Asymptotic behavior of gradient-like systems

(Habilitation Thesis)

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This habilitation thesis is based on the following research articles:

[**B1**] T. Bárta, R. Chill, and E. Fašangová, *Every ordinary differential equation with a strict Lyapunov function is a gradient system*, Monatsh. Math. **166** (2012), 57–72.

[**B2**] T. Bárta, *Convergence to equilibrium of relatively compact solutions to evolution equations*, Electron. J. Differential Equations **2014** (2014), No. 81, 1–9.

[**B3**] T. Bárta and E. Fašangová, *Convergence to equilibrium for solutions of an abstract wave equation with general damping function*, J. Differential Equations **260** (2016), no. 3, 2259–2274.

[**B4**] T. Bárta, *Rate of convergence and Łojasiewicz type estimates*, J. Dyn. Diff. Equat., online first, 16 pages.

[**B5**] T. Bárta, *Decay estimates for solutions of an abstract wave equation with general damping function*, submitted 2016.

All these articles study convergence to equilibrium of bounded solutions to gradient or gradient-like systems based on various generalizations of the Łojasiewicz gradient inequality. In [**B1**] we prove that every finite-dimensional gradient-like system is in fact a gradient system with respect to an appropriate Riemannian metric. This was an initial impulse to obtain new sufficient conditions for convergence to equilibrium of solutions to gradient-like systems that do not satisfy so called angle condition. In [**B1**] an abstract convergence result for ODEs on manifolds is proved and it is applied to second order equations with weak damping. Rate of convergence for abstract finite-dimensional problems and also second order ODEs with weak damping is estimated in [**B4**]. Papers [**B2**], [**B3**], [**B5**] are devoted mainly to infinite-dimensional problems (but they can also be applied to ODEs). Article [**B2**] contains several abstract convergence results. In [**B3**], resp. [**B5**] we show convergence to equilibrium resp. decay estimates for abstract wave equations with weak damping.

The collection of articles is supplemented by an introductory commentary. In Chapter 1 we present the studied problem with all the settings we consider — ordinary differential equations in $\mathbb{R}^n$, ordinary differential equations on manifolds and partial differential equations (evolution equations in Hilbert or Banach spaces). Chapter 2 is devoted to abstract convergence results (and corresponding decay estimates) in finite-dimensional and infinite-dimensional cases. In Chapter 3 we present the results on damped second order equations, both ordinary and partial.

Contents

Chapter 1

Introduction

Before we start with exact definitions and settings in Sections 1.1 – 1.3, let us write a few words without being precise to introduce the topic of this thesis. Let us consider a dynamical system governed by a differential equation (ordinary or partial)

$$\dot{u} + \mathcal{F}(u) = 0.$$

Here u is a function of time t, it has values in a state space X and $\dot{u}$ means time derivative of u. We assume that the system is dissipative in the sense that energy of any nonstationary solution is decreasing, i.e., there exists a strict Lyapunov function to the system. Such systems are called gradient-like. The main problem studied in this thesis is, whether (under what conditions) every bounded solution of such a system has a limit as time goes to infinity.

As a special case we consider so called gradient systems, where $\mathcal{F}$ is the gradient of a potential $\mathcal{E}$ and the equation is then in the form

$$\dot{u} + \nabla\mathcal{E}(u) = 0.$$

Since $-\nabla\mathcal{E}(u)$ is a vector pointing in the direction of the steepest decay of the function $\mathcal{E}$ in point u, we can say that solutions of a gradient system are maximizing loss of energy in every time t or that the system moves in the direction of maximal energy decay.

There are many mathematical models of real-life processes that can be written in the form of a gradient or gradient-like system. Let us mention several examples: second order equation describing an oscilating spring with nonlinear damping

$$\ddot{x} + g(\dot{x}) + kx = 0, \tag{1.1}$$

the heat equation

$$u_t - \Delta u = 0,$$

some semilinear heat equations, e.g. (see [24])

$$u_t - \Delta u + u - u^p = 0,$$

the wave equation with damping (see [24])

$$u_{tt} + \alpha u_t - \Delta u = 0,$$

some semilinear wave equations with nonlinear damping (see [13])

$$u_{tt} + g(u_t) - \Delta u + |u|^{p-1}u = 0, \tag{1.2}$$

all of them with appropriate boundary conditions and on appropriate domains.

In many examples, energy (the Lyapunov function) $\mathcal{E}$ of the system is coercive in the sense that the level sets

$$\{u \in X : \ \mathcal{E}(u) \le K\}$$

are bounded. Then any solution to the gradient-like system is bounded. If we are in a finite-dimensional space (and it is also true for some infinite-dimensional problems) then every bounded solution is relatively compact, i.e. the closure of

$$\{u(t) : \ t \in [0, +\infty)\}$$

is compact. Then the omega-limit set of u is nonempty, i.e. there exists

$$\varphi \in \omega(u) := \{a \in X : \ \exists\, t_n \nearrow +\infty, \ u(t_n) \to a\}.$$

Such φ is typically an equilibrium of the system. We ask (and this is the main task of this thesis), whether it neccessarily holds that

$$\lim_{t\to+\infty} u(t) = \varphi.$$

In general, the answer is negative. A simple counterexample in $\mathbb{R}^2$ is (in polar coordinates)

$$r' = (1-r)^3, \quad \varphi' = 1 - r.$$

Solutions of this system are spirals converging to the unit circle consisting of equilibria. A Lyapunov function to this system is $\mathcal{E}(r,\varphi)=(r-1)^2$. The answer is negative even for gradient systems. There is a famous example called 'mexican hat' by Palis and de Melo [55, page 14] and another one (which looks more difficult but is easier to handle) by Absil, Mahony and Andrews [1]. Further, Poláčik and Rybakowski gave counterexamples in $\mathbb{R}^2$ with any Riemannian metric and also for solutions of semilinear heat equations (see [56]). Jendoubi and Poláčik presented in [49] an example of a bounded solution u to a semilinear wave equation with $\omega(u)$ containing a continuum of functions.

So, an additional condition must be considered to obtain convergence to φ (i.e. $\omega(u)=\{\varphi\}$). It was observed by Łojasiewicz in 1962 (see [53]) that for gradient systems in $\mathbb{R}^n$ a gradient inequality can be such a condition. In particular, if the potential $\mathcal{E}$ of a gradient system satisfies

$$|\mathcal{E}(u)-\mathcal{E}(\varphi)|^{1-\theta}\le C\|\nabla\mathcal{E}(u)\| \tag{1.3}$$

for some $\theta\in(0,\frac{1}{2}]$ and for all u from a neighborhood of $\varphi\in\omega(u)$, then $\lim_{t\to+\infty}u(t)=\varphi$.

In this thesis we present some known generalizations to the Łojasiewicz inequality (conditions that imply $\lim_{t\to+\infty}u(t)=\varphi$ for $\varphi\in\omega(u)$) for gradient-like systems in $\mathbb{R}^n$, in Banach spaces, and on finitedimesional manifolds and we introduce some new generalizations. We also show how these conditions influence the speed of convergence to φ. Further we show that the new conditions/inequalities apply to second order equations with weak damping, i.e. partial differential equations of the type (1.2) or ordinary differential equations (1.1) with $g'(0)=0$ (so the damping is weaker than linear near zero). We show convergence and decay estimates for such equations.

For these results we do not need global (nor local) existence for every initial data, we neither need uniqueness. We only assume that we have one precompact solution $u:[0,+\infty)\to X$ to a gradient-like system and $\varphi\in\omega(u)$, then we show that $\lim_{t\to+\infty}u(t)=\varphi$. In some abstract results we even do not need that there is a differential equation behind.

1.1 Euclidean space setting

Let $M \subset \mathbb{R}^n$ be open and connected and let $\mathcal{F} : M \to \mathbb{R}^n$ be a continuous vector field and consider the following ordinary differential equation

$$\dot{u} + \mathcal{F}(u) = 0. \tag{1.4}$$

Let

$$\mathrm{Cr}(\mathcal{F}) = \{u \in \Omega : \ \mathcal{F}(u) = 0\} \tag{1.5}$$

be the set of stationary points of (1.4). A function $\mathcal{E} : M \to \mathbb{R}$ is called *a strict Lyapunov function* for (1.4) if

$$\langle \mathcal{E}'(u), \mathcal{F}(u) \rangle > 0 \qquad \text{for all } u \in M \setminus \mathrm{Cr}(\mathcal{F}) \tag{1.6}$$

where $\mathcal{E}'$ denotes the derivative of $\mathcal{E}$ and the brackets denote the duality between $(\mathbb{R}^n)'$ and $\mathbb{R}^n$. System (1.4) is called *a gradient-like system* if there exists *a strict Lyapunov function* $\mathcal{E}$ for (1.4). If $u : I \to M$ is a solution to (1.4), then

$$\frac{d}{dt}\mathcal{E}(u(t)) = \mathcal{E}'(u(t))\dot{u}(t) = -\mathcal{E}'(u(t))\mathcal{F}(u(t)) < 0, \tag{1.7}$$

whenever $u(t)$ is not a stationary point of (1.4). So, $\mathcal{E}$ is decreasing along any nonstationary solution.

We say that $\mathcal{E} : M \to \mathbb{R}$ is *a Lyapunov function* for (1.4) if $\mathcal{E}$ is nonincreasing along solutions, i.e. $t \mapsto \mathcal{E}(u(t))$ is nonincreasing for every solution u to (1.4). System (1.4) is called *weakly gradient-like* if there exists a Lyapunov function $\mathcal{E}$ for (1.4) satisfying

$$\text{if } \mathcal{E}(u(\cdot)) \text{ is constant on } [t_0, +\infty), \text{ then } u(\cdot) \text{ is constant on } [t_0, +\infty). \tag{1.8}$$

Important examples of gradient-like systems are gradient systems. Let $\mathcal{E} : M \to \mathbb{R}$ be a continuously differentiable function. The following ordinary differential equation is called *a gradient system*

$$\dot{u} + \nabla\mathcal{E}(u) = 0. \tag{1.9}$$

Of course, every gradient system is gradient-like. In fact, if $\mathcal{F} = \nabla\mathcal{E}$, then

$$\langle \mathcal{E}'(u), \mathcal{F}(u) \rangle = \langle \mathcal{E}'(u), \nabla\mathcal{E}(u) \rangle = \|\nabla\mathcal{E}(u)\|^2 = \|\mathcal{F}(u)\|^2 > 0$$

on $M \setminus \mathrm{Cr}(\mathcal{F})$.

An important notion for studying asymptotic behavior is the omega-limit set of a function $u : \mathbb{R}_+ \to \mathbb{R}^n$

$$\omega(u) := \{\varphi \in \mathbb{R}^n : \ \exists\, t_n \nearrow +\infty \text{ s.t. } \lim_{n\to\infty} \|u(t_n) - \varphi\| = 0\}.$$

Obviously, if $u : \mathbb{R}_+ \to \mathbb{R}^n$ is a bounded solution to (1.4), then $\omega(u)$ is nonempty, so there exists $\varphi \in \omega(u)$. Further, if $\mathcal{F}$ is locally Lipschitz continuous on $M \setminus \mathrm{Cr}(\mathcal{F})$, then $\omega(u) \subset \mathrm{Cr}(\mathcal{F})$. In fact, if $\varphi \in \omega(u) \setminus \mathrm{Cr}(\mathcal{F})$, then for the solution starting at φ we have (1.7) with $u(t)$ replaced by φ, so $\mathcal{E}$ is decreasing along this solution and this is a contradiction with the fact that $\mathcal{E}$ is constant on $\omega(u)$ and $\omega(u)$ is invariant. So, for gradient systems we have $\omega(u) \subset \{\varphi \in M : \ \mathcal{E}'(\varphi) = 0\}$.

Let $\varphi \in \omega(u)$. We are going to show that under additional conditions (gradient inequality) $\lim_{t\to+\infty} u(t) = \varphi$. In particular, for (weakly) gradient-like systems with a coercive Lyapunov function this means that every solution converges to an equilibrium. However, we often only assume that u is a solution to (1.4) and $\varphi \in \omega(u)$ to obtain minimal assumptions for the implication

$$\varphi \in \omega(u) \quad \Rightarrow \quad \lim_{t\to+\infty} u(t) = \varphi.$$

The results on (weakly) gradient-like systems can also be applied to second order equations. As an example let us mention the following ordinary differential equation describing damped oscilations of a spring (assume α, $k > 0$)

$$\ddot{x} + \alpha\dot{x} + kx = 0,$$

which can be rewritten as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} + \begin{pmatrix} -y \\ \alpha y + kx \end{pmatrix} = 0.$$

We can see that $\mathcal{E}(x, y) = kx^2 + y^2$ is a Lyapunov function satisfying condition (1.8) (so we have a weakly gradient-like system) since

$$\frac{d}{dt}\mathcal{E}(x(t), y(t)) = 2kx\dot{x} + 2y\dot{y} = -2\alpha y^2 \le 0.$$

On the other hand, $\mathcal{E}(x, y) = kx^2 + y^2 + \varepsilon xy$, $\varepsilon > 0$ small enough is a strict Lyapunov function (by easy computations), so the system is even gradient-like.

1.2 Manifold setting

Let (M, g) be a differentiable finite-dimensional Riemannian manifold with a Riemannian metric g. We denote by $\langle\cdot,\cdot\rangle$ the duality between tangent and cotangent vectors, $\langle\cdot,\cdot\rangle_{g(u)}$ the scalar product on the tangent space T_uM in point $u \in M$, and $\|\cdot\|_{g(u)}$ is the norm on T_uM generated by the scalar product. Sometimes, we write shortly $\langle\cdot,\cdot\rangle_g$ and $\|\cdot\|_g$. Also, if X, Y are tangent vector fields on M, we write $\langle X, Y\rangle_g$ meaning $\langle X(u), Y(u)\rangle_{g(u)}$ for every $u \in M$ and similarly we write $\|X\|_g$.

Let $\mathcal{F} : M \to TM$ be a continuous tangent vector field on M (where TM is the tangent bundle) and assume that the differential equation

$$\dot{u} + \mathcal{F}(u) = 0 \tag{1.10}$$

is *a gradient-like system*, i.e., that there exists a differentiable function $\mathcal{E} : M \to \mathbb{R}$ such that

$$\langle\mathcal{E}', \mathcal{F}\rangle > 0 \qquad \text{on } M \setminus \mathrm{Cr}(\mathcal{F}) = \{u \in M : \ \mathcal{F}(u) \neq 0\}. \tag{1.11}$$

As above, such function $\mathcal{E}$ is called *a strict Lyapunov function*. Definition of *a Lyapunov function* and *a weakly gradient-like system* is the same as in the Euklidean space.

For a scalar valued differentiable function $\mathcal{E} : M \to \mathbb{R}$ we define its gradient in $u \in M$ as a vector v representing the operator $\mathcal{E}'(u)$, i.e.,

$$\langle v, x\rangle_{g(u)} = \langle\mathcal{E}'(u), x\rangle \qquad \text{for all } x \in T_uM. \tag{1.12}$$

Since v depends on the scalar product $g(u)$, we write $\nabla_{g(u)}\mathcal{E}(u)$ and $\nabla_g\mathcal{E}$ for the corresponding gradient field.

As in Euclidean space, gradient systems

$$\dot{u} + \nabla_{g(u)}\mathcal{E}(u) = 0 \tag{1.13}$$

are important examples of gradient-like systems (with $\mathcal{F} = \nabla_{g(u)}\mathcal{E}(u)$ and $\mathcal{E}$ being a strict Lyapunov function).

1.3 Infinite-dimensional setting

We would like to generalize the concept of gradient-like systems and gradient systems to infinite-dimensional spaces to study convergence to equilibria for some partial differential equations.

