LABORATOIRE INTERDISCIPLINAIRE DES SCIENCES DU NUMÉRIQUE

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Object : Referee report on the PhD dissertation by Barbora Hudocvá

It is my great pleasure to write this report on the manuscript of Barbora Hudcová's PhD thesis. This thesis deals with discrete dynamical systems in general, with a strong focus on cellular automata (CA), although other models such as Turing machines or random Boolean networks are also investigated. There are two main parts in the manuscript: first sections 2 and 3 explore the complexity of discrete dynamical systems through their dynamics, second sections 4 and 5 focus on algebraic cellular automata.

The document begins with an introductory section (Section 1) of about twenty pages, which presents both the objects studied and the state of the art, and also offers a synthesis of the results presented in more detail in Sections 2 to 5. This introductory section is well written, easy to read and provides a concise overview of the rest of the document. In particular, Barbora Hudcová's personal contributions are clearly identified, which facilitates the writing of this report.

The point of view of the first part of the thesis is the following: it is a well-known fact that as soon as we consider even simple discrete dynamical systems over an uncountable set of configurations (e.g. $S^{\mathbb{Z}}$ where S is a finite alphabet for cellular automata or Turing machines), undecidability occurs. It is therefore a reasonable compromise to restrict ourselves to periodic configurations or finite configurations (for graphs cellular automata or random Boolean networks in Section 3) to study the dynamics of the system. For cellular automata in particular this makes even more sense since the set of periodic configurations is dense in $S^{\mathbb{Z}}$.

Section 2 of the manuscript proposes a classification of discrete dynamical systems acting on a finite space using transients. Since the phase space is finite, any orbit necessarily becomes periodic after a time bounded by the size of the phase space. The prefix of the orbit starting at u that constitutes the pre-period is called the transient of the initial configuration u. The main result of this section shows that classifying cellular automata according to the average length of their transients (bounded, logarithmic, linear, polynomial or exponential) allows to formalise their membership to classes of Wolfram's classification. The section concludes with a discussion of the drawbacks of the transient method.

In Section 3, the author takes a slightly different approach: instead of fixing *a priori* a set of periodic initial configurations and studying a whole class of dynamical systems on this space, a single dynamical system is chosen and it is the initial configurations that are varied. This makes it possible to identify phase transitions for a given system. In this section the dynamical systems considered are a variation of cellular automata, called graph cellular automata (GCA) which act on random d-regular graphs with n nodes, instead of acting on a cycle as CA in Section 2, and random Boolean networks. The main tools of this section are the backtracking dynamical cavity method and the classical method from statistical physics from which it is derived, the dynamical cavity method. The methods are used for general GCA to identify certain dynamical phases, and are illustrated using the example of the Anti-Conformist GCA to identify for which value of the initial density ρ_{init} the GCA changes its behaviour.

Section 4 poses the question of the computational capacity of a CA. Formulated differently, this question boils down to the question of which CAs can be simulated by a given CA. A natural related question is whether there exists a universal CA (which can simulate all CAs) or an intrinsically universal CA (which can simulate all CAs of a certain class to which the simulating CA belongs). There is no consensus in the literature on the *good* notion of simulation between CAs. Barbora Hudcová identifies two families of operations for the various existing notions of simulation: algebraic operations and geometric operations. She focuses on the former, and proposes a new, more general notion of algebraic simulation for CAs. To say it in a nutshell, CAs are seen as algebras so that standard operations on algebras (subalgebra, quotient, isomorphism) become operations on CAs. In addition, a notion of iterated algebra is derived from the iterations of a CA. A CA $\mathcal{B}$ is said to simulate another CA $\mathcal{A}$ if $\mathcal{A}$ can be obtained from $\mathcal{B}$ using these four operations. This simulation relation forms a pre-order (Corollary 29). In this algebraic setting, the natural notion that arises when combining additive CA (a well-studied class of algebraically defined CA) with this notion of simulation is the notion of affine CA. The main result of this section is Theorem 36, which states that, under some conditions of little practical constraint, an affine CA over a finite field $\mathbb{F}_p$ can only simulate affine CA over the same field. In other words, affine CAs have very low computational capacity. These results generalise known results about additive CAs. This section of the manuscript is, in my humble opinion, the strongest since it both generalises known results on additive CAs and opens the way to many other applications.

Finally, Section 5 proposes a more detailed study of affine CAs, initiated in Section 4. In this last section, Barbora Hudcová demonstrates several rigidity results culminating in a characterisation of additive CAs simulated by a given additive CA with radius 1. This section contains the most technical proofs of the manuscripts, and demonstrates the author's ability to bring complex demonstrations to a successful conclusion.

The work presented by Barbora Hudcová in her doctoral manuscript uses a wide variety of tools and methods, and the results obtained are of high quality and promising for the future. One of the remarkable qualities of this manuscript is that it systematically takes a step back: the results obtained are commented on, criticised and, where relevant, compared with those in the literature. I am convinced that Barbora Hudcová possesses all the qualities of a researcher, and that she will be able to demonstrate rigour and originality in her future research. In conclusion, I fully support the defence of the thesis.

On Thursday, March 14th 2024,
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