Referee's report on the Habilitation Thesis

Asymptotic Behavior of Gradient-like Systems

by Dr. Tomáš Bárta

The habiliation thesis of Dr. Bárta contains a quite wide collection of results mostly devoted to characterizing the long-time behavior of solutions $u = u(t)$ to dynamical systems of *gradient*, or *gradient-like* types, i.e., that admit a suitable notion of Lyapunov function. More specifically, one wants to describe the ω-limit set of u, namely the set of limit points (in the topology of a suitable phase space) of sequences of the form $u(t_n)$, where $t_n \nearrow \infty$. For dynamical systems having a gradient (or related) structure, one generally expects that these sets only contain stationary states (i.e., equilibria). Hence, whenever multiple equilibria exist, a basic question consists in establishing whether or not the ω-limit set of $u(t)$ consists of a single element ϕ and the whole trajectory converges to ϕ as $t \nearrow \infty$.

The dissertation is based on five recent research papers: two of them are still in press while the oldest one was published in 2012. Three of these articles are authored by Dr. Bárta only, while the other two are written together with international collaborators. The full collection of the five papers constitutes the second part of the thesis, whereas the first three chapters contain a fairly detailed explanation of the results proved in the articles together with a number of preliminary notions and tools. I now comment in some more detail about the results described in the thesis following the scheme of the first three chapters.

The first chapter has an introductory character and reports a number of general definitions, like those of gradient and gradient-like systems, Lyapunov functions, ω-limit sets, and a first description of the Łojasiewicz inequality, a tool that will play a key role in the sequel. These basic concepts are firstly presented in the Euclidean space setting; then it is explained how they can be extended to the case of Riemannian manifolds as well as to infinite-dimensional spaces.

The original part of the thesis starts with Chapter 2, which is mostly devoted to discussing convergence results for dynamical systems in an abstract setting. More precisely, one considers a trajectory $u(t)$ of a (generally autonomous) dynamical process (which is not necessarily related to an evolutionary equation, even though this holds in most examples), and wants to find sufficient conditions in order for $u(t)$ to converge to a stationary state ϕ as $t \nearrow \infty$. As mentioned, for gradient systems of the form $u_t + \nabla\mathcal{E}(u) = 0$, the basic tool permitting to achieve this property is the Łojasiewicz inequality, stating that the "energy" functional $\mathcal{E}$ generating the system satisfies in a suitable sense the bound

$$|\mathcal{E}(u) - \mathcal{E}(\phi)|^{1-\theta} \leq C\|\nabla\mathcal{E}(u)\|$$

with the exponent $\theta \in (0, 1/2]$ and for all u close enough to the point ϕ. Then, if ϕ belongs to the ω-limit of a trajectory $u(t)$ and the above inequality holds at ϕ, one may prove that the whole trajectory $u(t)$ tends to ϕ. The first original results discussed in this chapter are related to extensions of the inequality to the case of finite-dimensional manifolds. In particular, it is shown that modifications of the Riemannian metric g of the manifold may lead to interpret any gradient-like system as a true gradient system. This fact permits the author to generalize the Łojasiewicz method to manifolds in several directions and to obtain refined convergence results. The subsequent results reported in this chapter refer to the infinite-dimensional setting and are aimed at applying the Łojasiewicz method to second order evolutionary equations. In particular, the candidate can show that the convergence $u(t) \to \phi$ still holds by assuming weaker decay properties of the energy functional $\mathcal{E}$ along trajectories with respect to what was previously known. Further results are devoted to establishing the convergence rate of a trajectory $u(t)$ to an equilibrium ϕ, and also in this respect the author obtains improved estimates for the difference $\|u(t) - \phi\|$ in various different settings.

Chapter 3 is more devoted to specific applications, the most important of which refer to problems of the second order with respect to the time variable. Also in this case, the results are suitable both for finite and for infinite-dimensional problems, with the applications to PDE's being probably the more relevant ones. Entering details, the first series of results is devoted to generalizing, partly in the spirit of Chapter 2, some convergence and decay estimates for second order ODE's with nonlinear damping. Explicit examples are also presented in order to enlighten the theory. Subsequently, the candidate deals with infinite-dimensional problems, and, more precisely, with second order evolutionary PDE's also containing nonlinear damping terms. Here, the presentation can be seen, at least partially, as an infinite-dimensional extension of some of the results of Chapter 2, in the sense that the conditions on the damping function as well as the decay property of the energy functional are generalized in similar directions. Of course, in the PDE setting the situation is more involved in view of the fact that also the well-posedness and precompactness of trajectories are often nontrivial issues that may need a separate discussion. Also the results described in this part are mostly presented in an abstract form by making general structure and growth assumptions on the "energy" functional and on the damping term, but without referring directly to specific equations. This approach has the advantage that many of the theorems do not rely on the well-posedness theory for second order evolutionary PDE's (they have, indeed, a purely dynamical nature) and may be conditionally applied to any hypothetical trajectory satisfying suitable continuity and precompactness properties. Concrete examples illustrating possible applications of the theorems are actually presented in the last part of the chapter together with a discussion on the related literature.

Overall, the mathematical level of the dissertation is more than acceptable, and the five articles on which it is based have been published in scientific journals of generally good impact. I appreciated very much the fact that the thesis is not just constituted by a mere collection of the articles. Indeed, Chapters 1-3 provide a rather complete and well-written introduction that is not limited to presenting the results proved by the candidate, but also frames them at the light of the related literature. Up to my knowledge, all the results presented by the candidate are mathematically original and may be surely interesting for the scientific community working in the field. On the other hand, some of the applications are rather specific and somehow technical, in the sense that you need to be an expert of the field (and this is surely the case of Dr. Bárta) in order to appreciate the novelties. This is probably also the reason why the scientific work of the candidate has received, at least up to this moment, a somehow limited interest among the scientific community.

As a conclusion, I believe this a fairly good, though not outstanding, thesis, containing a number of original results and a well-written introductory part. Within this dissertation, the author proved to be able to address technically difficult problems by using nontrivial tools, and he also demonstrated a deep knowledge of the scientific literature in the field. Hence, in my opinion, the candidate deserves the Habilitation in Mathematical Analysis.

Prof. Giulio Schimperna

Department of Mathematics

University of Pavia