Before we introduce the settings let us start with some general notations. If X is a Banach space, we denote by $\|\cdot\|_X$ the norm in X, X' the dual of X and $\langle\cdot,\cdot\rangle_{X',X}$ the duality between X' and X. By $B_X(\varphi, r)$ we denote the closed ball in X with radius r centered in φ. For a function $u : \mathbb{R}_+ \to X$ we denote its omega-limit set in X by $\omega_X(u)$, i.e.,

$$\omega_X(u) := \{\varphi \in \mathbb{R}^n : \ \exists\, t_n \nearrow +\infty \text{ s.t. } \lim_{n\to\infty} \|u(t_n) - \varphi\|_X = 0\}.$$

We say that u has X-precompact range if $\{u(t) : \ t \geq 0\}$ is precompact in X. Obviously, X-precompact range implies that $\omega_X(u) \neq \emptyset$. If X is a Hilbert space, then we denote by $\langle\cdot,\cdot\rangle_X$ the scalar product in X.

Our settings will be as follows. Let $\mathcal{V} \subset \mathcal{H}$ be two Banach spaces, $\mathcal{V}$ continuously and densely embedded in $\mathcal{H}$. Let $M \subset \mathcal{V}$ be nonempty, open and connected and let $\mathcal{F} : M \to \mathcal{H}$ be a continuous map. We consider the following evolution equation

$$\dot{u} + \mathcal{F}(u) = 0. \tag{1.14}$$

We say that a function u is *a solution* to (1.14) if $u \in C(\mathbb{R}_+, \mathcal{V}) \cap C^1(\mathbb{R}_+, \mathcal{H})$ and (1.14) is satisfied (in $\mathcal{H}$) for every $t > 0$.

To define a strict Lyapunov function and gradient-like system we need to give a good sense to the computation

$$\frac{d}{dt}\mathcal{E}(u(t)) = \mathcal{E}'(u(t))\dot{u}(t) = -\mathcal{E}'(u(t))\mathcal{F}(u(t)) < 0.$$

A continuously differentiable function $\mathcal{E} : M \to \mathbb{R}$ is called *a strict Lyapunov function* for (1.14) if

$$\langle \mathcal{E}'(u), \mathcal{F}(u)\rangle > 0 \quad \text{for all } u \in M, \text{ s.t. } \mathcal{F}(u) \in \mathcal{V} \setminus \{0\}. \tag{1.15}$$

If there exists a strict Lyapunov function for (1.14) then (1.14) is called *a gradient-like system*. Definition of *a Lyapunov function* and *a weakly gradient-like system* is the same as in the Euklidean space.

Let us mention that if u is a solution to a weakly gradient-like system then $\mathcal{E}$ is constant on $\omega_{\mathcal{V}}(u)$. Moreover, if we have continuous dependence on initial values, then $\omega_{\mathcal{V}}(u)$ is positively invariant and as a consequence we have $\omega_{\mathcal{V}}(u) \subset \mathrm{Cr}(\mathcal{F})$.

We now define gradient systems. In the literature (see e.g. [24]), by a gradient system is often understood the equation (1.14) with $\mathcal{H} = \mathcal{V}'$ and

$\mathcal{F} = \mathcal{E}'$ for some $\mathcal{E} \in C^1(M)$. However, we follow [23] and define gradient to be a vector representing the linear functional $\mathcal{E}'(u)$ via scalar product. Let $\mathcal{H}$ be a Hilbert spaces and $\mathcal{E} \in C^1(M, \mathbb{R})$. For a fixed $u \in M$ let us assume that $\mathcal{E}'(u)$ extends to a bounded linear functional on $\mathcal{H}$. Then $v \in \mathcal{H}$ is called *gradient of* $\mathcal{E}$ *in* u if $\langle v, x\rangle_{\mathcal{H}} = \langle \mathcal{E}'(u), x\rangle_{\mathcal{H}',\mathcal{H}}$ for every $x \in \mathcal{H}$. We denote the gradient by $\nabla\mathcal{E}(u) = v$.

If $\mathcal{E} \in C^1(M, \mathbb{R})$ is such that $\mathcal{E}'(u)$ extends to a bounded linear functional on $\mathcal{H}$ for every $u \in M$, then the following equation is called a gradient system

$$\dot{u} + \nabla\mathcal{E}(u) = 0. \tag{1.16}$$

A simple example of an infinite-dimensional gradient system is the heat equation

$$u_t - \Delta u = 0$$

on a bounded domain $\Omega \subset \mathbb{R}^N$ with Dirichlet boundary conditions. Taking $\mathcal{V} = H_0^1(\Omega) \cap H^2(\Omega)$, $\mathcal{H} = L^2(\Omega)$ and $\mathcal{E}(u) = \frac{1}{2}\int_\Omega \|\nabla u\|^2$ we have $\mathcal{E}'(u)w = \int_\Omega \nabla u \cdot \nabla w$ and for $u \in \mathcal{V}$ this functional can be extended to $\mathcal{H}$ and represented via scalar product as $\mathcal{E}'(u)w = \langle -\Delta u, w\rangle$. It means that $\nabla\mathcal{E}(u) = -\Delta u = \mathcal{F}(u)$ for all $u \in \mathcal{V}$.

Chapter 2

Abstract convergence results

This chapter is devoted to abstract convergence results. The task is to find conditions (typically formulated in terms of a Lyapunov function $\mathcal{E}$) that imply convergence of u to some $\varphi \in \omega(u)$. We are not so much interested here, which differential equations satisfy these conditions. More about this question (applications of these abstract results) can be found in Chapter 3.

As we mentioned in the Introduction, the first convergence result based on a gradient inequality was formulated by Łojasiewicz [53] for gradient systems in $\mathbb{R}^n$. The convergence result reads as follows and the inequality (LI) is called *the Łojasiewicz gradient inequality.*

Theorem 2.0.1 (Łojasiewicz 1962)**.** *Let $M \subset \mathbb{R}^n$ be a nonempty open set and $\mathcal{E} \in C^1(M)$. Let $u : [0, +\infty) \to M$ be a solution to the gradient system (1.9) and $\varphi \in \omega(u)$. Assume that there exist $\theta \in (0, \frac{1}{2}]$ and $\eta > 0$ such that*

$$|\mathcal{E}(u) - \mathcal{E}(\varphi)|^{1-\theta} \leq C\|\nabla\mathcal{E}(u)\| \quad \textit{for all } u \in B(\varphi, \eta). \tag{LI}$$

Then $\|u(t) - \varphi\| \to 0$.

Although Łojasiewicz's main result was that inequality (LI) holds for any analytic function $\mathcal{E}$ in $\mathbb{R}^n$ and any φ, if we refer to Łojasiewicz's result in this work we always mean the convergence result, i.e. Theorem 2.0.1.

Since 1962, there are many works generalizing this result in many ways in finite-dimensional and also infinite-dimensional spaces. It was applied not only to semilinear heat or wave equations but also to Cahn–Hilliard equation [25], degenerate parabolic equations [28], or integrodifferential equations [63].

2.1 Finite-dimensional case

First generalization of the Łojasiewicz result we would like to mention is the result by Kurdyka [50], who observed that the function $s \mapsto s^{1-\theta}$ in (LI) can be replaced by a more general function $s \mapsto \Theta(s)$ and the convergence result remains valid. The inequality in [50] reads

$$\|\nabla(\Psi \circ \mathcal{E})(u)\| \geq C \quad \text{for all } u \in B(\varphi, \eta) \tag{2.1}$$

for a class of positive increasing functions $\Psi : \mathbb{R}_+ \to \mathbb{R}$. If Ψ is differentiable, then (2.1) can be rewritten as

$$\Theta(|\mathcal{E}(u) - \mathcal{E}(\varphi)|) \leq C\|\nabla\mathcal{E}(u)\| \quad \text{for all } u \in B(\varphi, \eta), \tag{KLI}$$

where $\frac{1}{\Theta} = \Psi'$ and $\mathcal{E}(\cdot)$ is replaced by $\mathcal{E}(\cdot) - \mathcal{E}(\varphi)$. Here, we need to assume that $1/\Theta$ is integrable on $(0, \varepsilon)$ if we want to get convergence to equilibrium. If we take $\Theta(s) = s^{1-\theta}$, then (KLI) becomes (LI). In fact, Kurdyka's main result was that (2.1) holds in $\mathbb{R}^n$ for a much larger class of functions than analytic functions (see [50] or Section 3.3.3 for more details).

Other generalizations go from gradient systems to gradient-like systems. If we consider a gradient-like system with a strict Lyapunov function $\mathcal{E}$ satisfying (KLI) or (LI), then an additional condition is needed to obtain convergence to equilibrium. This additional condition can be so called *angle condition*

$$\langle \mathcal{E}'(u), \mathcal{F}(u) \rangle \geq \alpha \|\mathcal{E}'(u)\| \cdot \|\mathcal{F}(u)\| \tag{AC}$$

for some $\alpha > 0$ and all $u \in M$. This was observed by Absil, Mahony and Andrews [1] and then generalized by Lageman [51] to gradient-like systems on Riemannian manifolds. In the following theorem, inequality (KLI) is hidden in the notion analytic-geometric category.

Theorem 2.1.1 ([51], Theorem 1.2)**.** *Let X be a Lipschitz continuous vector field on an analytic Riemannian manifold (M, g) with an associated Lyapunov function $\mathcal{E}$ satisfying: for every compact $K \subset M$ there exists $\alpha > 0$ such that*

$$\langle \nabla_{g(u)}\mathcal{E}(u), \mathcal{F}(u) \rangle_{g(u)} \geq \alpha \|\mathcal{E}'(u)\| \cdot \|\mathcal{F}(u)\|. \tag{2.2}$$

Assume that $\mathcal{E}$ belongs to an analytic-geometric category. Then the ω-limit set of any integral curve of X contains at most one point.

Neccessity of an additional condition (e.g. (AC)) is one reason why gradient systems are easier to handle than gradient-like systems. In **[B1]** we have shown that every gradient-like system is a gradient system if we change the Riemannian metric appropriately. Especially, every gradient-like system in Euklidean space becomes a gradient system if we deform the geometry of the space.

Theorem 2.1.2 ([**B1**], Theorem 1)**.** *Let M be a differentiable finite-dimensional Riemannian manifold, $\mathcal{F}$ a continuous tangent vector field on M, and let $\mathcal{E} : M \to \mathbb{R}$ be a continuously differentiable, strict Lyapunov function for* (1.10)*. Then there exists a Riemannian metric $\tilde{g}$ on the open set*

$$\tilde{M} := \{u \in M : \mathcal{F}(u) \neq 0\} \subseteq M$$

such that $\nabla_{\tilde{g}}\mathcal{E} = \mathcal{F}$. In particular, the differential equation (1.10) *is a gradient system on the Riemannian manifold $(\tilde{M}, \tilde{g})$.*

Let us call the Riemannian metric $\tilde{g}$ from Theorem 2.1.2 *a gradient Riemannian metric*. Let us mention that $\tilde{g}$ is not uniquely determined.

It seems to be possible to obtain convergence to equilibrium without the angle condition (AC) if we change the Riemannian metric and transform the gradient-like system to a gradient system. But then we need the Kurdyka-Łojasiewicz inequality (KLI) to be satisfied with respect to the new norm ($\tilde{g}$-norm) and we would obtain convergence in $\tilde{g}$-norm, which is not always equivalent to the original norm (g-norm on the tangent bundle TM or the Euclidean norm in $\mathbb{R}^n$) on a neigborhood of stationary points. In fact, the following theorem shows that equivalence of the new and the old norm is connected to the angle condition.

Theorem 2.1.3 ([**B1**], Theorem 2)**.** *The metrics g and $\tilde{g}$ are equivalent on $\tilde{M}$ if and only if $\mathcal{E}'$ and $\mathcal{F}$ satisfy the conditions* (AC) *and*

$$c\|\mathcal{E}'\|_g \leq \|\mathcal{F}\|_g \leq C\|\mathcal{E}'\|_g \tag{2.3}$$

holds with some c, $C > 0$.

As a consequence, if (AC) is not valid on a neighborhood of an equilibrium, then the new norm is not equivalent to the original norm. However, we can still obtain convergence to equilibrium even in the case when the angle condition is not satisfied. Since $\mathcal{F} = \nabla_{\tilde{g}}\mathcal{E}$ and by definition of a gradient $\langle \nabla_{\tilde{g}}\mathcal{E}, X\rangle_{\tilde{g}} = \langle \mathcal{E}', X\rangle$ for any continuous vector field X, we obtain

$$\|\nabla_{\tilde{g}}\mathcal{E}\|_{\tilde{g}} = \frac{1}{\|\mathcal{F}\|_{\tilde{g}}}\|\nabla_{\tilde{g}}\mathcal{E}\|_{\tilde{g}}^2 = \frac{1}{\|\mathcal{F}\|_{\tilde{g}}}\langle \mathcal{E}', \nabla_{\tilde{g}}\mathcal{E}\rangle = \frac{1}{\|\mathcal{F}\|_{\tilde{g}}}\langle \mathcal{E}', \mathcal{F}\rangle .$$

This computation leads us to a new condition (GenLI) under which we obtain convergence in the original norm $\|\cdot\|_g$.

Theorem 2.1.4 ([**B1**], Theorem 3)**.** *Let* (1.10) *be a gradient-like system on a Riemannian manifold* (M, g) *with a strict Lyapunov function* $\mathcal{E}$*. Let* $u : \mathbb{R}_+ \to M$ *be a global solution of* (1.10) *and* $\varphi \in \omega(u)$*. Assume that there exist a neighborhood* U *of* φ *and* $\Theta : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $1/\Theta \in L^1_{loc}([0, +\infty))$*,* $\Theta(s) > 0$ *for* $s > 0$*, and*

$$\Theta(|\mathcal{E}(v) - \mathcal{E}(\varphi)|) \leq \left\langle \mathcal{E}'(v), \frac{\mathcal{F}(v)}{\|\mathcal{F}(v)\|_g} \right\rangle \quad \text{for every } v \in U \cap \tilde{M}. \qquad \text{(GenLI)}$$

Then u *has finite length in* (M, g) *and, in particular,* $\lim_{t\to+\infty} u(t) = \varphi$ *in* (M, g)*.*

Remark 2.1.5. *The theorem remains valid with the same proof if we assume only that* (1.10) *is a weakly gradient-like system and* $\mathcal{E}$ *a Lyapunov function.*

A simple example in $\mathbb{R}^2$, where this result applies and the angle condition does not hold, is ([**B1**], Example 2)

$$\mathcal{F}(u) = \mathcal{F}(u_1, u_2) = (\|u\|^\alpha u_1 - u_2, u_1 + \|u\|^\alpha u_2), \quad \mathcal{E}(u) = \frac{1}{2}(u_1^2 + u_2^2).$$

For more details see Example 2.3.7 below where we also derive decay estimate for this equation. A more interesting application is a second order equation with weak nonlinear damping, which can be found in Chapter 3.

The gradient Riemannian metric $\tilde{g}$ from Theorem 2.1.2 is defined on $\tilde{M} = M \setminus \mathrm{Cr}(\mathcal{F})$. An interesting open question is, whether (under what conditions) it can be defined on the whole of M. It follows from Theorem 2.1.3 that $\tilde{g}$ can be continuously extended to a stationary point φ only if (AC) and (2.3) hold on a neighborhood of φ. So, not every $\tilde{g}$ can be extended continuously to a stationary point. In Example 3 in [**B1**] we have found two gradient Riemannian metrics for a gradient-like system in $\mathbb{R}^2$ such that one of them can be continuously extended to a stationary point φ and the other cannot.

