$ commutes with direct sums, i.e., the abelian groups Hom$(c, \bigoplus_\lambda d_\lambda)$ and $\bigoplus_\lambda$ Hom(c, d_λ) are canonically isomorphic for every family of objects d_λ. Compact objects in the category of all modules appear in the mathematical literature under a number of names: small modules, modules of type Σ, dually slender modules, Σ-compact modules, U-compact modules, ... Clearly, any finitely generated module is compact. The systematic research of compact objects in the context of module categories was started by Hyman Bass in the sixties.

A ring is called *right steady* if the class of compact right modules coincides with the class of finitely generated right modules. In this context, Zemlicka obtains very good results in the study of steadyness, in particular for semiartinian rings whose prime factor rings are artinian. In the sixties, Bass remarked that the covariant functor Hom$(M, -)$ commutes with direct sums if and only if the module M is not a union of a countably infinite increasing chain of proper submodules. The class of all finitely generated modules coincides with the class of all compact modules for some important classes of rings, for instance for right noetherian rings and perfect rings. Several constructions of compact modules that are not finitely generated are described in the mathematical literature, but one can classify all such constructions as belonging to essentially three ways only: (1) constructions of rings containing infinitely generated compact right ideals, (2) constructions of a directed system of right ideals whose direct limit is an infinitely generated dually slender module, and (3) constructions of rings over which all injective modules (both finitely generated and infinitely generated) are compact. A general ring-theoretical characterization of rings over which all compact right modules are finitely generated, i. e., of right steady rings, is still not known, except for some classes of rings. Zemlicka focuses, among other things, onto the class of regular semiartinian rings whose primitive factors are artinian. Notice that these notions were previously also studied by P. C. Eklof, K. R. Goodearl and J. Trlifaj, who constructed examples that showed that the classes of both steady and non-steady semiartinian regular rings are non-empty. In a joint paper with P. Ruzicka and J. Trlifaj, Zemlicka shows that a semiartinian abelian regular ring R is steady if and only if no factor of R contains an infinitely generated compact ideal. Zemlicka then generalizes this result to regular semiartinian rings with artinian primitive factors.

More precisely, using the notion of homogeneous ideal and an idea due to Kaplansky on regular rings with artinian primitive factors, Zemlicka shows that the structure of a non-steady regular semiartinian ring whose primitive factors are artinian is not far from the structure of an abelian regular semiartinian ring. Namely, Zemlicka finds enough central idempotents in a suitable factor of every non-steady ring that generate an infinitely generated compact ideal. This allows Zemlicka to generalize the proof of the characterization of steadiness that appears in the paper by P. Ruzicka, J. Trlifaj and J. Zemlicka already mentioned above. Zemlicka proves that a regular semiartinian ring whose primitive factors are artinian is right steady if and only if no factor of the ring contains an infinitely generated compact right ideal.

A last topic of research investigated by Zemlicka concerns modules with the restricted minimum condition. A module M is said to satisfy the *restricted minimum condition* if M/N is an artinian module for every essential submodule N of M. Recall that N is an essential submodule of M if there is no non-zero submodule K of M with $K \cap N = 0$. The class of all modules satisfying the restricted minimum condition is closed under submodules, factors and finite direct sums. A ring R is called a right RM-ring if the right R-module R_R satisfies the restricted minimum condition. RM-domains, that is, the integral domains R with the restricted minimum condition, are those for which R/I is artinian for all non-zero ideals I of R. A noetherian domain has Krull dimension 1 if and only if it is an RM-domain. Zemlicka studies various classes of RM-rings, their properties, and modules with the restricted minimum condition, in particular those which are singular, which is a very natural condition in this setting, because a module A is singular if and only if there exists a module M with an essential submodule N such that $A \cong M/N$. For instance, he shows (joint result with Tamer Kosan) that a commutative ring R is an RM-ring if and only if $R = \mathrm{Soc}(R)$ is Noetherian and every singular module is semiartinian. If M is a module with essential socle, M has the restricted minimum condition if and only if $M = \mathrm{Soc}(M)$ is Artinian.

In conclusion, I am convinced that Jan Zemlicka deserves to receive the habilitation. In my opinion, the results he has obtained until now and the thesis under examination are of very good quality and by far sufficient for success in the habilitation process. It is my opinion that the candidate deserves to become a docent.

Yours faithfully,

Alberto Facchini

Professor Alberto Facchini