Report on the habilitation thesis

Asymptotic behavior of gradient-like systems

by

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The presented thesis is based on the five research papers on the long-time behavior of solutions to gradient and gradient-like systems, namely convergence of bounded solutions to equilibria. The thesis consists of three main parts.

In the introduction, the problem is stated both in finite and infinite dimensional cases and also for ordinary differential equations on manifolds. Some counterexamples to convergence are given, and the main tool used for convergence proofs, the Łojasiewicz inequality, is introduced.

All the presented research papers deal with some generalizations of the Łojasiewicz, Łojasiewicz - Simon and Łojasiewicz - Kurdyka inequalities and their applications to gradient and gradient-like systems. Then known estimates of the rate of convergence of solutions to equilibria are improved.

In Section 2, abstract convergence results are given, and the abstract theory is then applied to some damped second order ordinary and also partial differential equations in Section 3.

The main result of the article [1] is that every gradient-like system is a gradient system if the Riemannian metric is appropriately changed. In particular, the article deals with an ordinary differential equation

$$u_t + F(u) = 0,$$

where F is a continuous tangent vector field on a manifold M. If E is a continuously differentiable, strict Lyapunov function, i.e., $\langle E'(u), F(u)\rangle > 0$ for all $\in M$ with $F(u) \neq 0$, then there exists a Riemannian metric $\tilde{g}$ on the set $\tilde{M} = \{u \in M : F(u) \neq 0\}$ such that $F = \nabla_{\tilde{g}} E$. The metric $\tilde{g}$ is constructed in the proof; however, it is not unique. Furthermore, as the question arises whether $\tilde{g}$ can be continuously extended to M, the equivalence of metrics is discussed. The authors show that the two metrics g and $\tilde{g}$ are equivalent on $\tilde{M}$ if and only if E and F satisfy both the angle condition

$$\langle E'(u), F(u)\rangle \geq \alpha \|E'(u)\| \|F(u)\|, \ u \in M,$$

and the comparability condition

$$c\|E'\|_g \leq \|F\}_g \leq C\|E'\|_g.$$

(See Theorems 2.1.2-2.1.4 in the thesis). Finally, sufficient conditions for the convergence of solutions to a singleton are given, which are of modified Kurdyka - Łojasiewicz type; one of these conditions is applied to a second-order problem, see Theorem 3.1.3 in Section 3 of the thesis.

The paper [2] generalizes the abstract convergence result from reference [24] in the sense that weaker Łojasiewicz type condition is needed, and that it is sufficient if energy estimates are satisfied only near the omega-limit set. (See Theorems 2.2.2 and 2.2.3 in the thesis) The abstract results are applied to several ordinary and partial differential equations of first and second order (Theorem 3.2.1).

In the paper [3], convergence to a stationary solution is proved for solutions to abstract nonlinear wave equation in a more general setting, assuming an abstract gradient operator $E'(u)$ (instead of, e.g., $\Delta u + f(x, u)$) which is supposed to satisfy the Kurdyka - Łojasiewicz - Simon gradient inequality. The main goal is to find conditions on the damping function g as general as possible, when the damping is a scalar multiple of the velocity, and this scalar depends on the norm of the velocity, i.e., $g(|u_t|)u_t$. Thus, the equation reeds

$$u_{tt}(t) + g(|u_t(t)|)u_t(t) = E'(u).$$

The analysis shows also the way how to generalize the result to a more general model, an anisotropic, inhomogeneous medium where the damping need not point into the direction of the velocity, that is the equation

$$u_{tt}(t) + G(u(t), u_t(t)) = E'(u).$$

In this formulation, the authors obtained a generalization of Theorem 3.1.3 of this thesis for ordinary differential equations.

Estimates on the rate of convergence to equilibrium of solutions to gradient-like systems satisfying certain angle condition and Kurdyka - Łojasiewicz inequality based on some generalizations of the Łojasiewicz gradient inequality are contained in the paper [4]. It is shown that it can be better to estimate the distance of $u(t)$ from the equilibrium ϕ directly by a function of E(u(t)) instead of, as usually, by the length of the remaining trajectory given by

$$\|u(t) - \phi\| \leq \int_t^\infty \|\dot{u}(s)\| ds.$$

This estimate is far from being optimal if the solution u has a shape of a spiral, which is exactly the case if a second order equation with a weak damping (smaller than linear) is considered. The results apply to a broad class of second order equations with damping, to abstract finite-dimensional problems and to even more general situation.

Generalizations of results from the reference [13] on abstract wave equations with weak dumping are contained in the last paper [5]. The decay estimates are shown for more general dumping functions and the energy E satisfying Kurdyka - Łojasiewicz - Simon inequality instead of Łojasiewicz - Simon inequality. Weaker assumptions on E similar to the previous results are assumed, see Theorems 3.2.3–3.2.7 in the thesis.

Some nonautonomous problems are also introduced, where the relation between the size of the forcing term and the damping is discussed. For completeness of the presentation, some well-posedness and global existence results for the second order problems are listed, and the assumption on precompactness of bounded solutions, which is crucial for convergence, is discussed in the Appendix. Finally, sufficient conditions on an energy functional to satisfy the Łojasiewicz inequality and its generalizations are given.

The theory is explained on several examples both in the text of the thesis and in the attached papers.

Summary

Long-time behavior of solutions to ordinary and partial differential equations, and, in particular, convergence of bounded solutions to equilibria has been recently extensively studied. The Łojasiewicz inequality was used for convergence proofs in many situations. The presented thesis extends the known results in several directions, especially, it brings many new results for second order problems. The topic is clearly explained. All results are of high quality published or accepted for publication in international peer reviewed journals.

To conclude, I have no doubts that the research profile as well as the high level results presented in his habilitation thesis qualify RNDr. Tomáš Bárta, Ph.D. to become scientist and excellent university teacher in his future career. I warmly recommend the present habilitation thesis to be *accepted* by the Faculty of Mathematics and Physics of Charles University in Prague.

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