2.2 Infinite-dimensional case

The first generalization of the Łojasiewicz result to infinite-dimensional setting is due to L. Simon [60], who proved that the gradient inequality

$$|\mathcal{E}(u)-\mathcal{E}(\varphi)|^{1-\theta} \leq C\|\nabla\mathcal{E}(u)\|_{L^2} \quad \text{for all } u \in B_{C^{2,\mu}}(\varphi,\eta) \tag{2.4}$$

holds for $\mathcal{E}(u)=\int_\Omega E(x,u,\nabla u)$, where E is analytic in the second and third variable, and used this inequality to show convergence (in C^2-norm) to equilibrium for solutions to the corresponding gradient system

$$\dot{u}+\nabla\mathcal{E}(u)=f$$

and also for solutions to the second order equation

$$\ddot{u}-\dot{u}-\nabla\mathcal{E}(u)=f,$$

which, in fact, becomes a gradient-like system. This fact was observed by Jendoubi who unified the approach to the first and second order problem in [46] and simplified significantly Simon's proof. Jendoubi and Haraux [37] finally came to the gradient inequality in most satisfactory setting

$$|\mathcal{E}(u)-\mathcal{E}(\varphi)|^{1-\theta} \leq C\|\mathcal{E}'(u)\|_{\mathcal{V}'} \quad \text{for all } u \in B_{\mathcal{V}}(\varphi,\eta), \tag{LSI}$$

in their case $\mathcal{V}=H^1_0(\Omega)$.

In [24], Chill, Haraux and Jendoubi proved the following abstract convergence result.

Theorem 2.2.1 ([24], Theorem 1)**.** *Let $u \in C(\mathbb{R}_+,\mathcal{V}) \cap C^1(\mathbb{R}_+,\mathcal{H})$ with $\mathcal{V}$-precompact range and $\varphi \in \omega_{\mathcal{V}}(u)$. Let $\rho>0$, $c>0$ and $\mathcal{E} \in C^2(\mathcal{V},\mathbb{R})$ be such that $t \mapsto \mathcal{E}(u(t))$ is differentiable almost everywhere and*

$$-\frac{d}{dt}\mathcal{E}(u(t)) \geq c\|\mathcal{E}'(u(t))\|_{\mathcal{V}'}\|\dot{u}(t)\|_{\mathcal{H}} \tag{2.5}$$

for almost every $t \in \mathbb{R}_+$. Assume in addition that (1.8) *holds and that $\mathcal{E}$ satisfies the Łojasiewicz–Simon gradient inequality* (LSI)*. Then $\lim_{t\to+\infty}\|u(t)-\varphi\|_{\mathcal{V}}=0$.*

We can see that u is not neccessarily connected to any differential equation, but if it is a solution to the evolution equation (1.14), then (1.14) is a weakly gradient-like system. Condition (2.5) is a kind of angle condition: if $u \in C^1(\mathbb{R}_+, \mathcal{V}')$ is a solution of a gradient-like system (1.14) with a strict Lyapunov function $\mathcal{E}$ satisfying the angle condition

$$\langle \mathcal{E}'(u), \mathcal{F}(u) \rangle_{\mathcal{V}',\mathcal{V}} \geq \alpha \|\mathcal{E}'(u)\|_{\mathcal{V}'} \|\mathcal{F}(u)\|_{\mathcal{H}} \tag{2.6}$$

for every $u \in \mathcal{V}$ s.t. $\mathcal{F}(u) \in \mathcal{V}$, then (2.5) is satisfied whenever $u \in C^1(\mathbb{R}_+, \mathcal{V})$ ([24], Proposition 5). For gradient systems in the sense of [24] (i.e. $\mathcal{H} = \mathcal{V}'$, $\mathcal{F} = \mathcal{E}'$) and also for gradient systems in the sense of our definition the angle condition (2.6) is satisfied automatically.

In **[B2]** we have generalized Theorem 2.2.1 in two ways. First, as was mentioned above, there are important cases where the angle inequality is not satisfied, e.g. second order equations with weak damping. In fact, what is really needed to show convergence is

$$-\frac{d}{dt}\mathcal{E}(u(t)) \geq \Theta(\mathcal{E}(u(t)))\|\dot{u}(t)\|_{\mathcal{H}} \tag{2.7}$$

for some positive function Θ s.t. $1/\Theta$ is integrable at zero or, equivalently,

$$-\frac{d}{dt}\mathcal{E}(u(t)) \geq \|\dot{u}(t)\|_{\mathcal{H}}. \tag{2.8}$$

Clearly, these two conditions are equivalent. In fact, if $\mathcal{E}$ satisfy (2.7), then (2.8) is satisfied with $\mathcal{E}$ replaced by $\tilde{\mathcal{E}} := \Phi_\Theta(\mathcal{E})$ where $\Phi_\Theta(t) = \int_0^t \frac{1}{\Theta(s)} ds$. The second implication is trivial.

So, conditions (2.5) and (LSI) can be replaced by more general condition (2.8). Inequality (2.8) follows from (2.5) and (LSI) by taking $\tilde{\mathcal{E}} = \mathcal{E}^{1-\theta}$:

$$-\frac{d}{dt}\tilde{\mathcal{E}}(u(t)) = -\frac{1}{\mathcal{E}(u(t))^\theta}\frac{d}{dt}\mathcal{E}(u(t)) \geq \frac{\|\mathcal{E}'(u)\|_{V'}}{\mathcal{E}(u(t))^\theta}\|\dot{u}\|_{\mathcal{H}} \geq c\|\dot{u}\|_{\mathcal{H}}.$$

Obviously, in Theorem 2.2.1 the Łojasiewicz–Simon inequality can be replaced by Kurdyka–Łojasiewicz–Simon inequality

$$\Theta(\mathcal{E}(u) - \mathcal{E}(\varphi)) \leq \|\mathcal{E}'(u)\|_{\mathcal{V}'} \tag{KLSI}$$

for any function $\Theta > 0$ with $\frac{1}{\Theta} \in L^1_{loc}([0, +\infty))$ (see **[B2]**, Theorem 3.2).

Second generalization is, that it is not neccessary to assume (2.8) on a whole halfline $t \geq t_0$. It is enough to assume that $\mathcal{E}$ is nonincreasing along solutions (e.g. $\frac{d}{dt}\mathcal{E}(u(t)) < 0$) for $t \geq t_0$ and the stronger estimate (2.8) holds whenever $u(t)$ is in a small neighborhood of φ. This assumption is easier to verify, e.g. for second order equations (ordinary or partial). We obtain the following result.

Theorem 2.2.2 (**[B2]**, Theorem 2.4)**.** *Let* $u \in C(\mathbb{R}_+, \mathcal{V}) \cap C^1(\mathbb{R}_+, \mathcal{H})$ *with* $\mathcal{V}$*-precompact range and* $\varphi \in \omega_{\mathcal{V}}(u)$*. Let* $\rho > 0$ *and* $\mathcal{E} \in C(\mathcal{V}, \mathbb{R})$ *be such that* $t \mapsto \mathcal{E}(u(t))$ *is nonincreasing on* $\mathbb{R}_+$ *and* (2.8) *holds for almost every* $t \in \mathbb{R}_+$ *such that* $u(t) \in B := B_{\mathcal{V}}(\varphi, \rho)$*.*

Then $\lim_{t\to+\infty} \|u(t) - \varphi\|_{\mathcal{V}} = 0$*.*

It was observed in [24] that the space $\mathcal{H}$ can be replaced by any larger space with a weaker norm. In other words, it is sufficient to verify the decay condition (2.8) or (2.5) for a very weak norm $\|\cdot\|_{\mathcal{H}}$ and the convergence is then obtained in the stronger norm $\|\cdot\|_{\mathcal{V}}$ by a compactness argument (due to precompact range). On the other hand, estimates of the speed of convergence are lost while using the compactness argument, so one can obtain these estimates in the norm of $\mathcal{H}$ only.

In applications to second order equations we work on a product space and we often need to control the first coordinate only. The following generalization of Theorem 2.2.2 is appropriate for such situations. If we need convergence of one coordinate, it is enough to assume (2.9) instead of (2.8). Unfortunately, the condition (2.9) does not imply any estimates of the speed of convergence.

Theorem 2.2.3 (**[B2]**, Theorem 2.6)**.** *Let* $u = (u_1, u_2)$ *be such that* $u_1 \in C(\mathbb{R}_+, V_1) \cap C^1(\mathbb{R}_+, H_1)$*,* $u_2 \in C(\mathbb{R}_+, V_2) \cap C^1(\mathbb{R}_+, H_2)$*,* $H_1 \hookrightarrow V_1$ *and let* $(u_1(\cdot), u_2(\cdot))$ *have a precompact range in* $V_1 \times V_2$*. Let* $\varphi \in \omega_{V_1}(u_1)$*,* $\rho > 0$ *and* $\mathcal{E} \in C(V_1 \times V_2, \mathbb{R})$ *be such that* $t \mapsto \mathcal{E}(u(t))$ *is nonincreasing on* $\mathbb{R}_+$ *and*

$$-\frac{d}{dt}\mathcal{E}(u(t)) \geq \|\dot{u}_1(t)\|_{H_1} \tag{2.9}$$

for almost every $t \in \mathbb{R}_+$ *such that* $u_1(t) \in B := B_{V_1}(\varphi, \rho)$*.*

Then $\lim_{t\to+\infty} \|u_1(t) - \varphi\|_{V_1} = 0$*.*

Remark 2.2.4. *Theorems 2.2.2 and 2.2.3 imply convergence to equilibrium for precompact solutions of weakly gradient-like systems with Lyapunov functions satisfying* (2.8)*, resp.* (2.9)*.*

Gradient systems.

As in finite-dimensional case, the situation is easier for gradient systems (no angle condition is needed). Therefore, it may be of interest that similarly to the finite-dimensional case, any gradient-like system can be transformed to a gradient system by taking an appropriate Riemannian metric.

It was mentioned above that gradient depends on the scalar product. Let $\mathcal{H}$ be a Hilbert space and let g be any scalar product on $\mathcal{H}$, we define *gradient of $\mathcal{E}$ in u with respect to g* as a vector $v \in \mathcal{H}$ (if such v exists) satisfying $\langle v, w\rangle_g = \mathcal{E}'(u)w$ for all $w \in \mathcal{H}$. Then we write $v = \nabla_g \mathcal{E}(u)$.

We define *a Riemannian metric on $\mathcal{H}$* to be a continuous mapping $r : \mathcal{V} \to \operatorname{Inner}\mathcal{H}$ where $\operatorname{Inner}\mathcal{H}$ is the space of all bounded scalar products on $\mathcal{H}$ equipped with strong convergence topology, i.e., $g_n \to g$ in $\operatorname{Inner}\mathcal{H}$ if $\langle u, v\rangle_{g_n} \to \langle u, v\rangle_g$ for every $u, v \in \mathcal{H}$. Then the equation

$$\dot{u} + \nabla_{r(u)}\mathcal{E}(u) = 0 \tag{2.10}$$

is called *a gradient system with respect to the Riemannian metric r*.

Let us call a Riemannian metric g *a gradient metric* for a gradient-like system (1.14) with a strict Lyapunov function $\mathcal{E}$, if $\mathcal{F}(v) = \nabla_{g(v)}\mathcal{E}(v)$ for all $v \in M$.

Theorem 2.2.5. *Let* (1.14) *be a gradient-like system with a strict Lyapunov function $\mathcal{E}$ such that $\nabla\mathcal{E}$ is continuous on M and $\langle \nabla\mathcal{E}, \mathcal{F}\rangle_{\mathcal{H}} > 0$ on $\tilde{M} = M \setminus \operatorname{Cr}(\mathcal{F})$. Then there exists a gradient metric g for* (1.14) *on $\tilde{M}$.*

Since this result was not published we present a proof here.

Proof. For any $w \in \tilde{M}$ we have $\langle \nabla\mathcal{E}(w), \mathcal{F}(w)\rangle_{\mathcal{H}} = \langle \mathcal{E}'(w), \mathcal{F}(w)\rangle > 0$ and therefore $0 \neq \mathcal{F}(w) \notin \ker \mathcal{E}'(w)$. As a consequence, for every $w \in \tilde{M}$ we have

$$\mathcal{H} = \ker \mathcal{E}'(w) \oplus \langle \mathcal{F}(w)\rangle. \tag{2.11}$$

For every $u \in \mathcal{H}$ and $w \in \tilde{M}$ let us define

$$u_{w0} := u - \frac{\langle \mathcal{E}'(w), u\rangle}{\langle \mathcal{E}'(w), \mathcal{F}(w)\rangle}\mathcal{F}(w) \text{ and } u_{w1} := \frac{\langle \mathcal{E}'(w), u\rangle}{\langle \mathcal{E}'(w), \mathcal{F}(w)\rangle}\mathcal{F}(w). \tag{2.12}$$

Then $u_{w0} \in \ker \mathcal{E}'(w)$, $u_{w1} \in \langle \mathcal{F}(w)\rangle$ and the mappings $w \mapsto u_{w0}$, $w \mapsto u_{w1}$ are continuous from $\mathcal{V}$ to $\mathcal{H}$ (the denominators are continuous since $\nabla\mathcal{E} : M \to \mathcal{H}$ is continuous).

Now we choose an arbitrary Riemannian metric r on $\mathcal{H}$. Starting from this metric, we define a new metric g on $\tilde{M}$ by setting

$$\begin{aligned} \langle u, v\rangle_{g(w)} &:= \langle u_{w0}, v_{w0}\rangle_{r(w)} + \frac{1}{\langle \mathcal{E}'(w), \mathcal{F}(w)\rangle}\langle \mathcal{E}'(w), u\rangle\langle \mathcal{E}'(w), v\rangle \\ &= \langle u_{w0}, v_{w0}\rangle_{r(w)} + \frac{1}{\langle \mathcal{E}'(w), \mathcal{F}(w)\rangle}\langle \mathcal{E}'(w), u_{w1}\rangle\langle \mathcal{E}'(w), v_{w1}\rangle. \end{aligned} \tag{2.13}$$

Clearly, $g(w)$ is a sesquilinear form on $\mathcal{H}$ and it is positive difinite due to $\langle \mathcal{E}', \mathcal{F} \rangle > 0$. Continuity of g follows from continuity of the mappings $w \mapsto u_{w0}$, $w \mapsto u_{w1}$, continuity of r, $\nabla\mathcal{E}$ and $\mathcal{F}$ and the fact (an easy 3ε argument) that $\langle u_n, v_n \rangle_{r(w_n)} \to \langle u, v \rangle_{r(w)}$ whenever $w_n \to w$ in $\mathcal{V}$, $v_n \to v$ in $\mathcal{H}$ and $u_n \to u$ in $\mathcal{H}$.

By definition of the metric g and by definition of the gradient $\nabla_g \mathcal{E}$, we have for every $v \in \mathcal{H}$, $w \in \tilde{M}$

$$\langle \mathcal{F}(w), v \rangle_{g(w)} = 0 + \langle \mathcal{E}'(w), v \rangle = \langle \nabla_{g(w)} \mathcal{E}(w), v \rangle_{g(w)},$$

so g is a gradient metric on $\tilde{M}$. □

For more about infinite-dimensional gradient systems see Chill and Fašangová [23] and an existence result by Boussandel [17]. These works assume the Riemannian metric r to be continuous in a stronger sense, in particular $r : W \to \operatorname{Inner} \mathcal{H}$ where W satisfying $\mathcal{V} \hookrightarrow W \hookrightarrow \mathcal{H}$ is a natural domain of the Lyapunov function $\mathcal{E}$ (i.e. W is the domain of the closure of $\mathcal{E}'$). It is not clear, whether one can find a gradient Riemannian metric g continuous in this sense for any gradient-like system. The gradient metric found in Theorem 2.2.5 is typically not continuous with respect to a weaker norm on $\mathcal{V}$.

2.3 Decay estimates

This section is devoted to decay estimates, i.e. estimates of the speed of convergence to equilibrium for a given solution (or for a given function to its limit). Such estimates usually follow from the proofs of convergence results based on gradient inequalities. The original convergence result by Łojasiewicz is accompanied by the following decay estimates proved by Haraux and Jendoubi in 2001.

Theorem 2.3.1 ([38], Theorem 2.2)**.** *Let u be a bounded solution to a gradient system with $\mathcal{E}$ satisfying the Łojasiewicz gradient inequality* (LI) *with some $\theta \in (0, \frac{1}{2}]$. Then there exists $\varphi \in M$ such that for $t \to +\infty$ we have*

$$\|u(t) - \varphi\| = \begin{cases} O(e^{-ct}) & \text{if } \theta = \frac{1}{2}, \\ O(t^{-\theta/(1-2\theta)}) & \text{if } \theta < \frac{1}{2}. \end{cases}$$

This result remains valid also for gradient systems in the infinite-dimensional setting. Chill and Fiorenza [22] proved decay estimates for an infinite-dimensional gradient system with $\mathcal{E}$ satisfying the Kurdyka–Łojasiewicz–Simon inequality (KLI). They formulated the result for a semilinear parabolic equation but in fact they proved the following abstract result.

Theorem 2.3.2 ([22], Theorem 2.1). *Let $\mathcal{E} \in C^2(\mathcal{V})$ satisfy the Kurdyka–Łojasiewicz–Simon gradient inequality* (KLI)*, u be a solution to the gradient system* (1.16) *with $\{u(t) : t \geq 1\}$ being relatively compact in $\mathcal{V}$. Then there exists $\varphi \in \omega_{\mathcal{V}}(u)$ and $t_0 > 0$ such that*

$$|\mathcal{E}(u(t)) - \mathcal{E}(\varphi)| = O(\psi^{-1}(t - t_0)), \tag{2.14}$$

$$\|u(t) - \varphi\|_{\mathcal{H}} = O(\Phi(\psi^{-1}(t - t_0))) \tag{2.15}$$

as $t \to +\infty$, where ψ is a primitive function to $-1/\Theta^2$, ψ^{-1} the inverse function to ψ and Φ a primitive function to $1/\Theta$.

An abstract result, which can be applied to gradient-like systems satisfying an angle inequality can be found in [24].

Theorem 2.3.3 ([24], Theorem 2). *If the assumptions of Theorem 2.2.1 hold and in addition*

$$-\frac{d}{dt}\mathcal{E}(u(t)) \geq \beta \|\mathcal{E}'(u(t))\|^2_{\mathcal{V}'} \quad \text{for a.e. } t \geq 0. \tag{2.16}$$

Then, for $t \to +\infty$ we have

$$\|u(t) - \varphi\|_{\mathcal{H}} = \begin{cases} O(e^{-ct}) & \text{if } \theta = \frac{1}{2}, \\ O(t^{-\theta/(1-2\theta)}) & \text{if } \theta < \frac{1}{2}. \end{cases}$$

This theorem can be easily modified for $\mathcal{E}$ satisfying the Kurdyka–Łojasiewicz–Simon inequality to obtain the estimates (2.14), (2.15) with the constants c, β from (2.5), (2.16) appearing somewhere. Estimates (2.14), (2.15) were also proved for finite-dimensional gradient and gradient-like systems by Begout, Bolte and Jendoubi in [10] (see Theorems 3.5, 3.7 therein) with condition (2.16) replaced by

$$\|\nabla \mathcal{E}(u)\| \leq \beta \|\mathcal{F}(u)\|. \tag{2.17}$$

This inequality and (2.16) are in fact comparability conditions that states the relation between $\|\mathcal{E}'\|$ and $\|\mathcal{F}\|$ (compare to (2.3)). Of course, some kind of comparability condition is needed, since if we change the size of $\mathcal{F}$ and keep the direction, then the solutions have the same trajectories but their speed is different. The previous theorems show that gradient-like systems satisfying (AC), (2.3) have the same speed of convergence as the corresponding gradient system. We can see below (Example 2.3.7) that if the angle condition is not valid, then the decay estimates become worse.

We generalize the finite-dimensional result to Riemannian manifolds and replace the Kurdyka–Łojasiewicz inequality (KLI) by the inequality (GenLI) introduced in Theorem 2.1.4. We assume that the relation between $\|\nabla\mathcal{E}\|$ and $\|\mathcal{F}\|$ is represented by a function α (see (2.18)), which may be arbitrary and appears also in the obtained decay estimates. Let us remark that this result can be applied to a second order equation with weak damping with $\alpha(s) = \Theta(s)h(\Theta(s))$ as we can see in the next chapter.

Theorem 2.3.4 (**[B4]**, Theorem 1)**.** *Let* (1.10) *be a gradient-like system on* (M, g) *with a strict Lyapunov function* $\mathcal{E}$*,* $u : [0, +\infty) \to M$ *be a solution to* (1.10) *and* $\varphi \in \omega(u)$*. Let* $\mathcal{E}$ *and* $\mathcal{F}$ *satisfy* (GenLI) *with a function* $\Theta : [0, 1) \to \mathbb{R}_+$ *such that* $\frac{1}{\Theta} \in L^1_{loc}([0, 1))$ *and* $\Theta(s) > 0$ *for* $s > 0$*. Then* u *has finite length in* (M, g) *and, in particular,* $\lim_{t\to+\infty} u(t) = \varphi$*. Moreover, if* $\alpha : (0, 1) \to (0, +\infty)$ *is nondecreasing and satisfies*

$$\alpha(\mathcal{E}(u(t)) - \mathcal{E}(\varphi)) \le \|\mathcal{F}(u(t))\| \quad \textit{for all } t \textit{ large enough}, \tag{2.18}$$

then there exists $t_0 > 0$ *such that*

$$|\mathcal{E}(u(t)) - \mathcal{E}(\varphi)| \le \psi^{-1}(t - t_0) \quad \textit{for all } t > t_0, \tag{2.19}$$

$$\|u(t) - \varphi\| \le \Phi(\psi^{-1}(t - t_0)) \quad \textit{for all } t > t_0, \tag{2.20}$$

where

$$\Phi(t) := \int_0^t \frac{1}{\Theta(s)} ds \quad \textit{and} \quad \psi(t) := \int_t^{1/2} \frac{1}{\Theta(s)\alpha(s)} ds$$

and $\|a - b\|$ *is for* a*,* $b \in M$ *the* g*-distance of* a *and* b*.*

Remark 2.3.5. *The theorem remains valid if* (1.10) *is a weakly gradient-like system and* $\mathcal{E}$ *is a Lyapunov function satisfying* (1.8)*.*

We can see that (2.17) together with (KLI) imply that one can take $\alpha = \Theta$ and the definition of ψ becomes the same as in Theorem 2.3.2 (and the same as in [10]).

All the decay estimates above are based on the inequality

$$\|u(t) - \varphi\| \leq \int_t^{+\infty} \|\dot{u}(s)\| ds, \tag{2.21}$$

so they estimate the length of the remaining trajectory which can be much longer than the distance $\|u(t) - \varphi\|$ (typically, for second order equations with weak damping, it is much longer). Often, there are direct estimates of $\|u - \varphi\|$ in the form

$$\|u - \varphi\| \leq \gamma(\mathcal{E}(u) - \mathcal{E}(\varphi)). \tag{2.22}$$

For example, $\mathcal{E}(u) = \sum_{i=1}^n |u_i|^p$, where $u = (u_1, u_2, \ldots, u_n)$, is a Lyapunov function for many ordinary differential equations and it satisfies (2.22) with $\gamma(s) = cs^{1/p}$. A similar estimate holds e.g. for

$$u_t - \Delta u + |u|^{p-1} u = 0$$

with

$$\mathcal{E}(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{1}{p+1} \int_\Omega |u|^{p+1}$$

(see Example 3.2.8 or [13]). Inequality (2.22) gives in many cases better decay estimates than (2.21).

Corollary 2.3.6 ([**B4**], Corollary 3)**.** *Let the assumptions of Theorem 2.3.4 hold and let* $\gamma : (0,1) \to (0, +\infty)$ *be a nondecreasing function such that* (2.22) *holds for all* u *in a neigborhood of* φ*. Then there exist* $t_0 > 0$ *such that*

$$\|u(t) - \varphi\| \leq \gamma(\psi^{-1}(t - t_0)) \quad \text{for all } t > t_0.$$

Application to some second order equations with weak damping can be found in the next chapter. Now we present a simple example where Corollary 2.3.6 yields optimal decay estimates and Theorem 2.3.4 does not. This means that estimating the distance from the equilibrium by the length of the remaining trajectory may be the only estimate which is not sharp in the whole process.

Example 2.3.7 ([**B4**], Example 4)**.** *Let $M \subseteq \mathbb{R}^2$ be the open unit disc equipped with the Euclidean metric. Let $\alpha \geq 0$, and let $F(u) = F(u_1, u_2) = (\|u\|^\alpha u_1 - u_2, u_1 + \|u\|^\alpha u_2)$ and $\mathcal{E}(u) = \frac{1}{2}(u_1^2 + u_2^2)$. Then one can show that $\mathcal{E}$ satisfies the Łojasiewicz inequality* (LI) *near the origin for $\theta = \frac{1}{2}$ but the angle condition* (AC) *does not hold (unless $\alpha = 0$). On the other hand,* (GenLI) *holds with $\Theta(s) = \frac{1}{\sqrt{2}} s^{1-\theta}$, $\theta = \frac{1-\alpha}{2}$ and* (2.18) *holds with $\alpha(s) = 2\sqrt{s}$. Then Theorem 2.3.4 yields*

$$\|u(t)\| \leq C(t - t_0)^{\frac{1}{\alpha} - 1}.$$

If the angle condition (AC) *were satisfied, the decay of u would be exponential due to the Łojasiewicz exponent equal to $\frac{1}{2}$. Since the* (AC) *condition is not satisfied, the decay is only polynomial. Further, we can apply Corollary 2.3.6 with $\gamma(s) = \sqrt{2s}$ and obtain*

$$\|u(t)\| \leq C(t - t_0)^{-\frac{1}{\alpha}}.$$

This is a better result since $-\frac{1}{\alpha} < \frac{1}{\alpha} - 1$. Moreover, transformation to polar coordinates show that this result is optimal. More details can be found in [**B4**].

Chapter 3

Second order problems

The main application of the abstract results from Chapter 2 are second order equations with damping. In this chapter, we assume that $V \hookrightarrow H \hookrightarrow V'$ are Hilbert spaces with embeddings being dense and continuous (we identify $H = H'$). We consider problems in the form

$$\ddot{u} + g(\dot{u}) + E'(u) = 0, \tag{3.1}$$

where $E : M \subset V \to \mathbb{R}$ and $g : H \to V'$ are two given functions, $E \in C^2(M)$. In the following, we write $\|\cdot\|$ instead of $\|\cdot\|_H$, $\|\cdot\|_*$ instead of $\|\cdot\|_{V'}$ and $\langle\cdot,\cdot\rangle$, $\langle\cdot,\cdot\rangle_*$ instead of $\langle\cdot,\cdot\rangle_H$, $\langle\cdot,\cdot\rangle_{V'}$ respectively. In a special case $V = H = V' = \mathbb{R}^N$ we have an ordinary differential equation of second order.

A typical example of such equation (and probably the most studied case) is a nonlinear wave equation with damping

$$u_{tt} + g(u_t) - \Delta u + f(x,u) = 0, \qquad t \geq 0,\ x \in \Omega \subset \mathbb{R}^n, \tag{3.2}$$

where $g : \mathbb{R} \to \mathbb{R}$, $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous, $H = L^2(\Omega)$, $V = H_0^1(\Omega)$. This equation can be rewritten as (3.1) with

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u(x)|^2 dx + \int_\Omega F(x,u(x))dx, \qquad F(x,s) = \int_0^s f(x,r)dr. \tag{3.3}$$

Equation (3.1) can be written as a first order problem

$$\dot{U} + \mathcal{F}(U) = 0, \tag{3.4}$$

where

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad \mathcal{F}(U) = \begin{pmatrix} -v \\ g(v) + E'(u) \end{pmatrix},$$

$\mathcal{F} : \mathcal{M} \subset \mathcal{V} \to \mathcal{H}$, $\mathcal{H} = H \times V'$, $\mathcal{V} = V \times H$, $\mathcal{M} = M \times H$. Our key assumption is

$$\langle g(v), v\rangle_{V',V} > 0 \quad \text{for all } v \in V,\ v \neq 0$$

which means that g has a damping effect. We denote by $S := \{u \in M : E'(u) = 0\}$ the set of stationary points of (3.1). Then $\mathrm{Cr}\,\mathcal{F} = S \times \{0\}$. We define

$$E_1(u, v) = \frac{1}{2}\|v\|^2 + E(u),$$

then for any solution to (3.4) such that $\dot{u} = v \in L^1_{loc}(\mathbb{R}_+, V)$ we have

$$\begin{aligned}\frac{d}{dt}E_1(U(t)) &= \langle E'(u), v\rangle_{V',V} + \langle v, \ddot{v}\rangle_{V,V'} \\ &= \langle E'(u), v\rangle_{V',V} - \langle v, g(v)\rangle_{V,V'} - \langle v, E'(u)\rangle_{V,V'} \\ &= -\langle v, g(v)\rangle_{V,V'} \\ &\leq 0.\end{aligned}$$

We can see that E_1 is a Lyapunov function (not neccessarily strict). Moreover, condition (1.8) holds: if $E_1(u(\cdot))$ is constant, then $\langle v, g(v)\rangle_{V,V'} = 0$, hence $v = 0$ and $u(\cdot)$, $v(\cdot)$ are constant. So, (3.1) is a weakly gradient-like system, but E_1 typically does not satisfy (GenLI), (2.9), so Theorems 2.1.4, 2.2.3 cannot be applied. The proofs of the convergence results in this chapter rest in finding another Lyapunov function $\mathcal{E}$ which satisfies these additional conditions. This $\mathcal{E}$ is usually in the form $\Phi(\tilde{E}_1)$, where $\tilde{E}_1$ is a small perturbation of E_1 and $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing function.

In Section 3.1 we consider the finite-dimensional case, i.e., $V = H = V' = \mathbb{R}^N$ and we obtain an ordinary differential equation with E' identified with ∇E. In Section 3.2 we consider the general infinite-dimensional case. The results of both sections (convergence results and also results on decay estimates) usually assume that we have a solution with a precompact range and that E satisfies the Kurdyka–Łojasiewicz–Simon inequality. Therefore we discuss in Section 3.3 the problem of precompact range of solutions and list some known sufficient conditions on E to satisfy the Kurdyka–Łojasiewicz–Simon inequality.

Finally, let us mention that we mostly focus on the damping term. Our intuition tells us that the smaller is the damping the slower is convergence to an equilibrium, and if the damping is too small for v small, then it may happen that it is not strong enough to stabilize the system and the system (e.g. an oscilating spring) may keep oscilating. The results of this chapter confirm this intuition, they show that convergence occurs if $g(v)$ is large enough for v near zero. We try to find as small lower bound for g as possible, in particular we focus on functions g with

$$\lim_{u\to 0}\frac{\|g(u)\|}{\|u\|}=0.$$

The most typical example is $g(u)=\tilde{g}(|u|)u$ (e.g. $g(u)=|u|^{\alpha}u$, $\alpha\in(0,1)$), which means that the damping force acts in the opposite direction to velocity and its size depends on the size of velocity only. But we also allow more general cases, e.g. damping depending on the direction of velocity (which corresponds to motions in an anisotropic environment) or damping depending not only on $\dot{u}$ but also on u (which corresponds to inhomogeneous environment).

We do not present any nonconvergence results in the next sections. The following nonconvergence example is due to Haraux. He proved in [34, Proposition 5.1.2] that the equation

$$\ddot{u}+(\dot{u})^2+f(u)=0$$

with $f=0$ on $[a,b]$, $f<0$ on $(-\infty,a)$ and $f>0$ on $(b,+\infty)$ has bounded solutions with $[a,b]\subset\omega(u)$. However, sharpness of the convergence results and optimality of decay estimates for weakly damped equations remain an open question.

3.1 Finite-dimensional case

In this section we consider the finite-dimensional case, i.e., second order ordinary differential equation with damping

$$\ddot{u}+g(\dot{u})+\nabla E(u)=0. \tag{3.5}$$

If $g\in C^1(\mathbb{R}^n,\mathbb{R}^n)$, $E\in C^2(\Omega)$, $\Omega\subset\mathbb{R}^n$, then there exists a unique maximal solution to (3.5) for any initial data $(u(0),v(0))\in\Omega\times\mathbb{R}^n$ and the solution depends continuously on the initial data. As was mention above, this implies $\omega(U)\subset\mathrm{Cr}(\mathcal{F})$, hence $\psi=0$ and $\nabla E(\varphi)=0$ for every $(\varphi,\psi)\in\omega(U)$. Then a standard argument yields $\lim_{t\to+\infty}\dot{u}(t)=0$ whenever $\omega(U)$ is nonempty (in particular for any bounded solution U).

Probably the first convergence result based on the Łojasiewicz inequality (so without assuming a special structure of E) in finite-dimensional setting is due to Haraux and Jendoubi [36].

Theorem 3.1.1 ([36], Theorem 1.1)**.** *Assume that $E : \mathbb{R}^n \to \mathbb{R}$ is analytic and $g : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz continuous and satisfies for all $v \in \mathbb{R}^N$*

$$\langle g(v), v\rangle \geq c\|v\|^2, \tag{3.6}$$

$$\|g(v)\| \leq C\|v\|, \tag{3.7}$$

with $0 < c \leq C < +\infty$ independent of v. Let $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$ be a solution to (3.5)*. Then there exists $\varphi \in S$ such that*

$$\lim_{t\to+\infty} \|\dot{u}(t)\| + \|u(t) - \varphi\| = 0.$$

In this theorem, analyticity of E can be replaced by $E \in C^2(\mathbb{R}^N)$ and E satisfies the Łojasiewicz inequality (LI). The damping function g is nonlinear, but it is larger than a linear function (in fact, it satisfies $c\|v\| \leq g(v) \leq C\|v\|$, so it is not a weakly damped equation. A similar result can be found in Alvarez et al. 2002 (see [3, Theorem 4.1]) for the equation

$$\ddot{u} + (\gamma + \beta\nabla^2 E(u))\dot{u} + \nabla E(u) = 0$$

with $E : \mathbb{R}^n \to \mathbb{R}$ analytic (hence satisfying (LI)).

In 2015, Bégout, Bolte and Jendoubi [10] consider linear damping $g(\dot{u}) = \gamma\dot{u}$ and more general potential F satisfying the Kurdyka–Łojasiewicz inequality with a function Θ s.t. $0 < \Theta(s) \leq c\sqrt{s}$ for $s \in (0, \tau)$ and $\frac{1}{\Theta} \in L^1_{loc}([0, 1))$. Under these assumptions, they obtain convergence to an equilibrium and also decay estimates

$$\|u(t) - \varphi\| = O(\Phi(\psi^{-1}(t - t_0))),$$

i.e. the same decay estimates as Chill and Fiorenza in [22] for first order problems. In fact, second order problems with linear damping are gradient-like systems satisfying conditions (AC), (2.3), so they have the same decay as corresponding gradient systems (first order problems).

Concerning weak damping, in 2008 Chergui [19] proved convergence and decay estimates for analytic E and $g(\dot{u}) = \|\dot{u}\|^\alpha\dot{u}$.

Theorem 3.1.2 ([19], Theorems 1.2, 1.3)**.** *Assume that $E \in C^2(\mathbb{R}^n)$ and that there exists $\theta \in [0, \frac{1}{2})$ such that for every $\psi \in S$ there exists $\eta > 0$ such that* (LI) *holds. Let $\alpha \in [0, \frac{\theta}{1-\theta})$ and let $u \in W^{2,\infty}(\mathbb{R}_+, \mathbb{R}^n)$ be a solution to* (3.5) *with $g(\dot{u}) = \|\dot{u}\|^\alpha\dot{u}$. Then there exists $\varphi \in S$ such that*

$$\lim_{t\to+\infty} \|\dot{u}(t)\| + \|u(t) - \varphi\| = 0.$$

Moreover, there exists $C > 0$ such that for all $t \in \mathbb{R}_+$

$$\|\dot{u}(t)\| + \|u(t) - \varphi\| \le C t^{-\frac{\theta-(1-\theta)\alpha}{1-2\theta+(1-\theta)\alpha}}. \tag{3.8}$$

We have generalized this convergence result in [**B1**, Theorem 4] (without decay estimates). We assumed $E \in C^2(\mathbb{R}^N)$ satisfying the Kurdyka–Łojasiewicz inequality (KLI) instead of the Łojasiewicz inequality and the damping function being dependent on u (not only $\dot{u}$) and having non-power-like growth, in particular we assumed

$$\ddot{u} + g(u, \dot{u}) + \nabla E(u) = 0, \tag{3.9}$$

with $g \in C^2(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ satisfying for all $u, v \in \mathbb{R}^n$

$$\begin{aligned} \langle g(u,v), v\rangle &\ge h(\|v\|)\,\|v\|^2, \\ \|g(u,v)\| &\le C h(\|v\|)\,\|v\|, \\ \|\nabla g(u,v)\| &\le C\, h(\|v\|), \end{aligned} \tag{3.10}$$

where $C \ge 0$ is a constant and $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a nonnegative, concave, nondecreasing function, $g(s) > 0$ for $s > 0$. In fact, $g(u, v) = \|v\|^\alpha v$ satisfies (3.10) with $h(s) = s^\alpha$ and condition $\alpha \in [0, \frac{\theta}{1-\theta})$ in Theorem 3.1.2 corresponds to the condition (3.11) below.

Theorem 3.1.3 ([**B1**], Theorem 4)**.** *Let $u \in W^{2,\infty}(\mathbb{R}_+; \mathbb{R}^n)$ be a global solution of* (3.9) *with g satisfying* (3.10)*. Assume that there exist $\varphi \in \omega(u)$, $\eta > 0$ and a nonnegative, concave, nondecreasing function $\Theta : \mathbb{R}_+ \to \mathbb{R}_+$ such that* (KLI) *holds. Assume that $\Theta(s) \le c\sqrt{s}$ for some $c > 0$ and all $s \ge 0$ small enough and that*

$$s \mapsto 1/\Theta(s) h(\Theta(s)) \in L^1_{loc}([0, +\infty)). \tag{3.11}$$

Then u has finite length and, in particular,

$$\lim_{t\to+\infty} \|\dot{u}(t)\| + \|u(t) - \varphi\| = 0.$$

The proof of this theorem follows the idea from [19] but we show that this problem fits in the abstract framework described in the previous chapter. In fact, we have rewritten (3.9) as a first order equation on the product space and we have shown that the function

$$\mathcal{E}(u, v) = \frac{1}{2}\|v\|^2 + E(u) + \varepsilon \langle g(u, \nabla E(u)), v\rangle$$

is a strict Lyapunov function for the equation and that the generalized Łojasiewicz inequality (GenLI) is satisfied with Θ replaced by the function $\tilde{\Theta}(s) = \Theta(s)h(\Theta(s))$.

In Theorem 3.1.3 we still assume that $\|g(u,v)\|$ lies between two multiples of a function $h(\|v\|)\|v\|$ for all $v \in \mathbb{R}^n$ (similarly to Haraux and Jendoubi [36]). In **[B3]** we further investigated which assumpions on the damping function g are important and which can be relaxed (primarily for a damped wave equation) and we ended up with different estimates from above and from below.

- **(e)** Let $E \in C^2(\mathbb{R}^n, \mathbb{R})$ satisfy (KLI) with a function $\Theta : [0,1) \to [0,+\infty)$ which is nondecreasing, sublinear ($\Theta(s+t) \leq \Theta(s)+\Theta(t)$), and it holds that $\frac{1}{\Theta} \in L^1_{loc}([0,1))$ and $0 < \Theta(s) \leq c\sqrt{s}$ for all $s \in (0,1)$ and some $c > 0$.
- **(g)** The function $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and there exists $\tau > 0$ such that
  - **(g1)** there exists $C_2 > 0$ such that $\|g(w,z)\| \leq C_2\|z\|$ for all $\|z\| < \tau$, $w \in \mathbb{R}^n$,
  - **(g2)** there exists $C_3 > 0$ such that $C_3\|z\| \leq \|g(w,z)\|$ for all $\|z\| \geq \tau$, $w \in \mathbb{R}^n$,
  - **(g3)** there exists $C_5 > 0$ such that $\langle g(w,z), z\rangle \geq C_5\|g(w,z)\|\|z\|$ for all $w, z \in \mathbb{R}^n$.
- **(h)** For τ from condition (g) there exists a function $h : [0,+\infty) \to [0,+\infty)$, which is concave and nondecreasing on $[0,\tau]$ and satisfies
  - **(h1)** $\|g(w,z)\| \geq h(\|z\|)\|z\|$ for all $\|z\| < \tau$, $w \in \mathbb{R}^n$,
  - **(h2)** the function $s \mapsto \frac{1}{\Theta(s)h(\Theta(s))}$ belongs to $L^1((0,\tau))$,
  - **(h3)** the function $\psi : s \mapsto sh(\sqrt{s})$ is convex on $[0,\tau^2]$.

In fact, condition (g2) can be weakened to '$g(w,z) \neq 0$ for all $z \neq 0$' which together with (g3) yields $\langle g(w,z), z\rangle > 0$ for all $z \neq 0$. This last condition implies that $\dot{u} \to 0$ for any bounded solution, so we do not need any further assumptions on $g(w,z)$ for $\|z\| > \tau$.

Theorem 3.1.4 ([**B3**], Theorem 6.1). *Let the functions E and g satisfy* (e), (g), *and* (h). *Let $u \in W^{1,\infty}((0,+\infty),\mathbb{R}^N) \cap W^{2,1}_{loc}([0,+\infty),\mathbb{R}^n)$ be a solution to* (3.9) *and let $\varphi \in \omega(u)$. Then $\lim_{t\to+\infty}(\|u(t)-\varphi\| + \|\dot{u}(t)\|) = 0$.*

Under the same assumptions we obtained decay estimates in [**B4**, Theorem 6]

Theorem 3.1.5 ([**B4**], Theorem 6). *Let the functions E and g satisfy* (e), (g), *and* (h). *Let $u \in W^{1,\infty}((0,+\infty),\mathbb{R}^n) \cap W^{2,1}_{loc}([0,+\infty),\mathbb{R}^n)$ be a solution to* (3.9) *and $\varphi \in \omega(u)$. Then there exists $t_0 > 0$ such that*

$$|\dot{u}(t)| + |u(t)-\varphi| + \int_t^{+\infty} |\dot{u}(s)|ds \le \Phi(\psi^{-1}(t-t_0)), \tag{3.12}$$

holds for all $t > t_0$, some C_1, $C_2 > 0$ and

$$\Phi(t) = C_1 \int_0^t \frac{1}{\Theta(s)h(\Theta(s))}ds, \quad \psi(t) = C_2 \int_t^{\frac{1}{2}} \frac{1}{\Theta^2(s)h(\Theta(s))}ds. \tag{3.13}$$

The proofs of these theorems are again based on the abstract results from the previous chapter. This time, we work with the energy function

$$\mathcal{E}(u,v) := \Phi(H(u,v)), \quad H(u,v) = \frac{1}{2}\|v\|^2 + E(u) + \varepsilon h(\|v\|)\langle \nabla E(u), v\rangle$$

and show that

$$-\frac{d}{dt}\mathcal{E}(u(t),v(t)) \ge \|\dot{u}(t)\|,$$

which is the condition (2.8) from Theorem 2.2.2. In fact, this inequality can be rewritten as

$$\tilde{\Theta}(H(u,v) - H(\varphi,0)) \le \left\langle H(u,v), \frac{\mathcal{F}(u,v)}{\|\mathcal{F}_1(u,v)\|}\right\rangle \tag{3.14}$$

which is almost (GenLI), in the denominator we have instead of $\mathcal{F}(u,v)$ its first coordinate $\mathcal{F}_1(u,v) = -v$ (we denote $\tilde{\Theta} = \Theta h(\Theta)$ as above). Further, we have shown that condition (2.18) from Theorem 2.3.4 is valid with $\alpha(s) = \Theta(s)$. To obtain decay estimates, we cannot apply Theorem 2.3.4 directly since $\|\mathcal{F}_1\|$ in (3.14) can be much smaller than $\|\mathcal{F}\|$. However, it holds ([**B4**, Lemma 8]) that

$$\int_{t_0}^t \frac{\|\mathcal{F}_1(u(s),v(s))\|}{\alpha(H(u(s),v(s)))}ds \ge L(t-t_0),$$

which is enough to do the remaining step in the decay estimates.

Let us remark that if $g(u,v) = \|v\|^\alpha v$ (then $h(s) = s^\alpha$) and $\Theta(s) = s^{1-\theta}$ then Theorem 3.1.5 yields the same decay estimate as Theorem 3.1.2. Further, we can obtain more delicate estimates in the logarithmic scale. By [**B5**, Example 5.3, Lemmas 6.5, 6.6], if $\Theta(s) = s^{1-\theta}$ and

$$h(s) = s^\alpha \ln^{r_1}(1/s)(\ln^{r_2}\ln(1/s))\dots(\ln^{r_k}\ln\dots\ln(1/s)), \tag{3.15}$$

on a neighborhood of zero, then for $a < \frac{\theta}{1-\theta}$ we obtain

$$\|u(t)-\varphi\| \le Ct^{-\frac{\theta-a(1-\theta)}{1-2\theta+a(1-\theta)}}\ln^{-q_1}(t)\ln^{-q_2}(\ln(t))\dots\ln^{-q_k}(\ln\dots\ln(t)), \tag{3.16}$$

where $q_k = \frac{r_k(1-\theta)}{1-2\theta+a(1-\theta)}$ and for $a = \frac{\theta}{1-\theta}$, $r_1 = \dots = r_{j-1} = 1$, $r_j > 1$, r_{j+1}, $\dots$, $r_k \in \mathbb{R}$ we obtain

$$\|u(t)-\varphi\| \le C\ln^{1-r_j}(\ln\dots\ln(t))\ln^{-r_{j+1}}(\ln\dots\ln(t))\dots\ln^{-r_k}(\ln\dots\ln(t)).$$

If we have a direct estimate of $\|u\|$ by the potential E due to special structure of E (see (3.17) and Example 3.1.7 below), we get better decay estimates by the following theorem.

Theorem 3.1.6 ([**B4**], Theorem 7)**.** *Let the assumptions of Theorem 3.1.5 hold and let*

$$\Theta(|E(u)-E(\varphi)|) \ge c\|\nabla E\| \quad \textit{for all } u \in B(\varphi,\eta).$$

Moreover, let γ be a nondecreasing function satisfying

$$\gamma(|E(u)-E(\varphi)|) \ge \|u-\varphi\| \quad \textit{for all } u \in N(\varphi). \tag{3.17}$$

Then

$$|\dot u(t)| \le C\sqrt{\psi^{-1}(t-t_0)} \quad \textit{and} \quad |u(t)-\varphi| \le C\gamma(\psi^{-1}(t-t_0)), \tag{3.18}$$

holds for all $t > t_0$ and some $C > 0$, ψ defined as in (3.13)*.*

Example 3.1.7. *If we consider*

$$\ddot u + g(u,v) + p\|u\|^{p-2}u = 0, \qquad p \ge 2$$

which corresponds to $E(u) = \|u\|^p$, then we have

$$\Theta(E(u)) \le c\|\nabla E(u)\| \le C\Theta(E(u))$$

with $\Theta(s) = s^{1-\frac{1}{p}}$*, i.e., the Łojasiewicz inequality holds with* $\theta = \frac{1}{p}$*. Moreover,* (3.17) *holds with* $\gamma(s) = s^{\frac{1}{p}}$*. Then, for* $\|g(u,v)\| \geq h(\|v\|)\|v\| = \|v\|^{\alpha+1}$ *we have*

$$\|u(t)\| \leq Ct^{-\frac{\theta}{1-2\theta+\alpha(1-\theta)}}, \tag{3.19}$$

which is better estimate than (3.8)*. For the function* h *given by* (3.15) *we obtain*

$$\|u(t) - \varphi\| \leq Ct^{-\frac{\theta}{1-2\theta+\alpha(1-\theta)}} \ln^{-q_1}(t) \ln^{-q_2}(\ln(t)) \dots \ln^{-q_k}(\ln \dots \ln(t))???, \tag{3.20}$$

where q_i *are as in* (3.16)*.*

In [35], Haraux has found optimal decay estimates for the damping function $g(\dot{u}) = |\dot{u}|^\beta \dot{u}$ and $E(u) = \|u\|^\alpha$. It follows that the estimate (3.19) is optimal only for $p = 2$. Optimality of the estimates for general E is open.

3.2 Infinite-dimensional case

In this section we study convergence of solutions to the second order evolution equation (3.1) in infinite-dimensional spaces. In contrast to finite-dimensional case, well-posedness of the problem is not always easy to proof. This is not a crucial problem for the following results since they consider only one trajectory. On the other hand, well-posedness implies that $\omega_{\mathcal{V}}(U) \subset \mathrm{Cr}(\mathcal{F}) = S \times \{0\}$, so it reduces the set of points where E should satisfy the Kurdyka–Łojasiewicz–Simon inequality and also some methods of verifying the precompact range condition need well-posedness. However, in many examples one can show $\omega_{\mathcal{V}}(U) \subset \mathrm{Cr}(\mathcal{F})$ or $\dot{u}(t) \to 0$ ad hoc. The following criterion applies in a large class of problems (in fact, the $*$-norm in (3.21) can be replaced by any weaker norm).

Theorem 3.2.1 ([**B2**], Theorem 2.8)**.** *Let* $g \in C(V \times H, V')$*,* $E \in C^2(V)$ *and assume that there exists a nondecreasing function* $h : (0, +\infty) \to (0, +\infty)$ *such that*

$$\langle g(u,v), v \rangle_{V',V} \geq h(\|v\|_*) \tag{3.21}$$

for all u*,* $v \in V$*,* $v \neq 0$*. Let* $u \in C^1(\mathbb{R}_+, V) \cap C^2(\mathbb{R}_+, H)$ *be a classical solution to*

$$\ddot{u}(t) + g(u(t), \dot{u}(t)) + E'(u(t)) = 0, \quad u(0) = u_0 \in V, \ \dot{u}(0) = u_1 \in H \tag{3.22}$$

such that $(u, \dot{u})$ *is precompact in* $V \times H$*. Then* $\lim_{t \to +\infty} \|\dot{u}(t)\| = 0$*.*

In the article [60] where Simon used for the first time the Łojasiewicz inequality in infinite-dimensional setting, he also proved a convergence result for a class of second order evolution equations with linear damping. Jendoubi [46] (see also [47]) proved convergence of solutions to

$$\ddot{u} + B\dot{u} + Au = f(x,u)$$

where A is a self-adjoint linear operator associated with a coercive bilinear form on $V \hookrightarrow L^2(\Omega)$ and B is a bounded linear operator. The function f is analytic and no global growth assumptions on f. On the other hand, precompactness of range in $W^{2,p} \cap W^{1,p}$ was needed. In 1999 Haraux and Jendoubi [37] extended the convergence result to weak solutions precompact in $V \times L^2$. Then the nonlinearity f is assumed to be analytic and satisfy $f \in C^1(B, V')$ for some ball $B \subset V$, which is in fact a growth condition. If $A = \Delta$, then f satisfying

$$|\partial_s f(x,s)| \le C(1 + |s|^\gamma) \qquad \text{with } \gamma \ge 0,\ (N-2)\gamma < 2 \tag{3.23}$$

are admissible. Moreover, they allowed the damping operator $B : V \to V'$ to be nonlinear (but still satisfying $\langle B(v), v\rangle_{V',V} \ge c\|v\|^2$, so the damping is not weak). Moreover, they have shown that every bounded solution has precompact range in $V \times L^2$.

An abstract wave equation with linear damping was cosidered in Chill, Haraux and Jendoubi [24], where convergence to equilibrium was proved for precompact solutions to (3.1) with $g(\dot{u}) = \gamma\dot{u}$ if E satisfies the Łojasiewicz–Simon inequality and $E''(u) = M'(u)$ satisfies condition (E2) below. Rate of convergence is also estimated by an exponential or a polynomial (depending on the Łojasiewicz exponent θ of the energy E).

A wave equation with weak damping was studied in 2009 by Chergui [20] and convergence to equilibrium was shown for $H_0^1 \times L^2(\Omega)$-bounded solutions to (3.2) with Dirichlet boundary conditions, $g(u_t) = |u_t|^\alpha u_t$ and f analytic satisfying $f(x,0) \in L^\infty(\Omega)$ and (3.23) if $N \ge 2$. The exponent α is assumed to belong to $[0, \frac{\theta}{1-\theta})$ (θ is again the Łojasiewicz exponent of the energy E) and to satisfy $\alpha < \frac{4}{N-2}$ if $N \ge 3$. In fact, Chergui shows that under these assumptions on f, the Łojasiewicz–Simon inequality holds with an appropriate θ and that bounded solutions are in fact precompact. Once he has these facts, he proves convergence to equilibrium.

In 2011, Ben Hassen and Haraux [13] proved convergence to equilibrium and decay estimates for an abstract wave equation (3.1) with E bounded from below and weak nonlocal damping $g : V \to V'$ with power-like behavior

$$\begin{aligned} \langle g(v), v\rangle_{V',V} &\geq c_1 \|v\|^{\alpha+2}, \qquad v \in V, \\ \|g(v)\|_{V'} &\leq c_2 \|v\|^{\alpha+1}, \qquad v \in H. \end{aligned} \tag{3.24}$$

It is assumed that E satisfies the Łojasiewicz–Simon inequality, condition (E2) below and a kind of inverse Łojasiewicz inequality

$$\|E'(u)\|_{V'} \leq cE(u)^\gamma \quad \text{for some } \gamma \in [1/2 - \alpha(1-\theta), 1-\theta] \tag{3.25}$$

in a ball $B \subset V$ containing the whole solution u (then no precompactness of the trajectory is needed).

In 2013, Aloui, Ben Hassen and Haraux [2] generalized Chergui's result from [20] to abstract wave equations (3.1) with a large class of damping operators g similar to [13], in fact they replaced the second condition in (3.24) with

$$\|g(v)\|_{V'} \leq \langle g(v), v\rangle_{V',V}^{\frac{\alpha+1}{\alpha+2}}, \quad v \in V,$$

which is a similar condition to ($\tilde{\mathrm{G}}1'$) below (here E is not bounded from below and precompactness of the range is asssumed).

In [**B3**] we have generalized the result by Chergui [20] to abstract wave equations and to more general damping functions. Our assumptions on E are the same as in [24] but we allow more general Kurdyka–Łojasiewicz–Simon inequality instead of Łojasiewicz–Simon inequality. Our assumptions on the damping function g are similar to those in finite-dimensional case. Basically, on a neighborhood of zero g is bigger than a concave function h which is related to the function Θ from the Kurdyka–Łojasiewicz–Simon inequality by condition (h2) below. This relation becomes $\alpha < \frac{\theta}{1-\theta}$ in Chergui's case. In contrast to the finite-dimensional case, a growth condition in infinity is needed, see (g3). This condition implies $g : V \to V'$.

In [**B3**, Theorem 2.1] we stated a convergence result for a 'scalar-valued' damping function $g(\dot{u})(x) = \tilde{g}(|\dot{u}(x)|)\dot{u}(x)$ and in [**B3**, Theorem 5.1] for a 'vector-valued' (but still pointwise) function $g(u, \dot{u})(x) = G(u(x), \dot{u}(x))$, which may depend on u (not only $\dot{u}$). Here are the assumptions and the result.

(E) Assume that $E \in C^2(V)$ satisfies:

(E1) there exists a function $\Theta : [0,1) \to [0,+\infty)$ which is nondecreasing, sublinear ($\Theta(s+t) \le \Theta(s) + \Theta(t)$), and it holds that $\frac{1}{\Theta} \in L^1_{loc}([0,1))$ and $0 < \Theta(s) \le c\sqrt{s}$ for all $s \in (0,1)$ and some $c > 0$ and such that E satisfies the Kurdyka–Łojasiewicz–Simon gradient inequality with the function Θ in a neighbourhood of the critical points of E, i.e., for each $\varphi \in \mathcal{N} := \{\psi \in V : \ E'(\psi) = 0\}$ there exist η, $C > 0$ such that

$$\|E'(u)\|_* \ge C\Theta(|E(u) - E(\varphi)|), \quad u \in B_V(\varphi, \eta);$$

(E2) for all $u \in V$, the operator $\mathcal{K}E''(u) \in L(V)$ extends to a bounded linear operator on H and $\sup \|\mathcal{K}E''(u)\|_{L(H)}$ is finite whenever u ranges over a compact subset of V.

(G) The function $G : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and there exists $\tau > 0$ such that

(G1) there exists $C_2 > 0$ such that $|G(w,z)| \le C_2|z|$ for all $z \in B_{\mathbb{R}^n}(0,\tau)$, $w \in \mathbb{R}^n$,

(G2) there exists $C_3 > 0$ such that $C_3|z| \le |G(w,z)|$ for all $z \in \mathbb{R}^n \setminus B_{\mathbb{R}^n}(0,\tau)$, $w \in \mathbb{R}^n$,

(G3) if $N = 2$ then there exist $C_4 > 0$, $\alpha > 0$ such that $|G(w,z)| \le C_4|z|^\alpha|z|$ for all $z \in \mathbb{R}^n \setminus B_{\mathbb{R}^n}(0,\tau)$, $w \in \mathbb{R}^n$; if $N > 2$ then the inequality holds with $\alpha = \frac{4}{N-2}$,

(G4) there exists $C_5 > 0$ such that $\langle G(w,z), z\rangle \ge C_5|G(w,z)||z|$ for all $w, z \in \mathbb{R}^n$.

(H) For τ from condition (G) there exists a function $h : [0,+\infty) \to [0,+\infty)$, which is concave and nondecreasing on $[0,\tau]$ and satisfies

(H1) $|G(w,z)| \ge h(|z|)|z|$ for all $z \in B_{\mathbb{R}^n}(0,\tau)$, $w \in \mathbb{R}^n$,

(H2) the function $s \mapsto \frac{1}{\Theta(s)h(\Theta(s))}$ belongs to $L^1((0,\tau))$,

(H3) the function $\psi : s \mapsto sh(\sqrt{s})$ is convex on $[0,\tau^2]$.

Theorem 3.2.2 ([**B3**], Theorem 5.1)**.** *Let E and G satisfy* (E)*,* (G) *and* (H)*. Let u be a strong solution to*

$$\ddot{u} + G(u,\dot{u}) + E'(u) = 0$$

such that $(u,\dot u)$ has $V\times H$-precompact range and let $\varphi\in\omega_V(u)$. Then

$$\lim_{t\to+\infty}(\|u(t)-\varphi\|_V+\|u_t(t)\|)=0.$$

In **[B5]** we have generalized the result from [13] to more general damping functions and E satisfying the Kurdyka–Łojasiewicz–Simon inequality instead of Łojasiewicz–Simon inequality and we combined this result with a previous one to weaken the assumptions on E (assuming on the other hand precompactness of the trajectory). Conditions ($\tilde{\mathrm{E}}1$), ($\tilde{\mathrm{E}}2$) are similar to (E1), (E2) (but on a larger set), condition ($\tilde{\mathrm{E}}3$) together with ($\tilde{\mathrm{G}}4$) generalizes (3.25) (let us mention, that in applications condition ($\tilde{\mathrm{E}}3$) is often satisfied with $G(s)=C\sqrt{s}$ and in this case ($\tilde{\mathrm{G}}4$) holds). Conditions ($\tilde{\mathrm{G}}3$), ($\tilde{\mathrm{G}}5$) are the same as (H2), (H3) and conditions ($\tilde{\mathrm{G}}1$), ($\tilde{\mathrm{G}}2$) correspond to (G), (H1).

Our hypothesis below use the notion of admissible functions, which weaken the assumptions on functions Θ and h from (E1), (H). We say that $f:\mathbb{R}_+\to\mathbb{R}_+$ is *admissible* if it is nondecreasing and there exists $c_A\geq 1$ such that for all $s>0$ we have $f(s)>0$ and $sf'(s)\leq c_Af(s)$. It holds that every nonnegative differentiable concave function is admissible with $c_A=1$. On the other hand, if f is admissible then f is C-sublinear, i.e. $f(t+s)\leq C(f(t)+f(s))$ for some $C\geq 0$ and all t, $s\geq 0$.

- (**$\tilde{\mathbf{E}}$**) Let $E\in C^2(V)$, $M=E'\in C^1(V,V^*)$ and let B be a fixed ball in V. Assume that:
  - (**$\tilde{\mathbf{E}}$1**) E is nonnegative on B and there exists an admissible function Θ such that $\Theta(s)\leq C_\Theta\sqrt{s}$ for all $s\geq 0$ and some $C_\Theta>0$, $\frac{1}{\Theta}$ is integrable in a neighbourhood of zero and

    $$\|M(u)\|_*\geq\Theta(E(u)),\quad\text{for all } u\in B, \tag{KLS}$$

    i.e., E satisfies the Kurdyka–Łojasiewicz–Simon gradient inequality with function Θ on B.
  - (**$\tilde{\mathbf{E}}$2**) There exists $C_M\geq 0$ such that

    $$|\langle M'(u)v,v\rangle_*|\leq C_M\|v\|^2\quad\text{for all } u\in B,\ v\in V,$$
  - (**$\tilde{\mathbf{E}}$3**) There exists a nondecreasing function $\Gamma:\mathbb{R}_+\to\mathbb{R}_+$ such that

    $$\|M(u)\|_*\leq\Gamma(E(u)),\quad\text{for all } u\in B. \tag{3.26}$$

(**$\tilde{\mathbf{G}}$**) The function $g : V \to V^*$ is continuous and there exists an admissible function h such that

- (**$\tilde{\mathbf{G}}$1**) there exists $C_2 > 0$ such that $\|g(v)\|_* \leq C_2\|v\|$ on $V \cap B(0, R)$ for any $R > 0$ with C_2 depending on R,
- (**$\tilde{\mathbf{G}}$2**) $\langle g(v), v\rangle_{V^*,V} \geq h(\|v\|)\|v\|^2$ on V,
- (**$\tilde{\mathbf{G}}$3**) the function $s \mapsto \frac{1}{\Theta(s)h(\Theta(s))}$ belongs to $L^1((0, 1))$,
- (**$\tilde{\mathbf{G}}$4**) there exists $C_\Gamma > 0$ such that $\Gamma(s) \leq C_\Gamma \frac{\sqrt{s}}{h(\Theta(s))}$ on $(0, K]$ for any $K > 0$ with C_G depending on K,
- (**$\tilde{\mathbf{G}}$5**) the function $\psi : s \mapsto sh(\sqrt{s})$ is convex for all $s > 0$.

Theorem 3.2.3 ([**B5**], Theorem 2.1)**.** *Let E and g satisfy* ($\tilde{\mathrm{E}}$) *and* ($\tilde{\mathrm{G}}$)*. Let u be a strong solution to* (3.1) *and there exists $t_1 > 0$ such that $u(t) \in B$ for all $t \geq t_1$. Then there exist $\varphi \in B$ and $t_0 \geq 0$ such that*

$$E(u(t)) \leq 2\Psi^{-1}(t - t_0), \tag{3.27}$$

$$\|u(t) - \varphi\| \leq \Phi(\Psi^{-1}(t - t_0)), \tag{3.28}$$

$$\|\dot{u}(t)\| \leq \sqrt{\Psi^{-1}(t - t_0))} \tag{3.29}$$

hold for all $t > t_0$, some C_1, $C_1 > 0$ and Φ, Ψ defined by (3.13)

Let us mention that if $\Theta(s) = cs^{1-\theta}$, $h(s) = s^\alpha$, then we are in the situation from [13] and the convergence rate we obtain is the same as in [13]. However, we can consider more general damping functions or we can get better decay estimates in the logarithmic scale as in finite-dimensional case. See Example 3.1.7 above or Example 3.2.8 below.

The next result combines the method from [20] (resp. [**B3**]) and [13] to obtain decay estimates for relatively compact solutions with (KLS) satisfied only on a small neighborhood of some $\varphi \in \omega_V(u)$.

Theorem 3.2.4 ([**B5**], Theorem 2.2)**.** *Let u be a strong solution to* (3.1) *with $(u, \dot{u})$ having $V \times H$-precompact range and $\varphi \in \omega_V(u)$ with $E(\varphi) = 0$. Let* ($\tilde{\mathrm{E}}$) *and* ($\tilde{\mathrm{G}}$) *hold with the following changes.*

- ($\tilde{\mathrm{E}}$1)*,* ($\tilde{\mathrm{E}}$3) *hold with B replaced by $B_V(\varphi, \delta)$ for some $\delta > 0$,*
- ($\tilde{\mathrm{E}}$2) *holds with B replaced by 'any compact subset of V with C_M depending on the subset',*

- *h is admissible with $c_A = 1$,*

Then $\lim_{t\to+\infty} \|u(t) - \varphi\|_V = 0$ *and there exists* $t_0 \geq 0$ *such that the decay estimates* (3.27), (3.28) *and* (3.29) *hold for all* $t > t_0$*, some* C_Φ, $C_\Psi > 0$ *and* Φ, Ψ *defined in* (3.13).

The assumption ($\tilde{\text{G}}$) on a nonlocal damping function are not met by all local damping functions satisfying (G). However, if we replace ($\tilde{\text{G}}1$) by ($\tilde{\text{G}}1'$)

(**$\tilde{\text{G}}$1′**) for every $R > 0$ there exists a convex function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ with property (K) and such that $\gamma(0) = 0$, $\lim_{s\to+\infty} \gamma(s) = +\infty$, $\gamma(s) \geq cs^2$ for some $c > 0$ and all s small enough, and $\gamma(\|g(v)\|_*) \leq \langle g(v), v\rangle_{V^*,V}$ on $V \cap B(0, R)$,

then Theorems 3.2.3, 3.2.4 remain valid and the new assumptions follow from (G) as states the following Proposition. Similar condition appears in Aloui, Ben Hassen and Haraux [2].

Proposition 3.2.5 ([**B5**], Proposition 3.1)**.** *Let* $G : \mathbb{R}^N \to \mathbb{R}^N$ *satisfy* (G), (H1) *and define* $(g(v))(x) := G(v(x))$ *for* $v \in V$*. Then* $g(V) \subset V^*$ *and* g *satisfies* ($\tilde{\text{G}}$) *with* ($\tilde{\text{G}}1$) *replaced by* ($\tilde{\text{G}}1'$).

In [**B5**] this Proposition is formulated and proved for G independent of u (depending on $\dot{u}$ only), but it remains valid with the same proof if G depends on u.

Theorem 3.2.6 ([**B5**], Theorem 2.3)**.** *Theorems 3.2.3 and 3.2.4 remain valid if we replace* ($\tilde{\text{G}}1$) *by* ($\tilde{\text{G}}1'$).

As in finite-dimensional case, if we have direct estimates for $\|u - \varphi\|$ by $E(u)$, then we can obtain better convergence rates.

Corollary 3.2.7 ([**B5**], Corollary 2.4)**.** *Suppose that the hypotheses of Theorems 3.2.3, 3.2.4 or 3.2.6 are satisfied. Let* $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ *be a nondecreasing function such that* $\alpha(E(u) - E(\varphi)) \geq \|u - \varphi\|$ *on a neighborhood of* φ*. Then*

$$\|u(t) - \varphi\| \leq \alpha(2\Psi^{-1}(t - t_0))$$

holds for some t_0 *and all* $t > t_0$.

Example 3.2.8. *It is shown in [13] that the following two problems fit into the framework considered in the theorems of this section: the Dirichlet problem*

$$\begin{cases} u_{tt} + g(u_t) - \Delta u - \lambda_1 u + |u|^{p-1}u = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\ u(t,x) = 0 & \text{on } \mathbb{R}_+ \times \partial\Omega, \end{cases} \tag{3.30}$$

and the Neumann problem

$$\begin{cases} u_{tt} + g(u_t) - \Delta u + |u|^{p-1}u = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\ \frac{\partial}{\partial n} u(t,x) = 0 & \text{on } \mathbb{R}_+ \times \partial\Omega, \end{cases} \tag{3.31}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, λ_1 is the first eigenvalue of $-\Delta$ and $p > 1$ with $(N-2)p < N+2$. The corresponding energy functions E satisfy the Lojasiewicz inequality with $\theta = \frac{1}{p+1}$ (so $\Theta(s) = Cs^{1-\theta}$) on any bounded subset of V and any strong solution to (3.30) is bounded in V. Moreover, $\Gamma(s) = C\sqrt{s}$ satisfies ($\tilde{\text{E}}3$) and ($\tilde{\text{G}}4$) and we have $E(u) \geq c\|u\|_V^{p+1}$.

In contrast to [13] we can obtain convergence for a larger class of damping functions, e.g. for $(g(v)) = G(v(x))$ with G having different growth/decay in each direction and also for $|s|$ large and $|s|$ small, e.g.

$$G(s) = \begin{cases} |s|^{b_1}s, & s > 1, \\ |s|^{a_1}s, & s \in [0,1], \\ |s|^{a_2}s, & s \in [-1,0), \\ |s|^{b_2}s, & s < -1, \end{cases} \tag{3.32}$$

with $0 \leq a_1 < a_2 < \frac{1}{p}$, $b_1, b_2 \leq \frac{4}{N-2}$ if $N > 2$. Denoting $A = \max\{a_1, a_2\}$ we have

$$\|u(t) - \varphi\|_V \leq Ct^{-\frac{1}{(A+1)p-1}}, \qquad t \geq t_0$$

by Corollary 3.2.7. We can see that the rate of decay depends on the growth of G near zero only.

We can also consider

$$G(s) = \begin{cases} |s|^a s \ln^r(1/|s|) & |s| \leq 1, \\ c|s|^b s & |s| > 1, \end{cases} \tag{3.33}$$

with $b < \frac{4}{N-2}$, $0 < a < \frac{1}{p}$, $r \in \mathbb{R}$ or $a = \frac{1}{p}$, $r > 1$. In this case we obtain more delicate decay estimates in the logarithmic scale, namely

$$\|u(t) - \varphi\|_V \leq C\left(\Psi^{-1}(t-t_0)\right)^{\frac{1}{p+1}} \leq Ct^{-\frac{1}{(a+1)p-1}} \ln^{-\frac{r}{(a+1)p-1}}(t), \qquad t \geq t_0.$$

Let us replace $|u|^{p-1}u$ *in* (3.31) *with* $f(u)$ *defined as*

$$f(u)=\begin{cases}(d_1-s)^{p-1}(s-d_1), & s<d_1,\\ 0, & s\in[d_1,d_2],\\ (s-d_2)^{p-1}(s-d_2), & s>d_2\end{cases}$$

for some $d_1<d_2$*. Then all the assumptions remain valid except that we do not have* $E(u)\geq c\|u\|_V^{p+1}$*. In this case, we apply Theorem 3.2.6 instead of Corollary 3.2.7 and obtain*

$$\|u(t)-\varphi\|\leq Ct^{-\frac{1-Ap}{(A+1)p-1}},\qquad t\geq t_0$$

for the damping function G *given by* (3.32)*. For* G *given by* (3.33) *we get in case* $a<\frac{1}{p}$

$$\|u(t)-\varphi\|\leq Ct^{-\frac{1-ap}{(a+1)p-1}}\ln^{-\frac{pr}{(a+1)p-1}}(t),\qquad t\geq t_0$$

and in case $a=\frac{1}{p}$

$$\|u(t)-\varphi\|\leq C\ln^{1-r}(t),\qquad t\geq t_0.$$

See [**B5**, Examples 5.2, 5.3] *for details.*

Nonautonomous case.

Further generalizations of the above results consider non-autonomous equations of the type

$$\ddot{u}+a(t)g(\dot{u})+E'(u)=f(t)\tag{3.34}$$

in finite-dimensional or infinite-dimensional settings. In order to obtain convergence to an equilibrium, we need to assume that f is not too large and a is not too small.

First results of this kind are due to Chill and Jendoubi [25] and Ben Hassen [12] for $g(s)=s$, $a(t)=1$ (in Hilbert spaces) and Cabot, Engler and Gadat for the case $a(t)\geq a_0$, $g(s)=s$, $f=0$ (in $\mathbb{R}^n$ assuming that $\mathrm{Cr}(\nabla E)$ is finite).

In 2013, Haraux and Jendoubi [40] proved convergence to equilibrium and decay estimates in $\mathbb{R}^n$ for $f=0$, g identity, E satisfying the Łojasiewicz inequality (LI) and $a(t) \geq (1+t)^{-\beta}$. Also vector-valued functions a were considered.

The case $a(t) \equiv 1$, $f \neq 0$, g nonlinear was studied in 2011 by Haraux [35] for g satisfying

$$\|g(v)\| \leq K\|v\|^{\alpha+1}, \quad \langle g(v), v\rangle \geq c\|v\|^{\alpha+2} \tag{3.35}$$

and $E \in W^{2,+\infty}(H)$ satisfying

$$\|E(u)\| \leq K\|u\|^{\beta+1}, \quad \langle \nabla E(u), u\rangle \geq c\|v\|^{\beta+2} \tag{3.36}$$

on bounded sets in a Hilbert space H. Haraux's result gives convergence and optimal decay estimates for exponentially decaying functions f and for arbitrarily large α (i.e. very weak damping).

Theorem 3.2.9 ([35], Theorem 6.1 and 6.2)**.** *Let B_1, B_2 be two closed balls in a Hilbert space H, $g \in W^{1,\infty}(B_1, H)$, $E \in W^{2,\infty}(B_2, R)$, $f \in C(\mathbb{R}_+, H)$ and $u \in C^2(\mathbb{R}_+, H)$ be a solution to* (3.34) *such that $(u(t), \dot{u}(t)) \in B_1 \times B_2$ for all $t > 0$. Assume that* (3.35) *and* (3.36) *hold for all $u \in B_1$, $v \in B_2$. Let us define*

$$\mathcal{E}(t) := \frac{1}{2}\|\dot{u}(t)\|^2 + E(u(t)).$$

If $\alpha < \frac{\beta}{\beta+2}$ and $\lambda \geq \frac{(\alpha+1)(\beta+1)}{\beta-\alpha}$ then

$$\|\mathcal{E}(t)\| \leq Ct^{-\frac{(\alpha+1)(\beta+1)}{\beta-\alpha}}, \quad t \geq 1.$$

If $\alpha \geq \frac{\beta}{\beta+2}$ and $\lambda \geq 1 + \frac{1}{\alpha}$ then

$$\|\mathcal{E}(t)\| \leq Ct^{-\frac{2}{\alpha}}, \quad t \geq 1.$$

Ben Hassen and Chergui [14, Theorem 1.6] showed convergence to equilibrium and decay estimates in $\mathbb{R}^n$ without assuming a special structure of E (assumption (3.36) replaced by the Łojasiewicz inequality (LI)) and for polynomially decaying f. Then the damping cannot be too small ($\alpha < \frac{\theta}{1-\theta}$). For $E(u) = \|u\|^{\beta+1}$, the decay estimates in [35] are better than those in [14].

Theorem 3.2.10 ([14], Theorem 6.1)**.** *Let $g \in C(\mathbb{R}^N, \mathbb{R}^N)$, $E \in C^2(\mathbb{R}^N, \mathbb{R})$, $f \in C(\mathbb{R}_+, \mathbb{R}^N)$ and $u \in W^{1,+\infty}(\mathbb{R}_+, \mathbb{R}^N)$ be any solution to* (3.34) *Assume that g satisfies* (3.35) *for all v in a bounded sets (with c, K depending on the* *set), E satisfies* (LI) *for any $\varphi \in S$ with $\theta \in (0, \frac{1}{2}]$, $\eta > 0$ independent of φ, and f satisfies*

$$\|f(t)\| \leq \frac{C}{(1+t)^{1+\delta+\alpha}}, \quad t \geq 0$$

for appropriate $C \geq 0$ and $\delta > 0$. If $\alpha < \frac{\theta}{1-\theta}$ then there exists $\varphi \in S$ such that $\lim_{t\to+\infty} \|\dot{u}(t)\| + \|u(t) - \varphi\| = 0$ and

$$\|u(t) - \varphi\| \leq C(1+t)^{-\mu}, \quad t \geq 0$$

with $\mu = \min\left\{\frac{1-(\alpha+1)(1-\theta)}{(\alpha+2)(1-\theta)-1}, \frac{\delta}{\alpha+1}\right\}$.

In [39], Haraux and Jendoubi proved weak convergence of solutions to

$$A\ddot{u} + aA\dot{u} + \nabla E(u) = f(t)$$

with a selfadjoint bounded linear operator A on a Hilbert space and a convex potential E.

The case of nonconstant a and $f \not\equiv 0$ was considered by Jendoubi and May [48] for convex potentials E on a Hilbert space and g being identity. Weak convergence was obtained for appropriate polynomial decays of f and a (see [48, Theorem 1.3]). The case of nonlinear g and (nonconvex) E satisfying the Łojasiewicz inequality (LI) in $\mathbb{R}^n$ was solved in 2015 by Balti [4].

Theorem 3.2.11 ([4], Theorem 1.2, Remark 1.7)**.** *Let $E \in W^{2,\infty}_{loc}(\mathbb{R}^N)$, $\gamma \in L^\infty$ be a positive function, $u \in W^{2,1}_{loc} \cap L^\infty(\mathbb{R}_+, \mathbb{R}^N)$ be a solution to* (3.34)*. Let $S = \arg\min E$ and E satisfies* (LI) *for all $\varphi \in S$ with a fixed $\theta \in (0, \frac{1}{2}]$ and $C > 0$, $\eta > 0$ depending on φ. Let g satisfies* (3.35) *on $\mathbb{R}^N$,*

$$\|f(t)\| \leq \frac{d}{(1+t)^{1+\delta}}, \quad \textit{for all } t \geq 0$$

for some d, $\delta > 0$ and let

$$\|a(t)\| \geq \frac{c}{(1+t)^{\beta}}, \quad \textit{for all } t \geq 0$$

for some $c > 0$ and $\beta \geq 0$ such that $\alpha + \beta \in (0, \min\{\frac{\theta}{1-\theta}, \delta\})$. Then there exist $\varphi \in S$ and $M \geq 0$ such that

$$\|u(t) - \varphi\| \leq Mt^{-\mu}, \quad \textit{for all } t \geq 0,$$

where

$$\mu = \min\left\{\frac{\theta - (\alpha+\beta)(1-\theta)}{(1-\theta)(\alpha+2)-1}, \frac{\delta - (\alpha+\beta)}{\alpha+1}\right\}.$$

3.3 Appendix to second order problems

3.3.1 Well-posedness and existence of global solutions

Although the convergence results in Section 3.2 hold for ill-posed problems as well, we list here some results on well-posedness and global existence for (3.1).

A well-posedness result which includes also nonmonotone damping functions can be found in Haraux '87 [33, Theorem II.2.2.1]. It concerns a problem

$$u_{tt} + Lu + g(u_t) + f(u) = cu_t + h(t,x) \tag{3.37}$$

with $V \hookrightarrow H = L^2(\Omega)$, $\Omega \subset \mathbb{R}^N$ bounded domain, $L : V \to V'$ being a linear operator associated with a coercive bilinear form on V, the nonlinearity $f \in C^1(\mathbb{R})$ is such that $u(\cdot) \mapsto f(u)(\cdot)$ maps V into H (i.e. so called subcritical case) and it is Lipschitz continuous on bounded subsets of V. Function $g : \mathbb{R} \to \mathbb{R}$ is assumed to be continuous and nondecreasing with $g(0) = 0$, $h \in L^1([0,T], H)$ and $c \geq 0$ (this means that the damping function $s \mapsto g(s) - sc$ is not neccessarily nondecreasing).

Global existence for (3.37) is shown in [33, Theorem II.2.2.2] under additional assumptions

$$C := \inf\left\{\int_0^u f(z)dz + Mu^2 : \ u \in \mathbb{R}\right\} > -\infty$$

for some $M \in \mathbb{R}$ and if Ω has infinite measure, then $C \geq 0$. This assumption means that the corresponding energy E given by (3.3) is bounded from below.

If the energy is not bounded from below, global existence depends on the interplay between the source term f and the damping term g. In 1994, Georgiev and Todorova [29] studied (3.37) with $L = -\Delta$, $V = H_0^1(\Omega)$, $f(u) = -|u|^{p-1}u$, $g(v) = |v|^{m-1}v$, $h \equiv 0$, $c = 0$ for $p \leq \frac{N}{N-2}$ (which corresponds to subcritical case). They have shown global existence for all initial values in $V \times H$ for $p \leq m$, blow-up of solutions with negative initial energy for $m < p$.

Levin, Park and Serrin in 1998 [52] studied global existence in subcritical case for source and damping terms depending on x (resp. x and t):

$$u_{tt} + g(t,x,u_t) - \Delta u = f(x,u) \tag{3.38}$$

with $\Omega = \mathbb{R}^N$, g satisfying $g(t,x,v)v \geq 0$ and some estimates from below (no monotonicity needed). They have shown that local existence implies global existence, whenever so called *continuation proprerty* holds, i.e. if every bounded local solution can be continued. However, this continuation property is known (according to [52]) only for $g(t, x, v) = a|v|^{m-2}v$.

In supercritical case, global existence for any initial data (and existence of a global attractor) was proved by Feireisl in 1995 [27] for any bounded regular domain $\Omega \subset \mathbb{R}^3$ and strictly increasing $g \in C^1(\mathbb{R})$ depending only on u_t satisfying $g(0) = 0$ (no additional assumptions at zero) and for appropriate growth conditions of g and f in infinity (f independent of x).

Serrin, Todorova and Vittilaro in 2003 (see [59]) proved local and global existence in a more general (supercritical) case (g depending on t, x, f depending on x, $\Omega \subset \mathbb{R}^N$) for compactly supported initial data, under some growth and regularity conditions on f and g, g increasing in the third variable and having the same power-like growth near zero and infinity. Further results are due to Radu (see [57], [58]).

Benaissa and Mokeddem in 2004 (see [11]) showed global existence for $\Omega = \mathbb{R}^n$, $f(x, u) = |u|^{p-1}u - \lambda^2(x)u$ (allowing also supercritical growth), g nondecreasing depending on u_t only and satisfying $c_1|v|^{1/m} \leq |g(v)| \leq c_2|v|^m$ for small v and $c_3|v| \leq |g(v)| \leq c_4|v|^r$ for large v (for appropriate positive constants c_i, r, m, p) and sufficiently small initial energy. They also prove decay estimates for the energy.

Global existence for an abstract problem $\ddot{u} + A(u) + B(t)\dot{u} + G(\dot{u}) = f(t)$ in Banach spaces was proved by Biazutti in 1995 ([15]). Here $B : V \to V'$ is an operator associated with a (uniformly in t) positive definite bilinear form, so this part of damping is stronger than linear. On the other hand, G is a nonlinear operator of lower order which can be negative near zero, so in fact there can be negative damping for small values of $\dot{u}$.

3.3.2 Precompactness of bounded solutions

In this subsection we discuss the assumption on precompactness of solutions. Of course, in finite-dimensional case, all bounded solutions have precompact range. The same implication holds in some infinite-dimensional problems.

An abstract result by Webb [61] states that if $(S(t))_{t\geq 0}$ is a dynamical system on a metric space X which can be written as a sum of $S_1(t)$, $S_2(t)$ with $\lim_{t\to+\infty} \|S_1(t)\| = 0$ and $S_2(t)$ compact for all t large enough, then all bounded orbits are precompact. Another result by Webb says that if T is a dynamical system which is bounded on bounded sets and satisfies

$$T(t) = S(t) + \int_0^t S(t-s)BT(s) \tag{3.39}$$

with S that can be splitted as above and B a bounded operator on X, then bounded orbits of T are precompact. In fact, the second results applies to perturbed problems — if A generates a linear semigroup S, then $A+B$ generates a semigroup T satisfying (3.39). Unfortunately, this result can be applied only if we have well-posedness.

In 1980 Webb applied this perturbation result to a damped wave equation

$$u_{tt} - \Delta u_t - \Delta u = f(u)$$

on a smooth bounded domain with Dirichlet boundary conditions with $f \in C^1(\mathbb{R})$, $|f'| \leq M$, $\limsup_{|x|\to+\infty} f(x)/x \geq 0$ and $f(0) = 0$ (see [62]). Let us mention that conditions on f imply that the energy is bounded from below. Webb proved that for any initial data, there exists a global solution and it has precompact range.

In 1999, Haraux and Jendoubi [37] showed precompactness of bounded solutions of a linearly damped equation

$$u_{tt} + cu_t - \Delta u = f(x,u)$$

with $\partial_u f$ only locally bouded and globally satisfying $|\partial_u f(x,u)| \leq C(1+|u|^\alpha)$ for some $C > 0$, $\alpha < \frac{2}{N-2}$ (which means that the energy is not neccessarily bounded from below but the growth is subcritical).

In 2009, Chergui [20] proved precompactness of bounded solutions under the same assumptions on f but with a nonlinear (possibly weak) damping $g(u_t)$, $g : \mathbb{R}_+ \to \mathbb{R}_+$ increasing, g^{-1} uniformly continuous. In fact, Chergui has shown that the right-hand side $h(t,x) = f(x,u(t,x))$ satisfies the assumptions of a criterion by Haraux [31, Theorem 4.1].

Ben Hassen and Chergui in 2011 [14] further generalized this result to a non-autonomous equation

$$u_{tt} + |u_t|^\alpha u_t - \Delta u + f(u) = h(t,x).$$

Aloui, Ben Hassen and Haraux 2013 [2] proved precompactness of bounded trajectories for a large class of abstract semilinear problems

$$\ddot{u} + g(\dot{u}) + Au + f(u) = h(t),$$

where $A : V \to V'$ is the duality mapping, $f = \nabla F : V \to V'$ is a gradient field and $g : V \to V'$ a nonlinear monotone damping operator with f being Lipschitz continuous on bounded sets from W to H for some W, $V \subset\subset W \subset H$.

The compactness results from [20] and [2] can be helpful for verifying the assumptions of Theorems 3.2.2 — 3.2.6, also for damping functions that are not covered by convergence results in [20] and [2].

3.3.3 Functions satisfying the Łojasiewicz inequality

In the results of Sections 3.1, 3.2 we assume that E satisfies the Łojasiewicz gradient inequality (or Łojasiewicz–Simon, Kurdyka–Łojasiewicz–Simon inequality). Now we present some sufficient conditions on E to satisfy these gradient inequalities.

By Łojasiewicz [53], any analytic function $E : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ satisfies (LI). Let us first stay in finite-dimensional spaces. In 1992, Kurdyka has shown that (KLI) is satisfied by any function E, whose graph is a set belonging to an o-minimal structure. An example of an o-minimal structure are semialgebraic sets, i.e. level sets of polynomials of several variables or their finite intersections or unions. In particular, the graph of $E(x, y) = \sqrt{x^4 + y^4}$ can be written as $\{(x, y, z) \in \mathbb{R}^3 : \ z^2 - x^4 - y^4 = 0, z \geq 0\}$, this set is semialgebraic. The set of polynomials can be replaced by other sets of functions to obtain other o-minimal structures, e.g. analytic functions. More on this topic can be found in [26]. Bolte et al. [16] gave some characterizations of functions satisfying (KLI) and also an example of a smooth convex function which does not satisfy (KLI) (several additional conditions to convexity are known that imply (KLI), see Section 4 in [16]).

In 2003, Chill [21] proved many sufficient conditions for one-dimensional case $E : \mathbb{R} \to \mathbb{R}$ to satisfy (LI), also with estimates of the Łojasiewicz exponent. These results are not interesting for the convergence of one-dimensional ODE's, but they are important since Chill showed that if (LI) is satisfied on a so called critical manifold (which is often finite-dimensional), then it holds on the whole neighborhood of a critical point (even in infinite-dimensional case). We come back to this result below. Concerning one-dimensional case, Chill proved that if $f'(x) = g(x) + o(|x-a|^p)$ with $|g(x)| = c|x-a|^p$, then (LI) holds with $\theta = \frac{1}{p+1}$ and if $f \in C^k(B(a, \delta))$, $f^{(k)}(a) \neq 0$ and $f^{(j)}(a) = 0$ for $j = 1, 2, \ldots, k-1$, then (LI) holds with $\theta = \frac{1}{k}$. Chill also gave some estimates of the Łojasiewicz exponent for products and compositions of functions (e.g. for $x \mapsto f(\varphi(x))$ where φ is a dipheomorphism on $\Omega \subset \mathbb{R}^n$ and f satisfy (LI)).

Concerning generalizations to infinite-dimensional spaces, in 1983 Simon [60] proved (LI) for a class of analytic functions in $C^{2,\mu}(\Omega)$, in particular $E(u) = \int_\Omega \tilde{E}(x, u(x), \nabla u(x))dx$ with $\tilde{E}$ analytic in the second and third variables and satisfying some further properties.

In 1998, Jendoubi [47] proved the Łojasiewicz–Simon inequality (LSI) in L^2-norm for $E(u) = \int_\Omega \frac{1}{2}\langle Au, u\rangle + F(x, u)dx$ with a linear operator A associated with a bilinear form on a subspace $V \subset L^2$ and an analytic nonlinearity F. In 1999, Haraux and Jendoubi [37] proved (LSI) for the same E in V'-norm, i.e.

$$|E(u) - E(\varphi)|^{1-\theta} \leq C\|E'(u)\|_{V'},$$

which allows to work with weak solutions of damped wave equations.

In 2001, Huang and Takáč [45] proved (LSI) in the abstract setting. In particular they showed that if $E : V \to \mathbb{R}$ is analytic and $E''(\varphi) : V \to V'$ is a Fredholm operator, then (LSI) holds on a neighborhood of φ. Haraux, Jendoubi and Kavian '03 [41] proved (LSI) for some nonanalytic energy functions. In particular for $Au + f(x, u)$ on $V \subset L^2(\Omega)$, $(A, D(A))$ linear self-adjoint with compact resolvent, $f(x, s) = \partial_s F(x, s)$ with $F \in C^2$ and $v \mapsto \int_\Omega F(x, v(x))dx \in C^2(V)$. If $\|u\|_V \leq C|E'(\varphi + u)|$, then (LSI) holds with $\theta = \frac{1}{2}$. Under some additional growth assumptions on f (no analyticity) (LSI) holds with $\theta = \frac{1}{p+1}$. They also considered an abstract setting: if $E''(\varphi) : V \to V'$ is an isomorphism, then (LSI) holds with $\theta = \frac{1}{2}$.

In 2003, Chill ([21]) proved that it is sufficient to verify (LI) on so called critical manifold. Here is the result.

Theorem 3.3.1 ([21], Theorem 3.10)**.** *Let V be a Banach space, $U \subset V$ an open subset and $E \in C^2(U)$. Let $\varphi \in V$ satisfy the following hypotheses:*

(L1) $E'(\varphi) = 0$ *and* $V_0 = \operatorname{Ker} E''(\varphi)$ *is a complemented subspace, i.e., there exists a bounded linear projection P on V, such that $V_0 = P(V)$. Denote $V_1 = \operatorname{Ker} P$.*

(L2) There exists $W \hookrightarrow V'$ such that the adjoint P' of P leaves W invariant, $E' \in C^1(U, W)$ and $E''(\varphi)(V) = V_1' \cap W$.

(L3) *E satisfies* (LSI) *on the critical manifold S_φ with a Łojasiewicz exponent $\theta \in (0, \frac{1}{2}]$, i.e. there exist C, $\eta > 0$ such that*

$$|E(u) - \mathcal{E}(\varphi)|^{1-\theta} \leq C\|E'(u)\|_W \tag{3.40}$$

holds for all $u \in S_\varphi$, where $S_\varphi = \{u \in B_V(\varphi, \eta) : \; E'(u) \in V_0'\}$.

Then E satisfies (3.40) *for all $u \in B_V(\varphi, \tilde{\eta})$ and a fixed $\tilde{\eta} > 0$ (with a different constant $C > 0$ but the same θ).*

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Chapter 4

Presented works

4.1 T. Bárta, R. Chill, and E. Fašangová, *Every ordinary differential equation with a strict Lyapunov function is a gradient system*, Monatsh. Math. 166 (2012), 57–72.

DOI: 10.1007/s00605-011-0322-4

4.2 T. Bárta, *Convergence to equilibrium of relatively compact solutions to evolution equations*, Electron. J. Differential Equations 2014 (2014), No. 81, 1–9.

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DOI: 10.1016/j.jde.2015.10.003

4.4 T. Bárta, *Rate of convergence and Łojasiewicz type estimates*, J. Dyn. Diff. Equat., online first, 16 pages.

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4.5 T. Bárta, *Decay estimates for solutions of an abstract wave equation with general damping function*, submitted 2016.

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